SPECULATION AND DISTRIBUTION OF RETURNS:
A SIMULATION AND EMPIRICAL STUDY

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ABSTRACT

The consequences of financial market speculation go far beyond the floors of stock and currency exchanges. In the last decade, speculative behaviour caused financial panics in South America, Southeast Asia and Russia. Questions about the reasons and nature of speculation once again attracted the attention of economists and financial practitioners. Although it seems to be rather impossible to provide a full explanation of speculation, partial evaluations are certainly possible.

This thesis concentrates on the impact of speculation on the distribution of market returns. The relationship between parameters of speculation processes and stable distribution is established. This allows conclusions to be drawn about the level of market speculation based on the estimated parameters of distribution of returns.

The thesis proposes an application of the minimum chi-squared methodology to estimate parameters of symmetric stable distribution. This approach offers substantial precision and flexibility when dealing with clustered, truncated and grouped data. It is applied to analyse Polish and Hungarian stock prices. In the main part of the thesis, the speculation processes that could produce heavy tailed returns are reviewed. The relationship between the parameters of the Diba-Grossman speculative process and those of the stable distribution of returns is evaluated. In particular, the positive dependence between the variance of the stochastic root of Diba-Grossman process and thickness of tails of the distribution is shown. In the empirical part, the thesis analyses returns to 66 stock indices. The tendency for mature markets to reveal slightly lower exposure to speculation than the emerging markets is found. Outlier tests show that some countries, e.g. Russia, Estonia and Bangladesh, are characterised by a lower level of speculation than is implied by the degrees of non-normality.
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Chapter I

INTRODUCTION

I.1. Motivation

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I.1. Motivation

The second half of the 20th century witnessed a great number of spectacular booms and crashes on the financial markets. In the last decade, panics on the currency and stock exchanges caused crises in South America, Southeast Asia and Russia. The economists usually failed to provide a satisfactory and convincing justification for these events and their analyses generally did not succeed to produce accurate prediction or warning mechanisms. Does this mean that the financial markets are completely unpredictable and their behaviour is governed solely by the information coming to the market? Even if this is not the case, financial models do not seem to describe the behaviour of stocks in a very accurate way.

The example below illustrates the benefits which could be gained from permanently accurate predictions of financial assets' behaviour. Farmer and Lo (1999) analyse an hypothetical investment fund, that in January 1926 was given a genuine prior knowledge, which of the following two assets: one-month U.S. Treasury bills or S&P 500, would yield a higher return in each month until December 1996. Assume that the fund took advantage of this information by switching at the end of each month the running total of its initial $1 investment into this asset, which gave higher returns in the subsequent month. Then, the final value of such perfectly foresight investment in December 1996 would be equal to $2,296,183,456. For comparison, a passive investment in U.S. Treasury bills would give $14 and a passive investment in S&P 500 would yield $1,370. Clearly, there was no market participant, who gained at least a significant percentage of the revenue provided by such perfect foresight strategy. However, even a very modest ability to forecast financial asset returns would be handsomely rewarded.

Hence, there is no surprise that investors apply a wide range of asset pricing models to predict at least a fraction of the changes in asset prices. These models usually belong to two main families. They concentrate on the approximations of the future discounted payments generated by assets or they rely on the analysis of patterns in the behaviour of market participants. The latter group rests on the assumption that the value of an asset is determined rather by what people want to pay for it than by the risky stream of future discounted payments. Consequently, it allows
markets to be driven by investor sentiments, which may result in herding, speculation, panics or other, often non-rational, forms of investor behaviour.

There are a wide range of models based on the above assumption. Two distinctive groups of such models are developed within the rational bubble and the behavioural finance frameworks. Rational bubbles can be described by non-standard solutions to the present value asset pricing relation. They satisfy period by period efficiency but not the transversality condition. Behavioural finance models rest on the assumptions of partial irrationality of agents and impossibility of complete arbitrage. Both theories allow departures from values defined by discounted streams of future payments from an asset. They are often criticised but at least in some cases they seem to provide a better description of financial market reality than traditional asset pricing models.

Rational bubbles and behavioural finance theories suggest that asset prices may be generated by speculative processes. If this is the case, the consequences for financial markets are at least nontrivial. For example, frequent and possibly large departures of prices from the fundamental values, defined by discounted streams of future payments from assets mean that, in some situations, the quality of firms' investments is not properly reflected by asset prices. This may lead to violations of the allocation efficiency of financial markets. Yet, there are many other problems regarding speculation. Does speculation violate the informational efficiency of financial markets? How does one measure the level of speculation and its impact on economic variables? If the theoretical speculative processes are to be accepted as an approximation of financial markets' reality, these and other questions have to be answered or at least seriously discussed.

This study does not try to provide a comprehensive evaluation of speculation. Such a task seems to be too ambitious for a volume of this size. Instead, the thesis chooses an area where structured and quantitative reasoning can be applied. It concentrates on the impact of speculation on the distribution of market returns. Establishing a relationship between speculation and the shape of distribution of market returns can provide a well-grounded measure of the level of speculation on different markets and its impact on the risk and dispersion of returns to asset prices. It is believed that comprehensive analysis of distributions generated by speculative
processes can provide valuable information about the nature of financial markets and processes generating the stock prices.

1.2. Main contributions

The thesis concentrates mainly on the family of Diba – Grossman speculative processes. The main aim of the analysis is to find a relationship between parameters of Diba – Grossman models and parameters of stable distributions of returns to series generated by such models. There are at least four distinctive ways in which this research contributes to the existing literature.

First, it is hypothesised that returns to Diba – Grossman price generating process can be approximated by a symmetric stable distribution. This assumption is confirmed in a series of $\chi^2$ tests performed on simulated data. This in turn motivates investigation of the character of the relationship between parameters of the stable distribution of returns and parameters of the Diba – Grossman processes. A new method of investigating such relationship is proposed. It is generally shown that the larger the variance of the stochastic root of the process, the thicker the tails of the distribution of returns. Similarly, the larger the variance of the random term of this process, called herein Diba – Grossman return, the bigger the dispersion of the returns to such process. In other words, there are general positive relationships between speculation and non-normality and between asset’s exposure to random shocks and dispersion of the distribution of returns. However, the analysed relationship is more complicated and a detailed analysis proves the necessity of bivariate approach. Such approach shows that high speculation slightly decreases dispersion and random shocks contribute towards tail thickness of the distribution. The proposed method allows for a more detailed analysis of market speculation and size of random impacts. Moreover, it offers the possibility of indirect estimation of Diba – Grossman process parameters.

Additionally, an empirical analysis of returns to 66 stock market indices is performed. The data is adjusted for possible effects of risk neutrality and the stable distributions are fitted to the series. The goodness of fit analysis proves positive. Then, the parameters of Diba – Grossman process are estimated through conversion
from stable parameters of the distribution of returns. The countries are grouped with respect to level of speculation and exposure to random shocks. A slight tendency for richer countries to reveal a lower level of speculation is shown. Additionally, the results allow for comparative analysis of relative levels of speculation and random shock exposure on individual markets. The outlier tests are performed in order to check the character of the relationship between speculation and non-normality. Few significant outliers are detected and hence the necessity of the applied bivariate approach in the empirical analysis is confirmed.

Third, the study proposes a new method to estimate parameters of stable distribution based on the minimum $\chi^2$ principle. It is suitable for regular series and also for non-standard, censored, clustered and grouped data. Monte Carlo simulations are performed to compare the efficiency of this method with the McCulloch quantile approach. The new method proves more accurate. It offers flexibility to model data with large clusters of returns around zero and in the tails of the distributions. The minimum $\chi^2$ algorithm is fast and resistant to irregularities in data.

Fourth, a comprehensive analysis of distributions of returns to assets on the Warsaw Stock Exchange and the Budapest Stock Exchange is performed. Returns on both markets are shown to be heavy tailed, with larger variations on the Hungarian market. The symmetric stable distributions are fitted to the data using quantile, maximum likelihood and minimum $\chi^2$ approaches. The advantage of the minimum $\chi^2$ method in the case of censored data from the Warsaw Stock Exchange is shown and possible reasons for the inaccuracy of traditional estimates are analysed. It is concluded that returns on the analysed markets are heavy tailed and do not have normal distribution.
I.3. Outline of chapters

Chapter II reviews the definition and major properties of stable distributions. It describes the traditional parameter estimation techniques, that is the quantile approach, the characteristic function type estimators, the tail-based methodology and the maximum likelihood techniques. Since these methods seem to be inadequate for some particular applications, the minimum $\chi^2$ framework is applied to estimate parameters of symmetric stable distributions. After providing the theoretical background, its small-sample properties for grouped and raw data are tested. Finally, an empirical analysis of Polish and Hungarian stock prices is performed. The last section discusses the recent theoretical developments in the stable distribution area, including the multivariate stable variables, stability with respect to different probability schemes and the asymptotic stable distributions.

Chapter III concentrates on the speculative market processes. It reviews the theoretical definitions and possible justifications of speculative behaviour of market participants. The differences and similarities between speculation and market efficiency are discussed. The concept of speculative bubbles as a tool for analysis of speculation is introduced and examples of real life bubbles are discussed. A clear distinction between concepts of speculative and rational bubbles is made. Formal derivation of the rational bubbles is provided. Examples of rational bubbles are given and appropriate tests are presented. The chapter discusses the theoretical plausibility of bubbles. Next, it reviews the behavioural approach to finance, which gives an alternative framework to model and interpret speculative processes. It describes the main motivation and ideas behind behavioural theory and analyses three representative asset pricing models. Finally, it introduces the Diba – Grossman price generating process, as an example of speculative process developed in the rational expectation framework. The process is formally derived and interpretations of its parameters are discussed. The appropriate estimation techniques are described.

Chapter IV provides the main results of the thesis. It deals with the relationship between the variances of the stochastic root and the random term of the Diba – Grossman process from one side and characteristic exponent and scale parameters of stable distribution of returns to such processes on the other side. First, the hypothesis that the Diba-Grossman process generates heavy-tailed returns is
verified and the accuracy of fit of other possible distributions is tested. Then, after a more detailed evaluation, appropriate conversion tables are produced and the tabulated bivariate function is analysed.

Chapter V concentrates on the empirical analysis of speculation and distribution of returns. First, it provides a comparative analysis of the US and the UK stock markets in the last five years. Then, the time varying levels of speculation and exposure to random shocks are evaluated for the New York Stock Exchange. This gives a possibility to analyse the changing degree of speculation and draw conclusions about investors’ behaviour in the last thirty years. Second, a detailed study of 66 stock market indices from different countries is presented. Data is filtered for possible heavy tailed effects connected with risk neutrality of agents. Then, the null hypothesis of stable distribution of returns is tested and appropriate parameters are fitted by the maximum likelihood method. The parameters of Diba – Grossman processes are estimated. The relative levels of speculation and exposures to random shocks are analysed for different countries. The relationship between speculation and non-normality is investigated and the presence of outliers is tested. Outliers can be defined as stock markets on which the level of speculation is significantly different than it is implied by the degree of non-normality, and hence their analysis allows conclusions about the character of the empirical relationship between speculation, random shocks and shape of the distribution of returns.

Chapter VI summarises the work presented in the thesis. It includes the main results of the study and provides suggestions for further research.
Chapter II

STABLE DISTRIBUTIONS

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II.1. Introduction

The most popular family of distributions is the Gaussian one, and therefore the majority of traditional financial theories are based either on the normal or on the log-normal distribution of returns (see e.g. Bachelier, 1900). However, in recent decades the assumption about the normality of the capital markets' returns has been seriously questioned. The excess kurtosis, found by Mandelbrot (1962, 1963a, 1963b, 1967) and Fama (1965), suggests rejection of the hypothesis about the normality of returns. One of the possible alternatives to model heavy tailed data is to apply the family of stable distributions.

The theory of univariate stable distributions is developed in the 1920s and 1930s by Paul Levy and Aleksander Khinchine\(^1\). It is covered in detail in such classical textbooks as Gnedenko and Kolmogorov (1954), Feller (1966) or more recently by Samorodnitsky and Taqqu (1994) and Janicki and Weron (1994). The stable distributions, also called heavy tailed, stable Paretian or alpha-stable, are defined as distributions which are invariant under addition with respect to one of their parameters, called the characteristic parameter of the distribution. The popularity of stable distributions is mainly because they generalise the central limit theorem to the case when the second moments of the underlying variables do not exist. Stable distributions are used to model data with a large number of extreme observations, where the classical distributions, as e.g. t-Student or \(\chi^2\) fail to capture the behaviour of the sample.

In this chapter, the definition and main properties of stable distributions are reviewed. Then, the most important estimation techniques are described. They are grouped, according to the underlying methodologies, into quantile, characteristic function, maximum likelihood and tail based estimations. In the third section, the minimum \(\chi^2\) algorithm is developed and its properties are compared to the characteristics of the McCulloch (1986) methodology. This algorithm can be applied both to regular time series and to non-standard, clustered, censored and grouped data. The fourth section deals with estimation of symmetric stable distribution parameters.

\(^1\) See e.g. Levy (1924,1937)
for truncated data from the Warsaw Stock Exchange and clustered data from the Budapest Stock Exchange. Both McCulloch and minimum $\chi^2$ methodologies are applied and the advantage of the latter for the irregular data is proven. Finally, the last section provides the most important information about recent developments in the stable distribution theory. It discusses multivariate stable variables, variables stable with respect to different probability schemes and the asymptotic stable distributions.

II.2. Definition and main properties

II.2.1. Definition of $\alpha$ stable distributions

There are four equivalent definitions of stable laws known in the literature (Feller, 1966 and Samorodnitsky and Taqqu, 1994). For the purpose of this study, the definition based on the characteristic function is chosen. Three alternative definitions are presented among the main properties of the stable laws in the next section. The characteristic function based definition seems to provide the most specific description of stable distributions. It explicitly includes all four stable distribution parameters, giving them clear intuitive meaning. This section discusses additionally the alternative parametrisations of stable distribution and defines strictly stable and symmetric stable variables.

