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Approximation on the Complex Sphere

by

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Abstract

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The aim of this thesis is to study approximation of multivariate functions on the complex sphere by spherical harmonic polynomials. Spherical harmonics arise naturally in many theoretical and practical applications. We consider different aspects of the approximation by spherical harmonic which play an important role in a wide range of topics. We study approximation on the spheres by spherical polynomials from the geometric point of view. In particular, we study and develop a generating function of Jacobi polynomials and its special cases which are of geometric nature and give a new representation for the left hand side of a well-known formulae for generating functions for Jacobi polynomials (of integer indices) in terms of associated Legendre functions. This representation arises as a consequence of the interpretation of projective spaces as quotient spaces of complex spheres. In addition, we develop new elements of harmonic analysis on the complex sphere, and use these to establish Jackson's and Kolmogorov's inequalities. We apply these results to get order sharp estimates for m-term approximation. The results obtained are a synthesis of new results on classical orthogonal polynomials geometric properties of Euclidean spaces. As another aspect of approximation, we consider interpolation by radial basis functions. In particular, we study interpolation on the spheres and its error estimate. We show that the improved error of convergence in n dimensional real sphere, given in [7], remain true in the case of the complex sphere.

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Lastly, I would like to dedicate this thesis to my family for their love, patience and encouragement. And most of all I would like to give a heartfelt, special thanks to my husband Bader who has faithfully supported me during this journey, thank you.

Contents

Symbols

$L_p(X)$	The space of measurable function f for which f^p is integrable.
$\|T\|_{X\longrightarrow Y}$	Norm of the operator $T : X \longrightarrow Y$.
T^*	The adjoint of the operatoe T.
$\mathbb{S}^{d-1}(\mathbb{R})$	Unit sphere in $\mathbb{R}^d$.
$\mathbb{S}^{d-1}(\mathbb{C})$	Unit sphere in $\mathbb{C}^d$.
B_d	Unit ball in $\mathbb{R}^d$.
Γ	Gamma function.
$\mathrm{Vol}_{\,d}(A)$	Volume of $A \subset \mathbb{R}^d$.
$f * g$	Convolution of f and g.
$P_n^{(\alpha,\beta)}$	Jacobi polynomials of degree n.
C_n^λ	Gegenbauer polynomials of degree n.
P_n	Legendre polynomials of degree n.
T_n	Chebyshev polynomials of degree n.
P_ν^μ	Associated Legendre function of the first kind.
Δ	Laplace operator.
H_m	Space of special harmonic polynomials of degree m in $\mathbb{R}^d$.
$H(m,n)$	Space of special harmonic polynomials of degree (m,n) in $\mathbb{C}^d$.
d_m	The dimension of the space H_m.
$d_{m,n}$	The dimension of the space $H(m,n)$.
$\mathrm{hom}(m)$	The set of all homogenous polynomials of degree m in $\mathbb{R}^d$.
$\mathrm{hom}(m,n)$	The set of all homogenous polynomials of degree (m,n) in $\mathbb{C}^d$.
b_m	The dimension of the space $\mathrm{hom}(m)$.
$b_{m,n}$	The dimension of the space $\mathrm{hom}(m,n)$.
Z_x^k	Zonal harmonic of degree k with pole x.
$F(a,b,c,x)$	Hypergeometric function.
$\mathcal{T}_n$	The space of polynomials of degree $\leqslant n$.
$\nu_m(\phi,\Xi,X)$	The best m-term approximation of $\phi \in X$ with regard to Ξ

$\nu_m(\mathcal{K}, \Xi, X)$	m-term approximation of a given set $\mathcal{K} \subset X$.
W_p^γ	Sobolev space.
C_n^δ	Cesàro number of order n and index δ.
K^o	Polar set of K.
N_φ	The native space for φ.
$h_{X,\Omega}$	The fill distance of X in the set Ω.
$R_{m,n}^\alpha$	The disk polynomial of degree (m, n).

To my family...

Introduction

The theory of orthogonal polynomials was first developed from a study of continued fractions in the 19th century. During the 20th century, many publications about this concept have been published to meet the mathematical needs of science. This evolution was made by many essential results and by mathematicians such as Gauss, Jacobi, Stieltjes and Markoff. Since then, orthogonal polynomial theory has played an important role in many applications in different areas of mathematics and physics, and it is still an active area of research. There are many branches of mathematics which have a connection with orthogonal polynomials. For instance, continued fractions, moment problems, approximation theory, interpolation, analytic functions and many other areas which in turn are a key tool for the analysis of many real phenomena in science and engineering. The most widely used families of orthogonal polynomials, which are related to many branches of analysis, include Jacobi, Laguerre and Hermite polynomials. They are considered as classical orthogonal polynomials and consequently, they are studied most extensively.

This thesis intends to provide a theoretical framework to the classical orthogonal polynomials on approximation on spheres with special attention given to Jacobi polynomials and their special cases, Gegenbauer and Legendre polynomials. They arise in the theory of orthogonal polynomials as complete orthogonal sets in Hilbert spaces associated with particular weight functions. However, the theory of spherical harmonic functions gives a different approach. From the point of view of spherical harmonic theory, these polynomials arise as eigenfunctions of the Laplace operator when it is represented in polar coordinates on the unit sphere in Euclidean spaces. These polynomials are orthogonal with respect to inner products defined by the integral taken over the surface of the unit sphere. This concept can be traced back to the 18th century when the mathematician Adrien-Marie Legendre investigated a power series for the Newtonian potential. The coefficients of the series are the eponymous Legendre polynomials. He worked on many areas, particularly in polynomials, number theory and elliptic functions, and his work was impressive for many years. Later, the Gegenbauer polynomials were introduced by the Austrian mathematician Leopold Gegenbauer (1849 - 1903). He was interested in many mathematical areas such as number theory, function theory and integration theory. Since then, spherical harmonics have become a very important concept in many fields of science and they have a widespread application. They are useful in physics, computer science, engineering and many other areas. They play an important role in atomic orbital, gravitational fields, magnetic fields, computer graphics and other different topics. The study of spherical harmonic functions in a complex vector space was developed by many theorists and analysts in the late 60's. Particularly, Koornwinder [29] studied functions on the unit sphere in a complex vector space with Hermitian inner product. He studied the finite dimension space $H(m,n)$ of spherical harmonic polynomials on the complex unit sphere $\mathbb{S}^{d-1}(\mathbb{C})$ which are homogeneous of degree (m,n). He also presented the explicit expression for the zonal functions in the spaces $H(m,n)$ in terms of Jacobi polynomials. Specifically, he shows that the zonal function in $H(m,n)$ is a multiple of

$$h_{m,n}(z) = e^{i(m-n)\phi}\,(\cos\theta)^{|m-n|}\,P_{m\wedge n}^{(d-2,|m-n|)}(\cos 2\theta)\,, \quad z \in \mathbb{S}^{d-1}(\mathbb{C}), \tag{1}$$

where $\langle z, e_1\rangle_{\mathbb{C}} = e^{i\phi}\cos\theta$, and $m \wedge n = \min(m,n)$. Here $P_i^{(\alpha,\beta)}$, $i \geq 0$, are the Jacobi polynomials which satisfy

$$\int_{-1}^{1}(1-s)^{\alpha}(1+s)^{\beta}P_i^{(\alpha,\beta)}(s)P_j^{(\alpha,\beta)}(s)ds \;=\; 0, \quad i \neq j.$$

On the other hand, Rudin [48] shows that the space H_{m+n} of homogeneous complex valued spherical harmonic polynomials on the unit sphere $\mathbb{S}^{d-1}(\mathbb{R})$ of degree $m+n$ is a direct sum of the pairwise orthogonal spaces $H(m,n)$.

This work concerns approximation theory on spheres. The objective of this thesis is to study different aspects of spherical approximation. The first part is devoted to study and development of a Jacobi generating function and its special cases. In fact, these generating functions have been proved in many different ways. Some mathematicians prove it as a potential function, and some others use algebraic properties of orthogonal polynomials to prove it (see for example MacRobert [41] and Szegö [55]). The well-known generating function for Gegenbauer polynomials is

$$\left(1+r^2-2r\cos\theta\right)^{-\lambda} \;=\; \sum_{n=0}^{\infty} r^n C_n^{(\lambda)}(\cos\theta), \quad 0<r<1, \tag{2}$$

where, for $\lambda > -1/2$, $C_n^{\lambda} = cP_n^{(\lambda-1/2,\lambda-1/2)}$, for a constant c and $n = 0, 1, \cdots$. The standard proof of (2) such as presented in [55] is analytic in nature, and offers little understanding why such a formula might be true. One of the goals in this thesis is to provide a more geometric proof using the harmonicity and symmetry of the potential

$$f(r,x) = \|x - rt\|^{-d+2}, \quad 0 < r < 1,$$

in $\mathbb{R}^d$, where $x, t \in \mathbb{S}^{d-1}(\mathbb{R})$ the unit sphere of $\mathbb{R}^d$. We interpret x as the fixed pole of $\mathbb{S}^{d-1}(\mathbb{R})$, and if we write $\cos\theta = \langle x, t\rangle_{\mathbb{R}}$ then

$$f(r,t) = \left(1+r^2-2r\cos\theta\right)^{-\lambda},$$

where $\lambda = (d-2)/2$. To generalise this geometric proof to the Jacobi generating function, we first present Jacobi polynomials (zonal functions for projective spaces) as spherical averages of Gegenbauer polynomials, the zonal kernel for the real sphere,

$$\int_{\mathbb{S}^{2j-1}(\mathbb{R})} C_{2n}^{\lambda}\left(\langle t\xi, \eta\rangle_{\mathbb{R}}\right) d\mu_{2j}(\eta) = \omega_{2j-2}\pi \frac{(\lambda)_n (1/2)_{j-1}}{(n+j-1)!} P_n^{(\lambda-j,j-1)}(2t^2-1)$$

where $\xi = (1, 0, \cdots, 0)$ belongs to the unit sphere $\mathbb{S}^{2j-1}(\mathbb{R})$ with respect to the Lebesgue measure μ_{2j} on $\mathbb{S}^{2j-1}(\mathbb{R})$.

In the case of the circle it was pointed out to us by Koornwinder [30] that we would obtain the generating function formula in Koekoek and Swarttouw [28, 1.8.13]. The key results of this part are a new way of proving the generating function formulae for Jacobi polynomials using this formula. This integral formula, was new to us and we arrived at it after studying the zonal kernel function in the complex projective space and the real space. Later, we found out that this result has been proved before by Dijksma and Koornwinder [15]. They proved an integral representation for the product $P_n^{(\alpha,\beta)}(1-2s^2)P_n^{(\alpha,\beta)}(1-2t^2)$ in term of Gegenbauer polynomials. However, this approach inspired a new presentation of this generating function formula which is of a geometric nature. This Jacobi generating function formula is given by:

$$(1+t)^{-\alpha-\beta-1}\,F\left(\frac{\alpha+\beta+1}{2},\ \frac{\alpha+\beta+2}{2},\ \beta+1,\ \frac{2t(x+1)}{(t+1)^2}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(\alpha+\beta+1)_n}{(\beta+1)_n}P_n^{(\alpha,\beta)}(x)t^n. \qquad (3)$$

Here F is the Gauss hypergeometric function.

We will show that the generating function side of (3), when $\alpha \in \mathbb{N},\ \beta = 0$, can be written as a Legendre polynomial (a Gegenbauer polynomial with $\lambda = 1/2$). We will extend this result to a more general case and prove that the generating function side of (3) can be written in general in term of associated Legendre functions of the first kind $P_v^{\mu}(z)$.

The best m-term approximation of ϕ in the Banach space X by the subspace $\Xi(\Omega_m) := \mathrm{lin}\{\xi_{k_l}\}_{l=1}^{m}$ where $\Omega_m := \{k_1 < \cdots < k_m\} \subset \mathbb{N}$, with regard to a dense subset $\Xi := \{\xi_k\}_{k\in\mathbb{N}}$ of X, is defined as

$$\nu_m(\phi, \Xi, X) := \inf_{\Omega_m \subset \mathbb{N}} \inf_{\xi \in \Xi(\Omega_m)} \|\phi - \xi\|_X.$$

This method is a nonlinear method of approximation. In m-term approximation the selection of elements in the approximation depends on the function that is being approximated and it depends on only the number of the element to be used and it does not recognise these frequency locations. This type of approximation has been introduced by Schmidt (1907), then studied by Oskolkov (1979) for multivariate splines. During recent years m-term approximations and n-widths have become very popular in numerical methods for PDEs. Also, the idea of so-called greedy algorithms has been inspired by m-term approximations (see e.g. [13] and [57]). One of the central problems of nonlinear approximation is to determine the approximation space, which is the space of all functions which have a specific approximation order (see e.g. [13] and [14]). Another challenging problem in this field is to estimate the rate of approximation for certain classes of functions and to determine the best m-term approximation. In the second part of this thesis, which includes joint work with Professor Alexander Kushpel, a new element of harmonic analysis on the complex sphere is developed. We establish Jackson's and Kolmogorov's inequalities and use these results to get error estimates for m-term approximation of functions belonging to Sobolev's classes.

The last method of approximation we consider in this work is radial basis interpolation on complex spheres. Recently, radial basis functions have been found to be a widely successful tool for multivariable interpolation of data. We intend to establish a theoretical foundation for interpolation by radial functions, and in particular, the error $f - s_{f,X}$ between a function f in the native space of a positive definite kernel and its interplant $s_{f,X}$ on a set $\Omega \subseteq \mathbb{R}^d$. Schaback [50], improved the well known approximation order by requiring more smoothness for the function f. He shows that for a suitable subspace of the native space the approximation order can be doubled. The authors in [7] adapted Schabacks error doubling trick to give error estimates for radial interpolation of functions which have less smoothness than Schaback requires. Their results are based on using a pseudo differential operator and estimate errors in pseudoderivatives of solutions to the interpolation problem. Considering the space $\mathbb{C}^d$ as a $2d$-dimensional real vector space, we will generalise the improved error for radial interpolation given in [7], which consider the real sphere, to the complex sphere. Thus the process given here is similar to a large extent to those in the real sphere case, with simple changes occuring due to using Bernstein's inequality in the complex sphere.

This work has been divided into six chapters;

- Chapter 1 briefly presents some background material which will be required in the next chapters. It does not contain any original work.
- Chapter 2 deals with orthogonal polynomials and classical orthogonal polynomials as a special case. The third section concentrates on these polynomials from the theory of a linear operator approach. In the last section we present a proof of Gegenbauer's generating function using Maclaurin series. The point of this proof is just to show how to prove it in analytic way.
- The first three sections of Chapter 3 are devoted to studing the theory of spherical harmonic functions with special emphasis on Gegenbauer polynomials. However, our purpose is not only to give a general description of spherical harmonics, but also to set the essential results which are the basic tool in our discussion in the next chapters. In Section 4, we present a geometric proof of Gegenbauer generating functions using their harmonicity and symmetry properties.
- Chapter 4 contains a study of the zonal spherical harmonics in a complex vector space which is given in the first section. Section 2 shows an integral

formula of the zonal spherical harmonics which will be used to give a new proof of a well known Jacobi generating function in section 3. In the last section a new presentation of the Jacobi generating function in terms of associated Legendre functions will be given.

- In Chapter 5 we turn our attention to m-term approximation. A general overview will be given in the first section and the error estimate will be proved in the second section.

- The last chapter looks at applications of the earlier developed theory. The first three sections give a short overview and survey about interpolation by radial basis functions. In Section 4 we show the improved error bound of radial interpolation on the complex sphere.

Chapter 1

Background

This introductory chapter reviews some material on inner product and Hilbert spaces and then discusses briefly some results in the concept of differential and integral calculus. The last section deals with some famous polynomials inequalities which have an important role in our work in Chapter 5. The inner product induces a norm which defines a metric space that could be a Hilbert space. Linear operators, and particularly self-adjoint operators, play an important role in the spherical harmonic concept which have been treated in the third chapter. The inner product can be used to introduce the concept of orthogonality of vectors, which is relevant to orthogonal polynomials. We are concerned partly with the classical orthogonal polynomials. These polynomials satisfy many properties with respect to L_2 norms. In what follows, general definitions and basic properties will be discussed and the orthogonal systems will take most of our attention. However, we limit the selection to the material which is needed later and for more details we refer the reader to [2], [18], [19], [20], [10], [11], [21], [27], [31], [53] and [60].

1.1 Hilbert Spaces and Linear Operators

1.1.1 Hilbert Spaces- Definition and Examples

An inner product on a complex vector space V is a function $\langle \cdot, \cdot \rangle$ from $V \times V$ to $\mathbb{C}$ that satisfies the following axioms for all v, u and w in V and all scalars $k \in \mathbb{C}$:

1. $\langle u, v\rangle = \overline{\langle v, u\rangle}$;

2. $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$;

3. $\langle ku, v\rangle = k\langle u, v\rangle$;

4. $\langle v, v\rangle \geqslant 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$.

From the definition we can deduce that $\langle u, u\rangle = \overline{\langle u, u\rangle}$, which means that $\langle u, u\rangle$ is a real number for all u in V. Moreover, it follows from (1) and (3) that

$$\langle u, kv\rangle = \overline{\langle kv, u\rangle} = \overline{k}\overline{\langle v, u\rangle} = \overline{k}\langle u, v\rangle .$$

A vector space V with an inner product is called an *inner product space*, and the norm is defined as $\|\cdot\| = \langle \cdot, \cdot\rangle^{\frac{1}{2}}$.

Example 1.1.

1. *The space* $\mathbb{C}^d$ *of ordered d-tuples* $x = (x_1, \cdots, x_d)$ *of complex numbers, with the inner product defined by*

$$\langle x, y\rangle_{\mathbb{C}} = \sum_{k=1}^{d} x_k \bar{y}_k,$$

is an inner product space.

2. *The space* ℓ^2 *of all infinite sequences of complex numbers* $x = (x_1, x_2, \cdots)$ *such that* $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ *with the inner product*

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_n \bar{y}_n$$

is an inner product space.

3. *Let* w *be a nonnegative integrable function in Lebesgue's sense on* $[a, b]$. *Let* $L_2([a, b], w)$ *be the space of equivalence classes of measurable functions for which*

$$\left(\int_a^b |f(x)|^2 w(x)dx\right)^{\frac{1}{2}} < \infty.$$

Then $L_2([a, b], w)$ *is a vector space with the inner product*

$$\langle f, g\rangle_{L_2([a,b],w)} = \int_a^b f(x)\overline{g(x)}w(x)dx, \tag{1.1}$$

and then the norm of a function f is given by

$$\|f\|_{L_2([a,b],w)} = \langle f, f\rangle^{\frac{1}{2}}_{L_2([a,b],w)} = \left(\int_b^a |f(x)|^2 \, w(x)dx\right)^{\frac{1}{2}}.$$

Definition 1.1. A normed space is a vector space equipped with a norm $\|\cdot\| : V \to \mathbb{R}$ that satisfies the properties:

1. $\|x\| \geqslant 0$ for all $x \in V$, and $\|x\| = 0$ if and only if $x = 0$;
2. $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{R}$ and all $x \in V$;
3. $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in V$.

Definition 1.2. A metric on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties:

1. $d(x, y) \geqslant 0$ and $d(x, x) = 0$ for all $x, y \in X$;
2. $d(x, y) = 0$ implies $x = y$;
3. $d(x, y) = d(y, x)$ for all $x, y \in X$;
4. $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If d is a metric on a set X, then the pair (X, d) is called metric space.

The following theorem gives the most important properties of inner product spaces.

Theorem 1.1. *If $(V, \langle\cdot,\cdot\rangle)$ is an inner product space equipped with the norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$ and $\lambda \in \mathbb{C}$, then for all f, $g \in V$,*

1. $\|f\| \geq 0$, $\|f\| = 0$ *if and only if* $f = 0$;
2. $\|\lambda f\| = |\lambda| \, \|f\|$;
3. $|\langle f, g\rangle| \leq \|f\| \, \|g\|$, *Cauchy-Schwarz Inequality;*
4. $\|f + g\| \leq \|f\| + \|g\|$, *Triangle Inequality;*
5. $\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2$, *Parallelogram Law;*

6. *The function* $d : V \times V \to \mathbb{R}$ *defined by*

$$d(f,g) = \|f-g\|,$$

is a metric.

Definition 1.3. Let (X,d) be a metric space.

1. A sequence (x_n) in X is a Cauchy sequence if, for every $\epsilon > 0$, there exists an integer k_0 such that $k, l \geq k_0$ implies that $d(x_k, x_l) < \epsilon$.

2. (X,d) is a complete metric space if every Cauchy sequence in X converges to a limit in X.

Definition 1.4.
1. A Hilbert space is an inner product space which is a complete metric space with respect to the metric induced by its inner product.

2. A Banach space is a normed space that is also a complete metric space under the metric induced by its norm.

One of the most important example of Hilbert spaces is the space of Lebesgue square integrable functions which is described in the following example (see e.g. [25]).

Example 1.2.

1. *The space* $L_2([a,b],w)$ *with the inner product*

$$\langle f,g\rangle_{L_2([a,b],w)} = \int_a^b f(t)\overline{g(t)}w(t)dt,$$

is a Hilbert space.

2. *On* $\mathbb{C}^d$*, if* $x = (x_1, \cdots, x_d)$ *and* $y = (y_1, \cdots, y_d)$ *belong to* $\mathbb{C}^d$*, then*

$$\langle x,y\rangle_{\mathbb{C}} = \sum_{k=1}^{d} x_k \bar{y}_k, \qquad \|x\| = \left[\sum_{k=1}^{d} |x_k|^2\right]^{\frac{1}{2}},$$

is an inner product and $\left(\mathbb{C}^d, \langle\cdot,\cdot\rangle_{\mathbb{C}}\right)$ *is a Hilbert space.*

An example of an inner product space which is not a Hilbert space is the space $C[0,1]$ of all continuous functions on the unit interval with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt.$$

To show this consider the following sequence of functions in $C[0,1]$:

$$f_n(x) = \begin{cases} 1, & 0 \leq x \leq 1/2, \\ 1 - 2n(x - 1/2), & 1/2 \leq x \leq 1/(2n) + 1/2, \\ 0, & 1/(2n) + 1/2 \leq x \leq 1. \end{cases}$$

Then $\|f_n - f_m\| \leq (1/n + 1/m)^{1/2} \to 0$ as $m, n \to \infty$. Thus $\{f_n\}$ is a Cauchy sequence. It is easy to check that the sequence has the limit

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1/2, \\ 0, & 1/2 \leq x \leq 1, \end{cases}$$

which is not continuous (see e.g. [25]).

Definition 1.5. Let $1 \leqslant p < \infty$. Then the ℓ^p-norm of a sequence $x = (x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ is defined by

$$\|x\|_p = \left\{\sum_{n=1}^{\infty} |x_n|^p\right\}^{\frac{1}{p}}.$$

The $\|\cdot\|_\infty$-norm of $x = (x_n)_{n=1}^{\infty}$ is defined by

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|.$$

The space ℓ^p, $1 \leqslant p \leqslant \infty$, is defined as the space of all sequence $x = (x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ such that $\|x\|_p < \infty$.

Theorem 1.2. *(Hölder's Inequality)*
Let $p > 1$, $q > 1$ and $1/p + 1/q = 1$. For any two sequences of complex numbers $\{x_n\}$ and $\{y_n\}$ we have

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |y_n|^q\right)^{\frac{1}{q}}.$$

Theorem 1.3. *(Minkowski's Inequality)*
Let $p > 1$. For any two sequences of complex numbers $\{x_n\}$ and $\{y_n\}$ we have

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}.$$

1.1.2 Orthogonal and Orthonormal Systems

Definition 1.6. A collection of vectors $\{v_i : i \in A \subset \mathbb{Z}\}$ in an inner product space $(V, \langle\cdot,\cdot\rangle)$ is said to be orthogonal if

$$\langle v_i, v_j\rangle = 0, \quad i, j \in A, \quad i \neq j,$$

and called orthonormal if in addition to being orthogonal

$$\langle v_i, v_i\rangle = 1, \quad i \in A.$$

Example 1.3.

1. *In ℓ^2, $\{e_n\}_{n=1}^{\infty}$ is an orthonormal sequence, where e_n is the sequence in ℓ^2 having 1 in the nth place and zeros everywhere else.*
2. *In $L_2([-\pi,\pi],1)$, an orthonormal sequence is $\{e_n\}_{n\in Z}$ where $e_n(t) = (2\pi)^{-1/2}\, e^{int}$, since*

$$\langle e_n, e_m\rangle_{L_2([-\pi,\pi],1)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)t}dt = \begin{cases} 1, & if \quad n = m, \\ 0, & otherwise. \end{cases}$$

Theorem 1.4. *(Pythagorean Theorem)*
Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space and $x, y \in V$. If x and y are orthogonal, then

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

In general,

$$\left\|\sum_{n=1}^{k} x_n\right\|^2 = \sum_{n=1}^{k} \|x_n\|^2,$$

whenever $\{x_1, \cdots, x_k\}$ are orthogonal vectors.

Proof. Let x and y be any orthogonal vectors, then from the definition of the norm we have

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle = \|x\|^2 + \|y\|^2,$$

which completes the proof. $\square$

Theorem 1.5. *(Bessel's Inequality) If $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal sequence in an inner product space V, then for any $x \in V$,*

$$\left\| x - \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2$$

and

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \|x\|^2 .$$

Definition 1.7. If $\{e_n\}_{n\in I}$ is an orthonormal sequence in a Hilbert space H then, for any $x \in H$, $\langle x, e_n\rangle$ is the nth Fourier coefficient of x with respect to $\{e_n\}$. The Fourier series of x with respect to the sequence $\{e_n\}$ is the series $\sum_{n\in I} \langle x, e_n\rangle e_n$.

Let V be a vector space and let $S = \{v_1, \cdots, v_n\} \subset V$. The subspace W of V consisting of all linear combinations of the vectors in S is called the *space spanned by* $v_1, v_2, \cdots, v_n$, and we write

$$W = \operatorname{span}(S) = \left\{ \sum_{i=1}^{n} c_i v_i : \quad c_i \in \mathbb{C} \right\}.$$

Theorem 1.6. *Let $e_1, \cdots, e_k$ be an orthonormal system in an inner product space V and let $x \in V$. The closest point y of span$\{e_1, \cdots, e_k\}$ to x is $y = \sum_{n=1}^{k} \langle x, e_n\rangle e_n$.*

Theorem 1.7. *(Riesz-Fischer)*
Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in a Hilbert space H and let $\lambda_n \in \mathbb{C}$, $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} \lambda_n e_n$ converges in H if and only if $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$, and in that case

$$\left\| \sum_{n=1}^{\infty} \lambda_n e_n \right\| = \sqrt{\sum_{n=1}^{\infty} |\lambda_n|^2}.$$

Note that Riesz-Fischer theorem and Bessel's inequality prove that the Fourier series converges for each $x \in V$, but they do not guarantee that it converges to x itself. To prove this result we need the following definition.

Definition 1.8. An orthonormal set $\{y_n\}_{n=1}^{\infty}$ is complete in a Hilbert space H if whenever

$$\langle x, y_n \rangle = 0, \quad n = 1, 2, \cdots,$$

for an element x in H, then $x = 0$.

Theorem 1.8. *Let $Y = \{y_n\}_{n\geq 1}$ be orthonormal set in a Hilbert space H. Then Y is complete if and only if for any $x \in H$, $x = \sum_{n=1}^{\infty} \langle x, y_n \rangle y_n$.*

Proof. Suppose $\{y_n\}_{n=1}^{\infty}$ is complete, and let $z = x - \sum_{n=1}^{\infty} \langle x, y_n \rangle y_n$. Then

$$\begin{aligned} \langle z, y_j \rangle &= \left\langle x - \sum_{n=1}^{\infty} \langle x, y_n \rangle y_n, y_j \right\rangle \\ &= \langle x, y_j \rangle - \langle x, y_j \rangle \|y_j\|^2 \\ &= 0 \quad for \;\; all \; j = 1, 2, \cdots. \end{aligned}$$

Thus $z = 0$ and hence $x = \sum_{n=1}^{\infty} \langle x, y_n \rangle y_n$. Conversely, suppose $x = \sum_{n=1}^{\infty} \langle x, y_n \rangle y_n$ for any $x \in H$. Then if $\langle w, y_n \rangle = 0$ for all n we have $w = 0$ and then $\{y_n\}_{n\geq 1}$ is a complete orthonormal set. ☐

Theorem 1.9. *(Parseval's Equality)*
$\{y_n\}_{n\geq 1}$ is a complete orthonormal set in a Hilbert space H if and only if for any $x \in H$, $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, y_n \rangle|^2$.

Definition 1.9. A finite collection of vectors $\{x_1, \cdots, x_n\}$ is called linearly independent if $a_1x_1 + \cdots + a_nx_n = 0$ only if $a_1 = \cdots = a_n = 0$. An infinite collection of vectors is called linearly independent if every finite subcollection is linearly independent.

Theorem 1.10.

1. *Every finite orthogonal set of nonzero vectors is linearly independent.*
2. *If $\{x_1, x_2, \cdots\}$ is an infinite orthogonal set of nonzero vectors in a Hilbert space, and $\sum_{n=1}^{\infty} a_nx_n = 0$, then $a_n = 0$ for all $n = 1, 2, \cdots$.*

Proof. Let $\{x_1, x_2, \cdots, x_m\}$ be any orthogonal set. If we have $\sum_{n=1}^{m} a_n x_n = 0$, then for $j = 1, 2, \cdots, m$,

$$0 = \left\langle \sum_{n=1}^{m} a_n x_k, x_j \right\rangle = a_j \|x_j\|^2 .$$

Thus $a_j = 0$ for all $j = 1, \cdots, m$. For (2) suppose we have a set $\{x_1, x_2, \cdots\}$ of orthogonal nonzero vectors in a Hilbert space H, and $\sum_{n=1}^{\infty} a_n x_n = 0$. Then $\{\frac{x_1}{\|x_1\|}, \cdots\}$ is an orthonormal set and

$$\sum_{n=1}^{\infty} a_n \|x_n\| \frac{x_n}{\|x_n\|} = 0.$$

Then by the Riesz-Fischer Theorem we get

$$0 = \sum_{n=1}^{\infty} |a_n|^2 \|x_n\|^2.$$

Thus $a_i = 0$ for all $i = 0, 1, \cdots$. □

In general, if $S = \{v_1, v_2, \cdots, v_n\}$ is an orthogonal collection of nonzero vectors then S is linearly independent and it can always be converted to an orthonormal set by multiplying every vector by the reciprocal of its length to obtain a vector of norm 1, i.e. the set

$$S' = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \cdots, \frac{v_n}{\|v_n\|} \right\},$$

is an orthonormal set. The following theorem shows that any linearly independent collection of vectors can be orthonormalized. This is called *The Gram-Schmidt Process.*

Theorem 1.11. *(Gram-Schmidt)*
Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space with norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$. Suppose $\{v_n\}_{n=1}^{\infty}$ is a linearly independent collection of vectors in V. Let

$$u_1 = \frac{v_1}{\|v_1\|}$$

and let

$$t_n = v_n - \sum_{k=1}^{n-1} \langle v_n, u_k \rangle u_k, \qquad u_n = \frac{t_n}{\|t_n\|}.$$

Then $\{u_n\}_{n=1}^{\infty}$ *is an orthonormal collection, and for each* n*,*

$$span\,\{u_1, u_2, \cdots, u_n\} = span\,\{v_1, v_2, \cdots, v_n\}\,.$$

1.1.3 Operators-Definition and Examples

This section is devoted to studying the linear operator in Hilbert spaces, with more attention focussed on an important class called the self-adjoint operators. Consequently, in the following subsection we consider the concept of eigenvalues and eigenfunctions of linear operator. These concepts play an important role in the theory of orthogonal polynomials which are discussed in the next chapter.

Definition 1.10. If X, Y are vector spaces over a field K, a linear operator from X to Y is a map $T : X \to Y$ such that,

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y),$$

for all $\lambda, \mu \in K$ and all $x, y \in X$.

If X, Y are normed spaces, a linear operator $T : X \to Y$ is said to be bounded if there exists $M \geq 0$ such that

$$\|Tx\| \leq M \|x\|\,, \quad \forall x \in X,$$

and we denote the collection of bounded linear operator on a Hilbert space H by ℓ_H.

Definition 1.11. Let T be a linear operator which maps a normed space X into another normed space Y. We define the operator norm of T, $\|T\|$, by setting

$$\|T\| = \|T\|_{X \to Y} = \sup_{\|x\|_X \leqslant 1} \|Tx\|_Y;$$

or

$$\|T\| = \inf\left\{M : \|Tx\|_Y \leq M\,\|x\|_X\,, \text{for all } \ x \in X\right\}.$$

From the definition we can see that $\|Tx\| \leq \|T\|\,\|x\|$ whenever T is a bounded operator. Consequently, if T and S are two bounded operators then

$$\|TSx\| \leq \|T\|\,\|Sx\| \leq \|T\|\,\|S\|\,\|x\|\,,$$

thus $\|TS\| \leq \|T\|\,\|S\|$.

Example 1.4. *These examples are given in [11] with more details.*

1. *Differential Operator:*
   The differential operator is given by

   $$(Df)(x) = \frac{df(x)}{dx} = f'(x),$$

   defined on the space of all differentiable functions on some interval $[a, b] \subset \mathbb{R}$.

2. *Integral Operator:*
   A integral operator with a kernel $k(\cdot,\cdot)$ *is defined by*

   $$(Tf)(s) = \int_a^b k(s,t)f(t)dt,$$

   where a *and* b *may be finite or infinite. The domain of an integral operator depends on* k.

Theorem 1.12. *(Riesz Representation theorem) Let* H *be a Hilbert space. For any bounded linear functional* $K : H \to \mathbb{C}$ *there exists a unique element* $y \in H$ *such that*

$$K(x) = \langle x, y\rangle\,, \quad \text{for all}\, x \in H.$$

Definition 1.12. 1. Let H be a Hilbert space and let $T : H \to H$ be a bounded operator. We define an operator T^* as follows,

$$\langle Tx, y\rangle = \langle x, T^*y\rangle\,, \quad \text{for all}\, x, y \in H.$$

T^* is called the adjoint of T.

2. Let T be a bounded operator in a Hilbert space H. T is called Hermitian or self-adjoint if $T = T^*$.

The existence of T^* is guaranteed by the Riesz Representation theorem, and it is unique. To show this fix $y \in H$ and let $K_y : H \to \mathbb{C}$ be a map defined as

$$K_y(x) = \langle Tx, y \rangle .$$

Since K_y is linear functional, by Riesz Representation theorem there exist $u \in H$ such that

$$K_y(x) = \langle x, u \rangle , \quad \forall x \in H.$$

Put $T^*(y) = u$ then $\langle Tx, y \rangle = \langle x, T^*y \rangle$. For the uniqueness suppose that there exist T_1^* and T_2^* such that

$$\langle Tx, y \rangle = \langle x, T_1^* y \rangle = \langle x, T_2^* y \rangle , \quad \forall x, y \in H.$$

Hence

$$\langle x, T_1^* y - T_2^* y \rangle = 0, \quad \forall x, y \in H.$$

This implies

$$T_1^* y - T_2^* y = 0,$$

for all y in H and then $T_1^* = T_2^*$.

For each self-adjoint operator there are three ways to compute the operator norm which are given in the following theorem.

Theorem 1.13. *Let H be a Hilbert space. Let T be self-adjoint in ℓ_H. Then $\|T\|$ can be computed from any one of the following:*

1. $\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$;
2. $\|T\| = \sup_{\|x\|=1} \|Tx\|$;
3. $\|T\| = \sup_{\|x\|=1} \|\langle Tx, x \rangle\|$.

Theorem 1.14. *Let H be a Hilbert space. Let T and S be in ℓ_H. Then*

1. *T^* and S^* are bounded.*
2. $(T^*)^* = T$.
3. $(T + S)^* = T^* + S^*$.
4. $(TS)^* = S^* T^*$.

5. *$I^* = I, \quad O^* = O$ (The identity and zero operator).*

6. *$\|T\| = \|T^*\|$.*

7. *$\|TT^*\| = \|T^*T\| = \|T\|^2$.*

8. *$T^*T = 0$ if and only if $T = 0$.*

9. *If T and S are self-adjoint operators then $T + S$ is self-adjoint.*

10. *If T is a self-adjoint operator then $a_nT^n + \cdots + a_1T + a_0$ is self-adjoint when a_i are real coefficients.*

11. *If T and S are self-adjoint operators then TS is self-adjoint if and only if $TS = ST$.*

Proof. We will prove some of the results and the rest of them are similar.

(1) By Cauchy-Schwarz's inequality, for any $x, y \in H$ we have

$$|\langle T^*x, y\rangle| = |\langle x, Ty\rangle| \leq \|x\| \, \|Ty\| \leq \|x\| \, \|y\| \, \|T\|$$

and hence for $y = T^*x$ we obtain

$$\|T^*x\|^2 = \langle T^*x, T^*x\rangle \leq \|T\| \, \|x\| \, \|T^*x\| \, .$$

Consequently,

$$\|T^*x\| \leq \|T\| \, \|x\| \, , \tag{1.2}$$

and hence T^* is bounded operator.

(3) Let $x \in H$. Then

$$\langle x, (T^* + S^*)y\rangle = \langle x, T^*y + S^*y\rangle = \langle x, T^*y\rangle + \langle x, S^*y\rangle$$

$$= \langle Tx, y\rangle + \langle Sx, y\rangle = \langle (T + S)x, y\rangle \, .$$

Thus $(T + S)^* = T^* + S^*$.

(6) From (1.2) we have

$$\frac{\|T^*x\|}{\|x\|} \leq \|T\| \,,$$

thus $\|T^*\| \leq \|T\|$. Use T^* instead of T, which gives

$$\|T\| = \|(T^*)^*\| \leq \|T^*\| \,.$$

Thus $\|T\| = \|T^*\|$.

(7) From the (1.2) and (6) we get

$$\|T^*T\| \leq \|T^*\| \, \|T\| = \|T\|^2 \,.$$

On the other hand, for every $x \in H$ we have

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle \leq \|T^*Tx\| \, \|x\| \leq \|T^*T\| \, \|x\|^2 \,.$$

Thus $\|T^*T\| = \|T\|^2$.

□

1.1.4 Eigenvalues and Eigenvectors

Definition 1.13. Let T be a linear operator in V, and let λ be a scaler. We say that λ is an eigenvalue of T if there is a nonzero vector $x \in V$ such that $Tx = \lambda x$. The vector x is called an eigenvector associated with λ. If V is a function space, eigenvectors are often called eigenfunctions.

Theorem 1.15. *Let T be a self-adjoint operator on a Hilbert space H. All eigenvalues of T are real, and eigenvectors of T corresponding to distinct eigenvalues are orthogonal.*

Proof. Suppose that λ is an eigenvalue of T and u is the corresponding eigenvector. Then

$$\langle u, Tu \rangle = \langle Tu, u \rangle \,,$$

so that

$$\langle u, \lambda u \rangle = \langle \lambda u, u \rangle \,.$$

Thus

$$\bar{\lambda}\langle u,u\rangle = \lambda\langle u,u\rangle .$$

Since $\langle u,u\rangle \neq 0$, we have $\bar{\lambda} = \lambda$, so that λ is real. Next let u and v be two eigenvectors corresponding to distinct eigenvalues λ and β. Then we have

$$\lambda\langle v,u\rangle = \langle v,Tu\rangle = \langle Tv,u\rangle = \beta\langle v,u\rangle .$$

Hence $(\lambda-\beta)(\langle v,u\rangle) = 0$, and since $\lambda \neq \beta$ then $\langle v,u\rangle = 0$. □

Theorem 1.16. *The collection of all eigenvectors corresponding to one particular eigenvalue of an operator is a vector space.*

The vector space corresponding to an eigenvalue λ is called the eigenspace of λ.

1.2 Differential and Integral calculus

The Lebesgue measure has played a fundamental role in this thesis and many results have been provided by it. Hence, we briefly describe the main idea and results about the Lebesgue measure and integration and L_p spaces. We also summarise the main properties of these concepts and state them without proof.

