Research Article

On the Neutrix Composition of the Delta and Inverse Hyperbolic Sine Functions

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Let $F$ be a distribution in $\mathcal{S}$ and let $f$ be a locally summable function. The composition $F(f(x))$ of $F$ and $f$ is said to exist and be equal to the distribution $h(x)$ if the limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) \ast \delta_n(x)$ for $n = 1, 2, \ldots$ and $\{\delta_n(x)\}$ is a certain regular sequence converging to the Dirac delta function. In the ordinary sense, the composition $\delta^{(s)}((\sinh^{-1}x, y))$ does not exist. In this study, it is proved that the neutrix composition $\delta^{(s)}((\sinh^{-1}x, y))$ exists and is given by $\delta^{(s)}((\sinh^{-1}x, y)) = \sum_{k=0}^{r-1} \sum_{i=0}^{k} c_{s,k,i}/2^{s+k} k!$ for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$, where $c_{s,k,i} = (-1)^s s! [(k - 2i + 1)^{r-1} + (k - 2i - 1)^{r-1}] / (2rs + r - 1))$. Further results are also proved.

1. Introduction

In the following, we let $\mathcal{S}$ be the space of infinitely differentiable functions with compact support, let $\mathcal{S}[a,b]$ be the space of infinitely differentiable functions with support contained in the interval $[a,b]$, and let $\mathcal{S}'$ be the space of distributions defined on $\mathcal{S}$.

Now, let $\rho(x)$ be a function in $\mathcal{S}[−1,1]$ having the following properties:

(i) $\rho(x) \geq 0$,
(ii) $\rho(x) = \rho(−x)$,
(iii) $\int_{−1}^{1} \rho(x) \, dx = 1$.

Putting $\delta_n(x) = n \rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if $F$ is
an arbitrary distribution in $\mathfrak{D}'$ and $F_n(x) = F(x) \ast \delta_n(x) = \langle F(x - t), \varphi(t) \rangle$, then $\{F_n(x)\}$ is a regular sequence converging to $F(x)$.

Since the theory of distributions is a linear theory, thus we can extend some of the operations which are valid for ordinary functions to the space of distributions and such operations are called regular operations such as: addition, multiplication by scalars; see [1]. Other operations can be defined only for a particular class of distributions or for certain restricted subclasses of distributions; these are called irregular operations such as: multiplication of distributions, convolution products, and composition of distributions; see [2–4]. Thus, there have been several attempts recently to define distributions of the form $F(f(x))$ in $\mathfrak{D}'$, where $F$ and $f$ are distributions in $\mathfrak{D}$; see for example [5–8]. In the following, we are going to consider an alternative approach. As a starting point, we look at the following definition which is a generalization of Gel’fand and Shilov’s definition of the composition involving the delta function [9], and was given in [6].

**Definition 1.1.** Let $F$ be a distribution in $\mathfrak{D}'$ and let $f$ be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to $h$ on the open interval $(a, b)$, with $-\infty < a < b < \infty$, if

$$
N - \lim_{n \to \infty} \int_{-\infty}^{\infty} F_n(f(x)) \varphi(x) dx = \langle h(x), \varphi(x) \rangle,
$$

(1.1)

for all $\varphi$ in $\mathfrak{D}[a, b]$, where $F_n(x) = F(x) \ast \delta_n(x)$ for $n = 1, 2, \ldots$ and $N$ is the neutrix, see [10], having domain $N'$ the positive and range $N''$ the real numbers, with negligible functions which are finite linear sums of the functions

$$
n^{-1} \ln^{r-1} n, \quad \ln^n n : \lambda > 0, \quad r = 1, 2, \ldots
$$

(1.2)

and all functions which converge to zero in the usual sense as $n$ tends to infinity.