The presented definition of stable distribution is due to Zolotarev (1957), who proposes to describe stable distributions by corresponding family of the characteristic functions.

A random variable $X$ is said to have a stable distribution if there are parameters $0 < \alpha \leq 2$, $c > 0$ and $-1 \leq \beta \leq 1$ and $\delta \in \mathbb{R}$ such that its log characteristic function has the form:

$$\psi(t) = \log E(e^{itX}) = \begin{cases} i\delta \ t - c |t|^{\alpha} & 1-i\beta \ \text{sgn}(t) \tan \frac{\pi \alpha}{2}, \ \alpha \neq 1, \\ i\delta \ t - c |t|^{\alpha} & 1+i\beta \ \frac{2}{\pi} \text{sgn}(t) \log |t|, \ \alpha = 1 \end{cases}$$

(2.1)
where \( i^2 = -1 \). The parameter \( \alpha \) is called the index of stability or characteristic exponent and:

\[
\text{sgn}(t) = \begin{cases} 
1 \text{ if } t > 0 \\
0 \text{ if } t = 0 \\
-1 \text{ if } t < 0
\end{cases}
\]

Parameter \( \beta \) is irrelevant when \( \alpha=2 \). Variable \( X \), that has a stable distribution, is called a stable variable (Zolotarev, 1957 and Samorodnitsky and Taqqu, 1994: p.3).

There are different variations of (2.1). One of them proposes a parametrisation that is often more convenient for analytical work. It says that \( X \) has a stable distribution with parameters \( \alpha, c, \beta \) and \( \delta \), denoted as \( X \sim S(\alpha,c,\beta,\delta) \), if its log characteristic function is given by (Nolan, 1999):

\[
\psi(t) = \log E(e^{itX}) = \begin{cases} 
i \delta t - c|t|^{\alpha}|t|^{\alpha} \left[ 1 + i \beta \text{sgn}(t) \tan \left( \frac{\pi}{2} \left( (c|t|)^{1-\alpha} - 1 \right) \right) \right], & \alpha \neq 1 \\
i \delta t - c|t|^{\alpha} \left[ 1 + i \beta \frac{2}{\pi} \text{sgn}(t)(\log|t| + \log(c)) \right], & \alpha = 1
\end{cases}
\]

The main advantage of the latter parametrisation is that the characteristic functions are jointly continuous in all four parameters (see Nolan, 1999). Some other possible parametrisations of stable distribution are discussed in McCulloch (1986) and Samorodnitsky and Taqqu (1994). Through this study, the parametrisation proposed in (2.1) is accepted.

The definitions of strictly stable and symmetric stable variables are adopted after Samorodnitsky and Taqqu (1994).

A stable variable \( X \sim S(\alpha,c,\beta,\delta) \) is called strictly stable if and only if:

a) \( \alpha \neq 1 \) and \( \delta = 0 \) or

b) \( \alpha = 1 \) and \( \beta = 0 \),

where \( \alpha, c, \beta, \delta \) are parameters of the characteristic function defined in (2.1) (Samorodnitsky and Taqqu, 1994: p.12).
A stable variable $X \sim S(\alpha, c, \beta, \delta)$ is called symmetric stable if and only if $\beta = 0$. A stable variable $X \sim S(\alpha, c, \beta, \delta)$ is called symmetric stable about 0 if and only if $\beta = 0$ and $\delta = 0$, where $\alpha, c, \beta, \delta$ are parameters of the characteristic function defined in (2.1) (Samorodnitsky and Taqqu, 1994: p.11).²

II.2.2. Properties of $\alpha$ stable distributions

This section presents the most important properties of stable distributions. First, interpretation of stable distribution parameters is given. Examples of stable distributions are discussed and the integral representation of the distribution function is presented. Then, descriptions of stable laws equivalent to (2.1) are discussed. Properties connected with sums of the stable variables and the consequences of multiplying and adding a real constant to a stable variable are reviewed. The domain of attraction is defined and the properties of symmetric stable variables are presented.

The full stable distribution class is characterised by four parameters, denoted by $\alpha, \beta, c$ and $\delta$. The location parameter $\delta$ shifts the distribution to the left or to the right. The scale parameter $c$ compresses or extends the distribution about $\delta$ in proportion to $c$. The characteristic exponent $\alpha$ lies in the range $(0,2]$ and determines the thickness of the tails of the distribution. It is almost universally agreed (McCulloch, 1986) that the skewness parameter $\beta$ should lie in the range $[-1,1]$. When $\beta$ is positive, the distribution is skewed to the right and for $\beta < 0$ it is skewed to the left. As $\alpha$ approaches 2, $\beta$ loses its effect and the distribution approaches the normal distribution regardless of $\beta$. The tail thickness of the distribution increases as the value of $\alpha$ decreases. Tails behave asymptotically like functions of the form $k^\alpha$, where $k$ is a real number. For $\alpha = 2$ a normal distribution results with mean $\delta$ and variance $2c^2$. When $\alpha < 2$ the variance is infinite. When $\alpha > 1$, the mean of the

² Note that these definitions vary slightly from Samorodnitsky and Taqqu (1994) terminology, who define symmetric stable variable as a stable variable with $\beta = 0$ and $\delta = 0$ and stable variable with $\beta = 0$ as stable variable symmetric about $\delta$. However, the proposed definition seems to be much more convenient for the purpose of this study.

¹ For other definitions of $\beta$ see Hall, 1981.
distribution exists and is equal to $\delta$. However for $\alpha \leq 1$ the tails are so heavy that the mean does not exist (see Zolotarev 1957, McCulloch 1986).

The only cases when the continuous densities of $\alpha$ stable distributions exist are (Samorodnitsky and Taqqu, 1994: p.10):

1) the normal distribution with $\alpha = 2$ and density function of the form:

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma}\right)^2},$$

2) the Cauchy distribution with $\alpha = 1$:

$$f(x) = \frac{1}{\pi} \frac{c}{c^2 + (x - \gamma)^2},$$

3) the Levy distribution with $\alpha = 1/2$:

$$f(x) = \frac{1}{\sqrt{2\pi} x^3} e^{-\frac{1}{2} x^2} \text{ for } x > 0.$$

There is no analytically derived closed density function for general stable distributions. Bergstrom (1952) develops a series expansion, which can be applied to approximate densities of the stable variables symmetric about 0. When $\alpha>1$, Bergstrom results yield the convergent series:

$$f_\alpha(u) = \frac{1}{\pi \alpha} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma \left( \frac{2k+1}{\alpha} \right)}{(2k)! \alpha^{2k}} u^{2k},$$

where $u = (x - \delta)/c$ is the standarized symmetric stable variable with characteristic exponent $\alpha$ and $f_\alpha(u)$ is its density function. For $\alpha > 1$ Bergstrom provides a finite series, which for $u > 0$ is:

$$f_\alpha(u) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^i}{k!} \frac{\Gamma(\alpha k + 1)}{\alpha^k} \sin \left( \frac{k\pi\alpha}{2} \right) + R(u),$$

where $R(u) = O(u^{(\alpha+1)/\alpha})$.

Term by term integration of the above expansions yields convergent and asymptotic series for the cumulative distribution function (c.d.f.) of a standarized symmetric stable variable.
Another approach is proposed by Paulson, Holcomb and Leitch (1975), who approximate stable density by evaluating the inverse Fourier transform of the characteristic function.

Finally, Zolotarev (1983) proves that the following function is the representation of the distribution function of stable variables:

1) If $\alpha \neq 1$ and $x > 0$:

$$ G(x, \alpha, \beta) = \begin{cases} 
\frac{1}{2} (1 - \beta) + \frac{1}{\pi} \int_{-\pi}^{\pi} \exp\{-V_\alpha(x, \varphi)\} d\varphi, & \text{if } \alpha < 1 \\
1 - \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\pi} \exp\{-V_\alpha(x, \varphi)\} d\varphi, & \text{if } \alpha > 1
\end{cases} $$

where

$$ K(\alpha) = 1 - |1 - \alpha|, $$

$$ V_\alpha(x, \varphi) = x^{\alpha-1} \left( \frac{\sin(\alpha \varphi + \frac{\pi}{2} \beta K(\alpha))}{\cos \varphi} \right)^{\frac{\alpha}{1-\alpha}} \frac{\cos((\alpha - 1)\varphi + \frac{\pi}{2} \beta K(\alpha))}{\cos \varphi}. $$

For $\alpha \neq 1$ and $x = 0$:

$$ G(0, \alpha, \beta) = \frac{1}{2} \left( 1 - \beta \frac{K(\alpha)}{\alpha} \right). $$

2) If $\alpha = 1$ and $\beta > 0$, then:

$$ G(x, 1, \beta) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\{-V_1(x, \varphi)\} d\varphi, $$

where

$$ V_1(x, \varphi) = \frac{\pi + \beta \varphi}{2 \cos \varphi} \exp\left\{ -\frac{x}{\beta} + \left( \varphi + \frac{\pi}{2\beta} \right) \tan \varphi \right\}. $$

and for $\alpha = 1$ and $\beta = 0$:

$$ G(x, 1, 0) = \frac{1}{2} + \frac{1}{\pi} \arctan x. $$
However, all of these methods are far from trivial and high precision in practical calculations is difficult to achieve (Nolan, 1999).

For the description of the main properties of the stable distributions, it is convenient to introduce a short hand notation "\(=d\)", which describes equality in distributions. Similarly as earlier, \(X, X_i, X_2, \ldots\) denote mutually independent random variables with a common distribution \(R\). Additionally:

\[
S_n = X_1 + \ldots + X_n. \tag{2.2}
\]

It is said that the distribution \(F\) of independent random variables \(X_i\) belongs to the domain of attraction of a distribution \(R\) if there exist norming constants \(a_n > 0\) and \(b_n\) such that the distribution of \(a_n^{-1} (S_n - b_n)\) tends to \(R\) as \(n\) increases (Feller, 1966: p.168).

Definition (2.1) implies the following property of stable variables (Samorodnitsky and Taqqu, 1994: p.6)

**Property 2.1** (Feller, 1966: p.166)

If a random variable \(X\) has a stable distribution then for any \(n \geq 2\) there exist constants \(c_n > 0\) and \(d_n\) such that:

\[
S_n = d c_n X + d_n.
\]

The only possible norming constants are of the form \(c_n = n^{1/\alpha}\), where \(\alpha \in (0,2]\) is the characteristic exponent of the distribution \(R\) (of random variable \(X\)) defined in (2.1).

Gnedenko and Kolmogorov (1954: Chapter 34) show that definition (2.1) can be implied from Property 2.1. Hence, Property 2.1 can be treated as equivalent to the definition of stable distribution.

Feller (1966: p.166) proves that the following, apparently more restrictive, statement is equivalent to Property 1.1.
Property 2.2 (Feller, 1966: p.166)

Let \( X, X_i, X_2, \ldots \) denote mutually independent random variables with a common stable distribution \( R \). Then, for any arbitrary constants \( c_1, c_2 \) there exist constants \( c > 0 \) and \( d \) such that

\[
c_1 X_i + c_2 X_2 = cX + d.
\]

The above means that a sum of independent stable variables with characteristic exponent \( \alpha \) is stable with the same exponent. Note that equivalence of Property 1.1 and Property 1.2 induces equivalence of Property 1.1 and the definition of stable distribution.

Moreover, it is known that Property 2.2, and hence definition (2.1), is equivalent to the following statement (Samorodnitsky and Taqqu, 1994: p.5).

Property 2.3 (Samorodnitsky and Taqqu, 1994: p.3)

If a random variable \( X \) has a stable distribution, then it has a domain of attraction, i.e. if there is a sequence of independently and identically distributed (i.i.d.) random variables \( Y_i, Y_2, \ldots \) and sequences of positive numbers \( \{d_n\} \) and real numbers \( \{a_n\} \), such that

\[
\frac{Y_1 + Y_2 + \ldots + Y_n}{d_n} + a_n \Rightarrow d X,
\]

where notation \( \Rightarrow d \) denotes convergence in distribution.

This means that stable distributions are the only distributions that can be obtained as limits of normalised sums of identically independently distributed (i.i.d.) random variables.

When summing the stable variables with the same characteristic exponents \( \alpha \), the values of the remaining three parameters can be easily derived. The appropriate formulae are given below.

Property 2.4 (Samorodnitsky and Taqqu, 1994: p.10)

Let \( X_i \) and \( X_2 \) be independent random variables with \( X_i \sim \text{S}(\alpha, \beta_i, c_i, \delta_i) \) and \( X_2 \sim \text{S}(\alpha, \beta_2, c_2, \delta_2) \). Then \( X_1 + X_2 \sim \text{S}(\alpha, \beta, c, \delta) \) with
The next property of the stable distribution is connected with adding a constant and multiplying a stable variable $X$ by a constant.

**Property 2.5** (Samorodnitsky and Taqqu, 1994: p.11)

Let $X \sim S(\alpha, c, \beta, \delta)$ and let $a$ be a real constant.

a) $(X + a) \sim S(\alpha, c, \beta, \delta + a)$,

b) $aX \sim S(\alpha, |a|c, \text{sign}(a)\beta, a\delta)$ for $\alpha \neq 1$, 
   
   $aX \sim S(\alpha, |a|c, \text{sign}(a)\beta, a\delta - 2a(\ln |a|c\beta / \pi))$ for $\alpha = 1$.

Properties 2.4 and 2.5 imply the following relationship.

**Property 2.6** (McCulloch, 1986)

If variable $X$ has the stable distribution $S(x; \alpha, \beta, c, \delta)$, the transformed variable $z = (x - \delta) / c$ has the same shaped distribution, but with location parameter 0 and scale parameter 1.

The following property describes stable distributions with location parameter equal to zero.

