An optimization approach to weak approximation of Lévy-driven stochastic differential equations with application to option pricing

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Abstract—We propose an optimization approach to weak approximation of Lévy-driven stochastic differential equations. We employ a mathematical programming framework to obtain numerically upper and lower bound estimates of the target expectation, where the optimization procedure ends up with a polynomial programming problem. An advantage of our approach is that all we need is a closed form of the Lévy measure, not the exact simulation knowledge of the increments or of a shot noise representation for the time discretization approximation. We also investigate methods for approximation at some different intermediate time points simultaneously.

I. INTRODUCTION

Stochastic differential equations have long been used to build realistic models in economics, finance, biology, the social sciences, chemistry, physics and other fields. In most active fields of application, dynamics with possible sudden shift have become more and more important. To model such shifts, one would like to employ stochastic differential equations where the underlying randomness contains jumps. For this purpose, the diffusion process is not sufficient since its sample paths are almost surely continuous. On the other hand, Lévy-driven stochastic differential equations, which contain diffusion as a special case, can formulate stochastic behavior with jumps. Regardless of its practical importance, however, the theory and the computational techniques of the Lévy processes have not been developed thoroughly as in the diffusion case. As nice references on the subject, we refer to Applebaum [1] and Bass [2].

From a practical point of view, the sample paths approximation of stochastic differential equations has been a central issue for the purpose of numerical evaluation and simulation on the computer. There are two notions of the approximation; strong and weak approximations. The strong approximation schemes provide pathwise approximations which can be employed in scenario analysis, filtering or hedge simulation. For applications such as derivative pricing, the computation of moments or expected utilities, the so-called weak approximation is sufficient, that is, we need to estimate the expected value of a function. Other applications of the weak approximation include the computation of functional integrals, invariant measures, and Lyapunov exponents.

The theoretical properties of time discretization schemes are mostly studied for the diffusion case. See [8], [9], [10], [12], [13] for detailed investigation. In fact, the weak approximation of the Lévy-driven stochastic differential equations via Monte Carlo type methods is still very difficult. Moreover, the other existing methods are applicable only to some of the simplest Lévy processes. The main purpose of this paper is to propose a new approach to weak approximation of Lévy-driven stochastic differential equations. Unlike Monte Carlo simulation with the time discretization approximation of sample paths, we employ a mathematical programming framework to obtain numerically upper and lower bounds of the target expectation, where the optimization procedure ends up with a polynomial programming.

To this end, we follow the methodologies1 that have been proposed and investigated in various fields of application by several authors, for example, Bertsimas, Popescu and Sethuraman [4], Helmes, Röhl and Stockbridge [6], Lasserre, Prieto-Rumeau and Zervos [11], to mention just a few. Note that these results deal only with the pure diffusion case (i.e., without jump component) for which standard Monte Carlo methods are sufficient. In this sense, it should be emphasized that our result is not a trivial extension. The main drawback is the complexity of the Ito formula for Lévy-driven stochastic differential equations. As such, we need to carefully examine whether or not the resulting optimization problems are practically solvable. Fortunately, as we show in the following sections, our approach covers various practically important Lévy-driven stochastic differential equations.

The rest of this paper is organized as follows. Section II gives mathematical definition of Lévy-driven stochastic differential equations. Section III introduces and studies our optimization approach to the weak approximation. Section IV provides a numerical example to illustrate that our method is able to efficiently capture the marginal distributions of Lévy-driven stochastic differential equations. Section V discusses a way to improve the approximation accuracy. Finally, Section VI concludes this paper.