In particular, we say that the composition $F(f(x))$ exists and is equal to $h$ on the open interval $(a, b)$ if

$$
\lim_{n \to \infty} \int_{-\infty}^{\infty} F_n(f(x)) \varphi(x) dx = \langle h(x), \varphi(x) \rangle,
$$

(1.3)

for all $\varphi$ in $\mathfrak{D}[a, b]$.

Note that taking the neutrix limit of a function $f(n)$ is equivalent to taking the usual limit of Hadamard’s finite part of $f(n)$. The definition of the neutrix composition of distributions was originally given in [10] but was then simply called the composition of distributions.

The following three theorems were proved in [11], [8], and [12], respectively.

**Theorem 1.2.** The neutrix composition $\delta^{(s)}(\text{sgn} x |x|^4)$ exists and

$$
\delta^{(s)}(\text{sgn} x |x|^4) = 0,
$$

(1.4)
Theorem 1.3. The neutrix compositions \( \delta^{(2s-1)}(\text{sgn} \, x^{|x|^1/s}) \) and \( \delta^{(s-1)}(|x|^{1/s}) \) exist and
\[
\delta^{(2s-1)}(\text{sgn} \, x^{|x|^1/s}) = \frac{1}{2} (2s)! \delta'(x),
\]
\[
\delta^{(s-1)}(|x|^{1/s}) = (-1)^{s-1} \delta(x),
\]
for \( s = 0, 1, 2, \ldots \).

Theorem 1.4. The neutrix composition \( \delta^{(s)}(\sinh^{-1}x_+^{1/r}) \) exists and
\[
\delta^{(s)} \left[ (\sinh^{-1}x_+)^{1/r} \right] = \sum_{k=0}^{(s+1)/r-1} \sum_{i=0}^{k} \frac{(-1)^{k-i} kr c_{s,k,i}}{2^{k+1} k!} \delta^{(k)}(x),
\]
for \( s = 0, 1, 2, \ldots \) and \( r = 1, 2, \ldots \), where
\[
c_{r,s,k,i} = \frac{(-1)^{s-1} [r^{s+1} + (2i-1)(s+1)]!}{2(rs+r-1)!},
\]
The next two theorems were proved in [13].

Theorem 1.5. The neutrix composition \( \delta^{(s)}[\ln^r(1+|x|)] \) exists and
\[
\delta^{(s)}[\ln^r(1+|x|)] = \sum_{k=0}^{r-1} \sum_{i=0}^{k} \frac{(-1)^{s-i} [1+(-1)^k] i! [i+1]^{r+s+r-1}}{2(r+s+r-1)! k!} \delta^{(i)}(x),
\]
for \( s = 0, 1, 2, \ldots \) and \( r = 1, 2, \ldots \).

In particular, the composition \( \delta[\ln(1+|x|)] \) exists and
\[
\delta[\ln(1+|x|)] = \delta(x).
\]

Theorem 1.6. The neutrix composition \( \delta^{(s)}[\ln(1+|x|^{1/r})] \) exists and
\[
\delta^{(s)}[\ln(1+|x|^{1/r})] = \sum_{k=0}^{m-1} \sum_{i=0}^{k} \frac{(-1)^{s+i-1} [1+(-1)^k] i! [i+1]^{r+s+r-1}}{2k!} \delta^{(i)}(x),
\]
for \( s = 0, 1, 2, \ldots \) and \( r = 2, 3, \ldots \), where \( m \) is the smallest non-negative integer greater than \( (s-r+1)r^{-1} \).
In particular, the composition $\delta^{(s)}[\ln(1 + |x^{1/r}|)]$ exists and
\[
\delta^{(s)}[\ln\left(1 + \left|x^{1/r}\right|\right)] = 0, \quad (1.12)
\]
for $s = 0, 1, 2, \ldots, r - 2$ and $r = 2, 3, \ldots$ and
\[
\delta^{(r-1)}[\ln\left(1 + \left|x^{1/r}\right|\right)] = (-1)^{r-1} r! \delta(x), \quad (1.13)
\]
for $r = 2, 3, \ldots$.