**Property 2.7** (Samorodnitsky and Taqqu, 1994: p.11)

For any $0 < \alpha < 2$,

$X \sim S(\alpha, c, \beta, 0) \iff -X \sim S(\alpha, c, -\beta, 0)$.

It is well known (Feller, 1966) that stable variables do not possess all moments. This can be formalised in the following rule.

**Property 2.8** (Feller, 1966: p.169)

A stable distributed variable with characteristic exponent $\alpha$ has absolute moments of all orders less than $\alpha$.

It means that only variables with $\alpha = 2$ have finite variance. Stable variables with $\alpha < 1$ have such heavy tails that even their means do not exist.
The next property concentrates on the domain of attraction of a distribution. It is a straightforward implication of Property 2.3.

**Property 2.9** (Feller, 1966: p.168)

A distribution possesses a domain of attraction, if and only if it is stable.

The next property states that any non-normal stable variable with location parameter equal to zero can be represented as a linear combination of two stable variables with skewness parameter equal to one.

**Property 2.10** (Samorodnitsky and Taqqu, 1994: p.16)

If \( X \sim S(\alpha, c, \beta, 0) \) with \( \alpha < 2 \), then there exist two i.i.d. random variables \( Y_i \) and \( Y_2 \) with common distribution \( S(\alpha, c, 1, 0) \) such that:

\[
X = \left(1 + \frac{\beta}{2}\right)^{\frac{1}{\alpha}} Y_1 - \left(1 - \frac{\beta}{2}\right)^{\frac{1}{\alpha}} Y_2 \quad \text{if } \alpha \neq 1,
\]

and

\[
X = \left(1 + \frac{\beta}{2}\right)^{\frac{1}{\alpha}} Y_1 - \left(1 - \frac{\beta}{2}\right)^{\frac{1}{\alpha}} Y_2 + \sigma \left( \frac{1 + \beta}{\pi} \ln \frac{1 + \beta}{2} - \frac{1 - \beta}{\pi} \ln \frac{1 - \beta}{2} \right) \quad \text{if } \alpha \neq 1.
\]

Hence, the totally skewed stable variables (stable variables with \( \beta = 1 \)) can be regarded as basic building blocks when constructing other stable variables. This property finds applications in the simulations of stable numbers.

The remaining properties concentrate on strictly stable and symmetric stable variables. The symmetric stable distributions form an important class of stable laws. This is mainly due to their relatively simple log characteristic function. Indeed, putting \( \beta = 0 \) and \( \delta = 0 \) in (2.2) results in the following log characteristic function of stable variable \( X \) symmetric about 0:

\[
\psi(\theta) = \log E(e^{i\theta}) = -c(\theta)^{\alpha}.
\]

This function is relatively easy to evaluate and therefore estimations and simulations are much simpler in the symmetric than in the asymmetric case.
Property 2.11 (Samorodnitsky and Taqqu, 1994: p.3)

If $X, X_1, X_2, \ldots$ denote mutually independent strictly stable random variables with a common distribution $R$, then, for any arbitrary constants $c_1, c_2$ there exists a constant $c > 0$ such that

$$c_1 X_1 + c_2 X_2 = d c X.$$

The above is equivalent to Property 2.2 with $d=0$.

Property 2.12 (Samorodnitsky and Taqqu, 1994: p.3)

If a random variable $X$ has symmetric stable distribution, then $X$ and $-X$ have the same distribution.

Property 2.13 (Samorodnitsky and Taqqu, 1994: p.3)

A symmetric stable variable is strictly stable.

II.3. Traditional estimation techniques

II.3.1. Fama and Roll approach

The estimators of stable distribution parameters can be sorted according to the underlying methodologies into four major groups: the quantile estimators, the tail estimators, the characteristic function estimators and the maximum likelihood based estimators. This section provides a brief description of all of these methods. Special attention is given to the McCulloch (1986) quantile algorithm. This method is computationally simple and has clear intuitive meaning. Additionally, it seems most convenient for empirical research and is frequently used in applied studies (see e.g. Charemza and Kominek, 1999).

Among the earliest approaches to estimate the stable distributions parameters (see e.g. Fama, 1965) the most attractive method is probably due to Fama and Roll (1968). Basing on Bergstrom (1952) approximations, they tabulate the c.d.f. of symmetric stable variables for $1 \leq \alpha \leq 2$. One of the applications of this result includes simple estimates of scale and location parameters in the symmetric case.
Fama and Roll (1971) suggest a procedure to compute quantile based estimates of the scale parameter and characteristic exponent in the symmetric case with $\alpha \in [1,2]$. They compute the .72 fractile of a standardised symmetric ($\beta=0$, $\delta=0$ and $\epsilon=1$) stable distribution, which for $\alpha \in [1,2]$ is in the range $0.827 \pm 3$. Hence, they propose to estimate the scale parameter $c$ from the following formula:

$$\hat{c} = \frac{1}{2 \cdot 0.827} (x_{72} - x_{32}) .$$

An estimate of $\alpha$ is obtained by searching a table of standardised symmetric stable c.d.f.'s for the value of

$$\hat{\alpha}_f = G(f, \hat{z}_f) ,$$

where

$$\hat{z}_f = 0.827 \frac{x_f - x_{1-f}}{x_{72} - x_{28}} .$$

The symbols of type $x_z$ denote the sample $z$ quantiles. It is proposed to choose $f$ in the range $0.95 < f < 0.97$ but specific suggestions are not given.

II.3.2. Tail estimators

A different approach is proposed by Pickands (1975) and Hill (1975). They look only at the tail behaviour of the analysed distributions and on this basis construct estimates of the characteristic exponent.

Let $Y_1, Y_2, \ldots, Y_n$ be a sequence of $n$ observations drawn from a stationary i.i.d. process with probability distribution function $F$, which is unknown. It is assumed that the distribution is heavy tailed. Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ be the descending order statistics from $Y_1, Y_2, \ldots, Y_n$. The extreme value theory states that the extreme value distribution of the ordered data must belong to one of three possible general families, regardless of the original distribution function $F$. In particular, if the original distribution is heavy tailed, there is only one general family it can belong to (Pictet, Dacarogna and Müller, 1998):

---

For parametric forms of these distributions see the max-stable and min-stable distributions in Chapter II.6.2.
where $G(y)$ is the probability that $Y_{(n)}$ exceeds $y$.

It is worth noting that all stable distributions, $t$-Student distributions and unconditional distributions of ARCH (autoregressive conditionally heteroscedastic processes) fall in the domain of attraction of distribution (2.3) (see e.g. Pictet, Dacarogna and Müller, 1998). However, the extreme value approach concentrates only on the estimation of one parameter, the characteristic exponent $\alpha$ defined in (2.3). Distribution (2.3) clearly permits $\alpha>2$, which is inconsistent with the accepted definition (2.1). Hence, the presented approach explores the estimates of the characteristic exponent of the generalised stable distribution (Chapter II.6) not these of the stable distribution defined in Chapter II.2. Nevertheless, the tail based methods belong to the most popular techniques of analysis of heavy tailed data (Pictet, Dacarogna and Müller, 1998). Conventionally, these methods concentrate on the estimation of $\gamma=1/\alpha$. Among many proposed estimators, the following attract the biggest attention of researchers:

1) The Pickands (1975) estimator:

$$\hat{\gamma}_{n,m}^P = \left[ \frac{\ln X_{(m)} - X_{(2m)}}{X_{(2m)} - X_{(4m)}} \right] / \ln 2. $$

2) The Hill (1975) estimator:

$$\hat{\gamma}_{n,m}^H = \frac{1}{m-1} \sum_{i=1}^{m-1} \ln X_{(i)} - \ln X_{(m)} \quad \text{where } m > 1. $$

3) The de Haan and Resnick (1980) estimator:

$$\hat{\gamma}_{n,m}^R = [\ln X_{(1)} - \ln X_{(m)}] / \ln m. $$

4) The Dekkers et al. (1990) estimator:

$$\hat{\gamma}_{n,m}^D = \hat{\gamma}_{n,m}^H + 1 - 0.5 \left( 1 - \frac{(\hat{\gamma}_{n,m}^H)^2}{\hat{\gamma}_{n,m}} \right)^{-1}. $$

where
\( \hat{\gamma}_{n,m}^{(1)} \) is the Hill estimator;
\[
\hat{\gamma}_{n,m}^{(2)} = \frac{1}{m-1} \sum_{i=1}^{m-1} [\ln X_{(i)} - \ln X_{(m)}]^2.
\]

In order to apply these estimators, the \( X_{(n)} \) ordered observation must be chosen from the sample. Hence, the most obvious problem connected with the above estimates is the choice of the appropriate \( m \). Clearly, for different values of \( m \), different estimates of \( \alpha \) may be obtained. Imagine, for example, a 1000 observation sample drawn from a normal distribution, with \( X_{(1)} = 3.35, X_{(2)} = 2.81, X_{(5)} = 2.74, X_{(10)} = 2.57 \) and \( X_{(50)} = 1.78 \). For \( m = 2 \), the value of the de Han and Resnick (1980) estimator of \( \gamma \) is equal to 0.253. The respective values for \( m = 5, m = 10 \) and \( m = 50 \) are 0.120, 0.115 and 0.161. There are many rules of thumb helping to choose \( m \) but no clear algorithm is given (see e.g. Picket, Dacorogna and Müller 1998).

Despite this problem, there are number of papers applying the tail estimates to distribution analysis, value-at-risk studies and many other fields (for literature review see e.g. Hols and de Vries, 1991 and Picket, Dacorogna and Müller, 1998).

II.3.3. Estimators based on characteristic function

The characteristic function is the Fourier transform of the p.d.f. and there is a one to one correspondence between these two representations of a given distribution (Feller, 1966: p.480). Consequently, in the absence of a closed density function, the characteristic function seems to be an appropriate tool for the parameter estimation problem. Such an approach is accepted by Press (1972) and Arad (1980) who use a linearisation of the sample characteristic function. Their results are extended by Koutrouvelis (1980, 1981).

Koutrouvelis (1980) analyses the characteristic function of stable variables, as defined in (2.1). He extracts the real part from the log-characteristic function \( \psi(t) \). The obtained log relationship is of the form:
\[
\log \{ -\text{Re}[\psi(t)] \} = \alpha \log |t| + \alpha \log c.
\]
Similarly, analysing the imaginary part of \( \psi(t) \), he gets for \( \alpha \neq 1 \):
\[
\text{Im}[\psi(t)] = \delta t + |ct|^\alpha \beta t \tan \frac{\pi \alpha}{2} = \delta t + \beta |ct|^\alpha f(t,\alpha,\beta).
\]

After normalising the observations with Fama and Roll (1971) estimates of scale and location parameters, the sample characteristic function is estimated from the formula:

\[
\hat{\Phi}(t) = \frac{1}{N} \sum_{n=1}^{N} \exp\{itx(n)\},
\]

where \(N\) is the number of data samples. The real and imaginary parts of the sample characteristic function can be computed in the following way:

\[
\text{Re}[\hat{\psi}(t)] = \log |\hat{\Phi}(t)|, \quad \text{Im}[\hat{\psi}(t)] = \arctan \left( \frac{\text{Im}[\hat{\Phi}(t)]}{\text{Re}[\hat{\Phi}(t)]} \right).
\]

Regressing the logarithms of the opposite of the real parts of the sample characteristic function on the logarithms of the absolute values of the particular moments yields the estimates of \(\alpha\) and \(c\) (the scale parameter should be corrected by the Fama and Roll estimates used in the normalisation). Similarly, using the formula for the imaginary part of the characteristic function, the estimates of \(\beta\) and \(\delta\) can be obtained.

The most obvious problem of the characteristic function based estimates is connected with the choice of moments, for which the sample characteristic function should be calculated. Koutrouvelis (1981) proposes the iterative weighted regression that improves the accuracy of estimates. However, even in this paper no clear estimation scheme is advised (for extensive review of characteristic function estimates see Kogon and Williams, 1998).

II.3.4. Maximum likelihood estimators

The maximum likelihood (ML) technique requires computation of exact values of probability density function in all sample points. Hence, lack of closed functional form of density function makes exact maximum likelihood impossible to implement.
Nevertheless, approximate methods applying the maximum likelihood principle can be developed.

DuMouchel (1975) shows that maximum likelihood may be used to estimate the four stable parameters, and that the ML estimates have the usual asymptotic normality governed by the information matrix, except in the boundary cases of $\alpha = 2$ and $\beta = \pm 1$. He tabulates the information matrix, which may be used for asymptotic hypothesis testing except in the boundary cases. However, full implementation of this methodology is a nontrivial task (see e.g. Mittnik and Rachev, 1993).

Recently, Brorsen and Yang (1990) and Liu and Brorsen (1995) employ the Zolotarev's (1983) integral representation to provide the maximum likelihood estimates of the stable distribution parameters. Another maximum likelihood algorithm, reported and applied by Mittnik and Rachev (1993), is proposed by Chen (1991). Buckle (1995) goes beyond the pure maximum likelihood theory and explores the Bayesian posterior distribution of the stable parameters. Finally, Nolan (1999) reports ML estimation algorithm and makes it available for empirical researchers. He maximises the following log likelihood function for an i.i.d. stable sample $X_1, X_2, ..., X_n$:

$$ l(\theta) = \prod_{i=1}^{n} \log f(X_i | \theta) $$

over the parameter space $\theta = (0,2] \times [-1,1] \times (0,\infty) \times (-\infty,\infty)$. Parameter vector $\theta$ consists of characteristic exponent, skewness, scale and location parameters. Nevertheless, not many details about the computational part of the procedure and the properties of the finite samples estimates are known.