II. PRELIMINARIES

Let us begin this section with general notations which will be used throughout the text. For \( k \in \mathbb{N} \), \( \partial_k \) indicates the

1It is known that there exist two dual formulation of this framework, both of which arrive at a semi-definite programming in the end. One is the so-called generalized moment problem that makes use of the semi-definiteness of (localizing) moment matrices. The other is a polynomial optimization approach for which sum-of-squares relaxation efficiently works. In this paper, our discussion is based on the latter formulation.
Moreover, throughout this study, we assume that the representation of the Poisson jump component.

\[ dX_t = a_0(t, X_t) dt + a_1(t, X_t) dW_t + \int_{\mathbb{R}_0} b(t, X_{t-}; \mu - \nu)(d\zeta, dt), \quad t \in [0, T], \]

where \( \{W_t: t \geq 0\} \) is a standard Brownian motion and \( \mu \) is a Poisson random measure on \( \mathbb{R}_0 \) whose compensator is given by the Lévy measure \( \nu \) satisfying \( \int_{|z|>1} |z| \nu(dz) < +\infty \) and \( \int_{\mathbb{R}_0} |z|^2 \nu(dz) < +\infty \). In order for the solution of (1) to be well defined, we impose the usual Lipschitz conditions and linear growth conditions on \( a_0, a_1 \) and \( b \).

We henceforth equip our underlying probability space with the natural filtration \( \mathcal{F}_t \) generated by \( \{X_t: t \in [0, T]\} \).

Moreover, throughout this study, we assume that \( b(t, x, z) \neq 0 \) and \( \nu \neq 0 \) to avoid triviality.

Let \( \tau \) be an \( (\mathcal{F}_t)_{t \in [0, T]} \)-stopping time taking its value in \([0, T]\). Our interest throughout this study is in approximating the expectation

\[ \mathbb{E}[V(\tau, X_{\tau})]. \]

Here, \( V \) is a function mapping from \([0, T] \times \mathbb{R} \) to \( \mathbb{R} \), piecewise polynomial in \( t \) and \( x \) and such that \( \mathbb{E}[V(\tau, X_{\tau})] < +\infty \). Note that the function \( V \) may have discontinuities. For the computation of \( \mathbb{E}[V(\tau, X_{\tau})] \), standard techniques include the Monte Carlo simulation of sample paths through the time discretization of stochastic differential equations, or even some exact knowledge of sample paths such as series representation of the Poisson jump component.

### III. Optimization Approach to Weak Approximation

#### A. Itô Formula and Supermartingale

Throughout this paper, we write

\[
E_0 := \text{supp} \{\tau, X_\tau\}, \\
E_1 := \text{sup} \{ (t, x_t): t \in [0, \tau) \}, \\
E_2 := E_0 \cup E_1.
\]

We are now in a position to introduce our optimization approach to the weak approximation. Let \( \mathcal{X} \subseteq \mathbb{R} \) be a support of \( \{X_t: t \in [0, T]\} \) defined in (1). For \( f \in C^{1,2}([0, T] \times \mathcal{X}; \mathbb{R}) \), the Itô formula yields

\[
df(t, x) = \mathcal{A}f(t, x) dt + \mathcal{B}f(t, x) a_1(x) dW_t + \int_{\mathbb{R}_0} \mathcal{C}f(t, x)(\mu - \nu)(d\zeta, dt), \quad a.s.,
\]

where

\[
\mathcal{A}f(t, x) := \partial_t f(t, x) + \partial_x f(t, x) a_0(t, x) \\
+ \frac{1}{2} \partial_{xx}^2 f(t, x) a_1(t, x)^2 \\
+ \int_{\mathbb{R}_0} (B_f(t, x) - \partial_x f(t, x) b(x, z)) \nu(dz),
\]

and for \( z \in \mathbb{R}_0 \),

\[
B_f(t, x) := f(t, x + b(x, z)) - f(t, x).
\]

One of important building blocks of our approach is the so-called Dynkin formula:

\[ \mathbb{E}[f(\tau, X_{\tau})] - \mathbb{E}[f(0, X_0)] = \mathbb{E}\left[ \int_0^\tau \mathcal{A}f(x, \xi) ds \right]. \]

See [8] for the conditions under which the formula makes sense.