2. Main Results

We now prove the following theorem.

**Theorem 2.1.** The neutrix composition $\delta^{(s)}[(\sinh^{-1} x,)^r]$ exists and
\[
\delta^{(s)}[(\sinh^{-1} x,)^r] = \sum_{k=0}^{s+r-1} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^k r^{c_{s,k,i}}}{2^{k+1} k!} \delta^{(k)}(x), \quad (2.1)
\]
for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$, where
\[
c_{r,s,k,i} = \frac{(-1)^s s! [(k - 2i + 1)^{r+r-1} + (k - 2i - 1)^{r+r-1}]}{2(rs + r - 1)!}. \quad (2.2)
\]

In particular, the neutrix composition $\delta(\sinh^{-1} x,)$ exists and
\[
\delta(\sinh^{-1} x,) = \frac{1}{2} \delta(x). \quad (2.3)
\]

**Proof.** To prove (2.1), we first of all evaluate
\[
\int_{-1}^{1} \delta^{(s)}[(\sinh^{-1} x,)^r] x^k \, dx. \quad (2.4)
\]
We have
\[
\int_{-1}^{1} \delta^{(s)}[(\sinh^{-1} x,)^r] x^k \, dx = n^{s+1} \int_{-1}^{1} \rho^{(s)}[(n \sinh^{-1} x,)^r] x^k \, dx
\]
\[
= n^{s+1} \int_{0}^{1} \rho^{(s)}[n(\sinh^{-1} x)] x^k \, dx
\]
\[
+ n^{s+1} \int_{-1}^{0} \rho^{(s)}(0) x^k \, dx \quad (2.5)
\]
\[
= I_1 + I_2.
\]
It is obvious that

\[
N - \lim_{n \to \infty} I_2 = N - \lim_{n \to \infty} \int_{-1}^{0} \delta_n^{(s)} \left[ \left( \sinh^{-1} x \right)^r \right] x^k \, dx = 0, \tag{2.6}
\]

for \( k = 0, 1, 2, \ldots \).

Making the substitution \( t = n(\sinh^{-1} x)^r \), we have for large enough \( n \)

\[
I_1 = \frac{n^{s-r+1}}{r} \int_0^1 t^{1/(r-1)} \sinh^k \left( \frac{t}{n} \right)^{1/r} \cosh \left( \frac{t}{n} \right)^{1/r} \rho^{(s)}(t) \, dt
\times \int_0^1 t^{1/(r-1)} \left\{ \exp \left[ (k-2i+1) \left( \frac{t}{n} \right)^{1/r} \right] + \exp \left[ (k-2i-1) \left( \frac{t}{n} \right)^{1/r} \right] \right\} \rho^{(s)}(t) \, dt, \tag{2.7}
\]

where

\[
n^{(s-1)/(r+1)} \int_0^1 t^{1/(r-1)} \left\{ \exp \left[ (k-2i+1) \left( \frac{t}{n} \right)^{1/r} \right] + \exp \left[ (k-2i-1) \left( \frac{t}{n} \right)^{1/r} \right] \right\} \rho^{(s)}(t) \, dt
= \sum_{p=0}^{\infty} \int_0^1 \frac{[(k-2i+1)^p + (k-2i-1)^p] t^{(p/r) + (1/r) - 1}}{p! n^{(p/r) + (1/r) - s-1}} \rho^{(s)}(t) \, dt. \tag{2.8}
\]