II.3.5. McCulloch quantile methodology

McCulloch (1986) generalises the Fama and Roll (1968, 1971) approach and provides estimators of all four stable parameters, with $\beta$ in its full permissible range $[-1,1]$ and $\alpha$ in the range $[0.6, 2]$. These estimators are based on functions of five predetermined sample quantiles and are asymptotically normal with calculable asymptotic standard error.
The main idea of the McCulloch's technique is relatively simple. In the first step, he proposes to calculate the values of the following functions:

\[ \psi_\alpha = \frac{x_{95} - x_{05}}{x_{75} - x_{25}} \]

\[ \psi_\beta = \frac{x_{95} + x_{05} - 2 \cdot x_{50}}{x_{95} - x_{05}} \]

where symbols of the type \( x_z \) denote, as earlier, the sample \( z \) quantiles. Both \( \psi_\alpha \) and \( \psi_\beta \) are monotonic functions of the parameters \( \alpha \) and \( \beta \). Using tabulated values of these functions, he estimates the values of \( \alpha \) and \( \beta \). If \( \psi_\alpha \) is smaller than 2.439 or bigger than 25, its values are not included in tables and one must proceed intuitively.

Next, the estimator of the scale parameter can be computed:

\[ \hat{c} = \frac{x_{25} - x_{25}}{\varphi_3(\hat{\alpha}, \hat{\beta})} \]

where the values of \( \varphi_3 \) are tabulated and depend on the obtained estimators of \( \alpha \) and \( \beta \) exclusively.

Finally, the location parameter is calculated from the equation:

\[ \hat{\delta} = x_{50} + \hat{c} \varphi_3(\hat{\alpha}, \hat{\beta}) - \hat{\beta} \hat{c} \tan \frac{\pi \hat{\alpha}}{2} \]

where as previously the values of \( \varphi_3 \) are tabulated and depend exclusively on the estimators of \( \alpha \) and \( \beta \).

McCulloch provides tables of all functions used in the estimation method. Their values can be found for the following arguments: \( \alpha = 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0 \) and \( \beta = 0.0, 0.25, 0.50, 0.75, 1.00 \). For intermediate cases, linear approximations are suggested. In a similar way the asymptotic standard deviations may be computed.

The McCulloch method is computationally simple and provides results with an acceptable level of accuracy, which makes it suitable for empirical analysis. However, difficulties in identifying particular quantiles may significantly weaken its efficiency. Such problems may occur, for example, when analysing censored, rounded and clustered data. In these situations, the identification of quantiles may be impossible.
and inference about the shape of the tails may be inaccurate. Imagine a series where 25 of 100 observations are concentrated in a censorship point. In such a case, it is impossible to distinguish between the 0.05 and 0.25 quantiles, and the whole estimation is rather inappropriate. Additionally, some systematic comparative studies show that the McCulloch algorithm is significantly less efficient than other known methods. For example, Akgiray and Lamoureux (1989), after comparing in a simulation study the iterative regression estimator of Koutrouvelis (1980,1981) and the McCulloch quantile estimator, find evidence in favour of the first method. Similarly, Mittnik and Rachev (1993) report that Chen (1991), comparing his maximum likelihood estimator with the McCulloc's quantile estimator, strongly recommends the maximum likelihood one.

It can be clearly seen that there exists a trade-off between accuracy and computational complication. Obviously, the best idea would be to find an easy to compute and relatively accurate estimator that would employ the maximum likelihood methodology. In some cases this goal can be achieved using the minimum $\chi^2$ estimation.

II.4. Minimum chi-squared estimation\(^5\)

II.4.1. Introduction

The traditional estimation techniques work reasonably well for the distributions of returns observed on the mature financial markets. However, even small irregularities in data may result in inaccurate estimates or even collapse of the computation algorithms. For example, truncations of the distribution and clusters of returns around zero may lead to difficulties in identifying individual quantiles. Similarly, discontinuities of the distribution of returns caused by rounding-off procedures raise problems in determining the shape of the distribution and maximising the likelihood function. Such events are not rare on the financial markets. Many of the emerging markets (e.g. Poland, Lithuania, China, Turkey) and some mature markets (e.g. France) impose limits on the daily changes in the stock prices, which might result

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\(^5\) The results presented in this section are based on Kominek (1999).
in the censored distributions of returns. High concentrations of zero returns are common on some non-liquid markets (e.g. Hungary) and non-continuities caused by rounding-off effects are present e.g. in Poland and Hungary.

One of the possible estimation techniques that can accommodate these problems is the minimum $\chi^2$ (MCS) algorithm. This technique is based on minimising one of the $\chi^2$ type statistics for the data grouped according to specific rules. The grouping causes some loses of the sample information, but offers exceptional flexibility when dealing with censored and clustered data. It is shown later that, when compared to the McCulloch quantile algorithm, the MCS method is advantageous not only for grouped data, but often also for ungrouped and uncensored series.

The main contribution of this section is to propose a new method to estimate the parameters of a symmetric stable distribution and test its relative efficiency. This method can be applied both to regular time series (e.g. from mature financial markets) and to non-standard, clustered, censored and grouped data. The section provides details about the minimum $\chi^2$ estimation. It shows the results of Monte Carlo simulations and comparisons of McCulloch's and minimum $\chi^2$ estimations accuracy for grouped and ungrouped data. In the case of grouped observations, evidence in favour of the MCS method is reported. When analysing ungrouped data the MCS estimation is generally better than the McCulloch's quantile method for samples larger than 400 observations and with large $\alpha's$.

II.4.2. Definition and properties of MCS estimator

The minimum $\chi^2$ estimation is grounded within the theory of the $\chi^2$ tests of fit. These tests are originally invented by Karl Pearson in 1900. Despite its long history, the $\chi^2$ statistics and their modifications still remain among the most useful statistical procedures. Modern developments have substantially increased their flexibility, among others to the situation when unknown parameters must be estimated for the hypothesised family of distributions.
Formal definition of minimum $\chi^2$ (MCS) estimators is presented following Stuart and Ord (1991). Let $s$ be a number of given samples of size $n_1$, $n_2$, ... , $n_s$. Suppose that each observation in the $i$th sample is classified into one of $k_i$ mutually exclusive and exhaustive classes and $\pi_{ij}$ is the probability of an observation from the $i$th sample falling into the $j$th class. It is true that:

$$\sum_{j=1}^{k_i} \pi_{ij} = 1.$$ 

Let $n_{ij}$ be the number of $i$th sample observations falling into the $j$th class and $p_{ij} = n_{ij} / n_i$ be the corresponding empirical frequency. The probabilities $\pi_{ij}$ are functions of a set of unknown parameters $\Theta$. The statistic is called a minimum $\chi^2$ estimator of $\Theta \in \Theta$, if it is obtained by minimising, with respect to $\theta$, the expression:

$$\chi^2 = \sum_{i=1}^{s} n_i \sum_{j=1}^{k_i} \left( \frac{p_{ij} - \pi_{ij}}{\pi_{ij}} \right)^2.$$ 

To minimise this expression the following equation must be solved:

$$\frac{\partial \chi^2}{\partial \theta} = -\sum_{i=1}^{s} n_i \sum_{j=1}^{k_i} \left( \frac{p_{ij}}{\pi_{ij}} \right)^2 \frac{\partial \pi_{ij}}{\partial \theta} = 0.$$ 

A root of this equation is the MCS estimator of parameter $\Theta$.

Fisher (1924) proves that the asymptotic properties of this estimator are similar to those of the grouped data maximum likelihood estimator. Additionally, he determines that the following log likelihood ratio statistic:

$$G^2 = \sum_{i=1}^{s} n_i \sum_{j=1}^{k_i} n_{ij} \log \frac{p_{ij}}{\pi_{ij}},$$

is asymptotically equivalent to the MCS.

Neyman (1949) proposes the MCS estimator obtained by minimising:

$$\chi^2 = \sum_{i=1}^{s} n_i \sum_{j=1}^{k_i} \left( \frac{p_{ij} - \pi_{ij}}{p_{ij}} \right)^2.$$ 

under the assumptions of positive $p_{ij}$. This minimising problem is more often solvable than the earlier propositions of Pearson and Fisher. Neyman proves additionally that his estimator is asymptotically normal.
Another similar MCS estimator can be obtained by minimising the so-called Freeman-Tukey statistic (see e.g. Stuart and Ord, 1991):

$$FT^2 = \sum_{i=1}^{g} \sum_{j=1}^{k} \left( \left( \frac{m_i}{m_j} \right)^{1/2} - \left( \frac{n_i}{n_j} \right)^{1/2} \right)^2 .$$

Stuart and Ord (1991) argue that since the maximum likelihood and the MCS methods have the same asymptotic properties, the choice between them should rest either on the grounds of computational convenience, or on those of superior sampling properties in small samples, or both. They suggest the maximum likelihood method for continuous data without giving any advice in case of discrete samples.

Following the principle of computational convenience in the case of stable distributions, it should be much easier to use the MCS procedure than the maximum likelihood algorithm. However, when using these methods attention should be given to the choice of the proper statistic for the minimisation procedure and to the choice of the optimal number of classes into which the data are divided.

The traditional $\chi^2$ approach uses the equal range cells, not taking into account the sample properties or the hypothesised distribution function. Mann and Wald (1942) recommend that the cells have equal probabilities under the hypothesised distribution. They find that for large sample of size $n$ and significance level $\alpha$ the number of cells should be approximately equal to:

$$M = 4 \left( \frac{2 \pi^2}{c(\alpha)^2} \right)^{1/2} ,$$

where $c(\alpha)$ is the upper $\alpha$-point of the standard normal distribution and $\alpha = 0.05$ is suggested. Mann and Wald use the large sample approximations and the complex minimax criterion, so $M$ should not be treated as a perfect proposal. Indeed, Schorr (1974) reports that the optimal $M$ is smaller than the Mann and Wald proposition. He recommends applying in the empirical research $2n^{0.4}$ equiprobable cells.

However, some difficulties may occur if equiprobable cells under the assumed distribution are used. For example, such cells can cause lack of continuity in the MCS values. In this situation, minimisation procedures fail very frequently resulting in the collapse of the whole estimation programs. Small changes in the cells' boundaries.
caused by small alteration of parameter values, might not change the number of
observations in each cell and, consequently, the corresponding frequencies. As the
theoretical probabilities of an observation falling into a given cell are constant (equal
to 1 divided by the number of cells), the MCS statistic remains unchanged. Then, the
estimated first derivative is equal to zero and many programs identify this situation
with reaching an extreme point, which obviously does not need to be true.

Therefore, in the case when the distribution parameters are to be estimated,
data dependant cells are highly recommended (Moore, 1986). Such cells are
constructed on the basis of the analysed data set. After choosing the optimal number
of cells, the boundaries are calculated to get similar numbers of observations in each
cell. As in the case of this study, the symmetric stable distribution is a priori assumed,
therefore the cells should also be approximately equiprobable under the estimated
distribution. Such a choice may damage the $\chi^2$ distribution of the MCS statistic but
should not effect the asymptotic properties of the estimates (Moore, 1986). The
second benefit comes from the continuity of the MCS statistic, because small changes
in the distribution parameters effect the theoretical probabilities of each cell and,
consequently, the values of the MCS statistics. The essential requirement connected
with the use of the data dependant cells is that as the sample size increases, the
random cell boundaries must converge in probability to a set of fixed boundaries.

The second problem is connected with the choice of the best MCS formula.
Usually the traditional Pearson statistic is suggested (Moore, 1986). However, in the
case of symmetric stable distributions, numerous Monte Carlo estimations, which
preceded this study, give evidence in favour of the Neyman statistic for ungrouped
data.

In the next two sections, the minimum $\chi^2$ estimation of symmetric stable
distribution is described and the most interesting results from Monte Carlo
experiments are reported.
II.4.3. MCS estimation for grouped data

The MCS estimators have similar properties to those of the grouped data maximum likelihood estimators (see Chapter II.3.2). Hence, the MCS estimation of grouped data is asymptotically equivalent to the maximum likelihood estimation (Fisher, 1924). As the MCS method is designed for grouped observations, it should be expected that it provides relatively accurate and precise estimates for all population parameters. In order to show this an exemplary Monte Carlo analysis is designed.

The symmetric stable distributed series are simulated using the Chambers, Mallows and Stuck (1976) method. Chambers, Mallows and Stuck propose an algorithm to simulation of stable variables with $\delta=0$, $c=1$ and $\beta=1$. By Property 2.10, this allows simulating stable numbers for all admissible values of $\beta$. Consequently, by Property 2.4, any stable variable with distribution function $S(\alpha, \beta, c, \delta)$ can be simulated. Such a simulation is encoded in GAUSS by J.H. McCulloch.

A series of 1000 observations grouped into 5, 10 and 20 subsets of equal range are analysed. The two extreme cells contain respectively the lower and upper 2.5%, 1.25% and 0.625% of sorted observations. Other cells are of equal ranges. The numbers of observations falling into each group are calculated. The theoretical frequencies in cells are computed by simulating the symmetric stable distribution function in 200 points in each cell and then calculating the corresponding theoretical probabilities of an observation falling into each group. This is done using McCulloch (1994) procedure. The computed theoretical and empirical frequencies are then put into the Neyman MCS criterion. This is minimised using the standard GAUSS optimisation library. Parameter alpha is restricted to the interval $[0.8, 2.0]$, scale can vary from 1/10000 to three initial values of scale, and location can change in the range of four initial values of scale parameter. The starting values are given by the McCulloch quantile method.

---

6 Respective formulas can be found in Chambers, Mallows and Stuck (1976).
7 All procedures by McCulloch are obtained from the American University GAUSS software archive on http://gurukul.uec.american.edu/econ/gaussres/GAUSSIDX.HTM.
8 For smaller $\alpha$'s serious difficulties using McCulloch (1994) procedure are encountered.
In order to check the properties of the MCS estimators, the parameters are calculated using both the McCulloch’s quantile method and the MCS algorithm. The McCulloch’s estimates are computed from grouped data. After 100 repeats of this experiment, the sums of squares of errors are calculated and the efficiency of both methods is compared. It should be noticed that the properties of the MCS estimators could be further improved if, instead of equal range groups, the equiprobable cells were chosen. The results for various levels of alpha are given in Tables 2.1-2.3.