1.2.1 Lebesgue Integral on the Euclidean Space $\mathbb{R}^d$

We begin with some preliminary structure and several notations. The Euclidean space $\mathbb{R}^d$ is an inner product space with the inner product defined as

$$\langle x,y\rangle_{\mathbb{R}} = x_1y_1 + x_2y_2 + \cdots + x_dy_d,$$

and then the norm (or length) is given by

$$\|x\| = (x_1^2 + x_2^2 + \cdots + x_d^2)^{1/2}.$$

We also have

$$\langle x,y\rangle_{\mathbb{R}} = \|x\|\,\|y\|\cos\vartheta$$

where ϑ is the angle between x and y.

The unit ball in $\mathbb{R}^d$ is defined by

$$B_d = \{x \in \mathbb{R}^d : \|x\| \leqslant 1\}.$$

The set of all vectors of length 1 is called the unit sphere, in other words, the unit sphere $\mathbb{S}^{d-1}(\mathbb{R})$ is defined as

$$\mathbb{S}^{d-1}(\mathbb{R}) = \left\{v \in \mathbb{R}^d : \|v\| = 1\right\},$$

and we call any vector $v \in \mathbb{S}^{d-1}(\mathbb{R})$ a unit vector. Furthermore, for any vector $x \in \mathbb{R}^d$, $x/\|x\| \in \mathbb{S}^{d-1}(\mathbb{R})$. A set $K \subset \mathbb{R}^d$ is said to be convex if for any two points x and y in K, the line segment joining them is contained in K.

Now we will follow [25] to define the Lebesgue measure μ_d in $\mathbb{R}^d$, which will happen in six steps.

Step1. Let R be the set of all rectangle subsets of $\mathbb{R}^d$ which are of the form

$$\begin{aligned} I &= [a_1, b_1] \times \cdots \times [a_d, b_d] \\ &= \{x \in \mathbb{R}^d | a_i \leqslant x_i \leqslant b_i, \text{for} \quad 1 \leqslant i \leqslant d\}. \end{aligned}$$

Then define

$$\mu_n(I) = (b_1 - a_1) \cdots (b_d - a_d),$$

and if $I = \emptyset$ let $\mu_d(I) = 0$.

Step 2. Let Θ be the set of all finite union of sets belonging to R which have nonempty measure. To measure a set $P \in \Theta$ first write it as a finite union of nonoverlapping (i.e., their interior are disjoint) rectangles

$$P = \bigcup_{n=1}^{k} I_n.$$

Then define

$$\mu_d(P) = \sum_{n=1}^{k} \mu_d(I_n).$$

This definition is well defined since each set P belongs to Θ can be expressed as a finite union of nonoverlaping rectangles, and the expression defining $\mu_d(P)$ is independent of the particular expression for P. For the proof of these facts see ([25]).

Step 3. Let $G \subseteq \mathbb{R}^d$ be an open set and $G \neq \emptyset$. We define

$$\mu_d(G) = \sup\{\mu_d(P) \mid P \subset G,\ P \in \Theta\}.$$

This is well defined since each nonempty open set has nonempty interior, and hence it contains a polygon.

Step 4. Let $K \subset \mathbb{R}^d$ be compact (closed and bounded) set, then the measure is defined as

$$\mu_d(K) = \inf\{\mu_d(G) |\ K \subset G,\ G \textit{ an open set}\}.$$

To complete defining the Lebesgue measure we need the following definitions.

Definition 1.14. Assume $A \subset \mathbb{R}^d$ is an arbitrary set. Then define

$$\begin{aligned} \mu_d^*(A) &= \inf\{\mu_d(G)|\ A \subset G,\ G \textit{ open set}\}, \\ \mu_{d*}(A) &= \sup\{\mu_d(K)|\ K \subset A,\ K \textit{ compact set}\}. \end{aligned}$$

The functions μ_d^* and μ_{d*} are called Lebesgue outer and inner measure, respectively.

Step 5. Let F_d be the set of all subsets A of $\mathbb{R}^d$ of finite outer measure such that $\mu_d^*(A) = \mu_{d*}(A)$. If $A \in F_d$ then define the measure as

$$\mu_d(A) = \mu_{d*}(A) = \mu_d^*(A).$$

From the above definition we notice that all open and compact sets of finite outer measure belong to F_d.

Proposition 1.1. *1. If $A, B \in F_n$ then also the sets $A \cup B$, $A \cap B$ and $A - B$ belong to F_d.*

2. *If A and B are in F_d and disjoint then*

$$\mu_d(A \cup B) = \mu_d(A) + \mu_d(B).$$

3. *If $A_n \in F_d$ for $n = 1, 2, \cdots$, and $A = \cup_{n=1}^{\infty} A_n$ with $\mu_d^*(A) < \infty$, then $A \in F_d$ and*

$$\mu_d(A) \leqslant \sum_{n=1}^{\infty} \mu_d(A_n).$$

In addition, if A_n are disjoint, then

$$\mu_d(A) = \sum_{n=1}^{\infty} \mu_d(A_n).$$

Step 6.

Definition 1.15. Let $A \subset \mathbb{R}^d$. Then A is called a measurable set if for any $M \in F_d$,

$$A \cap M \in F_d.$$

The set of all measurable sets in $\mathbb{R}^d$ is denoted by $\Im_d$. For a set $A \in \Im_d$ we define the measure

$$\mu_d(A) = \sup\{\mu_d(A \cap M)|\ M \in F_d\}.$$

Since measure theory extended the notions of length and volume to a large class of sets, some authors called the above measure a volume and denoted by $|A|$ or $\mathrm{Vol}_d(A)$. This terminology generalises the concept of computing the volume in three dimension to higher dimensional Euclidean spaces.

Proposition 1.2. 1. *If $A \in \Im_d$ then $A^c \in \Im_d$.*

2. *Countable unions and countable intersections of measurable sets are measureble.*

3. *If the sets $A_1, A_2, \cdots$ belongs to $\Im_d$ then*

$$\mu_d\left(\cup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} \mu_d(A_n).$$

In addition, if A_n are disjoint, then

$$\mu_d(A) = \sum_{n=1}^{\infty} \mu_d(A_n).$$

4. *All closed and open sets are measurable.*

5. *If $\mu_d^*(A) = 0$ then A is measurable and $\mu_d(A) = 0$.*

6. *If $A \subset \mathbb{R}^d$, $y \in \mathbb{R}^d$ and $0 < t < \infty$, then the following statements are equivalent*

   - *A is measurable,*
   - *$y + A = \{y + x|\ x \in A\}$ is measurable,*
   - *$tA = \{tx|\ x \in A\}$ is measurable.*

   If A is measurable then

$$\begin{aligned} \mu_d(A + y) &= \mu_d(A), \\ \mu_d(tA) &= t^n \mu_d(A). \end{aligned}$$

7. *If C is an $d \times d$ matrix and $A \subset \mathbb{R}^d$ measurable then $CA = \{Ca : a \in A\}$ is measurable and*

$$\mu_d(CA) = |\det(C)| \mu_d(A).$$

8. *If $E \subseteq \mathbb{R}^q$ and $F \subseteq \mathbb{R}^d$ are Lebesgue measurable sets, then $E \times F$ is Lebesgue measurable and $\mu_{q+d}(E \times F) = \mu_q(E)\mu_d(F)$.*

The above properties show some geometrical properties of Lebesgue measure. It is invariant under translation, reflection and rotation, since any reflection and rotation can be represented by an orthogonal matrix which has determinant 1 or -1; see Section 3.2 for the definition of orthogonal matrices.

In order to define the Lebesgue integral we have to define a large class of functions, called measurable function, that can be treated in the integral.

Definition 1.16. 1. Let f be a real valued function on $\mathbb{R}^d$. Then f is a Lebesgue measurable function if and only if for any $t \in [-\infty, \infty]$ the set $f^{-1}([t, \infty])$ belongs to $\Im_d$.

2. If f is a complex valued function on $\mathbb{R}^d$, then f is measurable if its real and imaginary parts are measurable.

Proposition 1.3. *Let f and g be Lebesgue measurable, then*

1. *If $f \neq 0$, then $\frac{1}{f}$ is Lebesgue measurable.*
2. *$|f|^p$ is Lebesgue measurable, where $0 < p < \infty$.*
3. *$f + g$ and fg are Lebesgue measurable.*
4. *Let f_n be Lebesgue measurable functions for all $n = 1, 2, \cdots$. Then all the following functions are measurable,*

$$\sup_n f_n, \quad \inf_n f_n, \quad \lim_{n\to\infty} \sup f_n, \quad \lim_{n\to\infty} \inf f_n,$$

$$\lim_{n\to\infty} f_n, \text{if it exists.}$$

5. *The functions*

$$f_+(x) = \begin{cases} f(x), & \text{when} \quad f(x) \geqslant 0, \\ 0, & \text{when} \quad f(x) < 0, \end{cases}$$

$$f_-(x) = \begin{cases} 0, & \text{when} \quad f(x) \geqslant 0, \\ -f(x), & \text{when} \quad f(x) < 0, \end{cases}$$

are measurable.

Now, the Lebesgue integral can be defined. First define the integral of a simple function and then the integral of any measurable function by using approximation by simple functions.

Definition 1.17. Let S be the set of all simple functions given by

$$s = \sum_{n=1}^{m} \alpha_n \chi_{A_n},$$

where $0 \leqslant \alpha_n < \infty$, the sets A_n are disjoint and χ_{A_n} is the characteristic function of the measurable set A_n. Then define the integral of s as follows

$$\int s d\mu_d = \sum_{n=1}^{m} \alpha_n \mu_d(A_n).$$

For any measurable function on $\mathbb{R}^d$, the Lebesgue integral denoted by $\int f d\mu_d$ is a number given in the following definition:

Definition 1.18. **(i)** Let f be nonnegative measurable function on $\mathbb{R}^d$. Then

$$\int f d\mu_d = \sup\left\{\int s\ d\mu_d : \ s \leqslant f,\ s \in S\right\}.$$

(ii) Let f be a real valued measurable function. If f_+ and f_- have finite integral, then f is said to be integrable and its integral is given by

$$\int f d\mu_n = \int f_+ d\mu_n - \int f_- d\mu_n.$$

The set of all integrable functions on $\mathbb{R}^n$ is denoted by $L_1(\mathbb{R}^n)$.

(iii) Let $E \subset \mathbb{R}^d$ be a measurable set and let f be a measurable function. The integral of f over the set E is defined as

$$\int_E f d\mu_d = \int f \chi_E d\mu_d.$$

An important fact about the Lebesgue integral is that the integral over a set of measure zero is zero. In other words, any two functions f and g which coincide except on a set of measure zero have the same integral, and this is clear from the definition. In general, we write $f = g$ *a.e.*, when f equals g except on a set of measure zero.

Proposition 1.4. *Let f and g be measurable functions.*

1. *If $f \in L_1(\mathbb{R}^d)$ then $|f| \in L_1(\mathbb{R}^d)$, and $|\int f d\mu_d| \leqslant \int |f| d\mu_d$.*

2. *If g belongs to $L_1(\mathbb{R}^d)$, and $|f| \leqslant |g|$ then $f \in L_1(\mathbb{R}^d)$.*

3. *If $f, g \in L_1(\mathbb{R}^d)$ and $a, b \in \mathbb{R}$, then $af + bg \in L_1(\mathbb{R}^d)$ and*

$$\int (af + bg) d\mu_d = a\int f d\mu_d + b\int g d\mu_d.$$

In order to differentiate under the integral sign, we shall need the following result.

Theorem 1.17. *Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is continuously differentiable, and that $g : \mathbb{R}^d \to \mathbb{R}$ is defined by*

$$g(x_1, \cdots, x_{d-1}) = \int_{a_d}^{b_d} f(x_1, \cdots, x_d) dx_d.$$

Then g is continuously differentiable, and for $j = 1, \cdots, d-1$,

$$\frac{\partial g}{\partial x_j} = \int_{a_d}^{b_d} \frac{\partial f}{\partial x_j}(x_1, \cdots x_d) dx_d.$$

Since our approaches consider approximation on a sphere from a geometric point of view, we will use the notation $\mathrm{Vol}_d(A)$ for the measure of the set A. The following theorems provide us with some interesting inequalities about volumes which can be found in [8] and [46].

Theorem 1.18. *Let A, B be two compact subsets of $\mathbb{R}^d$. Then for all $\lambda \in [0,1]$ we have*

$$\mathrm{Vol}_d(\lambda A + (1-\lambda)B) \geqslant \mathrm{Vol}_d(A)^\lambda \mathrm{Vol}_d(B)^{1-\lambda},$$

and

$$(\mathrm{Vol}_d(A+B))^{\frac{1}{d}} \geqslant (\mathrm{Vol}_d(A))^{\frac{1}{d}} + (\mathrm{Vol}_d(B))^{\frac{1}{d}}.$$

The next theorem requires some definitions which can be found in [1]. A set B is called centrally symmetric set if $B = -B$. Now we need to define the *polar set* of a set which is a subset of $\mathbb{R}^d$.

Definition 1.19. Let $K \subset \mathbb{R}^d$ be a convex, centrally symmetric and compact set. Then the polar set K^o is defined as $K^o := \left\{x \in \mathbb{R}^d : |\langle x, y\rangle| \leq 1, y \in K\right\}$.

Theorem 1.19. *(Urysohn's Inequality)*
Assume that K is a convex, centrally symmetric and compact subset of $\mathbb{R}^d$, and let

$$\|x\|_{K^o} = \sup\{\langle x, y\rangle, y \in K\}$$

for all $x \in \mathbb{R}^d$. Then

$$\left(\frac{\mathrm{Vol}_d(K)}{\mathrm{Vol}_d(B_d)}\right)^{\frac{1}{d}} \leqslant \int_{\mathbb{S}^{d-1}(\mathbb{R})} \|x\|_{K^o} d\mu_d(x)$$

where $\mathbb{S}^{d-1}(\mathbb{R})$ is the unit sphere in $\mathbb{R}^d$ and $d\mu_d$ its normalised area measure.

Proposition 1.5. *1. If A is a nonempty bounded subset of $\mathbb{R}^d$, then*

$$A^o := \left\{ x \in \mathbb{R}^d : \sup_{x' \in A} |\langle x, x' \rangle| \leq 1 \right\}.$$

2. If $A \subset B$, then $A^o \supset B^o$.

3. If $\epsilon \neq 0$, then $(\epsilon A)^o = \frac{1}{\epsilon} A^o$.

4. $\bigcap A_i^o = (\bigcup A_i)^o$.

5. Let A be convex, closed and circled (i.e. for any $x \in A$ and any $|\alpha| \leqslant 1$ then $\alpha x \in A$), then $A = A^{oo}$.

Theorem 1.20. *(Bourgain-Milman Inequality) If K is a convex centrally symmetric bounded set in $\mathbb{R}^d$, then there exist a constant $c > 0$ such that*

$$\mathrm{Vol}_d(K)\mathrm{Vol}_d(K^o) \geqslant c^n (\mathrm{Vol}_d(B_d))^2.$$

Before we embark on the study of spherical harmonic functions, we need some results about the unit sphere and polar coordinates in n dimensional Euclidean space. We will consider little detail, those of which we need later. The polar coordinates in $\mathbb{R}^d$, $r, \vartheta_1, \vartheta_2, \cdots, \vartheta_{d-2}, \varphi$, are defined by

$$\begin{aligned}
x_1 &= r \cos \vartheta_1 \\
x_2 &= r \sin \vartheta_1 \cos \vartheta_2 \\
x_3 &= r \sin \vartheta_1 \sin \vartheta_2 \cos \vartheta_3 \\
& \cdots \quad \cdots \quad \cdots \\
x_{d-2} &= r \sin \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{d-3} \cos \vartheta_{d-2} \\
x_{d-1} &= r \sin \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{d-2} \cos \varphi \\
x_d &= r \sin \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{d-2} \sin \varphi
\end{aligned}$$

where $r \geq 1$, $\|x\| = r$ and $0 \leq \vartheta_i \leq \pi$ $(i = 1, 2, \cdots, d-2)$, $0 \leq \varphi \leq 2\pi$.

It is well known that the Lebesgue integral can be computed as (see e.g. [19] and [25])

$$\int f d\mu_d = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_d) dx_1 \cdots dx_d.$$

The d-dimensional volume element in polar coordinates is given by

$$dv = r^{d-1}(\sin\vartheta_1)^{d-2}(\sin\vartheta_2)^{d-3}\cdots(\sin\vartheta_{d-2}) dr d\vartheta_1 \cdots d\vartheta_{d-2} d\varphi,$$

and the surface element $d\mu_d$ on the unit sphere becomes

$$d\mu_d = (\sin\vartheta_1)^{d-2}(\sin\vartheta_2)^{d-3}\cdots(\sin\vartheta_{d-2}) d\vartheta_1 \cdots d\vartheta_{d-2} d\varphi. \tag{1.3}$$

A function which is defined on $\mathbb{S}^{d-1}(\mathbb{R})$ can be considered as a function of $\vartheta_1, \vartheta_2, \cdots,$ ϑ_{d-2}, and φ. Consequently, the expression $\int_{\mathbb{S}^{d-1}(\mathbb{R})} f d\mu_d$ denotes the $(d-1)$-tuple integral of the variables $\vartheta_1, \vartheta_2, \cdots, \vartheta_{d-2}, \varphi$ and $d\mu_d$ the corresponding expression from (1.3).

It well known that the Lebesgue measure of a ball $B^r(x)$ of radius r and centre x is

$$\mu_d(B^r(x)) = r^d \mu_d(B^r(0)) = \frac{2\pi^{d/2} r^d}{d\Gamma(d/2)} = \frac{r^d}{d}\mu_d(\mathbb{S}^{d-1}(\mathbb{R})),$$

where $\omega_d = \mu_d(\mathbb{S}^{d-1}(\mathbb{R})) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$, is the area of the unit sphere in $\mathbb{R}^d$. Here the notation Γ represents the Gamma function, which may be defined by any of the following equivalent expressions:

(i)

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt = \int_0^1 (\log 1/t)^{z-1} dt,$$

(ii)

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)} = z^{-1} \prod_{n=1}^{\infty} \left[\left(1+\frac{1}{n}\right)^z \left(1+\frac{z}{n}\right)^{-1}\right].$$

There are many functional equations satisfied by $\Gamma(z)$, for example

1. $\Gamma(1+z) = z\Gamma(z)$,
   and hence if n is a positive integer, we get $\Gamma(n+1) = n!$.
2. $\frac{\Gamma(z)}{\Gamma(z-n)} = (z-1)(z-2)\cdots(z-n)$.

3. $\Gamma(1) = \int_0^\infty e^{-t}dt = 1.$

1.2.2 The Lebesgue Spaces L_p in $\mathbb{R}^d$

In this section we will concentrate on certain Banach spaces of measurable functions called L_p spaces. The basic structural facts about these spaces will be discussed here, and only those properties and inequalities that are extremely useful in this work. The L_2 space in a finite interval has been mentioned before. Here we will deal with a generalisation of this space.

Definition 1.20. Let A be a subset of $\mathbb{R}^n$. We define $L_p(A)$ for $1 \leqslant p < \infty$ to be the class of all measurable function f defined on A such that

$$\int_A |f|^p d\mu_n < \infty.$$

For $p = \infty$ the space $L_\infty(A)$ can be defined as the collection of all essentially (a.e.) bounded measurable functions on A. Here $d\mu_d$ is the Lebesgue measure on $\mathbb{R}^d$.

In fact, the element f of the class $L_p(A)$ can be considered as equivalence classes of measurable functions which are equal to f almost everywhere in A. Thus, $L_p(A)$ consists of equivalence classes of functions.

If f and g in $L_p(A)$ then, as an immediate consequence of the inequality $(a+b)^p \leqslant 2^{p-1}(a^p + b^p)$, we have $af + bg \in L_p(A)$, for any complex number a and b. Hence $L_p(A)$ is a vector space.

Define the functional $\|\cdot\|_{L_p(A)}$ of $f \in L_p(A)$ to be

$$\|f\|_{L_p(A)} = \left(\int_A |f|^p d\mu_n\right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

and

$$\|f\|_\infty = \inf\{M \mid |f(x)| \leqslant M \ \ a.e.\ for\ x \in A\}.$$

This functional on the vector space L_p constructs a normed space $(L_p(A), \|\cdot\|_{L_p(A)})$ since it satisfy the following axioms:

1. $0 \leqslant \|f\|_{L_p(A)} < \infty$,
2. $\|f\|_{L_p(A)} = 0$ if only if $f = 0$ *a.e.*,
3. $\|cf\|_{L_p(A)} = |c|\|f\|_{L_p(A)} \quad for\ c \in \mathbb{C}$,
4. $\|f+g\|_{L_p(A)} \leqslant \|f\|_{L_p(A)} + \|g\|_{L_p(A)}$.

The last inequality follows immediately from Minkowski's Inequality which can be proven by using Hölder's inequality, both of which are given below.

Theorem 1.21. *(Hölder's Inequality)*
Let $1 \leqslant p \leqslant \infty$ *and* $p' = \frac{p}{p-1}$, $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$. *If* $f \in L_p(A)$ *and* $g \in L_{p'}(A)$, *then* $fg \in L_1(A)$ *and*

$$\|fg\|_{L_1(A)} = \int_A |fg| d\mu_n \leqslant \|f\|_{L_p(A)} \|g\|_{L_{p'}(A)}.$$

Theorem 1.22. *(Minkowski's Inequality)*
Let $1 \leqslant p < \infty$ *and* $f, g \in L_p(A)$. *Then* $f + g \in L_p(A)$ *and*

$$\|f+g\|_{L_p(A)} \leqslant \|f\|_{L_p(A)} + \|g\|_{L_p(A)}.$$

For the proofs of these theorems we cite [25].

Theorem 1.23. *(An Imbedding Theorem for L_p Spaces)*
Assume that $\mathrm{Vol}_d(A) = \int_A 1 d\mu_d < \infty$ *and let* $1 \leqslant p \leqslant q < \infty$. *If* $f \in L_q(A)$, *then* $f \in L_p(A)$ *and*

$$\|f\|_{L_p(A)} \leqslant (\mathrm{Vol}_d(A))^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L_q(A)}.$$

In particular, if $\mathrm{Vol}_d(A) = 1$ *and* $f \in L_\infty(A)$, *then we have for* $1 \leqslant p < q \leqslant \infty$,

$$\|f\|_{L_1(A)} \leqslant \|f\|_{L_p(A)} \leqslant \|f\|_{L_q(A)} \leqslant \|f\|_\infty.$$

As we have seen the spaces L_p are normed spaces. In fact, they are Banach spaces. The completeness of L_p is given in a theorem known as the Riesz-Fischer Theorem, and since the proof is beyond the scope of this thesis we refer the reader again to [12]. Another famous theorem is the Interpolation Theorem of Riesz-Thorin which deals with L_p and bounded operators.

Theorem 1.24. *Let $1 \leqslant p_1, p_2, q_1, q_2 \leqslant \infty$, $0 \leqslant \theta \leqslant 1$ and let p, q be defined by*

$$\frac{1}{p} = (1-\theta)\frac{1}{p_1} + \theta\frac{1}{p_2}, \qquad \frac{1}{q} = (1-\theta)\frac{1}{q_1} + \theta\frac{1}{q_2}.$$

Assume that T is a bounded linear operator maping L_{p_i} into L_{q_i} with norm C_i, $i = 1, 2$. Then T can be extended to a linear bounded operator from L_p into L_q with norm

$$\|T\|_{L_p \to L_q} \leqslant C_1^{1-\theta} C_2^{\theta},$$

and hence

$$\|Tf\|_{L_q(A)} \leqslant C_1^{1-\theta} C_2^{\theta} \|f\|_{L_p(A)}.$$

Suppose $p_1 = q_1 = 1$ and $p_2 = q_2 = \infty$. Then the theorem implies that any linear bounded operator maps $L_1 \to L_1$ and $L_\infty \to L_\infty$ also maps $L_p \to L_p, 1 \leqslant p \leqslant \infty$ with norm less than $\max(C_1, C_2)$; see [12].

Theorem 1.25. *$L_2(A)$ is a Hilbert space with respect to the inner product*

$$\langle f, g \rangle_{L_2(A)} = \int_A f(x)\overline{g(x)} d\mu_d(x).$$

Hölder's inequality for L_2 is known as the Cauchy-Schwarz Inequality,

$$|\langle f, g \rangle_{L_2(A)}| \leqslant \|f\|_{L_2(A)} \|g\|_{L_2(A)}.$$

An important result in this direction is Young's Inequality, which depends on a notion known as convolution.

Definition 1.21. Let f and g be complex valued functions on $\mathbb{R}^d$. The convolution of f and g is the function denoted by $f * g$ defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y) g(x-y) d\mu_d(y), \qquad for\ x \in \mathbb{R}^d,$$

when the integral exists.

The above integral is well defined in various circumstances. For instance, if f and g belong to $L_1(\mathbb{R}^d)$ then $f * g$ exists a.e. and belongs to $L_1(\mathbb{R}^d)$. More is true, as the following theorem shows (see e.g., [39]).

Theorem 1.26. *(Young's Theorem)*
Let $p, q, r \geqslant 1$ and assume that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. Then

$$\left| \int_{\mathbb{R}^d} (u * v)(x) w(x) d\mu_d x \right| \leqslant c \|u\|_{L_p(\mathbb{R}^d)} \|v\|_{L_q(\mathbb{R}^d)} \|w\|_{L_r(\mathbb{R}^d)},$$

for a constant c, holds for all $u \in L_p(\mathbb{R}^d)$, $v \in L_q(\mathbb{R}^d)$ and $w \in L_r(\mathbb{R}^d)$.

Theorem 1.27. *(Young's Inequality) Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ and let $u \in L_p(\mathbb{R}^d)$, $v \in L_q(\mathbb{R}^d)$. Then $u * v \in L_r(\mathbb{R}^d)$ and*

$$\|u * v\|_{L_r(\mathbb{R}^d)} \leqslant \|u\|_{L_p(\mathbb{R}^d)} \|v\|_{L_q(\mathbb{R}^d)}.$$

1.3 Polynomial Inequalities

In this section we give a brief discussion of some inequalities which have a great importance for many problems in approximation theory. Markov-Bernstein type inequalities are fundamental to inverse theorem proofs, which provide some smoothness properties of a function from the approximation error by approximation classes of polynomials. These inequalities with Jackson type inequalities have an important role in characterising the corresponding approximation space (see DeVore and Lorentz [12]). Since Jackson type inequalities are not involved directly in our work, we decide to omit these inequalities. Another type of inequalities, which are also of great importance for approximation theory, are Landau-Kolmogorov type inequalities. These inequalities deal with different norms and, therefore, they are relevant to approximation of function classes in other norms. All the results in this section can be found in Borwein and Erdelyi [6], DeVore and Lorentz [12], Ditzian [16], Ditzian [17], Milovanovic, et al. [45], and Kwong and Zettl [37].

1.3.1 Markov-Bernstein Type Inequalities

Markov-Bernstein type inequalities can be considered in a general form given by

$$\|Dp\| \leqslant c\|p\|, \tag{1.4}$$

where p is a polynomial with derivative Dp and c a constant. For the maximum norm in the interval $[-1, 1]$,

$$\|f\| = \|f\|_\infty = \max_{-1 \leqslant t \leqslant 1} |f(t)|,$$

the inequality (1.4) is well known as the classical inequality of Markov with constant $c = n^2$, where n is the degree of the polynomial p. In the complex plane, this inequality goes back to Bernstein in 1912. He considers polynomials on the unit disk.

Theorem 1.28. *Let $\mathcal{P}_n$ be the set of all polynomials of degree n, and define $\|p\| = \max_{|z| \leqslant 1} |p(z)|$. The well known classical inequality of Bernstein on the unit disk $|z| \leqslant 1$ is*

$$\|Dp\| \leqslant n\|p\|, \quad p \in \mathcal{P}_n,$$

and the equality holds for $p(z) = cz$, c constant.

The Bernstein inequality for trigonometric polynomials defined on $[-\pi, \pi]$ is as follows:

$$|DT(\theta)| \leqslant n\|T\|_\infty, \quad -\pi \leqslant \theta \leqslant \pi,$$

where T is a trigonometric polynomial of degree n. For algebraic polynomials on the interval $[-1, 1]$ we have

$$|Dp(t)| \leqslant \frac{n}{\sqrt{1 - t^2}}, \quad -1 \leqslant t \leqslant 1,$$

where $p \in \mathcal{P}_n$ with $|p(t)| \leqslant 1$, $t \in [-1, 1]$.

There are many generalisations of the Markov-Bernstein inequality and numerous improved versions of it. Many investigations of this result concerned other norms. Higher derivatives and different differential operators also have been studied in many other papers. Since the Markov-Bernstein inequality can be represented as an estimation of the derivative norm, there are many studies of it to determine the best bound in the inequality for restricted polynomial classes. These problems have been elegantly treated in many sources, see for example Borwein and Erdelyi [6], DeVore and Lorentz [12] and Milovanovic, et al [45], which provide a huge number of references in this field. We will mention some cases presented in these books, without paying any attention to these inequalities including different norms, different operators or these for classes of polynomial with various constraints.

Consider the norm

$$\|p\|_{L_r([a,b])} = \left(\int_a^b |p(t)|^r dt\right)^{\frac{1}{r}}, \quad r \geqslant 1.$$

Theorem 1.29. *Let $r \geqslant 1$, $(a,b) = (0, 2\pi)$, then*

$$\|DT\|_{L_r([a,b])} \leqslant n\|T\|_{L_r([a,b])},$$

where T is a trigonometric polynomial of degree n.

The Markov-Bernstein inequality has also been generalised to polynomials in several variables. Let $E \subset \mathbb{R}^d$ be compact and satisfy the following condition. If $f \in C^\infty(\mathbb{R}^d)$ and the restriction $f|_E = 0$, then $D^\alpha f|_E = 0$, where

$$D^\alpha f|_E = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f|_E, \quad |\alpha| = \alpha_1 + \cdots + \alpha_d.$$

Define the norm $\|f\|_E = \max_{x \in E} |f(x)|$, and consider p as a polynomial of degree n.

Theorem 1.30. *If $E = [a,b]^n$ we have*

$$\|D^\alpha p\| \leqslant \left\{\frac{2n^2}{b-a}\right\}^{|\alpha|} \|p\|_E.$$

The Markov-Bernstein inequality in L_r norm, $r \geqslant 1$ have also been investigated using the norm

$$\|f\|_{L_r(E)} = \left(\int_E |f(x)|^r dx\right)^{\frac{1}{r}}, \quad 1 \leqslant r \leqslant \infty,$$

where $E \subset \mathbb{R}^d$. Then for any $r \geqslant 1$ and a set Ω, which satisfies some technical conditions (see Milovanovic, et al [45] p. 591), we have

$$\|D_i p\|_{L_r(\overline{\Omega})} \leqslant c(\Omega, r) n^2 \|p\|_{L_r(\overline{\Omega})},$$

where p is a polynomial in $\Omega \subset \mathbb{R}^d$ of dgree n and $D_i = \partial/\partial x_i$.

1.3.2 Landau-Kolmogorov Type Inequalities

Here we will discuss another type of inequality concerned with the norms of a function and its derivatives. These results are often associated with the names Landau, Kolmogorov and many others. This section includes a brief discussion of some of these inequalities. For more details and the corresponding references see DeVore and Lorentz [12] and Kwong and Zettl [37].

Consider the Landau-Kolmogorov type inequality

$$\|f^{(k)}\| \leqslant c_{k,n}\|f\|^{1-\frac{k}{n}}\|f^{(n)}\|^{\frac{k}{n}}.$$

Its study was initiated by Landau in 1913 on $\mathbb{R}^+$ and by Hadamard in 1914 in the whole line case, when $n = 2$ and $k = 1$ with supremum norm. Landau showed that in $\mathbb{R}^+$

$$\|Df\|_\infty \leqslant 2\|f\|_\infty^{1-\frac{1}{2}}\|f''\|_\infty^{\frac{1}{2}}, \tag{1.5}$$

and the constant $c_{1,2} = 2$ is the sharp constant in (1.5), and Hadamard proved that $c_{1,2} = \sqrt{2}$ in $\mathbb{R}$. Later, in 1932, Hardy and Littlewood investigated an extended version of Landau and Hadamard inequalities. They considered the spaces $L_2(\mathbb{R})$ and $L_2(\mathbb{R}^+)$, and they found the best constant $c_{1,2} = 1$ and $c_{1,2} = \sqrt{2}$, respectively.

In 1939, Kolmogorov generalised the Landau result to higher derivatives and he succeeded in finding the best constant $c_{k,n}$. He proved

Theorem 1.31. *Let $Df, \cdots, D^n f$ be bounded continuous functions on $\mathbb{R}$. Then for $m = 2, 3, \cdots$ and $0 \leqslant k \leqslant m$ we have*

$$\|f^{(k)}\|_\infty \leqslant c_{m,k}\|f\|_\infty^{1-\frac{k}{m}}\|f^{(m)}\|_\infty^{\frac{k}{m}},$$

where

$$c_{m,k} = k_{m-k} k_m^{-1+\frac{k}{m}}, \tag{1.6}$$

with

$$k_m = \begin{cases} \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{1}{(2j+1)^{m+1}}, & if \quad m \text{ is odd}, \\ \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^{m+1}}, & if \quad m \text{ is even}, \end{cases}$$

is the best constant.

In 1957, Stein extended the Kolmogorov inequality to the Lebesgue L_p space. Define the space $W^m(B) = \{f : \mathbb{R} \to \mathbb{R} | f^{(m-1)}$ is absolutely continuous and $f^{(m)} \in B\}$ where B is a Banach space. He proved the following theorem.

Theorem 1.32. *Let $f \in W^m(L_p(\mathbb{R}))$. For $1 \leqslant p \leqslant \infty$,*

$$\|f^{(k)}\|_{L_p(\mathbb{R})} \leqslant c_{m,k} \|f\|_{L_p(\mathbb{R})}^{1-\frac{k}{m}} \|f^{(m)}\|_{L_p(\mathbb{R})}^{\frac{k}{m}}, \quad k = 1, \cdots, m-1,$$

where $c_{m,k}$ is given in (1.6).

Similar to the Markov-Bernstein inequality, there are many generalisations of the Landau-Kolmogorov inequality concerned with different differential operators, different domains and classes of functions with various constraints. It also has been extended to functions in several variables. For instance, Ditzain [16] proved for $f \in C(\mathbb{R}^d)$ and $\triangle^n f \in L_\infty(\mathbb{R}^d)$,

$$\left\| \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_d} f \right\|_\infty \leqslant M_{m,k} \|f\|_\infty^{1-\frac{k}{2n}} \|\triangle^n f\|_\infty^{\frac{k}{2n}},$$

is valid for $0 < k < 2n$ and not valid for $k = 2n$, where $\triangle^n = \triangle(\triangle^{n-1})$, $\triangle$ is the Laplacian:

$$\triangle = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}.$$

Recently, Ditzian [17] gave a procedure to obtain a Kolmogorov-type inequality from Jackson-type inequality and Bernstein-type inequality. Since this result has an important role in our result given in Chapter 5, we will state it here and cite [17] for the proof.

Let $\varphi_{l,k}$ be a collection of functions such that span $\{\varphi_{l,k} : k = 0, 1, \cdots$ *and* $l = 1, \cdots, v(k) < \infty\}$ is dense in the Banach space B. Define the best approximation $E_n(f)_B$ by

$$E_n(f)_B = \min\{\|f - \varphi\|_B \ : \ \varphi \in \text{span}\,\{\varphi_{l,k} : k < n,\ l \leqslant v(k)\}\}.$$

A sequence σ_n is decreasing geometrically if

$$\sigma_n \geqslant \sigma_{n+1} > 0 \quad \textit{and} \quad 1 < A \leqslant \sigma_n/\sigma_{Ln} \leqslant C, \tag{1.7}$$

for all n and some $L > 0$. A sublinear operator Q is an operator satisfying

$$\|Q(\alpha f)\|_B = |\alpha|\|Q(f)\|_B \ \ \textit{and} \ \ \|Q(f+g)\|_B \leqslant \|Qf\|_B + \|Qg\|_B,$$

and define $D(Q) = \{f : Q(f) \in B\}$.

Theorem 1.33. *Suppose for a Banach space B, for $\{\varphi_{l,k}\}_{l \leqslant v(k)}$ whose span is dense in B, for σ_n satisfying (1.7), and the two sublinear operators P and Q, the following assertions are satisfied:*

1. *$\varphi_{l,k} \in D(P) \subset B$, $\varphi_{l,k} \in D(Q) \subset B$ for all l and k and $P\varphi_{l,0} = 0$ if $Q\varphi_{l,0} = 0$.*
2. *The Jackson-type inequality*

   $$E_n(f)_B \leqslant C_J \sigma_n^r \|Qf\|_B,$$

   is satisfied for all $f \in D(Q)$ with C_J independent of n and f .
3. *The Bernstein-type inequality*

   $$\|P\varphi\|_B \leqslant C_B \sigma_n^{-m} \|\varphi\|_B,$$

   for $\varphi \in$ span$\{\varphi_{l,k} : l \leqslant v(k),\ k \leqslant n\}$ with $m < r$ is satisfied with C_B independent of n and φ.

Then, $D(Q) \subset D(P)$, and the Kolmogorov-type inequality

$$\|Pf\|_B \leqslant C_K \|Qf\|_B^{m/r} \|f\|_B^{1-m/r},$$

is satisfied for all $f \in D(Q)$ *with* C_K *independent of* f.

This chapter provided an overview of the most important results in Hilbert space and linear operators which will be required to study spherical harmonic functions in the next chapters. The main reason of introducing and explaining the Lebesgue measure in this chapter is to introduce the geometric properties of it in the Euclidean space $\mathbb{R}^d$. These properties are essential in the proof of our main theorem on m-term approximation given in Chapter 5. Since we also used Bernstein's inequality in $\mathbb{C}^d$ and establish Jackson's and Kolmogorov's inequalities in the same space, we aimed in the last section of this chapter to enrich the knowledge of the reader with a summary of some published work in the same area.

Chapter 2

Orthogonal Polynomials

In this chapter we are concerned with the Hilbert spaces $L_2([a, b]; w)$, which are defined in Chapter 1, and orthogonal polynomials with respect to the inner products defined on these spaces. All orthogonal polynomials have a number of important properties in common. Therefore, the most basic properties of orthogonal polynomials that can be identified are examined in the first section. Next, we specialize the discussion to the classical orthogonal polynomials, and then we are going to discuss these polynomials from the point of view of the theory of linear operators. In the last section of this chapter the Gegenbauer generating function will be shown as a Maclaurin series expansion. The main point of this proof is to show that this series can be derived from many approaches which are beyond the scope of this thesis. For much of this material, with a slightly differing point of view, we cite [19], [6], [11], [31] and [55].

2.1 Definition and Properties

For the space $L_2([a, b]; w)$, the definition of orthogonal (orthonormal) functions will be as following;

Definition 2.1. 1. A collection of functions $\{f_\alpha : \alpha \in \Lambda\}$ is said to be orthogonal with respect to w if

$$\int_a^b f_\alpha(x) f_\beta(x) w(x) dx = 0, \quad \forall \alpha, \beta \in \Lambda \ , \ \alpha \neq \beta.$$

2. A collection of orthogonal functions $\{f_\alpha : \alpha \in \Lambda\}$ with respect to w is said to be orthonormal with respect to $w(x)$ if

$$\int_a^b (f_\alpha(x))^2 w(x) dx = 1, \quad \forall \alpha \in \Lambda.$$

Theorem 2.1. *For every nonnegative integrable function w on $[a, b]$, if the moments*

$$\int_a^b x^n w(x) dx$$

exist and are finite for $n = 0, 1, 2, \cdots$, then there is a unique sequence of polynomials $(p_n)_{n=0}^{\infty}$ with the following properties:

- $p_n(x) = \gamma_n x^n + q(x)$
  where $\gamma_n > 0$, and $q(x)$ is a polynomial of degree $n - 1$.
- *The collection $(p_n)_{n=0}^{\infty}$ is orthonormal.*

Proof. The set $S = \{1, x, x^2, \cdots\}$ is linearly independent set. Then by using Gram-Schmidt theorem there exists an orthonormal set $\{p_0, p_1, p_2, \cdots\}$ in $L_2([a, b], w)$ such that

$$\text{span}\left\{1, x, x^2, \cdots, x^m\right\} = \text{span}\left\{p_0, p_1, p_2, \cdots, p_m\right\}.$$

Therefore,

$$p_n(x) = \sum_{k=0}^{n} \gamma_k x^k = \gamma_n x^n + \sum_{k=0}^{n-1} \gamma_k x^k = \gamma_n x^n + q(x),$$

where q is a polynomial of degree $n - 1$. On the other hand, by following Gram Schmidt process the polynomial p_n has the form

$$p_n = \frac{t_n}{\|t_n\|_{L_2([a,b],w)}}, \quad t_n = x^n - \sum_{n=0}^{n-1} \langle x^n, p_k \rangle_{L_2([a,b],w)} p_k,$$

which means that $\gamma_n = \frac{1}{\|t_n\|} > 0$. $\square$

The polynomials p_n are called the orthogonal polynomials corresponding to w. The constant γ_n is called the leading coefficient and $p_n(x)/\gamma_n = x^n + \cdots$ is called the monic orthogonal polynomial.

