Adaptive Discontinuous Galerkin Methods for Fourth Order Problems

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This work is concerned with the derivation of adaptive methods for discontinuous Galerkin approximations of linear fourth order elliptic and parabolic partial differential equations.

Adaptive methods are usually based on a posteriori error estimates. To this end, a new residual-based a posteriori error estimator for discontinuous Galerkin approximations to the biharmonic equation with essential boundary conditions is presented. The estimator is shown to be both reliable and efficient with respect to the approximation error measured in terms of a natural energy norm, under minimal regularity assumptions. The reliability bound is based on a new recovery operator, which maps discontinuous finite element spaces to conforming finite element spaces (of two polynomial degrees higher), consisting of triangular or quadrilateral Hsieh-Clough-Tocher macroelements. The efficiency bound is based on bubble function techniques. The performance of the estimator within an $h$-adaptive mesh refinement procedure is validated through a series of numerical examples, verifying also its asymptotic exactness. Some remarks on the question of proof of convergence of adaptive algorithms for discontinuous Galerkin for fourth order elliptic problems are also presented.

Furthermore, we derive a new energy-norm a posteriori error bound for an implicit Euler time-stepping method combined with spatial discontinuous Galerkin scheme for linear fourth order parabolic problems. A key tool in the analysis is the elliptic reconstruction technique. A new challenge, compared to the case of conforming finite element methods for parabolic problems, is the control of the evolution of the error due to non-conformity. Based on the error estimators, we derive an adaptive numerical method and discuss its practical implementation and illustrate its performance in a series of numerical experiments.
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To Lumi and Marie
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Chapter I

Introduction

Fourth order initial and/or boundary-value problems arise in, among other disciplines, thin plate theories of elasticity, phase-field modelling, image processing and mathematical biology. For isotropic elastic behaviour of thin plates and membranes, popular models involve the biharmonic operator together with appropriate Dirichlet and Neumann boundary conditions. It is well-known that for most of these problems analytic solutions are not known explicitly.

This motivates use of numerical methods to approximate their solution. One class of such numerical methods, the Finite Element Methods (FEMs), are commonly used for solving partial differential equations (PDEs). FEMs are naturally applied when the PDE problem can be interpreted in variational form. Then, the FEM is a particular Galerkin projection, whereby the projection spaces are constructed as follows. The domain on which the partial differential equation is to be solved is divided into subregions (also commonly known as elements) and then on each subregion a function from a finite dimensional function space is used to approximate the exact solution. Polynomials constituting a finite dimensional basis on each element are frequently used to obtain the finite dimensional function space.

Various finite element methods have been proposed and tested for the fourth order elliptic problems: conforming methods in the 1960s (Argyris elements) and 1970s (HCT elements), to non-conforming and mixed FEMs proposed from the 1970s until recently. Conforming FEMs for fourth order problems require that the finite element space is a finite dimensional subspace of the Sobolev space $H^2(\Omega)$, where
$\Omega$ denotes the computational domain. To satisfy this conformity requirement, $C^1$-continuous elements have been, traditionally, introduced (see [40] for HCT elements or [35] and the references therein). The implementation of such finite element spaces is highly non-trivial, especially when high order basis functions are involved, or when $\Omega$ is a three-dimensional domain; consequently, they are rarely used in practice.

Another approach frequently employed in the literature is to rewrite the fourth order problem as a system of second-order problems and use mixed finite element methods (see, e.g., [26, 38, 42, 16, 61] and the references therein). Additionally, non-conforming FEMs have also been proposed for fourth order problems, cf. [35, 38] for a discussion of the classical (inconsistent) approaches, and the more recent works [42, 24], where $C^0$-elements are employed on consistent non-conforming discretizations.

In recent years, discontinuous Galerkin (DG) finite element methods, a certain class of non-conforming methods, have been receiving considerable attention as flexible and efficient discretizations for a large class of problems ranging from computational fluid dynamics to computational mechanics and electromagnetic theory; see, e.g., [36, 11, 64, 83] and the references therein. The practical interest in DG methods owes to their flexibility in mesh design and adaptivity, in that they cover meshes with hanging nodes and/or locally varying polynomial degrees. DG methods are thus ideally suited for $hp$-adaptivity and provide good local conservation properties of the state variable. Moreover, in DG methods the local elemental bases can be chosen freely for the absence of interelement continuity requirements, yielding very sparse—in many cases even diagonal—mass matrices even with high precision quadrature. Note also that DG methods are popular due to their very good stability properties in transport- or convection-dominated problems [36]. The first “modern” DG method for elliptic problems was presented in [12] by Baker (which included a formulation for fourth order boundary-value problems as a special case). Recently, $hp$-version interior penalty DG finite element methods for the biharmonic problem have been derived in [77, 91, 78, 55], where the stability and a priori error bounds are presented in various settings. The above interior penalty methods are based on the “divergence formulation”. Also, [42, 62, 24] and in the recent work [51] on DG methods for the Cahn-Hilliard problem, are concerned with the a priori analysis of interior penalty-type methods for fourth order problems employing “plate formulations”.

Discontinuous Galerkin methods have been studied widely in the literature in the context of parabolic problems (for general description of DG applied to the parabolic problems, see [93, 83]). We remark that the first discontinuous Galerkin finite element method by Reed and Hill, [82], was introduced for the numerical solution of first-order hyperbolic problems. Discontinuous Galerkin time discretization in the context of parabolic problems was first studied in [45] by Eriksson, Johnson and Thomée. Spatial discretization of second order parabolic problems by DG methods has been studied recently in work by Song, [87], where a priori error bounds for nonlinear parabolic problems are derived in various norms for a method consisting of forward Euler method in time and DG in space. Also recently, a priori error analysis for parabolic (and elliptic) second order problems is presented in work by Burman and Stamm, [29], for a method consisting of backward Euler method in time and DG method in space where bubble stabilisation techniques are utilised in the DG method.

There are at least two types of estimates for errors of finite element approximations. A priori error bounds for FEMs give an upper bound on the error which depends on the size of the subregions, $h$, the polynomial degree used, $p$, and the exact solution, $u$, of the equation. An a priori error bound could be described as,

$$ ||u - u_h||_A \leq C_0 \frac{h^\alpha}{p^\beta} ||u||_B, $$

where $u_h$ is the finite element solution, $C_0 > 0$ a constant depending only on the subdivision parameters, and the error (left hand side of (1.1)) is measured in some norm $A$ and is bounded from above by some norm $B$ of $u$ for some exponents $\alpha$ and $\beta$ of $h$ and $p$. As the exact solution is not known in general, the a priori error estimates do not give a directly computable upper bound on the error. However, the a priori error bounds are of theoretical importance in that they provide information on the approximation capabilities of the FEM in question. We refer reader to the books [90, 35, 23, 83] and the references therein for classical a priori error bounds for FEM. On the other hand, rigorous a posteriori error estimates for FEMs give computable upper bounds on the error (at least up to an unknown constant) which depend on the size of the subregions, $h$, on the polynomial degree used, $p$, on the data of the equation, $f$, and on the approximate solution, $u_h$. An a posteriori error
bound could be described as,

\[ ||u - u_h||_A \leq C_1 E(u_h, h, p, f), \] (1.2)

where \( u_h \) is the finite element solution, \( C_1 > 0 \) a (possibly unknown) constant depending only on the subdivision parameters, and the error (left hand side of (1.2)) is measured in some norm \( A \) and is bounded from above by some norm \( B \) of \( u \) for some exponents \( \alpha \) and \( \beta \) of \( h \) and \( p \). The a posteriori functional \( E \) may also depend on the boundary condition data but this consideration is left out for the sake of simplicity here. Most importantly, an \textit{a posteriori} error bound, \( C_1 E \), does not depend on the exact solution of the equation. Amongst various types of a posteriori error estimates (see Ainsworth and Oden [5]), the ones that give a \textit{local a posteriori error estimate} on each subregion are perhaps the most important ones from the point of view of using them in an adaptive algorithm. In an adaptive algorithm, \textit{local a posteriori error estimates} are used directly in an automatic procedure to control the error by determining which of the subregions contribute to the total error most and should be smaller and/or the polynomial degree should be increased in order to gain a more accurate solution with optimal computational efficiency.

A posteriori bounds for finite element approximations of fourth order problems have been considered in a number of different contexts: in [94, 79, 2] the case of \( C^1 \)-conforming elements is treated, while in [15], da Veiga, Niiranen and Stenberg presented a posteriori error bounds for classical non-conforming methods using the Morley element. Recently, in [63], Larson and Hansbo derived reliable a posteriori bounds for a \( C^0 \)-interior penalty method for Kirchoff-Love plates employing Helmholtz decompositions of the error. Finally, a posteriori bounds for a \( C^0 \)-interior penalty method for the biharmonic problem using quadratic basis functions are presented in the recent work [21] by Brenner, Gudi and Sung.

Adaptive methods for partial differential equations (PDE’s) of evolution type have become a staple in improving the efficiency in large scale computations. Since the 1980’s many adaptive methods have been increasingly based on \textit{a posteriori error estimates}, which provide a sound mathematical case for \textit{adaptive mesh refinement}, which can be decomposed in spatially and temporally local \textit{error indicators}. In the context of parabolic equations, a posteriori error estimates have been derived for
various norms since early 1990’s [44, 81]. Inspired by the milestones set recently for the mathematical theory of convergence for adaptive methods in elliptic second order problems [76, 18, 32], there has been a recent push for similar results for parabolic second order problems calling to a closer understanding of a posteriori error estimates [34, 95, 17, 72, e.g.]. Most results in this area cover simple time-stepping schemes and a conforming space discretization. The extant literature on a posteriori error control for nonconforming spatial methods for second order problems can be grouped in a handful of works [92, 46, 97, 80, 98]. In work by Sun and Wheeler [92], a posteriori $L^2(H^1)$-norm error bounds for a spatially semidiscrete method via interior penalty discontinuous Galerkin (IPDG) methods are derived and used heuristically in the implementation of the fully discrete scheme. In [46, 97] $L^2(L^2)$-norm error bounds for IPDG are obtained using duality techniques, while in [80], a posteriori error bounds are presented for a fully discrete method consisting of a backward Euler time-stepping and linear Crouzeix–Raviart elements in space.

Moreover, advances in research focusing on a posteriori error analysis of fully discrete schemes with a discontinuous Galerkin discretization in space for second order parabolic problems and design of adaptive algorithms in this context has been made recently in, for example, [49, 57] but an extension to fourth order problems seems a relatively new field with its own problems due to slightly more involved design of the elements concerned for conforming methods (for adaptivity with HCT elements, see [89]) as well as error analytical problems created by the lack of conforming subspace for nonconforming methods. Note that none of the papers in the literature, to our knowledge, cover the case of a posteriori energy-norm error bounds for fully discrete schemes with discontinuous Galerkin methods for fourth order problems, which is one of the main objectives in this thesis. Nevertheless, at least some results have been obtained for fourth order parabolic problems; Feng and Wu obtained local residual a posteriori error bounds for semi-discrete method based on conforming and mixed discretization in space in [53] for the Cahn-Hilliard equation and Hele-Shaw flow problems. They also derived an adaptive method based on their estimators.
1.1 Overview

In this thesis we derive and numerically test a posteriori error bounds for fourth order elliptic and parabolic PDEs (partial differential equations) discretized by the discontinuous Galerkin method. We also outline practical implementation of a posteriori error bounds in adaptive algorithms and discuss issues related to the convergence analysis of an adaptive algorithm for elliptic fourth order problems.

In chapter 2, we define discontinuous Galerkin method for a model problem and discuss some of its properties as well as a priori error estimates. We also give numerous other definitions necessary for the rest of this thesis.

In chapter 3, we present reliable and efficient a posteriori bounds in the energy norm for the interior penalty discontinuous Galerkin method (IPDG) proposed in [91] for the biharmonic problem with essential boundary conditions, under minimal regularity assumptions on the analytical solution. The reliability bound is based on a suitable recovery operator, which maps discontinuous finite element spaces to $H^2_0$-conforming finite element spaces (of two polynomial degrees higher), consisting of triangular or quadrilateral macro-elements defined in [40] (cf. also [24, 68, 65] for similar constructions). Using this recovery operator, in conjunction with the inconsistent formulation for the IPDG presented in [55] (which ensures that the weak formulation of the problem is defined under minimal regularity assumptions on the analytical solution), the efficient and reliable a posteriori estimates of residual type for the IPDG method in the corresponding energy norm are derived. We conclude this chapter by demonstrating efficiency of the a posteriori estimators via some numerical examples.

In chapter 4, we discuss issues related to the convergence analysis of an adaptive discontinuous Galerkin algorithm for the biharmonic problem. The a posteriori error bounds are naturally utilised in the construction of adaptive algorithms but it is only very recently that certain adaptive algorithms using discontinuous Galerkin method for the second order problems have been proven to be convergent (see [69],[32]). The author is not aware of such work in the context of interior penalty methods for fourth order problems. The purpose of this chapter is to discuss difficulties present in the convergence proof for the adaptive discontinuous Galerkin method for the biharmonic problem. The discussion is inspired and influenced by the works
of Karakashian and Pascal [69], Cascon, Kreuzer, Nochetto and Siebert [32] and Stevenson [88] in the context of second order problems. It appears that the main difficulty in the fourth order case is to prove that the penalty jumps are bounded from above by a posteriori error estimator quantities which are guaranteed to reduce under some marking strategy. This question, as well as the convergence proof of adaptive DG algorithm for biharmonic equation remains an open question after writing this thesis. The discussion in this chapter is issued in the hope that it will act as a guide for future work on this interesting question.

In chapter 5, we derive new energy-norm a posteriori error bound for an implicit Euler time-stepping method combined with spatial discontinuous Galerkin scheme for linear fourth order parabolic problems. A key tool in the analysis is the elliptic reconstruction technique. A new challenge, compared to the case of conforming finite element methods for parabolic problems, is the control of the evolution of the error due to non-conformity.

In chapter 6, we derive an adaptive numerical method based on the error estimators, discuss its practical implementation and illustrate its performance in a series of numerical experiments.

In chapter 7, we conclude the thesis by looking at some possible future directions of further research.

1.2 Thesis results

The main results of this thesis are contained in [56, 57, 58] of which [56] is accepted for publication, [57] is in review and [58] is in preparation. We give details on some of the important points below.

1) A posteriori error analysis of discontinuous Galerkin approximation of fourth order elliptic equations.

- Proofs of a posteriori error lower and upper bounds for discontinuous Galerkin approximation of fourth order elliptic problems are presented. In other words, we show that proposed a posteriori estimator is reliable and efficient under minimal regularity assumptions on the data of problem.
1.2 Thesis results

- New mapping between nonconforming and conforming spaces is established enabling the proof of the upper bound. This mapping is also used in the proof of a posteriori error bounds for the parabolic problem.
- New bubble function techniques are applied in the proof of the lower bound.
- Numerical experiments are presented confirming the results.

2) A posteriori error analysis of discontinuous Galerkin approximation of fourth order parabolic equations.

- Using elliptic reconstruction technique, a proof of upper a posteriori error bound for implicit Euler - discontinuous Galerkin approximation of fourth order parabolic problems is presented.
- An error identity, enabled by the elliptic reconstruction technique, is derived to prove a posteriori upper bound for fully discrete approximation in $L^2(H^2)$ - type norm.
- Numerical experiments confirming the efficiency of the derived a posteriori estimators are presented.
- An adaptive algorithm based on the estimators is derived and its practical performance is presented via numerical experiments.
Chapter II

Discontinuous Galerkin Method

In this chapter we introduce the discontinuous Galerkin method for a model problem, discuss existence and uniqueness of solutions and also recall some other common definitions used throughout the rest of this thesis. We conclude this chapter by stating a priori error bounds for the symmetric interior penalty discontinuous Galerkin (SIPDG) method.

2.1 Sobolev Spaces

Let $\omega \subset \mathbb{R}^2$ be an open domain and $p \in [1, \infty]$. Then the norms $\| \cdot \|_{L^p(\omega)}$ for $L^p$ spaces (Lebesgue spaces) of real valued measurable functions are given by

**Definition 2.1 (L^p norms).**

\[
\| v \|_{L^p(\omega)} := \left( \int_\omega v^p \, dx \right)^{1/p} < +\infty, \quad \text{for} \quad 1 \leq p < +\infty,
\]

\[
\| v \|_{L^\infty(\omega)} := \text{ess sup}_{x \in \omega} \| v(x) \| < +\infty, \quad \text{for} \quad p = +\infty.
\]  

(2.1)

The norm of $L^2(\omega)$ will be denoted by $\| \cdot \|_\omega$ for brevity.

**Definition 2.2 (L^p spaces).** We define,

\[
L^p(\omega) := \{ u : \| u \|_{L^p(\omega)} < \infty \},
\]  

(2.2)
to be the Lebesgue space of real valued measurable functions with exponent $p$, $1 \leq p \leq \infty$.

Using a multiindex of order $|\alpha| = \alpha_1 + \ldots + \alpha_n = k$, we denote the weak derivative $u$ (for definition see, for example, [50] pp. 242) of order $k$ or $|\alpha|$ in $\mathbb{R}^2$ by

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}.$$  \hspace{1cm} (2.3)

In this work we shall use Hilbertian Sobolev spaces. We define a norm and semi-norm on such spaces by

$$||u||_{H^k(\omega)} := \left( \sum_{0 \leq i \leq k} \sum_{|\alpha| = i} ||D^\alpha u||^2_{L^2(\omega)} \right)^{1/2}$$

and

$$|u|_{H^k(\omega)} := \left( \sum_{|\alpha| = k} ||D^\alpha u||^2_{L^2(\omega)} \right)^{1/2}$$

respectively. We also often write $|u|_{k,\omega} := |u|_{H^k(\omega)}$ for semi-norms.

**Definition 2.3 ($H^k$ spaces).** We define,

$$H^k(\omega) := \{ u : ||u||_{H^k(\omega)} < \infty \},$$  \hspace{1cm} (2.5)

to be the (Hilbertian) Sobolev space of order $k$.

**Definition 2.4 ($H^k_0$ spaces).** We define,

$$H^k_0(\omega) := \{ u : ||u||_{H^k(\omega)} < \infty, \text{ and } D^\alpha u|_{\partial \omega} = 0 \text{ for } |\alpha| \leq k - 1 \},$$

(2.6)

to be the (Hilbertian) Sobolev space of order $k$ of functions with vanishing traces.

$\partial \omega$ denotes the boundary of $\omega$.

We now give definition for dual norm of Sobolev spaces,

$$||u||_{H^{-k}(\omega)} := \sup_{v \in H^k_0(\omega)} \frac{\langle u, v \rangle_{L^2(\omega)}}{||v||_{H^k(\omega)}}$$

(2.7)
where \( \langle u, v \rangle_{L^2(\omega)} = \int_{\omega} u \, v \, dx \) denotes the standard \( L^2 \) inner product. We often denote the \( L^2 \) product and norm by \( \langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(\Omega)} \) and \( ||\cdot|| := ||\cdot||_{L^2(\Omega)} \) respectively on \( \Omega \).

**Definition 2.5 \((H^{-k} \text{ spaces})\).** We define,

\[
H^{-k}(\omega) := \{ u : ||u||_{H^{-k}(\omega)} < \infty \}. \tag{2.8}
\]

to be the dual space of a Sobolev space of order \( k \) of functions with vanishing traces.

We remark that it is also possible to define Sobolev spaces of fractional order based on function space interpolation. We refer reader to [1] for full details.

Finally, we describe norms and spaces needed for time dependent problems. For \( 1 \leq p \leq +\infty \), we define the spaces \( L^p(0, T; X) \), with \( X \) being a real Banach space with norm \( ||\cdot||_X \), consisting of all measurable functions \( v : [0, T] \to X \), for which

\[
||v||_{L^p(0, T; X)} := \left( \int_0^T ||v(t)||_X^p \, dt \right)^{1/p} < +\infty, \quad \text{for} \quad 1 \leq p < +\infty,
\]

\[
||v||_{L^\infty(0, T; X)} := \text{ess sup}_{0 \leq t \leq T} ||v(t)||_X < +\infty, \quad \text{for} \quad p = +\infty. \tag{2.9}
\]

We denote by \( C(0, T; X) \), and \( C^{0,1}(0, T; X) \), the spaces of continuous, respectively Lipschitz-continuous, functions \( v : [0, T] \to X \) with bounded norms

\[
||v||_{C(0, T; X)} := \max_{0 \leq t \leq T} ||v(t)||_X
\]

\[
||v||_{C^{0,1}(0, T; X)} := \max\{||v||_{C(0, T; X)}, ||v'||_{C(0, T; X)}\} \tag{2.10}
\]

where the time derivative \( v' \) is understood almost everywhere.

### 2.2 Model Problem

Let \( \Omega \) be a bounded open polygonal domain in \( \mathbb{R}^2 \), and let \( \Gamma_\partial \) denote its boundary. We consider the fourth order equation

\[
\Delta^2 u = f \quad \text{in} \quad \Omega, \tag{2.11}
\]
where \( f \in L^2(\Omega) \) with homogeneous essential boundary conditions

\[
\begin{align*}
u = 0 & \quad \text{on } \Gamma, \\
\nabla u \cdot n = 0 & \quad \text{on } \Gamma,
\end{align*}
\]

(2.12)

where \( n \) denotes the unit outward normal vector to \( \Gamma \). We multiply both sides of (2.11) with a function \( v \in H^0_0(\Omega) \), perform integration by parts twice on the left hand side and apply the boundary conditions to obtain weak formulation of the problem (2.11):

Find \( u \in H^2_0(\Omega) \) such that

\[
\langle \Delta u, \Delta v \rangle = \langle f, v \rangle \quad \text{for all } v \in H^2_0(\Omega).
\]

(2.13)

We denote the bilinear form on the left hand side of (2.13) by \( a(u, v) := \langle \Delta u, \Delta v \rangle \) and the linear functional on the right hand side of (2.13) by \( l(v) := \langle f, v \rangle \). Then, by Cauchy-Schwarz inequality we have that the bilinear form \( a(\cdot, \cdot) \) is continuous, i.e. there exists a constant \( C_1 > 0 \) such that,

\[
|a(u, v)| \leq C_1 ||u||_{H^2(\Omega)} ||v||_{H^2(\Omega)} \quad \text{for all } u, v \in H^2_0(\Omega),
\]

(2.14)

and by applying the Poincaré Friedrichs inequality (see Lemma C.4) together with the fact that \( |u|_{2,\Omega} \leq C ||\Delta u||_\Omega \) for all \( u \in H^2_0(\Omega) \) for some constant \( C \) independent of \( \Omega \) we get that the bilinear form \( a(\cdot, \cdot) \) is also coercive, i.e. there exists a constant \( C_2 > 0 \) such that,

\[
a(u, u) \geq C_2 ||u||_{H^2(\Omega)}^2 \quad \text{for all } u \in H^2_0(\Omega).
\]

(2.15)

Using Riesz Representation Theorem (see, for example, [50]) it can be shown that there exist unique \( u \in H^2_0(\Omega) \) that solves (2.13).

REMARK 2.1. Here the assumption that \( f \in L^2(\Omega) \) could be changed as it can be shown (see, for example, [59] or [54]) that solutions \( u \in H^{2+\alpha}(\Omega) \cap H^2_0(\Omega) \) to (2.13) exist and are unique whenever for the linear functional holds \( f \in H^{-2+\alpha} \) for \( \alpha \in [0, 1] \).

REMARK 2.2. For proofs of existence and uniqueness of solutions to (2.13) with
2.3 Trace Operators

Let $\kappa^+, \kappa^-$ be two (generic) elements sharing an edge $e := \partial \kappa^+ \cap \partial \kappa^- \subset \Gamma_{\text{int}}$. Define the outward normal unit vectors $\mathbf{n}^+$ and $\mathbf{n}^-$ on $e$ corresponding to $\partial \kappa^+$ and $\partial \kappa^-$, respectively. For functions $v : \Omega \to \mathbb{R}$, $q : \Omega \to \mathbb{R}^2$, and $Q : \Omega \to \mathbb{R}^{2 \times 2}$ that may be discontinuous across $\Gamma$, we define the following quantities. For $v^+ := v|_{e \subset \partial \kappa^+}$, $v^- := v|_{e \subset \partial \kappa^-}$, $q^+ := q|_{e \subset \partial \kappa^+}$, $q^- := q|_{e \subset \partial \kappa^-}$, and $Q^+ := Q|_{e \subset \partial \kappa^+}$, $Q^- := Q|_{e \subset \partial \kappa^-}$, we set

\[
\{v\} := \frac{1}{2} (v^+ + v^-), \quad \{q\} := \frac{1}{2} (q^+ + q^-), \quad \{Q\} := \frac{1}{2} (Q^+ + Q^-),
\]

and

\[
[v] := q^+ \mathbf{n}^+ + q^- \mathbf{n}^-, \quad [q] := q^+ \cdot \mathbf{n}^+ + q^- \cdot \mathbf{n}^-,
\]

and

\[
\|q\| := q^+ \otimes \mathbf{n}^+ + q^- \otimes \mathbf{n}^-, \quad [Q] := Q^+ \mathbf{n}^+ + Q^- \mathbf{n}^-,
\]

where $\otimes$ denotes the standard tensor product operator, whereby $q \otimes w = qw^T$; if $e \in \partial \kappa \cap \Gamma_{\partial}$, these definitions are modified as follows

\[
\{v\} := v^+, \quad \{q\} := q^+, \quad \{Q\} := Q^+, \quad [v] := v^+ \mathbf{n}, \quad [q] := q^+ \cdot \mathbf{n}, \quad \|q\| := q^+ \otimes \mathbf{n}, \quad [Q] := Q^+ \mathbf{n}.
\]

2.4 Finite Element Spaces And Meshes

Let $T$ be a conforming (i.e. contains no hanging nodes) subdivision of $\Omega$ into disjoint triangular or quadrilateral elements $\kappa \in T$. We assume that the subdivision $T$ is constructed via affine element mappings $F_\kappa$, where $F_\kappa : \hat{\kappa} \to \kappa$, with non-singular Jacobian, where $\hat{\kappa}$ is the reference triangle or quadrilateral. The above mappings are assumed to be constructed so as to ensure $\bar{\Omega} = \bigcup_{\kappa \in T} \bar{\kappa}$ and that the elemental edges are straight line segments.

Also, for a nonnegative integer $r$, we denote by $P_r(\hat{\kappa})$, the set of all polynomials
of total degree at most \( r \), if \( \hat{\kappa} \) is the reference triangle, or the set of all tensor-product polynomials on \( \hat{\kappa} \) of degree at most \( r \) in each coordinate direction, if \( \hat{\kappa} \) is the reference quadrilateral. For \( r \geq 2 \), we consider the finite element space

\[
S^r := \{ v \in L^2(\Omega) : v|_\kappa \circ F_\kappa \in P_r(\hat{\kappa}), \kappa \in T \}.
\]

The \( L^2 \)-projection operator \( \Pi : L^2 \to S^r \) is defined for any \( w \in L^1(\Omega) \) by,

\[
\langle \Pi w, v \rangle = \langle w, v \rangle \quad \text{for all } v \in S^r.
\]

By \( \Gamma \) we denote the union of all one-dimensional element edges associated with the subdivision \( T \) (including the boundary). Further we decompose \( \Gamma \) into two disjoint subsets \( \Gamma = \Gamma_{\partial} \cup \Gamma_{\text{int}} \), where \( \Gamma_{\text{int}} := \Gamma \setminus \Gamma_{\partial} \).

We define \( h_\kappa := \text{diam}(\kappa) \) and we collect them into the element-wise constant function \( h : \Omega \to \mathbb{R} \), with \( h|_\kappa = h_\kappa, \kappa \in T \) and \( h = \{ h \} \) on \( \Gamma \). We shall assume throughout that the families of meshes considered are locally quasiuniform, i.e., there exists constant \( c \geq 1 \), independent of \( h \), such that, for any pair of elements \( \kappa^+ \) and \( \kappa^- \) in \( T \) which share an edge,

\[
c^{-1} \leq h_{\kappa^+}/h_{\kappa^-} \leq c.
\]

Let \( \rho_\kappa := \sup\{ \text{diam}(S) : S \text{ is a ball contained in } \kappa \} \). We assume that the subdivision \( T \) is shape-regular (see, e.g., p.124 in [35]), i.e. there exists a constant \( C_{\text{reg}} \) such that

\[
\frac{h_\kappa}{\rho_\kappa} \leq C_{\text{reg}} \quad \text{for all } \kappa \in T,
\]

and also that the Jacobian determinant of the affine element mappings \( F_\kappa \) satisfy the following

\[
C_1 \rho_\kappa^2 \leq \det(DF_\kappa) \leq C_2 h_\kappa^2 \quad \text{for all } \kappa \in T.
\]

for constants \( C_1, C_2 > 0 \).
2.4.1 Broken Sobolev spaces

With the definition of the subdivision $T$, which is frequently also called the mesh, we define the broken Laplacian and the broken gradient, $\Delta_h u$ and $\nabla u$, given by,

\[
(\Delta_h u)|_{\kappa} = \Delta(u|_{\kappa}) \quad \forall \kappa \in T \quad \text{and} \\
(\nabla_h u)|_{\kappa} = \nabla(u|_{\kappa}) \quad \forall \kappa \in T,
\]

respectively.

**Definition 2.6 (Broken Sobolev spaces).** We define

\[
H^k(\omega, T) := \{ u : u|_{\kappa} \in H^k(\kappa) \quad \text{for all} \quad \kappa \in T \}.
\] (2.17)

to be the broken Sobolev spaces of order $k$.

2.5 The discontinuous Galerkin finite element method

Discontinuous Galerkin methods (DG methods) for biharmonic problem are derived in [77] and in [55] (see also [24] for derivation of $C^0$ - interior penalty methods for elliptic fourth order problems). We repeat here the derivation of the method given in [55] for the sake of completeness. Note that we derive the DG method here for non-homogeneous boundary conditions but we will consider the problem (2.11) with boundary conditions (2.12) throughout the thesis for simplicity. We begin by writing the boundary-value problem (3.1), with non-homogeneous boundary conditions

\[
u = g_D \quad \text{on} \quad \Gamma_\partial, \\
\nabla u \cdot n = g_N \quad \text{on} \quad \Gamma_\partial,
\] (2.18)
as a first-order system:

\[
t = \nabla u, \quad s = \nabla \cdot t, \quad q = \nabla s, \quad \nabla \cdot q = f, \quad \text{in} \ \Omega.
\] (2.19)

\[
u = g_D, \quad \nabla u \cdot n = g_N \quad \text{on} \quad \Gamma_\partial.
\] (2.20)
We multiply the first and third equations by vector test functions \( z \) and \( r \), respectively, and the second and fourth equations by scalar test functions \( w \) and \( v \), respectively; then, integrating over every \( \kappa \in T \), and upon formal integration by parts on every element \( \kappa \in T \) on each equation, we obtain

\[
\begin{align*}
\int_{\kappa} t \cdot z &= -\int_{\kappa} u \nabla \cdot z + \int_{\partial \kappa} u n \cdot z, \\
\int_{\kappa} sw &= -\int_{\kappa} t \cdot \nabla w + \int_{\partial \kappa} w n \cdot t, \\
\int_{\kappa} q \cdot r &= -\int_{\kappa} s \nabla \cdot r + \int_{\partial \kappa} s n \cdot r, \\
\int_{\kappa} fv &= -\int_{\kappa} q \cdot \nabla v + \int_{\partial \kappa} v n \cdot q,
\end{align*}
\]

where \( n \) denotes the unit outward normal unit vector to \( \partial \kappa \). We restrict the choice of the trial and test functions \( u, t, s, q \) and \( z, w, r, v \), respectively, to finite-dimensional subspaces. More specifically, we seek \( u_h, s_h \in S_1 \equiv S^r \) and \( t_h, q_h \in S_2 := [S_1]^2 \) such that

\[
\begin{align*}
\int_{\kappa} t_h \cdot z_h &= -\int_{\kappa} u_h \nabla \cdot z_h + \int_{\partial \kappa} \hat{u} n \cdot z_h, \\
\int_{\kappa} s_h w_h &= -\int_{\kappa} t_h \cdot \nabla w_h + \int_{\partial \kappa} w_h n \cdot \hat{t}, \\
\int_{\kappa} q_h \cdot r_h &= -\int_{\kappa} s_h \nabla \cdot r_h + \int_{\partial \kappa} \hat{s} n \cdot r_h, \\
\int_{\kappa} fv_h &= -\int_{\kappa} q_h \cdot \nabla v_h + \int_{\partial \kappa} v_h n \cdot \hat{q},
\end{align*}
\]

for \( w_h, v_h \in S_1 \) and \( z_h, r_h \in S_2 \). The numerical fluxes \( \hat{u}, \hat{t}, \hat{s}, \) and \( \hat{q} \) are approximations to \( u, \nabla u, \Delta u, \) and \( \nabla \Delta u \), respectively. These numerical fluxes can be chosen freely which gives rise to various discontinuous Galerkin methods.

We apply again the divergence theorem on (2.21), to obtain

\[
\int_{\kappa} t_h \cdot z_h = \int_{\kappa} \nabla u_h \cdot z_h + \int_{\partial \kappa} (\hat{u} - u_h) n \cdot z,
\]

(2.25)

Setting \( z_h = \nabla w_h \) in (2.25), and using the resulting equality to eliminate \( t_h \) from
2.5 The discontinuous Galerkin finite element method

\[ \int_{\kappa} s_h w_h = - \int_{\kappa} \nabla u_h \cdot \nabla w_h - \int_{\partial \kappa} (\hat{u} - u_h) \mathbf{n} \cdot \nabla w_h + \int_{\partial \kappa} w_h \mathbf{n} \cdot \hat{t} \quad (2.26) \]

Application of the divergence theorem to the first term on the right-hand side of (2.26), yields

\[ \int_{\kappa} s_h w_h = \int_{\kappa} \Delta_h u_h w_h - \int_{\partial \kappa} (\hat{u} - u_h) \mathbf{n} \cdot \nabla w_h + \int_{\partial \kappa} w_h \mathbf{n} \cdot (\hat{t} - \nabla u_h). \quad (2.27) \]

We recall the identity (see also Appendix A)

\[ \sum_{\kappa \in T} \int_{\partial \kappa} \psi \phi \cdot \mathbf{n} = \int_{\Gamma} [\psi] \cdot \{\phi\} \, ds + \int_{\Gamma_{\text{int}}} \{\psi\} [\phi] \, ds, \quad (2.28) \]

with \( \psi \in H^1(\Omega, T) \) and \( \phi \in [H^1(\Omega, T)]^2 \). Summing up over all elements \( \kappa \in T \), and using this identity on the second and third term on the right-hand side of (2.27), yields

\[ \int_{\Omega} s_h w_h = \int_{\Omega} \Delta_h u_h w_h - \int_{\Gamma} [\hat{u} - u_h] \cdot \{\nabla w_h\} \, ds + \int_{\Gamma_{\text{int}}} \{\hat{u} - u_h\} [\nabla w_h] \, ds \\
+ \int_{\Gamma} [w_h] \cdot (\hat{t} - \nabla u_h) \, ds + \int_{\Gamma_{\text{int}}} \{w_h\} (\hat{t} - \nabla u_h) \, ds. \quad (2.29) \]

Choosing

\[ \hat{u} = \begin{cases} 
\{u_h\}, & \text{if } e \subset \Gamma_{\text{int}}; \\
g_D, & \text{if } e \subset \Gamma_{\partial},
\end{cases} \quad \text{and} \quad \hat{t} = \begin{cases} 
\{\nabla u_h\}, & \text{if } e \subset \Gamma_{\text{int}}; \\
g_N \mathbf{n}, & \text{if } e \subset \Gamma_{\partial},
\end{cases} \]

equation (2.29) gives

\[ \int_{\Omega} s_h w_h = \int_{\Omega} \Delta_h u_h w_h + \int_{\Gamma_{\text{int}}} \left( [u_h] \cdot \{\nabla w_h\} - \{w_h\} [\nabla u_h] \right) \\
+ \int_{\Gamma_{\partial}} \left( (u_h - g_D)(\nabla w_h \cdot \mathbf{n}) + w_h (g_N - \nabla u_h \cdot \mathbf{n}) \right), \quad (2.30) \]

where \( \Delta_h \) denotes the broken Laplacian with respect to the subdivision \( T \).