For 5 and 10 groups, the Neyman MCS criterion is used. For samples divided into 20 groups the traditional Pearson MCS formula is employed. Symbol SSE denotes the sum of squares of errors of the estimates. These values are obtained by summing the squared differences between the population parameters and their estimates. The smaller the SSE, the better the properties of the analysed estimator. Mean denotes the mean value of the obtained estimates and should be as close to the real value of the parameter as possible.

For samples divided into 5 groups, the MCS estimates are more accurate for α’s less than 1.9. McCulloch’s quantile estimates are generally biased upwards, which may be the reason for its relatively better performance for highest α’s. The relative advantage of the MCS technique is more obvious for small values of the α parameter. In these cases, quality of both estimates is questionable, but MCS clearly dominates over the quantile algorithm.

For samples divided into 10 groups, the MCS estimates are more accurate than the McCulloch’s quantile estimates in all cases, although the difference is less significant for the larger values of parameter α. The situation is similar for samples divided into 20 groups, but in this case the increase of α is followed by improvement of relative performance of MCS estimators.
Table 2.1: Properties of MCS estimators for samples with 1000 observations divided into 5 groups

Table shows sums of squares of errors (SSE) of the estimates of characteristic exponent (Alpha), dispersion (Scale) and location (Location) parameters of symmetric stable distribution for samples with 1000 observations divided into 5 equal range groups. These values are obtained by summing 100 squared differences between the population parameters and their minimum $\chi^2$ (MCS) and quantile (McCulloch) estimates. Below SSE, the mean values of the obtained estimates (Mean) are presented. Above each section the values of population parameters are listed.

<table>
<thead>
<tr>
<th>Sample size: 1000</th>
<th>No of groups: 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population parameters: Alpha=1.0, Scale=1.0, Location=0</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>7.28733</td>
</tr>
<tr>
<td>Mean</td>
<td>1.26013</td>
</tr>
</tbody>
</table>

| Population parameters: Alpha=1.5, Scale=1.0, Location=0 |

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>1.38839</td>
</tr>
<tr>
<td>Mean</td>
<td>1.58157</td>
</tr>
</tbody>
</table>

| Population parameters: Alpha=1.6, Scale=1.0, Location=0 |

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>2.33405</td>
</tr>
<tr>
<td>Mean</td>
<td>1.69911</td>
</tr>
</tbody>
</table>

| Population parameters: Alpha=1.7, Scale=1.0, Location=0 |

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>3.58829</td>
</tr>
<tr>
<td>Mean</td>
<td>1.85000</td>
</tr>
</tbody>
</table>

| Population parameters: Alpha=1.8, Scale=1.0, Location=0 |

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>2.96520</td>
</tr>
<tr>
<td>Mean</td>
<td>1.94735</td>
</tr>
</tbody>
</table>

| Population parameters: Alpha=1.9, Scale=1.0, Location=0 |

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.91361</td>
</tr>
<tr>
<td>Mean</td>
<td>1.98758</td>
</tr>
</tbody>
</table>

| Population parameters: Alpha=1.95, Scale=1.0, Location=0 |

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.25923</td>
</tr>
<tr>
<td>Mean</td>
<td>1.99260</td>
</tr>
</tbody>
</table>

| Population parameters: Alpha=2.0, Scale=1.0, Location=0 |

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.01658</td>
</tr>
<tr>
<td>Mean</td>
<td>1.99871</td>
</tr>
</tbody>
</table>
Table 2.2: Properties of MCS estimators for samples with 1000 observations divided into 10 groups

Table shows sums of squares of errors (SSE) of the estimates of characteristic exponent (Alpha), dispersion (Scale) and location (Location) parameters of symmetric stable distribution for samples with 1000 observations divided into 10 equal range groups. These values are obtained by summing 100 squared differences between the population parameters and their minimum $\chi^2$ (MCS) and quantile (McCulloch) estimates. Below SSE, the mean values of the obtained estimates (Mean) are presented. Above each section the values of population parameters are listed.

<table>
<thead>
<tr>
<th>Sample size: 1000</th>
<th>No of groups: 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population parameters</td>
<td>Alpha=1.0 Scale=1.0 Location=0</td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>SSE</td>
<td>Mean</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>10.2014</td>
<td>60.0949</td>
</tr>
<tr>
<td>1.28744</td>
<td>1.68795</td>
</tr>
<tr>
<td>Population parameters</td>
<td>Alpha=1.5 Scale=1.0 Location=0</td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>SSE</td>
<td>Mean</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>0.66336</td>
<td>0.759438</td>
</tr>
<tr>
<td>1.54744</td>
<td>1.073272</td>
</tr>
<tr>
<td>Population parameters</td>
<td>Alpha=1.6 Scale=1.0 Location=0</td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>SSE</td>
<td>Mean</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>0.72392</td>
<td>0.391961</td>
</tr>
<tr>
<td>1.64064</td>
<td>1.046095</td>
</tr>
<tr>
<td>Population parameters</td>
<td>Alpha=1.7 Scale=1.0 Location=0</td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>SSE</td>
<td>Mean</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>0.71849</td>
<td>0.22645</td>
</tr>
<tr>
<td>1.73419</td>
<td>1.03014</td>
</tr>
<tr>
<td>Population parameters</td>
<td>Alpha=1.8 Scale=1.0 Location=0</td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>SSE</td>
<td>Mean</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>0.75209</td>
<td>0.27539</td>
</tr>
<tr>
<td>1.82554</td>
<td>1.03421</td>
</tr>
<tr>
<td>Population parameters</td>
<td>Alpha=1.9 Scale=1.0 Location=0</td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>SSE</td>
<td>Mean</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>0.62916</td>
<td>0.25188</td>
</tr>
<tr>
<td>1.91461</td>
<td>1.03306</td>
</tr>
<tr>
<td>Population parameters</td>
<td>Alpha=1.95 Scale=1.0 Location=0</td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>SSE</td>
<td>Mean</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>0.49313</td>
<td>0.26425</td>
</tr>
<tr>
<td>1.94546</td>
<td>1.03234</td>
</tr>
<tr>
<td>Population parameters</td>
<td>Alpha=2.0 Scale=1.0 Location=0</td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>SSE</td>
<td>Mean</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>0.55453</td>
<td>0.18259</td>
</tr>
<tr>
<td>1.96139</td>
<td>1.02349</td>
</tr>
</tbody>
</table>
Table 2.3: Properties of MCS estimators for samples with 1000 observations divided into 20 groups

Table shows sums of squares of errors (SSE) of the estimates of characteristic exponent (Alpha), dispersion (Scale) and location (Location) parameters of symmetric stable distribution for samples with 1000 observations divided into 20 equal range groups. These values are obtained by summing 100 squared differences between the population parameters and their minimum \( \chi^2 \) (MCS) and quantile (McCulloch) estimates. Below SSE, the mean values of the obtained estimates (Mean) are presented. Above each section the values of population parameters are listed.

<table>
<thead>
<tr>
<th>Sample size: 1000</th>
<th>No of groups: 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population parameters: Alpha=1.0 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td><strong>McCulloch estimators</strong></td>
<td><strong>MCS estimators</strong></td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>7.67735</td>
</tr>
<tr>
<td>Mean</td>
<td>1.24051</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.5 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td><strong>McCulloch estimators</strong></td>
<td><strong>MCS estimators</strong></td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.36269</td>
</tr>
<tr>
<td>Mean</td>
<td>1.52988</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.6 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td><strong>McCulloch estimators</strong></td>
<td><strong>MCS estimators</strong></td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.38719</td>
</tr>
<tr>
<td>Mean</td>
<td>1.61695</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.7 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td><strong>McCulloch estimators</strong></td>
<td><strong>MCS estimators</strong></td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.56825</td>
</tr>
<tr>
<td>Mean</td>
<td>1.70913</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.8 Scale=1.0 Location=0</td>
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<tr>
<td><strong>McCulloch estimators</strong></td>
<td><strong>MCS estimators</strong></td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
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<td>SSE</td>
<td>0.60675</td>
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<tr>
<td>Mean</td>
<td>1.80008</td>
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<tr>
<td>Population parameters: Alpha=1.9 Scale=1.0 Location=0</td>
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</tr>
<tr>
<td><strong>McCulloch estimators</strong></td>
<td><strong>MCS estimators</strong></td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.83949</td>
</tr>
<tr>
<td>Mean</td>
<td>1.90402</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.95 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td><strong>McCulloch estimators</strong></td>
<td><strong>MCS estimators</strong></td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.497855</td>
</tr>
<tr>
<td>Mean</td>
<td>1.930771</td>
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<tr>
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</tr>
<tr>
<td><strong>McCulloch estimators</strong></td>
<td><strong>MCS estimators</strong></td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.42443</td>
</tr>
<tr>
<td>Mean</td>
<td>1.96332</td>
</tr>
</tbody>
</table>
In all the cases, large errors of McCulloch estimates occur for $\alpha$'s around 1.0. It suggests that the McCulloch algorithm should not be applied for grouped data, when small $\alpha$'s are expected.

Generally, confirming the previous expectation, the relative advantage of MCS estimates is clearly shown. Based on these results the MCS estimation technique is strongly recommended. The MCS method gives significantly better results without unnecessary complication of the analysis.

II.4.4. MCS estimation for ungrouped data

The majority of empirical research is connected with analysis of ungrouped data. Hence, it is necessary to test the behaviour of the MCS estimates when applied to standard series. The estimation technique is very similar to that for grouped data, although some differences must be noted.

First, the observations are sorted and McCulloch estimates of stable distribution parameters are calculated. Then the sample data is divided into a specified number of cells, with approximately the same number of observations in each cell. As the number of cells is usually higher than for grouped samples, the theoretical frequencies in cells are calculated by simulating the symmetric stable distribution function only in 100 points in each cell\(^9\). Then, the computed theoretical and empirical frequencies are put into a specified $\chi^2$ criterion. Several criteria are employed and the most suitable is chosen. The restrictions on the parameters are similar to those for grouped data. The initial values are calculated using the McCulloch quantile method.

The parameters are calculated using both the McCulloch's quantile method and the MCS algorithm. After 100 repeats the sums of squares of errors are compared. The efficiency of MCS estimates strongly depends on the number of cells into which the data are divided.

\(^9\) Raising the number of points in each cell can increase the accuracy of the approximations, but 100 points seem enough for this analysis. Increased precision has very little impact on the results.
For the traditional MCS criterion and its log likelihood and Freeman-Tukey variations, problems concerning the convergence of the parameters and the precision of the obtained estimates are encountered. Consequently, the Neyman statistic is used. First, all of the estimations are performed using numbers of cell suggested by Man and Wald (M&W) and taking $c(\alpha) = 1.96$. For samples of size 100, 250 and 500 the MCS estimates are less accurate than the McCulloch quantile estimates. The results for samples of size 1000 are presented in Table 2.4.

For $\alpha$'s from the range [1.7; 1.95] and samples with 1000 observations, the MCS estimator is better than the McCulloch’s quantile one. However, the situation for smaller $\alpha$'s is still in favour of the McCulloch estimation method. Exemplary results for samples with 2000, 5000 and 10000 observations are presented in Table 2.5. For samples with 2000 observations and small $\alpha$'s the McCulloch’s estimator is still better, but for $\alpha$ equal to 1.8 the average square of error of MCS estimate of $\alpha$ is more than 50% less than the respective error of quantile estimates.

For samples with 5000 observations, the relative efficiency of the MCS estimators decreases. Analyses of series of 10000 observations provide results in favour of McCulloch quantile estimates. These facts raise serious doubts about the usefulness of MCS estimation for ungrouped data. Hence, an intensive search for the optimal number of cells is undertaken.
Table 2.4: Properties of MCS estimators for ungrouped samples with 1000 observations and Man and Wald number of cells

Table shows sums of squares of errors (SSE) of the estimates of characteristic exponent (Alpha), dispersion (Scale) and location (Location) parameters of symmetric stable distribution for samples with 1000 observations divided into equiprobable groups. These values are obtained by summing 100 squared differences between the population parameters and their minimum \( \chi^2 \) (MCS) and quantile (McCulloch) estimates. Below SSE, the mean values of the obtained estimates (Mean) are presented.

Above each section the values of population parameters are listed.

<table>
<thead>
<tr>
<th>Sample size: 1000</th>
<th>No of cells: M&amp;W</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population parameters: Alpha=1.0 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.166401</td>
</tr>
<tr>
<td>Mean</td>
<td>1.006929</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.5 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.372177</td>
</tr>
<tr>
<td>Mean</td>
<td>1.508222</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.6 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.406154</td>
</tr>
<tr>
<td>Mean</td>
<td>1.612523</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.7 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.806561</td>
</tr>
<tr>
<td>Mean</td>
<td>1.705622</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.8 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
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<td>Scale</td>
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<td>0.917776</td>
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<td>Mean</td>
<td>1.817808</td>
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<td>MCS estimators</td>
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<td>Scale</td>
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<td>Mean</td>
<td>1.90656</td>
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<tr>
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<td>MCS estimators</td>
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<td>Alpha</td>
<td>Scale</td>
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<td>0.782423</td>
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<tr>
<td>Mean</td>
<td>1.925041</td>
</tr>
<tr>
<td>Population parameters: Alpha=2.0 Scale=1.0 Location=0</td>
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<td>MCS estimators</td>
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<td>Alpha</td>
<td>Scale</td>
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<td>SSE</td>
<td>0.648273</td>
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<tr>
<td>Mean</td>
<td>1.952747</td>
</tr>
</tbody>
</table>
Table 2.5: Properties of MCS estimators for ungrouped samples with Man and Wald number of cells and various numbers of observations

Table shows sums of squares of errors (SSE) of the estimates of characteristic exponent (Alpha), dispersion (Scale) and location (Location) parameters of symmetric stable distribution for samples with 1000 observations divided into equiprobable groups. These values are obtained by summing 100 squared differences between the population parameters and their minimum $\chi^2$ (MCS) and quantile (McCulloch) estimates. Below SSE, the mean values of the obtained estimates (Mean) are presented.