Theorem 2.2. *Let $\{p_n\}_{n\geq 0}$ be a given orthonormal collection, finite or infinite, and $f \in L_2([a,b],w)$. Then the best approximation to f in terms of the polynomials $\{p_n\}_{n=0}^{l}$ for fixed integer l, has the form*

$$f(x) \sim f_0p_0 + f_1p_1 + \ldots + f_lp_l,$$

where $f_n = \int_a^b f(x)p_n(x)w(x)dx$. The coefficients f_n is called the Fourier coefficients of f with respect to the given system, and the expansion

$$f(x) \sim f_0p_0 + f_1p_1 + \ldots$$

is called Fourier expansion.

Proof. This proof is a special case of the general proof in the inner product space, which is available in many sources; see for example Borwein and Erdelyi [6]. Let $f \in L_2([a,b],w)$ and let

$$g = f_0p_0 + f_1p_1 + \cdots + f_np_n,$$

where $f_i = \langle f, p_i\rangle_{L_2([a,b],w)} = \int_a^b fp_iw$. Suppose $h \in \text{span}\,\{p_0, p_1, \cdots, p_n\}\,, \quad h \neq g$. In other words, the function h takes the form $h = \sum_{i=0}^{n} c_ip_i$. Note that,

$$\begin{aligned}
\langle g - f, p_j\rangle_{L_2([a,b],w)} &= \left\langle \left(\sum_{i=0}^{n} f_ip_i\right) - f, p_j \right\rangle_{L_2([a,b],w)} \\
&= \left\langle \left(\sum_{i=0}^{n} f_ip_i\right), p_j \right\rangle_{L_2([a,b],w)} - \langle f, p_j\rangle_{L_2([a,b],w)} \\
&= \sum_{i=0}^{n} f_i \langle p_i, p_j\rangle_{L_2([a,b],w)} - \langle f, p_j\rangle_{L_2([a,b],w)} \\
&= f_j - f_j = 0
\end{aligned}$$

as $\{p_0, p_1, \cdots, p_n\}$ is an orthonormal set. Consequently, we get

$$\begin{aligned}
\langle g-f, h-g\rangle_{L_2([a,b],w)} &= \langle g-f, h\rangle_{L_2([a,b],w)} - \langle g-f, g\rangle_{L_2([a,b],w)} \\
&= \left\langle g-f, \sum_{i=0}^{n} c_i p_i \right\rangle_{L_2([a,b],w)} - \left\langle g-f, \sum_{j=0}^{n} f_j p_j \right\rangle_{L_2([a,b],w)} \\
&= \sum_{i=0}^{n} c_i \langle g-f, p_i\rangle_{L_2([a,b],w)} - \sum_{j=0}^{n} f_j \langle g-f, p_j\rangle_{L_2([a,b],w)} = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|h-f\|^2_{L_2([a,b],w)} &= \|(h-g)+(g-f)\|^2_{L_2([a,b],w)} \\
&= \|h-g\|^2_{L_2([a,b],w)} + 2\langle h-g, g-f\rangle_{L_2([a,b],w)} + \|g-f\|^2_{L_2([a,b],w)} \\
&= \|h-g\|^2_{L_2([a,b],w)} + \|g-f\|^2_{L_2([a,b],w)} > \|g-f\|^2_{L_2([a,b],w)},
\end{aligned}$$

as $\|g-h\| \neq 0$, and this complete the proof. □

For more general properties of orthogonal polynomials we refer the reader to [55].

2.2 Classical Orthogonal Polynomials

There is no general definition of classical orthogonal polynomials, but most authors describe them by those satisfying a linear second order differential equation. Indeed, these polynomials have a great deal in common and possess many properties that no other set of orthogonal polynomials has, see e.g., [56] . These properties are:

- Their derivatives are again orthogonal polynomial sets.
- They all satisfy a *Rodrigues formula*;

  $$p_n(x) = \frac{1}{k_n w_n(x)} \frac{d^n}{dx^n} \left(w(x)\sigma^n(x)\right), \quad n = 1, 2, \cdots$$

  where $w(x)$ is a nonnegative function, k_n is a constant, and σ is a polynomial in x and independent of n.
- They are all orthogonal with respect to a weight function.

- They all satisfy a non linear equation of the form,

$$\sigma(x)\frac{d}{dx}\left\{p_n(x)p_{n-1}(x)\right\} = (\alpha_n x + \beta_n)p_n(x)p_{n-1}(x) + \gamma_n p_n^2(x) + \delta_n p_{n-1}^2(x),$$

  where $\alpha_n,\ \beta_n,\ \gamma_n$ and δ_n are constants independent of x.

- They all satisfy a differential equation of the form

$$A(x)y^{''} + B(x)y^{'} + \lambda_n y = 0,$$

  where $A(x)$ and $B(x)$ are independent of n, and λ_n is independent of x.

- They satisfy a three term recurrence formula of the form

$$p_n(x) = (A_n x + B_n)\, p_{n-1}(x) - C_n p_{n-2}(x), \quad n = 2, 3, \cdots$$

  where A_n, B_n and C_n are constants, $A_n > 0$ and $C_n > 0$.

Besides satisfying the above properties, any of these properties characterizes the classical orthogonal polynomials. In other words, any system of orthogonal polynomials which satisfies these properties can be reduced to a classical system (see Al-Salam [3]). They are the most widely used and they have many important applications. Here are some examples of the most common classical orthogonal polynomials with some powerful formulas:

1. **Hermite polynomials:**
   Rodrigues' Formula;

$$H_n(x) = \frac{(-1)^n}{\exp(-x^2)}\frac{d^n}{dx^n}\exp(-x^2).$$

   Then $\{H_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials on the real line $(-\infty, \infty)$ associated with the function $w(x) = \exp(-x^2)$. Furthermore,

$$H_{2n+1}(0) = 0, \quad H_{2n}(0) = (-1)^n\frac{(2n)!}{n!},$$

   and the explicit form

$$H_n(x) = n!\sum_{m=0}^{[n/2]}\frac{(-1)^m(2x)^{n-2m}}{m!(n-2m)!}.$$

The generating function is

$$\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{n!} = \exp(2xz - z^2).$$

Theorem 2.3. *The Hermite polynomials $\{H_n\}_{n=0}^{\infty}$ form a complete orthogonal set in the space $L_2\left((-\infty,\infty), e^{-x^2}\right)$. Then when f in $L_2\left((-\infty,\infty), e^{-x^2}\right)$,*

$$f(x) = \sum_{n=0}^{\infty} c_n H_n(x),$$

where

$$c_n = \frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} H_n(x) f(x) e^{-x^2} dx.$$

2. **Laguerre polynomials:**

The Rodrigues' Formula is given by;

$$L_n^{(\alpha)}(x) = \frac{1}{n!\exp(-x)x^{\alpha}} \frac{d^n}{dx^n}\left(\exp(-x)x^{\alpha+n}\right).$$

Then $\left\{L_n^{(\alpha)}\right\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials on $[0,\infty)$ associated with the function $w(x) = x^{\alpha}\exp(-x)$. Furthermore,

$$L_n^{(\alpha)}(0) = \begin{pmatrix} n+\alpha \\ n \end{pmatrix},$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!},$$

and $\binom{n}{0} = 1$. The explicit form is

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} \begin{pmatrix} n+\alpha \\ n-m \end{pmatrix} x^m.$$

The generating function is

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) z^n = \exp\left(\frac{xz}{z-1}\right)(1-z)^{-\alpha-1}.$$

Theorem 2.4. *The Laguerre polynomials $\{L_n^{\alpha}\}_{n=0}^{\infty}$ form a complete orthogonal set in the space $L_2\left((-\infty,\infty), x^{\alpha}e^{-x}\right)$.*

Consequently, if f in $L_2\left((-\infty,\infty), x^{\alpha}e^{-x}\right)$,

$$f(x) = \sum_{n=0}^{\infty} c_n L_n^{\alpha}(x),$$

where

$$c_n = \frac{1}{\Gamma(\alpha+n+1)/n!}\int_0^{\infty} L_n^{\alpha}(x) f(x) x^{\alpha} e^{-x} dx.$$

3. **Jacobi polynomials:**
The Rodrigues' formula is given by

$$P_n^{(\alpha,\beta)}(x) = (-1)^n \frac{2^{-n}}{n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha}(1+x)^{\beta}(1-x^2)^n\right].$$

Then $\left\{P_n^{(\alpha,\beta)}\right\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials on $[-1,1]$ associated with the function $w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad -1 < \alpha, \beta < \infty$. Furthermore, they satisfy

$$P_n^{(\alpha,\beta)}(1) = \left(\begin{array}{c} n+\alpha \\ n \end{array}\right),$$

and one of the explicit form is given by

$$P_n^{(\alpha,\beta)}(x) = \sum_{m=0}^{n} \frac{1}{2^n}\left(\begin{array}{c} n+\alpha \\ m \end{array}\right)\left(\begin{array}{c} n+\beta \\ n-m \end{array}\right)(x-1)^{n-m}(x+1)^m.$$

There are several generating functions of Jacobi polynomials, for example;

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) z^n = \frac{2^{\alpha+\beta}}{R(1-z+R)^{\alpha}(1+z+R)^{\beta}}$$

and

$$(1+w^2)^{-\lambda} F\left(\frac{\lambda}{2}, \frac{\lambda+1}{2}, j, \frac{4t^2w^2}{(1+w^2)^2}\right) = \sum_{n=0}^{\infty} w^{2n}\frac{(\lambda)_n}{(j)_n} P_n^{(\lambda-j,j-1)}\left(2t^2-1\right). \tag{2.1}$$

where $R = \sqrt{1-2xz+z^2}$, and F is the Gauss hypergeomteric function given in Chapter 4. The generating function (2.1) can be constructed geometrically. A consequence of the fact that the Jacobi polynomials are the zonal functions for the complex projective spaces, these polynomials can be realised as spherical averages of Gegenbauer polynomials which leads to the generating function (2.1). The geometric framework for all these results will be presented in Chapters 3 and 4.

Theorem 2.5. *The Jacobi polynomials* $\left\{P_n^{(\alpha,\beta)}\right\}_{n=0}^{\infty}$ *form a complete orthogonal set in the space* $L_2\left((-1,1),(1-x)^{\alpha}(1+x)^{\beta}\right)$.

Then if f in $L_2((-1,1),w)$ and $w=(1-x)^{\alpha}(1+x)^{\beta}$,

$$f(x)=\sum_{n=0}^{\infty}c_nP_n^{(\alpha,\beta)}(x),$$

where

$$c_n=\frac{1}{\left\|P_n^{(\alpha,\beta)}\right\|_{L_2((-1,1),w)}}\int_{-1}^{1}P_n^{(\alpha,\beta)}(x)f(x)(1-x)^{\alpha}(1+x)^{\beta}dx.$$

Also

$$\left\|P_n^{(\alpha,\beta)}\right\|_{L_2((-1,1),w)}=\frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(\alpha+\beta+2n+1)n!\Gamma(\alpha+\beta+n+1)}$$

There are numbers of important subclasses of Jacobi polynomials, related to specific value of α and β;

(i) The Legendre polynomials P_n is defined as

$$P_n=P_n^{(0,0)},\quad n=0,1,\cdots,$$

where

$$\int_{-1}^{1}P_n(x)P_m(x)dx=0 \tag{2.2}$$

(ii) The Chebyshev polynomials T_n

$$T_n=4^n\begin{pmatrix}2n\\n\end{pmatrix}^{-1}P_n^{(-1/2,-1/2)},\quad n=0,1,2,\cdots.$$

(iii) The Ultraspherical (Gegenbauer) polynomial C_n^{λ} is defined as follows

$$C_n^{\lambda}(x)=\frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)}\frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+1/2)}P_n^{(\lambda-1/2,\lambda-1/2)}(x),$$

where $\lambda > -1/2$. It also can be given by the following formulas, see e.g., [55];

$$C_n^{\lambda}(\cos\vartheta) = \sum_{j=0}^{[n/2]} \begin{pmatrix} j-1+\lambda \\ j \end{pmatrix} \begin{pmatrix} n-j+\lambda-1 \\ n-j \end{pmatrix} 2\cos(n-2j)\vartheta$$

where $\vartheta \in [0,\pi]$, and

$$C_n^{\lambda}(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m\Gamma(n-m+\lambda)}{\Gamma(\lambda)\Gamma(m+1)\Gamma(n-2m+1)}\,(2x)^{n-2m}.$$

which will be use to prove the generating function formula

$$\sum_{n=0}^{\infty} C_n^{\lambda}(x)z^n = (1-2xz+z^2)^{-\lambda},$$

at the end of this chapter.

The Gegenbauer polynomials will be discussed in more details in the following chapter. It will describe their relationship to the theory of harmonic functions which have a great importance in the central part of our work.

2.3 Differential Operators and Orthogonal Polynomials

The theory of linear operators gives an entirely different approach to the treatment of orthogonal polynomials. The classical orthogonal polynomials are considered as eigenfunctions of a special class of self-adjoint operators arising from second order differential equations. In this section we are going to concentrate on the classical orthogonal polynomials which were constructed from the theory of linear operators.
Consider equations of the form,

$$(Lu)(x) + \lambda u(x) = 0, \quad a \le x \le b, \tag{2.3}$$

where

$$L = \frac{1}{w(x)}\frac{d}{dx}\left(p\frac{d}{dx}\right) + q,$$

and λ is a constant parameter. We assume $w(x) > 0$ except possibly at isolated points where $w(x) = 0$. We notice that if u satisfies (2.3) for a parameter λ, then u is an eigenfunction of L corresponding to λ, which is the eigenvalue. In order to give very important examples we need to prove that L is a self-adjoint operator in $L_2((a,b),w)$ when $p(a) = p(b) = 0$, or in case of infinite interval we assume that $\lim_{x\to\infty} p(x) = \lim_{x\to-\infty} p(x) = 0$. To do this we need to show that $L = L^*$. In other words, we need to prove that $\langle Lu, v\rangle_{L_2((a,b),w)} = \langle u, Lv\rangle_{L_2((a,b),w)}$, where

$$\langle Lu, v\rangle_{L_2((a,b),w)} = \int_a^b (Lu)(x)v(x)w(x)dx.$$

Since

$$\langle Lu, v\rangle_{L_2((a,b),w)} = \int_a^b \frac{d}{dx}\left(p(x)\frac{du(x)}{dx}\right)v(x) + u(x)v(x)w(x)q(x)dx,$$

then by integration by parts, we get

$$\begin{aligned}
\langle Lu, v\rangle_{L_2((a,b),w)} &= v(x)p(x)u'(x)]_a^b - \int_a^b v'(x)p(x)u'(x)dx \\
&+ \int_a^b u(x)v(x)q(x)w(x)dx \\
&= v(b)p(b)u'(b) - v(a)p(a)u'(a) - \int_a^b v'(x)p(x)u'(x)dx \\
&+ \int_a^b u(x)v(x)q(x)w(x)dx \\
&= -\int_a^b v'(x)p(x)u'(x)dx + \int_a^b u(x)v(x)q(x)w(x)dx.
\end{aligned}$$

By using integration by parts again, we have

$$\begin{aligned}
\langle Lu, v\rangle_{L_2((a,b),w)} &= -v'(b)p(b)u(b) + v'(a)p(a)u(a) \\
&+ \int_a^b u(x)\left(v''(x)p(x) + v'(x)p'(x)\right)dx \\
&+ \int_a^b u(x)v(x)q(x)w(x)dx \\
&= \int_a^b u(x)\left(\frac{1}{w(x)}\frac{d}{dx}\left(p(x)\frac{dv(x)}{dx}\right) + q(x)v(x)\right)w(x)dx \\
&= \langle u, Lv\rangle_{L_2((a,b),w)},
\end{aligned}$$

so that L is a self-adjoint operator. Consequently, if there are two eigenfunctions corresponding to two different eigenvalues then they are orthogonal.

Example 2.1. *(Jacobi Operator)*
The Jacobi differential equation is

$$(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d}{dx}\left[(1-x)^{1+\alpha}(1+x)^{1+\beta}\frac{du}{dx}\right]+\lambda u=0, \qquad -1\le x\le 1,$$

and the Jacobi operator is

$$Lu=(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d}{dx}\left[(1-x)^{1+\alpha}(1+x)^{1+\beta}\frac{du}{dx}\right].$$

The eigenfunctions are the Jacobi polynomials

$$P_n^{(\alpha,\beta)}=\frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right],$$

corresponding to the eigenvalues $\lambda_n=n(n+\alpha+\beta+1), \quad n=1,2,\cdots$. *Consequently, the eigenvalues of the Gegenbauer, Legender and Chebyshev polynomials are* $\lambda_n=n(n+2\gamma), \lambda_n=n(n+1)$ *and* $\lambda_n=n^2$, *respectively, where* $\gamma-1/2=\alpha=\beta$ *and* $n=1,2,\cdots$.

Example 2.2. *(Laguerre Operator)*
The Laguerre differential equation is

$$e^x\frac{d}{dx}\left[xe^{-x}\frac{du}{dx}\right]+\lambda u=0, \qquad 0\le x\le\infty,$$

with the Laguerre operator

$$Lu=\frac{d}{dx}\left[xe^{-x}\frac{du}{dx}\right].$$

The eigenvalues are $\lambda_n=n$ *and the corresponding eigenfunction is the Laguerre polynomials.*

Example 2.3. *(Hermite Operator)*
The Hermite differential equation is

$$e^{x^2}\frac{d}{dx}\left[e^{-x^2}\frac{du}{dx}\right]+\lambda u=0, \qquad -\infty\le x\le\infty,$$

and in the same way, the Hermite operator is

$$Lu = \frac{d}{dx}\left[e^{-x^2}\frac{du}{dx}\right].$$

The eigenvalues are $\lambda_n = 2n$ and the corresponding eigenfunction are the Hermite polynomials.

2.4 Proof of the Generating Function of Gegenbauer Polynomials

One of our aims in the next chapters is to study and prove Gegenbauer's generating function for Gegenbauer polynomials C_n^λ:

$$\left(1-2xr+r^2\right)^{-\lambda} = \sum_{n=0}^{\infty} r^n C_n^\lambda(x), \quad x \in [-1,1],\ r \in (-1,1),$$

from the geometric point of view. In fact, in the literature there are several analytic proofs of this generating function, see for example [23] and [55]. Hence, in this section we present an analytic proof of it using Maclaurin series and the Binomial Theorem, which is given below.

Theorem 2.6. *(The Binomial Theorem)*
For any real number α and complex number x in $|x| < 1$ the following formula holds,

$$(1+x)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n.$$

Proof. The function $\left(1-2xr+r^2\right)^{-\lambda}$ has derivatives of all orders with respect to r throughout the interval $(-1,1)$. Then the Maclaurin series generated by this function at r is given by

$$\left(1-2xr+r^2\right)^{-\lambda} = \sum_{n=0}^{\infty} r^n \frac{1}{n!}\left(\frac{\partial^n}{\partial r^n}\left(1-2xr+r^2\right)^{-\lambda}\right)(0).$$

Since the function $\left(1-2xr+r^2\right)^{-\lambda}$ can be written as

$$\left(1-r\left(x+\sqrt{x^2-1}\right)\right)^{-\lambda}\left(1-r\left(x-\sqrt{x^2-1}\right)\right)^{-\lambda},$$

then,

$$\left(\frac{\partial^n}{\partial r^n}\left(1-2xr+r^2\right)^{-\lambda}\right)(0)=$$

$$\sum_{j=0}^{n}\binom{n}{j}\left(\left(1-r\left(x+\sqrt{x^2-1}\right)\right)^{-\lambda}\right)^{(j)}\left(\left(1-r\left(x-\sqrt{x^2-1}\right)\right)^{-\lambda}\right)^{(n-j)}(0).$$

Since $x\in[-1,1]$ and $|r|<1$, we can expand the above functions by using the Binomial theorem in the form

$$\left(1-r\left(x+\sqrt{x^2-1}\right)\right)^{-\lambda}=\sum_{k=0}^{\infty}\binom{-\lambda}{k}(-1)^k\left(x+\sqrt{x^2-1}\right)^k r^k. \tag{2.4}$$

From calculus we know that if the series

$$\sum_{k=0}^{\infty}\frac{\partial^m}{\partial r^m}\left(\binom{-\lambda}{k}(-1)^k\left(x+\sqrt{x^2-1}\right)^k r^k\right) \tag{2.5}$$

converges uniformly, then we can compute the m derivative of 2.4 by differentiate the series term by term. Since for $k\geqslant m$

$$\begin{aligned}&\frac{\partial^m}{\partial r^m}\left(\binom{-\lambda}{k}(-1)^k\left(x+\sqrt{x^2-1}\right)^k r^k\right)\\ =\ &\binom{-\lambda}{k}(-1)^k\left(x+\sqrt{x^2-1}\right)^k k(k-1)\cdots(k-(m-1))r^{k-m}\\ =\ &\binom{-\lambda}{k-m}(-1)^k\left(x+\sqrt{x^2-1}\right)^k r^{k-m}.\end{aligned}$$

Then for $|r|<1$ we have

$$\left|\frac{\partial^m}{\partial r^m}\left(\binom{-\lambda}{k}(-1)^k\left(x+\sqrt{x^2-1}\right)^k r^k\right)\right|\leqslant\binom{-\lambda}{k-m}\left(x+\sqrt{x^2-1}\right)^k,$$

and from the Binomial theorem the series

$$\sum_{k=m}^{\infty}\binom{-\lambda}{k-m}\left(x+\sqrt{x^2-1}\right)^k=\sum_{k=0}^{\infty}\binom{-\lambda}{k}\left(x+\sqrt{x^2-1}\right)^k\left(x+\sqrt{x^2-1}\right)^m$$

converges to $\left(x+\sqrt{x^2-1}\right)^m\left(1+x+\sqrt{x^2-1}\right)^{-\lambda}$. Hence, from Weierstrass M-test we can deduce that the series 2.5 converges uniformly. Thus the m

derivative of 2.4 when $r = 0$ is given by

$$\left(\frac{\partial^m}{\partial r^m}\left(1 - r\left(x + \sqrt{x^2-1}\right)\right)^{-\lambda}\right)(0) = \binom{-\lambda}{m}(-1)^m m!\left(x + \sqrt{x^2-1}\right)^m.$$

Similarly,

$$\left(\frac{\partial^m}{\partial r^m}\left(1 - r\left(x - \sqrt{x^2-1}\right)\right)^{-\lambda}\right)(0) = \binom{-\lambda}{m}(-1)^m m!\left(x - \sqrt{x^2-1}\right)^m.$$

Then we have

$$\begin{aligned}
& \frac{1}{n!}\left(\frac{\partial^n}{\partial r^n}\left(1 - 2xr + r^2\right)^{-\lambda}\right)(0) \\
= & \sum_{j=0}^{n}\binom{n}{j}\binom{-\lambda}{j}\binom{-\lambda}{n-j}\frac{j!(n-j)!}{(-1)^n n!}\left(x - \sqrt{x^2-1}\right)^j\left(x + \sqrt{x^2-1}\right)^{n-j} \\
= & \sum_{j=0}^{n}\binom{-\lambda}{j}\binom{-\lambda}{n-j}(-1)^n\left(x - \sqrt{x^2-1}\right)^j\left(x + \sqrt{x^2-1}\right)^{n-j} \\
= & \sum_{j=0}^{[n/2]}\binom{-\lambda}{j}\binom{-\lambda}{n-j}(-1)^n[\left(x - \sqrt{x^2-1}\right)^j\left(x + \sqrt{x^2-1}\right)^{n-j} \\
+ & \left(x - \sqrt{x^2-1}\right)^{n-j}\left(x + \sqrt{x^2-1}\right)^j] \\
= & \sum_{j=0}^{[n/2]}\binom{-\lambda}{j}\binom{-\lambda}{n-j}(-1)^n\left(x - \sqrt{x^2-1}\right)^j\left(x + \sqrt{x^2-1}\right)^j \\
\times & \left[\left(x - \sqrt{x^2-1}\right)^{n-2j} + \left(x + \sqrt{x^2-1}\right)^{n-2j}\right] \\
= & \sum_{j=0}^{[n/2]}\binom{-\lambda}{j}\binom{-\lambda}{n-j}(-1)^n\left[\left(x - \sqrt{x^2-1}\right)^{n-2j} + \left(x + \sqrt{x^2-1}\right)^{n-2j}\right].
\end{aligned}$$

By setting $x = \cos\vartheta$, we get

$$\left(\frac{\partial^n}{\partial r^n}\frac{\left(1 - 2xr + r^2\right)^{-\lambda}}{n!}\right)(0)$$

$$\begin{aligned}
= & \sum_{j=0}^{[n/2]}\binom{-\lambda}{j}\binom{-\lambda}{n-j}(-1)^n\left(e^{i(n-2j)\vartheta} + e^{-i(n-2j)\vartheta}\right) \\
= & \sum_{j=0}^{[n/2]}\binom{-\lambda}{j}\binom{-\lambda}{n-j}(-1)^n 2\cos(n-2j)\vartheta
\end{aligned}$$

$$\begin{aligned} &= \sum_{j=0}^{[n/2]} \binom{j-1+\lambda}{j} \binom{n-j+\lambda-1}{n-j} 2\cos(n-2j)\vartheta \\ &= C_n^\lambda(\cos\vartheta) = C_n^\lambda(x), \end{aligned}$$

since

$$\binom{-\lambda}{j}\binom{-\lambda}{n-j}(-1)^n = \binom{j-1+\lambda}{j}\binom{n-j-1+\lambda}{n-j}.$$

Therefore, the Taylor series becomes

$$\left(1-2xr+r^2\right)^{-\lambda} = \sum_{n=0}^{\infty} r^n C_n^\lambda(x), \quad x \in [-1,1],\ r \in (-1,1). \quad \square$$

Chapter 3

Spherical Harmonics on Euclidean Spaces

The last chapter shows the connection of orthogonal polynomials with operator theory, and how Jacobi and Gegenbauer polynomials naturally arise in that area. Another approach to generating these polynomials is the theory of spherical harmonics. They appear as particular solutions of Laplace's equation on the spheres. This chapter is designed to develop spherical harmonics on Euclidean spaces from a theoretical perspective, and to set out the critically important results for our geometric applications. We assume throughout this chapter that $n \geqslant 2$, since the theory is different when $n = 1$. For further background in this area we cite [2], [18], [19], [22], [41], [47], [52], [54] and [55].

3.1 Laplace Operator and Basic Properties of Harmonic Functions

Definition 3.1. Let $u : D \to \mathbb{R}$ be twice continuously differentiable function defined in a domain D in d-dimensional Euclidean space. The *Laplace operator*, or *Laplacian* Δ is defined as

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}.$$

If the equation $\Delta u = 0$ is satisfied at each point of a domain D, we say that u is harmonic in D or, simply, that u is a harmonic function.

According to the relationship between real valued and complex valued functions, a complex valued function is harmonic if its real and imaginary parts are harmonic. Moreover, since the Laplacian is linear on the space of all twice continuously differentiable functions on a domain, the sum and scaler multiplication of harmonic functions are harmonic on the same domain. Also, a translation of a harmonic function is harmonic. More precisely, if u is a harmonic function on D and $a \in \mathbb{R}^d$ then the translated function $u(x-a)$ defined on $D+a$ is harmonic.

According to the definition of polar coordinates in the n-dimensional Euclidean space, the Laplace operator in polar coordinates is given by

$$\begin{aligned}
\Delta u &= r^{-(d-1)}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}u\right)+r^{-2}\left(\sin\vartheta_1\right)^{-(d-2)}\frac{\partial}{\partial\vartheta_1}\left(\left(\sin\vartheta_1\right)^{d-2}\frac{\partial}{\partial\vartheta_1}u\right)\\
&+ r^{-2}\left(\sin\vartheta_1\right)^{-2}\left(\sin\vartheta_2\right)^{-(d-1)}\frac{\partial}{\partial\vartheta_2}\left(\left(\sin\vartheta_2\right)^{d-1}\frac{\partial}{\partial\vartheta_2}u\right)\\
&+ r^{-2}\left(\sin\vartheta_1\sin\vartheta_2\right)^{-2}\left(\sin\vartheta_3\right)^{-(d-4)}\frac{\partial}{\partial\vartheta_3}\left(\left(\sin\vartheta_3\right)^{d-4}\frac{\partial}{\partial\vartheta_3}u\right)+\cdots\\
&+ r^{-2}\left(\sin\vartheta_1\sin\vartheta_2\cdots\sin\vartheta_{d-3}\right)^{-2}\left(\sin\vartheta_{d-2}\right)^{-1}\frac{\partial}{\partial\vartheta_{d-2}}\left(\left(\sin\vartheta_{d-2}\right)^{1}\frac{\partial}{\partial\vartheta_{d-2}}u\right)\\
&+ r^{-2}\left(\sin\vartheta_1\sin\vartheta_2\cdots\sin\vartheta_{d-2}\right)^{-2}\frac{\partial^2}{\partial\varphi^2}u.
\end{aligned}$$

Consequently, in the 3-dimensional Euclidean space, the Laplacian becomes

$$\Delta u = r^{-2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}u\right)+r^{-2}\left(\sin\vartheta\right)^{-1}\frac{\partial}{\partial\vartheta}\left(\left(\sin\vartheta\right)\frac{\partial}{\partial\vartheta}u\right)+r^{-2}\left(\sin\vartheta\right)^{-2}\frac{\partial^2}{\partial\varphi^2}u.$$

The remaining of this section contains the basic properties of harmonic functions in the n dimensional Euclidean spaces.

Theorem 3.1. *(Mean Value Theorem)*
Let u be harmonic in $D \subset \mathbb{R}^d$. Then, for any ball $B^r(x) = \{y \in \mathbb{R}^d \mid \|x-y\| \leqslant r\} \subset D$ the value $u(x)$ is equal to the average value of u on the sphere $\partial B^r(x)$, that is

$$u(x) = \frac{1}{c_d r^{d-1}}\int_{\partial B^r(x)} u \; d\mu_d,$$

where c_d is an absolute constant depending only on d.

An important consequence of the mean value property is the maximum principle for harmonic functions.

Theorem 3.2. *(Maximum Principle)*
Let u be a real valued harmonic function on a connected open domain D in $\mathbb{R}^d$, and u have a maximum or minimum in D. Then u is constant.

The maximum principle can be rephrased as following: a nonconstant harmonic function can attain neither its maximum nor its minimum at any point of an open set D. On the other hand, from function analysis on Euclidean spaces, we know that a continuous function on a closed bounded set must attain its maximum and minimum on that set. These facts establish the following corollary.

Corollary 3.1. *Suppose D is bounded and u is continuous on $\overline{D}$, the closure of D, and harmonic on D. Then u attains its maximum and minimum over $\overline{D}$ on its boundary ∂D.*

The corollary above implies that if u and v are continuous functions on $\overline{D}$ and harmonic on D with $u = v$ on ∂D then $u = v$ on D. In other words, on a bounded domain a harmonic functions is uniquely determined by its value on the boundary.

3.2 Spherical Harmonic and Gegenbauer Polynomials

In this section, we are concerned with spherical harmonics and Gegenbauer polynomials as spherical harmonics polynomials in the Euclidean space $\mathbb{R}^d$. Spherical harmonics are very useful to study polynomials on the unit sphere. Theorem 3.3 establishes the most important relationship between an arbitrary polynomial on the unit sphere with spherical harmonic functions, which leads to a decomposition of $L^2(\mathbb{S}^{d-1}(\mathbb{R}))$, given in Theorem 3.4. At the end of this section we study a special class of spherical harmonics and its relation to Gegenbauer polynomials. These polynomials occur as eigenfunctions of Laplace's operator on the unit sphere. We shall work on the Euclidean space $\mathbb{R}^d$ and the unit sphere $\mathbb{S}^{d-1}(\mathbb{R})$.

Definition 3.2. A function f defined on $\mathbb{R}^d$ is said to be homogeneous of degree k if $f(ax) = a^k f(x)$ for all $x \in \mathbb{R}^d$ and $a \in \mathbb{R}$.

Let $\hom(k)$ be the set of all homogeneous polynomials of degree k on $\mathbb{R}^d$ with complex coefficients. Thus, if $p \in \hom(k)$ then p consists of terms of the form $c_\alpha x^\alpha$ where α denotes an d-tuple $(\alpha_1, \cdots, \alpha_d)$ of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_d = k$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$, i.e.,

$$p(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha.$$

In order to compute the dimension of the space $\hom(k)$ we need to calculate the number of the functions x^α which is the same as the number of ways an d-tuple $\alpha = (\alpha_1, \cdots, \alpha_d)$ of nonnegative integer can be chosen so that $|\alpha| = k$. So to determine the dimension b_k of $\hom(k)$ let us consider a linearly ordered array of $d+k-1$ boxes, and let us select $d-1$ boxes out of them. Consider the number of boxes preceding the first box we chose is α_1, and the number of boxes between the first and the second boxes we chose is α_2, and so on. It is clear that in this way we get nonnegative integers $\alpha_1, \cdots, \alpha_d$ satisfying $\alpha_1 + \cdots + \alpha_d = k$, see Figure 3.1. Thus the number of ways of choosing $d-1$ boxes out of a linearly ordered array of $d+k-1$ boxes, which is $\binom{d+k-1}{d-1}$ is the same as the ways of choosing $\alpha_1, \cdots, \alpha_d$ satisfying $\alpha_1 + \cdots + \alpha_d = k$. So the dimension b_k of $\hom(k)$ is

$$\binom{d+k-1}{d-1} = \binom{d+k-1}{k}$$

Definition 3.3. The restriction to the unit sphere $\mathbb{S}^{d-1}(\mathbb{R})$ of a homogeneous harmonic polynomial of degree k is called a spherical harmonic of degree k and the space of all spherical harmonics is denoted by H_k.

Recall that a polynomial $p \in \hom(k)$ is homogeneous of degree k, which can be written as

$$p(x) = \|x\|^k p(x/\|x\|) = \|x\|^k p|_{\mathbb{S}^{d-1}(\mathbb{R})}(x), \qquad \forall x \in \mathbb{R}^d - \{0\}.$$

However, the restriction $p|_{\mathbb{S}^{d-1}(\mathbb{R})}$ is the restriction of infinitely many homogeneous polynomials of different degrees of the form

$$f(x) = \|x\|^r p|_{\mathbb{S}^{d-1}(\mathbb{R})}(x), \qquad r \in \mathbb{N}.$$

Hence, we cannot identify a homogenous function by its restriction on the unit sphere unless we know its degree. On the other hand, the maximum principle guarantees that if p is a harmonic polynomial then we can uniquely identify it by its restriction to the unit sphere. Hence, there is a one to one correspondence between the spaces H_k and the class A_k of all polynomials in $\hom(k)$ that are harmonic which leads to the fact $\dim H_k = \dim A_k$.

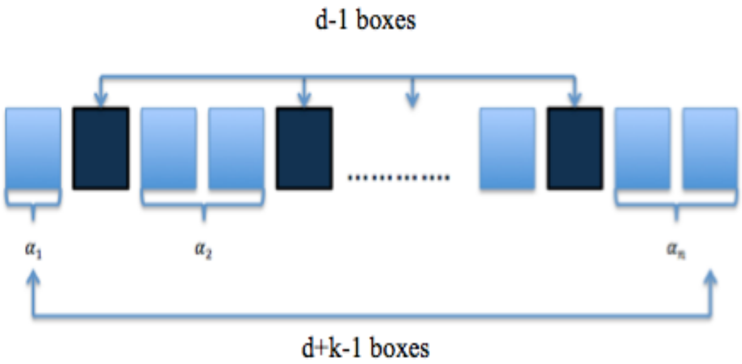

FIGURE 3.1: An example of selecting $d-1$ boxes from $d+k-1$ boxes.

$\|x\|$

The following decomposition theorem shows the relationship between polynomials and harmonic polynomials which can be found in [54].

Theorem 3.3. *If* $p \in \hom(k)$ *then*

$$p(x) = p_0(x) + \|x\|^2 p_1(x) + \cdots + \|x\|^{2l} p_l(x), \quad x \in \mathbb{R}^d,$$

where p_j *is a homogeneous harmonic polynomial of degree* $k-2j$, $j = 0, 1, \cdots, l$ *and* $l = [k/2]$.

To stress an important fact, suppose that

$$p(x) = \sum_{\alpha \in O} c_\alpha x^\alpha, \quad O \subseteq \mathbb{N}^d,$$

is a polynomial of degree m. Group all terms for which $|\alpha| = k$, where $k = 0, 1, \cdots, m$. Then the polynomial p can be written uniquely as a sum of homogeneous polynomials

$$p = p_1 + \cdots + p_m, \quad p_k \in \hom(k).$$

This result, when combined with Theorem 3.3 shows that every polynomial of degree m on the unit sphere can be written as a sum of homogeneous harmonic polynomials of degrees $0, 1, \cdots, m$. This fact with Weierstrass's approximation theorem, establishes the following remarkable theorem and the proof of which can be found in Stein and Weiss [54].

Theorem 3.4. *The collection of all finite linear combinations of elements of* $\cup_{k=0}^{\infty} H_k$ *is*

(a) *dense in the space of all continuous functions on* $\mathbb{S}^{d-1}(\mathbb{R})$ *with respect to the* $L_\infty(\mathbb{S}^{d-1}(\mathbb{R}))$ *norm.*

(b) *dense in* $L_2(\mathbb{S}^{d-1}(\mathbb{R}))$.

To compute the dimension of H_k we use our remarks prior to the Definition 3.3 and Theorem 3.3. It is obvious that the space $H_k \subset \hom(k)$ is a finite dimensional subspace of the Hilbert space $L_2(\mathbb{S}^{d-1}(\mathbb{R}))$ with the inner product given by

$$\langle f, g\rangle_{L_2\left(\mathbb{S}^{d-1}(\mathbb{R})\right)} = \int_{\mathbb{S}^{d-1}(\mathbb{R})} f(x)\overline{g(x)} d\mu_d. \tag{3.1}$$

From Theorem 3.3 we note that the number of ways of writing $p \in \hom(k)$ is the same as the number of ways of writing p_0 with the number of ways of writing the part $p_1(x) + \cdots + |x|^{2l-2} p_l(x)$ which belongs to $\hom(k-2)$, so

$$\dim \hom(k) = \dim \hom(k-2) + \dim A_k.$$

Therefore,

$$\begin{aligned} d_k = \dim H_k = \dim A_k &= \dim \hom(k) - \dim \hom(k-2) \\ = b_k - b_{k-2} &= \binom{d+k-1}{k} - \binom{d+k-3}{k-2}. \end{aligned}$$

The space A_k is called the space of homogeneous harmonic polynomials of degree k. Clearly, if h_k is any harmonic polynomial of degree k then $h_k(x)/\left\|x\right\|^k$ is a continuous function on the unit sphere $\mathbb{S}^{d-1}(\mathbb{R})$ and can also expressed as a trigonometric polynomial in $\vartheta_1, \vartheta_2, \cdots, \vartheta_{d-2}$, and φ.