It follows that

\[
N - \lim_{n \to \infty} n^{s-1/r+1} \int_0^1 t^{1/(r-1)} \left\{ \exp \left[ (k-2i+1) \left( \frac{t}{n} \right)^{1/r} \right] + \exp \left[ (k-2i-1) \left( \frac{t}{n} \right)^{1/r} \right] \right\} \rho^{(s)}(t) \, dt
= \frac{(-1)^s s! \left[ (k-2i+1)^{rs+r-1} + (k-2i-1)^{rs+r-1} \right]}{2(r s + r - 1)!}
\]

\[= c_{r,s,k,i}, \tag{2.9}\]

and by applying the neutrix limit we obtain

\[
N - \lim_{n \to \infty} I_1 = N - \lim_{n \to \infty} \int_0^1 \delta_n^{(s)} \left[ \left( \sinh^{-1} x \right)^r \right] x^k \, dx = \frac{1}{2^{k+1}} \sum_{i=0}^{k} \binom{k}{i} (-1)^i c_{r,s,k,i} \tag{2.10}
\]

for \( k = 0, 1, 2, \ldots \).
When $k = sr + r$, we have

$$|I_1| = \int_0^1 \delta_n^{(s)} \left[ \left( \sinh^{-1} x \right)' \right] x^{sr+r} dx$$

$$= n^{sr+1} \int_0^1 \rho_n^{(s)} \left[ n \left( \sinh^{-1} x \right)' \right] x^{sr+r} dx$$

$$\leq \frac{n^{(s-1)/(r+1)}}{2} \exp(sr + r + 1) \int_0^1 \left[ \left( \frac{t}{n} \right)^{1/r} \right] \rho^{(s)}(t) dt$$

$$= \frac{n^{(s-1)/(r+1)}}{2} \exp(sr + r + 1) \int_0^1 \left[ 2 \left( \frac{t}{n} \right)^{1/r} + O(n^{-2/r}) \right] \rho^{(s)}(t) dt$$

$$\leq n^{-1/r} \exp(sr + r + 1) \int_0^1 \left[ 1 + O(n^{-2/r}) \right] \rho^{(s)}(t) dt$$

$$= O\left(n^{-1/r}\right).$$

Thus, if $\varphi$ is an arbitrary continuous function, then

$$\lim_{n \to \infty} \int_0^1 \delta_n^{(s)} \left[ \left( \sinh^{-1} x \right)' \right] x^{sr+r} \varphi(x) dx = 0. \quad (2.12)$$

We also have

$$\int_{-1}^0 \delta_n^{(s)} \left[ \left( \sinh^{-1} x \right)' \right] \varphi(x) dx = n^{sr+1} \int_{-1}^0 \rho^{(s)}(0) \varphi(x) dx,$$  \quad (2.13)

and it follows that

$$N - \lim_{n \to \infty} \int_{-1}^0 \delta_n^{(s)} \left[ \left( \sinh^{-1} x \right)' \right] \varphi(x) dx = 0. \quad (2.14)$$

If now $\varphi$ is an arbitrary function in $\mathcal{D}[-1, 1]$, then by Taylor’s Theorem, we have

$$\varphi(x) = \sum_{k=0}^{sr+r-1} \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{x^{sr+r}}{(rs+r)!} \varphi^{(rs+r)}(\xi x),$$  \quad (2.15)
where $0 < \xi < 1$, and so

\[
N - \lim_{n \to \infty} \left\langle \delta_n^{(s)} \left[ \left( \sinh^{-1} x \right)^{1/r} \right], \varphi(x) \right\rangle \\
= N - \lim_{n \to \infty} \sum_{k=0}^{sr+r-1} \frac{\varphi^{(k)}(0)}{k!} \int_{0}^{1} \delta_n^{(s)} \left[ \left( \sinh^{-1} x \right)^{r} \right] x^k \, dx \\
+ N - \lim_{n \to \infty} \sum_{k=0}^{sr+r-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{0} \delta_n^{(s)} \left[ \left( \sinh^{-1} x \right)^{r} \right] x^k \, dx \\
+ \lim_{n \to \infty} \frac{1}{(sr + r)!} \int_{0}^{1} \delta_n^{(s)} \left[ \left( \sinh^{-1} x \right)^{r} \right] x^{sr+r} \varphi^{(sr+r)}(\xi x) \, dx \\
+ \lim_{n \to \infty} \frac{1}{(sr + r)!} \int_{-1}^{0} \delta_n^{(s)} \left[ \left( \sinh^{-1} x \right)^{r} \right] x^{sr+r} \varphi^{(sr+r)}(\xi x) \, dx
\]