Above each section the values of population parameters and sample size are listed.

<table>
<thead>
<tr>
<th>Sample size: 2000</th>
<th>No of cells: M&amp;W</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population parameters: Alpha=1.5 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>MCCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.231755</td>
</tr>
<tr>
<td>Mean</td>
<td>1.508222</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sample size: 2000</th>
<th>No of cells: M&amp;W</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population parameters: Alpha=1.8 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>MCCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.461816</td>
</tr>
<tr>
<td>Mean</td>
<td>1.81106</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sample size: 5000</th>
<th>No of cells: M&amp;W</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population parameters: Alpha=1.8 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>MCCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.247821</td>
</tr>
<tr>
<td>Mean</td>
<td>1.80635</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sample size: 10000</th>
<th>No of cells: M&amp;W</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population parameters: Alpha=1.8 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>MCCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.091148</td>
</tr>
<tr>
<td>Mean</td>
<td>1.80332</td>
</tr>
</tbody>
</table>

The Mann and Wald (1942) algorithm is abandoned and extensive simulations in order to find the optimal number of cells are undertaken. Monte Carlo analysis is performed for series with $\alpha$ equal to 1.8 and standard scale and location parameters ($c=1$, $\delta=0$). For each number of cells the MCS and McCulloch estimators are computed 100 times. Then, the sums of squares of errors (SSE) are calculated. The number of cells is considered as optimal for a given sample size when it has the smallest ratio of SSE for the MCS estimates to SSE for the McCulloch estimates. The analysis is performed for samples with 100, 250, 400, 600 and 1000 observations.
For samples of size 100, the optimal number of cells lies between 4 and 10 with the lowest SSE for 6 cells. However, the SSE is more than two times larger for the MCS estimates than for the McCulloch quantile estimates. For samples of size 250, the optimal number of cells lies between 20 and 29, with the best results for 26 cells. The MCS estimates are still in all cases outperformed by the McCulloch quantile estimates.

The situation changes for samples with 400 observations. For 40 and 41 cells, the MCS estimates are more precise than the McCulloch ones (the MCS estimates are slightly biased downwards). Samples with 400 observations seem to be the smallest for which the MCS estimation is desirable. The optimal number of cells is not sensitive to the values of scale and location parameters of the hypothesised stable distribution. However, it slightly changes with $\alpha$, as smaller numbers of cells are preferred for levels of $\alpha$ other than 1.8. The results of the MCS estimation with a modified number of cells for samples of 400, 600 and 1000 observations are presented in Tables 2.6-2.8.

For samples with 400 observations, the accuracy of the MCS estimation is comparable with that of the McCulloch’s method. The MCS estimator overcomes the quantile one for $\alpha$’s between 1.6 and 1.8. As such values of $\alpha$ appear very often on the financial markets, it may be desirable to use MCS method for analysing even short (about 400 observations) financial time series.

For samples with 600 observations, the MCS estimators are better than the McCulloch’s estimators for $\alpha$’s from 1.6 to 1.9. Their relative performance measured by the ratio of SSE of both estimators is significantly better than for samples with 400 observations.
Table 2.6: Properties of MCS estimators for ungrouped samples with 400 observations and 40 equiprobable cells

Table shows sums of squares of errors (SSE) of the estimates of characteristic exponent (Alpha), dispersion (Scale) and location (Location) parameters of symmetric stable distribution for samples with 400 observations divided into 40 equiprobable groups. These values are obtained by summing 100 squared differences between the population parameters and their minimum (MCS) and quantile (McCulloch) estimates. Below SSE, the mean values of the obtained estimates (Mean) are presented. Above each section the values of population parameters are listed.

<table>
<thead>
<tr>
<th>Sample size: 400</th>
<th>No of cells: 40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population parameters:</td>
<td>Alpha=1.0 Scale=1.0 Location=0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.483804</td>
</tr>
<tr>
<td>Mean</td>
<td>0.995429</td>
</tr>
</tbody>
</table>

| Population parameters | Alpha=1.5 Scale=1.0 Location=0 |

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>1.22231</td>
</tr>
<tr>
<td>Mean</td>
<td>1.51986</td>
</tr>
</tbody>
</table>

| Population parameters | Alpha=1.6 Scale=1.0 Location=0 |

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>1.58462</td>
</tr>
<tr>
<td>Mean</td>
<td>1.63012</td>
</tr>
</tbody>
</table>

| Population parameters | Alpha=1.7 Scale=1.0 Location=0 |

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>1.53701</td>
</tr>
<tr>
<td>Mean</td>
<td>1.70270</td>
</tr>
</tbody>
</table>

| Population parameters | Alpha=1.8 Scale=1.0 Location=0 |

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>1.62197</td>
</tr>
<tr>
<td>Mean</td>
<td>1.80847</td>
</tr>
</tbody>
</table>

| Population parameters | Alpha=1.9 Scale=1.0 Location=0 |

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>1.24197</td>
</tr>
<tr>
<td>Mean</td>
<td>1.89445</td>
</tr>
</tbody>
</table>

| Population parameters | Alpha=1.95 Scale=1.0 Location=0 |

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>1.11202</td>
</tr>
<tr>
<td>Mean</td>
<td>1.92287</td>
</tr>
</tbody>
</table>

| Population parameters | Alpha=2.0 Scale=1.0 Location=0 |

<table>
<thead>
<tr>
<th>McCulloch estimators</th>
<th>MCS estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>1.20509</td>
</tr>
<tr>
<td>Mean</td>
<td>1.93458</td>
</tr>
</tbody>
</table>
Table 2.7: Properties of MCS estimators for ungrouped samples with 600 observations and 48 equiprobable cells

Table shows sums of squares of errors (SSE) of the estimates of characteristic exponent (Alpha), dispersion (Scale) and location (Location) parameters of symmetric stable distribution for samples with 600 observations divided into 48 equiprobable groups. These values are obtained by summing 100 squared differences between the population parameters and their minimum $\chi^2$ (MCS) and quantile (McCulloch) estimates. Below SSE, the mean values of the obtained estimates (Mean) are presented. Above each section the values of population parameters are listed.

<table>
<thead>
<tr>
<th>Sample size: 600</th>
<th>No of cells: 48</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population parameters: Alpha=1.0 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>0.38511</td>
<td>0.46292</td>
</tr>
<tr>
<td>Mean</td>
<td>1.00801</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.5 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>0.66774</td>
<td>0.206414</td>
</tr>
<tr>
<td>Mean</td>
<td>1.49180</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.6 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>0.71589</td>
<td>0.32709</td>
</tr>
<tr>
<td>Mean</td>
<td>1.62009</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.7 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>1.33407</td>
<td>0.241699</td>
</tr>
<tr>
<td>Mean</td>
<td>1.72644</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.8 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>1.53421</td>
<td>0.316384</td>
</tr>
<tr>
<td>Mean</td>
<td>1.80668</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.9 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>1.55020</td>
<td>0.290662</td>
</tr>
<tr>
<td>Mean</td>
<td>1.87141</td>
</tr>
<tr>
<td>Population parameters: Alpha=1.95 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>0.84545</td>
<td>0.266667</td>
</tr>
<tr>
<td>Mean</td>
<td>1.91846</td>
</tr>
<tr>
<td>Population parameters: Alpha=2.0 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>0.64391</td>
<td>0.223198</td>
</tr>
<tr>
<td>Mean</td>
<td>1.95637</td>
</tr>
</tbody>
</table>
Table 2.8: Properties of MCS estimators for ungrouped samples with 1000 observations and 100 equiprobable cells

Table shows sums of squares of errors (SSE) of the estimates of characteristic exponent (Alpha), dispersion (Scale) and location (Location) parameters of symmetric stable distribution for samples with 1000 observations divided into 100 equiprobable groups. These values are obtained by summing 100 squared differences between the population parameters and their minimum \( \chi^2 \) (MCS) and quantile (McCulloch) estimates. Below SSE, the mean values of the obtained estimates (Mean) are presented. Above each section the values of population parameters are listed.

<table>
<thead>
<tr>
<th>Sample size: 1000</th>
<th>No of cells: 100</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Population parameters</strong>: Alpha=1.0 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.16077</td>
</tr>
<tr>
<td>Mean</td>
<td>1.01010</td>
</tr>
<tr>
<td><strong>Population parameters</strong>: Alpha=1.5 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.40568</td>
</tr>
<tr>
<td>Mean</td>
<td>1.51396</td>
</tr>
<tr>
<td><strong>Population parameters</strong>: Alpha=1.6 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.47623</td>
</tr>
<tr>
<td>Mean</td>
<td>1.60216</td>
</tr>
<tr>
<td><strong>Population parameters</strong>: Alpha=1.7 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.72069</td>
</tr>
<tr>
<td>Mean</td>
<td>1.70670</td>
</tr>
<tr>
<td><strong>Population parameters</strong>: Alpha=1.8 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.82892</td>
</tr>
<tr>
<td>Mean</td>
<td>1.80673</td>
</tr>
<tr>
<td><strong>Population parameters</strong>: Alpha=1.9 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.69673</td>
</tr>
<tr>
<td>Mean</td>
<td>1.89045</td>
</tr>
<tr>
<td><strong>Population parameters</strong>: Alpha=1.95 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.60193</td>
</tr>
<tr>
<td>Mean</td>
<td>1.93046</td>
</tr>
<tr>
<td><strong>Population parameters</strong>: Alpha=2.0 Scale=1.0 Location=0</td>
<td></td>
</tr>
<tr>
<td>McCulloch estimators</td>
<td>MCS estimators</td>
</tr>
<tr>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>SSE</td>
<td>0.864344</td>
</tr>
<tr>
<td>Mean</td>
<td>1.946011</td>
</tr>
</tbody>
</table>
For samples with 1000 observations, the MCS estimators are again better in the range of α's from 1.6 to 1.9. The SSE for α's equal to 1.5 and 1.95 are similar for both methods. Generally, it can be noticed that the larger the sample, the better the relative performance of the MCS estimator. The MCS estimators out-perform the McCulloch's ones especially for α's in range from 1.6. to 1.9. Such values of this parameter usually characterise the returns of single equities and indices on developed financial markets (see Chapter V). It entitles one to formulate the suggestion to employ the MCS method for returns' analysis in samples larger than 400. However, in all cases the optimal number of cells should be used.

The MCS method is relatively uncomplicated and requires only standard GAUSS software. The program uses the OPTMUM library. Estimation time using a reasonable Pentium II processor varies from approximately 1 minute for small samples to around 20 minutes for samples with 10 000 observations. It is slower than the McCulloch quantile algorithm but faster and easier than many of other methods\textsuperscript{10}. Thanks to these features the MCS method can be successfully used in the empirical analyses.

An additional variance analysis of the MCS estimators could help to define its properties. However, to perform this evaluation, longer series should be investigated and compared with asymptotic variance for maximum likelihood method. Some approximations of these values are given by DuMouchel (1975), but their precision is questionable (see Nolan, 1999).

Concluding, it should be remembered that the MCS estimators perform significantly better than the McCulloch's ones. The difference was especially clear for α's around 1 and between 1.7 and 1.9. Consequently, the MCS method is strongly suggested for analysing grouped data. In case of ungrouped data, the MCS method is suggested for all samples larger than 400 observations and with expected values of α ranging from 1.6 to 1.95.

\textsuperscript{10} Apart from e.g. Nolan (1999) maximum likelihood, which is almost instantaneous.
II.5. An empirical application of MCS: Analysis of the Warsaw and the Budapest stock markets

II.5.1. Markets and data

As it is stated in Section II.3, the traditional estimation techniques of stable distribution parameters may easily fail when certain irregularities occur in the analysed series. This may result in inaccurate estimates or even collapse of the computation algorithms. Some of the possible problems are connected with truncations of the distribution, clusters of returns around zero and discontinuities of the distribution of returns caused by rounding-off procedures. Classical examples of such anomalies are the emerging markets of Poland (WSE) and Hungary (BSE). Therefore, this part is devoted to estimation of the stable distribution parameters for returns to equities traded in Budapest and Warsaw.

These two markets are particularly interesting cases. WSE is the biggest stock exchange in Central Europe, whereas BSE operated with very limited administrative control. Poland and Hungary are two of the most advanced countries in terms of market oriented reforms in Central Europe. Recently, both of them joined NATO and they are due to be incorporated into the European Union. The markets are of relatively small size and attract strong interest of investors. Large demand for stocks results in substantial price changes, often reaching 10% per day. Such behaviour seems to be typical for the recently opened emerging markets (Charemza and Majerowska, 1999 and Zalewska-Mitura and Hall, 1999).

The remainder of this section gives the basic characteristics of Polish and Hungarian stock markets and describes the reasons of the existing disturbances. The second part deals with the MCS estimation of the stable parameters. The MCS based results are additionally compared to the McCulloch estimates.

The first session of the Warsaw Stock Exchange took place on 16 April 1991. During its early years, the market was characterised by big shifts in the dynamics,
incomparable to the changes on the mature markets of the EU and the USA. The highly volatile prices resulted in unusual distributions of returns. Some additional distortions were caused by various administrative regulations. The majority of them are connected with the 10% limits imposed on the price changes and with the procedure of rounding-off the quotations. The returns were especially volatile in the first years of existence of the WSE, resulting in big number of hits in the 10% boundary. Recent years were connected with much smaller shifts in prices. The decreasing variability was mainly due to the raising liquidity and a greater number of transactions.