With respect to the above inner product, the following theorem clarifies the orthogonality of the spaces H_k.

Theorem 3.5. *If h_k and h_l are spherical harmonics of degree k and l, with $k \neq l$, then*

$$\int_{\mathbb{S}^{d-1}(\mathbb{R})} h_k(x)\overline{h_l}(x)d\mu_d = 0.$$

We will now study a special class of spherical harmonic polynomials which are characterised, up to constant multiples, by a geometrical property. Here we follow the definition used by Stein and Weiss [54].

Let x' in $\mathbb{S}^{d-1}(\mathbb{R})$ and consider the linear functional L on H_k that assigns to each h_k in H_k the value $h_k(x')$. Since every linear functional on a finite dimensional space is bounded, see e.g. [4]. Then the functional L is also bounded. We can now use the Riesz Representation Theorem to prove that there exists a unique spherical harmonic $Z^k_{x'}$ such that

$$L(h_k) = h_k(x^{'}) = \int_{\mathbb{S}^{d-1}(\mathbb{R})} h_k(t)Z^k_{x'}(t)d\mu_d,$$

for all $h_k \in H_k$. This function $Z^k_{x'}$ is called the *zonal harmonic of degreek with pole x'*. The next theorems establish some elementary but basic properties of zonal harmonics.

Theorem 3.6. *(a) If $\{Y^k_1, \cdots, Y^k_{d_k}\}$ is an orthonormal basis of H_k then*

$$Z^k_{x'}(t') = \sum_{m=1}^{d_k} \overline{Y^k_m(x')}Y^k_m(t'),$$

(b) $Z^k_{x'}$ is real valued and $Z^k_{x'}(t') = Z^k_{t'}(x')$.

Proof. Since $\{Y^k_1, \cdots, Y^k_{d_k}\}$ is an orthonormal basis for H_k then

$$Z^k_{x'}(t') = \sum_{m=1}^{d_k} a_m Y^k_m(t').$$

By multiplying by $\overline{Y_j^k}$ and integrating over the surface of the unit sphere we get

$$\int_{\mathbb{S}^{d-1}(\mathbb{R})} Z_{x'}^k(t')\overline{Y_j^k(t')}d\mu_d(t') = \sum_{m=1}^{d_k} a_m \int_{\mathbb{S}^{d-1}(\mathbb{R})} \overline{Y_j^k(t')}Y_m^k(t')d\mu_d(t') = a_j,$$

and by the definition of the zonal harmonic we have

$$\overline{Y_j^k(x')} = a_j.$$

Thus

$$Z_{x'}^k(t') = \sum_{m=1}^{d_k} \overline{Y_m^k(x')}Y_m^k(t').$$

The second part is an immediate consequence of the definition and the first part.

□

In [42], the author gives a geometric description of zonal spherical harmonics where he uses the first part of the above theorem to define them. He studies these functions by using the group of orthogonal transformations which enables us to characterise the zonal spherical functions geometrically.
An $d\times d$ real matrix A is said to be orthogonal if $A^T = A^{-1}$. The set of all $d\times d$ real orthogonal matrices is called the orthogonal group and denoted by $O(d)$. The set of $d \times d$ orthogonal matrices with determinant one is called the special orthogonal group $SO(d)$. Clearly, the set $SO(d)$ is a subgroup of $O(d)$, and for any $A \in O(d)$

$$1 = \det(AA^{-1}) = \det(AA^T) = \det(A)\det(A^T) = (\det(A))^2.$$

Hence, $\det(A) = \pm 1$, for all $A \in O(d)$. Geometrically, the elements of $O(d)$ are either rotations or combinations of rotations and reflections. The elements of $SO(d)$ are just the rotations; see e.g. [9]. The following theorem is given by Müller [42], which provides us with some useful properties of the group $O(d)$.

Theorem 3.7. *(a) For any two vectors ξ and η, and for $A \in O(d)$ we have*

$$\langle \xi, \eta\rangle_{\mathbb{R}} = \langle A\xi, A\eta\rangle_{\mathbb{R}}.$$

(b) For any unit vector ξ there is a subgroup of orthogonal transformations, which keeps ξ fixed and which transforms a given unit vector η_0 into all

those vectors η for which

$$\langle \xi, \eta \rangle_{\mathbb{R}} = \langle \xi, \eta_0 \rangle_{\mathbb{R}} .$$

(c) $Z^k_{A\xi}(A\eta) = Z^k_\xi(\eta)$, *for all* $\xi, \eta \in \mathbb{S}^{d-1}(\mathbb{R})$ *and* $A \in SO(d)$.

It follows from the above theorem that $Z^k_\xi(A\eta) = Z^k_\xi(\eta)$, for all orthogonal matrices A for which $A\xi = \xi$. Also, if $\eta_1, \eta_2 \in \mathbb{S}^{d-1}(\mathbb{R})$ such that $\langle \xi, \eta_1 \rangle_{\mathbb{R}} = \langle \xi, \eta_2 \rangle_{\mathbb{R}}$ then there exists an orthogonal matrix such that $\eta_2 = A\eta_1$, and therefore

$$Z^k_\xi(\eta_1) = Z^k_{A\xi}(A\eta_1) = Z^k_\xi(\eta_2).$$

In other words, the zonal function $Z^k_\xi(.)$ depends on the inner product $\langle \xi, \cdot \rangle_{\mathbb{R}}$ only. Geometrically, if two vectors η_1, η_2 have the same inner product with ξ, then they belong to the intersection of the unit sphere $\mathbb{S}^{d-1}(\mathbb{R})$ with the hyperplane that is perpendicular to the line determined by the origin and ξ. Stein and Weiss [54], define this intersection as a parallel of $\mathbb{S}^{d-1}(\mathbb{R})$ orthogonal to the point ξ. Therefore, the function $Z^k_\xi(.)$ is fixed on this intersection and hence it is invariant under the rotation that fixes ξ. The next theorem shows that, up to a constant multiple, $Z^k_\xi(\cdot)$ are the only members of H_k that have this property.

Theorem 3.8. *Suppose η is a point of $\mathbb{S}^{d-1}(\mathbb{R})$. Then $h_k \in H_k$ is constant on parallels of $\mathbb{S}^{d-1}(\mathbb{R})$ orthogonal to ξ if and only if there exists a constant c such that $h_k = cZ^k_\xi$.*

Before we finish our discussion of spherical harmonics we give an essential result of the spherical harmonic theory, the *Funk-Hecke Theorem*.

Theorem 3.9. *(Funk-Hecke Theorem)*
Let $f(x)$ be a function of a real variable x which is continuous for $-1 \leq x \leq 1$, and let $h_m(\zeta)$ be any spherical harmonic of degree m. Then for any unit vector η

$$\int_{\mathbb{S}^{d-1}(\mathbb{R})} f\left(\langle \zeta, \eta \rangle_{\mathbb{R}}\right) h_m(\zeta) d\mu_d(\zeta) = \lambda_m h_m(\eta),$$

where the integral is taken over the whole area of the unit sphere $\mathbb{S}^{d-1}(\mathbb{R})$, and where

$$\lambda_m = \frac{\omega_d}{C_m^{(d-1)/2}(1)} \int_{-1}^{1} f(x) C_m^{(d-1)/2}(x) \left(1 - x^2\right)^{(d-2)/2} dx.$$

Here ω_d denote the total area of the unit sphere in d-dimensional space.

The next theorem gives a representation of zonal spherical harmonics and shows the connection between these polynomials and Gegenbauer polynomials.

Theorem 3.10. *Suppose $d > 2$ is an integer, $\lambda = (d-2)/2$ and $k = 0, 1, \cdots$. Then there exists a constant $c_{k,n}$ such that*

$$Z_y^k(x) = c_k C_k^\lambda(\langle x, y\rangle_{\mathbb{R}}),$$

for all $x, y \in \mathbb{S}^{d-1}(\mathbb{R})$.

Note that for $x \in \mathbb{S}^{d-1}(\mathbb{R})$, the expression for zonal spherical harmonics in the above theorem shows that Z_y^k is function of $\langle x, y\rangle_{\mathbb{R}}$ which can be predicted from Theorem 3.8. These zonal harmonics are constant on parallels of $\mathbb{S}^{d-1}(\mathbb{R})$ orthogonal to y. In the two dimensional sphere the zonal harmonic of degree k is the Legendre polynomial which is represented by

$$Z_x^k(y) = P_k(\langle x, y\rangle_{\mathbb{R}}) = P_k(\cos\theta),\ x, y \in \mathbb{S}^2(\mathbb{R}),$$

where θ is the angle between x and y. These polynomials were the first termed *zonal*. Since $P_k(x)$ has k distinct zeros between -1 and 1 arranged symmetrically around $x = 0$, see MacRobert [41], $P_k(\cos\theta)$ has k zeros between 0 and π arranged symmetrically around $\theta = \pi/2$. Accordingly, on the unit sphere $P_k(\cos\theta)$ vanishes on k circles and then these parallel circle divide the surface of the unit sphere into zones. This geometric property explains why these polynomials are named "*zonal*".

Another important consequence of Theorem 3.7 and Theorem 3.10 is that we can consider the zonal spherical harmonic $Z_y^k(x) = c_{k,n} C_k^\lambda(\langle x, y\rangle_{\mathbb{R}})$ as a function depending on θ_1 only, the first parameter in the spherical coordinates $\theta_1, \cdots, \theta_{d-2}, \varphi$. To show this let A be an orthogonal transformation such that $Ay = e_1$, and put $x' = Ax$. Then from Theorem 3.7, $\langle e_1, x'\rangle_{\mathbb{R}} = \langle Ay, Ax\rangle_{\mathbb{R}} = \langle x, y\rangle_{\mathbb{R}}$. Hence the function C_k^λ depends on the inner product $\langle e_1, x'\rangle_{\mathbb{R}} = \cos\theta_1,\ x' \in \mathbb{S}^{d-1}(\mathbb{R})$, or more precisely, depends on θ_1 only.
From the definition of spherical harmonic functions we have

$$\Delta r^k h_k = r^{-(d-1)} \frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r} r^k h_k\right) + r^{-2}\Delta_{\mathbb{S}^{d-1}(\mathbb{R})} r^k h_k = 0,$$

where $h_k \in H_k$ and $\Delta_{\mathbb{S}^{d-1}(\mathbb{R})}$ is the Laplace operator on the sphere $\mathbb{S}^{d-1}(\mathbb{R})$ given by

$$\Delta_{\mathbb{S}^{d-1}(\mathbb{R})} = (\sin\vartheta_1)^{-(d-2)} \frac{\partial}{\partial\vartheta_1}\left((\sin\vartheta_1)^{d-2}\frac{\partial}{\partial\vartheta_1}\right) + \cdots + (\sin\vartheta_1\cdots\sin\vartheta_{d-2})^{-2}\,\frac{\partial^2}{\partial\varphi^2}.$$

Hence,

$$h_k k(k+d-2)r^{k-2} + r^{k-2}\Delta_{\mathbb{S}^{d-1}(\mathbb{R})}h_k = 0,$$

which implies

$$\Delta_{\mathbb{S}^{d-1}(\mathbb{R})}h_k = -k(k+d-2)h_k. \tag{3.2}$$

The above argument shows that the spherical harmonic polynomial h_k is an eigenfunction of the Laplace operator on the sphere with eigenvalue $\gamma = -k(k+d-2)$, and hence H_k is the eigenspace of the Laplace operator corresponding to the eigenvalue $\gamma = -k(k+d-2)$. Now since C_k^λ depends on θ_1 only, (3.2) can be written in the form of the following theorem.

Theorem 3.11. *If C_m^λ is the Gegenbauer polynomial of degree m and $\lambda = (d-2)/2$, then*

$$\Delta_{\mathbb{S}^{d-1}(\mathbb{R})}C_m^\lambda(\cos\vartheta) = \frac{\frac{d}{d\vartheta}\left(\sin^{d-2}\vartheta\frac{d}{d\vartheta}C_m^\lambda(\cos\vartheta)\right)}{\sin^{d-2}\vartheta} = -m(m+d-2)C_m^\lambda(\cos\vartheta).$$

As we have seen in Section 2.3, Gegenbauer polynomials are eigenfunctions of the Gegenbauer operator,

$$Lu = (1-x^2)^{-\lambda+1/2}\frac{d}{dx}\left[(1-x^2)^{\lambda+1/2}\frac{du}{dx}\right],$$

on the real interval $-1 \leqslant x \leqslant 1$, with eigenvalue $\gamma = -m(m+2\lambda)$. To establish this result from the above theorem, put $x = \cos\theta$. Then $dx/d\theta = -\sin\theta$ and then

$$\begin{aligned}
&\sin^{-(n-2)}\vartheta\frac{d}{d\vartheta}\left(\sin^{n-2}\vartheta\frac{d}{d\vartheta}C_m^\lambda(\cos\vartheta)\right)\\
&= \left(\sqrt{1-x^2}\right)^{-(n-2)}(-\sin\theta)\frac{d}{dx}\left(-\left(\sqrt{1-x^2}\right)^{n-2}\sin\vartheta\frac{d}{dx}C_m^\lambda(x)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\sqrt{1-x^2}\right)^{-(n-3)} \frac{d}{dx}\left(\left(\sqrt{1-x^2}\right)^{n-1} \frac{d}{dx} C_m^{\lambda}(x)\right) \\
&= \left(1-x^2\right)^{-\lambda+1/2} \frac{d}{dx}\left(\left(1-x^2\right)^{\lambda+1/2} \frac{d}{dx} C_m^{\lambda}(x)\right),
\end{aligned}
$$

where $\lambda = (d-2)/2$. Hence

$$
LC_m^{\lambda}(x) = -m(m+2\lambda)C_m^{\lambda}(x).
$$

3.3 Proof of the Generating Function Formula of Gegenbauer Polynomials

Having now reviewed the concept of orthogonal polynomials and set out some basic results concerning spherical harmonic functions, we shall now introduce the first origine part of this thesis. This section is devoted to proving the formula for the generating function of Gegenbauer polynomials. The proof will enable us to show that this functions are spherical harmonic, and then use the theory of spherical harmonic to expand it in powers of r (see below) with Gegenbauer polynomials as coefficients. All the results that are required are given in the previous chapters.

Let $f(s) = (1+r^2-2rs)^{-\lambda}$ and consider the formula

$$
\sum_{i=0}^{\infty} C_i^{\lambda}(s) r^i = f(s). \tag{3.3}
$$

The geometric consideration of this formula is based on computing the distance between two vectors, one of them inside the unit sphere and the other on the unit sphere. We shall compute the norm $\|x-p\|$ which is related to the generating function of Gegenbauer polynomials. In particular, let $p \in \mathbb{R}^d$ be a fixed point such that $\|p\| = r < 1$, and let $x \in \mathbb{S}^{d-1}(\mathbb{R})$. The distance between the two vectors p and x is given by

$$
\|x-p\|^2 = \langle x-p, x-p\rangle_{\mathbb{R}} = \|x\|^2 + \|p\|^2 - 2\langle x, p\rangle_{\mathbb{R}} = 1 + r^2 - 2r\langle x, (p/r)\rangle_{\mathbb{R}},
$$

Since $(1 + r^2 - 2r \langle x, (p/r)\rangle_{\mathbb{R}})^{-\lambda}$ with $\lambda = (n-2)/2$, is the generating function of Gegenbauer polynomials then we have

$$f\left(\langle x, (p/r)\rangle_{\mathbb{R}}\right) = \|x - p\|^{-2\lambda} = \sum_{n=0}^{\infty} r^n C_n^{\lambda}(\langle x, (p/r)\rangle_{\mathbb{R}}).$$

Now we will give a geometric proof of the well known generating function of Gegenbauer polynomials which given in the following theorem.

Theorem 3.12. *Let* $r \in [0,1]$. *Then for* $s \in [-1,1]$ *the following formula holds,*

$$\left(1 + r^2 - 2rs\right)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{\lambda}(s) r^n.$$

Proof. Let us consider two vectors: p as a fixed vector with $\|p\| = r < 1$, and x, $\|x\| = 1$, in the Euclidean space $\mathbb{R}^d$, and let h be the function on $\mathbb{S}^{d-1}(\mathbb{R})$ defined by $h(x) = f\left(\langle x, (p/r)\rangle_{\mathbb{R}}\right) = (1 + r^2 - 2r \langle x, (p/r)\rangle_{\mathbb{R}})^{-\lambda}$. Notice that the function $g(x) = \|x\|^{-d+2}$, $x \neq 0$, is a harmonic function since $\triangle g = 0$. Hence the translated function $h(x) = \|x - p\|^{-d+2}$ is harmonic when $x \neq p$. Since h is continuous on $\mathbb{S}^{d-1}(\mathbb{R})$, by using Theorem 3.4, h can be written as

$$h(x) = \sum_{n=0}^{\infty} \sum_{k=1}^{d_n} a_{n,k} Y_k^n(x),$$

where $\left\{Y_1^n, Y_2^n, \cdots, Y_{d_n}^n\right\}$ is an orthonormal basis of the space H_n. Multiplying both sides by the zonal polynomial $Z_y^j(x)$, $y \in \mathbb{S}^{d-1}(\mathbb{R})$, and integrate over the area of the unit sphere, we have

$$\int_{\mathbb{S}^{d-1}(\mathbb{R})} f(\langle x, (p/r)\rangle_{\mathbb{R}}) Z_y^j(x) d\mu_d(x) = \sum_{k=1}^{d_j} a_{j,k} \int_{\mathbb{S}^{d-1}(\mathbb{R})} Z_y^j(x) Y_k^j(x) d\mu_d(x).$$

By using Funk-Hecke theorem, we get

$$\lambda_j(r) Z_y^j(p/r) = \sum_{k=1}^{d_j} a_{j,k} Y_k^j(y),$$

where

$$\lambda_j(r) = \frac{\omega_n}{C_j^{(n-1)/2}(1)} \int_{-1}^{1} f(s) C_j^{(n-1)/2}(s) \left(1 - s^2\right)^{(n-2)/2} ds.$$

Here ω_n denotes the total area of the unit sphere in n-dimensional space. Notice that λ_i is a function of r since f is a function of r. Thus

$$h(x) = \sum_{n=0}^{\infty} \lambda_n(r) Z_x^n(p/r) = \sum_{n=0}^{\infty} \lambda_n(r) c_n C_n^{\lambda}(\langle x, (p/r)\rangle_{\mathbb{R}}) = \sum_{n=0}^{\infty} \lambda_n(r) c_n C_n^{\lambda}(\cos\vartheta), \tag{3.4}$$

where ϑ is the angle between x and p, $\lambda = (d-2)/2$ and c_n is a constant depending on n and d only. To complete the proof we need to show that the function $\lambda_n(r)c_n = r^n$. By applying the Laplace operator on h we get

$$0 = \sum_{n=0}^{\infty} \Delta\left(\lambda_n(r) c_n C_n^{\lambda}(\cos\vartheta)\right)$$

$$= \sum_{n=0}^{\infty} \frac{c_n C_n^{\lambda}(\cos\vartheta)}{r^{(d-1)}} \frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\lambda_n(r)\right) + \frac{c_n\lambda_n(r)}{r^2(\sin\vartheta)^{(d-2)}}\frac{\partial}{\partial\vartheta}\left((\sin\vartheta)^{d-2}\frac{\partial C_n^{\lambda}(\cos\vartheta)}{\partial\vartheta}\right).$$

Since Gegenbauer polynomials satisfy the equation

$$\frac{\partial}{\partial\vartheta}\left((\sin\vartheta)^{d-2}\frac{\partial C_n^{\lambda}(\cos\vartheta)}{\partial\vartheta}\right) = -n(n+d-2)(\sin\vartheta)^{d-2} C_n^{\lambda}(\cos\vartheta),$$

it can be deduced that

$$0 = \sum_{n=0}^{\infty} c_n C_n^{\lambda}(\cos\vartheta)\left[r^{-(d-1)}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\lambda_n(r)\right) - r^{-2}\lambda_n(r) n(n+d-2)\right].$$

Since the set $\left\{C_n^{\lambda}\right\}$ is linearly independent, we have

$$r^{-(d-1)}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\lambda_n(r)\right) - r^{-2}\lambda_n(r) n(n+d-2) = 0, \quad n = 0, 1, 2, \cdots.$$

This equation can be reduced to

$$r^{-1}(d-1)\lambda_n'(r) + \lambda_n''(r) - n(n+d-2)r^{-2}\lambda_n(r) = 0, \quad n = 0, 1, 2, \cdots,$$

or

$$r^2\lambda_n''(r) + r(d-1)\lambda_n'(r) - n(n+d-2)\lambda_n(r) = 0, \quad n = 0, 1, 2, \cdots.$$

This is the Cauchy-Euler equation and the solution of this equation is

$$\lambda_n(r) = a_n r^{m_1} + b_n r^{m_2},$$

where m_1 and m_2 are the solution of the equation $x^2 + (d-1)x - n(n+d-1) = 0$.

Thus $m_1 = n$ and $m_2 = -(d+n)+1 < 0$. According to the domain of the function f, the function λ_n is defined at $r=0$, therefore b_n must be zero for all $n = 0, 1, 2, \cdots$. Hence,

$$\lambda_n(r) = a_n r^n,$$

and consequently we can write

$$h(x) = \sum_{n=0}^{\infty} a_n c_n r^n C_n^{\lambda}(\langle x, (p/r)\rangle_{\mathbb{R}}) = \sum_{n=0}^{\infty} a_n c_n r^n C_n^{\lambda}(\cos\vartheta).$$

By choosing $\vartheta = 0$ we get

$$(1-r)^{-2\lambda} = \sum_{n=0}^{\infty} a_n c_n r^n C_n^{\lambda}(1) = \sum_{n=0}^{\infty} a_n c_n r^n \binom{n+2\lambda-1}{n}.$$

On the other hand, from the Binomial theorem the function $(1-r)^{-2\lambda}$ can be written as

$$(1-r)^{-2\lambda} = \sum_{n=0}^{\infty} r^n \binom{n+2\lambda-1}{n}.$$

From the uniqueness of the power series we get $c_n a_n = 1$ for all $n = 0, 1, 2, \cdots$, and finally we have

$$\left(1 + r^2 - 2r\cos\vartheta\right)^{-\lambda} = \sum_{n=0}^{\infty} r^n C_n^{\lambda}(\cos\vartheta), \tag{3.5}$$

for $0 < r < 1, \vartheta \in [0, \pi]$. For $r = 0$ it is clear that the formula 3.5 is true. Now for $r = 1$, from 3.4 we get

$$h(x) = \sum_{n=0}^{\infty} \lambda_n(1) c_n C_n^{\lambda}(\cos\vartheta).$$

By chopsing $\vartheta = \pi$ and using the Binomial theorem we can show that $\lambda_n(1)c_n = 1$ for all $n \in \mathbb{N}$. Hence

$$\left(1 + r^2 - 2r\cos\vartheta\right)^{-\lambda} = \sum_{n=0}^{\infty} r^n C_n^{\lambda}(\cos\vartheta),$$

for $0 \leqslant r \leqslant 1, \vartheta \in [0, \pi]$. $\square$

In the next chapter we will see a similar sort of argument to develop a generating function for Jacobi polynomials.

Chapter 4

Zonal spherical harmonics in Complex Vector space

4.1 The Laplace Operator and Harmonic Functions in Complex space

In this section we are concerned with spherical harmonics in a complex space, and the zonal functions will take most of our attention. We develop the spherical harmonics in $\mathbb{C}^q$ in a similar way to the real case. Also we clarify the relationship between the real and complex case and then establish an addition formula which shows the connection between the zonal functions in these spaces. The main references for the material in this section are Koornwinder [29] and Rudin [48].
Let $\mathbb{C}^d$ be the complex vector space of dimension d, with the inner product given by

$$\langle z, w\rangle_{\mathbb{C}} = \langle (z_1, \cdots, z_d), (\ w_1, \cdots, w_d)\rangle_{\mathbb{C}} = z_1\overline{w_1} + \cdots + z_d\overline{w_d}, \quad z,\ w \in \mathbb{C}^d.$$

We can identify $\mathbb{C}^d$ with $\mathbb{R}^{2d}$ by writing each component of $z \in \mathbb{C}^d$ in terms of real and imaginary parts, $z_j = x_j + iy_j,\ x_j, y_j \in \mathbb{R}$. Hence $z = (x_1+iy_1, \cdots, x_d+iy_d) \in \mathbb{C}^d$ can be considered as a real vector given by $z = (x_1, y_1, \cdots, x_d, y_d) \in \mathbb{R}^{2d}$. Under this identification, the norm corresponding to the above inner product agrees with the usual Euclidean norm in $\mathbb{R}^{2d}$. Therefore, the unit sphere

$$\mathbb{S}^{d-1}(\mathbb{C}) = \{z \in \mathbb{C}^d : |z| = 1\}$$

can be also identified with the real unit sphere $\mathbb{S}^{2d-1}(\mathbb{R})$. The identities $x_k = (z_k + \overline{z}_k)/2$ and $y_k = (z_k - \overline{z}_k)/2i$ show that any complex valued polynomial on $\mathbb{R}^{2d}$ can be written as a complex polynomial on $\mathbb{C}^d$ and vice versa. Also, from the derivatives

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),$$

the Laplace operator on $\mathbb{C}^d$ can be expressed as

$$\triangle = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} = 4\sum_{i=1}^{d} \frac{\partial^2}{\partial z_i \partial \overline{z}_i}. \tag{4.1}$$

The above argument leads to the following definitions which correspond to the definitions of harmonics and spherical harmonic polynomials in $\mathbb{R}^{2d}$.

Definition 4.1. A function f on $\mathbb{S}^{d-1}(\mathbb{C}) \subset \mathbb{C}^d$ belongs to the space $\hom(m,n)$ if it is the restriction to $\mathbb{S}^{d-1}(\mathbb{C})$ of a polynomial

$$F(z,\bar{z}) = F(z_1,\cdots,z_d,\overline{z_1},\cdots,\overline{z_d})$$

which is homogeneous of degree m in z and of degree n in $\bar{z}$. In other words, $F(tz, r\bar{z}) = t^m \overline{r}^n F(z,\bar{z})$ where $t, r \in \mathbb{C},\ z \in \mathbb{C}^d$.

Definition 4.2. A harmonic of type (m,n) is a homogeneous polynomial $h(z,\bar{z})$ of degree (m,n) which satisfies

$$\left(\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \cdots + \frac{\partial^2}{\partial z_d \partial \bar{z}_d}\right) h(z,\bar{z}) = 0$$

Definition 4.3. A spherical harmonic function of type (m,n) is a function on $\mathbb{S}^{d-1}(\mathbb{C}) \subset \mathbb{C}^d$ which is a restriction to $\mathbb{S}^{d-1}(\mathbb{C})$ of a harmonic of type (m,n). The space of all spherical harmonic functions of type (m,n) is denoted by $H(m,n)$.

Continuing the analysis of homogenous polynomials, any function $f \in \hom(m,n)$ can be written as

$$f(z,\bar{z}) = \sum_{|\alpha|=m,|\beta|=n} c_{\alpha,\beta} z^{\alpha} \bar{z}^{\beta},$$

where α and β denote tuples $(\alpha_1,\cdots,\alpha_d)$ and $(\beta_1,\cdots,\beta_d)$ of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_d = m, |\beta| = \beta_1 + \cdots + \beta_d = n$ and $z^{\alpha} = z_1^{\alpha_1} \cdots z_d^{\alpha_d},\ \bar{z}^{\beta} =$

${\bar{z}_1}^{\beta_1} \cdots {\bar{z}_d}^{\beta_d}$. Clearly, $f \in \hom(m,n)$ if and only if $f(e^{i\theta}z) = e^{i(m-n)\theta}f(z)$. Hence, the function $f \in \hom(m,n)$ is real if and only if $m = n$. On the other hand, if $f \in \hom(m,n)$ then $\overline{f} \in \hom(n,m)$. As in the real case to compute the dimension of the space $\hom(m,n)$ we need to compute the number of the functions z^α, and $\bar{z}^\beta$. The number b_m of functions z^α is the same as the number of ways a d-tuple $\alpha = (\alpha_1, \cdots, \alpha_d)$ of non-negative integers can be chosen so that $|\alpha| = m$, which is calculated in the previous chapter. So the dimension $b_{m,n} = b_m b_n$ of the space $\hom(m,n)$ is given by

$$b_{m,n} = \dim(\hom(m,n)) = \binom{d+m-1}{m}\binom{d+n-1}{n}.$$

An important consequence of (4.1) is that the harmonicity in the real space coincides with harmonicity in the complex space. More precisely, a function f is harmonic in $\mathbb{C}^d$ if it is harmonic in $\mathbb{R}^{2d}$ considered as a function of $2d$ real variables.

Theorem 4.1. *(a)* $\hom(m,n) \subset \hom(m+1, n+1)$,

(b) $H(m,n) \subset \hom(m,n)$,

(c) $H(m,0) = \hom(m,0)$ *and* $H(0,n) = \hom(0,n)$.

Proof. For the first result we can use the fact that if f is the restriction of the polynomial F which is a homogenous of degree (m,n) then it is also the restriction of the polynomial $G(z,\bar{z}) = (z_1\bar{z}_1 + \cdots + z_d\bar{z}_d)F(z,\bar{z})$ which is homogeneous of degree $(m+1, n+1)$. The second result can be obtained directly from the definition of the space $\hom(m,n)$. The last results come from the fact that a polynomial p of degree 0 in its d complex variables $\bar{z}_1, \cdots, \bar{z}_d$ is a polynomial of the d variables $z_1, \cdots, z_d$ only. Hence $\partial p/\partial \overline{z}_j = 0$ for all $j = 0, \cdots, d$, and then $\triangle p = 0$. $\square$

The following discussion shows the orthogonality of the space $H(m,n)$. The inner product used here is given by

$$\langle f, g \rangle_{L_2(\mathbb{S}^{d-1}(\mathbb{C}))} \int_{\mathbb{S}^{q-1}(\mathbb{C})} f(z)\overline{g(z)}d\mu_{2d}.$$

with f and g square integrable functions on $\mathbb{S}^{d-1}(\mathbb{C})$ and the measure on $\mathbb{C}^d$ coincides with the Lebesgue measure on $\mathbb{R}^{2d}$, denoted by μ_{2d}.

Note that if f is a homogeneous harmonic function of degree (m,n), then f can be considered as a complex valued homogeneous harmonic function of degree $m+$

n on $\mathbb{R}^{2d}$, because of (4.1) which leads to $H(m,n) \subseteq H_{m+n}$. It follows that $H(m,n) \subseteq H_{m+n} \subseteq L_2(\mathbb{S}^{d-1}(\mathbb{C}))$ when $k = m+n$. Thus $H(m,n)$ is a finite dimensional subspace of the Hilbert space $L_2(\mathbb{S}^{d-1}(\mathbb{C}))$. On the other hand, from the orthogonality of the spaces H_k, $k \geqslant 0$, and the correspondance between the unit spheres $\mathbb{S}^{d-1}(\mathbb{C})$ and $\mathbb{S}^{2d-1}(\mathbb{R})$, $H(m,n)$ and $H(m',n')$ are orthogonal, with respect to the above inner product, when $m+n \neq m'+n'$. Moreover, Koornwinder [29] proved that spherical harmonics of different degree are orthogonal as given in the following theorem.

Theorem 4.2. *Let $h_1 \in H(m_1,n_1)$ and $h_2 \in H(m_2,n_2)$, and suppose that $(m_1,n_1) \neq (m_2,n_2)$. Then*

$$\int_{\mathbb{S}^{d-1}(\mathbb{C})} h_1(x)\overline{h_2(x)}d\mu_{2d} = 0.$$

By combining the fact that H_k is a vector space with the above argument we find $\sum_{n+m=k} H(m,n) \subseteq H_k$. In fact, more is proved in the following theorem, see e.g. Rudin [48] .

Theorem 4.3. *H_k is the direct sum of the pairwise orthogonal spaces $H(m,n)$, where $m+n=k$.*

Proof. Let $h_k \in H_k$. Then h_k can be written as

$$\begin{aligned} h_k(x_1,y_1,\cdots,x_d,y_d) &= \sum_{\sum_{i=1}^d \alpha_i+\beta_i=k} c_{\alpha,\beta}\ x_1^{\alpha_1}y_1^{\beta_1}\cdots x_d^{\alpha_d}y_d^{\beta_d} \\ &= \sum_{\sum_{i=1}^d \alpha_i+\beta_i=k} c_{\alpha,\beta}\left(\frac{z_1+\overline{z}_1}{2}\right)^{\alpha_1}\left(\frac{z_1-\overline{z}_1}{2i}\right)^{\beta_1}\cdots\left(\frac{z_d-\overline{z}_d}{2i}\right)^{\beta_d}. \end{aligned}$$

Since

$$(z_i+\overline{z}_i)^{\alpha_i}(z_i-\overline{z}_i)^{\beta_i} = \sum_{j=0}^{\alpha_i}\sum_{k=0}^{\beta_i}\binom{\alpha_i}{j}\binom{\beta_i}{k}(-1)^{\beta_i-k}(z_i)^{j+k}\ (\overline{z}_i)^{\alpha_i+\beta_i-(j+k)},$$

then each term $(z_i+\overline{z}_i)^{\alpha_i}(z_i-\overline{z}_i)^{\beta_i}$ can be written as a sum of terms of the form $z_i^{\gamma_i}\overline{z}_i^{\eta_i}$ such that $\gamma_i+\eta_i = \alpha_i+\beta_i$. Therefore

$$\begin{aligned} h_k(x_1,y_1,\cdots,x_d,y_d) &= \sum_{\sum_{i=1}^d \gamma_i+\eta_i=k} c_{\gamma,\eta}\ z_1^{\gamma_1}\overline{z}_1^{\eta_1}\cdots z_d^{\gamma_d}\overline{z}_d^{\eta_d} \\ &= \sum_{|\gamma|=m,\ |\eta|=n,\ n+m=k} c_{\gamma,\eta}\ z^{\gamma}\overline{z}^{\eta}, \end{aligned}$$

where $\gamma = (\gamma_1, \cdots, \gamma_d)$, $\eta = (\eta_1, \cdots, \eta_d)$, $z_1^{\gamma_1} \cdots z_d^{\gamma_d} = z^\gamma$, $\overline{z}_1^{\eta_1} \cdots \overline{z}_d^{\eta_d} = \overline{z}^\eta$, and $\sum_{i=1}^d \gamma_i = |\gamma|$, $\sum_{i=1}^d \eta_i = |\eta|$. Hence

$$h_k(x_1, y_1, \cdots, x_d, y_d) = \sum_{i=0}^{k} f_i(z, \overline{z}),$$

where f_i is homogeneous of degree $(i, k-i)$. Hence the function $\triangle f_i$ is homogeneous of degree $(i-1, k-i-1)$ and

$$0 = \triangle f = \sum_{i=1}^{k} \triangle f_i.$$

Since each $\triangle f_i$ has different degree, then $\triangle f_i = 0$. □

To compute the dimension $d_{m,n}$ of the space $H(m,n)$ let f in $\hom(m,n)$, then f lies in $\hom(k)$ where $m+n=k$. From Theorem 3.3 the function f can be written uniquely as

$$f = g + |\cdot|^2 h,$$

where $g \in H_k$ and $h \in \hom(k-2)$. Since f is homogeneous of degree (m,n) then g and h are homogenous of degree(m,n) and $(m-1, n-1)$ respectively. We can conclude that all f in $\hom(m,n)$ can be written uniquely in the form $f = g + |\cdot|^2 h$ where $g \in H(m,n)$ and $g \in \hom(m-1, n-1)$. Hence, similar to the real case, we have

$$\hom(m,n) = H(m,n) \oplus |\cdot|^2 \hom(m-1, n-1).$$

This formula enables us to compute the dimension of the space $H(m,n)$.

$$\begin{aligned} d_{m,n} = \dim H(m,n) &= \dim \hom(m,n) - \dim \hom(m-1, n-1) \\ &= \binom{d+m-1}{m}\binom{d+n-1}{n} - \binom{d+m-2}{m-1}\binom{d+n-2}{n-1} \\ &= \frac{(d+m-2)!(d+n-2)!(d+m+n-1)}{m!n!(d-1)!(d-2)!}. \end{aligned}$$

We can now use the correspondence between the real and complex spaces with Theorems 3.4 and 4.3 to conclude that $L_2(\mathbb{S}^{d-1}(\mathbb{C}))$ is the direct sum of the orthogonal spaces $H(m,n)$, see e.g.Rudin [48] .

Theorem 4.4. *$L_2(\mathbb{S}^{d-1}(\mathbb{C}))$ is the direct sum of the pairwise orthogonal spaces $H(m,n)$, $0 \leqslant m, n < \infty$.*

We will now study the zonal spherical harmonics which are a special class of spherical harmonic polynomials. Similar to the real case, let w be a fixed element in $\mathbb{S}^{d-1}(\mathbb{C})$ and let L be the linear functional defined on $H(m,n)$ as $L(h) = h(w)$. The space $H(m,n)$ is a finite dimensional Hilbert space and then the Riesz Representation Theorem implies the existence of a unique polynomial $Z_w^{(m,n)}$ in $H(m,n)$ such that

$$Lh = h(w) = \left\langle h, Z_w^{(m,n)} \right\rangle_{L_2(\mathbb{S}^{d-1}(\mathbb{C}))} = \int_{\mathbb{S}^{d-1}(\mathbb{C})} h\ \overline{Z_w^{(m,n)}} d\mu_{2d} \quad \forall h \in H(m,n).$$

The function $Z_w^{(m,n)}$ is called the zonal harmonic of degree (m,n) with pole w. From the above analysis on the space $\mathrm{hom}(m,n)$, the zonal $Z_w^{(m,n)}$ is not real unless $m = n$. Moreover, since $\overline{Z_w^{(m,n)}} \in H(n,m)$ then

$$\overline{Z_w^{(m,n)}}(z) = \left\langle \overline{Z_w^{(m,n)}}, Z_z^{(n,m)} \right\rangle_{L^2(\mathbb{S}^{d-1}(\mathbb{C}))} = \left\langle \overline{Z_z^{(n,m)}}, Z_w^{(m,n)} \right\rangle_{L^2(\mathbb{S}^{d-1}(\mathbb{C}))}.$$

On the other hand,

$$\overline{Z_w^{(m,n)}}(z) = \overline{\left\langle Z_w^{(m,n)}, Z_z^{(m,n)} \right\rangle}_{L^2(\mathbb{S}^{d-1}(\mathbb{C}))} = \int_{\mathbb{S}^{d-1}(\mathbb{C})} \overline{Z_w^{(m,n)}} Z_z^{(m,n)} d\mu_{2d}$$

$$= \left\langle Z_z^{(m,n)}, Z_w^{(m,n)} \right\rangle_{L^2(\mathbb{S}^{d-1}(\mathbb{C}))}.$$

From the uniqueness of the existence of the zonal harmonic, we conclude that

$$Z_z^{(m,n)} = \overline{Z_z^{(n,m)}} \quad \forall z \in \mathbb{C}^d, \quad n, m \in \mathbb{N}.$$

Koornwinder [29] was the first to establish the connection between the complex zonal harmonics and the group of unitary transformations. He used this group to define the spherical harmonics on the complex Euclidean space which leads to a geometric description of these polynomials on the unit sphere. To define the unitary group let A be a $d \times d$ complex matrix. A is said to be unitary if $A^* = A^{-1}$ where A^* is the adjoint of A, $(A^*)_{jk} = \overline{A_{kj}}$. Another equivalent definition is that A is unitary if $\langle z, w \rangle_{\mathbb{C}} = \langle Az, Aw \rangle_{\mathbb{C}}$ for all $z, w \in \mathbb{C}^d$. A linear operator T on $\mathbb{C}^d$ is unitary if and only if $T(z) = Az$, where A is unitary, or if it preserves the inner product defined on $\mathbb{C}^d$. The group of all $d \times d$ unitary matrices is denoted by $U(d)$. Now let e_k be the canonical unit vector. Then the subgroup $U(d-1)$ is considered as the set of all $A \in U(d)$ which leave e_1 fixed. To clarify this point let $A \in U(d)$

which leaves e_1 fixed. Clearly, from the multiplication $Ae_1 = e_1$, the first column must be e_1. On the other hand, since A is unitary then

$$e_1 = A^{-1}e_1 = A^* e_1.$$

Hence the first column of A^* is e_1 and the first column of A^* is the conjugate of the first row of A. Thus the first row of A is also e_1. In other words, the first row and first column of a unitary matrix A which leaves e_1 fixed are e_1. Thus A is determined by $d-1$ rows and $d-1$ columns.