Hence, as soon as one finds an \( f \in C^{1,2}([0, T] \times \mathcal{X}; \mathbb{R}) \) such that

\[
\begin{align*}
\mathcal{A}f(t, x) &\leq 0, \quad \text{on } E_0, \\
\mathcal{A}f(t, x) &\geq V(t, x), \quad \text{on } E_1,
\end{align*}
\]

it follows

\[ \mathbb{E}[V(\tau, X_{\tau})] \leq \mathbb{E}[f(\tau, X_{\tau})] \leq \mathbb{E}[f(0, X_0)]. \]

Clearly, \( f(0, X_0) \) serves as an upper bound of \( \mathbb{E}[V(\tau, X_{\tau})] \).

To minimize the upper bound \( f(0, X_0) \), we now turn to the optimization problem

\[
\min_{f \in C^{1,2}([0, T] \times \mathcal{X}; \mathbb{R})} f(0, X_0)
\]

s.t.

\[
\begin{align*}
f(t, x) &\geq V(t, x), \quad \text{on } E_1, \\
\mathcal{A}f(t, x) &\leq 0, \quad \text{on } E_0, \\
f &\in C^{1,2}([0, T] \times \mathcal{X}; \mathbb{R}).
\end{align*}
\]

#### B. Main Result

This optimization problem is very difficult to deal with since the class definitions of the functions \( f \) and \( V \) are too broad. To ease the above optimization problem, we restrict the class of the function \( f \) to be a polynomial both in \( t \) and \( x \), that is, in the form

\[
f(t, x) = \sum_{0 \leq k_1 + k_2 \leq K_2} c_{k_1, k_2} t^{k_1} x^{k_2},
\]

for some natural numbers \( K_1 \) and \( K_2 \) and for a sequence \( \{c_{k_1, k_2}\}_{k_1 \leq K_1, k_2 \leq K_2} \) of constants. For convenience in notation, we henceforth denote by \( C_p \) the class of polynomial functions in the form (6). We also need to set \( V \) to be a piecewise polynomial both in \( t \) and \( x \). Moreover, we assume that both \( a_0 \) and \( a_1 \) are polynomials. We are then instead to solve the following optimization problem

\[
\min_{f \in C_p} f(0, X_0)
\]

s.t.

\[
\begin{align*}
f(t, x) &\geq V(t, x), \quad \text{on } E_1, \\
\mathcal{A}f(t, x) &\leq 0, \quad \text{on } E_0, \\
f &\in C_p.
\end{align*}
\]

For the purpose of comparison, suppose that there is no jump in (1), that is, \( b \equiv 0 \) as in conventional results. This assumption clearly makes \( \mathcal{A}f \) a polynomial, and consequently (7) is a polynomial optimization problem. This is the main reason that the pure diffusion case is easier to deal with in this framework. In general, polynomial optimization problems are still NP hard. However, if the degrees of \( f \) are fixed, sum-of-squares relaxation enables us to solve the problem efficiently. For details, we refer to Parrilo [14]. On the other hand, this technique is not directly applicable to the
model with general stochastic jumps. This is because \( \mathcal{A} f \) is not necessarily a polynomial due to the additional integral term.

To circumvent this difficulty, we decompose the function \( b \) as follows:

**Assumption 1:** Functions \( a_0 \) and \( a_1 \) are polynomials, and \( b \) is decomposed as

\[
b(t, x, z) = b_1(t, x)b_2(z),
\]

where \( b_1 \) is a polynomial in \( x \) and where \( b_2 : \mathbb{R}_0 \rightarrow \mathbb{R} \) such that

\[
\int_{\mathbb{R}_0} |b_2(z)|^2 v(dz) < +\infty, \quad k = 2, \ldots, K_2.
\]

**Theorem 1:** Under Assumption 1, for any \( f \in C_p \), \( \mathcal{A} f \) is a polynomial in \( t \) and \( x \). Moreover, the coefficients of \( \mathcal{A} f \) are affine with respect to those of \( f \).