**Figure 2.1: WIG index from 1994 to 1998**

Figure shows values of Warsaw Stock Exchange WIG price index from 3 January 1994 to 17 April 1998.

Figure 2.1 shows the behaviour of WIG index in the analysed period 1994-1998. Throughout 1994, the stock prices were quoted three times a week and from 1995, five daily quotations per week were introduced. However, it is assumed that, as the frequency of sessions increased, the volume of trade on individual assets remained on approximately the same per session level (a similar assumption applies to the Budapest Stock Exchange). Between 1994 and 1998, WIG changed in the range from about 6000 to more than 20000, presenting very high volatility. The sharp decline in early 1994 reduced the wealth of investors by about 70%. Throughout 1996, the prices regained vast part of the previous loss, however the observed growth is far from being stable (Charemza and Majerowska, 1999).
The Budapest Stock Exchange was opened as a consequence of market-oriented reforms on 21 June 1990. At the beginning of April 1998 securities of 54 companies were traded. Since 1994 the stock exchange has noted growth (see Figure 2.2). This became very rapid in 1996. During this year market index BUX gained 133.5% in terms of USD. At the end of 1997, as an effect of the Asian Crisis, the prices of shares decreased. By the end of 1997 the market index recovered and reached the pre-crisis level. The year 1997 was the most successful for BSE. The total number of transactions increased 100 times comparing to its level in 1990 and 3 times comparing to the number of transaction in 1996 (Zalewska-Mitura and Hall, 1999).

The behaviour of returns on exemplary companies Danubius Hotels from the BSE and Zywiec from the WSE is shown in Figures 2.3 – 2.5. Figure 2.3 shows that the price limits effect was very strong and cannot be ignored in analysis of returns for WSE, while Figure 2.4 presents the decreasing variability of the BSE market. Departures from normality in the distributions of returns on both markets can be seen in Figure 2.5. Both distributions have heavy tails and big concentration of returns around zero.
Figure 2.3: Returns on Zywiec
Figure shows daily returns to Zywiec company quoted on the Warsaw Stock Exchange. Series starts on 3 January 1994 and finishes on 17 April 1998.

Figure 2.4: Returns on Danubius Hotels
Figure shows daily returns to Danubius Hotels company quoted on the Budapest Stock Exchange. Series starts on 4 January 1994 and finishes on 17 April 1998.
Figure 2.5: Distribution of returns on Danubius Hotels and Zywiec

Figure shows distributions of returns to Danubius Hotels (DAN) and Zywiec (ZYW) companies. The respective distributions are based on daily returns from 3 January 1994 to 17 April 1998.

Data analysed in this section are obtained from Charemza and Majerowska (1999). The sample contains twenty companies from the Warsaw Stock Exchange, nine companies from the Budapest Stock Exchange and the main indices from both markets. The WSE sample contains 1012 data points from 3 January 1994 to 17 April 1998. Additionally, the artificial Laspeyres type market index WIG21 is created, assuming that only the twenty-one largest companies exist on WSE. The BSE sample contains 9 companies that appeared on the market up to the end of 1993. The sample covers the same period as for the WSE, from 4 January 1994 to 17 April 1998, containing 1057 data points.

II.5.2. Estimation of symmetric stable distribution parameters

This part estimates the parameters of the stable distribution of returns on the Warsaw Stock Exchange and Budapest Stock Exchange. Both the McCulloch quantile method and the minimum $\chi^2$ (MCS) technique are applied. For the MCS estimation, the following 11 cells are used: (-$\infty$, -0.09], (-0.09, -0.07], (-0.07, -0.05], ..., (0.07,0.09], (0.09, $\infty$). Such choice provides convenient grouping of highly clustered returns.
Due to the price limits, in case of WSE the double censored symmetric stable distribution is fitted. The zero probabilities of returns higher than 10% and lower than -10% are imposed and the tails are concentrated at ±10% points. These restrictions determine the construction of marginal cells in the MCS procedure, adjusting this method to estimate the censored distributions parameters. Such alteration is impossible in the case of McCulloch methodology. However, the quantile based approach should not be affected by double censoring unless the concentration of probability in any of the censoring points is higher than 5%. In this situation, too small absolute values of 0.95 and too large absolute values of 0.05 quantiles may bias the estimates of alphas upwards.

The values of the estimated parameters of the double censored symmetric stable distributions for analysed returns from WSE are shown in Table 2.9. For all stocks, the estimated α's are far below two, which suggests that the returns are not normally distributed. T-tests of the null hypothesis that α=2, performed on the basis of McCulloch quantile estimators and their standard deviations, reject normality at 5% level of significance for all companies except Kable. However, Kable shows a much lower estimate of α when analysed by the MCS procedure, which suggest the inaccuracy of the quantile evaluation.12

The results in Table 2.9 allow sorting stocks according to their individual degrees of non-normality. Mostostal Warszawa, Mostostal Export and Exbud, having relatively high estimates of α’s in both methods, can be regarded as stocks with much smaller degrees of non-normality than e.g. Sokolow or Prochnik.

---

12 MCS method does not provide standard errors of estimates and hence significance of MCS estimates can not be verified in t-test.
Table 2.9: The double censored symmetric stable distribution parameters for returns on WSE

Table presents estimates of characteristic exponents (Alpha), scale parameters (Scale) and location parameters (Location) of double censored symmetric stable distribution fitted to the companies from Warsaw Stock Exchange. The companies' codes are listed in first column. Than the estimates obtained through minimum $\chi^2$ (MCS) and quantile methods (McCulloch) are shown.

<table>
<thead>
<tr>
<th>Company</th>
<th>MCS estimates</th>
<th>McCulloch estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>Efekt</td>
<td>1.01360</td>
<td>0.02680</td>
</tr>
<tr>
<td>Elektrim</td>
<td>0.99674</td>
<td>0.02996</td>
</tr>
<tr>
<td>Exbud</td>
<td>1.04256</td>
<td>0.02773</td>
</tr>
<tr>
<td>Irena</td>
<td>0.98927</td>
<td>0.03226</td>
</tr>
<tr>
<td>Kable</td>
<td>0.98845</td>
<td>0.03526</td>
</tr>
<tr>
<td>Krosno</td>
<td>0.95927</td>
<td>0.03175</td>
</tr>
<tr>
<td>Mostostal Export</td>
<td>1.00073</td>
<td>0.03018</td>
</tr>
<tr>
<td>Mostostal Warszawa</td>
<td>1.02539</td>
<td>0.02756</td>
</tr>
<tr>
<td>Okocim</td>
<td>0.99322</td>
<td>0.02820</td>
</tr>
<tr>
<td>Polifarb</td>
<td>0.96030</td>
<td>0.02527</td>
</tr>
<tr>
<td>Prochnik</td>
<td>0.87813</td>
<td>0.02495</td>
</tr>
<tr>
<td>Sokolow</td>
<td>0.88381</td>
<td>0.02714</td>
</tr>
<tr>
<td>Swarzedz</td>
<td>0.87644</td>
<td>0.03244</td>
</tr>
<tr>
<td>Tonsil</td>
<td>0.89627</td>
<td>0.03525</td>
</tr>
<tr>
<td>Universal</td>
<td>0.94224</td>
<td>0.03501</td>
</tr>
<tr>
<td>Vistula</td>
<td>0.96944</td>
<td>0.03089</td>
</tr>
<tr>
<td>WBK</td>
<td>1.05523</td>
<td>0.02574</td>
</tr>
<tr>
<td>Wedel</td>
<td>1.02810</td>
<td>0.02597</td>
</tr>
<tr>
<td>Wolczanka</td>
<td>0.96678</td>
<td>0.02171</td>
</tr>
<tr>
<td>Zywiec</td>
<td>1.09244</td>
<td>0.02154</td>
</tr>
<tr>
<td>WIG</td>
<td>1.18280</td>
<td>0.02058</td>
</tr>
<tr>
<td>WIG21</td>
<td>1.08533</td>
<td>0.02371</td>
</tr>
</tbody>
</table>

Ranking constructed using the McCulloch estimates of $\alpha$'s significantly differs from ranking constructed using the MCS estimates. This is connected with the upward bias of the McCulloch estimates. In all cases, the estimates of $\alpha$ obtained using the McCulloch method are higher than these obtained by the MCS method.
Figure 2.6 shows the relationship between the differences in the MCS and McCulloch estimates of $\alpha$'s and the number of returns, which hit the price limits. In order to control the effect of price rounding procedure all returns higher the 9.5% or lower than -9.5% have been treated as hits in the limits values. The presented relationship is almost linear. It confirms earlier expectations that high concentrations of returns at the censorship points bias the McCulloch estimates of characteristic exponent upwards. Indeed, the bigger the number of returns at ±10%, the higher the difference between two estimates of characteristic exponent $\alpha$. Therefore, we consider the MCS estimates much more reliable than the McCulloch ones.

The results of the estimation of symmetric stable distribution parameters for returns to the selected companies from the Budapest Stock Exchange are shown in Table 2.10. It includes nine companies together with the market index BUX. For 13 other companies traded on BSE during the analysed period the market noted more than 50% zero daily returns.
Table 2.10: The symmetric stable distribution parameters for the returns on BSE

Table presents estimates of characteristic exponents (Alpha), scale parameters (Scale) and location parameters (Location) of symmetric stable distribution fitted to the companies from Budapest Stock Exchange. The companies' codes are listed in first column. Than the estimates obtained through minimum \( \chi^2 \) (MCS) and quantile methods (McCulloch) are shown.

<table>
<thead>
<tr>
<th>Company</th>
<th>MCS estimates</th>
<th>McCulloch estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Alpha</td>
<td>Scale</td>
</tr>
<tr>
<td>Danubius Hotels</td>
<td>1.34376</td>
<td>0.01371</td>
</tr>
<tr>
<td>Domus</td>
<td>0.90451</td>
<td>0.01048</td>
</tr>
<tr>
<td>Fotex</td>
<td>1.20000</td>
<td>0.01120</td>
</tr>
<tr>
<td>Globus</td>
<td>1.01924</td>
<td>0.01162</td>
</tr>
<tr>
<td>Pannonflax</td>
<td>0.50170</td>
<td>0.00240</td>
</tr>
<tr>
<td>Pick Szeged</td>
<td>1.30711</td>
<td>0.00956</td>
</tr>
<tr>
<td>Primasz Hungary</td>
<td>1.08627</td>
<td>0.00924</td>
</tr>
<tr>
<td>Zalakeramia</td>
<td>1.09122</td>
<td>0.00666</td>
</tr>
<tr>
<td>Zwack Unicum</td>
<td>1.05391</td>
<td>0.00859</td>
</tr>
<tr>
<td>BUX</td>
<td>1.37531</td>
<td>0.00694</td>
</tr>
</tbody>
</table>

The results for BSE are much more diversified than those obtained for WSE. The estimates of alpha parameters vary form 0.50170 to 1.35626, being far below two. Indeed, the t-tests of the null hypothesis that \( \alpha = 2 \), performed on the basis of McCulloch quantile estimators rejects normality for all companies at 1% level of significance. This confirms the hypothesis about the non-normality of the BSE. The levels of \( \alpha \)'s suggest that returns to Danubius Hotel and Pick Szeged companies are closer to normal distribution than the rest of the market. The estimates of the symmetric stable parameters are similar for both methods.

Apart of the McCulloch (1986) and the MCS, the Nolan (1999) maximum likelihood method is applied to WSE and BSE series. For WSE the maximum likelihood estimates \( \alpha \)'s are slightly lower than the McCulloch's ones. However, for BSE the Nolan algorithm does not provide reasonable results and hence, as
appropriate comparison between WSE and BSE was not possible, maximum likelihood results are not reported.

II.5.3. Goodness of stable distributions

In this section, the fit of the symmetric stable distributions to the returns to assets on the Budapest Stock Exchange and on the Warsaw Stock Exchange is verified.

Figures 2.7 presents the best fitted McCulloch and MCS approximations of the double censored stable distributions to returns to companies from WSE. Figure 2.8 shows respective distributions for the worst fitted companies. In both cases, the distributions with estimated parameters fail to capture the high concentrations of returns around zero. For Zywiec, the approximations seem to be very accurate in the intervals (-10\%, -2\%) and (2\%, 10\%), whereas a much worse situation can be observed in the case of Krosno. The MCS fitted distribution manages to approximate the shape of distribution for high absolute values of returns, but the McCulloch fitted distribution has clearly too small a peak and too thin tails. This is connected with a too high value of characteristic exponent $\alpha$. 
**Figure 2.7: The double censored symmetric stable distributions, Zywiec**

Figure shows distribution of returns to Zywiec company along with its double censored symmetric stable approximations estimated using quantile (McCulloch) and minimum $\chi^2$ (MCS) estimates. This distribution is the best-fitted one from WSE.

![Figure 2.7: The double censored symmetric stable distributions, Zywiec](image)

**Figure 2.8: The double censored symmetric stable distributions, Krosno**

Figure shows distribution of returns to Krosno company along with its double censored symmetric stable approximations estimated using quantile (McCulloch) and minimum $\chi^2$ (MCS) estimates. This distribution is the worst-fitted one from WSE.

![Figure 2.8: The double censored symmetric stable distributions, Krosno](image)

Figures 2.9 and 2.10 present the best and the worst fitted McCulloch and MCS approximations of the stable distributions for companies from BSE.
The distribution of returns to Fotex is approximated almost perfectly, although it has slightly too low concentrations of zero returns in the fitted distributions. A much worse situation can be found in the case of Globus. The empirical distribution is very irregular with almost 40% of returns being equal to zero. Both approximations fail to capture such behaviour of returns.
Tables 2.11 and 2.12 show the values of $\chi^2$ statistics calculated for WSE and BSE using 11 cells. For 1%, 5% and 10% levels of significance, the critical values for the chi-squared statistics with 10 degrees of freedom are 23.209, 18.307 and 15.987. The double censored distribution of returns to assets on WSE cannot be rejected in 7 cases at 10% level of significance, in 11 cases at 5% level of significance and in 15 cases at 1% level of significance. Consequently, at 5% level of significance 50% of the analysed series from WSE have double censored symmetric stable distribution of returns with parameters estimated by the MCS method. In all cases the MCS estimated distribution fits the observed distribution of returns much better than the distribution with quantile based estimates of parameters.