Koornwinder [29] treats a special group of the zonal harmonics on the complex sphere which is the group of all zonal harmonics with pole e_1. He gives a precise expression of these polynomials where he uses the following proposition as a definition of zonal functions.

Proposition 4.1. *For all $A \in U(d-1)$ and $z \in \mathbb{S}^{d-1}(\mathbb{C})$, $Z_{e_1}^{(m,n)}(z) = Z_{e_1}^{(m,n)}(Az)$.*

Notice that if A is a unitary matrix which leaves e_1 fixed, then the first row in A is e_1 and hence

$$Az = [z_1, a_2^T z, \cdots, a_d^T z]^T \text{when } \ A = [e_1, a_2, \cdots, a_d]^T, \quad z = (z_1, \cdots z_d).$$

Then

$$Z_{e_1}^{(m,n)}(Az) = Z_{e_1}^{(m,n)}(z_1, a_2^T z, \cdots, a_d^T z) = Z_{e_1}^{(m,n)}(z_1, \cdots z_d), \quad \text{for all } A \in U(d-1).$$

Clearly, the function $Z_{e_1}^{(m,n)}$ depends on the inner product $\langle z, e_1 \rangle_{\mathbb{C}} = z_1$. In other words, $Z_{e_1}^{(m,n)}$ is fixed in the set $\{z \in \mathbb{C}^d : \langle z, e_1 \rangle = \lambda\}$ for $|\lambda| \leqslant 1$, $\lambda \in \mathbb{C}$. Then there is a continuous function f on the closed unit disk in the complex plane, such that

$$Z_{e_1}^{(m,n)}(z) = f(\langle z, e_1 \rangle_{\mathbb{C}}).$$

Koornwinder describes the zonal spherical harmonic for the space $H(m,n)$ which are given in the following theorem.

Theorem 4.5. *The function Φ on $\mathbb{S}^{d-1}(\mathbb{C})$ is a zonal function with pole e_1 in $H(m,n)$ if and only if*

$$\Phi(z) = c_{m,n} e^{i(m-n)\phi} \left(\cos\theta\right)^{|m-n|} P_{m\wedge n}^{(d-2,|m-n|)}\left(\cos 2\theta\right), \tag{4.2}$$

where $c_{m,n}$ is a constant and $z \in \mathbb{S}^{d-1}(\mathbb{C})$ written as $z = \cos\theta e^{i\phi} e_1 + \sin\theta z'$ where $z' \in \mathbb{S}^{d-2}(\mathbb{C})$, $0 < \theta < \pi/2$, $0 \le \varphi \le 2\pi$ and $P_{m\wedge n}$ is the Jacobi polynomials of degree $m \wedge n = min(m, n)$.

The above analysis provides us with a precise expression of the zonal harmonic $Z_z^{(m,n)}$. Since $Z_{e_1}^{(m,n)}(\cdot)$ depends only on $\langle \cdot, e_1\rangle_{\mathbb{C}}$, then for any two vectors z and w in $\mathbb{S}^{d-1}(\mathbb{C})$ there is a unitary matrix A such that $Az = e_1$ and put $y = Aw$. Hence

$$Z_z^{(m,n)}(w) = Z_{e_1}^{(m,n)}(y),$$

since $\langle y, e_1\rangle_{\mathbb{C}} = \langle Aw, Az\rangle_{\mathbb{C}} = \langle w, z\rangle_{\mathbb{C}}$. Therefore

$$Z_z^{(m,n)}(w) = c_{m,n} e^{i(m-n)\phi} (\cos\theta)^{|m-n|} P_{m\wedge n}^{(d-2,|m-n|)} (\cos 2\theta),$$

where $\langle w, z\rangle_{\mathbb{C}} = \cos\theta e^{i\phi}$.

Definition 4.4. The complex projective space $\mathbb{P}(\mathbb{C}^d)$ is the set of all complex lines in $\mathbb{C}^d$ passing through the origin.

Note that a complex line passing through the origin and $z \in \mathbb{C}^d$ is a one dimensional subspace given by $\{za : a \in \mathbb{C}\}$. This space may be alternatively realised as the quotient space (set of all equivalence classes) of $\mathbb{C}^d - \{0\}$ by the equivalence relation,

$$z_1 \sim z_2 \qquad if \text{ } and \text{ } only \text{ } if \text{ } z_1 = \lambda z_2, \ \lambda \in \mathbb{C} - \{0\}.$$

The restriction of this relation on the unit sphere is

$$z_1, z_2 \in \mathbb{S}^{d-1}(\mathbb{C}), \quad z_1 \sim z_2 \qquad if \text{ } and \text{ } only \text{ } if \text{ } z_1 = \lambda z_2, \ \lambda \in C.$$

where C is the unit circle in $\mathbb{C}$. Since each line through 0 intersects the unit sphere $\mathbb{S}^{d-1}(\mathbb{C})$ in a circle, the complex projective space may also be regarded as the set of all cosets $[z] = zC = \{az : |a| = 1\}$ on $\mathbb{S}^{d-1}(\mathbb{C})$.

Note that for $n = m$ the zonal function $\Phi(z) = P_l^{(d-2,0)} (\cos 2\theta)$ is invariant in each coset $[z], z \in \mathbb{S}^{d-1}(\mathbb{C})$, so it can be considered as a function defined on the complex projective space $\mathbb{P}(\mathbb{C}^d)$ and hence is the only zonal harmonic of degree l for this space, see Koornwinder [29]. On the other hand, viewing $\mathbb{S}^{d-1}(\mathbb{C})$ as $\mathbb{S}^{2d-1}(\mathbb{R})$ it is well known that the zonal function for H_l is

$$Z^l_{e_1}(z) = c_l C^{(d-1)}_l(\langle e_1, z\rangle_{\mathbb{R}}),$$

and since

$$\langle e_1, z\rangle_{\mathbb{R}} = \mathrm{Re}\ \langle e_1, z\rangle_{\mathbb{C}} = \mathrm{Re}\ \left\langle e_1, \cos\theta e^{i\phi} e_1 + \sin\theta z'\right\rangle_{\mathbb{C}} = \cos\theta\cos\phi,$$

we can deduce that the zonal function for H_l is

$$Z^l_{e_1}(z) = c_l C^{(d-1)}_l(\cos\theta\cos\phi). \tag{4.3}$$

So the average over the same coset of the complex sphere of the zonal harmonic function (4.3) would be expected to be a zonal harmonic of degree l for the complex projective space, which is the Jacobi polynomial. More precisely, since the sum of the zonal functions for the subspaces $H(m,n)$, $m+n=l$ must be the same as the zonal function for H_l, we arrive at the addition formula first observed in [33]:

$$\begin{aligned}\sum_{m+n=l} c_{m,n} e^{i(m-n)\phi} (\cos\theta)^{|m-n|} P^{(d-2,|m-n|)}_{m\wedge n}(\cos 2\theta) \\ = c_l C^{(d-1)}_l(\cos\theta\cos\phi).\end{aligned} \tag{4.4}$$

If we integrate this equation with respect to ϕ we get the following lemma which will be proved for more general cases in the following section.

Lemma 4.1.

$$\int_0^{2\pi} C^{(d-1)}_{2l}(\cos\theta\cos\phi)\, d\phi = c_{l,l} P^{(d-2,0)}_l(\cos 2\theta),$$

or

$$\int_0^{2\pi} C^{(d-1)}_{2l}(t\cos\phi)\, d\phi = c_{l,l} P^{(d-2,0)}_l(2t^2-1), \quad -1 \le t \le 1.$$

A direct consequence of Theorem 4.4 is that every φ in $L^2(\mathbb{S}^{d-1}(\mathbb{C}))$ can be expressed as

$$\varphi = \sum_{m,n=0}^{\infty} h_{m,n}, \quad h_{m,n} \in H(m,n).$$

Using the orthogonality of the spaces $H(m,n)$ we can deduce that

$$h_{m,n}(w) = \int_{\mathbb{S}^{d-1}(\mathbb{C})} \varphi(z)\overline{Z^{(m,n)}_w(z)}\, d\mu_{2d}(z).$$

From Theorem 4.3 and (4.4) we get

$$\varphi(w) = \sum_{k=0}^{\infty} h_k(w), \quad h_k \in H_k \tag{4.5}$$

where

$$h_k(w) = \int_{\mathbb{S}^{d-1}(\mathbb{C})} \varphi(z) \sum_{m+n=k} \overline{Z_w^{(m,n)}(z)} d\mu_{2d}(z) = \int_{\mathbb{S}^{d-1}(\mathbb{C})} \varphi(z) C_k^{(d-1)} \left(\langle z, w\rangle_{\mathbb{R}}\right) d\mu_{2d}(z).$$

4.2 The Integral formula of the Zonal Spherical harmonic

In this section we will show that the spherical averages of Gegenbauer polynomials are Jacobi polynomials. The result for averaging over the circle ($j = 1$) in the theorem is given as Lemma 4.1. We begin by defining the Pochhammer symbol $(a)_k = 1$ if $k = 0$, and $(a)_k = \Gamma(a+k)/\Gamma(a)$. Hence for an integers a and k, $(a)_k = a(a+1)\cdots(a+k-1)$, if $k \geq 0$. The Gauss hypergeomteric function is then

$$F(a,b;c;x) = 1 + \sum_{i=1}^{\infty} \frac{(a)_i (b)_i}{i!(c)_i} x^i.$$

We will also need a preliminary result. The Beta function is defined by the integral

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad \operatorname{Re} x > 0, \ \operatorname{Re} y > 0,$$

which also satisfies the following

$$\begin{aligned} B(x,y) &= 2\int_0^{\frac{\pi}{2}} (\sin t)^{2x-1}(\cos t)^{2y-1} dt, \\ B(x,y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \end{aligned}$$

Hence using the formula $\Gamma(2x) = 2^{2x-1}\pi^{-1/2}\Gamma(x)\Gamma(x+1/2)$, we obtain

$$\int_0^{\pi} \sin^{2m}\theta \cos^{2n}\theta d\theta = \frac{\Gamma(\frac{2m+1}{2})\Gamma(\frac{2n+1}{2})}{\Gamma(m+n+1)} = \frac{\pi}{2^{2m+2n}} \frac{(2m)!(2n)!}{m!n!(n+m)!}. \tag{4.6}$$

Also Jacobi and Gegenbauer polynomials can be defined in terms of Gauss hypergeometric function as follows

$$\begin{aligned} P_k^{(\alpha,\beta)}(x) &= \binom{k+\alpha}{k} F\left(-k, k+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right), \\ C_{2k}^{\lambda}(x) &= (-1)^k \binom{k+\lambda-1}{k} F(-k, k+\lambda; 1/2; x^2), \end{aligned}$$

see for example Erdélyi [18].

Theorem 4.6. *Let* $\xi = (1, 0, \cdots, 0)$ *belong to the unit sphere* $\mathbb{S}^{2j-1}(\mathbb{R})$*. Then*

$$\int_{\mathbb{S}^{2j-1}(\mathbb{R})} C_{2k}^{\lambda}\left(\langle t\xi, \eta\rangle_{\mathbb{R}}\right) d\mu_{2j}(\eta) = \omega_{2j-2}\pi \frac{(\lambda)_k (1/2)_{j-1}}{(k+j-1)!} P_k^{(\lambda-j, j-1)}(2t^2-1),$$

where ω_{2j-2} *denotes the total area of the unit sphere in* $\mathbb{R}^{2j-1}$*.*

Proof. Let

$$I = \int_{\mathbb{S}^{2j-1}(\mathbb{R})} C_{2k}^{\lambda}\left(\langle t\xi, \eta\rangle_{\mathbb{R}}\right) d\mu_{2j}(\eta) = \omega_{2j-2} \int_0^{\pi} C_{2k}^{\lambda}(t\cos\vartheta) \sin^{2j-2}\vartheta d\vartheta.$$

Since the Ultraspherical polynomial C_{2k}^{λ} can be written as

$$C_{2k}^{\lambda}(x) = (-1)^k \binom{k+\lambda-1}{k} F(-k, k+\lambda; 1/2; x^2)$$

we have

$$I = A_{k,j}^{\lambda} \left(\sum_{v=0}^{\infty} \frac{(-k)_v (k+\lambda)_v t^{2v}}{v!(1/2)_v} \int_0^{\pi} \cos^{2v}\vartheta \sin^{2(j-1)}\vartheta d\vartheta \right)$$

where $A_{k,j}^{\lambda} = \omega_{2j-2}(-1)^k \binom{k+\lambda-1}{k}$. By using (4.6) we get

$$I = A_{k,j}^{\lambda} \left(\sum_{v=0}^{\infty} \frac{(-k)_v (k+\lambda)_v}{v!(1/2)_v} \frac{(2j-2)!(2v)!\pi}{(v!)((j-1)!)(v+j-1)! 2^{2v+2j-2}} t^{2v} \right).$$

Since the following identities are true,

$$(2v)! = 2^{2v} v! (1/2)_v,$$

$$(v+j-1)! = (j)_v(j-1)!,$$

then we can see that

$$I = A^{\lambda}_{k,j}\left(\sum_{v=0}^{\infty}\frac{(-k)_v(k+\lambda)_v}{v!}\frac{(2j-2)!\pi}{((j-1)!)^2(j)_v 2^{2j-2}}t^{2v}\right)$$

$$= A^{\lambda}_{k,j}\frac{(2j-2)!\pi}{2^{2j-2}\left((j-1)!\right)^2}\left(\sum_{v=0}^{\infty}\frac{(-k)_v(k+\lambda)_v}{v!(j)_v}t^{2v}\right).$$

On the other hand, the Jacobi polynomials can be written as

$$\begin{aligned}
P_k^{(\lambda-j,j-1)}(2s^2-1) &= (-1)^k P_k^{(j-1,\lambda-j)}(1-2s^2) \\
&= (-1)^k\binom{k+j-1}{k}F(-k,k+\lambda;j;s^2) \\
&= (-1)^k\binom{k+j-1}{k}\left(\sum_{n=0}^{\infty}\frac{(-k)_n(k+\lambda)_n}{n!(j)_n}s^{2n}\right).
\end{aligned}$$

Therefore, we have

$$I = \omega_{2j-2}\binom{k+\lambda-1}{k}\binom{k+j-1}{k}^{-1}\frac{(2j-2)!\pi}{2^{2j-2}\left((j-1)!\right)^2}P_k^{(\lambda-j,j-1)}(2t^2-1)$$

$$= \omega_{2j-2}\pi\frac{(\lambda)_k(1/2)_{j-1}}{(k+j-1)!}P_k^{(\lambda-j,j-1)}(2t^2-1)$$

and this completes the proof. □

4.3 Jacobi Generating Function

In this section we reprove the Jacobi generating function given in [28] for a restricted set of indices. In the first subsection we consider the special case of $\lambda \in \mathbb{N}$ which arises from the study of complex spheres. For this case $j = 1$, we are considering identification of points on a circle. We will show that the generating function side of the equation can be written as a Legendre polynomial (a Gegenbauer polynomial with $\lambda = 1/2$).

4.3.1 Jacobi Generating Function when $j = 1$

We start with the Gegenbauer generating function which is given by

$$\left(1 - 2t\cos\vartheta w + w^2\right)^{-(d-1)} = \sum_{n=0}^{\infty} w^n C_n^{(d-1)}(t\cos\vartheta), \quad 0 < w < 1, \quad 0 < t < 1.$$

Integrating both sides then leads to

$$\int_0^{2\pi} \left(1 - 2t\cos\vartheta w + w^2\right)^{-(d-1)} d\vartheta = \sum_{n=0}^{\infty} w^n \int_0^{2\pi} C_n^{(d-1)}(t\cos\vartheta) d\vartheta$$

and by using the Lemma 4.1, we obtain

$$I = \frac{1}{2\pi}\int_0^{2\pi} \left(1 - 2t\cos\vartheta w + w^2\right)^{-(d-1)} d\vartheta = \sum_{n=0}^{\infty} w^{2n} \frac{(d-1)_n}{n!} P_n^{(d-2,0)}(2t^2 - 1).$$

To evaluate the real integral I we will apply the residue theorem, where the integral can be identified as a contour integral of a complex function around the positively oriented unit circle C. We parametrize C by

$$z = e^{i\vartheta}, \qquad 0 \le \vartheta \le 2\pi.$$

Hence

$$dz = ie^{i\vartheta} d\vartheta, \qquad \cos\vartheta = \frac{1}{2}\left(\frac{z^2+1}{z}\right).$$

Therefore,

$$I = \frac{1}{2\pi}\int_0^{2\pi} \left(1 - 2t\cos\vartheta w + w^2\right)^{-(d-1)} d\vartheta = \frac{1}{2\pi i}\oint \frac{z^{d-2}}{(z + zw^2 - twz^2 - tw)^{d-1}} dz$$

$$= \frac{1}{2\pi i(-tw)^{d-1}} \oint \frac{z^{d-2}}{\left(z^2 - \frac{(1+w^2)}{tw} z + 1\right)^{d-1}} dz,$$

since the function $f(z) = z^{d-2} / \left(z^2 - \frac{(1+w^2)}{tw} z + 1\right)^{d-1}$ has isolated singularities at the points

$$z_0 = \frac{(1+w^2)}{2tw} - \sqrt{\frac{(1+w^2)^2}{(2tw)^2} - 1}, \quad z_0' = \frac{(1+w^2)}{2tw} + \sqrt{\frac{(1+w^2)^2}{(2tw)^2} - 1},$$

with z_0 lying inside C. Thus, by using residue theory, we get

$$I = \frac{2\pi i}{2\pi i(-tw)^{d-1}} \frac{1}{(d-2)!} \left[\frac{d^{d-2}}{dz^{d-2}} \frac{z^{d-2}}{(z-z_0')^{d-1}} \right]_{z_0}$$

and obtain, after some algebra,

$$I = \frac{1}{(-tw)^{d-1}} \frac{1}{(d-2)!} \frac{(-1)^{-d+1}}{z_0'} \left[\frac{d^{d-2}}{dz^{d-2}} \frac{\left(\frac{z}{z_0'}\right)^{d-2}}{(1-\frac{z}{z_0'})^{d-1}} \right]_{z_0} .$$

Using the following identity,

$$\left[\frac{d^{d-2}}{dz^{d-2}} \frac{\left(\frac{z}{z_0'}\right)^{d-2}}{(1-\frac{z}{z_0'})^{d-1}} \right]_{z_0} = \left(\frac{1}{z_0'}\right)^{d-2} \left[\frac{d^{d-2}}{dz^{d-2}} \frac{(z)^{d-2}}{(1-z)^{d-1}} \right]_{\frac{z_0}{z_0'}}$$

the integral becomes,

$$I = \frac{1}{(tw)^{d-1}} \frac{1}{(d-2)!} \left(\frac{1}{z_0'}\right)^{d-1} \left[\frac{d^{d-2}}{dz^{d-2}} \frac{(z)^{d-2}}{(1-z)^{d-1}} \right]_{\frac{z_0}{z_0'}} .$$

To complete the proof we need the following hypergeometric equations (see e.g. [19]),

- **i.** $\frac{d^n}{dz^n} \left[z^{n+c-1} (1-z)^{b-c} \right] = (c)_n z^{c-1} (1-z)^{b-c-n} F(-n, b; c; z)$,
- **ii.** $F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z)$,
- **iii.** $F(a, b; 1+a-b; z) = (1-z)^{-a} F\left(\frac{a}{2}, \frac{a+1-2b}{2}; 1+a-b; -4z/(1-z)^2\right)$,
- **iv.** $F(a, b; c; z) = (1-z)^{-a} F(a, c-b; c; z/(z-1))$.

From **(i)**, with $n = d-2$, $b = 2-d$ and $c = 1$, and noting that $(c)_n = (d-2)!$, the integral I can be written as

$$I = \frac{1}{(tw)^{d-1}} \left(\frac{1}{z_0'}\right)^{d-1} \left(1 - \frac{z_0}{z_0'}\right)^{-2d+3} F\left(-d+2, -d+2; 1; \frac{z_0}{z_0'}\right) . \qquad (4.7)$$

After using **(ii)** and some simple computation we obtain

$$I = \frac{1}{(tw)^{d-1}} \left(\frac{1}{z_0'}\right)^{d-1} F\left(d-1, d-1; 1; \frac{z_0}{z_0'}\right) .$$

From **(iii)** we can deduce that

$$I=\frac{1}{(tw)^{d-1}}\left(\frac{1}{z_0'}\right)^{d-1}\left(1-\frac{z_0}{z_0'}\right)^{-d+1}F\left(\frac{d-1}{2},\frac{-d}{2}+1;1;-4\frac{z_0}{z_0'}\left(1-\frac{z_0}{z_0'}\right)^{-2}\right).$$

Finally, by applying **(iv)** the integral I can be written as

$$I=\frac{1}{(tw)^{d-1}}\left(\frac{1}{z_0'}\right)^{d-1}\left(1-\frac{z_0}{z_0'}\right)^{-d+1}\left(1+4\frac{z_0}{z_0'}\left(1-\frac{z_0}{z_0'}\right)^{-2}\right)^{-\frac{d-1}{2}}$$

$$\times F\left(\frac{d-1}{2},\frac{d}{2};1;-4\frac{z_0}{z_0'}\left(1-\frac{z_0}{z_0'}\right)^{-2}\left(-4\frac{z_0}{z_0'}\left(1-\frac{z_0}{z_0'}\right)^{-2}-1\right)^{-1}\right).$$

Equivalently, by substituting the value of z_0 and z_0',

$$\frac{1}{2\pi}\int_0^{2\pi}\left(1-2t\cos\vartheta w+w^2\right)^{-(d-1)}d\vartheta=\left(1+w^2\right)^{-d+1}F\left(\frac{d-1}{2},\frac{d}{2};1;\frac{4t^2w^2}{(1+w^2)^2}\right).$$

Thus the generating function of Jacobi polynomial is given by

$$\left(1+w^2\right)^{-d+1}F\left(\frac{d-1}{2},\frac{d}{2};1;\frac{4t^2w^2}{(1+w^2)^2}\right)=\sum_{n=0}^{\infty}w^{2n}\frac{(d-1)_n}{n!}P_n^{(d-2,0)}(2t^2-1). \tag{4.8}$$

Now, in order to present the generating function (4.8) as a Legendre polynomial, we write (see [55, 4.7.13] with $\lambda=1/2$) the Legendre polynomial as follows

$$P_n\left(\frac{1+x}{1-x}\right)=(1-x)^{-n}F(-n,-n;1;x).$$

From (4.7), we get

$$\begin{aligned}
I &= \frac{1}{(tw)^{d-1}}\left(\frac{1}{z_0'}\right)^{d-1}\left(1-\frac{z_0}{z_0'}\right)^{-2d+3}F\left(-d+2,-d+2;1;\frac{z_0}{z_0'}\right)\\
&= \frac{1}{(tw)^{d-1}}\left(\frac{1}{z_0'}\right)^{d-1}\left(1-\frac{z_0}{z_0'}\right)^{-2d+3}\left(1-\frac{z_0}{z_0'}\right)^{d-2}P_{d-2}\left((\frac{z_0}{z_0'}+1)(1-\frac{z_0}{z_0'})^{-1}\right)\\
&= \frac{1}{(tw)^{d-1}}\left(\frac{1}{z_0'-z_0}\right)^{d-1}P_{d-2}\left(\frac{z_0+z_0'}{z_0'-z_0}\right)\\
&= \left(\frac{1}{R}\right)^{d-1}P_{d-2}\left(\frac{w^2+1}{R}\right),
\end{aligned}$$

where $R = \sqrt{(1+w^2)^2 - 4t^2w^2}$.

Corollary 4.1. *For* $0 < r < 1$, $-1 < s < 1$,

$$\left(\frac{1}{R}\right)^{d-1} P_{d-2}\left(\frac{w^2+1}{R}\right) = \sum_{n=0}^{\infty} w^{2n} \frac{(d-1)_n}{n!} P_n^{(d-2,0)}(2t^2-1).$$

We can rewrite this as

$$\left(\frac{1}{S}\right)^{d-1} P_{d-2}\left(\frac{r+1}{S}\right) = \sum_{n=0}^{\infty} r^n \frac{(d-1)_n}{n!} P_n^{(d-2,0)}(s),$$

where $S = \sqrt{1+r^2-2sr}$.

4.3.2 Jacobi Generating Function in The General Case

In this section we will prove the Jacobi generating function formula in a more general case by integrating over higher dimensional spheres. The main complexity in this section is the evaluation of the integral of the potential over the higher dimensional spheres.

Recalling that x is the north pole of the sphere,

$$\begin{aligned} &\int_{\mathbb{S}^{2j-1}(\mathbb{R})} \left(1 - 2wt\langle x,y\rangle_{\mathbb{R}} + w^2\right)^{-\lambda} d\mu(y) \\ &= \sum_{n=0}^{\infty} r^n \int_{\mathbb{S}^{2j-1}(\mathbb{R})} C_n^{\lambda}\left(t\langle x,y\rangle_{\mathbb{R}}\right) d\mu(y) \\ &= \omega_{2j-2}\pi(1/2)_{j-1} \sum_{n=0}^{\infty} r^{2n} \frac{(\lambda)_n}{(n+j-1)!} P_n^{(\lambda-j,j-1)}\left(2t^2-1\right), \end{aligned}$$

using Theorem 4.6 (the odd powers integrate to 0).

Setting $\langle x,y\rangle_{\mathbb{R}} = \cos\theta$ we see that

$$\int_{\mathbb{S}^{2j-1}(\mathbb{R})} \left(1 - 2wt\langle x,y\rangle_{\mathbb{R}} + w^2\right)^{-\lambda} d\mu(y) = \omega_{2j-2} \int_0^{\pi} \frac{\sin^{2j-2}\theta}{\left(1-2wt\cos\theta + w^2\right)^{\lambda}} d\theta.$$

So that

$$\frac{(j-1)!}{\pi(1/2)_{j-1}} \int_0^{\pi} \frac{\sin^{2j-2}\theta}{\left(1-2wt\cos\theta+w^2\right)^{\lambda}} d\theta = \sum_{n=0}^{\infty} r^{2n} \frac{(\lambda)_n}{(j)_n} P_n^{(\lambda-j,j-1)}\left(2t^2-1\right). \quad (4.9)$$

We can expand the integral in power of t by using the Taylor series as follows:

$$\frac{(j-1)!}{2\pi(1/2)_{j-1}}\int_0^{2\pi}\left(1-2wt\cos\theta+w^2\right)^{-\lambda}\sin^{2j-2}\theta d\theta =$$

$$\sum_{k=0}^{\infty}\frac{d^k}{dt^k}\left[\frac{(j-1)!}{2\pi(1/2)_{j-1}}\int_0^{2\pi}\sin^{2j-2}\theta\left(1-2tw\cos\theta+w^2\right)^{-\lambda}d\theta\right]_{t=0}\frac{t^k}{k!}.$$

We compute the derivative of the even terms as following

$$\frac{d^{2k}}{dt^{2k}}\left[\frac{(j-1)!}{2\pi(1/2)_{j-1}}\int_0^{2\pi}\sin^{2j-2}\theta\left(1-2tw\cos\theta+w^2\right)^{-\lambda}d\theta\right]_{t=0}$$

$$= \frac{(j-1)!}{2\pi(1/2)_{j-1}}\int_0^{2\pi}\sin^{2j-2}\theta(\lambda)_{2k}\left(1+w^2\right)^{-\lambda-2k}(2w\cos\theta)^{2k}d\theta$$

$$= \frac{(j-1)!(2w)^{2k}(\lambda)_{2k}(1+w^2)^{-\lambda-2k}}{2\pi(1/2)_{j-1}}\int_0^{2\pi}\sin^{2j-2}\theta\cos^{2k}\theta d\theta,$$

by using (4.6). After some simple computation we obtain

$$\frac{d^{2k}}{dt^{2k}}\left[\frac{(j-1)!}{2\pi(1/2)_{j-1}}\int_0^{2\pi}\sin^{2j-2}\theta\left(1-2tw\cos\theta+w^2\right)^{-\lambda}d\theta\right]_{t=0} =$$

$$\frac{(2w)^{2k}(\lambda)_{2k}(1+w^2)^{-\lambda-2k}(2k)!}{k!(j)_k 2^{2k}}.$$

The odd terms integrate to zero since $\cos\theta$ is odd about $\pi/2$.

Thus

$$\begin{aligned}\frac{(j-1)!}{2\pi(1/2)_{j-1}}&\int_0^{2\pi}\left(1-2wt\cos\theta+w^2\right)^{-\lambda}\sin^{2j-2}\theta d\theta\\ &= \sum_{k=0}^{\infty}\frac{(2w)^{2k}(\lambda)_{2k}(1+w^2)^{-\lambda-2k}}{k!(j)_k 2^{2k}}t^{2k}\\ &= (1+w^2)^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda)_{2k}}{k!(j)_k 2^{2k}}\left(\frac{4w^2t^2}{(1+w^2)^2}\right)^k,\end{aligned}$$

and by using the identity, $(\lambda)_{2k} = 2^{2k}(\lambda/2)_k((\lambda+1)/2)_k$, the last result can be written as

$$\frac{(j-1)!}{2\pi(1/2)_{j-1}}\int_0^{2\pi}\frac{\sin^{2j-2}\theta}{(1-2wt\cos\theta+w^2)^{\lambda}}d\theta = (1+w^2)^{-\lambda}F\left(\frac{\lambda}{2},\ \frac{\lambda+1}{2};\ j;\ \frac{4t^2w^2}{(1+w^2)^2}\right).$$

Consequently, we obtain again the formula in Koekoek and Swarttouw [28, 1.8.13],

$$(1+w^2)^{-\lambda}F\left(\frac{\lambda}{2},\ \frac{\lambda+1}{2};\ j;\ \frac{4t^2w^2}{(1+w^2)^2}\right)=\sum_{n=0}^{\infty}w^{2n}\frac{(\lambda)_n}{(j)_n}P_n^{(\lambda-j,j-1)}\left(2t^2-1\right).$$

4.3.3 Jacobi Generating Function as Associated Legendre Function

In this section we show that the generating function of Jacobi polynomials, given in Koekoek and Swarttouw [28, 1.8.13], can be written in general in term of the associated Legendre function of the first kind which is defined as follows (see e.g. Erdélyi, A., et al. [18]).

Definition 4.5. The associated Legendre function of the first kind is a solution of the Legendre's differential equation

$$(1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\left(v(v+1)-\mu^2(1-z^2)^{-1}\right)w=0,$$

which is given by

$$P_v^{\mu}(z)=\frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{1/2\mu}F\left(-v,v+1;1-\mu;\frac{1}{2}-\frac{1}{2}z\right),\quad |1-z|<2,$$

$v\in\mathbb{R}$ and $\mu\neq 1,2,\cdots$.

Here F is the Gauss hypergeometric function. Clearly, this function is a generalisation of Legendre polynomial P_v to non integer degree. More precisely, if $\mu=0$ and v is a non negative integer then

$$P_v^0(z)=P_v(z),\quad v=0,1,2,\cdots,$$

where P_v is Legendre polynomial of degree v. Therefore, the last result in section 4.3.1 can be considered as a special case of the following theorem.

Theorem 4.7. *The Jacobi generating function which is given by the formula*

$$(1+t)^{-\alpha-\beta-1}F\left(\frac{\alpha+\beta+1}{2},\ \frac{\alpha+\beta+2}{2};\ \beta+1;\ \frac{2t(x+1)}{(t+1)^2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n}{(\beta+1)_n} P_n^{(\alpha,\beta)} t^n, \tag{4.10}$$

can be written as the associated Legendre function P_n^μ of the first kind as follows:

$$2^\beta R^{-\alpha-1} \left(-2(x+1)t\right)^{-\beta/2} P_\alpha^{-\beta}\left(\frac{1+t}{R}\right) = \sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n}{(\beta+1)_n} P_n^{(\alpha,\beta)} t^n,$$

where $R = \sqrt{t^2 - 2xt + 1}$, *and* $\alpha \in \mathbb{R}$, $\beta \neq -1, -2, \cdots$.

Proof. Let $I = (1+t)^{-\alpha-\beta-1} F\left(\frac{1}{2}(\alpha+\beta+1),\ \frac{1}{2}(\alpha+\beta+2);\ \beta+1;\ \frac{2t(x+1)}{(t+1)^2}\right)$,

and consider the following hypergeometric formulas,

- **i.** $F(a, \frac{a+1}{2}; b; z) = \left(\frac{1+(1-z)^{1/2}}{2}\right)^{-2a} F\left(2a, 2a-b+1; 1-a+b; \frac{1-(1-z)^{1/2}}{1+(1-z)^{1/2}}\right)$,
- **ii.** $F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z)$,
- **iii.** $F(a, b; c; z) = (1-z)^{-a} F(a, c-b; c; z/(z-1))$.

From **(i)**, with $a = (\alpha+\beta+1)/2,\ b = \beta+1$ the left hand side of (4.10) can be written as

$$I = \frac{2^{\alpha+\beta+1}}{(1+t)^{\alpha+\beta+1}} \left(1 + \left(1 - \frac{2(x+1)t}{(1+t)^2}\right)^{1/2}\right)^{-(\alpha+\beta+1)}$$

$$\times F\left(\alpha+\beta+1,\ \alpha+1;\ \beta+1;\ g(x,t)\right),$$

where

$$g(x,t) = \frac{1 - \left(1 - \frac{2(x+1)t}{(1+t)^2}\right)^{1/2}}{1 + \left(1 - \frac{2(x+1)t}{(1+t)^2}\right)^{1/2}}.$$

Using **(ii)** with $a = \alpha+\beta+1,\ b = \alpha+1,\ c = \beta+1$ we see that

$$I = \frac{2^{\alpha+\beta+1}}{(1+t)^{\alpha+\beta+1}} \left(1 + \left(1 - \frac{2(x+1)t}{(1+t)^2}\right)^{1/2}\right)^{-(\alpha+\beta+1)} (1-g(x,t))^{-2\alpha-1}$$

$$\times F\left(-\alpha,\ \beta-\alpha;\ \beta+1;\ g(x,t)\right)$$

$$= \frac{2^{\alpha+\beta+1}}{(1+t)^{\alpha+\beta+1}} \left(1 + \left(1 - \frac{2(x+1)t}{(1+t)^2}\right)^{1/2}\right)^{-(\alpha+\beta+1)} (1-g(x,t))^{-\alpha-1}$$

$$\times F\left(-\alpha,\ \alpha+1;\ \beta+1;\ \frac{g(x,t)}{g(x,t)-1}\right)$$

by applying **(iii)**. Since the associated Legender function of the first kind is given by

$$P_v^{\mu}(z) = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} F\left(-v,\ v+1;\ 1-\mu;\ \frac{1-z}{2}\right),\ |1-z| < 2,$$

then we equivalently have

$$\begin{aligned} I &= \frac{2^{\alpha+\beta+1}}{(1+t)^{\alpha+\beta+1}} \left(1 + \left(1 - \frac{2(x+1)t}{(1+t)^2}\right)^{1/2}\right)^{-(\alpha+\beta+1)} (1-g(x,t))^{-\alpha-1} \\ &\quad \times (-g(x,t))^{-\beta/2} P_\alpha^{-\beta}\left(\frac{g(x,t)+1}{1-g(x,t)}\right) \\ &= 2^{\beta} (1+t+R)^{-\beta} R^{-\alpha-1} \left(\frac{R-(1+t)}{R+1+t}\right)^{-\beta/2} P_\alpha^{-\beta}\left(\frac{1+t}{R}\right) \\ &= 2^{\beta} (-2(x+1)t)^{-\beta/2} R^{-\alpha-1} P_\alpha^{-\beta}\left(\frac{1+t}{R}\right) \end{aligned}$$

where $R = \sqrt{t^2 - 2tx + 1}$. □

The main result of this chapter, Theorem 4.6, was new to us and we arrived at it after studying the zonal kernels function in the complex projective space and real space. We later discovered that this result has been proved before by Dijksma and Koornwinder [15]. They proved an integral representation for the product $P_n^{(\alpha,\beta)}(1-2s^2)P_n^{(\alpha,\beta)}(1-2t^2)$ in term of Gegenbauer polynomials. In fact, they proved that the integral of Gegenbauer polynomials is invariant under the action of orthogonal transformations which leaves a subspace fixed, and then used a theorem due to Braaksma and Meulenbeld, which is beyond the scope of this thesis, to show that this integral is in fact the product of Jacobi polynomials.

Chapter 5

m-Term Approximation on The Complex Sphere

In this chapter we develop new elements of harmonic analysis on the complex sphere on the basis of which Bernstein's, Jackson's, and Kolmogorov's inequalities are established. We apply these results to get order sharp estimates of m-term approximations on the standard scale of Sobolev's sets of smooth functions in the standard L_p spaces on the complex spheres. In particular, when $1 < q < p \leq 2$, we present estimates of m-term approximation of general multiplier operators to underling the difference between harmonic analysis on the real d-sphere and the complex sphere. The results obtained are a synthesis of new results on classical orthogonal polynomials, harmonic analysis on manifolds and geometric properties of Euclidean spaces. It is natural to call the m-term approximations considered here harmonic m-widths by analogy with the known trigonometric m-widths. It is already known that Kolmogorov's n-widths, defined as

$$d_n(\mathcal{K}, X) := \inf_{L_n \subset X} \sup_{x \in \mathcal{K}} \inf_{y \in L_n} \|x - y\|_X,$$

where $\mathcal{K}$ is a subset of the normed space X and L_n is a subspace of X of dimension n, can be bigger, less or equal to the respective n-term approximations. Observe that m-term approximation is a highly nonlinear method of approximation. In particular, in this chapter we show that in the case of Sobolev's classes W_p^γ it is not possible to improve the rate of convergence in L_q, $1 \leq q \leq p \leq \infty$ by using m-term approximation instead of linear methods of polynomial approximation.

In this chapter there are several universal constants. These positive constants are mostly denoted by $C, C_1, \cdots$. We will only distinguish between the different constants where confusion is likely to arise. For ease of notation we will write $a_n \ll b_n$ for two sequences, if $a_n \leq C \cdot b_n \ \forall n \in \mathbb{N}$ for some constant C, and $a_n \asymp b_n$, if $C_1 \cdot b_n \leq a_n \leq C_2 \cdot b_n \ \forall n \in \mathbb{N}$ and some constants C_1 and C_2. Also, we shall put $(a)_+ := \max\{a, 0\}$. Since we are working on the unit sphere $\mathbb{S}^{d-1}(\mathbb{C})$, we will use $\|\cdot\|_p$ instead of $\|\cdot\|_{L_p(\mathbb{S}^{d-1}(\mathbb{C}))}$.