**Proof:** A simple algebra yields

\[
\mathcal{A} f(t, x) = \partial_t f(t, x) + \partial_x f(t, x)a_0(t, x)
\]

\[
+ \frac{1}{2} \partial_x^2 f(t, x)a_1(t, x)^2 + \sum_{0 \leq k_1 \leq k_2 \leq K_2} c_{k_1 k_2} \mathcal{P}^k \times
\]

\[
\sum_{k=0}^{k-2} q_{k_2} \mathcal{P}^k b_1(t, x)^{k-2-k} M_{k_2-k}
\]

where

\[
M_l := \int_{\mathbb{R}_0} |b_2(z)|^2 v(dz), \quad l = 2, \ldots, K_2.
\]

This completes the proof. \( \square \)

Clearly, the optimization (7) is now a polynomial programming problem. To be more precise, this problem is numerically tractable for any piecewise polynomial \( V \). Finally, to obtain a lower bound for \( \mathbb{E}[V(\tau, X_\tau)] \), we are to find a \( g \in C_p \) via the polynomial programming

\[
\max_{g(0, X_0)} \quad \text{s.t.} \quad g(t, x) \leq V(t, x), \quad \text{on } E_1,
\]

\[
\mathcal{A} g(t, x) \geq 0, \quad \text{on } E_0,
\]

\[
g \in C_p.
\]

Notice that our optimization approach does not require the sample paths simulation at all for the computation of the expectation \( \mathbb{E}[V(\tau, X_\tau)] \). It is a great advantage of our approach that all we need is the Lévy measure in closed form, not the exact knowledge of the increments or of a shot noise representation for sample paths simulation for the weak approximation with the sample paths discretization.

**C. Simultaneous approximation for homogeneous process**

In this section, we show that the optimal solution obtained through our approach provides some additional information, that are of direct practical use.

Firstly, note that the initial value \( X_0 \) does not appear in the constraints (4) in the previous section. Therefore, if \( f \) satisfies (4), \( f(0, x) \) automatically gives upper bounds for \( \mathbb{E}_x[V(\tau, X_\tau)] \), where the notation \( \mathbb{E}_x \) denotes the expectation taken under which the initial state of the stochastic differential equation (1) is given deterministically by \( X_0 = x \).

We next consider, for each \( \bar{T} \in [0, T] \),

\[
\tau_\bar{T} := \tau \land \bar{T},
\]

that is, the stopping time condensed into the shorter interval \([0, \bar{T}]\). Clearly, \( \tau_\bar{T} = \tau \). The next theorem indicates that functions satisfying (4) can also serve as bounds at arbitrary intermediate time points. We denote

\[
E_0(\bar{T}) := \{ \tau_\bar{T} \in [0, \bar{T}] \times \mathbb{R} \}
\]

\[
E_1(\bar{T}) := \{ \tau_\bar{T} \in [0, \bar{T}] \}
\]

**Theorem 2:** Let \( \bar{T} \in [0, T] \) and assume that (1) is time-homogeneous, i.e., \( a_0, a_1, a_2 \), and \( b \) are independent of \( t \). Suppose that \( f \in C^{1,2} \) satisfies (4). Suppose also that, for every \( (t, x) \in E_0(\bar{T}) \),

\[
(t + (T - \bar{T}), x) \in E_0
\]

and

\[
V(t, x) = V(t + (T - \bar{T}), x)
\]

(10) hold. Then,

\[
\mathbb{E}[V(\tau_\bar{T}, X_{\tau_\bar{T}})] \leq f(T - \bar{T}, X_0).
\]

**Proof:** Define

\[
f^\omega(t, x) := f(t + (T - \bar{T}), x).
\]