Now, we combine (2.23) and (2.24). Setting \( r_h = \nabla v_h \) in (2.23) and substituting
into (2.24), we deduce
\[ \int_{\kappa} f v_h = \int_{\kappa} s_h \Delta v_h - \int_{\partial \kappa} \hat{s} (\nabla v_h \cdot n) + \int_{\partial \kappa} v_h \hat{q} \cdot n. \] (2.31)

Summing over all elements \( \kappa \in \mathcal{T} \), setting \( w_h = \Delta v_h \) into (2.30), and inserting this into (2.31) we get
\[ \int_{\Omega} f v_h = \int_{\Omega} \Delta h u_h \Delta v_h + \int_{\Gamma_{\text{int}}} \left( [u_h] \cdot \{\nabla \Delta v_h \} - \{\Delta v_h\} [\nabla u_h]\right) \\
+ \int_{\Gamma_{\partial}} \left( (u_h - g_D) (\nabla \Delta v_h \cdot n) + \Delta v_h (g_N - \nabla u_h \cdot n) \right) \\
- \sum_{\kappa \in \mathcal{T}} \int_{\partial \kappa} \hat{s} (\nabla v_h \cdot n) + \sum_{\kappa \in \mathcal{T}} \int_{\partial \kappa} v_h \hat{q} \cdot n. \] (2.32)

Making use of (2.28) on the fourth and fifth terms on the right-hand side of (2.32), we deduce
\[ \int_{\Omega} f v_h = \int_{\Omega} \Delta h u_h \Delta v_h + \int_{\Gamma_{\text{int}}} \left( [u_h] \cdot \{\nabla \Delta v_h \} - \{\Delta v_h\} [\nabla u_h]\right) \\
+ \int_{\Gamma_{\partial}} \left( (u_h - g_D) (\nabla \Delta v_h \cdot n) + \Delta v_h (g_N - \nabla u_h \cdot n) \right) \\
- \int_{\Gamma} [\hat{s}] \cdot \{\nabla v_h\} \, ds - \int_{\Gamma_{\text{int}}} \{\hat{s}\} [\nabla v_h] \, ds \\
+ \int_{\Gamma} [v_h] \cdot \{\hat{q}\} \, ds + \int_{\Gamma_{\text{int}}} \{v_h\} [\hat{q}] \, ds. \] (2.33)

For what follows, we define the piecewise constant interior penalty parameters \( \sigma, \xi : \Gamma \rightarrow \mathbb{R} \) by
\[ \sigma|_e = \sigma_0 (h|_e)^{-3}, \quad \xi|_e = \xi_0 (h|_e)^{-1}, \] (2.35)
respectively, where \( \sigma_0 > 0 \) and \( \xi_0 > 0 \). To guarantee the stability of the IPDG method defined in (2.39), \( \sigma_0 \) and \( \xi_0 \) must be selected sufficiently large, see 2.2 for details.

Choosing
\[ \hat{s} = \begin{cases} \\
\{\Delta u_h\} - \xi [\nabla u_h], & \text{if } e \subset \Gamma_{\text{int}}; \\
\Delta u_h - \xi (\nabla u_h \cdot n - g_N), & \text{if } e \subset \Gamma_{\partial}, \\
\end{cases} \]
and
\[
\hat{q} = \begin{cases} 
\{ \nabla \Delta u_h \} + \sigma [u_h], & \text{if } e \subset \Gamma_{\text{int}}; \\
\nabla \Delta u_h + \sigma (u_h - g_D) \mathbf{n}, & \text{if } e \subset \Gamma_{\partial},
\end{cases}
\]
equation (2.33) gives
\[
\begin{align*}
\int_{\Omega} \Delta_h u_h \Delta_h v_h \\
+ \int_{\Gamma_{\partial}} \left( u_h (\nabla \Delta v_h \cdot \mathbf{n}) + v_h (\nabla \Delta u_h \cdot \mathbf{n}) - \Delta v_h (\nabla u_h \cdot \mathbf{n}) - \Delta u_h (\nabla v_h \cdot \mathbf{n}) \right) \\
+ \int_{\Gamma_{\partial}} \left( \sigma u_h v_h + \xi (\nabla u_h \cdot \mathbf{n}) (\nabla v_h \cdot \mathbf{n}) \right) + \int_{\Gamma_{\text{int}}} \left( \sigma [u_h] [v_h] + \xi [\nabla u_h] [\nabla v_h] \right) \\
+ \int_{\Gamma_{\text{int}}} \left( [u_h] \cdot \{ \nabla \Delta v_h \} + [v_h] \cdot \{ \nabla \Delta u_h \} - \{ \Delta v_h \} [\nabla u_h] - \{ \Delta u_h \} [\nabla v_h] \right)
\end{align*}
\]
= \int_{\Omega} f v_h + \int_{\Gamma_{\partial}} \left( g_D (\nabla \Delta v_h \cdot \mathbf{n} + \sigma v_h) + g_N (\xi \nabla v_h \cdot \mathbf{n} - \Delta v_h) \right).
\tag{2.36}
\]

Recalling the conventions \([v]_e = v \mathbf{n}, [r]_e = r \cdot \mathbf{n}, \{v\}_e = v, \{r\}_e = r\), (2.36) can be written in the compressed form
\[
\begin{align*}
\int_{\Omega} \Delta_h u_h \Delta_h v_h & \\
+ \int_{\Gamma} \left( [u_h] \cdot \{ \nabla \Delta v_h \} + [v_h] \cdot \{ \nabla \Delta u_h \} - \{ \Delta v_h \} [\nabla u_h] - \{ \Delta u_h \} [\nabla v_h] \right) \\
+ \int_{\Gamma} \left( \sigma [u_h] [v_h] + \xi [\nabla u_h] [\nabla v_h] \right)
\end{align*}
\]
= \int_{\Omega} f v_h + \int_{\Gamma_{\partial}} \left( g_D (\nabla \Delta v_h \cdot \mathbf{n} + \sigma v_h) + g_N (\xi \nabla v_h \cdot \mathbf{n} - \Delta v_h) \right).
\tag{2.37}
\]

Upon defining the lifting operator \( \mathcal{L} : S := S_1 + H^2(\Omega) \rightarrow S_1 \) by
\[
\int_{\Omega} \mathcal{L}(v) r = \int_{\Gamma} \left( [v] \cdot \{ \nabla r \} - \{ r \} [\nabla v] \right) \forall r \in S_1,
\tag{2.38}
\]
and the boundary lifting by \( \mathcal{G} \in S_1 \)
\[
\int_{\Omega} \mathcal{G} r = \int_{\Gamma_{\partial}} \left( g_D (\nabla r \cdot \mathbf{n}) - g_N r \right) \forall r \in S_1,
\]
equation (2.37) gives rise to the symmetric interior penalty DGFEM (SIPDG):

\[ \text{find } u_{DG} \in S_1 \text{ such that } B(u_{DG}, v) = l(v) \quad \forall v \in S_1, \]  

(2.39)

where the bilinear form \( B(\cdot, \cdot) \) and the linear form \( l(\cdot) \) are given by

\[
B(u_h, v_h) = \int_{\Omega} \left( \Delta_h u_h \Delta_h v_h + \mathcal{L}(u_h) \Delta_h v_h + \Delta_h u_h \mathcal{L}(v_h) \right) + \int_{\Gamma} \left( \sigma [u_h] [v_h] + \xi [\nabla u_h] [\nabla v_h] \right)
\]

and

\[
l(v_h) = \int_{\Omega} \left( f v_h + \mathcal{G} \Delta v_h \right) + \int_{\Gamma_D} \left( \sigma g_D v_h + \xi g_N (\nabla v_h \cdot n) \right).
\]

(2.40)

(2.41)

Note that this formulation is inconsistent for trial and test functions from the solution space \( S \). However, in comparison to the symmetric version interior penalty method presented in Süli & Mozolevski [91] for which the bilinear form in most general form is given by,

\[
\tilde{B}(u, v) = \int_{\Omega} \Delta_h u_h \Delta_h v_h \, dx + \int_{\Gamma} \left( \lambda_1 [u_h] \cdot \{ \nabla \Delta v_h \} + \{ v_h \} \cdot \{ \nabla \Delta u_h \} \right) ds
\]

\[
- \int_{\Gamma} \left( \lambda_2 \{ \Delta v_h \} [\nabla u_h] + \{ \Delta u_h \} [\nabla v_h] \right) ds + \int_{\Gamma} (\sigma [u] \cdot [v] + \xi [\nabla u] [\nabla v]) ds
\]

(2.42)

where \( \lambda_1, \lambda_2 \in [-1, 1] \), the bilinear form of the SIPDG method defined in (2.39) coincides with the symmetric version of this interior penalty method with \( \lambda_1 = \lambda_2 = 1 \), when the trial and test functions belong to the finite element space.

\textbf{Remark 2.3.} \textit{By using an alternative integration by parts formula (see Appendix...}
A) it is possible define a DG method with the following bilinear form:

\[
\hat{B}(u, v) = \int_{\Omega} D^2u : D^2v \, dx \\
+ \int_{\Gamma} ([u] \cdot \{\nabla \Delta v\} + [v] \cdot \{\nabla \Delta u\}) \, ds \\
- \int_{\Gamma} \{D^2u\} : [\nabla v] - \{D^2v\} : [\nabla u] \, ds \\
+ \int_{\Gamma} (\sigma [u] \cdot [v] + \xi [\nabla u] [\nabla v]) \, ds
\]  

(2.43)

where \(D^2u\) denotes the Hessian of \(u\). This formulation is often called “plate formulation” (Obviously, plate models in physical sciences are somewhat more complicated having various model parameters acting on terms present in the “plate” formulation given above). There are distinct benefits of using this formulation instead of the “divergence formulation” of definition (2.40) from the analytical point of view. One of them being that the associated term \(||D^2u||_\Omega\) defines a semi-norm on \(H^2(\Omega)\) whereby \(||\Delta u||_\Omega\) does not. However, most of the results and proofs (subject to minor modifications) in this thesis also hold for this formulation.

In the following, we define the energy norm associated with \(B(\cdot, \cdot)\) for any function \(w \in S\) by,

\[
|||w||| = (||\Delta_h w||^2_\Omega + ||\sqrt{\gamma}[w]||^2_\Gamma + ||\sqrt{\delta}[\nabla w]||^2_\Gamma)^{\frac{1}{2}}.
\]  

(2.44)

We now state various key results for the (SIPDG) method defined above proofs of which can be found in [55] or [91]. Firstly, we have the following stability result for the lifting operator \(L\) defined in (2.38):

**Lemma 2.1.** For any \(w \in S\),

\[
||L(w)||^2_\Omega \leq C (||\sqrt{\gamma_1}[w]||^2_\Gamma + ||\sqrt{\delta_1}[\nabla w]||^2_\Gamma),
\]  

(2.45)

where \(C\) is a positive constant, independent of \(u_h\), \(h\), and \(\gamma_1, \delta_1: \Gamma \to \mathbb{R}\) are piecewise constant functions defined by \(\gamma_1 = C_\gamma h^{-3}\) and \(\delta_1 = C_\delta h^{-1}\) where \(C_\gamma, C_\delta > 0\) depend only on mesh parameters (and polynomial degree, \(r\), of \(S^r\)).

**Proof.** The proof of this result can be found in [55].
2.5 The discontinuous Galerkin finite element method

For the bilinear form $B(\cdot, \cdot)$ given by 2.40 we have the continuity and coercivity;

**Lemma 2.2.** Let $\sigma_0 > 4\gamma_1$ and $\xi_0 > 4\delta_1$ (with $\gamma_1$ and $\delta_1$ as in Lemma 2.1 above). Then, there exist positive constants $C_{\text{cont}}$ and $C_{\text{coer}}$, depending only on the mesh parameters such that

$$|B(u, v)| \leq C_{\text{cont}} |||u||| |||v|||, \forall u, v \in S^r + H^2_0(\Omega) \quad \text{and} \quad (2.46)$$

$$B(u, u) \geq C_{\text{coer}} |||u|||^2, \forall u \in S^r + H^2_0(\Omega). \quad (2.47)$$

**Proof.** The proof of this result can be found in [55].

As the bilinear form is continuous and coercive and the linear functional $l(\cdot)$ is bounded, we can use the Lax-Milgram Theorem to show that there exist a unique solution $u_{\text{DG}} \in S^r$ to the problem (2.39).

The following Theorem is a key result for the (SIPDG) method as it shows that the method converges upon $h$-refinement and we state it here for fixed polynomial degrees $r$ in $S^r$. The original result is more general in that it concerns varying polynomial degrees over elements.

**Theorem 2.1 (A priori error bound).** Assume that for the solution $u$ of (2.13) holds $u|_{\kappa} \in H^{k_{\kappa}}(\kappa)$, $k_{\kappa} \geq 2$, $\kappa \in T$ and for the penalty parameters holds $\sigma_0 > 4\gamma_1$ and $\xi_0 > 4\delta_1$. Then, the following energy error bound holds,

$$|||u - u_{\text{DG}}|||^2 \leq C \sum_{\kappa \in T} \left( h_{\kappa}^{2s_{\kappa}} |u|_{s_{\kappa} + 2, \kappa}^2 + h_{\kappa}^{2t_{\kappa} + 2} |u|_{t_{\kappa} + 3, \kappa}^2 \right), \quad (2.48)$$

where $1 \leq s_{\kappa} \leq \min\{r - 1, k_{\kappa}\}$, $1 \leq t_{\kappa} \leq \min\{r, k_{\kappa} - 1\}$, $r \geq 2$, $\kappa \in T$ and the constant $C$ is independent of $u$ and $h_{\kappa}$.

**Proof.** The proof of this result can be found in [55].

Finally, we would like to remark that the important property of Galerkin orthogonality,

$$B(u - u_{\text{DG}}, v) = 0 \quad \text{for all } v \in S^r, \quad (2.49)$$

can be established (see [91]) upon assumption that for the solution $u$ of (2.13) holds $u \in H^4(\Omega) \cap H^2_0(\Omega)$. The above a priori bound (Theorem 2.1) gives an idea of the
rate of convergence expected by the discontinuous Galerkin method when sufficient regularity is assumed for the exact solution. However, fourth order problems on polygonal domains rarely satisfy these regularity assumptions up to the boundary, due to the presence of corner singularities in the solution (see Remark 2.1). To maintain the convergence rates in this case, adaptive algorithms are a desired method to resolve the issue whereby the mesh is refined towards the singularities, in an effort to equidistribute the error. This is the content of the next chapter.
Chapter III

A posteriori estimates for fourth order elliptic equations

In this Chapter we present reliable and efficient a posteriori bounds in the energy norm for the interior penalty discontinuous Galerkin method (IPDG) defined in section 2.5 for the biharmonic problem,

\[ \Delta^2 u = f \quad \text{in } \Omega, \quad (3.1) \]

where \( f \in L^2(\Omega) \). We consider the problem with homogeneous essential boundary conditions, under minimal regularity assumptions on the analytical solution,

\[ u = 0 \quad \text{on } \Gamma_\partial, \]
\[ \nabla u \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_\partial, \quad (3.2) \]

where \( \mathbf{n} \) denotes the unit outward normal vector to \( \Gamma_\partial \). The case of non-homogeneous boundary conditions may be treated analogously but has its own difficulties which we note in Remark 3.2 after the proof of reliability. We begin by introducing the problem and contribution of this work in relation to the previous results. We then give the definition of the recovery operator which is of essential importance to the proof of reliability. Followed by this, we prove the efficiency and reliability bounds under minimal regularity conditions. We conclude this Chapter with some numerical examples demonstrating the efficiency of the a posteriori estimators.
3.1 Recovery operator

We remark that with minor modifications only, the reliability a posteriori bounds presented here are also applicable to the case of other DG methods for fourth order problems (such as the interior penalty variants presented in [12, 77]) and to the case of various non-conforming methods with $C^0$-elements, such as the ones presented in [42, 24, 51, 63].

The reliability bound is based on a suitable recovery operator, which maps discontinuous finite element spaces to $H^2_0$-conforming finite element spaces (of two polynomial degrees higher), consisting of triangular or quadrilateral macro-elements defined in [40] (cf. also [24, 68, 65, 28] for similar constructions). Using this recovery operator, in conjunction with the inconsistent formulation for the IPDG presented in [55] (which ensures that the weak formulation of the problem is defined under minimal regularity assumptions on the analytical solution), we derive efficient and reliable a posteriori estimates of residual type for the IPDG method in the corresponding energy norm. Some ideas from a posteriori analyses for the Poisson problem presented in [14, 68, 65, 4, 31] are also implicitly utilized here in the context of fourth order problems. The results of this chapter have appeared in [56].

3.1 Recovery operator

In the a posteriori analysis below, we shall make use of a recovery operator mapping elements of $S^r$ onto a $C^1$-conforming space consisting of macro-elements of degree $r + 2$. The family of macro-elements considered below will be high order versions of the classical Hsieh-Clough-Tocher macro-element, constructed in [40]. This mapping is constructed via averages of the nodal basis functions; cf. [68, 65, 24, 27].

We recall the definition of the high-order $C^1$-conforming macro-elements constructed in [40].

**Definition 3.1.** Let element $\kappa \in \mathcal{T}$. For $m \geq 4$, a macro-element of degree $m$ is a nodal finite element $(\kappa, \tilde{P}_m, \tilde{N}_m)$, consisting of subtriangles $\kappa_i, i = 1, 2, \ldots, s$, with $s = 3$ if $\kappa$ is a triangle, or $s = 4$ if $\kappa$ is a quadrilateral (see Figures 3.1, 3.2, 3.3, 3.4 for an illustration). The local element space $\tilde{P}_m$ is defined by

$$\tilde{P}_m := \{ v \in C^1(\kappa) : v|_{\kappa_i} \in P_m(\kappa_i), \ i = 1, \ldots, s \}.$$
The degrees of freedom $\tilde{N}_m$ are defined as follows:

- the value and the first (partial) derivatives at the vertices of $\kappa$;
- the value at $m - 3$ distinct points in the interior of each exterior edge of $\kappa$;
- the normal derivative at $m - 2$ distinct points in the interior of each exterior edge of $\kappa$;
- the value and the first (partial) derivatives at the common vertex of all $\kappa_i$, $i = 1, \ldots, s$;
- the value at $m - 4$ distinct points in the interior of each edge of the $\kappa_i$, $i = 1, \ldots, s$ that is not an edge of $\kappa$;
- if $\kappa$ is a triangle, the normal derivative at $m - 4$ distinct points in the interior of each edge of the $\kappa_i$, $i = 1, \ldots, 3$ that is not an edge of $\kappa$; if $\kappa$ is a quadrilateral, the normal derivative at $m - 4$ distinct points in the interior of each edge of the $\kappa_i$, $i = 1, \ldots, 4$ that is not an edge of $\kappa$ and an extra normal derivative at a point in the interior of just one of the edges of the $\kappa_i$ that is not an edge of $\kappa$.
- the value at $(n - 4)(n - 5)/2$ distinct points in the interior of each $\kappa_i$ chosen so that, if a polynomial of degree $m - 6$ vanishes at those points, then it vanishes identically.

The corresponding finite element space consisting of the above macro-elements will be denoted by $\tilde{S}_h^m$.

The case $m = 3$, corresponding to the classical Hsieh-Clough-Tocher element (see, e.g., [35]), is not considered here as it is not relevant to the subsequent discussion.

Let us consider the standard Lagrange basis for a polynomial of degree $r$, $r \geq 2$ (depicted for $r = 2, 3$ for triangles and quadrilaterals on the left-hand side in Figures 3.1, 3.2, 3.3, 3.4, respectively). A crucial observation here is that the set of the nodal points of the Lagrange basis is a subset of the set of the nodal points of the macro-element of degree $r + 2$.
3.1 Recovery operator

Figure 3.1: $\mathcal{P}_2$ Lagrange triangular element and $\tilde{\mathcal{P}}_4$ $C^1$-conforming macro-element

Figure 3.2: $\mathcal{P}_3$ Lagrange triangular element and $\tilde{\mathcal{P}}_5$ $C^1$-conforming macro-element

Figure 3.3: $\mathcal{P}_2$ Lagrange quadrilateral element and $\tilde{\mathcal{P}}_4$ $C^1$-conforming macro-element
Lemma 3.1. Assume that the mesh $\mathcal{T}$ is constructed as in Section 2.4. Then, there exists an operator $E : S^r \to \tilde{S}_h^{r+2} \cap H_0^2(\Omega)$, satisfying the following bound

$$
\sum_{\kappa \in \mathcal{T}} |u_h - E(u_h)|_{a,\kappa}^2 \leq C \left( |h^{1/2-\alpha}[u_h]|_{\Gamma}^2 + |h^{3/2-\alpha}[\nabla u_h]|_{\Gamma}^2 \right),
$$

with $\alpha = 0, 1, 2$ and $C > 0$ a constant independent of $h$ and $u_h$. (The notational convention $\|w\|_{\kappa} \equiv |w|_{0,\kappa}$ for $w \in L^2(\kappa)$ is adopted here and in the sequel.)

Proof. For each nodal point $\nu$ of the $C^1$-conforming finite element space $\tilde{S}_h^{r+2}$, we define $\omega_\nu$ to be the set of $\kappa \in \mathcal{T}$ which share the nodal point $\nu$, i.e.,

$$\omega_\nu := \{ \kappa \in \mathcal{T} : \nu \in \kappa \};$$

also, $|\omega_\nu|$ will denote the cardinality of $\omega_\nu$. We notice that if $\nu$ is located in the interior of an element, we have $|\omega_\nu| = 1$. We define the operator $E : S^r \to \tilde{S}_h^{r+2} \cap H_0^2(\Omega)$, by

$$N_\nu(E(u_h)) = \begin{cases} 
\frac{1}{|\omega_\nu|} \sum_{\kappa \in \omega_\nu} N_\nu(u_h|_\kappa) & \text{if } \nu \notin \Gamma_\partial; \\
0 & \text{if } \nu \in \Gamma_\partial.
\end{cases}$$

here $N_\nu$ is any nodal variable at $\nu$ and $\nu$ is any nodal point of $\tilde{S}_h^{r+2}$. Note that $N_\nu(E(u_h)) = N_\nu(u_h)$ if $\nu$ is in the interior of an element. We denote by $\mathcal{N}$ the set of all nodal variables of $\tilde{S}_h^{r+2}$ defined on every element of $\mathcal{T}$ (i.e., they may be discontinuous across element boundaries) and we split $\mathcal{N}$ into two subsets $\mathcal{N}_0$ and $\mathcal{N}_1$ consisting of the nodal variables corresponding to function evaluations, and those
involving partial and normal derivatives of the function, respectively.

An inverse estimate gives,

$$\sum_{\kappa \in T} |u_h - E(u_h)|_{\alpha,\kappa}^2 \leq C \|h^{-\alpha}(u_h - E(u_h))\|_{\Omega}^2,$$

or a positive constant $C$, which is independent of $h$ and $u_h$. The equivalence of norms in a finite dimensional vector space, along with a scaling argument yields

$$\|h^{-\alpha}(u_h - E(u_h))\|_{\Omega}^2 \leq C \sum_{i=0}^{1} \sum_{N_{\nu} \in \mathcal{N}_i} h_{\kappa_i}^{2(i+1-\alpha)} (N_{\nu}(u_h - E(u_h)))^2.$$

For each nodal point $\nu$ that is not on the boundary $\Gamma_\partial$, we consider a local numbering $\kappa_1, \ldots, \kappa_{|\omega_{\nu}|-1}$ of the elements in $\omega_{\nu}$, so that each consecutive pair $\kappa_j, \kappa_{j+1}$ shares an edge. Recalling the arithmetic-geometric mean inequality (cf. Lemma 2.2 [68]), we have that

$$\sum_{N_{\nu} \in \mathcal{N}_0} h_{\kappa_i}^{2(i+1-\alpha)} (N_{\nu}(u_h - E(u_h)))^2 $$

$$= \sum_{N_{\nu} \in \mathcal{N}_0} h_{\kappa_i}^{2(i+1-\alpha)} \left( u_h(\nu)|_{\kappa_i} - \frac{1}{|\omega_{\nu}|} \sum_{\kappa \in \omega_{\nu}} u_h(\nu)|_{\kappa} \right)^2 + \sum_{N_{\nu} \in \mathcal{N}_0} h_{\kappa_i}^{2(i+1-\alpha)} \left( u_h(\nu)|_{\kappa_i} \right)^2 $$

$$\leq C \sum_{N_{\nu} \in \mathcal{N}_0} h_{\kappa_i}^{2(i+1-\alpha)} \left( \sum_{j=1}^{|\omega_{\nu}|-1} (u_h|_{\kappa_j}(\nu) - u_h|_{\kappa_{j+1}}(\nu))^2 \right) + \sum_{N_{\nu} \in \mathcal{N}_0} h_{\kappa_i}^{2(i+1-\alpha)} \left( u_h(\nu)|_{\kappa_i} \right)^2 $$

$$\leq C \sum_{e \in \Gamma} \|h^{1-\alpha}[u_h]\|_{L^\infty(e)}^2 \leq C \sum_{e \in \Gamma} \|h^{1/2-\alpha}[u_h]\|_{e}^2. \quad (3.4)$$

We now turn to the nodal variables in $\mathcal{N}_i$, which we further split into $\mathcal{N}_1^p$ and $\mathcal{N}_1^n$, the set of the nodal variables of normal derivatives across element edges and the set of nodal variables representing partial derivatives on elemental vertices, respectively.
For $N^n_1$, the argument is analogous to (3.4):

\[
\sum_{\nu \in \mathcal{N}^n_1} h^{2(2-\alpha)}_\kappa \left( N_\nu (u_h - E(u_h)) \right)^2 \\
\leq C \sum_{\nu \in \mathcal{N}^n_1} h^{2(2-\alpha)}_\kappa \left( (\nabla u_h \cdot n_{\kappa_1}) |_{\kappa_1}(\nu) - (\nabla u_h \cdot n_{\kappa_2}) |_{\kappa_2}(\nu) \right)^2 \\
+ \sum_{\nu \in \mathcal{N}^n_1} h^{2(2-\alpha)}_\kappa \left( (\nabla u_h \cdot n) |_{\kappa}(\nu) \right)^2 \\
\leq C \sum_{e \in \Gamma} ||h^{3/2-\alpha}[\nabla u_h]|_{e}||^2_{L^\infty(e)} \leq C \sum_{e \in \Gamma} ||h^{3/2-\alpha}[\nabla u_h]|_{e}||^2.
\]

For $N^n_p$, the first part of the argument is also analogous to (3.4), yielding

\[
\sum_{\nu \in \mathcal{N}^n_p} h^{2(2-\alpha)}_\kappa \left( N_\nu (u_h - E(u_h)) \right)^2 \leq C \sum_{e \in \Gamma} \sum_{z \in \{x,y\}} ||h^{3/2-\alpha}[u_h]|_{e}||^2. \tag{3.5}
\]

Splitting the partial derivatives on the right-hand side of (3.5) into normal and tangential components, and using an inverse estimate along each edge $e$ for the tangential derivative component, in conjunction with the fact that the edges $e$ are straight lines, gives

\[
||h^{3/2-\alpha}[u_h]|_{e}||^2 \leq 2||h^{3/2-\alpha}[\nabla u_h]|_{e}||^2 + 2||h^{3/2-\alpha}[u_h]|_{e}||^2 \\
\leq 2||h^{3/2-\alpha}[\nabla u_h]|_{e}||^2 + C||h^{1/2-\alpha}[u_h]|_{e}||^2,
\]

where $(\cdot)_t$ denotes the tangential derivative along the edge $e$.

The proof is completed by combining the above bounds. \qed

### 3.2 A posteriori error bounds

In the following theorem we establish a reliable a posteriori error bound in the case when the analytical solution $u$ of (3.1), (3.2) satisfies $u \in H^2_0(\Omega)$.

**Theorem 3.1.** Let $u \in H^2_0(\Omega)$ be the solution to (3.1), (3.2), $u_h \in S^r$ be the approximation obtained by the DG method and $\sigma$ and $\xi$ as in (2.35). Then there
exists a positive constant $C$, independent of $h$, $u$ and $u_h$, so that

$$
|||u - u_h|||^2 \leq C \left( ||h^2(f - \Delta_h u_h)||^2_\Omega + C_p \left( ||h^{-3/2}[u_h]||^2_\Gamma + ||h^{-1/2}[\nabla u_h]||^2_\Gamma \right) \right. \\
+ \left. ||h^{1/2}[\Delta u_h]||^2_\Gamma + ||h^{3/2}[\nabla \Delta u_h]||^2_\Gamma \right),
$$

(3.6)

with $C_p := \max\{1, \sigma_0, \xi_0, \sigma_0^2, \xi_0^2\}$.

Proof. Let $v_h \in S^r$, $v \in H^2_0(\Omega)$, $\eta = v - v_h$ and $E(u_h) \in \tilde{S}^r_h \cap H^2_0(\Omega)$ be as in Lemma 3.1. With this notation, we decompose the error as follows

$$
e := u - u_h = (u - E(u_h)) + (E(u_h) - u_h) \equiv e^c + e^d.
$$

Since $u$ is the solution to the weak problem, we have $B(u, v) = l(v)$, as $\mathcal{L}(u) = \mathcal{L}(v) = 0$. Hence,

$$
B(e, v) = B(u, v) - B(u_h, v) = l(v) - B(u_h, v - v_h) - B(u_h, v_h) = l(\eta) - B(u_h, \eta) \quad (3.7)
$$

and, thus

$$
B(e^c, v) = l(\eta) - B(u_h, \eta) - B(e^d, v). \quad (3.8)
$$

Next, we note that $\mathcal{L}(e^c) = 0$ as $e^c \in H^2_0(\Omega)$ and therefore, upon setting $v = e^c$ in (3.8), we deduce that

$$
|||\Delta e^c|||^2 = B(e^c, e^c) = l(\eta) - B(u_h, \eta) - B(e^d, e^c). \quad (3.9)
$$

First, we shall estimate the third term on the right hand side of (3.9). Since $\mathcal{L}(e^c) = [e^c] = [\nabla e^c] = 0$, we obtain

$$
|B(e^d, e^c)| = \left| \int_\Omega (\Delta_h e^d \Delta e^c + \mathcal{L}(e^d)\Delta e^c) \, dx \right| \\
\leq \left( ||\Delta_h e^d||^2_\Omega + C \left( ||\sqrt{\sigma}[u_h]||^2_\Gamma + ||\sqrt{\xi}[\nabla u_h]||^2_\Gamma \right) \right)^{1/2} ||\Delta e^c||_\Omega
$$

(3.10)

using the stability of the lifting operator (2.45).

Next, we shall estimate the first two terms on the right hand side of (3.9). After
integration by parts, we have

\[
\int_{\Omega} (f - \Delta_h^2 u_h) \eta \, dx - \int_{\Omega} (\mathcal{L}(\eta) \Delta_h u_h + \mathcal{L}(u_h) \Delta_h \eta) \, dx \\
- \sum_{\kappa \in \mathcal{T}} \int_{\partial \kappa} (\Delta_h u_h \cdot \nabla \eta - \nabla \Delta_h u_h \cdot n \eta) \, ds \\
- \int_{\Gamma} (\sigma[u_h] \cdot [\eta] + \xi[\nabla u_h][\nabla \eta]) \, ds.
\]

As \( u_h, v_h \in S^r \) and \( v \in H^2_0(\Omega) \), we can use the definition of the lifting operator, to obtain

\[
\int_{\Omega} \mathcal{L}(\eta) \Delta_h u_h \, dx = \int_{\Gamma} ([\eta] \cdot \{\nabla \Delta_h u_h\} - \{\Delta_h u_h\}[\nabla \eta]) \, ds.
\]

Employing the identity (A.12) on the third term on the right-hand side of (3.11) gives

\[
\sum_{\kappa \in \mathcal{T}} \int_{\partial \kappa} (\Delta_h u_h \cdot \nabla \eta - \nabla \Delta_h u_h \cdot n \eta) \, ds = \int_{\Gamma} [\nabla \eta] \{\Delta_h u_h\} \, ds + \int_{\Gamma_{int}} \{\nabla \eta\} \cdot [\Delta_h u_h] \, ds \\
- \int_{\Gamma} [\eta] \cdot \{\nabla \Delta_h u_h\} \, ds - \int_{\Gamma_{int}} \{\eta\}[\nabla \Delta_h u_h] \, ds.
\]

Inserting (3.12) and (3.13) into (3.11), we deduce that

\[
l(\eta) - B(u_h, \eta) = \int_{\Omega} ((f - \Delta_h^2 u_h) \eta - \mathcal{L}(u_h) \Delta_h \eta) \, dx + \int_{\Gamma_{int}} \{\nabla \eta\} \cdot [\Delta_h u_h] \, ds \\
- \int_{\Gamma_{int}} \{\nabla \eta\} \cdot [\Delta_h u_h] \, ds - \int_{\Gamma} (\sigma[u_h] \cdot [\eta] + \xi[\nabla u_h][\nabla \eta]) \, ds.
\]

Fix \( v_h \) to be the elementwise linear approximation to \( e^c \) such that

\[
|e^c - v_h|_{j, \kappa} \leq C h^{m-j} |e^c|_{m, \kappa}
\]

for \( C > 0 \), independent of \( \mathcal{T} \), for \( 0 \leq j \leq m \leq 2, \kappa \in \mathcal{T} \) (see, e.g., [35]). We shall use this to bound the terms on the right-hand side of (3.14) and (3.10).