Our lower bounds of $m-$term approximations are essentially based on Bernstein's inequality [33]

$$\|t_N^{(\gamma)}\|_r \leq N^\gamma \|t_N\|_r, \gamma > 0, \ 1 \leq r \leq \infty \quad \forall t_N \in \mathcal{T}_N, \tag{5.1}$$

where $\mathcal{T}_N$ is the direct sum of the spaces $H_k, \ k = 1, 2, \cdots, N$ defined in (5.4) and $t_N^{(\gamma)}$ is defined in Section 5.1. This inequality can be generalised to the more general case using different norms. Let $p, q \geqslant 1$ such that $q < p$, then

$$\|t_N\|_q \leqslant \|t_N\|_p,$$

and hence using (5.1), we get

$$\|t_N^{(\gamma)}\|_q \leq N^\gamma \|t_N\|_p, \gamma > 0.$$

Now, if $q > p$ consider the operator $\Lambda t_N = t_N^{(\gamma)}$, and consider

$$\Lambda : L_1 \longrightarrow L_q \qquad \Lambda : L_q \longrightarrow L_q$$

with norms C_1 and C_2 respectively. Then there exist $\theta \in (0, 1)$ such that

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{q} \quad \textit{and} \quad \frac{1}{q} = \frac{1-\theta}{q} + \frac{\theta}{q}, \tag{5.2}$$

and hence by Riesz-Thorin interpolation theorem

$$\|\Lambda\|_{L_p \longrightarrow L_q} \leqslant C_1^{1-\theta} C_2^{\theta}.$$

It is clear that $C_2 \leqslant N^\gamma$ and for C_1 we need to estimate the norm $\|\Lambda\|_{L_1 \longrightarrow L_q}$. By following the same process of the proof of Theorem 3.4 in [33], it is easy to show that $\|S_l^n\|_q \leqslant l^{\frac{2d(q-1)}{q}}$ where S_l^n is the Cesaro means of the reproducing kernels for H_k, $0 \leqslant k \leqslant l$. Consequently, $\|t_N^{(\gamma)}\|_q \leq N^{\gamma + 2d(1-1/q)} \|t_N\|_1$, (see [33] for more details). Thus $C_1 \leqslant N^{\gamma + \frac{2d(q-1)}{q}}$ and by using (5.2) with some algebra,

$$\|t_N^{(\gamma)}\|_q \leq N^{\gamma+2d(1/p-1/q)_+}\|t_N\|_p, \gamma > 0,\ 1 \leq p, q \leq \infty \quad \forall t_N \in \mathcal{T}_N. \tag{5.3}$$

We will need some general definitions.

Let $\Omega_m := \{k_1 < \cdots < k_m\} \subset \mathbb{N}$ and $\Xi_r(\Omega_m) := \mathrm{lin}\{\mathrm{H}_{k_l}\}_{l=1}^m$, where $r = \dim\{h_{k_l}\}_{l=1}^k$ and H_{k_l} is the space of spherical harmonic of degree k_l given in Chapter 3. In the special case $\Omega_m = \{1, 2, \cdots, m\}$ we shall write

$$\mathcal{T}_m := \mathrm{lin}\{\mathrm{H}_k\}_{k=1}^m. \tag{5.4}$$

It is known that, see e.g [33],

$$\dim(\mathcal{T}_m) \asymp m^{2d-1} = m^t. \tag{5.5}$$

Here d is the topological dimension of the complex sphere over reals, hence $t = 2d - 1$.

Let $\{\xi_k\}_{k\in\mathbb{N}}$ be a sequence of orthonormal functions on $\mathbb{S}^{d-1}(\mathbb{C})$. Let X be a Banach space of functions on $\mathbb{S}^{d-1}(\mathbb{C})$ with the norm $\|\cdot\|_X$ such that $\xi_k \in X$ $\forall k \in \mathbb{N}$. Clearly, $\Xi_r(X) := \mathrm{lin}\{\xi_1, \cdots, \xi_r\} \subset X$, $r \in \mathbb{N}$, is a sequence of closed subspaces of X with the norm induced by X. Consider the coordinate isomorphism J defined as

$$\begin{array}{rcl} J:\ \mathbb{R}^r & \longrightarrow & \Xi_r(X) \\ \alpha = (\alpha_1, \cdots, \alpha_r) & \longmapsto & \sum_{k=1}^r \alpha_k \cdot \xi_k. \end{array}$$

Hence, the definition

$$\|\alpha\|_{J^{-1}\Xi_r(X)} = \|J\alpha\|_X$$

induces a norm on $\mathbb{R}^r$. To be able to apply methods of geometry of Banach spaces to various open problems in different spaces of functions on $\mathbb{S}^{d-1}(\mathbb{C})$ we will need to find the Levy mean of the function $\rho_r(\alpha) := \|\alpha\|_{J^{-1}\Xi_r(X)}$ on the unit sphere $\mathbb{S}^{2d-1}(\mathbb{R})$:

$$M(\|\cdot\|_{J^{-1}\Xi_r(X)}) = \int_{\mathbb{S}^{r-1}(\mathbb{R})} \|\alpha\|_{J^{-1}\Xi_r(X)} d\mu_r(\alpha).$$

As a motivating example consider the case $X = L_p := L_p\left(\mathbb{S}^{d-1}(\mathbb{C})\right)$,

$$\|\phi\|_p := \begin{cases} \left(\int_{\mathbb{S}^{d-1}(\mathbb{C})} |\phi|^p \, d\mu_{2d}\right)^{1/p}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup} |\phi|, & p = \infty. \end{cases}$$

In this case we shall write $\|\alpha\|_{(p)} = \|J\alpha\|_p$. Our general result is stated in Lemma 5.3 which gives sharp order estimates for the Levy means which correspond to the norm induced on $\mathbb{R}^r$ by the subspace $\oplus_{s=1}^m \mathrm{H}_{k_s} \cap L_p$, dim $\oplus_{s=1}^m \mathrm{H}_{k_s} = r$ with an arbitrary index set $(k_1, \cdots, k_m)$. To extend our estimates to the case $p = \infty$ we apply Lemma 5.2 which gives a useful inequality between $1 \leq p, q \leq \infty$ norms of polynomials on $\mathbb{S}^{d-1}(\mathbb{C})$. We derive lower bounds for $m-$term approximation of Sobolev's classes (5.6) using Lemmas 5.2 and 5.3, Urysohn's inequality, the Bourgain-Milman inequality and estimates of Levy means given in (5.21). Upper bounds for $m-$term approximation contained in Theorem 5.2 where we establish a Jackson type inequality. All these results are in Section 5.2, and Section 5.1 introduces some notation and preliminary background.

5.1 Preliminaries

In m-term approximation fix the basis and approximate a function by a linear combination of m terms of this basis which can be taken in any order. Hence, the selection of elements in the approximation depends on the function that is being approximated and it depends only on the number of elements to be used. The set of all linear combination of m terms of this basis is nonlinear space. Thus, in this method of approximation the approximates come from nonlinear space and hence this approximation is a nonlinear approximation.

This type of approximation was introduced by Schmidt (1907), then studied by Oskolkov (1979) for multivariate splines. During recent years m-term approximation and n-widths have become very popular in numerical methods for PDE's. Also, the idea of so-called "greedy algorithms" has been inspired by m-term approximations. All the material in this section can be found in many places. We mention the papers, which investigate nonlinear approximation, [13] and [57].

Definition 5.1. Let X be a separable (contains a countable dense subset) real Banach space and $\Xi := \{\xi_k\}_{k\in\mathbb{N}}$ be a dense subset of X, i.e., the closure of Ξ is X. For a fixed $m \in \mathbb{N}$ let $\Omega_m := \{k_1 < \cdots < k_m\} \subset \mathbb{N}$ and $\Xi(\Omega_m) := \mathrm{lin}\{\xi_{k_l}\}_{l=1}^m$.

(a) The best approximation of an element $\phi \in X$ by the subspace $\Xi(\Omega_m)$ in X is

$$\nu(\phi, \Xi(\Omega_m), X) := \inf_{\xi \in \Xi(\Omega_m)} \|\phi - \xi\|_X = \inf_{(\alpha_1, \cdots, \alpha_m) \in \mathbb{R}^m} \left\| \phi - \sum_{l=1}^{m} \alpha_l \, \xi_{k_l} \right\|_X .$$

(b) The best m-term approximation of $\phi \in X$ with regard to the given system Ξ (frequently Ξ is called dictionary) is

$$\nu_m(\phi, \Xi, X) := \inf_{\Omega_m \subset \mathbb{N}} \nu(\phi, \Xi(\Omega_m), X).$$

(c) The m-term approximation of a given set $\mathcal{K} \subset X$ is

$$\nu_m := \nu_m(\mathcal{K}, \Xi, X) := \sup_{\phi \in \mathcal{K}} \nu_m(\phi, \Xi, X).$$

The above approximation problem can be formulated in a more general way as follows. Let $\sum_m(\Xi)$ denote the set of all function in X which can be written as a linear combination of at most m element of Ξ. Thus a function $g \in \sum_m(\Xi)$ can be written as

$$g = \sum_{h \in K} a_h h, \quad K \subset \Xi, \quad |K| \leqslant m.$$

We measure the m-term error of approximation of a function ϕ by

$$\nu_m(\phi, \Xi, X) := \inf_{g \in \sum_m(\Xi)} \|\phi - g\|.$$

This method of approximation, which approximates a function $f \in X$ by functions from $\sum_m(\Xi)$, is nonlinear since the space $\sum_m(\Xi)$ is not linear. A sum of two elements from $\sum_m(\Xi)$ is generally not in $\sum_m(\Xi)$. For linear approximation methods, the approximates come from a linear space. For instance, use the space $Y_m = \{\xi_k \; : 1 \leqslant k \leqslant m\}$ to approximate an element in X. The approximation error in this case is given by

$$E_m(f) = \inf_{g \in Y_m} \|f - g\|_X.$$

Since f can be written as $f = \sum_{k=1}^{\infty} a_k \xi_k$ then

$$E_m(f) = \inf_{c_k} \left\| \sum_{k=1}^{\infty} a_k \xi_k - \sum_{k=1}^{m} c_k \xi_k \right\|_X .$$

Thus, linear approximation depends on the frequency location of the coefficient, whereas nonlinear approximation does not recognise these locations. The best way to distinguish between linear and nonlinear approximation is to study the rate of approximation, or describe the properties of a function that specifies the rate of approximation. One of the central problems of nonlinear approximation is to determine the approximation space, which is the space of all functions which have a specific approximation order (see e.g. [13] and [14]). Another challenging problem in this field is to estimate the rate of approximation for certain classes of functions and to determine the best m-term approximation.

In order to embark on our theory, it is essential to make several definitions and remarks.

We define the convolution of $f \in L_1\left(\mathbb{S}^{d-1}(\mathbb{C})\right)$ with a kernel κ as

$$(f * \kappa)(x) = \int_{\mathbb{S}^{d-1}(\mathbb{C})} f(y) \cdot \kappa(x,y)\; d\mu_{2d}(y).$$

Notice that, from (4.5) in Chapter 4, any $\phi \in L_2(\mathbb{S}^{d-1}(\mathbb{C}))$ can be written as

$$\phi \sim \sum_{k=0}^{\infty} h_k, \qquad h_k \in H_k.$$

where

$$h_k(w) = \int_{\mathbb{S}^{d-1}(\mathbb{C})} \varphi(z) C_k^{(d-1)}\left(\langle z, w\rangle_{\mathbb{R}}\right) d\mu_{2d}(z).$$

Therefore, the orthogonal projector from $L_2(\mathbb{S}^{d-1}(\mathbb{C}))$ into H_k is given by the convolution $\phi \to C_k^{(d-1)} * \varphi$. We will call $C_k^{(d-1)}$ the kernel of the orthogonal projector $L_2(\mathbb{S}^{d-1}(\mathbb{C})) \to H_k$, and is denote it by M_k. Hence ϕ can be written as

$$\phi \sim \sum_{k \in \mathbb{N} \cup \{0\}} M_k * \phi.$$

Definition 5.2. Let $\{Y_m^k\}_{m=1}^{d_k}$ be an orthonormal basis of H_k, and let $\theta_k = k(k + d - 1)$ be the eighnvalue of the space H_k. Let also ϕ be an arbitrary function in

$L_p(\mathbb{S}^{d-1}(\mathbb{C}))$, $1 \le p \le \infty$ with the formal Fourier series

$$\phi \sim \sum_{k\in\mathbb{N}\cup\{0\}} \sum_{m=1}^{d_k} c_{k,m}(\phi)\cdot Y_m^k, \;\; c_{k,m}(\phi) = \int_{\mathbb{S}^{d-1}(\mathbb{C})} \phi \cdot \overline{Y_m^k} d\mu_{2d}.$$

(a) The γ-th fractional integral $I_\gamma \phi := \phi_\gamma$, $\gamma > 0$, is defined as

$$\phi_\gamma \sim \sum_{k\in\mathbb{N}\bigcup\{0\}} \theta_k^{-\gamma/2} \sum_{m=1}^{d_k} c_{k,m}(\phi)\cdot Y_m^k, \;\; C \in \mathbb{R}. \tag{5.6}$$

(b) The function $D_\gamma \phi := \phi^{(\gamma)}$ is called the γ-th fractional derivative of ϕ if

$$\phi^{(\gamma)} \sim \sum_{k\in\mathbb{N}\bigcup\{0\}} \theta_k^{\gamma/2} \sum_{m=1}^{d_k} c_{k,m}(\phi)\cdot Y_m^k.$$

(c) The Sobolev classes W_p^γ are defined as sets of functions given by

$$W_p^\gamma = \{\varphi \in L_p\left(\mathbb{S}^{d-1}(\mathbb{C})\right) : \varphi^{(\gamma)} \in L_p\left(\mathbb{S}^{d-1}(\mathbb{C})\right) \;\; and \;\; \|\varphi^{(\gamma)}\|_p \leqslant 1\}.$$

The Sobolev classes given in the definition is equal to (see [34])

$$W_p^\gamma = \{c + \varphi_\gamma : c \in \mathbb{R}, \;\; \varphi \in B_p\},$$

where $B_p = \{f \in L_p(\mathbb{S}^{d-1}(\mathbb{C})) : \|f\|_p \leqslant 1\}$. For $\mathbb{S}^{d-1}(\mathbb{C})$ the following addition formula is straightforward to show (see [24]):

$$\sum_{m=1}^{d_k} |Y_m^k(x)|^2 = d_k \;\; \forall x \in \mathbb{S}^{n-1}(\mathbb{C}). \tag{5.7}$$

Corresponding to the above definition, we define a multiplier operator which acts on the harmonic spaces $H(m,n)$.

Definition 5.3. Let $\{Y_s^{(m,n)}\}_{s=1}^{d_{m,n}}$ be an orthonormal basis of $H(m,n)$, and let ϕ be an arbitrary function in $L_2(\mathbb{S}^{d-1}(\mathbb{C}))$ with the formal Fourier series

$$\phi \sim \sum_{n,m\geqslant 0} \sum_{s=1}^{d_{m,n}} c_s^{m,n}(\phi)\cdot Y_s^{(m,n)}, \tag{5.8}$$

where

$$c_s^{m,n}(\phi) = \int_{\mathbb{S}^{d-1}(\mathbb{C})} \phi \cdot \overline{Y_s^{(m,n)}} d\mu_{2d}.$$

Let $\Lambda = \{\lambda_{m,n}\}_{m,n\geqslant 0}$ be a fixed sequence of real numbers. We say that Λ is of (p,q) type, $1 \leq p, q \leq \infty$, if for any $\phi \in L_p(\mathbb{S}^{d-1}(\mathbb{C}))$ with the formal Fourier series (5.8) there is a function $f = \Lambda\phi \in L_q(\mathbb{S}^{d-1}(\mathbb{C}))$ with formal Fourier series

$$f \sim \sum_{n,m\geqslant 0} \lambda_{m,n} \sum_{s=1}^{d_{m,n}} c_s^{m,n}(\phi) \cdot Y_s^{(m,n)}, \tag{5.9}$$

Finally, denote by ΛB_p the set of functions f representable in the form (5.9), where $\phi \in B_p$.

Clearly, Sobolev's classes are cases of function classes ΛB_p.
Now since

$$d_{m,n} = \dim H(m,n) = \frac{(m+n+d-1)(m+d-2)!(n+d-2)!}{m!n!(d-1)!(d-2)!},$$

we can deduce that $d_{m,n} \asymp (mn)^{d-2}(m+n)$, and we have

$$\sum_{s=1}^{d_{m,n}} |Y_s^{(m,n)}(x)|^2 = d_{m,n}, \ \ \forall x \in \mathbb{S}^{d-1}(\mathbb{C}),$$

To give the estimate of the upper bounds for $m-$term approximation we need to use Cesàro means and Abel's transformation which are defined below. For these definitions we cite [63] and [5].

Given a sequence $s_0, s_1, \cdots$ we define the sequence $T_0^k, T_1^k, \cdots$ as follows

$$\begin{aligned} T_n^0 &= s_0 + s_1 + \cdots + s_n \\ T_q^m &= T_0^{m-1} + T_1^{m-1} + \cdots + T_q^{m-1}, \end{aligned}$$

for $m = 1, 2, \cdots,\ n = 0, 1, \cdots$. We can express T_q^m explicitly in terms of s_n as follows

$$T_q^m = \sum_{k=0}^{q} s_k \begin{pmatrix} q-k+m \\ q-k \end{pmatrix}.$$

Define

$$S_q^m = \frac{1}{G_q^m} T_q^m = \frac{1}{G_q^m} \sum_{k=0}^{q} s_k G_{q-k}^m,$$

where $G_q^m = \begin{pmatrix} q+m \\ q \end{pmatrix}$.

T_q^m and S_q^m are known as Cesàro sum and Cesàro means of the sequence $\{s_1, s_2, \cdots\}$ of order m, respectively, and the number G_q^m is called Cesàro number of order q and index m.

Theorem 5.1. *(Abel's Transformation) Let $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ be any real sequences, and let us define $B(n) = b_0 + \cdots + b_n$. Then for any values of m and n we find that*

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n-1} (a_k - a_{k+1}) B(k) + a_n B(n) - a_m B(m-1).$$

For $m = 0$

$$\sum_{k=0}^{n} a_k b_k = \sum_{k=0}^{n-1} (a_k - a_{k+1}) B(k) + a_n B(n).$$

Let us have $\sum_{k=0}^{n} a_k b_k$, by applying Abel's transformation we get

$$\sum_{k=0}^{n} a_k b_k = a_n B(n) + \sum_{k=0}^{n-1} B(k)(a_k - a_{k+1})$$

where $B(s) = b_0 + \cdots + b_s$, for $s = 0, \cdots, n$. Now apply Abel's transformation again on the right hand side

$$\begin{aligned} \sum_{k=0}^{n} a_k b_k &= a_n B(n) + (B(0) + \cdots + B(n-1))(a_{n-1} - a_n) \\ &+ \sum_{k=0}^{n-2} (B(0) + \cdots + B(k)) \left((a_k - a_{k+1}) - (a_{k+1} - a_{k+2}) \right). \end{aligned}$$

Performing Abel's transformation r times we get

$$\sum_{k=0}^{n} a_k b_k = \sum_{k=0}^{r-1} \Delta^k a_{n-k} T_{n-k}^k + \sum_{k=0}^{n-r} T_k^{r-1} \Delta^r a_k, \tag{5.10}$$

where $\Delta^0 a_k := a_k$, $\Delta^1 a_k = a_k - a_{k+1}$ and $\Delta^{s+1} a_k = \Delta^s a_k - \Delta^s a_{k+1}$, $k, s \in \mathbb{N}$ and T_q^m is the Cesàro sum of the sequence $\{b_1, b_2, \cdots\}$. Thus this process will be useful in order to estimate a partial sum of a sequence by using Cesàro means.

5.2 m-Term Approximation

Our upper bounds come from Jackson's type inequality.

Theorem 5.2. *Let $f \in L_p(\mathbb{S}^{n-1}(\mathbb{C}))$ and*

$$E(f, \mathcal{T}_N, L_p(\mathbb{S}^{n-1}(\mathbb{C}))) := \inf_{t_N \in \mathcal{T}_N} \|f - t_N\|_p$$

be the best approximation of f by $\mathcal{T}_N$. If $f^{(\gamma)} \in L_p(\mathbb{S}^{n-1}(\mathbb{C}))$ and $\gamma > (d-1)/2$ then

$$E(f, \mathcal{T}_N, L_p(\mathbb{S}^{n-1}(\mathbb{C}))) \leq C \cdot N^{-\gamma} \cdot E(f^{(\gamma)}, \mathcal{T}_N, L_p(\mathbb{S}^{n-1}(\mathbb{C}))), \;\; 1 \leq p \leq \infty.$$

Proof. To produce our estimates we will need some information concerning Cesàro means. The Cesàro means for the kernels M_m is

$$S_n^s := \frac{1}{G_n^s} \sum_{m=0}^{n} G_{n-m}^s \cdot M_m,$$

where

$$G_n^s \asymp n^s. \tag{5.11}$$

It is known [33] that for $0 \leq s \leq (t+1)/2$,

$$\|S_n^s\|_1 \leq C \begin{cases} n^{(t-1)/2-s}, & s \leq (t-3)/2, \\ (\log n)^2, & s = (t-1)/2, \\ 1, & s = (t+1)/2. \end{cases} \tag{5.12}$$

Fix a polynomial $\phi_M \in \mathcal{T}_M$ with $\|\phi_M\|_p \leq 1$, and let

$$K_N := \sum_{k=1}^{N} \lambda_k \cdot M_k,$$

where $\lambda_k = \theta_k^{-\gamma/2}$. Applying Abel's transform $s+1$ times, see (5.10), where $s := (t+1)/2$ we see that, for $N > s+1$,

$$K_N * \phi_M = (K^1_{N-s-1} + K^2_N) * \phi_M,$$

where

$$K^1_N := \sum_{k=1}^{N} \Delta^{s+1}\lambda_k \cdot G^s_k \cdot S^s_k,$$

$$K^2_N := \sum_{k=0}^{(t+1)/2} \Delta^k \lambda_{N-k} \cdot G^k_{N-k} \cdot S^k_{N-k},$$

$\Delta^0\lambda_k := \lambda_k$, $\Delta^1\lambda_k = \lambda_k - \lambda_{k+1}$ and $\Delta^{s+1}\lambda_k = \Delta^s\lambda_k - \Delta^s\lambda_{k+1}$, $k, s \in \mathbb{N}$. Using (5.11) and (5.12) we get

$$\left\|K^1_N\right\|_1 \leq \sum_{k=1}^{N} \left|\Delta^{s+1}\lambda_k\right| \cdot G^s_k \cdot \|S^s_K\|_1 \leq C \cdot \sum_{k=1}^{N} \left|\Delta^{s+1}\lambda_k\right| \cdot k^s \cdot \|S^s_k\|_1$$

$$\leq C \cdot \sum_{k=1}^{N} \left|\Delta^{s+1}\lambda_k\right| \cdot k^s = C \cdot \sum_{k=1}^{N} \left|\Delta^{(t+3)/2}\lambda_k\right| \cdot k^{(t+1)/2}.$$

It is easy to show that

$$|\Delta^{(t+3)/2}\lambda_k| \leqslant k^{-\gamma-(t+3)/2}$$

and hence

$$\left\|K^1_N\right\|_1 \leq C \cdot \sum_{k=1}^{N} k^{-\gamma-(t+3)/2} \cdot k^{(t+1)/2} = C \sum_{k=1}^{N} k^{-\gamma-1}, \;\; \gamma > 0. \tag{5.13}$$

Since $\sum_{k=1}^{N} k^{-\gamma-1}$ is a p-series with $\gamma + 1 > 1$ then it convergence to a constant. Hence $\|K^1_N\|_1 \leq C$.

Similarly, using (5.12) we get

$$\begin{aligned}
\|K^2_N\|_1 &\leq \sum_{k=0}^{(t+1)/2} |\Delta^k \lambda_{N-k}| \cdot G^k_{N-k} \cdot \|S^k_{N-k}\|_1 \\
&\leq C \cdot \sum_{k=0}^{(t+1)/2} |\Delta^k \lambda_{N-k}| \cdot (N-k)^k \cdot (N-k)^{(t+1)/2-1-k} \\
&\leq (N-k)^{-\gamma-k}(N-k)^{(t-1)/2} \\
&\leq C \cdot N^{-\gamma+(t-1)/2}.
\end{aligned} \tag{5.14}$$

From (5.14) follows that if $\gamma > (t-1)/2$ then

$$\lim_{N\to\infty} \|K_N^2\|_1 = 0. \tag{5.15}$$

Comparing (5.13) and (5.15) we get that for any fixed polynomial $\phi_M \in \mathcal{T}_M$, $M \in \mathbb{N}$ the sequence of functions $K_N^1 * \phi_M$ converges in L_1 to the function

$$K * \phi_M = \left(\sum_{k=1}^{\infty} \theta_k^{-\gamma/2} \cdot M_k \right) * \phi_M.$$

We remark

$$K = \lim_{N\to\infty} K_N = \lim_{N\to\infty} K_N^1 + \lim_{N\to\infty} K_N^2 = \lim_{N\to\infty} K_N^1 + 0 = \sum_{k=1}^{\infty} \Delta^{s+1}\lambda_k \cdot G_k^s \cdot S_k^s.$$

Then

$$\begin{aligned} \left\|K - K_N^1\right\|_1 &\le \sum_{k=N+1}^{\infty} \left|\Delta^{s+1}\lambda_k\right| \cdot G_k^s \cdot \|S_k^s\|_1 \\ &\le \sum_{k=N+1}^{\infty} k^{-\gamma-1} \le \sum_{k=N}^{\infty} k^{-\gamma-1} = \lim_{m\to\infty} \int_N^m x^{-\gamma-1} dx \\ &= cN^{-\gamma}. \end{aligned} \tag{5.16}$$

Fix an arbitrary polynomial $\psi_N \in \mathcal{T}_N$. For any $f = K * f^{(\gamma)}$ such that $f^{(\gamma)} \in L_p$ we have

$$\begin{aligned} E(f, \mathcal{T}_N, L_p) &\le \left\|K * f^{(\gamma)} - K * \psi_N + K_N^1 * \psi_N - K_N^1 * f^{(\gamma)}\right\|_p \\ &= \left\|K * \left(f^{(\gamma)} - \psi_N\right) - K_N^1 * \left(f^{(\gamma)} - \psi_N\right)\right\|_p \\ &\le \left\|(K - K_N^1) * \left(f^{(\gamma)} - \psi_N\right)\right\|_p \le \left\|(K - K_N^1)\right\|_1 \cdot \left\|\left(f^{(\gamma)} - \psi_N\right)\right\|_p \\ &\le C \cdot N^{-\gamma} \cdot E(f^{(\gamma)}, \mathcal{T}_N, L_p), \end{aligned} \tag{5.17}$$

where in the last line we used (5.16) and the fact that ψ_N is an arbitrary polynomial. $\square$

Remark 5.1. From Theorem 5.2, Theorem 1.33 and (5.3) we get Kolmogorov's type inequality,

$$\|f^{(\alpha)}\|_p \le C\|f^{(\beta)}\|_p^{\alpha/\beta} \cdot \|f\|_p^{1-\alpha/\beta},$$

where $1 \le p \le \infty$ and $(d-1)/2 \le \alpha \le \beta$.

To prove our lower bounds we will need several lemmas.

Lemma 5.1. *There is a sequence of functions $Q_{2N} \in \mathcal{T}_{2N}, N \in \mathbb{N}$ such that for any $t_N \in \mathcal{T}_N$ we have*

$$Q_{2N} * t_N = t_N$$

and

$$\|Q_{2N}\|_1 \le C, \ \ \forall N \in \mathbb{N}.$$

Proof. The proof of this statement is based on the norm estimates for the Cesàro means (5.12) and the line of arguments used in [34] (see Example 2.6).
Let

$$Q_{2N} = \sum_{k=0}^{2N} \lambda_k M_k,$$

where M_k is the reproducing kernel in the space H_k and λ_k is defined as follows.
Let

$$\chi_0(x) = \begin{cases} 1 & 0 \leqslant x \leqslant 1, \\ 0 & \quad x > 1, \end{cases}$$

and for $1 \leqslant s \leqslant d$,

$$\chi_s(x) = 2d \int_x^{x+\frac{1}{2d}} \chi_{s-1}(u) du.$$

The function χ_d is $d-1$ times cotinuously differentiable and positive on $[0, \infty)$, and χ_d^{d-1} is Lipschitz continuous (there is a constant c such that $|\chi_d^{d-1}(x) - \chi_d^{d-1}(y)| \leqslant |x-y|$ for all $x, y \in [0, \infty)$). Furthermore, $\chi_d(x) = 1$ for $0 \leqslant x \leqslant 1/2$, and

$$\chi_d(x) = g_d(x) = \frac{(2d)^d}{d!}(1-x)^d, \quad 1 - \frac{1}{2d} \leqslant x \leqslant 1,$$

and in each interval $[x_s, x_{s-1}]$, $1 \leqslant s \leqslant d$, where $x_s = 1 - s/(2d)$, χ_d is a polynomial of degree d.
Put

$$\lambda_k = \chi_d\left(\frac{k}{2N}\right), \quad 0 \leqslant k \leqslant 2N.$$

Then

$$\lambda_k = \chi_d\left(\frac{k}{2N}\right) = 1 \quad for \;\; 0 \leqslant k \leqslant N,$$

and therefore

$$Q_{2N} * t_N = t_N \quad for\ all \;\; t_N \in \mathcal{T}_N.$$

To complete the proof we need to show that $\|Q_{2N}\|_1 \leq C,\ \forall N \in \mathbb{N}$ for some constant C. Apply Abel's transform $d+1$ times to Q_{2N}, where $d = (t+1)/2$. Then we get, for $2N > d$, that

$$Q_{2N} = \sum_{k=0}^{2N-d-1} (\Delta^{d+1}\lambda_k) G_k^d S_k^d + \sum_{s=0}^{d} (\Delta^s \lambda_{2N-s}) G_{2N-s}^s S_{2N-s}^s,$$

where S_k^s is the Cesaro mean of M_k. By applying triangle inequality,

$$\|Q_{2N}\|_1 \leqslant \sum_{k=0}^{2N-d-1} |\Delta^{d+1}\lambda_k| G_k^d \|S_k^d\|_1 + \sum_{s=0}^{d} |\Delta^s \lambda_{2N-s}| G_{2N-s}^s \|S_{2N-s}^s\|_1$$

$$\leqslant \sum_{k=0}^{2N-d-1} |\Delta^{d+1}\lambda_k| G_k^d + \sum_{s=0}^{d} |\Delta^s \lambda_{2N-s}| G_{2N-s}^s \|S_{2N-s}^s\|_1,$$

using (5.12). Next we need to get some bounds for the differences $|\Delta^{d+1}\lambda_k|$ and $|\Delta^s \lambda_{2N-s}|$. First for $0 \leqslant s \leqslant d$ and $N > d^2$, we get $1 - 1/(2d) < 1 - s/(2N) < 1$ and $\lambda_{2N-s} = \chi_d(1 - s/2N) = g_d(1 - s/2N)$. It is easy to prove that, see [34],

$$\Delta^s \lambda_k = \Delta^s_{(2N)^{-1}} g_d\left(\frac{k}{2N}\right),$$

where

$$\Delta_h^k g_d(x) = (-1)^k \int_0^h \cdots \int_0^h g_d^{(k)}(x + u_1 + \cdots + u_k) du_1 \cdots du_k.$$

Then

$$|\Delta^s \lambda_{2N-s}| = |\Delta^s_{(2N)^{-1}} g_d(1 - \frac{s}{2N})|$$

$$\begin{aligned}
&\leqslant \int_0^{(2N)^{-1}} \cdots \int_0^{(2N)^{-1}} |g_d^{(s)}(1-\frac{s}{2N}+u_1+\cdots+u_s)|du_1\cdots du_s \\
&\leqslant (2N)^{-s} \max_{1-s/(2N)\leqslant u\leqslant 1} |g_d^{(s)}(u)| = (2N)^{-s}\frac{(2d)^d}{(d-s)!}(\frac{s}{2N})^{d-s} \\
&\leqslant 2^d d^{2d}(2N)^{-d},
\end{aligned}$$

for $0\leqslant s\leqslant d,\quad N>d^2$. On the other hand, since $G^s_{2N-s}\asymp(2N-s)^s$, we have

$$\begin{aligned}
(2N)^{-d}G^s_{2N-s}\|S^s_{2N-s}\|_1 &\leqslant c_0\begin{cases}(2N)^{-1}, & s\leqslant d-2,\\ (2N)^{-d}(2N-s)^s(\log(2N-s))^2, & s=d-1,\\ (\frac{2N-s}{2N})^d, & s=d,\end{cases}\\
&\leqslant c_0\begin{cases}(2N)^{-1}, & s\leqslant d-2,\\ (2N)^{-d+s}(\log(2N-s))^2, & s=d-1,\\ 1, & s=d.\end{cases}
\end{aligned}$$

Since

$$\lim_{N\to\infty}\sum_{s=0}^{d-1}|\Delta^s\lambda_{2N-s}|G^s_{2N-s}\|S^s_{2N-s}\|_1=0$$

we then have

$$\sum_{s=0}^{d-1}|\Delta^s\lambda_{2N-s}|G^s_{2N-s}\|S^s_{2N-s}\|_1+|\Delta^n\lambda_{2N-n}|G^n_{2N-n}\|S^n_{2N-n}\|_1\leqslant c', \tag{5.18}$$

for some constant c'.

Secondly, we need to find an upper bound for $|\Delta^{d+1}\lambda_k|$, for $0\leqslant k\leqslant 2N-d-1$. Since χ_d is $d-1$ times continuously differentiable, it follows from the definition of differences that

$$\begin{aligned}
\left|\Delta^{d+1}\lambda_k\right| &= \left|\Delta^d\lambda_k-\Delta^d\lambda_{k+1}\right|\leqslant\left|\Delta^d\lambda_k\right|+\left|\Delta^d\lambda_{k+1}\right|\\
&= \left|\Delta^{d-1}\lambda_k-\Delta^{d-1}\lambda_{k+1}\right|+\left|\Delta^{d-1}\lambda_{k+1}-\Delta^{d-1}\lambda_{k+2}\right|\\
&= \left|\Delta^{d-1}_{(2N)^{-1}}\chi_d(\frac{k}{2N})-\Delta^{d-1}_{(2N)^{-1}}\chi_d(\frac{k+1}{2N})\right|\\
&+ \left|\Delta^{d-1}_{(2N)^{-1}}\chi_d(\frac{k+1}{2N})-\Delta^{d-1}_{(2N)^{-1}}\chi_d(\frac{k+2}{2N})\right|
\end{aligned}$$

Since $\chi_d^{(d-1)}$ is Lipschitz continuous, we get

$$|\Delta^{d+1}\lambda_k| \leqslant \frac{c''}{2}(2N)^{-d} + \frac{c''}{2}(2N)^{-d} = c''(2N)^{-d},$$

for some constant c''. Therefore, from [34] (see Example 2.6), we have

$$\sum_{k=0}^{2N-d-1} |\Delta^{d+1}\lambda_k| G_k^d \| S_k^d \|_1 \leqslant \sum_{k=0}^{2N-d-1} |\Delta^{d+1}\lambda_k| k^d \leqslant c''(d+1)(d+2). \tag{5.19}$$

Finally, from (5.18) and (5.19) it follows that $\|Q_{2N}\|_1 \leq C, \ \forall N \in \mathbb{N}$. $\square$

Lemma 5.2. *For any Ω_m and any $\xi \in \Xi_r(\Omega_m)$, $m \in \mathbb{N}$ we have*

$$\|\xi\|_q \leq r^{(1/p-1/q)_+} \cdot \|\xi\|_p,$$

where $1 \leq p, q \leq \infty$ and $r := \dim \Xi_r(\Omega_m)$.

Proof. Let $\{\eta_i\}_{i=1}^r$ be an orthonormal basis of $\Xi_r(\Omega_m)$ and let

$$K_r(x,y) := \sum_{i=1}^r \eta_i(x) \cdot \overline{\eta_i(y)},$$

be the reproducing kernel for $\Xi_r(\Omega_m)$. Clearly,

$$K_r(x,y) = \int_{\mathbb{S}^{d-1}(\mathbb{C})} K_r(x,z) \cdot K_r(z,y) \cdot d\mu_{2d}(z),$$

and $K_r(x,y) = \overline{K_r(y,x)}$. Hence, using the Hölder inequality,

$$\|K_r(\cdot,\cdot)\|_\infty \leq \|K_r(y,\cdot)\|_2 \cdot \|K_r(x,\cdot)\|_2$$

for any $x, y \in \mathbb{S}^{d-1}(\mathbb{C})$. Due to the addition formula (5.7), we have

$$\begin{aligned} \|K_r(x,\cdot)\|_2^2 &= \int_{\mathbb{S}^{d-1}(\mathbb{C})} K_r(x,z) K_r(x,z) d\mu_{2d}(z) \\ &= K_r(x,x) = \sum_{i=1}^r \eta_i(x) \cdot \overline{\eta_i(x)} \\ &= \sum_{i=1}^r |\eta_i(x)|^2 = r. \end{aligned}$$

Hence, $\|K_r(x,\cdot)\|_2 = r^{1/2}$, and then

$$\|K_r(\cdot,\cdot)\|_\infty \leq r. \tag{5.20}$$

Let $\xi \in \Xi_r(\Omega_m)$. Then applying Young's inequality and (5.20) we get

$$\|\xi\|_\infty = \|\xi * K_r\|_\infty \leq \|K_r(\cdot,\cdot)\|_\infty \cdot \|\xi\|_1 \leq r \cdot \|\xi\|_1,$$

and hence

$$\|I\|_{L_1(\mathbb{S}^{d-1}(\mathbb{C}))\cap\Xi_r(\Omega_m)\to L_\infty(\mathbb{S}^{d-1}(\mathbb{C}))\cap\Xi_r(\Omega_m)} \leq r,$$

where $I : L_p \to L_q$ is the embedding operator. Trivially,

$$\|I\|_{L_p(\mathbb{S}^{d-1}(\mathbb{C}))\cap\Xi_r(\Omega_m)\to L_p(\mathbb{S}^{d-1}(\mathbb{C}))\cap\Xi_r(\Omega_m)} = 1,$$

where $1 \leq p \leq \infty$. Hence, using the Riesz-Thorin interpolation theorem and embedding arguments we obtain

$$\|\xi\|_q \leq r^{(1/p-1/q)_+} \cdot \|\xi\|_p \ \ \forall \xi \in \Xi_r(\Omega_m),\ 1 \leq p,q \leq \infty. \quad \square$$

Let us fix a norm $\|\cdot\|$ on $\mathbb{R}^r$ and let $E = (\mathbb{R}^r, \|\cdot\|)$ with the unit ball B_E. The dual space E^o is the Banach space of all bounded operators from E to $\mathbb{R}$, and the dual norm is

$$\|x'\|^o = \sup_{\|x\|\leqslant 1} |x'(x)| = \sup_{x\in B_E} |x'(x)|,\ x' \in E^o,\ x \in E.$$

If E is a Hilbert space then by Riesz theorem for any $x' \in E'$ there exist $x \in E$ such that $x'(y) = \langle x, y\rangle$. Therefore, we may identify the dual space as the space $E^o = (\mathbb{R}^r, \|\cdot\|^o)$ endowed with the norm

$$\|x\|^o = \|x'\| = \sup_{y\in B_E} |x'(y)| = \sup_{y\in B_E} |\langle x, y\rangle|,$$

and hence the unit ball $B_{E^o} = \{x \in \mathbb{R}^r : \|x\|^o \leqslant 1\} = \{x \in \mathbb{R}^r : \sup_{y\in B_E} |\langle x, y\rangle| \leq 1\} = (B_E)^o$ (see e.g., [1]). With this notations the Levy mean M_{B_E} is

$$M_{B_E} = \int_{\mathbb{S}^{r-1}(\mathbb{R})} \|\xi\| d\mu_r,$$

where $d\mu_r$ denotes the normalized invariant measure on $\mathbb{S}^{r-1}(\mathbb{R})$. We are interested in the case where $\|\cdot\| = \|\cdot\|_{(p)}$. In this case we shall write $J^{-1}B_{L_p\cap\Xi_r(\Omega_m)} = B^r_{(p)}$. In the case $\Omega_m = \{1,\cdots,m\}$ the estimates of the associated Levy means were obtained in [36]. This result can be easily generalized to an arbitrary index set $\Omega_m = \{k_1 < \cdots < k_m\}$. Hence we omit the details of the proof and give the estimates of the associated Levy means for an arbitrary index set Ω_m in the following lemma.