(2.16)

\[
= \sum_{k=0}^{sr+r-1} \sum_{i=0}^{k} \binom{k}{i} \frac{r^{sr+r} \varphi^{(k)}(0)}{2^{k+1} k!} + 0
\]

\[
= \sum_{k=0}^{sr+r-1} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^k r^{sr+r} \varphi^{(k)}(0)}{2^{k+1} k!} \left\langle \delta^{(k)}(x), \varphi(x) \right\rangle,
\]

on using (2.3) to (2.14). This proves (2.1) on the interval $(-1, 1)$.

It is clear that $\delta^{(s)}[(\sinh^{-1} x)^{r}] = 0$ for $x > 0$ and so (2.1) holds for $x > -1$.

Now, suppose that $\varphi$ is an arbitrary function in $\mathfrak{D}[a, b]$, where $a < b < 0$. Then,

\[
\int_{a}^{b} \delta_n^{(s)} \left[ \left( \sinh^{-1} x \right)^{r} \right] \varphi(x) \, dx = n^{sr+1} \int_{a}^{b} \varphi^{(s)}(0) \varphi(x) \, dx
\]

(2.17)

and so

\[
N - \lim_{n \to \infty} \int_{a}^{b} \delta_n^{(s)} \left[ \left( \sinh^{-1} x \right)^{r} \right] \varphi(x) \, dx = 0.
\]

(2.18)

It follows that $\delta^{(s)}[(\sinh^{-1} x)^{r}] = 0$ on the interval $(a, b)$. Since $a$ and $b$ are arbitrary, we see that (2.1) holds on the real line. This completes the proof of the theorem.

\[\square\]

**Corollary 2.2.** The neutrix composition $\delta^{(s)}[(\sinh^{-1} |x|)^{r}]$ exists and

\[
\delta^{(s)} \left[ \left( \sinh^{-1} |x| \right)^{r} \right] = \sum_{k=0}^{sr+r-1} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^k + 1}{2^{k+1} k!} c_{r,s,k,i} \delta^{(k)}(x),
\]

(2.19)

for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$
In particular, the composition $\delta(\sinh^{-1}|x|)$ exists and

$$
\delta(\sinh^{-1}|x|) = \frac{1}{2} \delta(x).
$$

(2.20)

Proof. To prove (2.19), we note that

$$
\int_{-1}^{1} \delta^{(s)} \left[ (\sinh^{-1}|x|)^{r} \right] x^{k} dx = n^{s+1} \int_{-1}^{1} \rho^{(s)} \left[ (n \sinh^{-1}|x|)^{r} \right] x^{k} dx
$$

(2.21)

$$
= n^{s+1} \left[ 1 + (-1)^{k} \right] \int_{0}^{1} \rho^{(s)} \left[ n \left( \sinh^{-1}x \right)^{r} \right] x^{k} dx,
$$

and (2.19) now follows as above.

Equation (2.20) follows on noting that in the particular case $s = 0$, the usual limit holds in (2.10). This completes the proof of the corollary.