For the first term on the right-hand side of (3.14), we use (3.15) and the stability
of the lifting operator \((2.45)\), to deduce

\[
\left| \int_\Omega (f - \Delta_h^2 u_h) \eta - \mathcal{L}(u_h) \Delta_h \eta \right| dx \leq C \left( \left\| h^2 (f - \Delta_h^2 u_h) \right\|_{1, \Omega}^2 + \left| \sqrt{\sigma}[u_h] \right|_{1}^2 + \left| \sqrt{\xi}[\nabla u_h] \right|_{1}^2 \right)^{1/2} |e^c|_{2, \Omega}.
\]

Using the bound

\[
\left\| h^{-3/2} \{\eta\} \right\|_{1, \text{int}}^2 \leq C \sum_{\kappa \in T} h^{-3}_\kappa \left\| \eta \right\|_{\partial \kappa}^2 \leq C \sum_{\kappa \in T} h^{-3}_\kappa (h^{-1}_\kappa \left\| \eta \right\|_{1, \kappa}^2 + h_\kappa \left\| \eta \right\|_{2, \kappa}^2) \leq C |e^c|_{2, \Omega}^2,
\]

the second term on the right-hand side of (3.14) can be bounded as follows

\[
\left| \int_{\Gamma_{\text{int}}} \{\eta\} [\nabla \Delta u_h] ds \right| \leq C \left\| h^{3/2} [\nabla \Delta u_h] \right\|_{1, \text{int}} |e^c|_{2, \Omega}.
\]

Similarly, using the bound

\[
\left\| h^{-1/2} \{\nabla \eta\} \right\|_{1, \text{int}}^2 \leq C \sum_{\kappa \in T} h^{-1}_\kappa \left\| \nabla \eta \right\|_{\partial \kappa}^2 \leq C \sum_{\kappa \in T} h^{-1}_\kappa (h^{-1}_\kappa \left\| \nabla \eta \right\|_{1, \kappa}^2 + h_\kappa \left\| \nabla \eta \right\|_{2, \kappa}^2) \leq C |e^c|_{2, \Omega}^2,
\]

the third term on the right-hand side of (3.14) can be bounded as follows

\[
\left| \int_{\Gamma_{\text{int}}} \{\nabla \eta\} \cdot [\Delta u_h] ds \right| \leq \left\| h^{1/2} [\Delta u_h] \right\|_{1, \text{int}} |e^c|_{2, \Omega}.
\]

For the penalty terms we work in a similar fashion, to deduce

\[
\left| \int_{\Gamma} (\sigma[u_h] \cdot [\eta] + \xi[\nabla u_h][\nabla \eta]) ds \right| \leq C C_\sigma^{1/2} \left( \left\| h^{-3/2} [u_h] \right\|_{1}^2 + \left\| h^{-1/2} [\nabla u_h] \right\|_{1}^2 \right)^{1/2} |e^c|_{2, \Omega}.
\]

Observing that \( |e^c|_{2, \Omega} \leq C \left\| \Delta e^c \right\|_{\Omega} \), since \( e^c \in H^2_0(\Omega) \), and using the bounds (3.10), (3.16), (3.17), (3.18) and (3.19) on (3.9) gives

\[
\left\| \Delta e^c \right\|_{\Omega}^2 \leq C \left( \left\| h^2 (f - \Delta_h^2 u_h) \right\|_{1, \Omega}^2 + C_p \left( \left\| h^{-3/2} [u_h] \right\|_{1}^2 + \left\| h^{-1/2} [\nabla u_h] \right\|_{1}^2 \right) + \left\| h^{1/2} [\Delta u_h] \right\|_{1}^2 + \left\| h^{3/2} [\nabla \Delta u_h] \right\|_{1}^2 + \left\| \Delta_h e^d \right\|_{\Omega}^2 \right).
\]
Having estimated $||\Delta e^c||_\Omega$, we now focus on estimating the non-conforming part of the error $||\Delta e^d||_\Omega$. We have, respectively,

$$||\Delta e^d||^2_\Omega \leq \sum_{\kappa \in T} |u_h - E(u_h)|^2_\kappa \leq C \left(||h^{1/2-\alpha}[u_h]||^2_f + ||h^{3/2-\alpha}[\nabla u_h]||^2_\Gamma\right),$$

using Lemma 3.1. Employing the triangle inequality

$$||\Delta_h e||_\Omega \leq ||\Delta e^c||_\Omega + ||\Delta_h e^d||_\Omega,$$

already yields the result. \hfill \Box

**Remark 3.1.** The above theorem can be extended to the case of non-homogeneous boundary conditions $u = g_1^d \in H^{3/2}(\partial \Omega)$ and $\nabla u \cdot n = g_2^d \in H^{1/2}(\partial \Omega)$, in an analogous fashion to [65], where the second order elliptic problem for general Dirichlet boundary conditions is considered. The conforming subspace of the finite element space up to the boundary can be constructed by projecting the non-homogeneous boundary data onto the space of traces of finite element functions and use the stability of the boundary value problem with respect to the boundary conditions.

**Remark 3.2.** Additional difficulties in the extension of the reliability bound proof to the case of non-homogeneous boundary conditions are created by the fact that $||\Delta e^c||$ is not a semi-norm on $H^2(\Omega)$. This difficulty can be overcome by using the so called “plate” formulation defined in (2.43) where the corresponding term $||D^2 e^c||$ is a semi-norm on $H^2(\Omega)$. Similar procedure to overcome this difficulty is outlined in [52].

**Remark 3.3.** The residual term $||h^2(f - \Delta^2 u_h)||_\kappa$ in the bound (3.6) of the above theorem is in practice often approximated by projecting $f$ onto the finite element space $\Pi : L^2(\Omega) \to S^r$ (where $\Pi$ denotes the orthogonal $L^2$-projection operator onto $S^r$) thereby necessitating further estimation of the residual term:

$$||h^2(f - \Delta^2 u_h)||_\kappa \leq ||h^2(\Pi f - \Delta^2 u_h)||_\kappa + ||h^2(f - \Pi f)||_\kappa.$$  

However, the so-called data oscillation term $||h^2(f - \Pi f)||_\kappa$ is dominated by the
estimator term

\[ \| h^2(f - \Pi f) \|_\kappa = \| h^2(f - \Delta^2 u_h + \Pi(\Delta^2 u_h - f)) \|_\kappa \leq C \| h^2(f - \Delta^2 u_h) \|_\kappa, \]

and hence does not influence the reliability of the upper bound.

Next we prove the efficiency of the estimator.

**Theorem 3.2.** Under the foregoing assumptions stated in Theorem 3.1, there are positive constants \( c_1, c_2 \) and \( c_3 \), independent of \( h \) and \( u_h \), such that for each element \( \kappa \in T \), we have

\[ \| h^2(f - \Delta^2 u_h) \|_\kappa^2 \leq c_1 (\| \Delta e \|_\kappa^2 + \| h^2(f - \Pi f) \|_\kappa^2), \tag{3.21} \]

and for each edge \( e \in \Gamma_{\text{int}} \) we have

\[ \| h^{1/2}[\Delta u_h] \|_e^2 \leq c_2 (\| \Delta e \|_{\kappa_1 \cup \kappa_2}^2 + \| h^2(f - \Pi f) \|_{\kappa_1 \cup \kappa_2}^2), \tag{3.22} \]

and

\[ \| h^{3/2}[\nabla \Delta u_h] \|_e^2 \leq c_3 (\| \Delta e \|_{\kappa_1 \cup \kappa_2}^2 + \| h^2(f - \Pi f) \|_{\kappa_1 \cup \kappa_2}^2), \tag{3.23} \]

where \( \kappa_1 \) and \( \kappa_2 \) denote two elements such that \( e = \kappa_1 \cap \kappa_2 \).

**Proof.** Fix \( \kappa \in T \) and let \( v \in H^2_0(\Omega) \cap H^2_0(\kappa) \), with \( v = 0 \) on \( \Omega \setminus \kappa \), be a polynomial function on \( \kappa \) (to be defined later). Setting \( v_h = 0 \) and taking \( v \) as above in (3.7) yields

\[ \int_\kappa \Delta e \Delta v \, dx = \int_\kappa (f - \Delta^2 u_h)v \, dx = \int_\kappa (\Pi f - \Delta^2 u_h)v \, dx + \int_\kappa (f - \Pi f)v \, dx, \tag{3.24} \]

noting that \([v] = [\nabla v] = \{v\} = \{\nabla v\} = 0\) on \( \Gamma \) and that \( L(u) = L(v) = 0 \) on \( \Omega \). Hence, we have that

\[ \int_\kappa (\Pi f - \Delta^2 u_h)v \, dx \leq \| \Delta e \|_\kappa \| \Delta v \|_\kappa + \| \Pi f - f \|_\kappa \| v \|_\kappa \]

\[ \leq C (\| \Delta e \|_\kappa + \| h^2(f - \Pi f) \|_\kappa) \| h^{-2}v \|_\kappa. \tag{3.25} \]

We set \( v|_\kappa = (\Pi f - \Delta^2 u_h)b_\kappa^2 \), where \( b_\kappa : \kappa \to \mathbb{R} \) is the standard interior “bubble” function (which is defined by \( b_\kappa := b_\kappa \circ F_\kappa \), where \( b_\kappa := 27\lambda_1\lambda_2\lambda_3 \), if \( \kappa \) is the reference...
triangle with barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$, and $b_\kappa := (1 - \lambda_1^2)(1 - \lambda_2^2)$ if $\kappa$ is the reference rectangle with $\lambda_1$ and $\lambda_2$ its corresponding coordinates. We note that $\|\cdot b_\kappa\|_\kappa$ defines a norm on the finite dimensional space $P_{r+2}(\kappa)$. This norm is, therefore, equivalent to $\|\cdot\|_\kappa$ on $P_{r+2}(\kappa)$. A scaling argument reveals that the equivalence constants are independent of $h$ (see Appendix B); in particular, we have

$$\|\Pi f - \Delta^2 u_h\|_\kappa^2 \leq C \int_\kappa (\Pi f - \Delta^2 u_h)^2 b_\kappa^2 \, dx = C \int_\kappa (\Pi f - \Delta^2 u_h) v \, dx. \quad \text{(3.26)}$$

Combining (3.25) with (3.26) and using the triangle inequality

$$\|f - \Delta^2 u_h\|_\kappa \leq \|\Pi f - \Delta^2 u_h\|_\kappa + \|f - \Pi f\|_\kappa,$$

(3.27)
gives (3.21).

Figure 3.5: Inscribed rhombus $\tilde{\kappa}$ for: (a) Triangular elements; (b) Quadrilateral elements.

Next, for each internal edge $e \in \Gamma_{\text{int}}$, we define $\tilde{\kappa} \subset \kappa_1 \cup \kappa_2$, to be the largest rhombus contained in $\kappa_1 \cup \kappa_2$ that has $e$ as one diagonal (see Figure 3.5). Also, we define $b_\tilde{\kappa} : \tilde{\kappa} \to \mathbb{R}$ to be the “bubble” function on the rhombus $\tilde{\kappa}$. Let also $b_\ell : \tilde{\kappa} \to \mathbb{R}$ to be an affine function (i.e., a piece of a plane) having value zero along the edge $e$, such that $(\nabla b_\ell \cdot \mathbf{n})|_e = h^{-2}|e$, thereby defining $b_\ell$ completely up to a sign, which is not relevant to the discussion. Using the above definitions, we consider the function $b_e : \Omega \to \mathbb{R}$, with $b_e|_\tilde{\kappa} := b_\ell b_\tilde{\kappa}^3$, and $b_e := 0$ on $\Omega \setminus \tilde{\kappa}$, which has the following
properties:

\[ b_e \in C^2(\Omega) \cap H_0^2(\Omega), \quad [b_e] = [\nabla b_e] = \{b_e\} = 0 \text{ on } \Gamma, \]
\[ ([\nabla b_e] \cdot \mathbf{n})|_e = (h^{-1} b^2_\kappa)|_e \quad \text{and} \quad \{\nabla b_e\} = 0 \text{ on } \Gamma \setminus e, \quad (3.28) \]

observing that \( \nabla b_\kappa \cdot \mathbf{n} = 0 \) along the edge \( e \).

We define \( v = \phi b_e \) where \( \phi \) is a constant function in the normal direction to \( e \) (i.e., \( (\nabla \phi \cdot \mathbf{n})|_e = 0 \)). For this \( v \) and for \( v_h = 0 \), (3.7) yields

\[
\int_{\kappa_1 \cup \kappa_2} (f - \Delta^2 u_h) v \, dx - \int_{\kappa_1 \cup \kappa_2} \Delta_h e \Delta_h v \, dx = \int_e [\Delta u_h] \cdot \{\nabla v\} \, ds. \quad (3.29)
\]

Setting \( \phi|_e = (h^{-1}[\Delta u_h] \cdot \mathbf{n})|_e \) in (3.29), we have

\[
\int_e [\Delta u_h] \cdot \{\nabla v\} \, ds = ||b^3_\kappa h^{-1}[\Delta u_h]|_e||^2 \geq C ||h^{-1}[\Delta u_h]|_e||_e^2, \quad (3.30)
\]

from a norm-equivalence and scaling argument (see Appendix B).

Let then \( l : e \to \mathbb{R}, \) where \( l(s) \) denotes the length of the intersection of the line normal to \( e \), crossing \( e \) at the point \( s \in e \), and \( \bar{k} \). Then, we have

\[
||v||_{\kappa_1 \cup \kappa_2} \leq C ||\phi||_{\kappa_1 \cup \kappa_2} = C \left( \int_e \phi^2(s) l(s) \, ds \right)^{1/2} \leq C ||h^{1/2} \phi||_e = C ||h^{-1/2}[\Delta u_h]|_e||. \quad (3.31)
\]

From (3.29) and (3.30), we arrive to

\[
||h^{-1}[\Delta u_h]|_e||^2 \leq (||h^{1/2}(f - \Delta^2 u_h)||_{\kappa_1 \cup \kappa_2} + h^{-3/2}||\Delta_h e||_{\kappa_1 \cup \kappa_2}) ||h^{-1/2}v||_{\kappa_1 \cup \kappa_2}. \quad (3.32)
\]

Using (3.31) and (3.21) in (3.32) and multiplying both sides of (3.31) by \( h^3 \), the bound stated in (3.22) follows.

To estimate \([\nabla \Delta u_h]\), we first note that we have

\[ b^3_\kappa \in C^2(\Omega) \cap H_0^2(\Omega), \quad [b^3_\kappa] = [\nabla b^3_\kappa] = \{\nabla b^3_\kappa\} \cdot \mathbf{n} = 0 \text{ on } \Gamma \quad \text{and} \quad \{b^3_\kappa\} = 0 \text{ on } \Gamma \setminus e. \quad (3.33)\]

We set \( v = \psi b^3_\kappa \), where \( \psi \) is a constant function in the normal direction to \( e \). For
this $v$ and for $v_h = 0$, (3.7) yields

$$\int_{\kappa_1 \cup \kappa_2} (f - \Delta_h^2 u_h) v \, dx - \int_{\kappa_1 \cup \kappa_2} \Delta_h e \Delta_h v \, dx = \int_e [\nabla \Delta u_h] \{v\} \, ds.$$  \hspace{1cm} (3.34)

Setting $\psi|_e = [\nabla \Delta u_h]|_e$ in (3.34), and working similarly as above, the result follows.

\section*{3.3 Numerical Experiments}

In this section we present a series of two-dimensional numerical examples to illustrate the practical performance of the proposed a posteriori error estimator within an automatic adaptive refinement procedure. In each of the examples shown below, we set the polynomial degree $r$ equal to 2. The DG solution of (2.39) is computed using the interior penalty parameters $\sigma_0 = \xi_0 = 10$. The adaptive meshes are constructed by employing the fixed fraction strategy, with refinement and derefinement fractions set to 20\% and 10\%, respectively. The algorithm utilised here is a variant of the so-called Maximum Strategy algorithm which is of interest later in the thesis and which is described in detail in chapter 6. In the next chapter, we use another marking strategy, the so-called Dörfler marking strategy. Here, the emphasis will be to demonstrate that the proposed a posteriori error indicator converges to zero at the same asymptotic rate as the energy norm of the actual error on a sequence of non-uniform adaptively refined meshes. With this mind, as in [14], we set the constant $C$ arising in Theorem 3.1 equal to one and ensure that the corresponding effectivity indices are roughly constant on all of the meshes employed; here, the effectivity index is defined as the ratio of the a posteriori error bound and the energy norm of the actual error. In general, to ensure the reliability of the error estimator, $C$ must be determined numerically for the underlying problem at hand, cf. [43], for example.

\textbf{Remark 3.4.} Quadrilateral elements are used in each of the examples which are refined via longest edge bisection. By doing this, one is inevitably left with hanging nodes. We emphasize that our results in this chapter are still confirmed through the numerical results above even though hanging nodes are not dealt with theoretically.
3.3 Numerical Experiments

Figure 3.6: Example 1. Computational mesh with: (a) 7300 elements, after 7 adaptive refinements; (b) 30208 elements, after 10 adaptive refinements;

*Similar results for triangular elements without hanging nodes are achieved in the numerical experiments in the next chapter.*

![Figure 3.6: Example 1. Computational mesh with: (a) 7300 elements, after 7 adaptive refinements; (b) 30208 elements, after 10 adaptive refinements.](image)

Figure 3.7: Example 1. (a) Comparison of the actual and estimated energy norm of the error with respect to the number of degrees of freedom; (b) Effectivity Indices.

![Figure 3.7: Example 1. (a) Comparison of the actual and estimated energy norm of the error with respect to the number of degrees of freedom; (b) Effectivity Indices.](image)
3.3 Numerical Experiments

3.3.1 Example 1

Here, we let $\Omega = (0, 1)^2$ and select $f$ so that the analytical solution to (3.1) and (3.2) is given by

$$u(x, y) = \sin(2\pi x)^2 \sin(2\pi y)^2;$$

this is a variant of the model problem considered in [91], cf. also [55].

In Figure 3.6 we show the mesh generated using the proposed a posteriori error indicator after 7 and 10 adaptive refinement steps. Here, we see that while the mesh has been largely uniformly refined throughout the entire computational domain, additional refinement has been performed where the gradient/curvature of the analytical solution is relatively large.

Finally, in Figure 3.7 we present a comparison of the actual and estimated energy norm of the error versus the number of degrees of freedom in the finite element space $S^r$, on the sequence of meshes generated by our adaptive algorithm. Here, we observe that there is an initial transient, whereby the effectivity index is relatively large. However, as the refinement algorithm proceeds, the error bound (asymptotically) over-estimates the true error by a consistent factor; indeed, from Figure 3.7(b), we see that the computed effectivity indices tend to a value of just over 4.

3.3.2 Example 2

In this second example, we investigate the performance of the interior penalty DG method (2.39) for a problem with a corner singularity in $u$. To this end, we let $\Omega$ be the L-shaped domain $(-1, 1)^2 \setminus [0, 1) \times (-1, 0]$, and select $f = 0$. Then, writing $(r, \varphi)$ to denote the system of polar coordinates, we impose an appropriate inhomogeneous boundary condition for $u$ so that

$$u = r^{5/3} \sin(5\varphi/3).$$

The analytical solution $u$ contains a singularity at the corner located at the origin of $\Omega$; here, we only have $u \in H^{8/3-\varepsilon}(\Omega), \varepsilon > 0$.

In Figure 3.8 we show the mesh generated using the local error indicators after 7 and 9 adaptive refinement steps. Here, we see that the mesh has been largely refined in the vicinity of the re-entrant corner located at the origin, as well as in
3.3 Numerical Experiments

Figure 3.8: Example 2. Computational mesh with: (a) 1248 elements, after 7 adaptive refinements; (b) 3198 elements, after 9 adaptive refinements;

the region adjacent to this singular point. Finally, Figure 3.9 shows the history of the actual and estimated energy norm of the error on each of the meshes generated by our adaptive algorithm, together with their corresponding effectivity indices. As in the previous example, we observe that the a posteriori bound over-estimates the true error by a consistent factor around 4.

The numerical examples presented above have been also considered in [55] where uniform meshes were employed. In view of the results in [55], we conclude the superior performance of the adaptive algorithm based on the a posteriori error indicator presented in this work when compared to uniform refinement.
Figure 3.9: Example 2. (a) Comparison of the actual and estimated energy norm of the error with respect to the number of degrees of freedom; (b) Effectivity Indices.
Chapter IV

On the convergence of adaptive discontinuous Galerkin methods for elliptic fourth order problems

The a posteriori error bounds are naturally utilised in the construction of adaptive algorithms but it is only very recently that the adaptive discontinuous Galerkin method for the second order problems have been proven to be convergent in the sense of $h$-adaptivity. Even though there are many papers dealing with convergence analysis of adaptive algorithms for the second order problems in the literature (see [69, 19] for DG and [32] for conforming methods), the authors are not aware of such work in the context of interior penalty discretizations of elliptic fourth order problems. The purpose of this chapter is to present some ideas towards a convergence proof for the adaptive discontinuous Galerkin (ADG) method for the biharmonic problem. However, we would like to emphasize that the ideas presented here do not constitute a proof of convergence due to the Assumption 4.1 below but underline some technical details which may be improved in future work. The discussion here is inspired and influenced by the works of Bonito and Nochetto [19], Karakashian and Pascal [69], Cascon, Kreuzer, Nochetto and Siebert [32] and Stevenson [88] in the context of second order problems. At the end of this chapter, we present
numerical examples which support Assumption 4.1. The basic idea of convergence of a $h$-adaptive method is that the refinement of elements which are picked by using an estimator always leads to error being reduced in the subsequent iteration of the solution on the refined mesh. We pursue this idea further here.

We begin by defining an abbreviation for the penalty jumps,\[ F_p(u_h) := \left( ||h^{-3/2}[u_h]||_T^2 + ||h^{-1/2}[\nabla u_h]||_T^2 \right)^{1/2}, \]
and for the global estimator,\[ \eta_T^2(u_h) := \sum_{\kappa \in T} h_\kappa^4 ||\Pi f - \Delta^2 u_h||_\kappa^2 \quad \text{and} \quad \eta_T^2(u_h) := \sum_{e \in \Gamma_{\text{int}} \cup \partial \kappa} \left( h_\kappa^2 ||[\Delta u_h]||_e^2 + h_\kappa^2 ||[\nabla \Delta u_h]||_e^2 \right). \]

Using this notation for the penalty jumps and the global estimator, we can abbreviate the a posteriori error bound from previous chapter as,\[ |||u - u_h|||^2 \leq C (\eta_T^2(u_h) + C_p^2 F_p^2(u_h)). \quad (4.1) \]
where $C_p := \max\{\sigma_0, \xi_0\}$.

Now, we make the important assumption which for this chapter.

**Assumption 4.1.** Assume that $\min\{\sigma_0, \xi_0\} \geq \gamma_0 \geq 1$, for some constant $\gamma_0 \equiv \gamma_0(\tilde{\sigma}_0, \tilde{\xi}_0, c, r)$. Let also $0 < \beta \leq 1$ such that $\min\{\sigma_0, \xi_0\} \geq \beta C_p$. Then there exists a positive constant $C_{4.1} \equiv C_{4.1}(r, \beta)$ such that\[ C_p^2 F_p^2(u_h) \leq C_{4.1} \eta_T^2(u_h). \quad (4.2) \]

**Remark 4.1.** This assumption establishes that the estimator $\eta_T$ dominates the penalty terms. Such a result is proven in [19, 69] for DG methods for second order elliptic problems, so it constitutes a reasonable assumption. The validity of (4.2) is numerically assessed in the numerical experiments later in this chapter.
4.1 Adaptive algorithm

The description of the adaptive algorithm is completely analogous to the one presented in [32]. Each iteration round of the adaptive algorithm consists of 4 parts and from now on, all objects indexed by $m$ refer to the object in the $m$-th iteration round of the algorithm. Given a conforming mesh $\mathcal{T}_m$ (i.e. $\mathcal{T}_m$ does not contain hanging nodes), we calculate the solution to the problem:

$$
\text{find } u_m \in S^r_m \text{ such that } B_m(u_m, v_m) = l(v_m) \quad \forall v_m \in S^r_m. \quad (4.3)
$$

This is the first part of the algorithm. Note that finite element space $S^r_m$ and the bilinear form $B_m$ depend on the mesh $\mathcal{T}_m$. In the second part of the adaptive algorithm, the local estimator

$$
\eta^2_{\mathcal{T}_m}(u_m, \kappa) = h_\kappa ^4 ||\Pi f - \Delta^2 u_m||^2_\kappa + \sum_{e \in \Gamma_{\text{int}, m} \cap \partial \kappa} \left( h_\kappa ||[\Delta u_m]||^2_e + h_\kappa ^3 ||[\nabla \Delta u_m]||^2_e \right),
$$

is calculated for each mesh element $\kappa \in \mathcal{T}_m$. In the third part of the algorithm, mesh elements are marked using the so-called Dörfler marking strategy (see [39]): for a user-defined number $0 < \theta \leq 1$ (from now on termed the Dörfler marking parameter), find $\mathcal{M}_m$, subset of $\mathcal{T}_m$, such that

$$
\eta^2_{\mathcal{T}_m}(u_m, \mathcal{M}_m) \geq \theta \eta^2_{\mathcal{T}_m}(u_m, \mathcal{T}_m); \quad (4.4)
$$

the $\kappa \in \mathcal{M}_m$ are called marked elements. In the fourth and the last part of the algorithm, the marked elements are refined by bisection; if the resulting mesh is not conforming, it is made into a conforming mesh by sufficient additional refinement (again by bisection) of elements possessing hanging nodes. These elements are then added to the marked elements $\mathcal{M}_m$, thereby arriving to the new mesh $\mathcal{T}_{m+1}$, and returning back to the first part of the algorithm.

The adaptive algorithm constructs a sequence of objects $\{\mathcal{T}_m, S^r_m, u_m\}_{m \geq 1}$, starting with a (given) conforming mesh $\mathcal{T}_0$ and it is summarised as follows. Based on the Assumption 4.1, the main result of this chapter is a proof that the above algorithm converges. This is the content of the next section.

**Remark 4.2.** Note that the algorithm in this chapter is different to the one studied
4.2 Convergence of the ADG method

Set parameter \(0 < \theta \leq 1\) and \(m = 0\);

1. obtain the solution \(u_m \in S_m^c\) of (4.3) \((S_m^c\) is associated with \(T_m)\);
2. use \(u_m\) to calculate the local estimators \(\eta_{T_m}(u_m, \kappa), \kappa \in T_m\);
3. find the set of marked elements \(M_m\) for refinement in \(T_m\) using (4.4);
4. obtain new mesh \(T_{m+1}\) by refining the marked elements \(M_m\) and eliminating hanging nodes, increment \(m\) and return to 1.

In numerical examples of the previous chapter. The algorithm here does not use any coarsening and the marking is done using the Dörfler marking strategy (4.4) as outlined above.

4.2 Convergence of the ADG method

The following lemma is inspired by a corresponding result in [32, Corollary 4.4].

**Lemma 4.1.** We have

\[
\eta_{T_{m+1}}^2(u_{m+1}) \leq (1 - \frac{\lambda \theta}{2}) \eta_{T_m}^2(u_m) + (1 + \frac{2}{\lambda \theta}) C_{4.1} ||\Delta_h(u_{m+1} - u_m)||^2;
\]

where \(\theta \in (0, 1)\) is the Dörfler marking parameter, \(C_{4.1} = C_{4.1}(c, r)\), and \(\lambda = 1 - 1/\sqrt{2}\). (We have adopted the notational convention that the broken Laplacian \(\Delta_h\) always corresponds to the mesh its argument is defined on.)

**Proof.** Using standard inverse estimates, one can show that

\[
\begin{align*}
||h_{m+1}^2 \Delta_h(u_{m+1} - u_m)||^2 &+ ||h_{m+1}^{1/2} [\Delta(u_{m+1} - u_m)]||^2_{\Gamma_{int,m+1}} \\
&+ ||h_{m+1}^{3/2} \nabla \Delta(u_{m+1} - u_m)||^2_{\Gamma_{int,m+1}} \leq C ||\Delta_h(u_{m+1} - u_m)||^2;
\end{align*}
\]

for a positive constant \(C \equiv C(r, c)\). Using the bound (4.6) and Young’s inequality,
we have

\[
\eta_{T_{m+1}}^2(u_{m+1}) = ||h_{m+1}^2(f - \Delta_h^2 u_{m+1})||^2 + ||h_{m+1}^{1/2}[\Delta u_{m+1}]||^2_{T_{m+1}} + ||h_{m+1}^{3/2}[\nabla \Delta u_{m+1}]||^2_{T_{m+1}}
\leq (1 + \delta) (||h_{m+1}^2(f - \Delta_h^2 u_m)||^2 + ||h_{m+1}^{1/2}[\Delta u_m]||^2_{T_{m+1}}
+ ||h_{m+1}^{3/2}[\nabla \Delta u_m]||^2_{T_{m+1}}) + (1 + \delta^{-1})C||\Delta_h(u_{m+1} - u_m)||^2,
\]

for any \(\delta > 0\).

As a result of the refinement part of the adaptive algorithm, an element \(\kappa \in \mathcal{M}_m\) is bisected into elements

\[\mathcal{R}_\kappa := \{\kappa' \in T_{m+1} : \kappa' \subset \kappa\}\]

and we observe that for all \(\kappa' \in \mathcal{R}_\kappa\) holds, \(h_{\kappa'} \leq 2^{-1/2}h_\kappa\). Hence, we have, respectively,

\[
\eta_{T_{m+1}}^2(u_{m}) = \eta_{T_{m}}^2(u_{m}, T_{m} \setminus \mathcal{M}_m) + \eta_{T_{m+1}}^2(u_{m}, \{\mathcal{R}_\kappa : \kappa \in \mathcal{M}_m\})
\leq \eta_{T_{m}}^2(u_{m}, T_{m} \setminus \mathcal{M}_m) + 2^{-1/2}\eta_{T_{m}}^2(u_{m}, \mathcal{M}_m),
\]

observing that \(\|\Delta u_{m}\| = \|\nabla \Delta u_{m}\| = 0\) almost everywhere on \(\Gamma_{\text{int}, m+1} \setminus \Gamma_{\text{int}, m}\). Combining (4.7) with (4.8), the result already follows by making use of the Dörfler marking property (4.4) and setting \(\delta = \lambda \theta/2\).

**Lemma 4.2.** Let \(u_m \in S_m\) and \(u_{m+1} \in S_{m+1}\) be two DG solutions and denote by \(e_m := u - u_m\) the error at the \(m\)-th iteration of the adaptive algorithm. Then, assuming that \(C_p \geq C_{(3.1)}\) and that \(C_p > 4 \max\{\tilde{\sigma}_0, \tilde{\xi}_0\}\), we have

\[
B_{m+1}(e_{m+1}, e_m) \leq (1 + C_p^{-1})B_m(e_m, e_m) - \frac{1}{4}||u_{m+1} - u_m||^2
+ C_{(4.2)}C_p^{-1}(\eta_{T_{m}}^2(u_{m}) + \eta_{T_{m+1}}^2(u_{m+1}))
\]

for any \(\epsilon > 0\), with \(C_{(4.2)} = C(c, C_{3.1}, C_{4.1})\). (We have adopted the notational convention that the DG-norm \(||\cdot||\) always corresponds to the mesh its argument is defined on.)

**Proof.** For brevity, we set \(v_{m+1} := u_m - u_{m+1}\) and \(w_{m+1} := v_{m+1} - E(v_{m+1})\), with the recovery operator \(E\) understood to be associated with the mesh \(T_{m+1}\). A straight-
4.2 Convergence of the ADG method

forward calculation reveals the identity:

\[
B_{m+1}(e_{m+1}, e_m) = B_{m+1}(e_m, e_m) - B_{m+1}(v_{m+1}, v_m) + 2B_{m+1}(e_{m+1}, w_{m+1}) \\
- 2\left(B_{m+1}(u_{m+1}, w_{m+1}) - l(w_{m+1})\right),
\]

(4.10)

using the method (2.39). First, we observe that

\[
B_{m+1}(e_m, e_m) = B_m(e_m, e_m) + B_{p,m+1}(u_m, u_m) - B_{p,m}(u_m, u_m) \\
\leq B_m(e_m, e_m) + 8C_p F_p^2(u_m),
\]

(4.11)
as the penalty parameters in \(B_{p,m+1}\) depend on \(h_{m+1}\), and the corresponding ones in \(B_{p,m}\) which depend on \(h_m\).