Due to the time homogeneity, we have

\[
\mathcal{A} f^\omega(t, x) = \mathcal{A} f(t + (T - \bar{T}), x) \leq 0, \quad \text{in } E_1(\bar{T})
\]

where the last inequality follows from \( E_1(\bar{T}) \subset E_1 \) and (4). We also have

\[
f^\omega(t, x) \leq V(t + (T - \bar{T}), x) = V(t, x), \quad \text{on } E_0(\bar{T}),
\]

where the first inequality follows from (9) and (4). By combining these inequalities and Dynkin formula, we obtain

\[
\mathbb{E}[V(\tau_\bar{T}, X_{\tau_\bar{T}})] \leq \mathbb{E}[f^\omega(\tau_\bar{T}, X_{\tau_\bar{T}})]
\]

\[
= f^\omega(0, X_0) + \mathbb{E} \left[ \int_{0}^{\tau_\bar{T}} \mathcal{A} f^\omega(s, X_s)ds \right]
\]

\[
\leq f(T - \bar{T}, x).
\]

This completes the proof. \( \square \)

Roughly speaking, (9) and (10) represent the time-invariance of the stopping strategy of \( \tau \) and the value function \( V \), respectively. In the 2 examples in the next section, all assumptions in this theorem are satisfied. Assumptions (9) and (10) can be easily relaxed to some extent. However, in that case, the resulting approximation is expected to be too conservative.

We here make a brief comment on the choice of the cost function in the optimization problem. When we attempt to as tight bounds for (2) as possible, we should solve (7) and (8). However, when we need to approximate \( V(\tau_\bar{T}, X_{\tau_\bar{T}}) \) for some different time points \( \bar{T} \in [0, T] \) and also different initial value \( X_0 \), it is useful to suitably change the cost function.
Fortunately, for suitable measure \( \phi \) on \([0,T] \times \mathbb{R} \), we can similarly optimize
\[
\int f(t,s)\phi(dt,ds), \quad \int g(t,s)\phi(dt,ds),
\]
since these are linear combination of the decision variables (the coefficients of \( f \) and \( g \)).

IV. NUMERICAL EXAMPLES

In this section we give some approximation examples. In the numerical examples presented hereafter, we utilize MATLAB SOSTOOLS combined with SeDuMi [15], [19], using a computer with a Pentium 4 3.2GHz processor and 2 GB memory. In all cases, it took at most 1 second to obtain a bound.

A. Moment estimation of truncated stable subordinator

Set \( X_0 = 0, a_1(t,x) = 0, b_1(t,x) = 1, b_2(z) = z \), and
\[
v(z) = \frac{1}{z^{1+\alpha}}1(z \leq \eta)dz, \quad z \in \mathbb{R}^+,
\]
for some \( \alpha \in (0,1) \) and \( \eta > 0 \), and \( a_0(t,x) = \int_{\mathbb{R}^+} v(z)dz \). Then, the stochastic differential equation (1) reduces to
\[
X_t = \int_0^t \int_{\mathbb{R}^+} z\mu(dz,ds),
\]
that is, an one-sided truncated stable Lévy process. To the best of our knowledge, a truncation of the series representation of the (non-truncated) stable process is the only method for the sample paths generation, which is extremely expensive.

Here, in view of Houdré and Kawai [7] and Rosiński [17], we test our method on the short- and long-range behavior of truncated stable processes, by looking at the convergence of the first and the second moments, it is sufficient to first compute moments of \( X_{ht} \), and to multiply next a scaling related to \( h \). Note that \( \mathcal{L} = \mathbb{R}_+ \) and \( \int_{\mathbb{R}^+} z^\alpha v(dz) = \eta^{1-\alpha}/(k - \alpha) \). We have for \( f \in C_p([0,T] \times \mathbb{R}; \mathbb{R}) \),
\[
\mathcal{A}f(\xi, x) = \sum_{1 \leq k_1 \leq k_2 \leq K_2} c_{k_1} c_{k_2} (\xi h_{k_1-1}^{1-x_2})
+ \sum_{0 \leq k_1 \leq k_2 \leq K_2} c_{k_1} c_{k_2} (\xi h_{k_2-1}^{1-x_2})
\]
On the other hand, in the long run, as \( h \uparrow +\infty \),
\[
\left\{ h^{-1/2}(X_{ht} - \mathbb{E}[X_{ht}]) : t \geq 0 \right\} \subseteq \left\{ W_t : t \geq 0 \right\},
\]
where \( [W_t : t \geq 0] \) is a centered Brownian motion with \( \mathbb{E}[W_t]^2 = \eta^{2-\alpha}/(2 - \alpha) \). In this case, \( \mathcal{L} = \mathbb{R} \), \( a_0(t,x) = 0 \), and we take the form \( f(X_{ht}, X_{ht} - \mathbb{E}[X_{ht}]) \). Hence, we have
\[
\mathcal{A}f(\xi, x) = \sum_{1 \leq k_1 \leq k_2 \leq K_2} c_{k_1} c_{k_2} (\xi h_{k_1-1}^{1-x_2})
+ \sum_{0 \leq k_1 \leq k_2 \leq K_2} c_{k_1} c_{k_2} (\xi h_{k_2-1}^{1-x_2})
\]
We present numerical results in Table I and in Table II. Similarly to the previous example, for the estimation of the \( p \)-th moment, we set \( K_1 = K_2 = p \).