\hfill \Box

**Theorem 2.3.** The neutrix composition $\delta^{(2s-1)} \left[ \sinh^{-1}(\text{sgn} \cdot x^2) \right]$ exists and

$$
\delta^{(2s-1)} \left[ \sinh^{-1}(\text{sgn} \cdot x^2) \right] = \sum_{k=0}^{2s-1+i} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k} b_{s,k,i} \delta^{(k)}(x),
$$

(2.22)

for $s = 1, 2, \ldots$, where

$$
b_{s,k,i} = (k - 2i + 1)^{2s-1} + (k - 2i - 1)^{2s-1}.
$$

(2.23)

Proof. To prove (2.22), we now have to evaluate

$$
\int_{-1}^{1} \delta^{(2s-1)} \left[ \sinh^{-1}(\text{sgn} \cdot x^2) \right] x^{k} dx.
$$

(2.24)

We have

$$
\int_{-1}^{1} \delta^{(2s-1)} \left[ \sinh^{-1}(\text{sgn} \cdot x^2) \right] x^{k} dx = n^{2s} \int_{-1}^{1} \rho^{(2s-1)} \left[ n \sinh^{-1}(\text{sgn} \cdot x^2) \right] x^{k} dx
$$

$$
= \begin{cases} 
2n^{2s} \int_{0}^{1} \rho^{(2s-1)} \left[ n \left( \sinh^{-1}x^2 \right) \right] x^{k} dx, & k \text{ odd}, \\
0, & k \text{ even}. 
\end{cases}
$$

(2.25)
Making the substitution $t = n(\sinh^{-1} x^2)$, we have for large enough $n$

$$
\int_{-1}^{1} \delta^{(2s-1)}_n \left[ \sinh^{-1} \left( \text{sgn} \ x \cdot x^2 \right) \right] x^k dx
$$

$$
= 2n^2 \int_{0}^{1} \rho^{(2s-1)}(n(\sinh^{-1} x^2)) x^{2k+1} dx
$$

$$
= \frac{n^{2s-1}}{2^{k+1}} \sum_{i=0}^{k} (\frac{n}{i!}) \int_{0}^{1} \left\{ \exp \left( \frac{(k-2i+1)t}{n} \right) + \exp \left( \frac{(k-2i-1)t}{n} \right) \right\} \rho^{(s)}(t) dt,
$$

where

$$
\int_{0}^{1} \left\{ \exp \left( \frac{(k-2i+1)t}{n} \right) + \exp \left( \frac{(k-2i-1)t}{n} \right) \right\} \rho^{(s)}(t) dt
$$

$$
= \sum_{p=0}^{\infty} \int_{0}^{1} \frac{\left( (k-2i+1)^p + (k-2i-1)^p \right)}{p! n^{p\cdot 2s+1}} t^p \rho^{(2s-1)}(t) dt.
$$

It follows that

$$
N - \lim_{n \to \infty} n^{2s-1} \int_{0}^{1} \left\{ \exp \left( \frac{(k-2i+1)t}{n} \right) + \exp \left( \frac{(k-2i-1)t}{n} \right) \right\} \rho^{(s)}(t) dt
$$

$$
= N - \lim_{n \to \infty} \sum_{p=0}^{\infty} \int_{0}^{1} \frac{\left( (k-2i+1)^p + (k-2i-1)^p \right)}{p! n^{p\cdot 2s+1}} t^p \rho^{(2s-1)}(t) dt
$$

$$
= -\frac{(k-2i+1)^{2s-1} + (k-2i-1)^{2s-1}}{2}
$$

$$
= \frac{b_{s,k,i}}{2},
$$

and so by using the neutrix limit, we have

$$
N - \lim_{n \to \infty} \int_{-1}^{1} \delta^{(2s-1)}_n \left[ \sinh^{-1} \left( \text{sgn} \ x \cdot x^2 \right) \right] x^{2k+1} dx = \sum_{i=0}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) \frac{(-1)^i b_{s,k,i}}{2^{k+1}},
$$