Also, the coercivity of the bilinear form implies

\[
B_{m+1}(v_{m+1}, v_{m+1}) \geq \frac{1}{2}|||v_{m+1}|||^2.
\]

(4.12)

To estimate the third term on the right-hand side of (4.10) we work as follows. First we observe that

\[
2B_{m+1}(e_{m+1}, w_{m+1}) = 2B_{m+1}(e_m, w_{m+1}) + 2B_{m+1}(v_{m+1}, w_{m+1}).
\]

(4.13)

We now estimate the first term on the right-hand side of (4.13):

\[
B_{m+1}(e_m, w_{m+1}) \leq ||\Delta_h e_m|| ||\Delta_h w_{m+1}|| + ||\Delta_h e_m|| ||\mathcal{L}_{m+1}(w_{m+1})|| \\
+ ||\Delta_h w_{m+1}|| ||\mathcal{L}_{m+1}(e_m)|| + |B_{p,m+1}(e_m, w_{m+1})| \\
= ||\Delta_h e_m|| ||\Delta_h w_{m+1}|| + ||\Delta_h e_m|| ||\mathcal{L}_{m+1}(v_{m+1})|| \\
+ ||\Delta_h w_{m+1}|| ||\mathcal{L}_{m+1}(u_m)|| + |B_{p,m+1}(u_m, v_{m+1})|.
\]

(4.14)

Using the Cauchy-Schwarz inequality together with (2.45) and Lemma 3.1, we de-
duce

\[ B_{m+1}(e_m, w_{m+1}) \leq (C_{3.1}^{1/2} + (\max\{\tilde{\sigma}_0, \tilde{\xi}_0\})^{1/2})||\Delta_h e_m||F_{p,m+1}(v_{m+1}) \]

\[ + \left( (C_{3.1} \max\{\tilde{\sigma}_0, \tilde{\xi}_0\})^{1/2} + C_p \right) F_{p,m+1}(u_m) F_{p,m+1}(v_{m+1}) \]

\[ \leq (4C_p)^{-1}||\Delta_h e_m||^2 + (C_{3.1}^{1/2} + (\max\{\tilde{\sigma}_0, \tilde{\xi}_0\})^{1/2})C_p F_{p,m+1}^2(v_{m+1}) \]

\[ + \left( \frac{1}{2}C_{3.1}^{1/2} + 1 \right) C_p F_{p,m+1}(u_m) F_{p,m+1}(v_{m+1}), \]

(4.15)

recalling that \( C_p > 4 \max\{\tilde{\sigma}_0, \tilde{\xi}_0\} \); this implies

\[ 2B_{m+1}(e_m, w_{m+1}) \leq C_{p}^{-1}B_m(e_m, e_m) + C_1 C_p F_{p,m+1}^2(v_{m+1}), \]

(4.16)

for some positive constant \( C_1 \equiv C_1(C_{3.1}, \tilde{\sigma}_0, \tilde{\xi}_0) \), using coercivity. For the second term on the right-hand side of (4.13), we use the continuity of the bilinear form to obtain

\[ 2B_{m+1}(v_{m+1}, w_{m+1}) \leq \frac{1}{4}||v_{m+1}||^2 + 4C_{cont}^2||w_{m+1}||^2 \]

\[ \leq \frac{1}{4}||v_{m+1}||^2 + 4C_{cont}^2(C_{3.1} + C_p) F_{p,m+1}^2(v_{m+1}), \]

(4.17)

using Lemma 3.1. Applying (4.16) and (4.17) in (4.13), we arrive to

\[ 2B_{m+1}(e_{m+1}, w_{m+1}) \leq C_{p}^{-1}B_m(e_m, e_m) + \frac{1}{4}||v_{m+1}||^2 + C_2 C_p F_{p,m+1}^2(v_{m+1}), \]

for some positive constant \( C_2 \equiv C_2(C_{3.1}, \tilde{\sigma}_0, \tilde{\xi}_0) \). We derive the following identity using integration by parts,

\[ \sum_{\kappa \in T} \int_{\partial\kappa} (\Delta u_h \mathbf{n} \cdot \nabla v - \nabla \Delta u_h \cdot \mathbf{n} v) \, ds = \int_{\Gamma} [\nabla v] \{\Delta u_h\} \, ds + \int_{\Gamma_{int}} \{\nabla v\} \cdot [\Delta u_h] \, ds \]

\[ - \int_{\Gamma} [v] \cdot \{\nabla \Delta u_h\} \, ds - \int_{\Gamma_{int}} \{v\} [\nabla \Delta u_h] \, ds. \]

(4.19)

For the last term on the right-hand side of (4.10), we perform integration by
parts twice and use (4.19), to arrive to the identity

\[
 l(w_{m+1}) - B_{m+1}(u_{m+1}, w_{m+1}) = \int_{\Omega} \left( (f - \Delta_h^2 u_{m+1}) w_{m+1} - \mathcal{L}_{m+1}(u_{m+1}) \Delta_h w_{m+1} \right) \, dx \\
+ \int_{\Gamma_{\text{int},m+1}} \left( [\nabla \Delta u_{m+1}] \{ w_{m+1} \} + [\Delta u_{m+1}] \cdot \{ \nabla w_{m+1} \} \right) \, ds \\
- B_{p,m+1}(u_{m+1}, v_{m+1}),
\]

(4.20)

observing that \( B_{p,m+1}(u_{m+1}, w_{m+1}) = B_{p,m+1}(u_{m+1}, v_{m+1}) \); the lifting operator \( \mathcal{L}_{m+1} \) is understood to be associated with the mesh \( T_{m+1} \). Application of the Cauchy-Schwarz inequality results to

\[
|l(w_{m+1}) - B_{m+1}(u_{m+1}, w_{m+1})| \leq ||h_{m+1}^2 (f - \Delta_h^2 u_{m+1})|| ||h_{m+1} w_{m+1}|| \\
+ ||\mathcal{L}_{m+1}(u_{m+1})|| ||\Delta_h w_{m+1}|| \\
+ ||h_{m+1}^{3/2} [\nabla \Delta u_{m+1}]||_{\Gamma_{\text{int},m+1}} ||h_{m+1}^{-3/2} \{ w_{m+1} \}||_{\Gamma_{\text{int},m+1}} \\
+ ||h_{m+1}^{1/2} [\Delta u_{m+1}]||_{\Gamma_{\text{int},m+1}} ||h_{m+1}^{-1/2} \{ \nabla w_{m+1} \}||_{\Gamma_{\text{int},m+1}} \\
+ C_p F_{p,m+1}(u_{m+1}) F_{p,m+1}(v_{m+1}).
\]

(4.21)

Standard trace estimates and application of Lemma 3.1 imply

\[
||h_{m+1}^{-2} w_{m+1}||^2 + ||\Delta_h w_{m+1}||^2 + ||h_{m+1}^{3/2} \{ w_{m+1} \}||_{\Gamma_{\text{int},m+1}}^2 \\
+ ||h_{m+1}^{-1/2} \{ \nabla w_{m+1} \}||_{\Gamma_{\text{int},m+1}}^2 \leq C' F_{p,m+1}^2 (v_{m+1}),
\]

(4.22)

where \( C' \) depends only on \( c \) and on \( C_{3,1} \), and the stability of the lifting operator implies

\[
||\mathcal{L}_{m+1}(u_{m+1})||^2 \leq \max\{ \tilde{\sigma}_0, \tilde{\xi}_0 \} F_{p,m+1}^2 (u_{m+1}).
\]

(4.23)

Through a (discrete) Cauchy-Schwarz inequality, use of (4.22) and (4.23), and recalling that \( C_p > 4 \max\{ \tilde{\sigma}_0, \tilde{\xi}_0 \}, \) (4.21) gives

\[
|l(w_{m+1}) - B_{m+1}(u_{m+1}, w_{m+1})| \leq C' \left( \eta_{T_{m+1}}(u_{m+1}) + 2 C_p F_{p,m+1}(u_{m+1}) \right) F_{p,m+1}(v_{m+1}) \\
\leq C' \left( 1 + 2 C_{4,1}^{1/2} \right) \eta_{T_{m+1}}(u_{m+1}) F_{p,m+1}(v_{m+1}),
\]

(4.24)

using Assumption 4.1.
Finally, we derive a bound for $F_{p,m+1}(v_{m+1})$. The triangle inequality and another use of Assumption 4.1, imply
\begin{equation}
F_{p,m+1}(v_{m+1}) \leq 2^{3/2} F_{p,m}(u_m) + F_{p,m+1}(u_{m+1})
\leq 2^{3/2} C_{4.1}^{1/2} C_p^{-1} (\eta_{T_m}(u_m) + \eta_{T_{m+1}}(u_{m+1})).
\end{equation}

The result follows by using (4.25) on (4.11), (4.18) and (4.24) and by applying the resulting bounds, together with (4.12) on (4.10).

**Remark 4.3.** The constant $C_{4.2}$ depends on the maximum constant $C_{4.1}$ taken from meshes $T_m$ and $T_{m+1}$.

We are now in position to prove the main Theorem of this chapter.

**Theorem 4.1.** Let $u \in H^2_0(\Omega)$ and $C_p(\sigma_0, \xi_0) \geq \gamma_0$. Then, there exist constants $\zeta > 0$ and $0 < \alpha < 1$ depending only on the shape regularity of $T_0$ and on the marking parameter $\theta$ such that
\begin{equation}
B_{m+1}(e_{m+1}, e_{m+1}) + \zeta \eta_{T_{m+1}}^2(u_{m+1}) \leq \alpha (B_m(e_m, e_m) + \zeta \eta_{T_m}^2(u_m))
\end{equation}

**Proof.** Combining the a posteriori bound (4.1) with (4.2), and using the continuity of the bilinear form, we deduce
\begin{equation}
B_m(e_m, e_m) \leq \overline{C} \eta_{T_m}^2(u_m),
\end{equation}
with $\overline{C} := C_{cont} C_{3.1}(1 + C_{4.1})$.

Using Lemmas 4.2 and 4.1, we have, respectively
\begin{align}
B_{m+1}(e_{m+1}, e_{m+1}) + \zeta \eta_{T_{m+1}}^2(u_{m+1}) &\leq (1 + C_p^{-1}) B_m(e_m, e_m) - \frac{1}{4} |||u_{m+1} - u_m|||^2 \\
&+ C_{4.2} C_p^{-1} \eta_{T_m}^2(u_m) + (C_{4.2} C_p^{-1} + \zeta) \eta_{T_{m+1}}^2(u_{m+1}) \\
&\leq (1 + C_p^{-1}) B_m(e_m, e_m) - \frac{1}{4} |||u_{m+1} - u_m|||^2 \\
&+ \left( C_{4.2} C_p^{-1} + \left( C_{4.2} C_p^{-1} + \zeta \left(1 - \frac{\lambda \theta}{2}\right)\right) \right) \eta_{T_m}^2(u_m) \\
&+ \left( C_{4.2} C_p^{-1} + \zeta \left(1 + \frac{2}{\lambda \theta}\right)\right) |||\Delta_h(u_{m+1} - u_m)|||^2.
\end{align}
We annihilate the second term on the right-hand side of (4.28) with the last component of the fourth term on the right-hand side of (4.28) by setting

\[ \zeta = \left( 4 \left( 1 + \frac{2}{\lambda \theta} C_{(4.1)} \right) \right)^{-1} - C_{(4.2)} C_p^{-1}, \]

and we set \( C_p = C_{(4.2)}/(\epsilon \zeta) \) for some \( \epsilon > 0 \), thereby yielding

\[
B_{m+1}(e_{m+1}, e_{m+1}) + \zeta h_{m+1}^2(u_{m+1}) \leq (1 + \epsilon \zeta) B_m(e_m, e_m) + \left( 2\epsilon + 1 - \frac{\lambda \theta}{2} \right) \zeta h_{m+1}^2(u_m) \\
\leq (1 - \epsilon \zeta) B_m(e_m, e_m) + \left( 2\epsilon (\bar{C} + 1) + 1 - \frac{\lambda \theta}{2} \right) \zeta h_{m+1}^2(u_m),
\]

using (4.27). Set \( \epsilon = \lambda \theta/(8(\bar{C} + 1)) \) (and \( C_p \) large enough to make \( \zeta > 0 \)) to arrive to

\[
B_{m+1}(e_{m+1}, e_{m+1}) + \zeta h_{m+1}^2(u_{m+1}) \leq \left( 1 - \frac{\lambda \theta \zeta}{8(\bar{C} + 1)} \right) B_m(e_m, e_m) + \left( 1 - \frac{\lambda \theta}{4} \right) \zeta h_{m+1}^2(u_m).
\]

The result follows by choosing

\[ \alpha := \max \left\{ 1 - \frac{\lambda \theta \zeta}{8(\bar{C} + 1)}, 1 - \frac{\lambda \theta}{4} \right\}. \]

\[ \square \]

**Remark 4.4.** From the proof of Theorem 4.1, it is evident that the continuity-penalisation parameter \( C_p \) is of order \( (\lambda \theta)^{-2} \). This choice of \( C_p \) may not be practical for small values of \( \theta \). The numerical examples, however, indicate that in practice smaller values of \( C_p \) appear to be sufficient for convergence.

### 4.3 Numerical Experiments

In this section we verify our theoretical results with a practical implementation of the adaptive algorithm applied to two example problems. We use the same example 1 as in the previous chapter to enable comparison of results between these two different adaptive approaches. In each of the examples shown below, we set the polynomial...
degree \( r \) equal to 2 and, for the sake of simplicity, the DG solution of (2.39) is computed using the interior penalty parameters of equal values \( \sigma_0 = \xi_0 = C_p \).

Note that the adaptive algorithm here is essentially different to the one studied numerically in the previous chapter: 1) marking is done by Dörfler strategy 2) no coarsening is performed 3) conforming meshes based on triangular elements are used throughout the adaptive process. The DG method was implemented using Fenics C++ library (www.fenics.org) and the linear systems were solved using UMFPACK (www.cise.ufl.edu/research/sparse/umfpack/) direct solver. It is evident from the two examples that the adaptive algorithm described in section 4.1 does indeed lead to consistent error reduction. In the examples below, we are interested in the behaviour of the effectivity index \( \frac{||e||}{\eta} \), error reduction properties of the estimators \( \eta(u_m) \) (given in (4)) and \( (\eta(u_m)^2 + C_p^2 F_p^2(u_m))^{1/2} \) (given by the right hand side of Theorem 3.1) and the constant in Assumption 4.1. For the latter, we monitor the quantity jump effectivity index, \( EI_{jumps} := \frac{C_p^2 F_p^2(u_m)}{\eta(u_m)} \), over adaptive loops. Results for both examples indicate that the use of estimator \( \eta(u_m) \) in marking step leads to very similar error reduction as with the estimator \( (\eta(u_m)^2 + C_p^2 F_p^2(u_m))^{1/2} \). Perhaps surprisingly, the results seem to indicate that the jump effectivity index does indeed remain bounded by a constant from above throughout the adaptive process and also its average value tends towards a constant value asymptotically which is in accordance with our assumption. Obviously, only a limited amount of parameter values and examples are studied here.

### 4.3.1 Example 1

We recall the solution for example 1 from 3.3.1; let \( \Omega = (0, 1)^2 \) and select \( f \) so that the analytical solution to (3.1) and (3.2) is given by

\[
  u(x, y) = \sin(2\pi x)^2 \sin(2\pi y)^2;
\]

this is a variant of the model problem considered in [91], cf. also [55].

In Figure 4.1 we show the mesh generated by the adaptive algorithm after 23 and 21 adaptive refinement steps with penalties \( \sigma_0 = \xi_0 = 10 \) and \( \sigma_0 = \xi_0 = 15 \) and with the marking parameter \( \theta = 0.25 \) and \( \theta = 0.35 \) respectively. In this example, most of the refinement has been performed where the gradient/curvature
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Figure 4.1: Example 1. Computational mesh with: (a) $\theta = 0.25$, $\sigma_0 = \xi_0 = 10$, 21488 elements, after 23 adaptive refinements; (b) $\theta = 0.35$, $\sigma_0 = \xi_0 = 15$, 55400 elements, after 21 adaptive refinements;

of the analytical solution is relatively large. Refinement has concentrated on areas similar to the ones in numerical example in the previous chapter, see Figure 3.6 but without coarsening in the adaptive iteration one is left with considerably more refined elements whereas the adaptive algorithm in section 3.3 maintains “tidier” meshes with more optimal distribution of degrees of freedom. In Figure 4.2 we present a comparison of the estimators, $\eta(u_m)$ and $(\eta(u_m)^2 + C_p^2 F_p^2(u_m))^{1/2}$ and the energy norm errors which result from adaptive algorithm 4.1 where either $\eta(u_m)$ or $(\eta(u_m)^2 + C_p^2 F_p^2(u_m))^{1/2}$ is used in Dörfler marking respectively. The results are illustrated in a log-log plot of the energy norm errors and corresponding estimators against the number of degrees of freedom. The penalty and marking parameter values here were chosen from the range which was observed to guarantee convergence. The results show that both estimators lead to very similar error reduction. In Figure 4.3, dependency of the effectivity index $\frac{\eta}{\|e\|}$ on penalty parameter is illustrated again using a range of penalty parameter values representing some typical settings of these values. Here, we observe that there is an initial transient, whereby the effectivity indexes are relatively large followed by a descent to a fairly constant value, i.e., as the refinement algorithm proceeds, effectivity indexes are asymptotically approaching constant values. Furthermore, the estimator $\eta$ seems to show better
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Figure 4.2: Example 1. Error reduction by adaptive strategy with $C_p = 80$. Dörfler marking in adaptive algorithm according to $\eta(u_m)$ and $(\eta(u_m)^2 + C_p^2 F_p^2(u_m))^{1/2}$, $\theta = 0.5$. Plotted are the corresponding energy errors and estimator values against degrees of freedom.

effectivity index behaviour in that it reaches the asymptotic value slightly faster than $(\eta(u_m)^2 + C_p^2 F_p^2(u_m))^{1/2}$. Also, the effectivity index of $\eta$ does not depend as much on the penalty parameter $C_p$ as $(\eta(u_m)^2 + C_p^2 F_p^2(u_m))^{1/2}$. In Figure 4.4, we monitor the jump effectivity index for varying values of the penalty parameter a) and the marking parameter $\theta$ b). The values with $\theta = 0.6$ in Figure 4.4 b) show that after an initial transient, the values oscillate boundedly between 7 and 11 with asymptotic average approximately 9 and this trend seems to continue with a slight damping on the oscillation upon progress of adaptive steps.

4.3.2 Example 2

In this second example, we investigate the performance of the adaptive interior penalty DG method (2.39) for a problem with a corner singularity in $u$. To this end, we let $\Omega$ be the L-shaped domain $(-1, 1)^2 \setminus [0, 1) \times (-1, 0]$, and select $f$ so that the
analytical solution to (3.1) and (3.2) is given by (see [60, 21])

\[ u(r, \phi) = (r^2 \cos^2(\phi) - 1)^2(r^2 \sin^2(\phi) - 1)^2r^{1+z_0}g(\phi); \]

where \( z_0 = 0.544483736782464 \) is a root of the equation

\[ \sin^2(z_0 \omega) = z_0^2 \sin^2(\omega) \]

with \( \omega = \frac{3\pi}{2} \) and the function \( g \) is given by

\[
g(\phi) = \left( \frac{1}{z_0 - 1} \sin((z_0 - 1)\phi) - \frac{1}{z_0 + 1} \sin((z_0 + 1)\phi) \right) \\
\times (\cos((z_0 - 1)\phi) - \cos((z_0 + 1)\phi)) \\
- \left( \frac{1}{z_0 - 1} \sin((z_0 - 1)\phi) - \frac{1}{z_0 + 1} \sin((z_0 + 1)\phi) \right) \\
\times (\cos((z_0 - 1)\omega) - \cos((z_0 + 1)\omega)) \quad (4.31)\]

The Laplacian \( \Delta u \) of the analytical solution \( u \) has a singularity at the corner located at the origin of \( \Omega \);

In Figure 4.5 we show the mesh generated by the adaptive algorithm after 24 and 19 adaptive refinement steps with penalties \( \sigma_0 = \xi_0 = 40 \) and \( \sigma_0 = \xi_0 = 25 \) and with the marking parameter \( \theta = 0.2 \) and \( \theta = 0.35 \) respectively. In this example, most of the refinement has been concentrated in the vicinity of the re-entrant corner located at the origin i.e. near the singularity of the Laplacian of the solution, as well as in the regions where the gradient/curvature of the analytical solution is relatively large. As expected, refinement near the origin is very similar to the numerical example in the previous chapter, see Figure 3.8. In Figure 4.6 we present a comparison of the estimators, \( \eta(u_m) \) and \( (\eta(u_m)^2 + C_p^2 F_p^2(u_m))^{1/2} \) and the energy norm errors which result from adaptive algorithm 4.1 where either \( \eta(u_m) \) or \( (\eta(u_m)^2 + C_p^2 F_p^2(u_m))^{1/2} \) is used in Dörfler marking respectively. The results are illustrated in a log-log plot of the energy norm errors and corresponding estimators against the number of degrees of freedom. The penalty and marking parameter values here were chosen from the range which was observed to guarantee convergence. As in the previous example, the results show that both estimators lead to very similar error reduction. In Figure 4.7, dependency of the effectivity index \( \frac{\eta}{\|e\|} \) on penalty parameter is
illustrated again using a range of penalty parameter values representing some typical settings of the values. Here, we observe like in the previous example that there is an initial transient, whereby the effectivity indexes are relatively large followed by a quick descent to a fairly constant value, i.e. as the refinement algorithm proceeds, effectivity indexes are asymptotically approaching constant values. Again, like in the previous example, the estimator $\eta$ seems to exhibit better effectivity index behaviour in that it reaches the asymptotic value slightly faster than $(\eta(u_m)^2 + C_p^2 F_p^2(u_m))^{1/2}$. Also, the effectivity index of $\eta$ does not depend as much on the penalty parameter $C_p$ as $(\eta(u_m)^2 + C_p^2 F_p^2(u_m))^{1/2}$. In Figure 4.8, we monitor the jump effectivity index for varying values of the penalty parameter $a$) and the marking parameter $\theta$ b). Here the asymptotic tendency of the jump effectivity index values is demonstrated after an initial transient; the values oscillate boundedly between 10 and 20 with asymptotic average approximately 14 and this trend seems to continue with a damping on the oscillation upon progress of adaptive steps.
Figure 4.3: Example 1. Effectivity indexes against degrees of freedom with $\theta = 0.35$, over 20 adaptive loops.

(a) Effectivity indexes for estimator $(\eta(u_m))^2 + C_p^2 F_p^2(u_m))^{1/2}$ against degrees of freedom with $\theta = 0.35$, over 20 adaptive loops; Asymptotic values range between 4 and 13 depending on the penalty parameter value, $C_p \in \{10, 20, 40, 80, 320, 1280\}$.

(b) Effectivity indexes for estimator $\eta(u_m)$ against degrees of freedom with $\theta = 0.35$, over 20 adaptive loops; Asymptotic values range between 1.5 and 4 depending on the penalty parameter value, $C_p \in \{10, 20, 40, 80, 320, 1280\}$. 
Figure 4.4: Example 1. Jump bound 4.1 constant estimation, monitoring the quantity, jump effectivity index $E_I^{\text{jumps}} := C_p^2 E_p^2(u_m)/\eta(u_m)$, over adaptive steps.

(a) Jump effectivity, $E_I^{\text{jumps}}$, values plotted with $\theta = 0.5$ for $C_p \in \{200, 1000, 2000\}$ over 17 adaptive steps.

(b) Jump effectivity, $E_I^{\text{jumps}}$, values plotted with $C_p = 80$ for varying values of $\theta \in \{0.4, 0.5, 0.6\}$ over 17 adaptive loops.
Figure 4.5: *Example 2.* Computational mesh with: (a) $\theta = 0.2$, $\sigma_0 = \xi_0 = 40$, 6310 elements, after 24 adaptive refinements; (b) $\theta = 0.35$, $\sigma_0 = \xi_0 = 25$, 20420 elements, after 19 adaptive refinements;

Figure 4.6: *Example 2.* Error reduction by adaptive strategy with $C_p = 80$. Dörfler marking in adaptive algorithm according to $\eta(u_m)$ and $(\eta(u_m)^2 + C_p^2 F_p(u_m))^{1/2}$, $\theta = 0.5$. Plotted are the corresponding energy errors and estimator values against degrees of freedom.
4.3 Numerical Experiments

Figure 4.7: Example 2. Effectivity indexes against degrees of freedom with $\theta = 0.35$ and over 20 adaptive loops.

(a) Effectivity indexes for estimator $(\eta(u_m))^2 + C_p^2 r_p^2(u_m))^{1/2}$ against degrees of freedom with $\theta = 0.35$ and over 20 adaptive loops; Asymptotic values range between 3.5 and 12 depending on the penalty parameter value, $C_p \in \{10, 20, 40, 80, 320, 1280\}$.

(b) Effectivity indexes for estimator $\eta(u_m)$ against degrees of freedom with $\theta = 0.35$ and over 20 adaptive loops; Asymptotic values range between 1 and 3.5 depending on the penalty parameter value, $C_p \in \{10, 20, 40, 80, 320, 1280\}$.
Figure 4.8: *Example 2.* Jump bound 4.1 constant estimation, monitoring the quantity, jump effectivity index $EI_{\text{jumps}} := C_p^2 F_p^2 (u_m)/\eta(u_m)$, over adaptive steps.

(a) Jump effectivity, $EI_{\text{jumps}}$, values plotted with $\theta = 0.5$ for $C_p \in \{200, 1000, 2000\}$ over 17 adaptive steps.

(b) Jump effectivity, $EI_{\text{jumps}}$, values plotted with $C_p = 80$ for varying values of $\theta \in \{0.4, 0.5, 0.6\}$ over 17 adaptive loops.
Our main results in this chapter are a posteriori error bounds in the energy norm for a family of fully discrete approximations of the following PDE problem.

**Problem 5.1 (linear parabolic boundary-initial value problem).** Given an open polygonal domain $\Omega \subseteq \mathbb{R}^d$, $d = 2$ a real number $T > 0$, a function

$$f \in L^\infty(0, T; L^2(\Omega)), \quad \text{and} \quad a \in L^\infty(\Omega \times (0, T))^{d \times d},$$

(5.1)

find a function $u \in L^2(0, T; \mathcal{H})$,

$$u_t \in L^\infty(0, T; \mathcal{H}'),$$

(5.2)

and such that

$$u_t + \hat{\gamma}_1 \Delta^2 u - \hat{\gamma}_2 \nabla \cdot (a \nabla u) = f \quad \text{on} \quad \Omega \times [0, T],$$

$$u(0) = u_0 \quad \text{on} \quad \Omega, \quad \text{and} \quad u|_{\Gamma^0}(t) = 0, \quad \hat{\gamma}_1 \nabla u \cdot n|_{\Gamma^0}(t) = 0, \quad \text{for} \ t \in [0, T]$$

(5.3)

and $(\hat{\gamma}_1, \hat{\gamma}_2) \in \{(1, 1), (1, 0), (0, 1)\}$.

where the space $\mathcal{H}$ and its dual, $\mathcal{H}'$, depend on the parameter $\hat{\gamma}_1$, see definition (5.5).
Remark 5.1. Note that if $\hat{\gamma}_1 = 0$, i.e. the equation is purely a second order equation, the analysis in this chapter is valid also for three dimensional domains, $\Omega \subseteq \mathbb{R}^d$ with $d = 3$.

In section 5.1 we propose a class of numerical methods for solving this problem. These methods consist in a backward Euler time-stepping scheme in combination with symmetric interior penalty method introduced in (2.39) for discretization of the fourth order part and various choices of spatial DG methods for the discretization of the second order part of the equation. Our emphasis for the second order part is on the widely applied interior penalty discontinuous Galerkin method [10, 84, 66]. Although our focus here is on a simple time-stepping such as implicit Euler, different time-stepping methods, of higher-order, may be studied using the reconstruction approach. This has been successfully used with spatially conforming methods [6, 7] and we plan to explore this issue in the future.

We consider the notation of Problem 5.1 to be valid throughout the chapter and $u$ denotes the solution of problem (5.3). Although the assumption $f \in L^\infty(0, T; L^2(\Omega))$ may be weakened, we refrain from doing it for simplicity’s sake. In fact, we consider $f$ to be piecewise continuous in time with a finite number of time-discontinuities and with the implied constraints on the time partition, to be discussed in section 5.1.2. The matrix-valued function $\mathbf{a}(\cdot, t)$, for each $t \in (0, T)$ is allowed to have jump discontinuities; the set of spatial discontinuities of $\mathbf{a}$ will be considered to be aligned with the finite element meshes. For simplicity, we shall assume that $\mathbf{a}$ is continuous in time, but a finite number of discontinuities can be accounted for easily, as long as these occur at the points of the time partition in the fully discrete scheme. The PDE (5.3) is assumed to be uniformly elliptic in the sense that the supremum and the infimum of the set

$$\{ (\mathbf{a}(x, t)\zeta) \cdot \zeta / |\zeta|^2 : \zeta \in \mathbb{R}^d, (x, t) \in \Omega \times (0, T) \}$$

are both positive real numbers. As for the boundary values, we remark that our approach can be appropriately modified in order to extend homogeneous to general time-dependent Dirichlet boundary values for the second order case. Note, however, that some additional conditions have to be met for the extension of the fourth order case to the general Dirichlet boundary conditions which are achieved using slightly
different formulation (see remarks 2.3 and 3.2). With these assumptions, we have for the case \( \hat{\gamma}_1 = 0 \) that the solution to (5.3) exists and satisfies \( u \in C^0(0, T; H^1_0(\Omega)) \) and \( u_t \in L^2(0, T; L^2(\Omega)) \), see [71]. Similarly, for the case \( \hat{\gamma}_1 > 0 \), we assume that the solution to (5.3) exists and satisfies \( u \in C^0(0, T; H^2_0(\Omega)) \) and \( u_t \in L^2(0, T; L^2(\Omega)) \).

We refer the reader to the works of Solonnikov [86], Yao and Zhou [99], King, Stein and Winkler [70] and Danumjaya and Pani [37] for the existence and uniqueness of solutions to the fourth order parabolic problems.

In line with a unified approach to a posteriori error analysis for elliptic-problem DG methods [3, 4, 31] our discussion will be presented first in an abstract setting. Our results are then shown to be applicable to a wide class of DG methods provided that a posteriori error bounds for the corresponding steady-state problem are available.

A key tool in our a posteriori error analysis is the elliptic reconstruction technique [75]. Roughly speaking, the elliptic reconstruction technique, as far as energy estimates are concerned, allows neat separation of the time discretization analysis from the spatial one. This technique, which has been adapted to tackle fully-discrete schemes via energy methods for conforming methods [72], is extended in this work to the nonconforming setting, to all methods that meet the two requirements above. Briefly said, the idea of elliptic reconstruction—denoting by \( u \) the solution of (5.3) and by \( U \) that of the discrete problem—consists in building an auxiliary function \( w \), called the elliptic reconstruction of \( U \), which satisfies two key properties: (a) a PDE-like relation binds the parabolic error \( u - w \) with data quantities only involving \( w - U \) and the problem’s data, \( f, a, \) and \( u_0 \), (b) the function \( U \) is the Ritz projection of \( w \) onto \( S^r \). Note that \( w \) is an analysis-only device that, despite its name, it is not a computable object. Fortunately, computing \( w \) is not needed in practice, as it does not appear in the resulting a posteriori bounds.

In purely second order case, \( \hat{\gamma}_1 = 0 \), it is possible to obtain similar a posteriori error estimates, for each single method at hand, by working directly, i.e., without using the elliptic reconstruction technique. For example, in a recently published paper [49], flux-reconstruction-based estimators are used to build the elliptic-error estimators. By measuring the error in a more sophisticated norm in the spirit of [95], explicit-constant upper and lower bounds are derived. In contrast we use simpler residual estimators inspired from [68] and work directly with the energy norm. A
feature of our approach, is that it allows the derivation of “abstract” results that can accommodate any type of spatial-estimators in parabolic problems. Furthermore, it is not obvious how the flux-reconstruction-based approach in [49] can be extended to the fourth order parabolic problems which is the main objective here. To the best of our knowledge, the results presented here are the first and only results of their kind for the fourth order parabolic problems utilising DG method in space.

We remark that new estimators arise in the derivation of fully discrete a posteriori error bounds, due to the time-dependent diffusion tensor considered in this work, compared to [72] where only time-independent diffusion coefficients are addressed. Finally, we point out that it would be possible to follow a similar approach to ours, i.e., using the elliptic reconstruction, in order to derive a posteriori error estimates in lower order functional spaces such as $L^\infty(0,T;L^2(\Omega))$, but this will not be pursued further in this thesis.

The following is an outline of this chapter. After introducing some notation needed especially for this chapter and the method in 5.1, the elliptic reconstruction is used to develop an abstract framework for spatially semidiscrete schemes in 5.2 and their fully discrete counterpart in 5.4. The actual error estimators for each particular method are then consequences of our abstract framework and specific elliptic error estimators such as the ones presented in [14, 68, 30, 65, 67, 47]. Moreover, in section 5.3 we prove a posteriori bounds for the corresponding steady state problem of (5.1) for interior penalty discontinuous Galerkin methods, thus extending existing results [14, 68, 65, 67] to the case of general (non-diagonal) diffusion tensor, with minimal regularity assumptions on the exact solution [47, cf.] in the second order case. Furthermore, we extend these results to the fourth order equations using the proof from chapter 3 in combination with the second order part. Due to the analysis for this elliptic steady state problem being directly related to the specific parabolic problem setting, we present the corresponding a posteriori error analysis in this chapter (and not in chapter 3) for cohesion and to avoid confusion. The elliptic a posteriori bounds are then combined with the general framework presented in 5.2 and 5.4 to deduce fully computable bounds for the DG-approximation error of the parabolic problem. Last in 5.5 we summarize results from computer experiments aimed at exhibiting the reliability (derived theoretically) and efficiency of the error estimators in the special case of the (SIPDG) method.
5.1 Preliminaries

We consider the space
\[ \mathcal{H} := \begin{cases} H^1_0(\Omega) & \hat{\gamma}_1 = 0 \\ H^2_0(\Omega) & \hat{\gamma}_1 > 0 \end{cases}, \]  
and we denote its dual by,
\[ \mathcal{H}' := \begin{cases} H^{-1}(\Omega) & \hat{\gamma}_1 = 0 \\ H^{-2}(\Omega) & \hat{\gamma}_1 > 0 \end{cases}. \]  
See definitions (2.6) and (2.8) for details. We remark that the Poincaré-Friedrichs inequality, see Appendix Lemma C.4, turns the \( H^2(\Omega) \) seminorm into a norm on \( H^1_0(\Omega) \). We consider thus \( ||\Delta v||_{L^2(\Omega)} \) to be the norm on \( H^2_0(\Omega) \) (due to the \( L^2 \) norm of mixed derivatives of \( v \) being bounded by the \( L^2 \)-norm of \( \Delta v \)) and, similarly, we consider \( ||\nabla v||_{L^2(\Omega)} \) to be the norm on \( H^1_0(\Omega) \). Throughout the chapter, we use the shorthand \( f(t) = f(\cdot, t) \), for a function \( f : [0, T] \times \Omega \to \mathbb{R} \). We consider some shorthand notation for quantities involving the diffusion tensor \( a \). In particular, we define the elementwise constant functions \( a_\#, a_\flat : \Omega \times [0, T] \to \mathbb{R} \) by
\[ a_\#(\cdot, t) |_{\kappa} := |||\sqrt{a}(\cdot, t)||^2_{L^\infty(\kappa)} \text{ and } a_\flat(\cdot, t) |_{\kappa} := |||(\sqrt{a}(\cdot, t))^{-1}||^2_{L^\infty(\kappa)}, \]  
for \( \kappa \in \mathcal{T} \), and \( a_\# = \{a_\#\}, \ a_\flat = (\{1/a_\#\})^{-1} \), on \( \Gamma \), where \( |\cdot|_2 \) denotes the Euclidean-induced matrix norm. Finally, let
\[ \alpha_\#(t) := \max_{x \in \Omega} a_\#(x, t) \text{ and } \alpha_\flat(t) = \min_{x \in \Omega} a_\flat(x, t). \]  

The weak formulation of the problem 5.3 reads,

Find \( u \in L^2(0, T; \mathcal{H}) \) with \( \partial_t u \in L^\infty(0, T; \mathcal{H}') \) such that
\[ \langle \partial_t u, v \rangle + \hat{\gamma}_1 \langle \Delta u, \Delta v \rangle + \hat{\gamma}_2 \langle a \nabla u, \nabla v \rangle = \langle f, v \rangle \quad \text{for all } v \in \mathcal{H}, t \in (0, T], \]  
and \( u(0) = u_0 \).

We denote the bilinear form arising from the weak formulation by,
\[ \hat{B}(t; u, v) := \hat{\gamma}_1 \langle \Delta u, \Delta v \rangle + \hat{\gamma}_2 \langle a(\cdot, t) \nabla u, \nabla v \rangle. \]
5.1 Preliminaries

5.1.1 Spatial DG discretization

Introduce the DG space $\mathcal{S} := S + H$ and a corresponding DG bilinear form $B : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$, which is assumed to be an extension of the bilinear form in weak formulation, 5.10,

$$B(t; v, z) = \hat{B}(t; v, z) \text{ for all } v, z \in H, \ t \in (0, T].$$  \hfill (5.11)

Though $B$ is time-dependent, we do not write it explicitly in the semidiscrete case and omit the $t$.

The space $\mathcal{S}$ is equipped with a DG norm, denoted $||\cdot||$ and depending on the method at hand, which extends the energy norm, i.e.,

$$||v|| = \left(\hat{\gamma}_1 ||\Delta v||^2 + \hat{\gamma}_2 \sqrt{a(\cdot, t)\nabla v||}^2\right)^{1/2} \text{ for all } v \in H,$$  \hfill (5.12)

for $t \in [0, T]$. Also here, the norm is time-dependent, but this dependence is not explicitly written. A norm equivalence between the energy norm $||\cdot||$ and $\left(\hat{\gamma}_1 ||\Delta v||^2 + \hat{\gamma}_2 \sqrt{a(\cdot, t)\nabla v||}^2\right)^{1/2}$ in $\mathcal{H}$, uniformly with respect to $t$, suffices for all the bounds presented below to hold, modulo a multiplicative constant; but, we eschew this much generality for clarity’s sake.

The semidiscrete DG method in space for problem (5.3), reads as follows:

Find $U \in C^{0,1}(0, T; S^r)$ such that

$$\langle \partial_t U, V \rangle + B(U, V) = \langle f, V \rangle \quad \text{for } V \in S^r, t \in [0, T].$$  \hfill (5.13)

We stress that these assumptions are satisfied by many DG methods for second order elliptic problems available in the literature, possibly by using inconsistent formulations [11]. Also note that these assumptions are satisfied by the method (2.39) for fourth order equations. Moreover, (5.11) and (5.12) are satisfied by interior penalty discontinuous Galerkin (along with the corresponding energy norm) considered below as a paradigm.

Assumption (5.11) implies the consistency of the bilinear form $B$ on $\mathcal{H}$, i.e.,

$$\langle \partial_t u, v \rangle + B(u, v) = \langle f, v \rangle, \quad \text{for all } v \in \mathcal{H},$$  \hfill (5.14)
where \( u \) is the exact (weak) solution to the initial-boundary value problem \((5.9)\).