B. Barrier option pricing

In this section, we test numerically our method on a Doléans-Dade stochastic exponential without the diffusion component, that is, \( a_1(t,x) \equiv 0 \). Set \( X_0 > 0 \), \( a_0(t,x) = a_1(t,x) = 0, b_1(t,x) = x, b_2(z) = z \), and
\[
v(z) = a^{1-\alpha}/\zeta dz, \quad z > 0,
\]
for \( a > 0 \) and \( b > 0 \), that is a gamma Lévy measure. In this setting, (1) reduces to a Doléans-Dade stochastic exponential driven by gamma process
\[
dx_t = X_{t-} \int_{\mathbb{R}^+} (\mu - v)(dz, dt), \quad X_0 > 0,
\]
which is a martingale with respect to its natural filtration. We take the stopping time
\[
\tau := \inf\{ t \geq 0 : X_t \notin [0, B_u] \} \land T,
\]
with positive constant \( B_u \). Then, defining
\[
V(t,x) := \left\{ \begin{array}{ll}
(x-K)_+, & \text{if } x < B_u \\
0, & \text{otherwise}
\end{array} \right.
\]
makes (2) the European call option premium with barrier \( B_u \) and strike price \( K \) at maturity \( T \). The definition of \( \tau \) and the unboundedness of the jump size gives
\[
E_0 = [0,T] \times [0,B_u], \quad E_1 = [0,T] \times [B_u, \infty) \cup \{ T \} \times [0,B_u].
\]
Let us take \( a = 0.1, b = 1.5, X_0 = 1.0, B_u = 2.0, T = 1.0 \) and \( K = 0.8 \). The obtained lower and upper bounds for the barrier options are
\[
g^*(0,X_0) = 0.16534, \quad f^*(0,X_0) = 0.17156,
\]
where $f^*$ and $g^*$ are optimal functions in (7) and (8) within the degree constraints $c_{k_1, k_2} = 0$ for $k_1 + k_2 > 10$.

Recall that the current model is time-homogeneous. Hence, these bounding functions also give upper and lower bounds for intermediate time points without solving other optimization problem. To put it precisely, for every maturity $T' \in [0, T)$, $f^*(T - T')$ and $g^*(T - T')$ give the upper and lower bounds for the barrier option price; see solid lines in Fig. 1. In this sense, we can provide a parametric bounding function with respect to maturities.

Though we need to compromise the tightness of the bound at time $T$, we can take the intermediate approximation error into account. To see this, we adopt the following optimization criteria:

$$f(0, X_0) + f(0.8, X_0), \quad g(0, X_0) + g(0.8, X_0),$$

where the latter terms represent the gap at time $\bar{T} = 0.2$.

The dashed line shows the resulting bounds. We can see this optimization can provide tighter bounds for intermediate time points with minor increase of the gap at the final time $T$:

$$g^*(0, X_0) = 0.16534, \quad f^*(0, X_0) = 0.17238.$$

**Remark 1:** In this example, we considered the constant barrier. If we need to deal with time-varying barrier such as partial barrier, assumption (9) does not hold, and consequently such an approximation at intermediate time points is not available straightforwardly.