Lemma 5.3.

$$M_{B^r_{(p)}} \le C\cdot p^{1/2},\ p<\infty.$$

Applying Lemmas 5.2 and 5.3 with $p = \log r$ we get

$$\begin{aligned} M_{B^r_{(\infty)}} &= \int_{\mathbb{S}^{r-1}(\mathbb{R})} \|\xi\|_{(\infty)}\cdot d\mu_r \le r^{1/p}\cdot \int_{\mathbb{S}^{r-1}(\mathbb{R})} \|\xi\|_{(p)}\cdot d\mu_r \\ &\le C\cdot p^{1/2}\cdot r^{1/p} = C\cdot(\log r)^{1/2}\cdot r^{1/(\log r)} \le C\cdot(\log r)^{1/2}. \end{aligned} \tag{5.21}$$

Our lower bounds for m-term approximation are contained in

Theorem 5.3.

$$\nu_m(W^\gamma_p,\Xi,L_q\left(\mathbb{S}^{d-1}(\mathbb{C})\right)) \ge C\cdot m^{-\gamma/t}\cdot\vartheta_m,$$

where

$$\vartheta_m \ge \begin{cases} 1, & 1<q,p<\infty, \\ (\log m)^{-C'}, & p=\infty, q>1, \end{cases}.$$

Proof. First we will suppose that there is $N\in\mathbb{N}$ such that $m = \dim T_{[N/2]}$ and let $n = \dim T_N$ and we will write in this proof L_q instead of $L_q\left(\mathbb{S}^{d-1}(\mathbb{R})\right)$. It is sufficient to consider the case $p\ge 2$ and $1\le q\le 2$ since all other cases follow by embedding arguments. By Bernstein's inequality (5.3),

$$(\dim \mathcal{T}_N)^{-\gamma/t}B_p\cap\mathcal{T}_N \subset W^\gamma_p.$$

Hence, from the definition of ν_m it follows

$$\nu_m(W^\gamma_p,\Xi,L_q) \ge \nu_m((\dim\mathcal{T}_N)^{-\gamma/t}\cdot B_p\cap\mathcal{T}_N,\Xi,L_q)$$

$$= (\dim \mathcal{T}_N)^{-\gamma/t} \cdot \nu_m(B_p \cap \mathcal{T}_N, \Xi, L_q). \tag{5.22}$$

Let $\phi \in B_p \cap \mathcal{T}_N$ and $\xi \in \Xi_m$. Then applying Lemma 5.1 we get

$$\|Q_{2N} * (\phi - \xi)\|_q = \|Q_{2N} * \phi - Q_{2N} * \xi\|_q \le \|Q_{2N}\|_1 \cdot \|\phi - \xi\|_q,$$

where $Q_{2N} * \xi \in \mathcal{T}_{2N} \cap \Xi_m$ and $\phi = Q_{2N} * \phi$ for any $\phi \in \mathcal{T}_N$. Consequently,

$$\inf_{\xi \in \Xi(\Omega_m)} \|\phi - \xi\|_q \ge \frac{1}{\|Q_{2N}\|_1} \inf_{Q_{2N} * \xi \in \Xi(\Omega_m) \cap \mathcal{T}_{2N}} \|\phi - Q_{2N} * \xi\|_q \ge C \inf_{\eta \in \Xi(\Omega_m) \cap \mathcal{T}_{2N}} \|\phi - \eta\|_q, \tag{5.23}$$

for any $\phi \in B_p \cap \mathcal{T}_N$. Comparing (5.22) and (5.23) we find

$$\nu_m(W_p^\gamma, \Xi, L_q) \ge C(\dim \mathcal{T}_N)^{-\gamma/t} \cdot \vartheta_m, \tag{5.24}$$

where

$$\vartheta_m := \nu_m(B_p \cap \mathcal{T}_N, \Xi \cap \mathcal{T}_{2N}, L_q \cap \mathcal{T}_{2N}).$$

Let $e_1, \cdots, e_k$ be the canonic basis in $\mathbb{R}^k$, $k := \dim \mathcal{T}_{2N}$. Let $\mathbf{l} = \{k_1, \cdots, k_m\} \subset \mathbb{N}$, $k_s \le k$, $1 \le s \le m$ and $X_{\mathbf{l}}^m = \operatorname{lin}\{e_{k_s}\}_{s=1}^m$. Since $p \ge 2$, then by Hölder's inequality $J^{-1}(B_p \cap \mathcal{T}_N) \subset J^{-1}(B_2 \cap \mathcal{T}_N)$ and, therefore,

$$\begin{aligned} J^{-1}(B_p \cap \mathcal{T}_N) &\subset \cup_{\mathbf{l}}(X_{\mathbf{l}}^m + J^{-1}((\vartheta_m \cdot B_q) \cap \mathcal{T}_{2N})) \cap J^{-1}(B_p \cap \mathcal{T}_N) \\ &\subset \cup_{\mathbf{l}}(X_{\mathbf{l}}^m + J^{-1}((\vartheta_m \cdot B_q) \cap \mathcal{T}_{2N})) \cap J^{-1}(B_2 \cap \mathcal{T}_N). \end{aligned} \tag{5.25}$$

Let $P^\perp(X_{\mathbf{l}}^m)$ be the orthoprojector onto $(X_{\mathbf{l}}^m)^\perp$ in $J^{-1}\mathcal{T}_N$. Observe that for any $\mathbf{l}$

$$\begin{aligned} &(X_{\mathbf{l}}^m + J^{-1}((\vartheta_m \cdot B_q) \cap \mathcal{T}_{2N})) \cap J^{-1}(B_2 \cap \mathcal{T}_N) \\ &= X_{\mathbf{l}}^m \cap J^{-1}(B_2 \cap \mathcal{T}_N) + J^{-1}\left(((\vartheta_m \cdot B_q) \cap \mathcal{T}_{2N}) \cap J^{-1}(B_2 \cap \mathcal{T}_N)\right) \\ &= X_{\mathbf{l}}^m \cap J^{-1}(B_2 \cap \mathcal{T}_N) + J^{-1}(((\vartheta_m \cdot B_q) \cap \mathcal{T}_N) \cap (B_2 \cap \mathcal{T}_N)) \\ &\subset X_{\mathbf{l}}^m \cap J^{-1}(B_2 \cap \mathcal{T}_N) + P^\perp(X_{\mathbf{l}}^m) \circ J^{-1}(((\vartheta_m \cdot B_q) \cap \mathcal{T}_N) \cap (B_2 \cap \mathcal{T}_N)). \end{aligned}$$

Let

$$l := \dim J(X_{\mathbf{l}}^m \cap J^{-1}\mathcal{T}_N)^\perp_{\mathcal{T}_N}, \quad s := n - l,$$

where $(L)^\perp_{\mathcal{T}_N}$ denotes the orthogonal complement of a subspace $L \subset \mathcal{T}_N$ in $\mathcal{T}_N$. Taking volumes we get

$$\begin{aligned}
& \mathrm{Vol}_n((X_{\mathbf{I}}^m + J^{-1}((\vartheta_m \cdot B_q) \cap \mathcal{T}_{2N})) \cap J^{-1}(B_2 \cap \mathcal{T}_N)) \\
\leq \; & \mathrm{Vol}_n \left(X_{\mathbf{I}}^m \cap J^{-1}(B_2 \cap \mathcal{T}_N) + P^{\perp}\left(X_{\mathbf{I}}^m\right) \circ J^{-1}(((\vartheta_m \cdot B_q) \cap \mathcal{T}_N) \cap (B_2 \cap \mathcal{T}_N)) \right) \\
= \; & \mathrm{Vol}_s \left(B_{(2)}^s \right) \cdot \mathrm{Vol}_l \left(P^{\perp}\left(X_{\mathbf{I}}^m\right) \circ J^{-1}(((\vartheta_m \cdot B_q) \cap \mathcal{T}_N) \cap (B_2 \cap \mathcal{T}_N)) \right).
\end{aligned}$$

To get an upper bound for

$$\mathrm{Vol}_l \left(P^{\perp}\left(X_{\mathbf{I}}^m\right) \circ J^{-1}(((\vartheta_m \cdot B_q) \cap \mathcal{T}_N) \cap (B_2 \cap \mathcal{T}_N)) \right)$$

we proceed as follows. Let $x_1, \cdots, x_{M'}$ be a 1-net for $J^{-1}\left(B_q \cap \mathcal{T}_N\right)$ in the norm induced by $J^{-1}\left(B_2 \cap \mathcal{T}_N\right)$ where we assume that M' is the minimal number of points such that the following inclusion holds.

$$J^{-1}\left(B_q \cap \mathcal{T}_N\right) \subset \bigcup_{k=1}^{M'} \left(x_k + J^{-1}\left(B_2 \cap \mathcal{T}_N\right) \right).$$

Therefore,

$$\begin{aligned}
& \mathrm{Vol}_l \left(P^{\perp}\left(X_{\mathbf{I}}^m\right) \circ J^{-1}\left(\vartheta_m \cdot B_q \cap \mathcal{T}_N\right) \cap \left(B_2 \cap \mathcal{T}_N\right) \right) \\
\leq \; & \mathrm{Vol}_l \left(P^{\perp}\left(X_{\mathbf{I}}^m\right) \circ J^{-1}\left(\vartheta_m \cdot B_q \cap \mathcal{T}_N\right) \right) \\
= \; & \vartheta_m^l \cdot \mathrm{Vol}_l \left(P^{\perp}\left(X_{\mathbf{I}}^m\right) \circ J^{-1}\left(B_q \cap \mathcal{T}_N\right) \right) \\
\leq \; & \vartheta_m^l \cdot \mathrm{Vol}_l P^{\perp}\left(X_{\mathbf{I}}^m\right) \left(\bigcup_{k=1}^{M'} \left(x_k + J^{-1}\left(B_2 \cap \mathcal{T}_N\right) \right) \right) \\
= \; & \vartheta_m^l \cdot \mathrm{Vol}_l \left(\bigcup_{k=1}^{M'} P^{\perp}\left(X_{\mathbf{I}}^m\right) \left(x_k + J^{-1}\left(B_2 \cap \mathcal{T}_N\right) \right) \right) \\
= \; & \vartheta_m^l \cdot \mathrm{Vol}_l \left(\bigcup_{k=1}^{M'} \left(P^{\perp}\left(X_{\mathbf{I}}^m\right) x_k + P^{\perp}\left(X_{\mathbf{I}}^m\right) \circ J^{-1}\left(B_2 \cap \mathcal{T}_N\right) \right) \right) \\
\leq \; & \vartheta_m^l \cdot \sum_{k=1}^{M'} \mathrm{Vol}_l \left(P^{\perp}\left(X_{\mathbf{I}}^m\right) \circ J^{-1}\left(B_2 \cap \mathcal{T}_N\right) \right) \\
= \; & \vartheta_m^l \cdot M' \cdot \mathrm{Vol}_l \left(B_{(2)}^l \right),
\end{aligned}$$

since $P^{\perp}\left(X_{\mathbf{I}}^m\right) \circ J^{-1}\left(B_2 \cap \mathcal{T}_N\right) = B_{(2)}^l$. The t-th entropy number e_t of a set $\mathcal{K}$ in a Banach space X with the unit ball B_X is defined as the infimum of $\varepsilon > 0$ such that there exist an ε-net $x_1, \cdots, x_{2^{a-1}}$ in X such that

$$\mathcal{K} \subset \bigcup_{k=1}^{2^{a-1}} (x_k + \varepsilon \cdot B_X) .$$

To get an upper bound for M' we use the estimate [32]

$$\sup_{a \geq 1} a^{1/2} \cdot e_a \leq C \cdot n^{1/2} \cdot M_{V^o},$$

which is valid for any convex symmetric body $V \subset \mathbb{R}^n$. Put $e_a = 1$, then the minimal cardinality M' of $1-$net for $J^{-1}(B_q \cap \mathcal{T}_N)$ in the norm induced by $J^{-1}(B_2 \cap \mathcal{T}_N)$ can be estimated as

$$M' \leq 2^{CnM^2_{V^o}},$$

where $V := J^{-1}(B_q \cap \mathcal{T}_N)$. Finally, we get

$$\mathrm{Vol}_n((X_{\mathbf{l}}^m + J^{-1}((\vartheta_m \cdot B_q) \cap \mathcal{T}_{2N}) \cap J^{-1}(B_2 \cap \mathcal{T}_N))$$

$$\leq 2^{CnM^2_{V^o}} \cdot \vartheta^l_m \cdot \mathrm{Vol}_s \left(B^s_{(2)}\right) \cdot \mathrm{Vol}_l \left(B^l_{(2)}\right) .$$

Observe that the number of terms in (5.25) is bounded by

$$\sum_{m=0}^{\dim \mathcal{T}_{2N}} \begin{pmatrix} \dim \mathcal{T}_{2N} \\ m \end{pmatrix} = 2^{\dim \mathcal{T}_{2N}}.$$

Hence, from (5.25) it follows that

$$\mathrm{Vol}_n(J^{-1}(B_p \cap \mathcal{T}_N)) \leq 2^k \max_{\mathbf{l}} \omega^{\mathbf{l}}_m, \tag{5.26}$$

where

$$\begin{aligned} \omega^{\mathbf{l}}_m &= \mathrm{Vol}_n \left((X_{\mathbf{l}}^m + J^{-1}((\vartheta_m \cdot B_q) \cap \mathcal{T}_{2N}) \cap J^{-1}(B_2 \cap \mathcal{T}_N)\right) \\ &\leq \vartheta^l_m \cdot 2^{CnM^2_{(J^{-1}(B_q \cap \mathcal{T}_N))^o}} \times \mathrm{Vol}_s \left(B^s_{(2)}\right) \cdot \mathrm{Vol}_l \left(B^l_{(2)}\right) . \end{aligned} \tag{5.27}$$

By Hölder's inequality and Lemma 5.3,

$$\begin{aligned} M_{(J^{-1}(B_q \cap \mathcal{T}_N))^o} &\leq M_{J^{-1}\left(B_{q'} \cap \mathcal{T}_N\right)} \\ &\leq C \cdot (q^{'})^{1/2}, 1/q + 1/q^{'} = 1, 1 < q < \infty. \end{aligned} \tag{5.28}$$

Comparing (5.26), Lemma 5.3, (5.27) and (5.2) we get

$$\omega_m^{\mathbf{l}} \leq \vartheta_m^{'l} \cdot 2^{C\cdot(q')\cdot n} \cdot \mathrm{Vol}_s\left(B^s_{(2)}\right) \cdot \mathrm{Vol}_l\left(B^l_{(2)}\right). \tag{5.29}$$

Now we turn to the lower bounds for $\mathrm{Vol}_n(J^{-1}(B_p \cap \mathcal{T}_N))$. From the Bourgain-Milman inequality, Theorem 1.20,

$$\left(\frac{\mathrm{Vol}_n V \cdot \mathrm{Vol}_n V^o}{\left(\mathrm{Vol}_n B^n_{(2)}\right)^2}\right)^{1/n} \geq C,$$

which is valid for any convex symmetric body $V \subset \mathbb{R}^n$, it follows that

$$\mathrm{Vol}_n(J^{-1}(B_p \cap \mathcal{T}_N)) \geq C^n \cdot \left(\frac{\mathrm{Vol}_n\left(B^n_{(2)}\right)}{\mathrm{Vol}_n\left(B^n_{(p)}\right)^o}\right) \cdot \mathrm{Vol}_n\left(B^n_{(2)}\right).$$

Comparing this estimate with Lemma 5.3 and Urysohn's inequality , Theorem 1.19,

$$\left(\frac{\mathrm{Vol}_n(V)}{\mathrm{Vol}_n\left(B^n_{(2)}\right)}\right)^{1/n} \leq \int_{\mathbb{S}^{n-1}\mathbb{R}} \|\alpha\|_{V^o}\, d\mu,$$

which is valid for any convex symmetric body $V \subset \mathbb{R}^n$ we get

$$\begin{aligned}
\mathrm{Vol}_n(J^{-1}(B_p \cap \mathcal{T}_N)) &\geq \left(\frac{\mathrm{Vol}_n\left(B^n_{(2)}\right)}{\mathrm{Vol}_n\left(B^n_{(p)}\right)^o}\right) \cdot \mathrm{Vol}_n\left(B^n_{(2)}\right) \\
&\geq C^n \cdot \left(M_{B^n_{(p)}}\right)^{-n} \mathrm{Vol}_n\left(B^n_{(2)}\right) \\
&\geq C^n \cdot \left\{\begin{array}{ll} (p)^{1/2}, & p < \infty \\ (\log n)^{1/2}, & p = \infty \end{array}\right\}^{-n} \cdot \mathrm{Vol}_n\left(B^n_{(2)}\right)
\end{aligned} \tag{5.30}$$

Applying (5.26), (5.29), and (5.2) we obtain

$$\begin{aligned}
& C^n \cdot \left\{\begin{array}{ll} (p)^{1/2}, & p < \infty \\ (\log n)^{1/2}, & p = \infty \end{array}\right\}^{-n} \cdot \mathrm{Vol}_n\left(B^n_{(2)}\right) \\
\leq\ & 2^k \cdot \max_{\mathbf{l}} \vartheta_m^{'l} \cdot 2^{Cq'n} \cdot \mathrm{Vol}_s\left(B^s_{(2)}\right) \cdot \mathrm{Vol}_l\left(B^l_{(2)}\right),
\end{aligned}$$

which means that

$$\begin{aligned}
\vartheta_m &\geq 2^{-k/l} \cdot 2^{-C\cdot(q')\cdot n/l} \cdot C^{n/l} \cdot \left\{ \begin{array}{cc} (p)^{1/2}, & p<\infty \\ (\log n)^{1/2}, & p=\infty \end{array} \right\}^{-n/l} \\
&\quad \times \left(\frac{\mathrm{Vol}_n\left(B^n_{(2)}\right)}{\mathrm{Vol}_s\left(B^s_{(2)}\right)\cdot \mathrm{Vol}_l\left(B^l_{(2)}\right)} \right)^{1/l} . \qquad (5.31)
\end{aligned}$$

Observe that $0 \leq \dim J(X_{\mathbf{1}} \cap J^{-1}\mathcal{T}_N) \leq \dim \mathcal{T}_{[N/2]}$ and $\dim \mathcal{T}_{[N/2]} \leq \dim(JX_{\mathbf{1}} \cap \mathcal{T}_N)^{\perp} \leq \dim \mathcal{T}_N$. This implies that $\dim \mathcal{T}_{[N/2]} \leq l \leq \dim \mathcal{T}_N$, or $cn \leq l \leq n$, where $0 < c \leq 1$. Let us put for convenience $\mathrm{Vol}_0\left(B^0_{(2)}\right) = 1$. Since

$$\mathrm{Vol}_n\left(B^n_{(2)}\right) = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$$

and

$$\Gamma(z) = z^{z-1/2}\cdot e^{-z}\cdot (2\pi)^{1/2}\cdot\left(1+O\left(z^{-1}\right)\right),$$

then

$$\begin{aligned}
r_{l,s,n} &= \left(\frac{\mathrm{Vol}_s\left(B^s_{(2)}\right)\cdot \mathrm{Vol}_l\left(B^l_{(2)}\right)}{\mathrm{Vol}_n\left(B^n_{(2)}\right)} \right)^{1/l} \\
&= \left(\frac{\Gamma(n/2+1)\cdot \pi^{(n-l)/2}\cdot \pi^{l/2}}{\pi^{n/2}\cdot \Gamma((n-l)/2+1)\cdot \Gamma(l/2+1)} \right)^{1/l} \\
&= \left(\frac{\Gamma(n/2+1)}{\Gamma((n-l)/2+1)\cdot \Gamma(l/2+1)} \right)^{1/l} \\
&= \left(\frac{e^{-n/2-1}\cdot\left(\frac{n}{2}+1\right)^{n/2+1-1/2}}{e^{-(n-l)/2-1}\cdot\left(\frac{n-l}{2}+1\right)^{(n-l)/2+1-1/2}\cdot e^{-l/2-1}\cdot\left(\frac{l}{2}+1\right)^{l/2+1-1/2}} \right)^{1/l} \\
&\quad\times \left(\frac{\left(1+O\left(\frac{1}{n}\right)\right)}{\left(1+O\left(\frac{1}{n-l}\right)\right)\cdot\left(1+O\left(\frac{1}{l}\right)\right)} \right)^{1/l} \\
&\leq C\left(\frac{\left(\frac{n}{2}+1\right)^{n/2+1-1/2}}{\left(\frac{n-l}{2}+1\right)^{(n-l)/2+1-1/2}\cdot\left(\frac{l}{2}+1\right)^{l/2+1-1/2}} \right)^{1/l} \\
&= C\left(\frac{(n+2)^{n/2+1-1/2}}{(n-l+2)^{(n-l)/2+1-1/2}\cdot (l+2)^{l/2+1-1/2}} \right)^{1/l} \\
&\leq C\left(\frac{n^{n/2+1/2}}{(n-l)^{(n-l)/2+1/2}\, l^{l/2+1/2}} \right)^{1/l}
\end{aligned}$$

$$
\begin{aligned}
&= C\frac{n^{n/(2l)+1/(2l)}}{(n-l)^{(n-l)/(2l)+1/(2l)}\, l^{1/2+1/(2l)}} \\
&\leq C\frac{n^{n/(2l)}}{(n-l)^{(n-l)/(2l)}\, l^{1/2}} \\
&\leq C\frac{n^{n/(2l)}}{(n-l)^{n/(2l)-1/2}\, l^{1/2}} = \left(\frac{1}{1-\frac{l}{n}}\right)^{\frac{n}{2l}} \left(\frac{n-l}{l}\right)^{\frac{1}{2}} \\
&\leq C\left(\frac{1}{1-\frac{l}{n}}\right)^{\frac{n}{2l}},
\end{aligned}
$$

where the penultimate and ultimate steps are justified by the condition $cn \leq l < n$ (see, e.g., [35]). Since the sequence $(1-l/n)^n$ converges to e^{-l}, it is bounded by a constant independent of n. Consequently, $r_{l,s,n} \leq C$ for any $n \in \mathbb{N}$ and using (5.31) and (5.5) we get

$$
\vartheta_m \geq C \left\{ \begin{array}{cc} p^{1/2} \cdot 2^{Cq'}, & p < \infty, q > 1, \\ (\log m)^{1/2} \cdot 2^{Cq'}, & p = \infty, q > 1, \end{array} \right\}^{-n/m}.
$$

Since $m \asymp n$ then $n/m \asymp 1$, and hence

$$
\vartheta_m \geq C \left\{ \begin{array}{cc} 1, & p < \infty, q > 1, \\ (\log m)^{-C'}, & p = \infty, q > 1, \end{array} \right. .
$$

Finally, from (5.24) and the last line it follows

$$
\nu_m(W_p^\gamma, \Xi, L_q) \geq C m^{-\gamma/t} \vartheta_m.
$$

Next we will show that the theorem is true for all $m \in \mathbb{N}$. Suppose that there is no $N \in \mathbb{N}$ such that $m = \dim T_{[N/2]}$. Then there is $N \in \mathbb{N}$ such that $m_1 = \dim T_{[N/2]} < m < \dim T_N = m_2$. Hence the theorem is true for m_1 and m_2, i.e.,

$$
\nu_{m_1}(W_p^\gamma, \Xi, L_q) \geq C m_1^{-\gamma/d} \vartheta_{m_1}, \qquad \nu_{m_2}(W_p^\gamma, \Xi, L_q) \geq C m_2^{-\gamma/d} \vartheta_{m_2}.
$$

From the definition of ν_m we have $\nu_{m_1} \leqslant \nu_m \leqslant \nu_{m_2}$. On the other hand since $m_1 < m$ then $m_1^{-\gamma/d}\vartheta_{m_1} \geqslant m^{-\gamma/d}\vartheta_m$. Hence, $\nu_m(W_p^\gamma, \Xi, L_q) \geq C m^{-\gamma/d}\vartheta_m$ and then the theorem is true for all $m \in \mathbb{N}$ $\square$

Remark 5.2. Comparing Theorem 1 and Theorem 2 we get

$$\nu_m(W_p^\gamma, \Xi, L_q) \asymp m^{-\gamma/t}, \ \gamma > (t-1)/2, \ 1 < q \leq p < \infty.$$

The following statement contains lower bounds for m-term approximation of general multiplier operator in the case $1 < q \leqslant p \leqslant 2$.

Theorem 5.4. *Let* $1 < q < p \leqslant 2$, $\Xi = \{H(s,l), \ s,l \in \mathbb{N}\}, \ \Lambda = \{\lambda_{s,l}\}_{s,l>0}, \ \lambda_{s,l} > 0$ *for any* $s,l \geqslant 0$ *and* $m = \dim \mathcal{T}_N = \frac{(2n+N-2)!(2n+2N-1)}{N!(2n-1)!}$, *where* $N \in \mathbb{N}$. *Then*

$$\nu_m(\Lambda B_p, \Xi, L_q) \geqslant c(2^{-cq'}) \min\{|\lambda_{s,l}|, \ s+l \leqslant N, \ s,l \geqslant 0\}.$$

Proof. Let $\mathcal{T}_N = \oplus\{H(s,l)|s+l \leqslant N, \ s,l \geqslant 0\}$. Since $\Lambda B_{p_1} \subset \Lambda B_{p_2}$ for any $p_1 \geqslant p_2$ then it is sufficient to consider the case $p = 2$. Clearly,

$$\min\{|\lambda_{s,l}|, \ s+l \leqslant N, \ s,l \geqslant 0\} B_2 \cap \mathcal{T}_N \subset \Lambda B_2.$$

Now, it is sufficient to repeat the proof of the last theorem. $\square$

Chapter 6

Scattered Data Interpolation by Radial Basis Function

The real multivariate interpolation problem is the means to construct a continuous function which interpolates a data value at the data sites. Suppose that $X = \{x_1, \cdots, x_n\} \subseteq \mathbb{R}^d$, $X \subseteq \Omega \subseteq \mathbb{R}^d$, and real scalars $\{f_i\}_{i \leqslant n} \subseteq \mathbb{R}$. Then we seek to construct an interpolant $s_{f,X} : \Omega \to \mathbb{R}$ for which

$$s_{f,X}(x_i) = f_i, \qquad for\ i = 1, \cdots, n.$$

If the interpolation does not require any conditions on the data X, except that $X \subseteq \Omega$, then it is often called scattered interpolation.

In the radial basis function approach, the interpolants are required to be an element of the linear span of the radial functions $\Phi(\| \cdot - x_i\|)$, i.e.,

$$s_{f,X}(x) = \sum_{i=1}^{n} a_i \Phi(\|x - x_i\|),$$

where Φ is a univariate function, and $\| \cdot \|$ is a norm on Ω. The coefficients a_i are determined by the interpolation condition

$$s_{f,X}(x_i) = f(x_i), \quad i = 1, \cdots, n.$$

This defines a linear system $A_{\Phi,X} a = f$, where

$$A_{\Phi,X} = (\Phi(\|x_j - x_i\|))_{1 \leqslant i,j \leqslant n},$$

and $a = (a_1, \cdots, a_n)^T$, $f = (f(x_1), \cdots, f(x_n))^T$. Hence, if the interpolation matrix $A_{\Phi,X}$ is nonsingular, then s and a exist uniquely for all data and for all n.

In this particular case, the interpolation problem is equivalent to asking whether the interpolation matrix is invertible. Well-known examples of nonsingular interpolation matrix are Gaussians $\Phi(r) = e^{-\alpha r^2}$, $\alpha > 0$, the inverse multi-quadric $\Phi(r) = \frac{1}{\sqrt{c^2+r^2}}$ and the multi quadric $\Phi(r) = \sqrt{c^2 + r^2}$, $c > 0$, where in the first two cases the matrix $A_{\Phi,X}$ is always positive definite, see e.g. [58]. In fact, most of the recent work in interpolation problems focus on basis functions that generate positive definite matrices which can be handled by many efficient numerical algorithms.

Schoenberg [51] was first to study positive semi-definite functions on the sphere. He characterised the family of all positive semi-definite functions on $\mathbb{S}^{d-1}(\mathbb{R})$ as

$$f(t) = \sum_{n=0}^{\infty} a_n C_n^{\lambda}(\cos(t)), \qquad \lambda = \frac{d-2}{2}, \tag{6.1}$$

where $a_n \geqslant 0$, $\sum_{n=0}^{\infty} a_n < \infty$ and C_n^{λ} are the Gegenbauer polynomials. The first paper concerned with conditions for the invertibility of the interpolation matrix on the unit sphere $\mathbb{S}(\mathbb{R}) \subset \mathbb{R}^2$ was by Light and Cheney [40]. On the n-dimensional sphere Xu and Cheney [59] were the first who gave a sufficient condition for positive definiteness. They show that whenever $a_n > 0$ in (6.1) the function f is positive definite in $\mathbb{S}^{d-1}(\mathbb{R})$.

Since positive definiteness is crucial for interpolation, the first section of this chapter is devoted to study these functions and a related space, called the native space. The second section deals with error estimates of interpolation by radial basis functions, and the application of these functions on the sphere is given in the third section. In the last section, we give an improved error bound for radial interpolation on the complex sphere, based on a generalised result on the real sphere which is given by Brownlee, Georgoulis and Levesley [7], and Bernstein's inequality in the complex sphere given by Kushpel and Levesley [33]. Most of the materials presented in the first three sections can be found in [44] and [58], which also contains a survey of results concerning scattered data approximation and recent results on radial basis functions.

6.1 Positive Definite Kernel and Native Spaces

Definition 6.1. Let A be a subset of $\mathbb{R}^d$, or $\mathbb{C}^d$. A continuous function $\varphi : A \to \mathbb{C}$ is called positive definite if for all $N \in \mathbb{N}$ and all sets of pairwise distinct points $x_1, \cdots, x_N \in A$, the quadratic form

$$\sum_{j=1}^{N}\sum_{k=1}^{N} \alpha_j \overline{\alpha_k} \varphi(x_j - x_k)$$

is positive for all $\alpha \in \mathbb{C}^N - \{0\}$. The function φ is called positive semi-definite if the quadratic form is nonnegative.

In other words, the function φ is positive definite if and only if every matrix of the form $(\varphi(x_j - x_i))_{1\leqslant i,j\leqslant N}$, $\{x_i, \cdots, x_N\} \subset A$ is positive definite.

Theorem 6.1. *Let φ be a positive semi-definite function. Then the following properties are satisfied.*

- $\varphi(0) \geqslant 0$.
- $\varphi(-x) = \overline{\varphi(x)}$.
- $|\varphi(x)| \leqslant \varphi(0)$, *for all* $x \in \mathbb{R}^d$.
- $\varphi(0) = 0$ *if and only if* $\varphi = 0$.
- *Further, If $\varphi_1, \cdots, \varphi_n$ are positive semi-definite and $c_j \geqslant 0$, $1 \leqslant j \leqslant n$, then $\sum_{j=1}^{n} c_j \varphi_j$ is also positive semi-definite. If one of the φ_j is positive definite and the corresponding c_j is positive then φ is also positive definite.*
- *The product of two positive definite functions is positive definite.*

More generally, let $\Omega \subseteq \mathbb{R}^d$, or $\Omega \subseteq \mathbb{C}^d$. A function $\varphi : \Omega\times\Omega \to \mathbb{C}$ is called positive definite kernel on Ω if for all $N \in \mathbb{N}$, all $x_1, \cdots, x_N \in \Omega$, and all $\alpha \in \mathbb{C}^N - \{0\}$ we have

$$\sum_{j=1}^{N}\sum_{k=1}^{N} \alpha_j \overline{\alpha_k} \varphi(x_j, x_k) > 0.$$

Definition 6.2. A function $\varphi : A \subseteq \mathbb{R}^d \to \mathbb{R}$ is said to be radial if there exists a function $\phi : [0, \infty) \to \mathbb{R}$ such that $\varphi(x) = \phi(\|x\|)$ for all $x \in \mathbb{R}^d$.

The norm is usually the Euclidean distance, but we do not place any restrictions on the norm $\|\cdot\|$ at this point. A function $\phi : [0, \infty) \to \mathbb{R}$ is said to be positive definite on $\mathbb{R}^d$ if the corresponding function $\varphi(x) = \phi(\|x\|)$, $x \in \mathbb{R}^d$, is positive definite. Moreover, for a more general setting the radial basis kernels can be defined as a composition of a univariate function with a distance function from a point x, called centre. In other words, a kernel $\varphi : A \times A \to \mathbb{R}$ is said to be radial if $\varphi(x, y) = \phi(\|x - y\|)$ for a function $\phi : [0, \infty) \to \mathbb{R}$.

Next, we will introduce two Hilbert spaces, the reproducing kernel Hilbert spaces and the native spaces. The first space is a Hilbert space that have a reproducing kernel which can be positive definite. The second consideration of these spaces is starting with a positive definite kernel φ, and construct a Hilbert space N_φ, and φ acts as a reproducing kernel in this space. This space is known as *the native space* corresponding to the kernel φ. These spaces have a special importance in radial basis function research and are used as a tool in studying error estimates.

Let Ω be an arbitrary subset of $\mathbb{R}^d$ which contains at least one point.

Definition 6.3. Let Υ be a real Hilbert space of functions $f : \Omega \to \mathbb{R}$. A function $\varphi : \Omega \times \Omega \to \mathbb{R}$ is called a reproducing kernel for Υ if

(a) $\varphi(\cdot, y) \in \Upsilon$ for all $y \in \Omega$.

(b) $f(y) = (f, \varphi(\cdot, y))_\Upsilon$ for all $f \in \Upsilon$ and all $y \in \Omega$.

The second condition in the definition guarantees the uniqueness of the reproducing kernel. If we suppose that there are two reproducing kernels φ_1 and φ_2 then $\varphi_1 = \varphi_2$ since

$$\varphi_1(x, y) = (\varphi_1(\cdot, y), \varphi_2(\cdot, x))_\Upsilon = \varphi_2(y, x)$$

where $\varphi_2(x, y) = \varphi_2(y, x)$, as the following theorem state.

Theorem 6.2. *Suppose Υ is a Hilbert space of functions $f : \Omega \to \mathbb{R}$ with reproducing kernel φ. Then*

(a) $\varphi(x, y) = (\varphi(\cdot, x), \varphi(\cdot, y))_\Upsilon$ for $x, y \in \Omega$.

(b) φ is symmetric, i.e, $\varphi(x, y) = \varphi(y, x)$ for all $x, y \in \Omega$.

(c) If $f, f_n \in \Upsilon$, $n \in \mathbb{N}$, are given such that f_n converges to f in the Hilbert space norm, i.e. $\|f - f_n\|_\Upsilon \to 0$, as $n \to \infty$. Then f_n also converges point-wise to f.

The evaluation functional S_y at y is defined as

$$S_y(f) = (f, \varphi_y)_\Upsilon \quad \text{where} \quad \varphi_y(x) = \varphi(x, y).$$

Not all Hilbert spaces have a reproducing kernel. The evaluation functionals can give us a characterisation of a Hilbert space with a reproducing kernel and when the reproducing kernel is positive definite.

Theorem 6.3. *Let Υ be a Hilbert space of functions $f : \Omega \to \mathbb{R}$. Then the following statements are equivalent.*

(a) The point evaluation functionals are continuous.

(b) Υ has a reproducing kernel.

Theorem 6.4. *Suppose that Υ is a reproducing kernel Hilbert function space with reproducing kernel $\varphi : \Omega \times \Omega \to \mathbb{R}$. Then φ is positive semi-definite. Moreover, φ is positive definite if and only if the point evaluation functionals are linearly independent in the dual space Υ^*.*

If the function space Υ is a complex vector space containing complex valued functions, then everything said so far remains true with special care to be taken with complex conjugate sign, Wendland [58]. In this case the reproducing kernel is a complex valued positive semi-definite function.
The notion of native space was introduced by Madych and Nelson [43]. For a positive definite kernel, the corresponding native space can be constructed in different ways. In [49] Schaback gave a new formulation of the theory of radial basis functions using integral operators instead of Fourier transforms. In Theorem 6.6 he use Mercer theorem to gives a new characterisation of the native spaces on a domain $\Omega \subseteq \mathbb{R}^d$ using eigenfunction expansions of the operators. The native space can be characterised by the following results (see e.g. Wendland [58]), we omit details here.

Theorem 6.5. *(Mercer Theorem)*
Let φ be a symmetric positive definite kernel on $\Omega \subseteq \mathbb{R}^d$. Then φ possesses the absolutely and uniformly convergent representation

$$\varphi(x, y) = \sum_{n=1}^{\infty} \rho_n \varphi_n(x) \varphi_n(y)$$

where $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq L_2(\Omega)$ are eigenfunctions of the integral operator $T : L_2(\Omega) \to L_2(\Omega)$ given by

$$Tv(x) = \int_\Omega \varphi(x, y) v(y) dy \quad v \in L_2(\Omega), \; x \in \Omega,$$

with eigenvalue $\rho_1 \geqslant \rho_2 \geqslant \cdots > 0$.

Theorem 6.6. *Suppose* φ *is a symmetric positive definite kernel on a compact set* $\Omega \subseteq \mathbb{R}^d$. *Then it is reproducing kernel in its native space which is given by*

$$N_\varphi(\Omega) = \left\{ f \in L_2(\Omega) : \sum_{n=1}^{\infty} \frac{|\left\langle f, \varphi_n \right\rangle_{L_2(\Omega)}|^2}{\rho_n} < \infty \right\},$$

and the inner product has the representation

$$(f, g)_\varphi = \sum_{n=1}^{\infty} \frac{\left\langle f, \varphi_n \right\rangle_{L_2(\Omega)} \left\langle g, \varphi_n \right\rangle_{L_2(\Omega)}}{\rho_n}, \quad f, g \in N_\varphi(\Omega).$$

A subspace of this native space has a special importance in improving error estimates of interpolation by radial functions. This subspace is given in the following theorem.

Theorem 6.7. *Let* φ *be a symmetric positive definite kernel on a compact set* $\Omega \subseteq \mathbb{R}^d$. *Then the range of the integral operator is given by*

$$T(L_2(\Omega)) = \left\{ f \in L_2(\Omega) : \sum_{n=1}^{\infty} \frac{|\left\langle f, \varphi_n \right\rangle_{L_2(\Omega)}|^2}{\rho_n^2} < \infty \right\}.$$

6.2 Error Estimates for Interpolation by Radial Basis Functions

In this section, suppose that the data value $\{f_i\}$ come from a continuous function f, i.e. $f(x_i) = f_i$ for all $x_i \in X \subseteq \Omega$. We will give estimates for the error $f - s_{f,X}$ between a function f in the native space of a positive definite kernel and its interplant $s_{f,X}$ on a set $\Omega \subseteq \mathbb{R}^d$. These estimates involves an expression called power function, which in turn can be given in terms of the maximum separation distance of the points X known as *fill distance*, $h_{X,\Omega}$, which is given by

$$h_{X,\Omega} = \sup_{x \in \Omega} \min_{x_i \in X} \|x - x_i\|.$$

Since $h_{X,\Omega}$ measure the density of the points X in Ω, these estimates show how the interpolant $s_{f,X}$ approximates the function f when X is dense in Ω. To construct the power function, let φ be a positive definite kernel with the native space N_φ

and its interpolant $s_{f,X}$. Define the associated function by

$$u_j^* = \sum_{i=1}^{N} \alpha_i^{(j)} \varphi(\cdot, x_i)$$

where $\alpha^{(j)} = (\alpha_1^{(j)}, \cdots, \alpha_N^{(j)})$ satisfy $A\alpha^{(j)} = e^{(j)}$ the jth unit vector and $A = (\varphi(x_i, x_j))_{1\leqslant i,j\leqslant N}$.

Definition 6.4. Suppose that $\Omega \subseteq \mathbb{R}^d$ and that $\varphi \in C^{2k}(\Omega\times\Omega)$ is positive definite symmetric kernel on Ω. If $X = \{x_1, \cdots, x_n\} \subseteq \Omega$ then for every $x \in \Omega$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d \leqslant k$ the power function is defined by

$$\left[P_{\varphi,X}^{(\alpha)}\right]^2(x) = D_1^\alpha D_2^\alpha \varphi(x,x) - 2\sum_{j=1}^{N} D^\alpha u_j^*(x) D_1^\alpha \varphi(x, x_j)$$

$$+ \sum_{i,j=1}^{N} D^\alpha u_i^*(x) D^\alpha u_j^*(x) \varphi(x_i, x_j),$$

where $D^\alpha f(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$, $x = (x_1, \cdots, x_d)$ and D_i^α denotes the derivative with respect to the ith argument.