### 5.1.2 Fully discrete solution

To further discretize in time, consider an increasing time partition \( \{ t_n \}_{n=0}^{N} \), and the corresponding time-steps \( \tau_n = t_n - t_{n-1} \), for \( n = 1, \ldots, N \). For each \( n = 0, \ldots, N \), \( S^r_n \) is a DG finite element space of fixed degree \( r \) built on a partition \( \mathcal{T}_n \), which may be different from \( \mathcal{T}_{n-1} \) when \( n \geq 1 \). In 5.4.2, we will further discuss the sequence of meshes and compatibility relations among them.

Let \( f_n(x) := f(x, t_n) \) and let \( U_0 \) be the projection (or an interpolation) of \( u_0 \) onto the finite element space \( S^r_0 \). We say that \( \{ U_n \}_{n=0}^{N} \) is a **fully discrete solution** of \((5.3)\) if, for each \( n = 0, \ldots, N \) we have that \( U_n \in S^r_n \) satisfies

\[
\langle (U_n - U_{n-1})/\tau_n, V \rangle + B^n(U_n, V) = \langle f_n, V \rangle \quad \text{for all } V \in S^r_n,
\]

\((5.15)\)

Since the bilinear form \( B \) depend on time, in the fully discrete setting, we denote its value at time \( t \) by \( B(t) \) and when \( t = t_n \) we take \( B^n := B(t_n) \).

Noting that the term \( U_{n-1} \) can be replaced by \( \Pi^n U_{n-1} \), where \( \Pi^n : L^2(\Omega) \rightarrow S^r_n \) is the orthogonal projection, we consider a slightly more general situation where \( \Pi^n U_{n-1} \) in \((5.15)\) is replaced by \( I^n U_{n-1} \); here \( I^n : S^r_{n-1} \rightarrow S^r_n \) is a general data transfer operator, depending on the particular implementation. The operator \( I^n \) may coincide with \( \Pi^n \), but it may be an interpolation operator for example. The general fully discrete Euler scheme then reads

\[
\langle (U_n - I^n U_{n-1})/\tau_n, V \rangle + B^n(U_n, V) = \langle f_n, V \rangle \quad \text{for all } V \in S^r_n,
\]

\((5.16)\)

We have taken \( f^n = f(t_n) \), but we could take a more general approximation than \( f(t_n) \), for example, a good choice is also given by \( f^n := \int_{t_{n-1}}^{t_n} f(s) ds \), for which a suitable modification of our arguments leads to similar results. However, we remark that, in practise, the appropriate approximation should be carefully chosen for the adaptive algorithm if the corresponding estimator, \( \beta_n \) is included in the time estimators.
5.2 Abstract a posteriori bounds for the semidiscrete problem

We derive next an abstract a posteriori error bound for the quantity

$$\|u - U\|_{L^2(0,T;\mathcal{V})} := \left( \int_0^T |||u(t, \cdot) - U(t, \cdot)|||^2 \right)^{1/2},$$  \hspace{1cm} (5.17)

where $||| \cdot |||$ denotes the appropriate (space) energy norm.

In the a posteriori error analysis below, we shall make use of the idea of elliptic reconstruction operators introduced in [75] for the semidiscrete problem and extended to fully discrete (conforming-in-space) methods in [72] in the context of second order problems.

**Definition 5.1 (elliptic reconstruction and discrete operator).** Let $U$ be the (semidiscrete) DG solution to the problem (5.13). We define the elliptic reconstruction $w \in \mathcal{H}$ of $U$ to be the solution of the elliptic problem

$$B(w, v) = \langle AU - \Pi f + f, v \rangle \text{ for all } v \in \mathcal{H},$$  \hspace{1cm} (5.18)

where $\Pi : L^2(\Omega) \to S^r$ denotes the orthogonal $L^2$-projection on the finite element space $S^r$, and $A : S^r \to S^r$ denotes the discrete DG operator defined by

$$\langle AZ, V \rangle = B(Z, V) \text{ for all } V \in S^r,$$  \hspace{1cm} (5.19)

for each $Z \in S$. Note that this is valid on $[0, T]$.

**Remark 5.2 (the role of the elliptic reconstruction).** The elliptic reconstruction is well defined. Indeed, $AU \in S^r$ is the unique $L^2$-Riesz representation of a linear functional on the finite dimensional space $S^r$ and the existence and uniqueness of (weak) solution of (5.18), with data $AU - \Pi f + f \in L^2(\Omega)$, follows from the Lax–Milgram Theorem and from the Lemma 2.2 for the bilinear form.

The key property of $w$ is that the DG solution $U$ of the semidiscrete time-dependent problem (5.13) is also the DG solution of the steady-state boundary-value problem (5.18). Indeed, let $W \in S^r$ be the DG-approximation to $w$, defined by the
finite dimensional linear system

\[ B(W, V) = \langle AU - \Pi f + f, V \rangle, \quad (5.20) \]

for all \( V \in S^r \), which implies \( B(W, V) = \langle AU, V \rangle = B(U, V) \) for all \( V \in S^r \), i.e., \( W = U \).

**Definition 5.2 (error, elliptic and parabolic parts).** We shall decompose the error as follows:

\[ e := U - u = \rho - \epsilon, \text{ where } \epsilon := w - U, \text{ and } \rho := w - u, \quad (5.21) \]

where \( w = w(t) \) denotes the elliptic reconstruction of \( U = U(t) \) at time \( t \in [0, T] \). We call \( \epsilon \) the elliptic error and \( \rho \) the parabolic error.

**Lemma 5.1 (semidiscrete error relation).** Let \( u \) be the solution of Problem 5.1, \( U \) denote the solution of the DG scheme (5.13). Then, we have

\[ \langle \partial_t e, v \rangle + B(\rho, v) = 0 \text{ for all } v \in \mathcal{H}. \quad (5.22) \]

Proof. For each \( v \in \mathcal{H} \) we have

\[
\langle \partial_t e, v \rangle + B(\rho, v) = \langle \partial_t U, v \rangle + B(w, v) - \langle f, v \rangle \\
= \langle \partial_t U, v \rangle + \langle AU - \Pi f + f, v \rangle - \langle f, v \rangle \\
= \langle \partial_t U, \Pi v \rangle + \langle AU, \Pi v \rangle - \langle f, \Pi v \rangle = 0, \quad (5.23)
\]

where in the first equality we used (5.14); in the second and fourth equalities, we made use of Definition 5.1; in the third equality the properties of the orthogonal \( L^2 \)-projection onto \( S^r \) are used; finally, the last equality follows from (5.13).

**Definition 5.3 (conforming-nonconforming decomposition).** In the theory developed below, we shall consider the decomposition of the DG solution \( U \in S \) into conforming (continuous) and nonconforming (discontinuous) parts as follows

\[ U = U_c + U_d, \quad (5.24) \]
5.2 Abstract a posteriori bounds for the semidiscrete problem

where \( U_c \in S_c \),

\[
S_c := \begin{cases} 
H^1_0(\Omega) \cap S^r & \hat{\gamma}_1 = 0 \\
H^2_0(\Omega) \cap \tilde{S}^{r+2}_h & \hat{\gamma}_1 > 0
\end{cases}
\] (5.25)

and \( U_d := U - U_c \). Let

\[
e_c := e - U_d = U_c - u \in \mathcal{H}, \quad \text{and} \quad \epsilon_c := \epsilon + U_d = w - U_c \in \mathcal{H}.
\] (5.26)

**Theorem 5.1** (long-time a posteriori error bound for DG). Let \( u \) and \( U \) be the exact weak solution of (5.3) and the DG solution of the problem (5.13), respectively. Let \( w \) be the elliptic reconstruction of \( U \) as in Definition 5.1. Assuming the consistency of the bilinear form within space \( \mathcal{H} \), Lemma 2.2, and that a decomposition of the form (5.24) is available, then the following error bound holds

\[
\|U - u\|_{L^2(0,T;\mathcal{H})} \leq 3\|w - U\|_{L^2(0,T;\mathcal{H})} + 2\|U_d\|_{L^2(0,T;\mathcal{H})} + 2\|\partial_t U_d C_{\alpha}\|_{L^2(0,T;\mathcal{H}')} + 2\|u_0 - U_c(0)\|,
\] (5.27)

where

\[
C_{\alpha} := (\hat{\gamma}_2(1 - \hat{\gamma}_1) + \hat{\gamma}_1 \sqrt{\alpha_\flat})/(\sqrt{\alpha_\flat}).
\] (5.28)

**Proof.** Set \( v = e_c \) in (5.22); then, in view of (5.26), we have

\[
\langle \partial_t e_c, e_c \rangle + B(\rho, \rho) = B(\rho, \epsilon_c) - \langle \partial_t U_d, e_c \rangle.
\] (5.29)

Recalling (2.47), (2.46) and that \( \rho, e_c \in \mathcal{H} \) for every \( t \in [0, T] \), using the Cauchy–Schwarz inequality, and the duality pairing \((\mathcal{H}', \mathcal{H})\), we arrive to

\[
\frac{d}{dt}\|e_c\|^2 + |||\rho|||^2 \leq |||\rho||| |||e_c||| + \|\partial_t U_d\|_{\mathcal{H}'}\|e_c\|_{\mathcal{H}}.
\] (5.30)

Also, (5.26) with (5.21) implies

\[
\|e_c\|_{\mathcal{H}} \leq (|||e_c||| + |||\rho|||) C_{\alpha}.
\] (5.31)

Setting \( I_1 := |||e_c|||, I_2 := |||C_{\alpha}\partial_t U_d\|_{\mathcal{H}'} \) in (5.30) and rearranging lead to

\[
\frac{d}{dt}\|e_c\|^2 + |||\rho|||^2 \leq |||\rho|||(I_1 + I_2) + I_1 I_2,
\] (5.32)
which implies
\[
\frac{d}{dt} ||e_c||^2 + ||\rho||^2 \leq 4(I_1^2 + I_2^2).
\] (5.33)
Integration on \([0,T]\) and taking square roots yields
\[
||\rho||_{L^2(0,T;\mathcal{H})} \leq ||e_c(0)|| + 2||e_c||_{L^2(0,T;\mathcal{H})} + 2\|C\alpha\partial_t U_d\|_{L^2(0,T;\mathcal{H})}.
\] (5.34)

The assertion follows using triangle inequality on (5.21) and (5.26).

**Theorem 5.2** (short-time a posteriori error bound for DG). *Let the assumptions of Theorem 5.1 hold. Then the following error bound holds:
\[
||u - U||_{L^2(0,T;\mathcal{H})} \leq \sqrt{\frac{3}{2}}||w - U||_{L^2(0,T;\mathcal{H})} + \sqrt{\frac{3}{2}}||U_d||_{L^2(0,T;\mathcal{H})} + 2\|\partial_t U_d\|_{L^1(0,T;L^2(\Omega))} + \sqrt{2}||u_0 - U_c(0)||.
\] (5.35)
Proof. Let \(T_0 \in [0,T]\) be such that
\[
||e_c(T_0)|| = \max_{0 \leq t \leq T} ||e_c(t)|| =: E_c.
\] (5.36)
Then, (5.29) implies
\[
\frac{d}{dt} ||e_c||^2 + ||\rho||^2 \leq ||\rho|| \||e_c|| + E_c\|\partial_t U_d\|,
\] (5.37)
which, after integration on \([0,T_0]\), yields
\[
\frac{1}{2}E_c^2 + ||\rho||_{L^2(0,T_0;\mathcal{H})}^2 \leq \frac{1}{2}||e_c(0)||^2 + ||\rho||_{L^2(0,T_0;\mathcal{H})}||e_c||_{L^2(0,T_0;\mathcal{H})} + E_c\|\partial_t U_d\|_{L^1(0,T;L^2(\Omega))},
\] (5.38)
or
\[
\frac{1}{4}E_c^2 \leq \frac{1}{2}||e_c(0)||^2 + \frac{1}{2}||e_c||_{L^2(0,T;\mathcal{H})}^2 + \|\partial_t U_d\|_{L^1(0,T;L^2(\Omega))}^2.
\] (5.39)
Going back to (5.37), upon integration with respect to \(t\) between \([0,T]\), we obtain
\[
\frac{1}{2}||\rho||_{L^2(0,T;\mathcal{H})}^2 \leq \frac{1}{2}||e_c(0)||^2 + \frac{1}{2}||e_c||_{L^2(0,T;\mathcal{H})}^2 + \frac{1}{4}E_c^2 + \|\partial_t U_d\|_{L^1(0,T;L^2(\Omega))}^2.
\] (5.40)
which, in conjunction with (5.39) gives
\[
\|\rho\|^2_{L^2(0,T;\mathcal{X})} \leq 2\|\epsilon_c(0)\|^2 + \frac{3}{2}\|\epsilon_c\|^2_{L^2(0,T;\mathcal{X})} + 4\|\partial_t U_d\|^2_{L^1(0,T;L^2(\Omega))},
\]
the final bound now follows using the triangle inequality on (5.21) and (5.26).

**Remark 5.3 (long- versus short-time bounds).** We note that the crucial difference between bounds (5.27) and (5.35) is that in the latter the \(L^1\)-accumulation term \(\|\partial_t U_d\|_{L^1(0,T;\mathcal{X})}\) is present; this implies
\[
\|\partial_t U_d\|_{L^1(0,T;L^2(\Omega))} \leq \sqrt{T}\|\partial_t U_d\|_{L^2(0,T;L^2(\Omega))},
\]
which may be preferable if \(T < 1\), but can be inefficient for long-time integration. On the other hand, the corresponding term in (5.27) is \(\|C_\alpha \partial_t U_d\|_{L^2(0,T;\mathcal{H}')}\), which can be a bit less inefficient when the diffusion tensor \(a\) varies substantially on \(\Omega\) in the purely second order case \(\hat{\gamma}_1 = 0\) and hence \(C_\alpha = 1/\sqrt{\alpha_\gamma}\) (note that if \(\hat{\gamma}_1 > 0\) then \(C_\alpha = 1\)). We note, however, that in case \(\hat{\gamma}_1 = 0\) it is possible to avoid dividing by the factor \(\alpha_\gamma\), by equipping \(\mathcal{H}'\) with the dual norm of the energy norm (5.12) in \(\mathcal{H}\). In practice, however, this improvement is relevant only if the dual norm is calculated explicitly, like in [73]. Alternatively, one can apply a Poincaré–Friedrichs inequality to bound the dual norm by the \(L^2\)-norm, which results into the reappearance of the factor \(\alpha_\gamma\).

**Remark 5.4 (elliptic a posteriori error estimates).** The bounds (5.27) and (5.35) are not (yet) explicitly a posteriori bounds: \(\|w - U\|_{L^2(0,T;\mathcal{X})}\) still needs to be bounded by a computable quantity. To this end, given \(g \in L^2(\Omega)\), consider the elliptic problem:

\[
\begin{align*}
\hat{\gamma}_1 \Delta^2 u - \hat{\gamma}_2 \nabla \cdot (a \nabla u) &= g \quad \text{in } \Omega, \\
\hat{\gamma}_1 \nabla z \cdot n &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
and \((\hat{\gamma}_1, \hat{\gamma}_2) \in \{(1,1), (1,0), (0,1)\}\).

whose solution can be approximated by the following DG method:

\[
\begin{align*}
\text{find } Z \in S^r \text{ such that } \\
B(Z,V) &= \langle g, V \rangle \quad \text{for all } V \in S^r.
\end{align*}
\]
If we assume that an a posteriori estimator functional $\mathcal{E}$ exists, i.e.,

$$|||z - Z||| \leq \mathcal{E}(Z, g, \mathcal{T}), \quad (5.45)$$

then we can computably bound $\|w - U\|_{L^2(0,T;\mathcal{V})}$ in (5.27) and (5.35) through

$$\|w - U\|_{L^2(0,T;\mathcal{V})} \leq \left( \int_0^T \mathcal{E}(U, AU - \Pi f + f, \mathcal{T})^2 \right)^{1/2}. \quad (5.46)$$

A posteriori bounds for various DG methods in context of second order problems (i.e. case $\gamma_1 = 0$) have been studied, under different assumptions on data and admissible finite element spaces, by many authors [14, 68, 65, 4, 67, 47, 31]. Thus Theorems 5.1 and 5.2 can be applied to any DG—and more generally to any non-conforming—method satisfying (5.11) and (5.12), and for which (5.45) is available. The object of section 5.3 is to address this for various interior penalty methods which are used to discretize the second order part of the equation (5.43) and symmetric interior penalty discontinuous Galerkin discretization of the fourth order part defined in (2.39).

5.3 A posteriori error bounds for the interior penalty DG method

Here we extend the energy-norm a posteriori bounds for the family interior penalty DG methods cf.[14, 68, 65] for the biharmonic/Poisson problem, to the case of the general diffusion problem (5.43) combined with a fourth order problem. A similar analysis has recently appeared also in [47] for a second order problem, while a related DG method based on weighted averages for anisotropic and high-contrast second order diffusion problems can be found in [48]. We remark that our results can be generalized as to allow for inhomogeneous or mixed boundary conditions following [65, 68, resp.] at least in the purely second order case.

DEFINITION 5.4 (interior penalty DG method). For $z, v \in \mathcal{S}$, the bilinear form $B : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ for the interior penalty DG method for the problem (5.43) can be
written as

\[
B(z, v) := \hat{\gamma}_1 \left( \int_\Omega \left( \Delta_h z \Delta_h v + \mathcal{L}(z) \Delta_h v + \Delta_h z \mathcal{L}(v) \right) \right) \\
+ \hat{\gamma}_1 \left( \int_\Gamma \left( \sigma[z][v] + \xi[\nabla z][\nabla v] \right) \right) \\
+ \hat{\gamma}_2 \left( \int_\Omega (a \nabla_h z) \cdot \nabla_h v \right) \\
+ \hat{\gamma}_2 \left( \int_\Gamma (\theta\{a\Pi \nabla_h v\} \cdot [z] - \{a\Pi \nabla_h z\} \cdot [v] + \chi[z] \cdot [v]) \right),
\]

(5.47)

for \( \theta \in \{-1, 0, 1\} \), where \( \Pi : [L^2(\Omega)]^d \to [\{S^r\}]^d \) denotes also the orthogonal \( L^2 \)-projection operator onto \([\{S^r\}]^d\), and the penalty function \( \chi : \Gamma \to \mathbb{R} \) is defined by

\[
\chi := \frac{C_{a,\mu(T)} \{a_z\} \{a_z\}}{h},
\]

(5.48)

where the constant \( C_{a,\mu(T)} > 0 \) depends on the shape-regularity of the mesh \( T \) and on the smallest possible \( C_a > 1 \) such that

\[
C_a^{-1} \leq \frac{a_z|_{\kappa^+}}{a_z|_{\kappa^-}} \leq C_a,
\]

(5.49)

for every pair of elements \( \kappa^+ \) and \( \kappa^- \) sharing a common side.

We also define the corresponding energy norm \( \|\cdot\| \) for the interior penalty DG method by

\[
\|v\| := \left( \hat{\gamma}_1 \left( \|\Delta_h w\|_\Omega^2 + \|\sqrt{\sigma}[w]\|_\Gamma^2 + \|\xi[\nabla w]\|_\Gamma^2 \right) \\
+ \hat{\gamma}_2 \left( \|\sqrt{a\nabla_h v}\|_\Omega^2 + \|\sqrt{\chi[v]}\|_\Gamma^2 \right) \right)^{1/2},
\]

(5.50)

for \( v \in \mathcal{S} \). Note that for \( v \in S^r \), we have \( \Pi \nabla v = \nabla v \) and, therefore, \( B \) can be
reduced to the more familiar form

\[
B(z, v) := \hat{\gamma}_1 \left( \int_\Omega \left( \Delta_h z \Delta_h v + \mathcal{L}(z) \Delta_h v + \Delta_h z \mathcal{L}(v) \right) \right) \\
+ \hat{\gamma}_1 \left( \int_\Gamma \left( \sigma[z][v] + \xi[\nabla z][\nabla v] \right) \right) \\
+ \hat{\gamma}_2 \left( \int_\Omega (a \nabla_h z ) \cdot \nabla_h v \right) \\
+ \hat{\gamma}_2 \left( \int_\Gamma (\theta \{a \nabla_h v\} \cdot [z] - \{a \nabla_h z\} \cdot [v] + \chi[z] \cdot [v]) \right),
\]

for \( z, v \in S^r \) \( [10, 11, 84, 66, \text{cf.}] \). Observe that both (5.11) and (5.12) hold for the particular \( B \) and \( \|\cdot\| \) defined above.

**Remark 5.5** (conforming part of a nonconforming finite element function, second order case). The space-discontinuous finite element space \( S^r \) contains the conforming (continuous) finite element space \( S_c = S^r \cap H_0^1(\Omega) \) as a subspace. The approximation of functions in \( S^r \) by functions in \( S_c \) plays an important role in our derivation of the a posteriori bounds in the purely second order case, \( \hat{\gamma}_1 = 0 \). This can be quantified in the following result, which is an extension of [69, Thm. 2.1]. For other similar results we refer to [92, 14, 65, 28].

**Lemma 5.2** (bounding the nonconforming part via jumps, second order case). Suppose \( \mathcal{T} \) is a regular mesh and \( a \) is elementwise (weakly) differentiable. Then, for any function \( Z \in S^r \) there exists a function \( Z_c \in S_c \) such that

\[
\|Z - Z_c\| \leq C_1 \|\sqrt{h}[Z]\|_\Gamma,
\]

and

\[
\|\sqrt{\mathbf{a}} \nabla_h (Z - Z_c)\| \leq C_2 \|\sqrt{\chi}[Z]\|_\Gamma,
\]

where the constants \( C_1, C_2 > 0 \) depend only on the shape-regularity, on the maximum polynomial degree of the local basis and on \( C_a \).

The proof, omitted here, follows closely that of [68, Thm. 2.2]. Lemma 5.2 can be proved for irregular (i.e., with hanging-nodes) meshes [68, Thm. 2.3], in which case \( C_1 \) and \( C_2 \) depend on the maximum refinement and coarsening levels \( L_{\text{max}} \).
Remark 5.6 (conforming part of a nonconforming finite element function, fourth order case). In the fourth order case, $\hat{\gamma}_1 > 0$, we utilise the conforming space $S_c = \tilde{S}_h^{r+2} \cap H^2_0(\Omega)$ defined in section 3.1 whereby $S_c$ is not a subspace of the space $S^r$ and use Lemma 3.1 to bound the nonconforming part by function and normal derivative jumps.

Lemma 5.3 (a posteriori bounds for interior penalty DG method for elliptic problem). Let $\mathcal{T}$ be a regular and $a$ is elementwise (weakly) differentiable. Let $z$ and $Z$ be given by (5.43) and (5.44). Then

$$\|z - Z\| \leq \mathcal{E}(Z, a, g, \mathcal{T}),$$

(5.54)

where

$$\mathcal{E}(Z, a, g, \mathcal{T}) := C \left( \hat{\gamma}_1 \left( \|h^2 (g - \hat{\gamma}_1 \Delta_h^2 Z + \hat{\gamma}_2 \nabla_h \cdot (a \nabla_h Z))\|^2 \right. \\
+ C_p \left( \|h^{-3/2} [Z]\|^2_{1, \text{int}} + \|h^{-1/2} [\nabla Z]\|^2_{1, \text{int}} \right) \\
+ \|h^{1/2} [\Delta Z]\|^2_{1, \text{int}} + \|h^{3/2} [\nabla \Delta Z]\|^2_{1, \text{int}} \right) + \hat{\gamma}_2 \left( K_a^\mathcal{T} \right)^2 \left( \|h/\sqrt{a} (g - \hat{\gamma}_1 \Delta_h^2 Z + \hat{\gamma}_2 \nabla_h \cdot (a \nabla_h Z))\|^2 \\
+ \|\sqrt{h/a} [a \nabla_h Z]\|^2_{1, \text{int}} + \|\chi [Z]\|^2_{1, \text{int}} \right) \right)^{1/2},$$

(5.55)

and $K_a^\mathcal{T} := \max_{\Omega} \sqrt{a_2/\sigma_2}$, $C_p := \max \{1, \sigma_0, \xi_0, \sigma_0^2, \xi_0^2\}$, where $C > 0$ depends only on $\mu(T)$ and $C_a$.

Proof. Denoting by $Z_c \in S_c$ the conforming part of $Z$ as in Lemma 3.1 (or 5.2, in case $\hat{\gamma}_1 = 0$), we have

$$e := z - Z = e_c + e_d,$$

where $e_c := z - Z_c$ and $e_d := Z_c - Z$,

(5.56)

yielding $e_c \in \mathcal{H}$. Thus, we have $B(z, e_c) = \langle g, e_c \rangle$. Let $\Pi_1 : L^2(\Omega) \rightarrow \mathbb{R}$ denote the orthogonal $L^2$-projection onto the elementwise linear functions; then $\Pi_1 e_c \in S^r$ and
we define $\eta := e_c - \Pi_1 e_c$. We also have

$$B(e, e_c) = B(z, e_c) - B(Z, e_c) = \langle g, e_c \rangle - B(Z, \eta) - B(Z, \Pi_1 e_c) = \langle g, \eta \rangle - B(Z, \eta),$$

which implies

$$|||e_c|||^2 = B(e_c, e_c) = \langle g, \eta \rangle - B(Z, \eta) - B(e_d, e_c).$$

For the last term on the right-hand side of (5.58), we have, using estimate (3.10),

$$|B(e_d, e_c)| \leq \hat{\gamma}_1 \left( ||\Delta_h e_d||^2_{\Omega} + C \left( ||\sqrt{\sigma}(Z)||^2_{\Gamma^0} + ||\sqrt{\xi}(\nabla Z)||^2_{\Gamma} \right) \right)^{1/2} ||\Delta e_c||_{\Omega}
+ \hat{\gamma}_2 \left( ||\sqrt{a\nabla h} e_d|| ||\sqrt{a\nabla e_c}|| + \frac{1}{2} \sum_{s \subset \Gamma} \sum_{\kappa=\kappa^+,\kappa^-} a_{s|\kappa} \theta \sqrt{h}(\Pi h e_c)_{|\kappa} ||s|| ||e_d||_{\sqrt{h}/s} \right),$$

where $\kappa^+$ and $\kappa^-$ are the (generic) elements having $e$ as common side. Using the inverse estimate of the form $||\sqrt{h}V||_e \leq C ||V||_\kappa$ for $V = \Pi h e_c$, and the stability of the $L^2$-projection, we arrive to

$$|B(e_d, e_c)| \leq \hat{\gamma}_1 \left( ||\Delta_h e_d||^2_{\Omega} + C \left( ||\sqrt{\sigma}(Z)||^2_{\Gamma^0} + ||\sqrt{\xi}(\nabla Z)||^2_{\Gamma} \right) \right)^{1/2} ||\Delta e_c||_{\Omega}
+ \hat{\gamma}_2 \left( ||\sqrt{a\nabla h} e_d|| ||\sqrt{a\nabla e_c}|| + CK_{a} \sqrt{\chi} \Pi h e_c || \sqrt{\chi}[e_d] ||_{\Gamma} \right).$$

Finally, noting that $[e_d] = [Z]$, and making use of (5.53) we conclude that

$$|B(e_d, e_c)| \leq \hat{\gamma}_1 \left( ||\Delta_h e_d||^2_{\Omega} + C \left( ||\sqrt{\sigma}(Z)||^2_{\Gamma^0} + ||\sqrt{\xi}(\nabla Z)||^2_{\Gamma} \right) \right)^{1/2} ||\Delta e_c||_{\Omega}
+ \hat{\gamma}_2 \left( CK_{a} \sqrt{\chi} \Pi h e_c || \sqrt{\chi}[Z] ||_{\Gamma} \right).$$

To bound the first two terms on the right-hand side of (5.58), we begin by an
elementwise integration by parts yielding

\[
\langle g, \eta \rangle - B(Z, \eta) = \int_{\Omega} \left( (g - \hat{\gamma}_1 \Delta^2 Z + \hat{\gamma}_2 \nabla_h \cdot (a \nabla_h Z)) \eta - \hat{\gamma}_1 \mathcal{L}(Z) \Delta h \eta \right) \, dx
\]

\[
+ \hat{\gamma}_1 \int_{\Gamma_{\text{int}}} \{\eta\} [\nabla \Delta Z] \, ds - \hat{\gamma}_1 \int_{\Gamma_{\text{int}}} \{\nabla \eta\} \cdot [\Delta Z] \, ds
\]

\[
- \hat{\gamma}_2 \int_{\Gamma} (\sigma [Z] \cdot [\eta] + \xi [\nabla Z][\nabla \eta]) \, ds - \hat{\gamma}_2 \int_{\Gamma_{\text{int}}} \{\eta\} [a \nabla Z] \, ds
\]

\[
+ \hat{\gamma}_2 \int_{\Gamma} \theta \{a \Pi \nabla \eta\} \cdot [Z] \, ds - \hat{\gamma}_2 \int_{\Gamma} \chi [Z] \cdot [\eta] \, ds.
\]

(5.62)

The first term on the right-hand side of (5.62) can be bounded as follows:

\[
\left| \int_{\Omega} (g - \hat{\gamma}_1 \Delta^2 Z + \hat{\gamma}_2 \nabla_h \cdot (a \nabla_h Z)) \, \eta \right|
\]

\[
\leq \frac{1}{\hat{\gamma}_1 + \hat{\gamma}_2} (\hat{\gamma}_1 ||h^2 (g - \hat{\gamma}_1 \Delta^2 Z + \hat{\gamma}_2 \nabla_h \cdot (a \nabla_h Z))|| \|h^{-2} \eta\|)
\]

(5.63)

\[
\hat{\gamma}_2 ||h/ \sqrt{a_0} (g - \hat{\gamma}_1 \Delta^2 Z + \hat{\gamma}_2 \nabla_h \cdot (a \nabla_h Z))|| \|\sqrt{a_0} h^{-1} \eta\|);
\]

upon observing that \(\|h^{-1} \eta\|_{\kappa} \leq C \| \nabla e_c \|_{\kappa}\) and \(\|h^{-2} \eta\|_{\kappa} \leq C \| \Delta e_c \|_{\kappa}\), this becomes

\[
\left| \int_{\Omega} (g - \hat{\gamma}_1 \Delta^2 Z + \hat{\gamma}_2 \nabla_h \cdot (a \nabla_h Z)) \, \eta \right|
\]

\[
\leq C (\hat{\gamma}_1 ||h^2 (g - \hat{\gamma}_1 \Delta^2 Z + \hat{\gamma}_2 \nabla_h \cdot (a \nabla_h Z))|| \|h^{-2} \eta\|)
\]

\[
+ K_a^{\beta} \gamma_2 ||h/ \sqrt{a_0} (g - \hat{\gamma}_1 \Delta^2 Z + \hat{\gamma}_2 \nabla_h \cdot (a \nabla_h Z))|| \|\sqrt{a} \nabla e_c\|).
\]

(5.64)

For the sixth term on the right-hand side of (5.62), we use a trace estimate, the bound \(\|h^{-1} \eta\|_{\kappa} \leq C \| \nabla e_c \|_{\kappa}\), to deduce

\[
\left| \int_{\Gamma_{\text{int}}} \{\eta\} [a \nabla Z] \, ds \right| \leq C K_a^{\beta} ||\sqrt{a} \nabla e_c|| \|\sqrt{h/ a_0} [a \nabla_h Z]\|_{\Gamma_{\text{int}}}.
\]

(5.65)

For the seventh term on the right-hand side of (5.62), working alike to (5.59), we obtain

\[
\left| \int_{\Gamma} \theta \{a \Pi \nabla \eta\} \cdot [Z] \right| \leq C K_a^{\beta} ||\theta|| \|\sqrt{a} \nabla e_c|| \|\sqrt{\chi} [Z]\|_{\Gamma}.
\]

(5.66)
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and finally, for the last term on the right-hand side of (5.62), we get

\[ \left| \int_{\Gamma} \chi[\eta] \cdot [Z] \right| \leq C K_{\alpha}^a \| \sqrt{\alpha} \nabla e_c \| \| \chi[Z] \|_{\Gamma}. \]  

(5.67)

The result follows combining the above relations, using bounds (3.16), (3.17), (3.18) and (3.19) on second, third, fourth and fifth term on the right-hand side of (5.62) and applying triangle inequality on \|\|e\|\| \leq \|\|e_c\|\| + \|\|e_d\|\|.

\[\text{Theorem 5.3 (A posteriori bounds for interior penalty DG method for parabolic problem). Let } u,U \text{ be the exact weak solution of (5.3), and the interior penalty DG solution of the problem (5.13), respectively, and let } a \text{ be elementwise (weakly) differentiable. Then, the following error bound holds:} \]

\[ \| u - U \|_{L^2(0,T;\mathcal{V})}^2 \leq C \int_0^T \mathcal{E}^2(U, a, AU - \Pi f + f, \mathcal{T}) + \| C_{\alpha} \sqrt{h} \partial_t U \|_{\Gamma}^2 \]

\[ + \hat{\gamma}_1 \int_0^T \| h^{3/2} [\nabla \partial_t U] \|_{\Gamma}^2 \]

\[ + C \left( \| u_0 - U(0) \| + \| \sqrt{h}[U(0)] \| + \hat{\gamma}_1 \| h^{3/2} [\nabla U(0)] \| \right)^2. \]  

(5.68)

If we assume also that \( u, U \in C(0,T;\mathcal{H}) \cap H^1(0,T;L^2(\Omega)) \), then the following bound also holds:

\[ \| u - U \|_{L^2(0,T;\mathcal{V})}^2 \leq C \int_0^T \mathcal{E}^2(U, a, AU - \Pi f + f, \mathcal{T}) + C \left( \int_0^T \| \sqrt{h} [\partial_t U] \|_{\Gamma} \right)^2 \]

\[ + C \gamma_1 \left( \int_0^T \| h^{3/2} [\nabla \partial_t U] \|_{\Gamma} \right)^2 \]

\[ + C \left( \| u_0 - U(0) \| + \| \sqrt{h}[U(0)] \| + \gamma_1 \| h^{3/2} [\nabla U(0)] \| \right)^2. \]  

(5.69)

Proof. The results follow immediately from combining Theorems 5.1 and 5.2 with Lemma 5.3, in conjunction with (5.52).