(2.29)

for $k = 0, 1, 2, \ldots$
When \( k = 2s \), we have
\[
\int_{-1}^{1} \left( \frac{\delta_n^{(2s-1)}}{n} \left[ \sinh^{-1} \left( \text{sgn} \cdot x^2 \right) \right] x^{4s+1} \right) dx = n^{2s-1} \int_{-1}^{1} \rho^{(2s-1)} \left[ n \left( \sinh^{-1} x^2 \right) \right] x^{4s+1} dx
\]
\[
\leq \frac{n^{2s-1}}{2s-1} \exp(s + 1) \int_{-1}^{1} \left| 1 - \exp \left( -\frac{2t}{n} \right) \right|^{2s} \rho^{(2s-1)}(t) dt
\]
\[
= \frac{n^{2s-1}}{2s-1} \exp(s + 1) \int_{-1}^{1} \left| \frac{2t}{n} + O \left( n^{-2} \right) \right|^{2s} \rho^{(2s-1)}(t) dt
\]
\[
\leq 2^{2s+1} n^{-1} \exp(s + 1) \int_{-1}^{1} \left[ 1 + O \left( n^{-2} \right) \right] |\rho^{(2s-1)}(t)| dt
\]
\[
= O \left( n^{-1} \right). \quad (2.30)
\]

Thus, if \( \psi \) is an arbitrary continuous function, then
\[
\lim_{n \to \infty} \int_{-1}^{1} \delta_n^{(2s-1)} \left[ \sinh^{-1} \left( \text{sgn} \cdot x^2 \right) \right] x^{4s+1} \psi(x) dx = 0. \quad (2.31)
\]

If now \( \psi \) is an arbitrary function in \( \mathfrak{D}[-1,1] \), then by Taylor’s Theorem, we have
\[
\psi(x) = \sum_{k=0}^{4s} \frac{\psi^{(k)}(0)}{k!} x^k + \frac{x^{4s+1}}{(4s+1)!} \psi^{(4s+1)}(\xi x), \quad (2.32)
\]
where \( 0 < \xi < 1 \), and so
\[
N - \lim_{n \to \infty} \left\langle \delta_n^{(2s-1)} \left[ \sinh^{-1} \left( \text{sgn} \cdot x^2 \right) \right], \psi(x) \right\rangle
\]
\[
= N - \lim_{n \to \infty} \sum_{k=0}^{2s} \frac{\psi^{(2k+1)}(0)}{(2k+1)!} \int_{-1}^{1} \delta_n^{(2s-1)} \left[ \sinh^{-1} \left( \text{sgn} \cdot x^2 \right) \right] x^{2k+1} dx
\]
\[
+ \lim_{n \to \infty} \frac{1}{(4s+1)!} \int_{-1}^{1} \delta_n^{(4s+1)} \left[ \sinh^{-1} \left( \text{sgn} \cdot x^2 \right) \right] x^{4s+1} \psi^{(4s+1)}(\xi x) dx
\]
\[
= \sum_{k=0}^{2s} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^{l+1} b_{s,k,l} \psi^{(k)}(0)}{2^{k+1}(2k+1)!} + 0
\]
\[
= \sum_{k=0}^{2s} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^{l+k+1} b_{s,k,l}}{2^{k+1}(2k+1)!} \left\langle \delta^{(k)}(x), \psi(x) \right\rangle,
\]

on using (2.25) to (2.31), proving (2.22) on the interval \((-1,1)\). However, it is clear that \( \delta_n^{(2s-1)} \left[ \sinh^{-1} \left( \text{sgn} x \cdot x^2 \right) \right] = 0 \) for \(|x| > 0 \) and so (2.22) holds on the real line, completing the proof of the theorem.
Corollary 2.4. The composition \( \delta'[\sinh^{-1}\sgn x \cdot x^2] \) exists and
\[
\delta'[\sinh^{-1}\left(\sgn x \cdot x^2\right)] = \frac{\delta'(x)}{4.3!} - 2\delta(x).
\] (2.34)

Proof. To prove (2.34) note that in the particular case \( s = 1 \), the usual limits hold and then (2.34) is a particular case of (2.22). This completes the proof of the corollary.

For further related results on the neutrix operation of distributions, see [12–22] and [2, 3, 23].

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