Theorem 6.8. *Let $\Omega \subseteq \mathbb{R}^d$ be open. Suppose that $\varphi \in C^{2k}(\Omega \times \Omega)$ is a positive definite kernel on Ω. Denote the interplant of $f \in N_\varphi(\Omega)$ by $s_{f,X}$. Then for every $x \in \Omega$ and every $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leqslant k$ the error between f and $s_{f,X}$ can be bounded by*

$$|D^\alpha f(x) - D^\alpha s_{f,X}(x)| \leqslant P_{\varphi,X}^{(\alpha)}(x) |f|_{N_\varphi(\Omega)}.$$

The condition that the region $\Omega \subseteq \mathbb{R}^d$ is open, is only necessary for estimates on the derivatives. If Ω is not open, the estimates on the derivatives holds in every interior point, Wendland [58].

This approximation order can be improved by requiring more smoothness for the function f. Schaback [50], shows that for a suitable subspace of the native space the approximation order can be doubled, Theorem 6.9. The basic idea of this improvement depends on the fact that if $f \in N_\varphi(\Omega)$ and $s_{f,X}$ its interpolant then the function $g = f - s_{f,X} \in N_\varphi(\Omega)$ and then the error estimate in Theorem 6.8 becomes

$$|f(x) - s_{f,X}| \leqslant P_{\varphi,X}^{(0)}(x) |f - s_{f,X}|_{N_\varphi(\Omega)},$$

where the unique interpolant function of g is the zero function. Then by adding some assumption on the function f to get an upper bound of the term $|f - s_{f,X}|_{N_\varphi(\Omega)}$ in term of the power function.

Theorem 6.9. *Suppose that φ is a symmetric positive definite kernel on a compact set $\Omega \subseteq \mathbb{R}^d$. Then for avery $f \in T(L_2(\Omega))$ we have*

$$|f(x) - s_{f,X}| \leqslant P_{\varphi,X}(x)\|P_{\varphi,X}\|_{L_2(\Omega)}\|T^{-1}f\|_{N_\varphi(\Omega)}, \quad x \in \Omega.$$

6.3 Interpolation on The Sphere

Let f be a function on the unit sphere $\mathbb{S}^{d-1}(\mathbb{R})$ belonging to $L_2(\mathbb{S}^{d-1})$. With respect to the inner product defined in $L_2(\mathbb{S}^{d-1})$. The function f has the Fourier representation of the form

$$f = \sum_{l=0}^{\infty} f_l = \sum_{l=0}^{\infty}\sum_{k=1}^{d_l} f_{l,k}Y_k^l \quad \textit{with} \quad f_{l,k} = \left\langle f, Y_k^l \right\rangle_{L_2(\mathbb{S}^{d-1})}$$

where $\{Y_k^l : 1 \leqslant k \leqslant d_l\}$ are an orthonormal basis for H_l, the space of spherical harmonic polynomials of degree l. We will consider a positive definite kernel on $\mathbb{S}^{d-1}(\mathbb{R})$ of the form

$$\varphi(x, y) = \sum_{l=0}^{\infty}\sum_{k=1}^{d_l} a_{l,k}Y_k^l(x)Y_k^l(y), \tag{6.2}$$

and we are concerned with radial functions with respect to the geodesic distance which defined in the following definition.

Definition 6.5. A kernel φ on $\mathbb{S}^d(\mathbb{R}) \times \mathbb{S}^d(\mathbb{R})$ is called radial or zonal if $\varphi(x, y) = \psi(d(x, y)) = \phi(x^T y)$ with univariate functions ϕ and ψ. Here $d(x, y) = \cos^{-1}(\langle x, y\rangle_{\mathbb{R}})$ is the geodesic distance between x and y.

Theorem 6.10. *A kernel φ of the form (6.2) is radial if and only if $a_{l,k} = a_l$, $1 \leqslant k \leqslant d_l$.*

The above result provides us with a form of the radial kernel on $\mathbb{S}^{d-1}(\mathbb{R})$ which is given by

$$\varphi(x, y) = \sum_{l=0}^{\infty} a_l \frac{d_l}{w_{d-1}} C_l^{\frac{d-2}{2}}(\langle x, y\rangle_{\mathbb{R}}), \tag{6.3}$$

where $w_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface area.

Lemma 6.1. *Let $Y \in span\{Y_k^l : 1 \leqslant k \leqslant d_l\}$. Then*

$$|Y(x)| \leqslant \sqrt{\frac{d_l}{w_{d-1}}}\|Y\|_{L_2(\mathbb{S}^{d-1})}, \quad x \in \mathbb{S}^{d-1}(\mathbb{R}).$$

Moreover, the Gegenbauer polynomials satisfy $|C_l^{(d-2)/2}(t)| \leqslant 1$ for $t \in [-1,1]$.

To prove continuity for a radial kernel given in an expansion of the form (6.3) it is sufficient to assume that

$$\sum_{l=0}^{\infty} |a_l| d_l < \infty,$$

since then from the above lemma

$$\sum_{l=0}^{\infty} |a_l| \frac{d_l}{w_{d-1}} |C_l^{\frac{d-2}{2}}(\langle x,y\rangle_{\mathbb{R}})| \leqslant \frac{1}{\omega_{d-1}} \sum_{l=0}^{\infty} |a_l| d_l < \infty.$$

Then from Weierstrass M-test the given expansion (6.3) converges uniformly to the kernel φ and then it is continuous since the Gegenbauer polynomials are continuous. Moreover, we can characterise a positive definite kernel on the sphere by it's Fourier coefficients.

Theorem 6.11. *Suppose that the kernel (6.2) is continuous. Then it is positive definite if all coefficients are positive.*

Corollary 6.1. *A radial function $\varphi(x,y)$, $x,y \in \mathbb{S}^{d-1}(\mathbb{R})$ is positive definite if*

$$\varphi(x,y) = \sum_{l=0}^{\infty} a_l C_l^{\frac{d-2}{2}}(x^T y)$$

with $a_l > 0$ for all $l \in \mathbb{N}_0$ and $\sum_{l=0}^{\infty} a_l < \infty$.

In [44], Menegatto and Peron generalise Schoenberg's [51] result on positive definite functions on the real sphere. They show that positive definite kernels on the spheres in $\mathbb{C}^d$ are series of disk polynomials, which are introduced by Zernike and Brinkman [61], with nonnegative coefficients. More precisely, let $\mathbb{S}^{d-1}(\mathbb{C})$ be the unit sphere in $\mathbb{C}^d$, and let $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ denote the usual inner product in $\mathbb{C}^d$. Let f be a function on the closed unit disk B in $\mathbb{C}$. Then f is called positive definite whenever the corresponding zonal kernel $f(\langle\cdot,\cdot\rangle_{\mathbb{C}})$ is positive definite on $\mathbb{S}^{d-1}(\mathbb{C}) \times \mathbb{S}^{d-1}(\mathbb{C})$.

Recall that the orthogonal decomposition of $L_2(\mathbb{S}^{d-1}(\mathbb{C}))$ is given by

$$L_2(\mathbb{S}^{d-1}(\mathbb{C})) = \bigoplus_{m,n=0}^{\infty} \mathrm{H}(m,n),$$

where $\mathrm{H}(m,n)$ is the space of spherical harmonic homogeneous polynomials of degree m in z and n in $\overline{z}$. Moreover, the disk polynomial of degree $m+n$ associated to a positive real number α is the polynomial $R^{\alpha}_{m,n}$ given by

$$R^{\alpha}_{m,n}(z) = r^{|m-n|} e^{i(m-n)\theta} P^{(\alpha,|m-n|)}_{m\wedge n}(2r^2-1), \quad z = re^{i\theta} = x + iy.$$

Due to orthogonality relation for Jacobi polynomials, the set $\{R^{\alpha}_{m,n} : 0 \leqslant m, n \leqslant \infty\}$ is a complete orthogonal system in $L_2(\mathbb{S}^{d-1}(\mathbb{C}))$, see Koornwinder [29].

The result of Menegatto and Peron, [44] is given in the following theorem.

Theorem 6.12. *A continuous function f is positive semi-definite on B if and only if it is of the form*

$$f(z) = \sum_{(m,n)\in\mathbb{N}^2} a_{m,n} R^{d-2}_{m,n}(z), \quad z \in B$$

in which $\sum_{(m,n)\in\mathbb{N}^2} a_{m,n} < \infty$ and $a_{m,n} \geqslant 0$ for all (m,n).

Another way to phrase Theorem 6.12 is that a continuouse zonal kernel $f(\langle\cdot,\cdot\rangle_{\mathbb{C}})$ is positive semi-definite on $\mathbb{S}^{d-1}(\mathbb{C}) \times \mathbb{S}^{d-1}(\mathbb{C})$ if and only if it is of the form

$$f(\langle w, w'\rangle_{\mathbb{C}}) = \sum_{(m,n)\in\mathbb{N}^2} a_{m,n} R^{d-2}_{m,n}(z),$$

where $z = \langle w, w'\rangle_{\mathbb{C}}$, $\sum_{(m,n)\in\mathbb{N}^2} a_{m,n} < \infty$ and $a_{m,n} \geqslant 0$ for all (m,n).

Now to define the native space on the real spheres, let a given $\varphi \in C(\mathbb{S}^{d-1}(\mathbb{R}) \times \mathbb{S}^{d-1}(\mathbb{R}))$ be a positive definite kernel given by (6.2) where $a_{l,k} > 0$ and $\sum_{l=0}^{\infty} \tilde{a}_l d_l < \infty$, $\tilde{a} = \max_{1\leqslant k\leqslant d_l} |a_{l,k}|$. The native space $N_{\varphi}(\mathbb{S}^{d-1}(\mathbb{R}))$ can be characterised as

$$N_{\varphi}(\mathbb{S}^{d-1}(\mathbb{R})) = \left\{ f = \sum_{l=0}^{\infty}\sum_{k=1}^{d_l} f_{l,k} Y^l_k : \sum_{l=1}^{\infty}\sum_{k=1}^{d_l} \frac{|f_{l,k}|^2}{a_{l,k}} < \infty \right\},$$

and the inner product takes the form

$$(f,g)_{\varphi} = \sum_{l=1}^{\infty}\sum_{k=1}^{d_l} \frac{f_{l,k} g_{l,k}}{a_{l,k}}.$$

The fill distance $h_{X,\mathbb{S}^{d-1}(\mathbb{R})}$ takes the form

$$h_{X,\mathbb{S}^{d-1}(\mathbb{R})} = \sup_{x\in\mathbb{S}^{d-1}(\mathbb{R})} \inf_{x_i\in X} d(x, x_i)$$

The error estimates for interpolation using positive definite kernels on d-dimensional sphere were done in [26]. These estimates are given in the following two results.

Theorem 6.13. *Suppose that the kernel φ has only positive Fourier coefficients $a_{l,k}$ which satisfy the condition $\sum_{l=0}^{\infty} \tilde{a}_l d_l < \infty$. Suppose further that $X = \{x_1, \cdots, x_N\} \subseteq \mathbb{S}^{d-1}(\mathbb{R})$ has fill distance $1/(2m+2) \leqslant h_X \leqslant 1/(2m)$. The error between $f \in N_\varphi(\mathbb{S}^{d-1}(\mathbb{R}))$ and its interplant $s_{f,X}$ can be bounded by*

$$|f(x) - s_{f,X}|^2 \leqslant \frac{9}{w_{d-1}} \sum_{l=m+1}^{\infty} \tilde{a}_l d_l \|f\|^2_{N_\varphi(\mathbb{S}^{d-1}(\mathbb{R}))}, \quad x \in \mathbb{S}^{d-1}(\mathbb{R}).$$

Corollary 6.2. *Suppose the assumptions of Theorem 6.13 hold.*

(a) If $\tilde{a}_l d_l \leqslant c(1+l)^{-\alpha}$ with $\alpha > 1$ then

$$\|f(x) - s_{f,X}\|_\infty \leqslant c h^{(\alpha-1)/2}_{X,\mathbb{S}^{d-1}(\mathbb{R})} \|f\|_{N_\varphi(\mathbb{S}^{d-1}(\mathbb{R}))}.$$

(b) If $\tilde{a}_l d_l \leqslant c(1+l)^{-\alpha}$ with $\alpha > 0$ then

$$\|f(x) - s_{f,X}\|_\infty \leqslant c e^{-\alpha/(4h_X)} \|f\|_{N_\varphi(\mathbb{S}^{d-1}(\mathbb{R}))}.$$

The improved error given in Theorem 6.9 in the last section is valid for the case $\Omega = \mathbb{S}^{d-1}(\mathbb{R})$ since the sphere is compact. This estimate is true for a subspace of the native space which requires double smoothness of the functions. In [7] the authors extend the range of applicability of the error estimates on sphere by adapting Schaback's error doubling trick to functions that have less smoothness than Schaback requires. Their results are based on using a pseudo differential operator and estimate errors in pseudoderivatives of the solution to the interpolation problem. To show the main result let the φ spline interplant $s_{f,Y}$ given by

$$s_{f,Y}(x) = \sum_{y\in Y} \alpha_y \varphi(d(x,y)),$$

with interpolation conditions

$$s_{f,Y}(y) = f(y), \quad y \in Y,$$

where $Y \subset \mathbb{S}^{d-1}(\mathbb{R})$ be a finite set of point and $d(x,y)$ denote the geodesic distance between x and y. Suppose that the function φ has an expansion

$$\varphi(d(x,y)) = \sum_{n\geqslant 0} a_n C_n^{\lambda}(\langle x,y\rangle_{\mathbb{R}}),$$

where $\lambda = (d/2)-1$, $a_n > 0$ for $n = 0,1,\cdots$, and

$$\sum_{n\geqslant 0} a_n < \infty.$$

The native space of φ will be given by

$$N_{\varphi}(\mathbb{S}^{d-1}(\mathbb{R})) = \left\{ f = \sum_{n\geqslant 0} f_n \in L_2(\mathbb{S}^{d-1}(\mathbb{R})) : \|f\|_{\varphi} = \left(\sum_{n\geqslant 0} \frac{\|f_n\|^2_{L_2(\mathbb{S}^{d-1})}}{a_n} \right)^{\frac{1}{2}} < \infty \right\} \quad (6.4)$$

Now, for some $s > 0$ define a pseudodifferential operator Λ on $\mathbb{S}^{d-1}(\mathbb{R})$ by

$$\Lambda f = \sum_{n\geqslant 0} \lambda_n f_n, \quad \lambda_n = (n(d+n-2))^s, \quad n = 0,1,\cdots.$$

The operator Λ acts via multiplication by λ_n on each eigenspace H_n. The improved error estimate is given for a subspace of N_{φ} defined as

$$N_{\Lambda\varphi}(\mathbb{S}^{d-1}(\mathbb{R})) = \left\{ f \in N_{\varphi}(\mathbb{S}^{d-1}(\mathbb{R})) : \|f\|_{\Lambda\varphi} = \left(\sum_{n\geqslant 0} \frac{\|f_n\|^2_{L_2(\mathbb{S}^{d-1})}}{(\lambda_n a_n)^2} \right)^{\frac{1}{2}} < \infty \right\}.$$

The representation given in (6.4) demonstrates the relationship between the smoothness of the functions, in some sense, and the space based on the rate of decay of its orthogonal projection from $L_2(\mathbb{S}^{d-1}(\mathbb{R}))$ onto H_l. Since a function $f \in L_2(\mathbb{S}^{d-1}(\mathbb{R}))$ can be written as

$$f = \sum_{l=0}^{\infty} \sum_{k=1}^{d_l} f_{l,k} Y_k^l \quad with \quad f_{l,k} = \left\langle f, Y_k^l \right\rangle_{L_2(\mathbb{S}^{d-1}(\mathbb{R}))},$$

then if f is twice continuously differentiable, or if it satisfy

$$\nabla f(x) = \sum_{l=0}^{\infty} \lambda_l \sum_{k=1}^{d_l} f_{l,k} Y_k^l < \infty,$$

then the Fourier coefficients rapidly decaying in l. Here λ_l is the eigenvalue corresponding to H_l. In other words, we can consider the function f smoother if it has more rapidly decaying Fourier coefficients. Hence, the space $N_{\Lambda\varphi}(\mathbb{S}^{d-1}(\mathbb{R}))$ can be considered as a space of functions that having less smoothness than the functions in the space $T(L_2(\mathbb{S}^{d-1}(\mathbb{R})))$, since $\frac{1}{(\lambda_n a_n)^2} < \frac{1}{a_n^2}$ for sufficient large n.

Theorem 6.14. *Let $s_{f,Y}$ be the φ spline interplant to $f \in N_{\Lambda\varphi}(\mathbb{S}^{d-1}(\mathbb{R}))$ on the point set $Y \subset \mathbb{S}^{d-1}(\mathbb{R})$ where $h_{Y,\mathbb{S}^{d-1}(\mathbb{R})} \leqslant 1/(2N)$, for some fixed $N \in \mathbb{N}$. Suppose that $\sum_{n\geqslant 0} d_n a_n \lambda_n^2 < \infty$. Then for $f \in N_{\Lambda\varphi}(\mathbb{S}^{d-1}(\mathbb{R}))$ and for all $x \in \mathbb{S}^{d-1}(\mathbb{R})$,*

$$|f(x) - s_{f,Y}(x)| \leqslant (1+2E)^2 \left(\sum_{n\geqslant 0} d_n a_n \lambda_n^2\right)^{\frac{1}{2}} \left(\sum_{n\geqslant 0} d_n a_n\right)^{\frac{1}{2}} \|f\|_{\Lambda\varphi},$$

for some constant E.

6.4 Improved Error Bound for Radial Interpolation on Complex Sphere

This section is devoted to generalising the improved error for radial interpolation given in [7], which considers the real sphere, to the complex sphere. The essential fact in this section is on considering the space $\mathbb{C}^d$ as a $2d$ dimensional real vector space. Thus the process given here is similar to a large extent to those in the real sphere case, with simple changes occuring due to using Bernstein's inequality in the complex sphere. Here the normalised Lebesgue measure on $\mathbb{S}^{d-1}(\mathbb{C})$ is denoted by $d\mu_{2d}$.

Let $d(w,z) = \cos^{-1}(Re\langle w,z\rangle_{\mathbb{C}})$ denote the geodesic distance between x and y where $Re\langle w,z\rangle_{\mathbb{C}}$ is the real part of the inner product. Let $X \subset \mathbb{S}^{d-1}(\mathbb{C})$ be a finite set of points, and the fill distance of X on $\mathbb{S}^{d-1}(\mathbb{C})$ will be given by

$$h = h_{X,\mathbb{S}^{d-1}(\mathbb{C})} = \sup_{z\in\mathbb{S}^{d-1}(\mathbb{C})} \min_{w\in X} d(z,w).$$

Consider the φ-spline interplant to a function f in the form

$$s_{f,X}(z) = \sum_{w\in X} \alpha_w \varphi(d(w,z)),$$

where the function φ is a univariate function and the interpolation conditions

$$s_{f,X}(w) = f(w) \quad \text{for } \ w \in X.$$

According to the results given in the previous sections, the function φ is positive semi-definite if it can be written as a sum of the disk polynomials with nonnegative coefficients, i.e.,

$$\varphi(d(w,z)) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n,k} R_{n,k}^{d-2}(z') = \sum_{l=0}^{\infty}\sum_{k+n=l} a_{n,k} R_{n,k}^{d-2}(z'),$$

where $a_{n,k} \geqslant 0$, $\sum_{(n,k)\in\mathbb{N}^2} a_{n,k} < \infty$ and $z' = \langle w, z\rangle_{\mathbb{C}}$. Therefore, because of the result (4.4), the function φ is positive definite if it has the form

$$\varphi(d(w,z)) = \sum_{l=0}^{\infty} a_l C_l^{(d-1)}(Re\langle w,z\rangle_{\mathbb{C}}) = \sum_{l=0}^{\infty} a_l C_l^{(d-1)}(\langle w,z\rangle_{\mathbb{R}}), \ \ a_l > 0, \ \sum_{l=0}^{\infty} a_l < \infty,$$

where $\langle w, z\rangle_{\mathbb{R}}$ is the usual real inner product of the points w and z as points in the real sphere $\mathbb{S}^{d-1}(\mathbb{R})$.

The function $f \in L_2(\mathbb{S}^{d-1}(\mathbb{C})) = \bigoplus_{n=0}^{\infty} \mathrm{H}_n$, can be expanded in the form

$$f(z) = \sum_{n=0}^{\infty} T_n f(z),$$

where

$$T_n f(z) = \langle C_n^{(d-1)}(\langle w,z\rangle_{\mathbb{R}}), f(w)\rangle_{L_2(\mathbb{S}^{d-1}(\mathbb{R}))} = \int_{\mathbb{S}^{d-1}(\mathbb{C})} \overline{f(w)} C_n^{(d-1)}(\langle w,z\rangle_{\mathbb{R}}) d\mu_{2d}(w).$$

From [38], we get

$$\|T_n f\|_{L_\infty(\mathbb{S}^{d-1}(\mathbb{C}))} \leqslant \sqrt{d_n}\|T_n f\|_{L_2(\mathbb{S}^{d-1}(\mathbb{C}))}, \quad \text{for all } \ f \in L_2(\mathbb{S}^{d-1}(\mathbb{C})). \tag{6.5}$$

The native space for φ is defined by

$$N_\varphi(\mathbb{S}^{d-1}(\mathbb{R})) = \left\{ f \in L_2(\mathbb{S}^{d-1}(\mathbb{C})) : \|f\|_\varphi = \left(\sum_{n\geqslant 0} \frac{\|T_n f\|^2_{L_2(\mathbb{S}^{d-1}(\mathbb{C}))}}{a_n}\right)^{\frac{1}{2}} < \infty \right\},$$

where the φ inner product defined in $L_2(\mathbb{S}^{d-1}(\mathbb{C}))$ is given by

$$(f,g)_\varphi = \sum_{n=0}^{\infty} \frac{\langle T_n f, T_n g\rangle_{L_2(\mathbb{S}^{d-1}(\mathbb{C}))}}{a_n}.$$

Before we start our analysis, we will show the Pythagorean property, see e.g., [7].

Proposition 6.1. *Let $s_{f,X}$ be the φ-spline interplant of f on the point set $X \subset \mathbb{S}^{d-1}(\mathbb{C})$. Then, for all $f \in N_\varphi(\mathbb{S}^{d-1}(\mathbb{R}))$*

(a) $(f - s_{f,X}, s_{f,X})_\varphi = 0,$

(b) $\|f\|_\varphi^2 = \|f - s_{f,X}\|_\varphi^2 + \|s_{f,X}\|_\varphi^2.$

Proof. Since we have

$$(f - s_{f,X}, s_{f,X})_\varphi = (f, s_{f,X})_\varphi - (s_{f,X}, s_{f,X})_\varphi\,.$$

Then from the properties of reproducing kernels in Hilbert spaces we get

$$\begin{aligned}
(f - s_{f,X}, s_{f,X})_\varphi &= \sum_{w\in X} \overline{\alpha_w}\,(f, \varphi(d(w,\cdot)))_\varphi - \sum_{w\in X} \overline{\alpha_w}\,(s_{f,X}(\cdot), \varphi(d(w,\cdot)))_\varphi \\
&= \sum_{w\in X} \overline{\alpha_w} f(w) - \sum_{w\in X} \overline{\alpha_w} s_{f,X}(w) \\
&= \sum_{w\in X} \overline{\alpha_w} f(w) - \sum_{w\in X} \overline{\alpha_w} f(w) = 0,
\end{aligned}$$

since $f(w) = s_{f,X}$, for all $w \in X$. For the second result of the proposition, using the first result we get

$$\begin{aligned}
\|f - s_{f,X}\|_\varphi^2 &= (f - s_{f,X}, f - s_{f,X})_\varphi = (f - s_{f,X}, f)_\varphi \\
&= \|f\|_\varphi^2 - (s_{f,X}, f)_\varphi = \|f\|_\varphi^2 - \|s_{f,X}\|_\varphi^2,
\end{aligned}$$

since $(s_{f,X}, f)_\varphi = \|s_{f,X}\|_\varphi^2$. Therefore,

$$\|f\|_\varphi^2 = \|f - s_{f,X}\|_\varphi^2 + \|s_{f,X}\|_\varphi^2. \quad \square$$

We also here assumed for some $s > 0$ a pseudo differential operator Λ on $\mathbb{S}^{d-1}(\mathbb{C})$ which is defined for the homogeneous space H_n as

$$\Lambda p_n = \lambda_n p_n, \quad p_n \in H_n, \ \lambda_n = (n(2d+n-2))^s,$$

which satisfies Bernstein's inequality given in [33],

$$\|\Lambda p\|_\infty \leqslant cn^{2s}\|p\|_\infty, \quad p \in \mathcal{T}_N = \bigoplus_{l=0}^{n} H_l \tag{6.6}$$

for some constant c independent of n.

Proposition 2.2 in [7] details the error when applying pseudo derivatives on solutions to the interpolation problem. It is also true for the complex case, with a little change regarding conjugate sign, since the proof depends on only the interpolation conditions. Therefore, this estimate is given in the following corollary without proof.

Corollary 6.3. *Let $s_{f,X}$ be the φ-spline interplant to $f \in N_\varphi\left(\mathbb{S}^{d-1}(\mathbb{C})\right)$ on the point set $X \subset \mathbb{S}^{d-1}(\mathbb{C})$. Let Λ be a pseudo differential operator. Then for each $z \in \mathbb{S}^{d-1}(\mathbb{C})$*

$$|\Lambda(f - s_{f,X})(z)| \leqslant \inf_{\mu \in X^*} \sup_{v \in N_\varphi\left(\mathbb{S}^{d-1}(\mathbb{C})\right), \|v\|_\varphi = 1} |\Lambda v(z) - \mu(\overline{v})| \|f - s_{f,X}\|_\varphi.$$

Here X^* denotes the span of the point evaluation functionals supported on X, where the point evaluation functional is given by

$$S_z(f) = (\varphi(z,\cdot), f(\cdot))_\varphi = \sum_{n=0}^{\infty} \overline{T_n f}(z).$$

Hence the space X^* is given by

$$\begin{aligned}
X^* &= \operatorname{span}\{S_x(\cdot) : x \in X\} = span\left\{\sum_{n=0}^{\infty} \overline{T_n(\cdot)}(x) : x \in X\right\} \\
&= \left\{v : v(f) = \sum_{x \in X} a_x \left(\sum_{n=0}^{\infty} \overline{T_n(f)}(x)\right), a_x \in \mathbb{C}\right\} \\
&= \left\{v : v(f) = \sum_{x \in X} a_x \overline{f(x)}, a_x \in \mathbb{C}\right\}.
\end{aligned}$$

Proposition (2) in [26] gives an important result on the linear functional on the real sphere based on the fill distance $h_{X,\mathbb{S}^{d-1}(\mathbb{R})}$. Considering $\mathbb{C}^d$ as a real $2d$-dimensional real vector space with $h_{X,\mathbb{S}^{d-1}(\mathbb{R})} = h_{X,\mathbb{S}^{d-1}(\mathbb{C})}$ and $S_x(p) = p(x)$ for all $p \in \mathcal{T}_N$, this result remains true on the complex sphere $\mathbb{S}^{d-1}(\mathbb{C})$. The following lemma is a special case of this result.

Lemma 6.2. *Let X be a finite set of points with fill distance $h_{X,\mathbb{S}^d(\mathbb{C})} \leqslant 1/(2N)$ for some $N \in \mathbb{Z}_+$. Then, for any linear functional γ on $\mathcal{T}_N$ with*

$$\sup_{p\in\mathcal{T}_N, \|p\|_{L_\infty(\mathbb{S}^{d-1}(\mathbb{C}))}=1} |\gamma p| \leqslant 1,$$

there is a set of real numbers $\{l_x\}_{x\in X}$ with $\sum_{x\in X} |l_x| \leqslant 2$ such that

$$\gamma p = \sum_{x\in X} l_x p(x), \quad \textit{for all} \quad p \in \mathcal{T}_N.$$

.

Define the linear functional γ_z for $z \in \mathbb{S}^{d-1}(\mathbb{C})$ by

$$\gamma_z p = \frac{\Lambda p(z)}{cN^{2s}}, \quad for \;\; all \; p \in \mathcal{T}_N.$$

Then by Bernstein's inequality 6.6, for any $p \in \mathcal{T}_N$,

$$|\gamma_z p| = \left|\frac{\Lambda p(z)}{cN^{2s}}\right| \leqslant \frac{\|\Lambda p\|_\infty}{cN^{2s}} \leqslant \|p\|_\infty = 1$$

if $\|p\|_\infty = 1$. Then there is a set of real numbers $\{l_x\}_{x\in X}$ such that

$$\gamma_z p = \sum_{x\in X} l_x p(x),$$

where $\sum_{x\in X} |l_x| \leqslant 2$. Hence we have

$$\Lambda p(z) = \sum_{x\in X} l_x cN^{2s} p(x) = \sum_{x\in X} c_x p(x), \quad \text{for all} \;\; p \in \mathcal{T}_N, \tag{6.7}$$

where $c_x = l_x cN^{2s}$ and

$$\sum_{x\in X} |c_x| \leqslant 2cN^{2s}. \tag{6.8}$$

The next theorem is the first main theorem of this section which is similar to the result given in [7], and we will follow the same stages of its proof but with simple changes occuring using Bernstein's inequality for the complex sphere.

Theorem 6.15. *Let $s_{f,X}$ be the φ spline interplant to $f \in N_\varphi\left(\mathbb{S}^{d-1}(\mathbb{C})\right)$, on the point set $X \subset \mathbb{S}^{d-1}(\mathbb{C})$ where $h_{X,\mathbb{S}^{d-1}(\mathbb{C})} \leqslant 1/(2N)$, for some $N \in \mathbb{Z}_+$. Suppose that*

$$\sum_{n\geqslant 0} d_n \lambda_n^2 a_n < \infty.$$

Then, for $z \in \mathbb{S}^{d-1}(\mathbb{C})$

$$|\Lambda(f - s_{f,X})(z)| \leqslant (1+2c)\left(\sum_{n>N} d_n \lambda_n^2 a_n\right)^{1/2} \|f - s_{f,X}\|_\varphi.$$

Proof. Let $\{c_x\}_{x\in X}$ be the coefficients given in (6.8), and let $v \in N_\varphi$ with $\|v\|_\varphi = 1$. Then from (6.7) and (6.5) we get

$$\begin{aligned}
\inf_{\mu\in X^*} |\Lambda v(z) - \mu(\overline{v})| &\leqslant \left|\sum_{n\geqslant 0}\left(\Lambda T_n v(z) - \sum_{x\in X} c_x T_n v(x)\right)\right| \\
&= \left|\sum_{n>N}\left(\lambda_n T_n v(z) - \sum_{x\in X} c_x T_n v(x)\right)\right| \\
&\leqslant \sum_{n>N} |\lambda_n T_n v(z) - \sum_{x\in X} c_x T_n v(x)| \\
&\leqslant \sum_{n>N}\left(\lambda_n + \sum_{x\in X}|c_x|\right)\|T_n v\|_{L_\infty(\mathbb{S}^{d-1}(\mathbb{C}))} \\
&\leqslant \sum_{n>N}\left(\lambda_n + \sum_{x\in X}|c_x|\right)\sqrt{d_n}\|T_n v\|_{L_2(\mathbb{S}^{d-1}(\mathbb{C}))} \\
&\leqslant \left(\sum_{n>N}\left(\lambda_n + \sum_{x\in X}|c_x|\sqrt{d_n} a_n^{1/2}\right)^2\right)^{\frac{1}{2}} \\
&\times \left(\sum_{n>N}\left(a_n^{-1/2}\|T_n v\|_{L_2(\mathbb{S}^{d-1}(\mathbb{C}))}\right)^2\right)^{\frac{1}{2}} \\
&\leqslant \left(\sum_{n>N}\left(\lambda_n + \sum_{x\in X}|c_x|\sqrt{d_n} a_n^{1/2}\right)^2\right)^{\frac{1}{2}} \|v\|_\varphi
\end{aligned}$$

by Hölder's inequality. Then with $\|v\|_\varphi = 1$

$$\inf_{\mu \in X^*} |\Lambda v(z) - \mu(\overline{v})| \leqslant \left(\sum_{n>N} \left(\lambda_n + 2cN^{2s}\right)^2 d_n a_n \right)^{\frac{1}{2}},$$

since $\lambda_N = (N^2 + (2d-1)N)^s \geqslant N^{2s}$. Then we get

$$\begin{aligned}
\inf_{\mu \in X^*} |\Lambda v(z) - \mu(\overline{v})| &\leqslant \left(\sum_{n>N} (\lambda_n + 2c\lambda_N)^2 d_n a_n \right)^{\frac{1}{2}} \\
&\leqslant \left(\sum_{n>N} (\lambda_n + 2c\lambda_n)^2 d_n a_n \right)^{\frac{1}{2}} \\
&= (1+2c) \left(\sum_{n>N} \lambda_n^2 d_n a_n \right)^{\frac{1}{2}}
\end{aligned}$$

for all $v \in N_\varphi$, $\|v\|_\varphi = 1$. Thus

$$\inf_{\mu \in X^*} \sup_{v \in N_\varphi, \|v\|_\varphi = 1} |\Lambda v(z) - \mu(\overline{v})| \leqslant (1+2c) \left(\sum_{n>N} \lambda_n^2 d_n a_n \right)^{\frac{1}{2}},$$

and from Corollary 6.3

$$|\Lambda(f - s_{f,X})(z)| \leqslant (1+2c) \left(\sum_{n>N} d_n \lambda_n^2 a_n \right)^{1/2} \|f - s_{f,X}\|_\varphi. \quad \square$$

Under the same hypotheses of the above theorem, we have

$$\|\Lambda(f - s_{f,X})\|_{L_2(\mathbb{S}^{d-1}(\mathbb{C}))} \leqslant (1+2c) \left(\sum_{n>N} d_n \lambda_n^2 a_n \right)^{1/2} \|f - s_{f,X}\|_\varphi.$$

Notice that when the points in X become more dense, or $h_{X,\mathbb{S}^{d-1}(\mathbb{C})} \to 0$, the variable N will increase so that $\sum_{n>N} d_n \lambda_n^2 a_n \to 0$, which leads to $\|\Lambda(f - s_{f,X})(z)\|_{L_2(\mathbb{S}^{d-1}(\mathbb{C}))} \to 0$. The next theorem gives double error estimates by adding another term $\sum_{n>N} d_n a_n$ for functions on a subspace of the native space which have smoothness more than the typical function and less than double smoothness on the complex sphere. This theorem is a generalisation of the main result in [7] which considers the real sphere. The proof is again the same and depends on all the facts given before.

The new native space is given by

$$N_{\Lambda\varphi*\Lambda\varphi}=\left\{f\in N_\varphi\left(\mathbb{S}^{d-1}(\mathbb{C})\right):\|f\|_{\Lambda\varphi*\Lambda\varphi}=\left(\sum_{n\geqslant 0}\frac{\|T_nf\|^2_{L_2(\mathbb{S}^{d-1}(\mathbb{C}))}}{(\lambda_na_n)^2}\right)^{\frac{1}{2}}<\infty\right\}.$$

Theorem 6.16. *Let $s_{f,X}$ be the φ spline interplant to $f\in N_\varphi\left(\mathbb{S}^{d-1}(\mathbb{C})\right)$, on the point set $X\subset\mathbb{S}^{d-1}(\mathbb{C})$ where $h_{X,\mathbb{S}^d(\mathbb{C})}\leqslant 1/(2N)$, for some $N\in\mathbb{Z}_+$. Suppose that*

$$\sum_{n\geqslant 0}d_n\lambda_n^2a_n<\infty.$$

Then, for $f\in N_{\Lambda\varphi*\Lambda\varphi}$ and $z\in\mathbb{S}^{d-1}(\mathbb{C})$

$$|(f-s_{f,X})(z)|\leqslant(1+2c)^2\left(\sum_{n>N}d_n\lambda_n^2a_n\right)^{1/2}\left(\sum_{n>N}d_na_n\right)^{1/2}\|f\|_{\Lambda\varphi*\Lambda\varphi}.$$

Proof. Using the Pythagorean property in Proposition 6.1 and Cauchy Schwarz inequality, we have

$$\begin{aligned}\|f-s_{f,X}\|^2_\varphi &= (f-s_{f,X},f)_\varphi\\ &= \sum_{n\geqslant 0}a_n^{-1}\langle T_n(f-s_{f,X}),T_nf\rangle_{L_2(\mathbb{S}^{d-1}(\mathbb{C}))}\\ &\leqslant \left(\sum_{n\geqslant 0}\lambda_n^2\|T_n(f-s_{f,X})\|^2_{L_2(\mathbb{S}^{d-1}(\mathbb{C}))}\right)^{\frac{1}{2}}\\ &\times \left(\sum_{n\geqslant 0}(\lambda_na_n)^{-2}\|T_nf\|^2_{L_2(\mathbb{S}^{d-1}(\mathbb{C}))}\right)^{\frac{1}{2}}\\ &= \|\Lambda(f-s_{f,X})\|_{L_2(\mathbb{S}^{d-1}(\mathbb{C}))}\|f\|_{\Lambda\varphi*\Lambda\varphi}\\ &\leqslant (1+2c)\left(\sum_{n>N}d_n\lambda_n^2a_n\right)^{1/2}\|f-s_{f,X}\|_\varphi\|f\|_{\Lambda\varphi*\Lambda\varphi}.\end{aligned}$$

Therefore,

$$\|f-s_{f,X}\|_\varphi\leqslant(1+2c)\left(\sum_{n>N}d_n\lambda_n^2a_n\right)^{1/2}\|f\|_{\Lambda\varphi*\Lambda\varphi},$$

and hence from Theorem 6.15 when $\lambda_n = 1$ for all n

$$|(f - s_{f,X})(z)| \leqslant (1+2c)^2 \left(\sum_{n>N} d_n \lambda_n^2 a_n \right)^{1/2} \left(\sum_{n>N} d_n a_n \right)^{1/2} \|f\|_{\Lambda\varphi * \Lambda\varphi},$$

for some constant c independent of n. $\square$

Conclusion

This thesis has focused on orthogonal polynomials and spherical harmonic functions in the Euclidean space. It includes a presentation of some results describing the approximation of multivariate functions in spheres by spherical polynomials. In particular, we attempted to study different aspects of approximations on the complex sphere from the geometric point of view.

This work has been initiated by studying the connection between zonal polynomials and harmonic functions on the unit sphere and using this connection to present new proofs of the generating functions of Jacobi polynomial and its special cases. The main result of Chapter 4 is the integral formula, Theorem 4.6. This was new to us and we arrived at it after studying the zonal kernels function in the complex projective space and real space. Later, we found out that this result has been proved before by Dijksma and Koornwinder [15], for more general cases. Further, in Chapter 4 we gave a new representation for the left hand side of well known formula for generating function for Jacobi polynomial in term of associated Legender function. This result was inspired by a proof approach of Jacobi generating function in the special case $\alpha \in \mathbb{N},\ \beta = 0$ which based on the integral formula, Theorem 4.6.

In the second part of this work, we investigated the m-term approximation which is a highly nonlinear method of approximation. We get order sharp estimates of m-term approximation, for functions on Sobolev's spaces, which are based on Bernstein's inequality and geometric properties of Euclidean spaces. In particular, we show that in the case of Sobolev's classes it is not possible to improve the rate of convergence in L_q, $1 \leqslant q \leqslant p \leqslant \infty$ using m-term approximation instead of linear polynomial approximation.

The final part of this thesis has been devoted to studying interpolation by radial basis functions and studying the error estimates. The improved error in [7] for functions in the real sphere raises questions concerning the improved error for functions on the complex sphere. The last section of Chapter 6 has been very successful in addressing this problem. We generalised the improved error estimate for radial interpolation given in [7], which consider the real sphere, to the complex sphere. The essential fact in this section considers the space $\mathbb{C}^d$ as a $2d$ dimensional real vector space. Thus the process given is similar to a large extent to those in the real sphere case, with simple changes occuring due to using Bernstein's inequality in the complex sphere.

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