Finally, we give a result on useful properties of the interior penalty DG bilinear form and of the norm, which will be useful in section 5.4.

\[\text{Lemma 5.4 (continuity of } B \text{ and stability of the } L^2\text{-projection). Consider the notation of section 5.3 and let } B \text{ and } |||\cdot||| \text{ denote the interior penalty DG bilinear form} \]
(5.51) and the DG-norm (5.50). Then for $Z,V \in S$ we have

$$B(Z,V) \leq C(\hat{\gamma}_1 + \hat{\gamma}_2 K_a) ||Z|| ||V||.$$  \hfill (5.70)

Moreover, for $v \in \mathcal{H}$, the $L^2$-projection is DG-norm-stable, i.e.,

$$||\Pi v|| \leq C(\hat{\gamma}_1 C_p + \hat{\gamma}_2 K_a) ||v||.$$  \hfill (5.71)

**Proof.** We omit the proof of (5.70) which mimics that of (5.61). For stability, note

\[
\|\sqrt{a} \nabla_h (\Pi v - \Pi_0 v)\|^2 + \|\sqrt{\chi} [v - \Pi v] \|_I^2 \\
\leq C \left( \|a_\sharp^{-\frac{1}{2}} h^{-1} (\Pi v - \Pi_0 v)\|^2 + \|a_\sharp^\frac{1}{2} h^{-1} (v - \Pi v)\|^2 + \|a_\sharp^\frac{1}{2} \nabla_h (v - \Pi v)\|^2 \right) \\
\leq C \left( \|a_\sharp^{-\frac{1}{2}} h^{-1} (v - \Pi_0 v)\|^2 + \|a_\sharp^\frac{1}{2} \nabla_h (v - \Pi v)\|^2 \right) \\
\leq C \| (a_\sharp^{-\frac{1}{2}} / a) \frac{1}{\sqrt{a}} \nabla \nabla_h v\|^2,
\]

and,

\[
\|\Delta_h (\Pi v - \Pi_1 v)\|^2 + \|\sqrt{\sigma} [v - \Pi v] \|_I^2 + \|\sqrt{\xi} [\nabla (v - \Pi v)] \|_I^2 \\
\leq C C_p \left( \|h^{-2} (\Pi v - \Pi_0 v)\|^2 + \|h^{-2} (v - \Pi v)\|^2 + \|h^{-1} \nabla_h (v - \Pi v)\|^2 + |v - \Pi v|_2^2 \Omega \right) \\
\leq C C_p \|\Delta_h v\|^2
\]

which implies (5.71). \qed

### 5.4 A posteriori error bound for the fully discrete scheme

In this section we discuss the abstract error analysis for the fully discrete scheme defined in 5.1.2.
5.4.1 The elliptic reconstruction and the basic error relation

Extend the sequence \( \{U_n\} \) into a continuous piecewise linear function of time:

\[
U(0) = U_0 \quad \text{and} \quad U(t) = l_n(t)U_n + l_{n-1}(t)U_{n-1},
\]

for \( t \in [t_{n-1}, t_n] \), and \( n = 1, \ldots, N \), where the functions \( l_n(t) \) and \( l_{n-1}(t) \) are the Lagrange basis functions

\[
l_n(t) := \frac{t-t_{n-1}}{\tau_n} 1_{[t_{n-1},t_n]} + \frac{t_{n+1}-t}{\tau_{n+1}} 1_{[t_n,t_{n+1}]}.
\]

Using these time extensions and the (time and mesh dependent) discrete elliptic operator \( A^n \) of Definition 5.1 with respect to \( S^r_n \), defined by

\[
A^n Z \in S^r_n \text{ such that } \langle A^n Z, V \rangle = B^n(Z, V) \quad \text{for all } V \in S^r_n,
\]

we can write the scheme (5.16) in the following form:

\[
\langle \partial_t U(t), V \rangle + \langle A^n U_n, V \rangle = \langle (I^n U_{n-1} - U_{n-1})/\tau_n, V \rangle \\
+ \langle \Pi f_n, V \rangle \quad \text{for all } V \in S^r_n,
\]

and for all \( n \in \{1, \ldots, N\} \).

We like to warn at this point that we use the same symbol \( U(t) \) to indicate the fully discrete solution time-extension in this section, and the semidiscrete solution in 5.2. This should cause no confusion as long as the two cases are kept in separate sections.

For each fixed \( t \in [0,T] \), and the corresponding \( n = 1, \ldots, N \) such that \( t \in [t_{n-1}, t_n] \), we define the time-dependent elliptic reconstruction to be the function \( w(t) \in \mathcal{H} \) satisfying

\[
w(t) = l_n(t)w_n + l_{n-1}(t)w_{n-1}^+,
\]

where \( w_n \in \mathcal{H} \) is the elliptic reconstruction of \( U_n \) defined implicitly as the (weak) solution of the elliptic problem with data \( A^n U_n \), i.e., \( w_n \) satisfies

\[
B^n(w_n, v) = \langle A^n U_n, v \rangle \quad \text{for all } v \in \mathcal{H}
\]
and \( w_{n-1}^+ \) is the *forward elliptic reconstruction* of \( I^n U_{n-1} \), defined as the solution of the problem
\[
B_{+}^{n-1}(w_{n-1}^+, v) = \langle A_{+}^{n-1} I^n U_n, v \rangle \quad \text{for all } v \in \mathcal{H}
\]
where the operator \( A_{+}^{n-1} : S_n^r \to S_n^r \) is defined by
\[
A_{+}^{n-1} Z \in S_n^r \text{ such that } \langle A_{+}^{n-1} Z, V \rangle = B_{+}^{n-1}(Z, V) \quad \text{for all } V \in S_n^r,
\]
\( B_{+}^{n-1} \) being the nonconforming bilinear form corresponding to \((\cdot, t_{n-1})\), but with respect to the space \( S_n^r \) (in contrast to \( B_{+}^{n-1} \) which is defined with respect to \( S_n^r \)).

Using properties of \( L^2 \)-projection operator \( \Pi^n \), we arrive at
\[
\langle \partial_t U(t), \Pi^n v \rangle + \langle A^n U_n, \Pi^n v \rangle = \langle (I^n U_{n-1} - U_{n-1})/\tau_n, \Pi^n v \rangle + \langle \Pi^n f_n, \Pi^n v \rangle \quad \text{for all } v \in \mathcal{H}.
\]
and using the definition of \( w \) on \([t_{n-1}, t_n]\), the equation (5.82), implies
\[
\langle \partial_t U(t), v \rangle + B(w(t), v) = \langle (I^n U_{n-1} - U_{n-1})/\tau_n, v \rangle + \langle \Pi^n f_n, v \rangle + B(w(t) - w_n, v) \quad \text{for all } v \in \mathcal{H}.
\]
Subtracting the exact equation from this identity we obtain
\[
\langle \partial_t (U(t) - u(t)), v \rangle + B((w(t) - u), v) = \langle (I^n U_{n-1} - U_{n-1})/\tau_n, v \rangle + \langle \Pi^n f_n - f, v \rangle + B(w(t) - w_n, v) \quad \text{for all } v \in \mathcal{H}.
\]
for all \( t \in [t_{n-1}, t_n] \) and \( n = 1, \ldots, N \), this leads to the following technical basis of this section.

**Lemma 5.5 (fully discrete error relation).** With the notation introduced in this section, let \( e = U - u \) (full error), \( \rho := w - u \) (parabolic error) and \( \epsilon := w - U \) (elliptic error). Then we have
\[
\langle \partial_t e, v \rangle + B(\rho, v) = \langle (I^n U_{n-1} - U_{n-1})/\tau_n, v \rangle + \langle \Pi^n f_n - f, v \rangle + B(w(t) - w_n, v) \quad \text{for all } v \in \mathcal{H}.
\]
on \([t_{n-1}, t_n]\) and \( n = 1, \ldots, N \).
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Proof. Replace the new notation for the errors in (5.84). \qed

5.4.2 Mesh interaction, DG spaces and decomposition

The domain \( \Omega \)'s subdivisions, also known as meshes, \( \{ \mathcal{T}_n \}_{n=0,\ldots,N} \) are assumed to be compatible in the sense that for any two consecutive meshes, say \( \mathcal{T}_n \) and \( \mathcal{T}_{n-1} \), we have that \( \mathcal{T}_n \) is a constructed from \( \mathcal{T}_{n-1} \) in two main steps: (1) \( \mathcal{T}_{n-1} \) is locally coarsened by merging a chosen subset of elements then (2) the resulting coarsened mesh is locally refined [72, 73]. This procedure leads to meshes which are locally a refinement of one another.

For each \( n = 1, \ldots, N \), we denote by \( \mathcal{T}_n^* \) the coarsest common refinement of \( \mathcal{T}_{n-1} \) and \( \mathcal{T}_n \). The finite element space corresponding to \( \mathcal{T}_n \) being \( S^r_n \), we shall be using the space \( S^r_n^* \) which is the finite element space with respect to \( \mathcal{T}_n^* \). Furthermore we denote by \( S_n := S^r_n + \mathcal{H} \) and by \( S := \sum_{n=0}^N S_n \), the minimal space that contains all these spaces. These spaces are equipped with the same type of norms given as in Section 5.1. The conforming-nonconforming decomposition of \( U \), that we shall be using is performed as follows:

i) For each given \( t_n \), with \( n = 1, \ldots, N-1 \) we assume that the following two decompositions exist for \( U_n \),

\[
U_n = U_n^d + U_n^c \quad \text{with respect to the mesh} \quad \mathcal{T}_n^*,
\]

and \( U_n = U_n^{d+} + U_n^{c+} \) with respect to the mesh \( \mathcal{T}_{n+1}^* \).

ii) For each \( t \in [t_{n-1}, t_n] \), \( n = 1, \ldots, N \), we define

\[
U^d(t) = l_{n-1}(t)U_{n-1}^{d+} + l_n(t)U_n^d
\]

and

\[
U^c(t) = l_{n-1}(t)U_{n-1}^{c+} + l_n(t)U_n^c
\]

(5.87)

For the following, we recall the definition of the constant due to diffusion tensor,

\[
C_{\alpha} := (\hat{\gamma}_2(1 - \hat{\gamma}_1) + \hat{\gamma}_1 \sqrt{\alpha}_b)/(\sqrt{\alpha}_b)).
\]

(5.88)

and we write \( C_{\alpha}^n := C_{\alpha}(t_n, \cdot) \) for short.
**Definition 5.5 (A posteriori error indicators).** We set here some notation that is useful to state the main results concisely. We make some assumptions in the process. For each time interval $t \in [t_{n-1}, t_n]$, with $n = 1, \ldots, N$, we introduce a posteriori error indicators as follows.

i) We assume that there exist $C_{els}, C_{dgc} > 0$ such that

$$|||\Pi^n v||| \leq C_{els}||v||, \text{ for all } v \in \mathcal{H},$$

(5.89) and

$$B^n(Z, V) \leq C_{dgc}|||Z||| ||V||, \text{ for all } Z, V \in S^r_n.$$  

(5.90)

The time-stepping indicator is given by

$$\theta_n := \frac{C_{els}C_{dgc}}{\sqrt{3}}|||I^n U_{n-1} - U_n|||.$$  

(5.91)

ii) The time data-approximation indicator is

$$\beta_n := \left( \int_{t_{n-1}}^{t_n} \frac{||f(t_n) - f(s)||_{L^2}}{\tau_n} ds \right)^{1/2}.$$  

(5.92)

iii) The mesh-change (or coarsening) indicators are defined as

$$\gamma'_n := \frac{||I^n U_{n-1} - U_{n-1}||_{H^r}}{\tau_n} \left( \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} C_{\alpha_o}^2 \right)^{1/2}$$

and

$$\gamma''_n := |||I^n U_{n-1} - U_{n-1}|||.$$  

(5.93)

(5.94)

iv) The parabolic nonconforming part indicator is given by

$$\delta_n := \frac{||U^d_n - U^d_{n-1}||_{H^r}}{\tau_n} \left( \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} C_{\alpha_o}^2 \right)^{1/2},$$

(5.95)

and the elliptic nonconforming part indicator defined as

$$\tilde{\delta}_n := \left( |||U^d_n|||^2 + |||U^d_{n-1}|||^2 \right)^{1/2}.$$  

(5.96)
v) The space (or elliptic) error indicator is given by

\[ \varepsilon_n := \mathcal{E}(U_n, a(t_n), A^n U_n, T_n), \]  

(5.97)

where \( \mathcal{E} \) is the energy-norm elliptic error estimator given by 3.1. Furthermore, the forward elliptic error indicator, due to mesh change, is given by

\[ \varepsilon^+_{n-1} := \mathcal{E}(I^n U_{n-1}, a(t_{n-1}), A_{n-1} A^n U_n, T_{n-1}). \]  

(5.98)

vi) Consider first the auxiliary function of time

\[ \lambda_{a,n}(s) := ||| \sqrt{a(s) a(t_n)^{-1}} - \sqrt{a(s)^{-1} a(t_n)} |||_{L^\infty(\Omega)}, s \in [t_{n-1}, t_n] \]  

(5.99)

where the inner matrix norm is the Euclidean-induced one. This definition is possible thanks to \( a \)'s being symmetric positive definite. The function \( \lambda_{a,n} \) is identically zero if the operator is time-independent, otherwise it acts like the numerator of \( a \)'s normalized Hölder-continuity ratio. Then we may define the following operator approximation indicators

\[ \zeta_{n} := ||| C_2^2 a_n^{-1} A^n U_n |||_{\mathcal{H}'} \left( \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} \lambda_{a,n}^2 \right)^{1/2}, \]  

(5.100)

\[ \zeta''_{n} := ||| C_{a_n+1}^2 A_{n-1} A^n U_{n-1} |||_{\mathcal{H}'} \left( \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} \lambda_{a_{n-1},n}^2 \right)^{1/2}, \]  

(5.101)

\[ \zeta^\circ_{n} := ||| (A_{n-1}^+ - A^n) I^n U_{n-1} |||_{\mathcal{H}'} \left( \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} C_{a_n+1}^2 \lambda_{a_{n-1}}^2 \right)^{1/2}, \]  

(5.102)

\[ \zeta_{n} := \zeta^\circ_{n} + \zeta'_{n} + \zeta''_{n}. \]  

(5.103)

vii) Finally, the parabolic nonconforming part indicator of higher order is given by

\[ \kappa_{n} := \frac{||| U_{n}^d - U_{n+}^d |||_{\mathcal{H}'}}{\tau_n}. \]  

(5.104)

Remark 5.7 (computing the \( \mathcal{H}^l \) norms). The \( H^{-1}(\Omega) \) norms appearing in the indicators in the purely second order case \( \hat{\gamma}_1 = 0 \) are easily estimated at the cost of inverting a stiffness matrix, for details see [73]. For many practical purposes and
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in case $\hat{\gamma}_1 > 0$ where estimation of $H^{-2}(\Omega)$ norm requires construction and implementation of $H^2$ conforming elements, this has to be replaced by the $L^2(\Omega)$ norm times the Poincaré–Friedrichs constant $C_{PF}$ defined in Lemma C.4 which implies the dual inequality

$$||v||_{H^{-2}(\Omega)} \leq C_{PF}||v||,$$

for all $v \in L^2(\Omega)$. \hfill (5.105)

Note that this will not deteriorate most of the indicators. The only indicators that may be affected by this change are $\beta_n$, $\gamma'_n$ and $\delta_n$, and it may be possible to provide a sharp bound for negative Sobolev norms of $U_d$, but this seems to remain an open question at the time of writing. In the numerical experiments below, both the use of $H^{-1}(\Omega)$ and of $L^2(\Omega)$ norms are used for comparison. The resulting effectivity indices differ only by a constant that appears to be comparable to the Poincaré–Friedrichs constant $C_{PF}$, as expected (cf. figure 5.11).

**Remark 5.8 (computing $A^n U_n$ and similar terms).** The operator $A^n$ appearing in Definition 5.5, can be realized in two ways in practice:

- To save time, one can use the fully discrete scheme in pointwise form (5.77) to evaluate some of these terms. For example

$$A^n U_n = \Pi^n f_n - (U_n - I^n U_{n-1}) \tau_n. \hfill (5.106)$$

- The corresponding stiffness matrix could be computed and applied to the argument. This seems to be necessary for $A^0$.

**Remark 5.9 (mesh-change prediction).** The mesh-change indicators $\gamma'_n$ and $\gamma''_n$ can be precomputed in a given computation. Indeed, this term does not use explicitly any quantity deriving from the solution of the $n$-th Euler time-step (5.16). This term is usually computable when a precise operator $I^n$ is available and it involves only local matrix-vector operations on each group of elements to be coarsened.

**Remark 5.10 (an alternative time-stepping indicator).** An alternative definition for the time-stepping estimator $\theta_n$ can be given by

$$\hat{\theta}_n := \frac{1}{\sqrt{3}} ||A_{+1}^{-1} P^n U_{n-1} - A^n U_n||_{V'}. \hfill (5.107)$$
This alternative definition has the advantage of having no constants, but it is more complicated to compute and it must be reduced to the $L^2(\Omega)$ norm by using the Poincaré–Friedrichs inequality. A good side effect of this alternative choice is that in this case the indicator $\zeta_n$ vanishes; all other estimators remain unchanged.

**Theorem 5.4** (abstract a posteriori energy-error bound for Euler–DG). Let $\{U_n\}_n$ be the solution of (5.16) and $U$ its time-extension as defined by (5.74) and $w$ the elliptic reconstruction as defined by (5.78). Then, with reference to definition 5.5, for $t_m \in \{t_n\}_{n=1,...,N}$ with $m \in \{1,\ldots,N\}$, we have

$$
\|u - U\|_{L^2(0,t_m;\mathcal{V})} \leq \|u(0) - U^c(0)\| + 3\eta_{p,m} + 3\eta_{e,m}
$$

\[+ 2 \left( \sum_{n=1}^{m} \delta_n^2 \tau_n \right)^{1/2} + \sqrt{\frac{3}{2}} \sum_{n=1}^{m-1} \kappa_n \tau_n + 2 \left( \sum_{n=1}^{m-1} \gamma_n^2 \tau_n \right)^{1/2},
\]

where the parabolic-error estimator is defined as

$$
\eta_{p,m} := \left( \sum_{n=1}^{m} \left( \theta_n + \zeta_n + \beta_n + \gamma'_n + \delta_n \right)^2 \tau_n \right)^{1/2}
$$

and the elliptic estimator is defined by

$$
\eta_{e,m} := \left( \sum_{n=1}^{m} \left( \varepsilon_n^2 + \varepsilon_{n-1}^2 \right) \tau_n \right)^{1/2}.
$$

We spread the proof in paragraphs 5.4.3–5.4.6.

### 5.4.3 The energy identity

As in the proof of Theorem 5.1 to get an energy identity out of (5.85), we will test with the error’s conforming part

$$
e_c := e - U^d = \rho + \varepsilon_c.
$$
5.4 A posteriori error bound for the fully discrete scheme

Start with combining (5.85) and definition (5.78) to get

\[ \langle \partial_t e_c, v \rangle + B(\rho, v) = \langle \partial_t U^d, v \rangle + \langle (I^n U_{n-1} - U_{n-1})/\tau_n, v \rangle + \langle \Pi^n f_n - f, v \rangle + B(w(t) - w_n, v) \quad \text{for all } v \in \mathcal{H}. \]  

(5.112)

Setting \( v = e_c \in \mathcal{H} \) in the above relation, we obtain the following energy identity:

\[ \frac{1}{2} \frac{d}{dt} \| e_c \|^2 + \| \rho \|^2 = \langle \partial_t e_c, e_c \rangle + B(\rho, \rho) + \langle \partial_t U^d, e_c \rangle + B(w - w_n, e_c) + \langle (I^n U_{n-1} - U_{n-1})/\tau_n, e_c \rangle + \langle \Pi^n f_n - f, e_c \rangle. \]

(5.113)

Integrating (5.113) from 0 to \( t \in [t_{m-1}, t_m] \), for an integer \( m, 1 \leq m \leq N \) fixed, we may write the integral form of the energy identity

\[ \frac{1}{2} \| e_c(t) \|^2 + \int_0^t \| \rho \|^2 = \frac{1}{2} \| e_c(0) \|^2 + \int_0^t B(\rho, e_c) \\
+ \sum_{n=1}^m \left( \int_{t_{n-1}}^{t_n} B(w - w_n, e_c) + \int_{t_{n-1}}^{t_n} \langle (I^n U_{n-1} - U_{n-1})/\tau_n + \Pi^n f_n - f, e_c \rangle \right) \\
+ \int_0^t \langle \partial_t U^d, e_c \rangle + \frac{1}{2} \sum_{n=1}^{m-1} \left( \| u(t_n) - U^c_n \|^2 - \| e_c(t_n) \|^2 \right) \\
:= I_0 + I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t). \]

(5.114)

To obtain the a posteriori error bound for scheme (5.16), we now bound each of \( I_i(t), i = 1, \ldots, 4 \) (\( I_0 \) needs no bounding) appearing in relation (5.114), in terms of either a-posteriori-computable or left-hand-side quantities.

A term that substantially distinguishes the fully discrete case from the semidiscrete one discussed in 5.2 is the time-discretization term \( I_2(t) \), so we start by bounding this term.
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5.4.4 Time discretization estimate

To bound $I_2(t)$ we start by working out the first factor of the integrand as follows

$$B(w(s) - w_n, e_c) = B\left((l_n(s)w_n + l_n^{-1}(s)w_{n-1}^+) - w_n, e_c\right)$$

$$= l_n(s)(B(t; w_n, e_c) - B(t_n; w_n, e_c))$$

$$+ l_n^{-1}(s)(B(t; w_n, e_c) - B(t_{n-1}; w_{n-1}^+, e_c))$$

$$+ l_n^{-1}(s)(B(t_{n-1}; w_{n-1}^+, e_c) - B(t_n; w_n, e_c)).$$

(5.115)

Since $w_n$ and $e_c$ are both in $\mathcal{H}$, we may bound the first term with

$$\|B(t; w_n, e_c) - B(t_n; w_n, e_c)\| = \int_\Omega \| (a(s) - a(t_n)) \nabla w_n \cdot \nabla e_c(s) \| ds$$

$$= \int_\Omega \sqrt{\alpha(s)} \left( \sqrt{\alpha(s)} a(t_n)^{-1} - \sqrt{\alpha(s)} a(t_n) \right) \sqrt{\alpha(t_n)} \nabla w_n \cdot \nabla e_c(s)$$

$$\leq \| \| \sqrt{\alpha(s)} a(t_n)^{-1} - \sqrt{\alpha(s)} a(t_n) \|_2 \| L_{\infty}(\Omega) \| \sqrt{\alpha(s)} \nabla e_c(s) \| \| \sqrt{\alpha(t_n)} \nabla w_n \|$$

$$= \lambda_{a_n}(s) \| w_n \| \| e_c(s) \|.$$  

(5.116)

The second factor above can be bounded as follows

$$\| |w_n| |^2 = B(w_n, w_n) = \langle A^n U^n, w_n \rangle \leq \| A^n U^n \|_{\mathcal{H}} \| w_n \|_{\mathcal{H}}$$

$$\leq C_{\alpha_n}^2 \| A^n U^n \|_{\mathcal{H}} \| w_n \|,$$  

(5.117)

where the last step owes to the fact that

$$\frac{1}{C_{\alpha_n}^2} \| w_n \|_{\mathcal{H}} \leq B(w_n, w_n) = \| |w_n| |^2.$$  

(5.118)

Thus $\| |w_n| | \leq C_{\alpha_n}^2 \| A^n U^n \|_{\mathcal{H}}$ and we obtain

$$\int_{t_{n-1}}^{t_n} l_n(s)(B(s; w_n, e_c) - B(t_n; w_n, e_c)) \, ds$$

$$\leq C_{\alpha_n}^2 \| A^n U^n \|_{\mathcal{H}} \left( \int_{t_{n-1}}^{t_n} l_n^2(s) \lambda_{a_n}(s)^2 \, ds \right)^{1/2} \left( \int_{t_{n-1}}^{t_n} \| e_c(s) \|^2 \, ds \right)^{1/2}$$

$$= \zeta_n \sqrt{\tau_n} \| e_c \|_{L^2([t_{n-1}, t_n]; \mathcal{V})}.$$  

(5.119)
Similarly, we obtain

\[ \int_{t_{n-1}}^{t_n} l_n(s)(B(s; w_n, e_c) - B(t_{n-1}; w_{n-1}^+, e_c)) \, ds \]

\[ \leq C_{\alpha_n^{-1}}^2 I_n^{n-1} U_n^{n-1} \| H \| \left( \int_{t_{n-1}}^{t_n} l_{n-1}(s) \lambda_{\alpha_n^{-1}}(s)^2 \right)^{1/2} \left( \int_{t_{n-1}}^{t_n} \| e_c \|^2 \right)^{1/2} \]

\[ = \zeta'_n \sqrt{\tau_n} \| e_c \|_{L^2([t_{n-1}, t_n]; \mathcal{V})}. \]

(5.120)

To estimate the resultant of the integrand’s third term in (5.115), we recall the elliptic reconstruction’s definition and note that in view of (5.77) we may write, for \( n \geq 1 \), that

\[ B(t_{n-1}; w_{n-1}^+, e_c) - B(t_n; w_n, e_c) = \left\langle \left( A_{\alpha_n^{-1}}^{n-1} I_n U_{n-1} - A^n U_n \right), e_c \right\rangle \]

(5.121)

given that \( A^n (I^n U_{n-1} - U_n) \in S^r_n \). The right-hand side of (5.121) can be simply bounded by

\[ \left\langle \left( A_{\alpha_n^{-1}}^{n-1} I_n U_{n-1} - A^n U_n \right), e_c \right\rangle \leq \| A_{\alpha_n^{-1}}^{n-1} I_n U_{n-1} - A^n U_n \| \| e_c \|_{H}, \]

(5.122)

Yielding alternative timestepping indicator \( \hat{\theta}_n \).

Otherwise, the first term on the right-hand side of (5.121) can be bounded by

\[ \left\langle \left( A_{\alpha_n^{-1}}^{n-1} I_n U_{n-1} - A^n U_n \right), e_c \right\rangle \leq \| (A_{\alpha_n^{-1}}^{n-1} - A^n) I^n U_{n-1} \| \| e_c \|_{H}, \]

(5.123)

and thus, recalling definition (5.102), we have

\[ \int_{t_{n-1}}^{t_n} l_{n-1}(s) \left\langle \left( A_{\alpha_n^{-1}}^{n-1} - A^n \right) I^n U_{n-1}, e_c \right\rangle \, ds \leq \zeta'_n \sqrt{\tau_n} \| e_c \|_{L^2([t_{n-1}, t_n]; \mathcal{V})}. \]

(5.124)

The second term on the right-hand side of (5.121) can be given a simpler expres-
sion as follows:

$$\langle A^n (I^n U_{n-1} - U_n), \Pi^n e_c \rangle = B(t_n; I^n U_{n-1} - U_n, \Pi^n e_c)$$

$$\leq C_{dgc} |||I^n U_{n-1} - U_n||| |||\Pi^n e_c|||$$

$$\leq C_{dgc} C_{els} |||I^n U_{n-1} - U_n||| |||e_c|||$$

(5.125)

thanks to the stability of $\Pi^n$ with respect to the energy norm $||| \cdot |||$ assumed in (5.89). Therefore, recalling definition (5.91), we obtain

$$\int_{t_{n-1}}^{t_n} l_{n-1}(s) \langle A^n (I^n U_{n-1} - U_n), \Pi^n e_c \rangle \ ds \leq \theta_n \sqrt{\tau_n} |||e_c|||_{L^2([t_{n-1}, t_n]; \mathbb{R})}.$$  (5.126)

The time error estimate follows:

$$\mathcal{I}_2(t) \leq \sum_{n=1}^{m} (\zeta_n + \theta_n) \sqrt{\tau_n} (|||\rho|||_{L^2([t_{n-1}, t_n]; \mathbb{R})} + |||e_c|||_{L^2([t_{n-1}, t_n]; \mathbb{R})}).$$  (5.127)

### 5.4.5 Other error estimates

To bound the spatial error term, $\mathcal{I}_1(t)$ in (5.114), we simply consider

$$\mathcal{I}_1(t) = \int_0^t B(\rho, e_c) \leq \int_0^t |||\rho||| |||e_c|||,$$  (5.128)

with the aim of absorbing the first factor in the left-hand side of (5.114) and using an elliptic error estimator to bound the second term.

The term $\mathcal{I}_3(t)$ in (5.114) which takes into account data approximation:

$$\mathcal{I}_3(t) = \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \langle (I^n U_{n-1} - U_{n-1})/\tau_n + \Pi^n f_n - f, e_c \rangle.$$  (5.129)

The first term can be bounded by using the $(\mathcal{H}', \mathcal{H})$ pairing, as we did with the
5.4 A posteriori error bound for the fully discrete scheme

\[ \sum_{n=1}^{m} \int_{t_{n-1}}^{t_{n}} \langle (I^n U_{n-1} - U_{n-1})/\tau_{n} + \Pi^n f_n - f, e_c \rangle dt \]

\[ \leq \sum_{n=1}^{m} (\gamma_n' + \beta_n) \sqrt{\tau_n} \| e_c \|_{L^2(t_{n-1}, t_n; \mathcal{X})}, \quad (5.130) \]

where we have used definitions (5.92) and (5.93). Hence we obtain the bound

\[ I_3(t) \leq \sum_{n=1}^{m} (\beta_n + \gamma_n') \sqrt{\tau_n} \left( \| \rho \|_{L^2(t_{n-1}, t_n; \mathcal{X})} + \| e_c \|_{L^2(t_{n-1}, t_n; \mathcal{X})} \right). \quad (5.131) \]

We estimate the second-last term on the right-hand side of (5.114). This term can be bounded in two different ways. For concision’s sake we expose only the estimate that yields smaller accumulation over long integration-times:

\[ I_4(t) = \int_{0}^{t} \langle \partial_t U^d, e_c \rangle dt \leq \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \| \partial_t U^d \|_{\mathcal{X}} \| e_c \|_{\mathcal{X}} dt \]

\[ \leq \sum_{n=1}^{m} \delta_n \sqrt{\tau_n} \left( \| \rho \|_{L^2(t_{n-1}, t_n; \mathcal{X})} + \| e_c \|_{L^2(t_{n-1}, t_n; \mathcal{X})} \right), \quad (5.132) \]

by recalling (5.95).

Observing the identity

\[ \| u(t_n) - U_{n+}^c \|^2 - \| e_c(t_n) \|^2 = \| U_{n+}^d - U_{n+}^d \|^2 + \langle U_{n+}^d - U_{n+}^d, e_c(t_n) \rangle, \quad (5.133) \]

we estimate the last term on the right-hand side of (5.114), as follows:

\[ I_5(t) = \frac{1}{2} \sum_{n=1}^{m-1} \left( \| U_{n+}^d - U_{n+}^d \|^2 + \langle U_{n+}^d - U_{n+}^d, e_c(t_n) \rangle \right) \]

\[ \leq \frac{1}{2} \sum_{n=1}^{m-1} \left( \kappa_n^2 \tau_n^2 + \max_{1 \leq l \leq m-1} \| e_c(t_l) \| \kappa_n \tau_n \right) \]

\[ \leq \frac{3}{4} \left( \sum_{n=1}^{m} \kappa_n \tau_n \right)^2 + \frac{1}{4} \max_{1 \leq l \leq m-1} \| e_c(t_l) \|^2 \quad (5.134) \]
5.4.6 Concluding the proof of Theorem 5.4

Combining the energy relation (5.114) with the bounds (5.127), (5.128), (5.131), (5.132) and (5.134), we obtain

\[
\frac{1}{2}||e_c(t)||^2 + ||\rho||^2_{L^2(0,t;\mathcal{Y})} \\
\leq \frac{1}{2}||e_c(0)||^2 + \frac{3}{4} \left( \sum_{n=1}^{m-1} \kappa_n \tau_n \right)^2 + \frac{1}{4} \max_{1 \leq l \leq m-1} ||e_c(t_l)||^2 \\
+ \sum_{n=1}^{m} (\theta_n + \zeta_n + \beta_n + \gamma_n' + \delta_n) \sqrt{\tau_n} ||e_c||_{L^2(t_{n-1},t_n;\mathcal{Y})} \\
+ \sum_{n=1}^{m} ((\theta_n + \zeta_n + \beta_n + \gamma_n' + \delta_n) \sqrt{\tau_n} + ||e_c||_{L^2(t_{n-1},t_n;\mathcal{Y})}) ||\rho||_{L^2(0,t;\mathcal{Y})}.
\]

(5.135)

Choosing \( t = t_{m_\ast} \) so that \( ||e_c(t_{m_\ast})|| = \max_{1 \leq l \leq m-1} ||e_c(t_l)|| \) in (5.135), yields a bound on \( \max_{1 \leq l \leq m-1} ||e_c(t_l)||^2 / 4 \), which is then used again to bound the third term on the right-hand of (5.135), resulting to

\[
||\rho||^2_{L^2(0,t;\mathcal{Y})} \leq ||e_c(0)||^2 + \frac{3}{2} \left( \sum_{n=1}^{m-1} \kappa_n \tau_n \right)^2 \\
+ 2 \sum_{n=1}^{m} (\theta_n + \zeta_n + \beta_n + \gamma_n' + \delta_n) \sqrt{\tau_n} ||e_c||_{L^2(t_{n-1},t_n;\mathcal{Y})} \\
+ 2 \sum_{n=1}^{m} ((\theta_n + \zeta_n + \beta_n + \gamma_n' + \delta_n) \sqrt{\tau_n} + ||e_c||_{L^2(t_{n-1},t_n;\mathcal{Y})}) ||\rho||_{L^2(0,t;\mathcal{Y})},
\]

(5.136)

which is an inequality of the form

\[
|a|^2 \leq c^2 + d \cdot b + (d + b) \cdot a,
\]

(5.137)

where \( a, b, d \in \mathbb{R}^{m+1} \) and \( c \in \mathbb{R} \) are appropriately chosen.

Inequality (5.137) implies

\[
|a| \leq \max \{|c|, |d|\} + |d| + |b|.
\]

(5.138)
This is seen by excluding the case where $|a| \leq |d| + |b|/2$ (a trivial case of (5.138)), taking $m := \max \{|c|, |d|\}$ and completing squares in combination with basic Hilbert space inequalities as follows:

$$
\left( |a| - \frac{|d| + |b|}{2} \right)^2 \leq |a|^2 - a \cdot (d + b) + \left( \frac{|d| + |b|}{2} \right)^2 \\
\leq c^2 + d \cdot b + \frac{1}{4} |d + b|^2 \leq m^2 + \frac{2m |d| + |b|}{2} + \left( \frac{|d| + |b|}{2} \right)^2 \\
\leq \left( m + \frac{|d| + |b|}{2} \right)^2.
$$

(5.139)

Using the notation introduced in the statement of Theorem 5.4, inequality (5.138) implies

$$
||\rho||_{L^2(0,t;\mathcal{H})} \leq ||e_c(0)|| + \sqrt{\frac{3}{2}} \sum_{n=1}^{m-1} \kappa_n \tau_n + 3\eta_{p,m} + ||\epsilon_c||_{L^2(0,t;\mathcal{H})}.
$$

(5.140)

To close the estimate, the last term on the right-hand side of (5.140) is bounded by

$$
||\epsilon_c||_{L^2(0,t;\mathcal{H})} \leq ||\epsilon||_{L^2(0,t;\mathcal{H})} + ||U^d||_{L^2(0,t;\mathcal{H})}.
$$

(5.141)

The first term yields

$$
||\epsilon||_{L^2(0,t;\mathcal{H})}^2 = \sum_{n=1}^{m} \int_{t_{n-1}}^{t_{n}} ||l_n(w_n - U_n) + l_{n-1}(w_{n-1}^+ - U_{n-1}) + l_{n-1}(U_n^U_{n-1} - U_{n-1})||^2 \\
\leq 2 \sum_{n=1}^{m} \int_{t_{n-1}}^{t_{n}} (l_n \varepsilon_{n-1}^2 + l_n \varepsilon_n^2) + \sum_{n=1}^{m} \int_{t_{n-1}}^{t_{n}} l_n \gamma_{n}^2 \\
\leq 2 \left( \sum_{n=1}^{m} (\varepsilon_{n-1}^2 + \varepsilon_n^2) \tau_n + \sum_{n=1}^{m} \gamma_{n}^2 \tau_n \right) = 2 \left( \eta_{e,m} + \sum_{n=1}^{m} \gamma_{n}^2 \tau_n \right).
$$

(5.142)
Similarly, the second term yields
\[
||U^d||^2_{L^2(0,t;\mathcal{X})} \leq 2 \sum_{n=1}^{m} \tilde{\delta}_n^2 \tau_n. \tag{5.143}
\]

Merging these inequalities with \((5.140)\) and using the triangle inequality we obtain
\[
||e||_{L^2(0,t;\mathcal{X})} \leq ||\epsilon||_{L^2(0,t;\mathcal{X})} + ||\rho||_{L^2(0,t;\mathcal{X})} \tag{5.144}
\]
we obtain \((5.108)\).