**V. DISCUSSION**

When $V$ is a smooth function as in the moment estimation in Section IV, fairly tight bounds are obtained in most of numerical experiments which we have performed. On the other hand, the tightness of upper and lower bounds is lost when $V$ is not smooth and when $V$ is a quickly decaying function of $x$.

We conjecture that the gap between upper and lower bounds stems mainly from the polynomial constraints, that

$$f(0, X_0) \geq E[f(T, X_T)].$$

This means that $f(0, X_0)$ cannot give a good approximation if $f(T, x) - V(T, x)$ is too large (more precisely, in the sense of expectation). In this sense, $f(T, x)$ needs to approximate $V(T, x)$ to a high accuracy. However, even if we are allowed to take sufficiently large degrees, it is still difficult to approximate non-smooth function by polynomials in unbounded regions. This can limit the effect of the use of higher order
polynomials, especially in applications such as probability tail estimation.

A. Piecewise polynomials

Recall that \( V(t,x) \) is assumed to be a piecewise polynomial. We conjecture that the issue mentioned above may be overcome by relaxing the functions \( f,g \in C_p \) also to a piecewise polynomial. The main drawback is the non-locality of the infinitesimal generator. To see this, let us assume \( \mathcal{X} := \mathbb{R}_+ \) and

\[
 f(t,x) = \begin{cases} 
 f_1(t,x), & x \leq \alpha \\
 f_2(t,x), & x > \alpha 
\end{cases}
\]

where \( \alpha > 0 \) is a constant and where \( f_i \)'s are polynomials that make \( f \in C^{1,2} \). If \( b = 0 \) (no stochastic jumps) in (1) the infinitesimal generator of \( f \) is determined only by its local property:

\[
\mathcal{A} f(t,x) = \partial_t f_t(t,x) + \partial_x f_1(t,x)a_0(t,x) + \frac{1}{2} \partial_x^2 f_1(t,x)a_1(t,x)^2
\]

for \( (t,x) \in [0,T] \times \mathcal{X}_1 \). This implies that we can search piecewise polynomials in pure diffusion model as in [16]. On the other hand, when \( b = z \) as in the gamma process, we have

\[
B_z f(t,x) = \begin{cases} 
 f_1(t,x+z) - f_1(t,x), & 0 < z \leq \alpha - x \\
 f_2(t,x+z) - f_2(t,x), & z > \alpha - x 
\end{cases}
\]

for \( x \in \mathcal{X}_1 \). As a result, additional terms appear in

\[
\mathcal{A} f(t,x) = \partial_t f_t(t,x) + \partial_x f_1(t,x)a_0(t,x) + \frac{1}{2} \partial_x^2 f_1(t,x)a_1(t,x)^2
\]

\[
\int_0^\alpha \left( f_1(t,z) - f_1(t,x) - \partial_x f_1(t,x)z \right) v(dz)
\]

\[
\int_{\alpha}^\infty \left( f_2(t,z) - f_1(t,x) - \partial_x f_2(t,x)z \right) v(dz).
\]

Unfortunately, it is known that these integrals have no analytical expression. We are currently constructing an approximation procedure that makes the proposed method applicable.

VI. CONCLUSION

In this paper, we have developed a new approach to the weak approximation of Lévy-driven stochastic differential equations via an optimization problem yielding upper and lower bounds on the target expectation. We need neither the exact knowledge of the increments nor a shot noise representation for sample path simulation for the weak approximation with the sample path discretization. We have also investigated how we can obtain accurate approximation at transient times.

The most important remaining work is the improvement of the approximation accuracy. In addition to the piecewise polynomial approach stated in Section V, it is a good direction to pursue to use exponentially tempered polynomials [8]. Other currently ongoing work is

- application to calibration in finance,
- construction of finite-horizon control theory for Lévy driven stochastic systems.

REFERENCES