**Remark 5.11 (short-time integration).** In the spirit of Theorem 5.2, it is possible to modify Theorem 5.4 and the appropriate indicators as to accommodate a short time-integration version of this result where \(L^1\)-accumulation in time replaces the \(L^2\)-accumulation for certain estimators. Over shorter time-intervals this provides a tighter bound.

**Theorem 5.5 (a posteriori energy-error bound for Euler–SIPDG).** Under the same assumptions of Theorem 5.4, assuming we employ the SIPDG method in space as described in \((2.39)\), the error bound \((5.108)\) holds with the estimators \(\eta_{p,m}, \eta_{e,m}\) and \(\sum_{n=1}^{m} \tilde{\delta}_n^2 \tau_n\) explicitly computable as follows:

i) for the nonconforming part indicators \(\delta_n\) and \(\tilde{\delta}_n\), respectively, (cf. 5.4.2 and Lemma 3.1), we replace
\[
||U^d_n - U^d_{n-1}||_{\mathcal{H}'} \quad ||U^d_n|| \quad \text{and} \quad ||U^d_{n-1}||, \tag{5.145}
\]
respectively, by
\[
C_{PF}C_1 ||\sqrt{h^*_n}[U_n - U_{n-1}]||,
\]
\[
C_2 \left( \hat{\gamma}_1 \left( ||\sqrt{\sigma^*_n}[U_n]|| + ||\sqrt{\xi^*_n}[\nabla U_n]|| \right) + \hat{\gamma}_2 \left( ||\sqrt{\chi^*_n}[U_n]|| \right) \right)
\]
and
\[
C_3 \left( \hat{\gamma}_1 \left( ||\sqrt{\sigma^*_n}[U_{n-1}]|| + ||\sqrt{\xi^*_n}[\nabla U_{n-1}]|| \right) + \hat{\gamma}_2 \left( ||\sqrt{\chi^*_n}[U_{n-1}]|| \right) \right), \tag{5.146}
\]
where \(h^*_n\) is the mesh-size function of \(T^*_n\) and \(\sigma^*_n, \xi^*_n, \chi^*_n\) are related to it via \((2.44)\);
5.4 A posteriori error bound for the fully discrete scheme

ii) replace all $\mathcal{H}'$ norms by $C_{PF}$ times the $L^2(\Omega)$ norm.

Remark 5.12. We stress that, although we focus on a posteriori error bounds for spatial DG methods, our abstract results can be applied to a wider class of non-conforming methods (other than DG methods such as, for example, the $C^0$ method derived by Brenner and Sung in [24]), provided they satisfy certain requirements. More specifically, given a particular nonconforming finite element space $S^r$, assume that:

1. With $\hat{\gamma}_1 = 0$, for each $Z \in S^r$ it is possible to decompose it as

$$Z = Z^c + Z^d \text{ such that } Z^c \in H^1_0(\Omega) \cap S^r,$$  \hspace{1cm} (5.147)

(see 3) where $Z^c$ and $Z^d$ are called $Z$’s conforming part and the nonconforming part, respectively. This decomposition is an analytic device and is not needed for computational purposes.

2. With $\hat{\gamma}_1 > 0$, for each $Z \in S^r$ it is possible to decompose it as

$$Z = Z^c + Z^d \text{ such that } Z^c \in H^2_0(\Omega) \cap S^{r+2}_{HCT},$$  \hspace{1cm} (5.148)

(see 3) where $Z^c$ and $Z^d$ are called $Z$’s conforming part and the nonconforming part, respectively. Note that throughout the analysis $Z^d$ is not needed to be in $S^r$. The key observation is that $Z^c \in H^2_0(\Omega)$. Again, this decomposition is an analytic device and is not needed for computational purposes.

3. With $\hat{\gamma}_1 = 0$, given a function $z \in H^1_0(\Omega)$, and let $Z \in S^r$ be the corresponding Ritz-projection via the finite element method, it is possible to bound the norm of the error $Z - z$, using a posteriori error estimators for the steady state problem.

4. With $\hat{\gamma}_1 > 0$, given a function $z \in H^2_0(\Omega)$, and let $Z \in S^r$ be the corresponding Ritz-projection via the finite element method, it is possible to bound the norm of the error $Z - z$, using a posteriori error estimators for the steady state problem.
5.5 Computer experiments

In this final section we summarize the results of computer experiments aimed at testing the efficiency and reliability of the fully discrete estimators derived in § 5.4. We built our code upon the free finite element software FEniCS [74] while Matlab was used as an end-tool to visualize the time-behavior of various estimators.

All the computational examples are in space dimension \( d = 2 \) and their choice is such as to illustrate as many aspect as possible of practical convergence rate, also known as experimental order of convergence, (in short EOC) and the effectivity index (EI), on uniform space-time meshes, of the proposed a posteriori error indicators defined in § 5.5.

5.5.1 Benchmark solutions

We consider six benchmark problems for which \( u_0 \) and \( f \) are chosen so that the exact solution \( u \) of problem (5.3) with \( \hat{\gamma}_1 = 1, \hat{\gamma}_2 = 0 \) coincides with one of the following three benchmark solutions:

\[
\begin{align*}
    u_1(x, y, t) &= \sin(\pi t) \sin^2(\pi x) \sin^2(\pi y) e^{-10(x^2+y^2)}, \\
    u_2(x, y, t) &= \hat{u}_2(r, \phi, t) = \sin(\pi t)(r^2 \cos^2(\phi) - 1)^2(r^2 \sin^2(\phi) - 1)^2 r^{\frac{3}{2}} g(\phi), \\
    u_3(x, y, t) &= \sin(20\pi t) \sin^2(\pi x) \sin^2(\pi y) e^{-10(x^2+y^2)},
\end{align*}
\]

and so that the exact solution \( u \) of problem (5.3) with \( \hat{\gamma}_1 = 0, \hat{\gamma}_2 = 1 \) coincides with one of the following three benchmark solutions:

\[
\begin{align*}
    u_4(x, y, t) &= \sin(\pi t) e^{-40(x-0.5)^2+(y-0.5)^2}, \\
    u_5(x, y, t) &= \hat{u}_5(r, \phi, t) = \sin(\pi t)(r^2 \cos^2 \phi - 1)(r^2 \sin^2 \phi - 1)r^{\frac{3}{2}} \sin \frac{2\phi}{3}, \\
    u_6(x, y, t) &= \sin(20\pi t) e^{-40(x-0.5)^2+(y-0.5)^2}
\end{align*}
\]
for \( t \in [0, 1] \) and

\[
(x, y) \in \begin{cases} 
(0, 1) \times (0, 1) & \text{in (5.149),(5.151),(5.152) and (5.154)} \\
(-1, 1)^2 \setminus [0, 1) \times (-1, 0] & \text{in (5.150) and (5.153).} 
\end{cases}
\]

To complete the definition of \( u_2 \) in (5.150) we consider

\[
z_0 := 0.544483736782464 \text{ such that } \sin^2(z_0 \omega) = z_0^2 \sin^2(\omega), \text{ with } \omega = \frac{3\pi}{2},
\]

and

\[
g(\phi) := \left( \frac{1}{z_0 - 1} \sin((z_0 - 1)\omega) - \frac{1}{z_0 + 1} \sin((z_0 + 1)\omega) \right) \\
\times (\cos((z_0 - 1)\phi) - \cos((z_0 + 1)\phi)) \\
- \left( \frac{1}{z_0 - 1} \sin((z_0 - 1)\phi) - \frac{1}{z_0 + 1} \sin((z_0 + 1)\phi) \right) \\
\times (\cos((z_0 - 1)\omega) - \cos((z_0 + 1)\omega)).
\]

It is well-known \([60, 21]\) that the gradient of \( u_5 \) in (5.153) and Laplacian of \( u_2 \) in (5.150) has a singularity at the reentrant corner located at the origin of \( \Omega \).

Solutions \( u_1 \) and \( u_4 \) are smooth and vary “slowly” in time. Solutions \( u_3 \) and \( u_6 \) are also smooth but oscillate much faster and are used to emphasize the time-error indicator appearing in the parabolic error estimator \( \eta_{p,m} \), defined in (5.109).

Similar examples have been studied elsewhere, for example in \([72, 73]\).

Note that the diffusion tensor, \( a(\vec{x}, t) \), is a constant function (equal to 1) of space-time and that the initial error \( ||u(0) - U(0)|| = 0 \) in all examples.

### 5.5.2 Computed quantities

In each of the examples, we compute the solution of (5.16) using finite element spaces consisting of polynomials of degree \( p \) equal to 2 and 3 for examples (5.149) - (5.151) and polynomials of degree \( p \) equal to 1, 2 and 3 for examples (5.152) - (5.154) with interior penalty parameters \( \sigma_1 \) and \( \tau_1 \) in (2.40) both having values 500 and 1000 for polynomial degrees 2 and 3 and examples (5.149) - (5.151) and values 10, 40 and 90 for polynomial degrees 1, 2 and 3 and examples (5.152) and (5.154) respectively and value 1000 for example (5.153) which are sufficient to guarantee stability of the
numerical scheme as well as good effectivity index behaviour.

We study the asymptotic behavior of the indicators by setting all constants appearing in Theorem 5.5 equal to 1 and monitoring the evolution of the values and experimental order of convergence of the estimators and the error as well as effectivity index over time on a sequence of uniformly refined meshes with a fixed time step \( \tau \) and polynomial degree \( p \). For this purpose, we define experimental order of convergence, in symbols \( EOC \), of a given sequence of positive quantities \( a(i) \) defined on a sequence of meshes of size \( h(i) \) by

\[
EOC(a, i) = \frac{\log(a(i + 1)/a(i))}{\log(h(i + 1)/h(i))}
\]

and the inverse effectivity index, \( IEI \), by

\[
IEI = \frac{||e||_{L^2(0,t_m; T)}}{\eta_{p,m} + \eta_{e,m}}.
\]

We use the inverse effectivity index, instead of the (direct) effectivity index, because it is easier to visualize while conveying the same information. It also has the advantage of relating directly to the constants appearing in Theorem 5.5.

### 5.5.3 Conclusions

The numerical experiments clearly indicate that the error estimators are reliable (as expected from the theory) and efficient. This is clearly seen from the effectivity index behaviour and the EOC of the error and the estimators \( \eta_{e,m} \) and \( \eta_{p,m} \) for each \( m \). Since we have chosen to use varying polynomial degrees to illustrate the behaviour of indicators in a wider range of possibilities, we observe a known effect [69, e.g.] on the inverse effectivity indexes. Moreover, the effectivity index also depends on the penalty parameter used in the discontinuous Galerkin. These effects are illustrated in figure 5.10, (in the context of elliptic problems see, for example, [69]).

The \( H^{-1} \) norms arising in the \( \eta_{p,m} \) for examples (5.152) - (5.154) with \( \hat{\gamma}_1 = 0 \) are estimated using a method for which implementation details can be found in [73], Lemma 3.9. We also illustrate the effect of using \( L^2 \) norms in Theorem 5.5 instead of the \( H^{-1} \) norms in Theorem 5.4 for the indicators in figure 5.11.
Since we use time-invariant finite element spaces, the mesh-change estimators are null and do not influence the estimators.

The nonconforming indicator \( \left( \frac{1}{2} \sum_{n=1}^{m} \tilde{\delta}_{n} \tau_{n} \right)^{1/2} \) was found to be of higher order with respect the elliptic estimator, \( \eta_{e,m} \). This is most likely to be an effect of equivalence relation of type (4.1). Results for problem (5.149 - 5.151) with \( p = 2 \) and \( p = 3 \) are depicted and commented further in figures 5.1, 5.3 and 5.5 and 5.2, 5.4 and 5.6 respectively. Results for problem (5.152) with \( p = 1 \), problem (5.153) with \( p = 2 \) and problem (5.154) with \( p = 3 \) are depicted and commented further in figures 5.7, 5.8 and 5.9 respectively.

Adding space-time-dependent diffusion \( a \) to our numerical experiments will exhibit more properties of the estimators but we eschew deeper numerical experiments in this work for concision’s sake.

The derivation of adaptive methods based on our indicators is presented in the next chapter.
Figure 5.1: Example with exact solution $u_1$, given by (5.149), approximated with piecewise polynomials of degree $p = 2$, penalty parameters have value 500.

(a) Mesh-size $h \in \{\frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}}, \frac{1}{8}\}$, and timestep $\tau = 0.1 h^2$. On top we plot the EOC of the single cumulative indicators $\eta_{p,m}$ and $\eta_{e,m}$. Effectivity index tends towards 18.

(b) Mesh-size $h \in \{\frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}}, \frac{1}{8}\}$, and timestep $\tau = 0.1 h^3$. On top we plot the EOC of the single cumulative indicators $\eta_{p,m}$ and $\eta_{e,m}$. Effectivity index tends towards 18.
Figure 5.2: Example with exact solution $u_1$, given by (5.149), approximated with piecewise polynomials of degree $p = 3$, penalty parameters have value 1000.

(a) Mesh-size $h \in \{\frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}}, \frac{1}{8}\}$, and timestep $\tau = 0.1 \, h^3$. On top we plot the EOC of the single cumulative indicators $\eta_{p,m}$ and $\eta_{e,m}$. Effectivity index tends towards 57.

(b) Mesh-size $h \in \{\frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}}, \frac{1}{8}\}$, and timestep $\tau = 0.1 \, h^4$. On top we plot the EOC of the single cumulative indicators $\eta_{p,m}$ and $\eta_{e,m}$. Effectivity index tends towards 57.
5.5 Computer experiments

Figure 5.3: Example with exact solution \( u_2 \), given by (5.150), approximated with piecewise polynomials of degree \( p = 2 \), penalty parameters have value 500.

(a) Mesh-size \( h \in \{ \frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}} \} \), and timestep \( \tau = 0.1 h^2 \). On top we plot the EOC of the single cumulative indicators \( \eta_{p,m} \) and \( \eta_{e,m} \). Effectivity index tends towards 18.

(b) Mesh-size \( h \in \{ \frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}} \} \), and timestep \( \tau = 0.1 h^3 \). On top we plot the EOC of the single cumulative indicators \( \eta_{p,m} \) and \( \eta_{e,m} \). Effectivity index tends towards 18.
Figure 5.4: Example with exact solution $u_2$, given by (5.150), approximated with piecewise polynomials of degree $p = 3$, penalty parameters have value 1000.

(a) Mesh-size $h \in \{\frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}}\}$, and timestep $\tau = 0.1 h^3$. On top we plot the EOC of the single cumulative indicators $\eta_{p,m}$ and $\eta_{e,m}$. Effectivity index tends towards 57.

(b) Mesh-size $h \in \{\frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}}\}$, and timestep $\tau = 0.1 h^4$. On top we plot the EOC of the single cumulative indicators $\eta_{p,m}$ and $\eta_{e,m}$. Effectivity index tends towards 57.
Figure 5.5: Example with exact solution \( u_3 \), given by (5.151), approximated with piecewise polynomials of degree \( p = 2 \), penalty parameters have value 500.

(a) Mesh-size \( h \in \left\{ \frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}}, \frac{1}{8} \right\} \), and timestep \( \tau = 0.1 h^2 \). On top we plot the EOC of the single cumulative indicators \( \eta_{p,m} \) and \( \eta_{e,m} \). Effectivity index tends towards 30.

(b) Mesh-size \( h \in \left\{ \frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}}, \frac{1}{8} \right\} \), and timestep \( \tau = 0.1 h^3 \). On top we plot the EOC of the single cumulative indicators \( \eta_{p,m} \) and \( \eta_{e,m} \). Effectivity index tends towards 18.
5.5 Computer experiments

Figure 5.6: Example with exact solution $u_3$, given by (5.151), approximated with piecewise polynomials of degree $p = 3$, penalty parameters have value 1000.

(a) Mesh-size $h \in \{\frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}}, \frac{1}{8}\}$, and timestep $\tau = 0.1 h^3$. On top we plot the EOC of the single cumulative indicators $\eta_{p,m}$ and $\eta_{e,m}$. Effectivity index tends towards 83.

(b) Mesh-size $h \in \{\frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}}, \frac{1}{8}\}$, and timestep $\tau = 0.1 h^4$. On top we plot the EOC of the single cumulative indicators $\eta_{p,m}$ and $\eta_{e,m}$. Effectivity index tends towards 57.
Figure 5.7: Example with exact solution $u_1$, given by (5.152), approximated with piecewise polynomials of degree $p = 1$, penalty parameter 10, and mesh-size $h$ ranging in $\left\{ \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4} \right\}$, color-coded (or grey-shaded) with $\{c, g, y, r, m\}$. We test with two different time-step to mesh-size relations.

(a) Time-step to mesh-size $\tau = 0.1h$. On top we plot the EOC of the single cumulative indicators $\eta_{p,m}$ and $\eta_{e,m}$. Both indicators have the same asymptotic EOC $\approx 1$ as has the error. The effectivity index tends towards 7.1.

(b) Time-step to mesh-size $\tau = 0.1h^2$. On top we plot the EOC of the single cumulative indicators $\eta_{p,m}$ and $\eta_{e,m}$. Both indicators have the same asymptotic EOC $\approx 1$ as has the error. The effectivity index tends towards 7.1.
Figure 5.8: Example (5.153) with piecewise polynomials of degree $p = 2$, penalty parameter 1000, and mesh-size $h$ ranging in $\left\{ \frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}} \right\}$; color-coded (or grey-shaded) with $\{c, g, y, r\}$. We test with two different time-step to mesh-size relations.

(a) Time-step to mesh-size $\tau = 0.1 h^2$. On top we plot the EOC of the single cumulative indicators $\eta_{p,m}$ and $\eta_{e,m}$. $EOC < 1$ for the error is due to lack of $H^2$-regularity. Note that the elliptic estimator has asymptotically the same EOC as the error. Effectivity index tends towards value 13.5.

(b) Time-step to mesh-size $\tau = 0.1 h^3$. On top we plot the EOC of the single cumulative indicators $\eta_{p,m}$ and $\eta_{e,m}$. $EOC < 1$ for the error is due to lack of $H^2$-regularity. Note that the elliptic estimator has asymptotically the same EOC as the error. Effectivity index tends towards value 13.5.
Figure 5.9: Example (5.154) with discontinuous piecewise polynomials of degree \( p = 3 \), penalty parameter 90, and mesh-size \( h \) ranging in \( \left\{ \frac{1}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4\sqrt{2}} \right\} \), color-coded (or grey-shaded) with \{c, g, y, r\}. We test with two different time-step to mesh-size relations.

(a) Time-step to mesh-size \( \tau = 0.1 \, h^3 \). On top we plot the EOC of the single cumulative indicators \( \eta_{p,m} \) and \( \eta_{e,m} \). Both indicators have the same asymptotic EOC \( \approx 3 \) as has the error. The effectivity index tends towards asymptotic value 25.

(b) Time-step to mesh-size \( \tau = 0.1 \, h^4 \). On top we plot the EOC of the single cumulative indicators \( \eta_{p,m} \) and \( \eta_{e,m} \). Both indicators have the same asymptotic EOC \( \approx 3 \) as has the error. The effectivity index tends towards asymptotic value 25.
Figure 5.10: Penalty parameter effect on effectivity index. Example (5.153) with discontinuous piecewise polynomials of degree $p = 2$.

(a) Inverse effectivity indexes with penalty parameter 40 left and 120 right.

(b) Inverse effectivity indexes with penalty parameter 520 left and 1000 right.
Figure 5.11: Effect of using $L^2$ norms on effectivity index.

(a) Example (5.149) with discontinuous piecewise polynomials of degree $p = 2$, penalty parameter 40. Left effectivity index using $L^2$ norms and right using $H^{-1}$ norms. The parabolic estimator is not significant in this example, there is no noticeable difference in asymptotic effectivity indexes both approaching value $1/0.07$ but effectivity indexes are much sharper when using $H^{-1}$ norms.

(b) Example (5.151) with discontinuous piecewise polynomials of degree $p = 2$, penalty parameter 40. Left effectivity index using $L^2$ norms and right using $H^{-1}$ norms. As the parabolic estimator plays significant role in this example, one can clearly see a difference in asymptotic effectivity indexes. Note that here asymptotic effectivity index using $H^{-1}$ norms approaches value $1/0.07$. 
Chapter VI

Adaptive algorithm for parabolic problems

In this chapter we describe an adaptive algorithm which utilises estimators defined in the previous chapter. First we introduce the concept of adaptivity in the context of parabolic problems and then we present the implementation of the algorithm followed by demonstration of its practical use which is illustrated via model problems defined in previous chapter.

6.1 Introduction

The goal of the adaptive algorithm here is to reduce the total number of degrees of freedom (and hence the computational cost) needed to reach a desired error tolerance. To achieve this goal we follow commonly applied method for time-dependent problems (see, for example [34, 81]) and perform space and time adaptivity separately. Timesteps and mesh are controlled adaptively via the estimators \( \hat{\theta}_n + \delta_n + \beta_n \) (or \( \theta_n \)), \( \gamma_n' \) and \( \eta_{e,n} \), see Lemma 5.4 and Theorem 5.5. At each timestep \( t_{n-1} \leftarrow t_n \), we employ an adaptive strategy for elliptic problems to minimise the estimator \( \eta_{e,n} \).

Also, certain strategies can be applied to perform timestep adaptivity with the goal of minimizing \( \eta_{p,n} \) over time. The timestep adaptivity strategy that is applied here is the most efficient in the sense that it does not require re-iteration upon determination of acceptable timestep size. This procedure, however attractive from the
point of view of CPU efficiency, does lead to an algorithm which does not guarantee convergence to a tolerance. For spatially conforming methods in general, it is possible to define a so-called implicit timestepping algorithm for which the estimator $\eta_{p,n}$ is guaranteed to be under certain predefined tolerance (see, for example, [73, 85]) but in the case of the a posteriori error estimators given by Theorem 5.5 this is unfortunately not possible. This issue is related to the mesh change dependency of the timestep estimators and is explained in more detail in next section. The coarsening estimator $\gamma'_n$ is kept under a tolerance by precomputing it and mesh coarsening is performed only once at the beginning of each timestep. It is not the purpose of this text to prove any rigorous result about the adaptive algorithm but to demonstrate possible uses of the estimators in heuristic adaptive algorithms.

### 6.2 Adaptive algorithm

An adaptive algorithm for parabolic problems can be seen as consisting of three main parts: spatial coarsening, spatial refinement and temporal refinement/coarsening. The idea of adaptivity is to guarantee that the given estimators are kept under tolerances resulting into total error being kept under a tolerance whilst distributing the error more equally over the time-space domain. Also, the adaptive procedure should be automatic in the sense that it requires minimal additional information, i.e. only the data of the problem and a previous solution before the timestep, to perform an iteration on the solution. We begin by recalling the definitions of various estimators given by Theorem 5.5 which are used in the algorithm:

**Definition 6.1 (A posteriori error estimators for adaptive algorithm).**

\[ i) \text{ Time-stepping estimator is given by} \]
\[ \theta_n := \frac{C_{els}C_{dgc}}{\sqrt{3}} |||I^n U_{n-1} - U_n|||, \]  \hspace{1cm} (6.1)
ii) which can be used in conjunction with the operator approximation estimator due to mesh change given by

\[ \zeta_n := \zeta'_n + \zeta''_n + \| (A_{n-1} - A^n) U_{n-1} \|_{L^2(\Omega)}. \] (6.2)

Note that estimators \( \zeta'_n \) and \( \zeta''_n \) are both zero in the numerical examples ((5.149)) - ((5.154)) due to diffusion tensor being a constant over time.

iii) Another way to define the time-stepping estimator is given by \( \hat{\theta}_n \) can be given by

\[ \hat{\theta}_n := \| A_{n-1}^{-1} J_n U_{n-1} - A^n U_n \|_{L^2(\Omega)}. \] (6.3)

Benefit of using this estimator is that it can be used instead of the estimators \( \theta_n \) and \( \zeta_n \) leading to a possibly more efficient algorithm.

iv) The time data-approximation estimator is

\[ \beta_n := \left( \int_{t_{n-1}}^{t_n} \frac{\| f(t_n) - f(s) \|_{L^2(\Omega)}^2}{\tau_n} ds \right)^{1/2}. \] (6.4)

v) The mesh-change (or coarsening) estimator is defined as

\[ \gamma'_n := \| I^n U_{n-1} - U_{n-1} \|_{L^2(\Omega)}. \] (6.5)

vi) The parabolic nonconforming part estimator is given by

\[ \delta_n := \| U^d_n - U^d_{n-1} \|_{L^2(\Omega)}, \] (6.6)

and the elliptic nonconforming part estimator defined as

\[ \tilde{\delta}_n := \left( \| U^d_n \|^2 + \| U^d_{n-1} \|^2 \right)^{1/2}. \] (6.7)

vii) The space (or elliptic) error estimator is given by

\[ \varepsilon_n := \varepsilon(U_n, A^n U_n, T_n), \] (6.8)
where $\mathcal{E}$ is the energy-norm elliptic error estimator given by 3.1. Furthermore, the forward elliptic error indicator, due to mesh change, is given by

$$
\varepsilon_{n-1}^+ := \mathcal{E}(I^n U_{n-1}, a(t_{n-1}), A_{n-1}^{-1} I^n U_n, T_{n-1}).
$$

(viii) Finally, the operator approximation estimator due to mesh change is given by

$$
\zeta_n := ||A_{n-1}^{-1} U_{n-1} - A^n I^n U_{n-1}||_{L^2(\Omega)}
$$

Specifically, for our adaptive algorithm we define the following,

$$
\begin{align*}
\text{Est}_{\text{space}}^2 &:= \varepsilon_n^2 + \varepsilon_{n-1}^+^2 \\
\text{Est}_{\text{time}}^2 &:= \theta_n^2 \quad \text{or} \quad (\hat{\theta}_n + \delta_n + \beta_n)^2 \\
\text{Est}_{\text{coarse}}^2 &:= \gamma_n'
\end{align*}
$$

and then require that

$$
\begin{align*}
\text{Est}_{\text{space}} &\leq \text{TOL}_{\text{space}} \\
\text{Est}_{\text{time}} &\leq \text{TOL}_{\text{time}} \\
\text{Est}_{\text{coarse}} &\leq \text{TOL}_{\text{coarse}}
\end{align*}
$$

The total tolerance is then given by

$$
\text{TOL}_{\text{total}} := \sqrt{T} \ast (\text{TOL}_{\text{space}} + \text{TOL}_{\text{time}} + \text{TOL}_{\text{coarse}})
$$

Note that we study two different time estimators: the time estimator with $(\hat{\theta}_n + \delta_n + \beta_n)$ arises naturally from the a posteriori error bound but the, perhaps, more ad-hoc time estimator $\theta_n$ also yields good results when used in timestep controlling algorithm as is indicated by the numerical results. It should also be noted there are no systematic ways to predetermine settings for coarsening tolerance. As a rule, we used $\text{TOL}_{\text{coarse}} := 10^{-5} \ast \text{TOL}_{\text{space}}$. In general, when setting coarsening tolerances, one should avoid too much coarsening for more space dominated problem, such as (5.150), but for time dominated (oscillating) problems such as (5.151) there should be sufficient amount of coarsening. The mesh coarsening is an important part of
the adaptive algorithm whereby degrees of freedom that are no longer needed are removed and, hence, computational cost of the discrete problem is reduced. Also, note that we are using $\gamma'_n$ for coarsening only because the estimator $\gamma''_n$ is of higher order in time. However, we remark that by adding this estimator one could obtain more restricted coarsening behaviour but the value for $\text{TOL}_{\text{coarse}}$ should be adjusted accordingly.

### 6.2.1 Spatial Coarsening and Refinement

At each timestep, $t_n$, we solve the elliptic problem given by (5.16). Then, using residual based estimator $\text{Est}_{\text{space}}$, we get local estimate of estimator contributions per element. Using this information, we refine elements which are marked for refinement using either the so-called Maximum Strategy (see, for example, [76]) or Dörfler marking strategy. Dörfler marking strategy approach was studied in chapter 4 and Maximum Strategy was used in the numerical examples in section 3.3 earlier in the thesis. We also employ Dörfler marking strategy in the numerical examples to enable comparison between the two different strategies. The Maximum Strategy is often used in practice even though has not been shown to be convergent for second order elliptic problems.

In Maximum Strategy all elements which contribute to the total estimator more than some percentage of the maximum over single element contributions are marked for refinement. Mesh refinement is done by bisecting the longest edge of the marked element (see figure 6.1) and, to keep the mesh conforming (i.e. without hanging nodes), the adjacent (to the longest edge of the marked element) element is also marked and refined. The added edge is stored into a look-up data structure to be used in coarsening part of the space adaptivity. We outline the Maximum Strategy space adaptivity algorithm, MaximumStrategySpaceAdaptivity, in pseudocode below.
6.2 Adaptive algorithm

MaximumStrategySpaceAdaptivity

**Input:** $U_{n-1}, f, TOL_{space}, TOL_{coarse}, \tau_n, t_n, T, T_{n-1}, \xi_{\text{refine}}$

**Set:** $T_n := T_{n-1}$.

$T_n := \text{SpaceCoarsening}(U_{n-1}, TOL_{coarse}, \tau_n, T_n)$

{Refinement}

compute local elliptic estimators, $(\varepsilon_{n,\kappa})_{\kappa \in T_n}$, and so $\text{Est}_{space}$.

**While** $(\text{Est}_{space} > TOL_{space})$

sort $(\varepsilon_{n,\kappa})_{\kappa \in T_n}$ in descending order, set $Q := \emptyset$.

{Maximum Strategy}

**While** $((\varepsilon_{n,\kappa} > \xi_{\text{refine}} \times \max_{\kappa \in T_n}\{\varepsilon_{n,\kappa}\})$ and $(\kappa \in T_n))$

Mark $\kappa$ for refinement; $Q := \{\kappa\} \cup Q$.

**End While**

Refine all elements in $Q$ to obtain new mesh $T_n$.

Solve $I^n U_{n-1}$.

Solve (5.16) for $U_n$ with $\Pi^n U_{n-1}, \Pi^n f^n, \tau_n$ and $t_n$ on $T_n$.

compute local elliptic estimators, $(\varepsilon_{n,\kappa})_{\kappa \in T_n}$, and $\text{Est}_{space}$.

**End While**

**Output:** $U_n, T_n$

In Dörfler marking strategy all elements which contribute to the total estimator as a sum less than some percentage times the the total sum over single element contributions are marked for refinement. After obtaining all local estimators per each element, the elements are ordered in an array by the magnitude of their local estimator value so that elements with largest local estimator value are first. Then the local element estimators are summed over, starting from the begin of the array, and as long as the sum is less than some percentage of the total sum over single element contributions, the elements are marked for refinement. Mesh refinement is done similarly to the Maximum Strategy space adaptivity. We outline the Dörfler marking strategy space adaptivity algorithm, DörflerSpaceAdaptivity, in pseudocode below.
6.2 Adaptive algorithm

**DörflerSpaceAdaptivity**

**Input:** $U_{n-1}, f, TOL_{\text{space}}, TOL_{\text{coarse}}, \tau_n, t_n, T, T_{n-1}, \xi_{\text{refine}}$

**Set:** $T_n := T_{n-1}$.

$T_n := \text{SpaceCoarsening}(U_{n-1}, TOL_{\text{coarse}}, \tau_n, T_n)$

{Refinement}

compute local elliptic estimators, $(\varepsilon_{n,\kappa})_{\kappa \in T_n}$.

sum up local estimators and set $\text{Sum}_{\text{total}} := \sum_{\kappa \in T_n} \varepsilon_{n,\kappa}$, and compute $\text{Est}_{\text{space}}$.

**While** $(\text{Est}_{\text{space}} > TOL_{\text{space}})$

sort $(\varepsilon_{n,\kappa})_{\kappa \in T_n}$ in descending order, set $Q := \emptyset$.

**Set:** Sum = 0.

**While** $((\text{Sum} < \xi_{\text{refine}} \cdot \text{Sum}_{\text{total}}) \text{ and } (\kappa \in T_n))$

{Dörfler marking}

Sum := Sum + $\varepsilon_{n,\kappa}$

if $(\text{Sum} < \xi_{\text{refine}} \cdot \text{Sum}_{\text{total}})$

Mark $\kappa$ for refinement; $Q := \{\kappa\} \cup Q$.

**End While**

Refine all elements in $Q$ to obtain new mesh $T_n$.

Solve $I^n U_{n-1}$.

Solve (5.16) for $U_n$ with $\Pi^n U_{n-1}, \Pi^n f^n, \tau_n$ and $t_n$ on $T_n$.

compute local elliptic estimators, $(\varepsilon_{n,\kappa})_{\kappa \in T_n}$.

sum up local estimators and set $\text{Sum}_{\text{total}} := \sum_{\kappa \in T_n} \varepsilon_{n,\kappa}$, and compute $\text{Est}_{\text{space}}$.

**End While**

**Output:** $U_n, T_n$

Mesh coarsening is performed in two stages. Firstly, mesh is scanned for interior vertices that share four edges or boundary vertices that share three edges and which lie on a straight bit of the boundary (i.e. they are not corner nodes). Each vertex found constitutes a coarsenable patch of elements. Secondly, coarsening estimator $\gamma_n'$ is computed locally on each of the coarsenable patches and in order to get the largest number of patches marked for coarsening, this information is sorted in reverse order (smallest first) of local contributions of coarsenable patches within $\gamma_n'$. Starting from the smallest value of $\gamma_n'$ for a patch, the patch is marked to be coarsened if the sum of the marked patches local contributions in $\gamma_n'$ is less than $TOL_{\text{coarse}}$. By doing this,
the coarsening estimator is precomputed within desired tolerance. Then a look-up data structure is used to determine how a marked vertex and which related edges within the patch should be removed to force coarsening to be a reverse operation to refinement. In case that such information does not exist, arbitrary edges are removed from the coarsenable patch defined by the marked vertex (see Figure 6.1) so that the coarsened patch does not contain hanging nodes. We remark that this mesh conformity requirement (i.e. we do not allow hanging nodes in the mesh) is to test theoretical results obtained for the fourth order part in chapter 5. However, in practice, this condition can be relaxed and we believe our theoretical findings are still true as is indicated by the numerical examples in section 3.3. For purely second order parabolic problems, the mesh conformity requirement is not even theoretically necessary. We outline the algorithm for mesh coarsening, \textit{SpaceCoarsening}, in pseudocode below.

\begin{verbatim}
SpaceCoarsening
Input: $U_{n-1}, TOL_{\text{coarse}}, \tau_n, T_n$
{Coarsening}
get set of coarsenable patches, $L$.
For each patch, $M$, solve $I^n U_{n-1}$ and get local coarsening estimator $\gamma'_{m,M}$.
Resolve maximum subset $X$ of $L$ such that $\sum_{M \in X} \gamma'_{m,M} < TOL_{\text{coarse}}$.
Coarsen all patches in $X$ and obtain new mesh $T_n$.
Output: $T_n$
\end{verbatim}

\subsection*{6.2.2 Timestep Control}

Timestep control for algorithms which determine timestep size locally in time and are guaranteed to reach a tolerance is achieved by so called implicit timestep control. A good description of this approach with implementation details can be found in [34]. We outline this algorithm in pseudocode, \textit{ImplicitTimeStepControl}, below. However, as the estimator $Est_{\text{time}}$ was not found to exhibit guaranteed reduction in practice utilising the \textit{ImplicitTimeStepControl} upon mesh refinement when timestep size approaches zero, the implicit timestep control mechanism is not suited for the algorithm here. Nevertheless, it is possible to define a variant of timestep controlling based on a simple argument that the forward mesh change effect is not dominating the estimator $Est_{\text{time}}$ and the estimator primarily describes the error
distribution in time. The explicit timestep control algorithm used here does not require $\text{Est}_{\text{time}}$ to reduce when timestep size is decreased and mesh is refined; the value of $\text{Est}_{\text{time}}$ is merely used to steer the timestep size to the right direction upon each timestep in order to keep the total error within tolerance. This is realised by shortening the timestep by a constant factor $2^{-1/2}$ if $\text{Est}_{\text{time}}$ greater than tolerance and if $\text{Est}_{\text{time}}$ less than some minimum tolerance (we used $\text{TOL}_{\text{time, min}} := 0.5 \times \text{TOL}_{\text{time}}$ in the numerical examples) the timestep is increased by a constant factor $\sqrt{2}$. By doing this, one also avoids the repeated solution loops within the timestep control (which cannot be avoided with implicit timestep control algorithms) resulting into significant improvement in computation time. Also, even though the explicit timestep controlling algorithm is not guaranteed to reach a tolerance, it was found to converge to predefined tolerances for the numerical test problems. To complete our definition of adaptive algorithm, we follow the recipe for explicit timestep control given in [73] and we outline this algorithm in pseudocode, $\text{ExplicitTimeStepControl}$, below.
6.2 Adaptive algorithm

ImplicitTimeStepControl

Input: $U_0$, $f$, TOL$_{time,\text{min}}$, TOL$_{time}$, TOL$_{space}$, TOL$_{coarse}$, $\tau_0$, $t_0$, $T$

$T_0$, $\xi_{\text{refine}}$, SpaceAdaptivity, InitialSpaceAdaptivity

{ Initial condition interpolation and mesh refinement }

$(U_0, T_0)$ := InitialSpaceAdaptivity($U_0$, $f$, $T_0$, $\xi_{\text{refine}}$).

{Initialize.}

Set: $n = 1$, $\tau_n = \tau_{n-1}$.

While ($t_n \leq T$)

Set: $\tilde{\text{Est}}_{\text{time}} := \text{TOL}_{\text{time}} + 1$

While ($\tilde{\text{Est}}_{\text{time}} > \text{TOL}_{\text{time}}$)

$t_n := t_{n-1} + \tau_n$

Set: $T_n := T_n$

$(U_n, T_n)$ := SpaceAdaptivity($U_{n-1}$, $f$, TOL$_{space}$, TOL$_{coarse}$, $\tau_n$, $t_n$, $T$, $T_{n-1}$, $\xi_{\text{refine}}$) compute $\tilde{\text{Est}}_{\text{time}}$.

if ($\tilde{\text{Est}}_{\text{time}} > \text{TOL}_{\text{time}}$) then

{Shorten timestep.}

$\tau_n := \tau_n / 2$

Set: $T_n := T_n$

endif

End While

$\tau_{n+1} := \tau_n * 2$

$n := n + 1$

End While

Output: $U_n$

Naturally, one is interested in the possibility of forcing all of $\eta_{p,n}$ under a certain tolerance but unfortunately this cannot be guaranteed by the algorithms here as already mentioned earlier. Indeed, the fact that the estimators $\delta_n$ and $\zeta_n$ are very sensitive to the forward mesh change makes their use in practical implicit timestep size search algorithm unfeasible (implicit timestep control guarantees reaching a tolerance for time estimators, see [85, 34, 73]). Also, the alternative timestepping estimator (defined in (5.107)) was found to exhibit the same mesh change sensitivity as $\zeta_n$. By mesh change sensitivity of an estimator, we mean that upon timestep size
refinement $(t_n - t_{n-1}) \to 0$, the estimator is not guaranteed to reduce in practice if
the mesh on the current time level $t_n$ is refined. To this end, the best and the most
mesh-change insensitive estimator was found to be $\theta_n$ (defined in (6.1)) and explicit
timestep size determining algorithm based on just this estimator was found to give
convergence to a tolerance for the example problems considered. The estimator $\theta_n$
also exhibits good effectivity index behaviour which also partly justifies its use as a
sole estimator for timestep control. However, in practice the estimator $(\hat{\theta}_n + \delta_n + \beta_n)$
was found to be the best in explicit timestep control. The use of this estimator is
directly supported by the a posteriori error bound. It is also worth noting that the
tolerances set for this estimator were reached within the explicit timestep controlling
scheme.

\begin{verbatim}
ExplicitTimeStepControl
Input: $U_0$, $f$, TOL\text{time.min}, TOL\text{time}, TOL\text{space}, TOL\text{coarse}, \tau_0, t_0, T$
\hspace{1cm} $T_0, \xi_{\text{refine}}, \text{SpaceAdaptivity, InitialSpaceAdaptivity}$
\hspace{1cm} \{ Initial condition interpolation and mesh refinement \}
\hspace{1cm} $(U_0, T_0) := \text{InitialSpaceAdaptivity}(U_0, f, T, T_0, \xi_{\text{refine}}).$
\hspace{1cm} \{Initialize.\}

Set: $n = 1$, $\tau_n = \tau_{n-1}$ and $t_n = t_{n-1} + \tau_n$.

While $(t_n \leq T)$
\hspace{1cm} $(U_n, T_n) := \text{SpaceAdaptivity}(U_{n-1}, f, TOL\text{space}, TOL\text{coarse}, \tau_n, t_n, T, T_{n-1}, \xi_{\text{refine}}).$
\hspace{1cm} compute $\text{Est.time}.$
\hspace{1cm} if $(\text{Est.time} > \text{TOL}_\text{time})$ then
\hspace{2cm} $\tau_{n+1} := \tau_n / \sqrt{2}$
\hspace{1cm} elseif $(\text{Est.time} < \text{TOL}_\text{time.min})$ then
\hspace{2cm} $\tau_{n+1} := \tau_n \ast \sqrt{2}$
\hspace{1cm} endif
\hspace{1cm} $t_{n+1} := t_n + \tau_n$
\hspace{1cm} $n := n + 1$

End While

Output: $U_n$
\end{verbatim}
To summarize, the problem with good timestep size determining estimator within the space nonconforming spatial method framework is to find/derive an estimator which not too sensitive to mesh change. We would like to emphasize that this problem is not concerned with fourth order problems only but is present in lower order problems also. If such good estimators do not exist then the issue would have to be resolved via algorithm design so that convergence to a tolerance for time estimators could be guaranteed. Otherwise, one has to resort to explicit timestep controlling algorithms, as is done here, but at a cost of losing the guaranteed convergence to a tolerance. On the other hand, by using explicit timestep control one can guarantee termination of the algorithm within reasonable computation times. Implicit timestep controlling is naturally much slower computationally than explicit control due to repeated solving for finding an acceptable timestep size. However, this thesis is not concerned with answering this relatively large question of design of adaptive algorithms based on nonconforming discretization in space that guarantee convergence to a tolerance. We merely state this point that has risen from this research. This is currently an open problem.

Figure 6.1: Conforming mesh refinement and coarsening.
6.2.3 Numerical examples

We ran the adaptive algorithm, ExplicitTimeStepControl, for time estimators $\theta_n$ and $(\hat{\theta}_n + \delta_n + \beta_n)$ to the tolerances given by the uniform strategy in chapter 5 for polynomial degree $p = 2$ with penalty parameter values set to 500 in examples (5.149)-(5.151) and with penalty parameter value set to 40 in examples (5.152)-(5.154). Results on accumulated degrees of freedom for examples (5.149)-(5.151) are listed in tables 6.1 and 6.2 for spatial Maximum Principle refinement ratios $\xi = 0.65$ and $\xi = 0.75$ for time estimators $\theta_n$ and $(\hat{\theta}_n + \delta_n + \beta_n)$ respectively and results for examples (5.152)-(5.154) are listed in table 6.3 with Dörfler marking strategy refinement ratios $\xi = 0.65$ and $\xi = 0.75$ and $\theta_n$ as a sole time-estimator. Timestep size and degrees of freedom evolution for the adaptive algorithm are monitored and compared to the uniform timestep and mesh strategy in Figures 6.2.3 and 6.2.3 for examples (5.149)-(5.151) for each time estimator respectively and in Figure 6.4 for examples (5.152)-(5.154). The results clearly indicate that the use of the adaptive algorithm does lead to significant reduction of accumulated degrees of freedom whilst reaching the desired tolerances. This was achieved most efficiently when the complete time-estimator $(\hat{\theta}_n + \delta_n + \beta_n)$ was used to control the time steps in conjunction with the Maximum Strategy in space. For this reason, adaptive algorithm enables solutions to greater accuracy with less computational effort. Some typical meshes resulting from the adaptive algorithm are depicted in figures 6.5 - 6.8 for example problems (5.150) and (5.151) at various times.

However, the implementation of the mesh projection operator $(I^n : S^r_{n-1} \rightarrow S^r_n)$ used in the numerical examples is currently not optimised for efficiency but for accuracy using highly accurate quadratures. Also, an implementation of this operator was not technically CPU efficient via the FEniCS libraries framework provided at the time of writing this thesis. As this technical setting does not give an accurate description of the possible gains in terms of the CPU efficiency, this question is not discussed in more detail here.
Table 6.1: Adaptive algorithm results with $\theta_n$ as time-stepping estimator for example problems (5.149)-(5.151). The adaptive algorithm is tested on the tolerances given by the uniform strategy for $p = 2$. Results on accumulated degrees of freedom are listed in table 6.1 for spatial refinement ratios used in Maximum Strategy marking strategy $\xi = 0.65$ and $\xi = 0.75$.  

<table>
<thead>
<tr>
<th>Example</th>
<th>Tolerance</th>
<th>Degrees of freedom</th>
<th>Uniform</th>
<th>Adaptive $\xi = 0.65$</th>
<th>Adaptive $\xi = 0.75$</th>
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<tr>
<td></td>
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<td>2 487 456</td>
<td>2 563 014</td>
<td></td>
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<tr>
<td>(5.150)</td>
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<td>12 222</td>
<td>9 216</td>
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<tr>
<td></td>
<td>32.46</td>
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<td>61 500</td>
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<td>6 090</td>
<td>6 090</td>
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<tr>
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### Table 6.2: Adaptive algorithm results with $(\hat{\theta}_n + \delta_n + \beta_n)$ as time-stepping estimator for example problems (5.149)-(5.151). The adaptive algorithm is tested on the tolerances given by the uniform strategy for $p = 2$. Results on accumulated degrees of freedom are listed in table 6.2 for spatial refinement ratios used in Maximum Strategy marking strategy $\xi = 0.65$ and $\xi = 0.75$.

<table>
<thead>
<tr>
<th>Example</th>
<th>Tolerance</th>
<th>Degrees of freedom</th>
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Table 6.2: Adaptive algorithm results with $(\hat{\theta}_n + \delta_n + \beta_n)$ as time-stepping estimator for example problems (5.149)-(5.151). The adaptive algorithm is tested on the tolerances given by the uniform strategy for $p = 2$. Results on accumulated degrees of freedom are listed in table 6.2 for spatial refinement ratios used in Maximum Strategy marking strategy $\xi = 0.65$ and $\xi = 0.75$.  


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<thead>
<tr>
<th>Example</th>
<th>Tolerance</th>
<th>Degrees of freedom</th>
<th>Uniform</th>
<th>Adaptive</th>
<th>( \xi = 0.65 )</th>
<th>( \xi = 0.75 )</th>
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</thead>
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<td>0.168</td>
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<td>0.077</td>
<td>7 864 320</td>
<td>2 821 272</td>
<td>3 025 536</td>
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<td>(5.153)</td>
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<td>130 752</td>
<td>15 732</td>
<td>12 792</td>
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<td>737 280</td>
<td>54 588</td>
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<td>4 172 544</td>
<td>169 212</td>
<td>169 212</td>
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<tr>
<td>(5.154)</td>
<td>0.51</td>
<td>43 584</td>
<td>21 072</td>
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<td>3 187 008</td>
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Table 6.3: Adaptive algorithm results for example problems (5.152)-(5.154). The adaptive algorithm is tested on the tolerances given by the uniform strategy for \( p = 2 \). Results on accumulated degrees of freedom are listed in table 6.3 for spatial refinement ratios used in *Dörfler marking* strategy \( \xi = 0.65 \) and \( \xi = 0.75 \). Note that the estimator \( \theta_n \) only was used here for timestep control.
Figure 6.2: Adaptive algorithm vs uniform strategy, refinement ratio = 0.65, $\theta_n$ as time-stepping estimator.

(a) Example (5.149), tolerance 0.2, left timestep sizes $\Delta t$ and right degrees of freedom.

(b) Example (5.150), tolerance 8.703, left timestep sizes $\Delta t$ and right degrees of freedom.

(c) Example (5.151), tolerance 0.2116, left timestep sizes $\Delta t$ and right degrees of freedom.
Figure 6.3: Adaptive algorithm vs uniform strategy, refinement ratio = 0.75, \((\hat{\theta}_n + \delta_n + \beta_n)\) as time-stepping estimator.

(a) Example (5.149), tolerance 0.2, left timestep sizes \(\Delta t\) and right degrees of freedom.

(b) Example (5.150), tolerance 8.703, left timestep sizes \(\Delta t\) and right degrees of freedom.

(c) Example (5.151), tolerance 0.2116, left timestep sizes \(\Delta t\) and right degrees of freedom.
Figure 6.4: Adaptive algorithm vs uniform strategy, refinement ratio = 0.65.

(a) Example (5.152), adaptivity with tolerance 0.077. The ordinate is the timestep size \( \tau_n \), on the left, and the spatial degrees of freedom \( \dim S^n \), on the right, with \( t_n \) in abscissa.

(b) Example (5.153), adaptivity with tolerance 0.036. The ordinate is the timestep size \( \tau_n \), on the left, and the spatial degrees of freedom \( \dim S^n \), on the right, with \( t_n \) in abscissa.

(c) Example (5.154), adaptivity with tolerance 0.168. The ordinate is the timestep size \( \tau_n \), on the left, and the spatial degrees of freedom \( \dim S^n \), on the right, with \( t_n \) in abscissa.
6.2 Adaptive algorithm

Figure 6.5: *Example (5.150)*. Meshes resulting from adaptive algorithm, $p = 2$, penalty parameter=500, refinement ratio = 0.65, (a) $t = 0.0$; (b) $t = 0.37678$

Figure 6.6: *Example (5.150)*. Meshes resulting from adaptive algorithm, $p = 2$, penalty parameter=500, refinement ratio = 0.65, (c) $t = 0.49079$; (d) $t = 0.97438$
Figure 6.7: *Example (5.151).* Meshes resulting from adaptive algorithm, \( p = 2 \), penalty parameter \( = 500 \), refinement ratio \( = 0.65 \), (a) \( t = 0.030661 \); (b) \( t = 0.20172 \).

Figure 6.8: *Example (5.151).* Meshes resulting from adaptive algorithm, \( p = 2 \), penalty parameter \( = 500 \), refinement ratio \( = 0.65 \), (c) \( t = 0.36185 \); (d) \( t = 0.91046 \).
Chapter VII

Summary and outlook

7.1 Summary

In this thesis we have discussed the problem of error control for fourth order PDEs discretized by the discontinuous Galerkin method in detail. To this end, we derived new residual based a posteriori bounds and outlined practical implementation of a posteriori error bounds in adaptive algorithms. Also, we discussed the convergence analysis of an adaptive algorithm for elliptic fourth order equations. We also illustrated efficiency and reliability of theoretical findings via numerical tests.

In chapter 3, we presented new reliable and efficient a posteriori bounds in the energy norm for the interior penalty discontinuous Galerkin method (IPDG) proposed in [91] for the biharmonic problem with essential boundary conditions, under minimal regularity assumptions on the analytical solution. The novelty of the proof presented here is that it does not require a conforming subspace of the finite element space, unlike the previous results for second order problems. The reliability bound was based on a suitable recovery operator, which maps discontinuous finite element spaces to $H^2_0$-conforming finite element spaces (of two polynomial degrees higher), consisting of triangular or quadrilateral macro-elements defined in [40] (cf. also [24, 68, 65] for similar constructions). Using this recovery operator, in conjunction with the inconsistent formulation for the IPDG presented in [55] (which ensures that the weak formulation of the problem is defined under minimal regularity assumptions on the analytical solution), the efficient and reliable a posteriori estimates...
of residual type for the IPDG method in the corresponding energy norm were derived for finite element spaces consisting of triangular or quadrilateral element of polynomial degree, $r \geq 2$.

In chapter 4, we discussed issues related to the convergence analysis of an adaptive discontinuous Galerkin algorithm for the biharmonic equation. The a posteriori error bounds are naturally utilised in the construction of adaptive algorithms but it is only very recently that the adaptive discontinuous Galerkin method for the second order problems have been proven to be convergent in the sense of $h$-adaptivity. The purpose of this chapter was to point to difficulties present in the convergence proof for the adaptive discontinuous Galerkin (ADG) method for the biharmonic problem. The discussion was inspired and influenced by the works of Bonito and Nochetto [19], Karakashian and Pascal [69], Cascon, Kreuzer, Nochetto and Siebert [32] and Stevenson [88] in the context of second order problems. The conclusion was that the difficulty with the approach presented for the fourth order convergence proof, using the state-of-the-art technical tools available from the context of second order problems, was to show that the penalty jumps are bounded from above by a posteriori error estimator quantities which are guaranteed to reduce under some marking strategy. This question, as well as the convergence proof of adaptive DG algorithm for biharmonic equation still remains an open question after writing this thesis, but the discussion in this chapter is issued in the hope that it will help future work on this interesting question.

In chapters 5 and 6, we derived new energy-norm a posteriori error bound for an implicit Euler time-stepping method combined with spatial discontinuous Galerkin scheme for linear fourth (and second) order parabolic problems. We believe that these results are completely new in that no previous work has considered the energy norm a posteriori error analysis for fully discrete noncoforming approximation of fourth order parabolic problems before and also that no previous work has numerically tested the a posteriori error estimators for a fully discrete time-stepping method in an adaptive algorithm with spatial discontinuous Galerkin discretization for both fourth and second order parabolic problems. Conclusion of these results was an adaptive algorithm which, despite of not necessarily guaranteeing convergence to a tolerance, was shown to perform well for the test problems and most importantly lead to efficiency improvement in terms of degrees of freedoms needed to achieve a
desired tolerance.

7.2 Outlook and future work

The main goal of this thesis was to develop adaptive algorithms based on new a posteriori bounds derived for the discontinuous Galerkin approximation of fourth order elliptic and parabolic equations. We outline some possible future directions that could naturally arise from this work:

- As already discussed in chapter 3, non-homogeneous Dirichlet boundary conditions (i.e. boundary conditions for function as well as normal derivative across the boundary of the domain) can be dealt with using the “plate” formulation but additional questions arise from the Neumann boundary condition (or more complex mixed boundary condition) from the point of view of design of dg methods as well as corresponding a posteriori error bounds which are far from resolved. The question of design of appropriate fully discontinuous dg methods for these boundary conditions is, to the best of our knowledge, not solved, though work to this direction has been made in [96].

- The meshes in this work were assumed to be conforming, in practice however, this is not necessarily a feasible option and one is left with hanging nodes resulting from refinement procedures. We believe that one can resolve this issue by following ideas presented in work by Karakashian and Pascal [68] for hanging nodes in case of meshes consisting of triangles. Indeed, the necessary requirement on the mesh in their work is that a nonconforming mesh can be made into a conforming mesh by finite number of mesh refinement and coarsening operations. More importantly, this feature is independent of the dual basis of the conforming element against which the averaging is performed, enabling its use in the context of our recovery operator (See section 3.1) based on HCT elements.

- Another very interesting question to consider could be the issues regarding a possible convergence proof of adaptive discontinuous Galerkin methods for biharmonic equation. One way to go about the problem, raised by the assumption 4.1 in chapter 4, could be a derivation of a bound for the resulting
Galerkin orthogonality residual \( B(u - u_m, E(u_m)) \) by terms which are a posteriori “controllable” and do reduce by some carefully chosen marking strategy. This is obviously a completely heuristic argument without any mathematical justification as of yet. As a recent and perhaps related development to this, it is interesting to note the bounds derived in [22] for the corresponding similar recovery error term \( u - E(u_m) \) in various norms.

- Throughout this thesis we have considered \( h \)-adaptive discontinuous Galerkin methods. To fully appreciate the flexibility and power of the discontinuous Galerkin methods which are best suited for the \( hp \)-adaptivity whereby refinement and coarsening takes place not only through mesh refinement or coarsening but via locally bounded variation (by increasing or decreasing) of polynomial degrees over elements. From the approximation point of view, it makes sense to increase the polynomial degree (and hence also approximation power) for an element where the solution to be approximated is smooth. On the other hand, it makes perhaps more sense to refine the mesh where solution exhibits nonsmooth behaviour. A challenging question could be the derivation of \( hp \)-version a posteriori error bounds in the light of work concluded in chapter 3. In the context of second order problems such analysis for DG methods is presented in [65].

- Further design of adaptive algorithms for parabolic problems utilising DG in space. Solving issues on timestep controlling estimators as outlined in chapter 6 to enable implicit timestep control leading to guaranteed error reduction. This question is perhaps not motivated from the practical point of view as the explicit timestep controlling algorithm described in section 6.2.2 works well in practice but could be of theoretical importance.

- Extension of results for parabolic problems to nonlinear problems. One possible direction could be to follow the ideas established in [13] for Ginzburg-Landau equations with the nonlinearity \( f(u) = u - u^3 \) resulting into fourth order equations usually termed the extended Kolmogorv-Fisher equation (see, for example, [37]). Amongst different approaches to tackle nonlinear problems, we also mention the approach by Feng and Wu [53] where continuous dependence estimate is used to derive a posteriori bounds (see also [9, 33]), for the
Cahn-Hillard equation.

- Extension of the results to three dimensional domains; an extension of the recovery operator results in chapter 3 could already be possible for second order polynomials due to recent work by Alfeld and Sorokina [8] where cubic tetrahedral elements are constructed which utilize first order derivatives and function evaluations in the nodal basis.
APPENDIX A

INTEGRATION BY PARTS FORMULAS

Lemma A.1. Let \( u,v \in H^1(\kappa) \) and \( \kappa \subset \mathbb{R}^n \) a polygonal domain then

\[
\int_{\kappa} \Delta^2 uv \, dx = \int_{\kappa} D^2 u : D^2 v \, dx + \int_{\partial \kappa} \partial_n(\Delta u) \, v \, ds - \int_{\partial \kappa} \partial_n(\nabla u) \cdot \nabla v \, ds. \quad (A.1)
\]

Proof. Suppose first that \( u,v \in C^\infty(\overline{\kappa}) \) then Gauss’ formula gives

\[
\int_{\kappa} \nabla \cdot (\nabla \Delta u) \, dx = \int_{\kappa} \nabla \cdot (\nabla u) v \, dx + \int_{\kappa} \nabla \Delta u \cdot \nabla v \, dx = \int_{\partial \kappa} \nabla \Delta u \cdot n_\kappa v \, ds \quad (A.2)
\]

and so

\[
\int_{\kappa} \Delta^2 uv \, dx = -\int_{\kappa} \nabla \Delta u \cdot \nabla v \, dx + \int_{\partial \kappa} \nabla \Delta u \cdot n_\kappa v \, ds. \quad (A.3)
\]

Writing

\[
-\int_{\kappa} \nabla \Delta u \cdot \nabla v \, dx = -\sum_{i=1}^{n} \int_{\kappa} (\nabla \cdot \nabla u_{x_i}) v_{x_i} \, dx \quad (A.4)
\]

which is possible as \( u \in C^\infty(\overline{\kappa}) \) and applying Gauss’ formula

\[
\int_{\kappa} \nabla \cdot (\nabla uv) \, dx = \int_{\kappa} \Delta uv \, dx + \int_{\partial \kappa} \nabla u \cdot \nabla v \, dx = \int_{\partial \kappa} \nabla u \cdot n_\kappa v \, ds \quad (A.5)
\]

\[
\Rightarrow \int_{\kappa} \Delta uv \, dx = -\int_{\kappa} \nabla u \cdot \nabla v \, dx + \int_{\partial \kappa} \nabla u \cdot n_\kappa v \, ds \quad (A.6)
\]
on each term on the right-hand side of (A.4), we get
\[- \sum_{i=1}^{n} \int_{\kappa} (\nabla \cdot \nabla u_{x_i}) v_{x_i} \, dx = \sum_{i=1}^{n} \left( \int_{\kappa} \nabla u_{x_i} \cdot \nabla v_{x_i} \, dx - \int_{\partial \kappa} \nabla u_{x_i} \cdot n_{\kappa} v_{x_i} \, ds \right) \]  (A.7)

The terms on the right-hand side of (A.7) can be re-written as
\[\sum_{i=1}^{n} \int_{\kappa} \nabla u_{x_i} \cdot \nabla v_{x_i} \, dx = \int_{\kappa} D^2 u : D^2 v \, dx \]  (A.8)
and
\[\sum_{i=1}^{n} - \int_{\partial \kappa} \nabla u_{x_i} \cdot n_{\kappa} v_{x_i} \, ds = - \int_{\partial \kappa} \partial_{n_{\kappa}} (\nabla u) \cdot \nabla v \, ds \]  (A.9)
which can be used with (A.3) to obtain
\[\int_{\kappa} \Delta^2 uv \, dx = \int_{\kappa} D^2 u : D^2 v \, dx - \int_{\partial \kappa} \partial_{n_{\kappa}} (\nabla u) \cdot \nabla v \, ds + \int_{\partial \kappa} \partial_{n_{\kappa}} (\Delta u) v \, ds \]  (A.10)

As \(C^\infty(\kappa)\) is dense in \(H^4(\kappa)\) the assertion follows. □

**Lemma A.2.** Let \(u, v \in H^4(\kappa)\) and \(\kappa \subset \mathbb{R}^n\) a polygonal domain then
\[\int_{\kappa} \Delta^2 uv \, dx = \int_{\kappa} \Delta u \Delta v \, dx + \int_{\partial \kappa} \nabla \Delta u \cdot n_{\kappa} v \, ds - \int_{\partial \kappa} \Delta u \cdot \nabla v \cdot n_{\kappa} \, ds \]  (A.11)

*Proof.* Analogous to the previous proof of Lemma A.1. □

**Lemma A.3.** Let \(T\) be a subdivision of \(\Omega\), \(\Omega \subset \mathbb{R}^n\) with \(2 \leq n \leq 3\) and \(u, v \in H^4(T, \Omega)\) then
\[\sum_{\kappa \in T} \int_{\partial \kappa} v \cdot n \, ds = \int_{\Gamma} [v] \cdot \{q\} \, ds + \int_{\Gamma_{int}} \{v\} [q] \, ds. \]  (A.12)

*Proof.* See [91] for details. □
Appendix B

Bubble functions

In the following, $P^m(A)$ denotes the set of m-th degree polynomials defined on $A \subset \Omega$.

Lemma B.1. Let $T$ be a subdivision of $\Omega \subset \mathbb{R}^2$ and $e = \kappa^+ \cap \kappa^- \in \Gamma_{\text{int}}$ an interior edge and $\kappa^{+,-} \subseteq \kappa^+ \cup \kappa^-$ is assumed to be a quadrilateral for any elements $\kappa^+$ and $\kappa^-$ sharing an edge. Let also $b_\kappa : \Omega \to \mathbb{R}_+$ and $b_\kappa : \Omega \to \mathbb{R}_+$ be defined by

$$b_\kappa(x,y) = \begin{cases} 27(1-x-y)yx & \text{for all } x, y \in \hat{\kappa} \\ 0 & x, y \notin \hat{\kappa} \end{cases}$$ \hspace{1cm} (B.1)

and

$$b_{\kappa^{+,-}}(x,y) = \begin{cases} (1-x^2)(1-y^2) & \text{for all } x, y \in \hat{\kappa}^{+,-} \\ 0 & x, y \notin \hat{\kappa}^{+,-} \end{cases}$$ \hspace{1cm} (B.2)

where $\hat{\kappa}^{+,-} = \{(x,y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$ and $\hat{\kappa} = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1-x\}$.

Then, the functions $p_1 : P^m(\kappa) \to \mathbb{R}_+$ and $p_2 : P^m(e) \to \mathbb{R}_+$ given by

$$p_1(v) = \left( \int_\kappa v^2 b_\kappa^r dx \right)^\frac{1}{2}$$ \hspace{1cm} (B.3)

and

$$p_2(v) = \left( \int_e v^2 b_{\kappa^{+,-}}^r ds \right)^\frac{1}{2}$$ \hspace{1cm} (B.4)
define norms on $P^m(\kappa)$ and $P^m(e)$ (m fixed) respectively for any fixed $r \in \mathbb{N}$. For $p_1$ and $p_2$ holds,

$$C_1\|v\|_{\kappa} \leq p_1(v) \leq C_2\|v\|_{\kappa} \quad (B.5)$$

and

$$D_1\|v\|_e \leq p_2(v) \leq D_2\|v\|_e \quad (B.6)$$

where the constants $C_1$ and $C_2$ are independent of $\kappa$ and the constants $D_1$ and $D_2$ are independent of $e$.

Proof. Let $u, v \in P^m(\kappa)$ and $\hat{u}, \hat{v} \in P^m(\hat{\kappa})$ with $\hat{u}(\hat{x}) = u(F_\kappa(\hat{x}))$ and $\hat{v}(\hat{x}) = v(F_\kappa(\hat{x}))$ for all $\hat{x} \in \hat{\kappa}$. Clearly, $p_1(\hat{u}) \geq 0$, and

$$p_1(\lambda \hat{u}) = (\lambda^2)^{\frac{1}{2}} p_1(\hat{u}) = |\lambda| p_1(\hat{u}) \quad (B.7)$$

and also

$$p_1(\hat{u} + \hat{v})^2 = p_1(\hat{u})^2 + p_1(\hat{v})^2 + 2 \int_\kappa \hat{u} \hat{v} b_\kappa^2 \text{d}x \quad \leq \quad p_1(\hat{u})^2 + p_1(\hat{v})^2 + 2 p_1(\hat{u}) p_1(\hat{v}) \quad (B.8)$$

Taking square root of both sides of (B.8) yields $p_1(\hat{u} + \hat{v}) \leq p_1(\hat{u}) + p_1(\hat{v})$. If $p_1(\hat{u}) = 0$ then $\hat{u} = 0$ almost everywhere because $b_\kappa^2 > 0$ almost everywhere on $\kappa$. So, $p_1$ is a norm on $P^m(\hat{\kappa})$. The same way it can be shown that $p_1$ defines a norm on $\kappa$ and that $p_2$ defines a norm on $e$ (as well as on some reference quadrilateral).

As $\dim(P^m(\hat{\kappa})) < \infty$, we have that all norms on $P^m(\hat{\kappa})$ are equivalent. This implies that there are $\hat{C}_1 > 0$ and $\hat{C}_2 > 0$ (depending on $\hat{\kappa}$) such that

$$\hat{C}_1\|\hat{v}\|_{\hat{\kappa}} \leq p_1(\hat{v}) \leq \hat{C}_2\|\hat{v}\|_{\hat{\kappa}} \quad \forall v \in P^m(\hat{\kappa}) \quad . \quad (B.9)$$

On the other hand, the transformation rule for integrals gives,

$$\int_\hat{\kappa} \hat{v}^2 b_\kappa^2 \text{d}\hat{x} = \int_\kappa v^2 b_\kappa^2 \text{det}(DF_\kappa^{-1}) \text{d}x \quad (B.10)$$

and also

$$\int_\hat{\kappa} \hat{v}^2 \text{d}\hat{x} = \int_\kappa v^2 \text{det}(DF_\kappa^{-1}) \text{d}x \quad . \quad (B.11)$$

Using the assumption (2.17) and (2.17) and the results (B.9) and (B.10), we get
\[ \tilde{C}_1 \left( \frac{C_1}{C_2} \right)^{\frac{1}{p}} \|v\|_{\kappa} \leq p_1(v) \leq \tilde{C}_2 \left( \frac{C_2}{C_1} \right)^{\frac{1}{p}} \|v\|_{\kappa}. \]  

(B.12)

The proof of the case \( p_2 \) is analogous. □
APPENDIX C

APPROXIMATION PROPERTY, INVERSE ESTIMATES, TRACE AND BROKEN POINCARE-FRIEDRICHS INEQUALITY

Here we state some Theorems frequently used in this thesis. Below $P^r(\kappa)$ denotes space of polynomials of degree $r$ defined on $\kappa$ and $h_\kappa := \text{diam}(\kappa)$.

**Lemma C.1.** (Approximation Property) Let $0 \leq m \leq r + 1$ and $T$ be a subdivision of $\Omega$, $\Omega \subset \mathbb{R}^2$. Then there exists a constant $C_{\text{APP}}$, independent of $h_\kappa$, such that for any $u \in H^m(\Omega)$ and $\kappa \in T$, there exists $p \in P^r(\kappa)$ (space of polynomials of degree $r$)

$$|u - p|_{j,\kappa} \leq C_{\text{APP}} h_\kappa^{m-j} |u|_{m,\kappa} , 0 \leq j \leq m$$

(C.1)

*Proof.* See, e.g. [40] for HCT elements or [41] or [35] pp. 122 for details.

**Lemma C.2.** (Inverse Estimate) There exists a constant $C_{\text{INV}}$, independent of $T \subset \mathbb{R}^2$, such that

$$|u|_{j,\kappa} \leq C_{\text{INV}} h_\kappa^{i-j} |u|_{i,\kappa} , 0 \leq i \leq j \leq 2 \text{ for all } u \in P^r(\kappa)$$

(C.2)

*Proof.* See [35], pp. 140 for details.
**Lemma C.3.** *(Trace Inequality)* Let \( \kappa \subset \mathbb{R}^2 \) and \( \partial \kappa \) denote its boundary. Then for any \( u \in H^1(\kappa) \) holds

\[
||u||^2_{0,\partial \kappa} \leq C_{TR}(h_D^{-1}||u||^2_{0,\kappa} + h_D||u||^2_{1,\kappa})
\]  
*(C.3)*


**Lemma C.4.** *(Poincaré-Friedrichs Inequality)* Let \( T \) be a subdivision of \( \Omega \), \( \Gamma \) a collection of edges of elements in \( T \) and \( h_e \) denotes a length of an edge \( e \in \Gamma \). Then there exists a constant \( C_{PF} \), independent of \( h_\kappa \) such that for any \( u \in H^2(\Omega,T) \) there holds

\[
||u||^2_{0,\Omega} + ||u||^2_{1,\Omega} \leq C_{PF}\left(||u||^2_{2,\Omega} + \sum_{e \in \Gamma} \left(\frac{1}{h_e^3}||[u]||^2_{0,e} + \frac{1}{h_e}||[\nabla u]||^2_{0,e}\right)\right).
\]  
*(C.4)*


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