Table 2.11: Calculated $\chi^2$ statistics for WSE

Table shows computed values of $\chi^2$ test of fit statistics. The statistics are calculated to verify the null hypothesis about double censored symmetric stable distributions of returns to companies on Warsaw Stock Exchange. Table lists code of the companies along with $\chi^2$ statistics for distributions estimated by minimum $\chi^2$ (MCS) and quantile (McCulloch) methods.

<table>
<thead>
<tr>
<th>Company</th>
<th>MCS</th>
<th>McCulloch</th>
<th>Company</th>
<th>MCS</th>
<th>McCulloch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Efekt</td>
<td>12.222</td>
<td>75.4123</td>
<td>Sokolow</td>
<td>15.735</td>
<td>139.582</td>
</tr>
<tr>
<td>Elektrim</td>
<td>17.024</td>
<td>120.029</td>
<td>Swarzedz</td>
<td>9.401</td>
<td>211.302</td>
</tr>
<tr>
<td>Exbud</td>
<td>21.959</td>
<td>83.536</td>
<td>Tonsil</td>
<td>16.281</td>
<td>237.979</td>
</tr>
<tr>
<td>Irena</td>
<td>22.011</td>
<td>157.162</td>
<td>Universal</td>
<td>14.909</td>
<td>201.984</td>
</tr>
<tr>
<td>Kable</td>
<td>69.073</td>
<td>166.155</td>
<td>Vistula</td>
<td>19.823</td>
<td>144.203</td>
</tr>
<tr>
<td>Krosno</td>
<td>16.728</td>
<td>156.110</td>
<td>WBK</td>
<td>23.423</td>
<td>60.464</td>
</tr>
<tr>
<td>Mostostal Export</td>
<td>24.603</td>
<td>126.690</td>
<td>Wedel</td>
<td>30.605</td>
<td>75.920</td>
</tr>
<tr>
<td>Mostostal Warszawa</td>
<td>19.873</td>
<td>87.776</td>
<td>Wolczanka</td>
<td>18.091</td>
<td>55.878</td>
</tr>
<tr>
<td>Okocim</td>
<td>31.258</td>
<td>118.341</td>
<td>Zywiec</td>
<td>9.414</td>
<td>17.105</td>
</tr>
<tr>
<td>Polifarb</td>
<td>27.905</td>
<td>92.497</td>
<td>WIG</td>
<td>6.560</td>
<td>9.361</td>
</tr>
<tr>
<td>Prochnik</td>
<td>24.087</td>
<td>128.068</td>
<td>WIG21</td>
<td>6.433</td>
<td>23.115</td>
</tr>
</tbody>
</table>
Table 2.12: Calculated $\chi^2$ statistics for BSE

Table shows computed values of $\chi^2$ test of fit statistics. The statistics are calculated to verify the null hypothesis about symmetric stable distributions of returns to companies on Budapest Stock Exchange. Table lists code of the companies along with $\chi^2$ statistics for distributions estimated by minimum $\chi^2$ (MCS) and quantile (McCulloch) methods.

<table>
<thead>
<tr>
<th>Company</th>
<th>MCS</th>
<th>McCulloch</th>
<th>Company</th>
<th>MCS</th>
<th>McCulloch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Danubius Hotels</td>
<td>41.229</td>
<td>43.004</td>
<td>Pick Szeged</td>
<td>23.485</td>
<td>28.138</td>
</tr>
<tr>
<td>Domus</td>
<td>49.982</td>
<td>62.344</td>
<td>Primasz Hungaria</td>
<td>53.645</td>
<td>73.674</td>
</tr>
<tr>
<td>Fotex</td>
<td>17.664</td>
<td>21.389</td>
<td>Zalakeramia</td>
<td>21.797</td>
<td>78.304</td>
</tr>
<tr>
<td>Globus</td>
<td>53.802</td>
<td>62.278</td>
<td>Zwack Unicum</td>
<td>29.199</td>
<td>97.571</td>
</tr>
<tr>
<td>Pannonflax</td>
<td>97.485</td>
<td>257.400</td>
<td>BUX</td>
<td>11.223</td>
<td>29.890</td>
</tr>
</tbody>
</table>

In case of BSE, the null hypothesis about the symmetric stable distribution of returns cannot be rejected at 10% level of significance for one series. The stable returns cannot be rejected at the 5% level for 2 series and at the 1% level for 3.

It can be seen that the symmetric stable distribution approximate the returns from WSE much better than the returns from BSE. Generally, $\chi^2$ statistics for the symmetric stable distributions are much lower than those for the normal distributions (see Table 1).

The analysis shows that the MCS technique is appropriate for the double censored returns on the Polish stock market, whereas both methods give similar results for the Hungarian stock exchange. Consequently, the MCS method is recommended for analysis of censored data. The possibility of convenient grouping allows one to avoid the inaccurate estimations of the extreme quantiles and the nonlinearities connected with truncations in the maximum likelihood methods.
II.6. Further developments in stable variables

II.6.1. Multivariate stable variables

In this section, three generalisations of the stable distribution theory are discussed. First, the multivariate stable variables are presented. Then the max-stable, min-stable, multiplicative-stable and geometric-stable variables are described. Finally, some information about asymptotic stable variables is provided.

The most obvious generalisation of the one-dimensional stable variables is the multivariate stable distribution. The appropriate definition can be constructed by extending to \( R^d \) the description from the univariate case. This section provides definition, major properties and the characteristic function of the multidimensional stable variables (for more extensive information see e.g. Samorodnitsky and Taqqu, 1994: Chapter II).

A random vector \( X = (X_1, X_2, \ldots, X_d) \) is a stable random vector in \( R^d \) if for positive numbers \( A \) and \( B \) there is a positive number \( C \) and a vector \( D \in R^d \) such that

\[
AX^{(1)} + BX^{(2)} = CX + D
\]  

(2.4)

where \( X^{(1)} \) and \( X^{(2)} \) are independent copies of \( X \) (Samorodnitsky and Taqqu, 1994: p.57).

Similarly as in the univariate case, the vector is called strictly stable if (2.4) holds with \( D = 0 \). The vector is called symmetric stable about 0 if it is stable and satisfies the following relation:

\[
P\{X \in W\} = P\{-X \in W\}
\]

for any Borel set \( W \) of \( R^d \). A symmetric stable vector is strictly stable (Samorodnitsky and Taqqu, 1994: p.57).

If \( x \in R^d \) is a stable vector, then there exists a constant \( \alpha \in (0,2] \) such that in (2.4) \( C = (A^\alpha + B^\alpha)^{1/\alpha} \). Moreover, any linear combination of the components of \( X \) is a stable variable. The same holds for strictly stable variables and stable variables, which are symmetric about 0 (Samorodnitsky and Taqqu, 1994: p.59).

The second popular definition of the multivariate stable variable reflects the principle of invariance under addition. A random vector \( X = (X_1, X_2, \ldots, X_d) \) is a
stable random vector in $\mathbb{R}^d$ if for any $n \geq 2$ there is an $\alpha \in (0,2]$ and a vector $D_n$ such that.

$$X^{(1)} + X^{(2)} + \ldots + X^{(n)} = n^{1/\alpha} + D_n,$$

where $X^{(1)}, \ldots, X^{(n)}$ are independent copies of $X$. The index $\alpha$ is called the index of stability or the characteristic exponent of vector $X$ (Samorodnitsky and Taqqu, 1994: p.58).

It should be noted that if all linear combinations of the components of a random variable $Y \in \mathbb{R}^d$ are stable, then variable $Y$ does not need to be stable itself. Such a counterexample for $0 < \alpha < 1$ is given e.g. by Marcus (1983).

Generally it is true for any random variable $X \in \mathbb{R}^d$ that:

a) If all linear combinations of the components of $X$ have strictly stable distributions, then $X$ is a strictly stable random vector in $\mathbb{R}^d$.

b) If all linear combinations of the components of $X$ have stable distributions, which are symmetric about 0, then $X$ is a stable random vector in $\mathbb{R}^d$ and $X$ is symmetric about 0.

c) If all linear combinations of the components of $X$ have stable distributions with characteristic exponent greater than or equal to one, then $X$ is a stable random vector in $\mathbb{R}^d$ (Samorodnitsky and Taqqu, 1994: p.59).

The characteristic function of an $\alpha$-stable random vector can be introduced in the following way. Let $X = (X_1, X_2, \ldots, X_d)$ be an $\alpha$-stable random vector in $\mathbb{R}^d$ and let

$$\Phi_\alpha(\theta) = \Phi_\alpha(\theta_1, \theta_2, \ldots, \theta_d) = E \exp\left\{i(\theta, X)\right\} = E \exp\left\{i \sum_{k=1}^{d} \theta_k X_k\right\},$$

denotes its characteristic function. Let $\alpha \in (0,2)$. Then $X = (X_1, X_2, \ldots, X_d)$ is an alpha-stable random vector in $\mathbb{R}^d$ if and only if there exists a finite measure $I$ on the unit sphere $S_d$ of $\mathbb{R}^d$ and a vector $\mu^0$ in $\mathbb{R}^d$ such that:

1) If $\alpha \neq 1$

$$\Phi_\alpha(\theta) = \exp\left\{-\int_{S_d} ||(\theta, s)||^{\alpha} \left(1 - i \text{sgn}((\theta, s)) \tan \frac{\pi \alpha}{2} \right) |(ds) + i(\theta, \mu^0)\right\}.$$
2) If \( \alpha = 1 \)

\[
\Phi_\alpha(\theta) = \exp\left\{-\int_{\mathbb{S}}^{\mathbb{T}} (\theta, s) \left( 1 - i \frac{2}{\pi} \text{sgn}((\theta, s)) \ln((\theta, s)) \right) \Gamma(ds) + i(\theta, \mu^0) \right\}.
\]

The pair \((\Gamma, \mu^0)\) is unique (Samorodnitsky and Taqqu, 1994: p.65).

Multivariate stable distributions can be applied to model \(n\)-dimensional systems of time series exposed to stable random shocks. Such an approach seems to be feasible for relatively small values of \(n\). Large values of \(n\) cause technical problems in estimation, which are mainly due to lack of closed functional representation of the multivariate stable distribution function. Commensurate with a large number of estimated parameters, the whole procedure becomes infeasible from the computational point of view. Therefore, practitioners usually analyse the multivariate normal systems (see e.g. Levin, 1999).

The multivariate stable distributions are, however, an important theoretical generalisation and their description should not be omitted in any serious study about the distributions of market returns. It should be expected that the multivariate stable distribution finds application to real data in the near future. Generalisation of Nolan (1999) maximum likelihood for larger values of \(n\) should open one of the possible paths for such development.

II.6.2. Stability with respect to different probability schemes

The original stability concept arises from the non-random summation scheme. This idea can be extended by considering a variety of different probabilistic schemes. Such alternatives, summarised by Mittnik and Rachev (1993), lead to a wide range of distributions that are stable with respect to their underlying schemes. Generally, it is said that mutually independent random variables \(X, X_1, X_2, \ldots\) with a common distribution \(R\) are stable with respect to probability scheme \(\circ\) if

\[
X \stackrel{\circ}{=} a_n (X_1 \circ X_2 \circ \ldots \circ X_n) + b_n,
\]

where \(a_n > 0, b_n \in \mathbb{R}\) and \(n\) is an integer.
In Mittnik and Rachev (1993) overview, ° stands for summation, minimisation, maximisation or multiplication. The standard non-random summation scheme, where ° stands for + and \( n \) is a deterministic integer, produces the stable Pareto distribution. The maximum and minimum schemes yield the extreme value distributions, of which the Weibull distribution is the basic example. The multiplication scheme leads to the multiplication distributions, including the log-normal ones. The intuitive justification to apply such nontrivial schemes is based on the lack of clear evidence on how the new information aggregates. Hence, any of the above schemes may be more suitable to model stock prices than the summation.

In the remainder of this section, following Mittnik and Rachev (1993), the min-stable, max-stable and multiplication-stable distribution are discussed. The concept of geometric summation is introduced on the example of geometric summation stable distributions.

Let \( Y, Y_1, Y_2, \ldots \) be the i.i.d. random variables with common distribution function \( H \) and let:

\[
M_n = \max(Y_1, Y_2, \ldots Y_n), \quad m_n = \min(Y_1, Y_2, \ldots Y_n).
\]

A random variable \( X \) is called max-stable with distribution \( M \) if

\[
a_n M_n + b_n \Rightarrow^d X
\]

where \( a_n > 0 \) and \( b_n \in \mathbb{R} \) are suitable normalising constants. The set of all distribution functions \( H \) for which the above holds is called the max-domain of attraction of the max-stable distribution \( M \) (Mittnik and Rachev, 1993).

There exists a location parameter \( b \in \mathbb{R} \) and a scale parameter \( \alpha > 0 \) such that the max-stable distribution admits one of the following three parametric forms:

1) max-stable distribution of Type I

\[
M(x) = \exp\{-e^{-\alpha x + b}\}, \quad x \in \mathbb{R},
\]

2) max-stable distribution of Type II

\[
M(x) = \begin{cases} 
0, & \text{if } x \leq b/\alpha \\
\exp\{-((x-b)/\alpha)\}, & \text{if } x > b/\alpha \text{ for some } \alpha > 0.
\end{cases}
\]
3) max-stable distribution of Type III

\[ M(x) = \begin{cases} 
\exp\{-(ax + b)^{\alpha}\}, & \text{if } x \leq b/a, \text{ for some } \alpha > 0 \\
1, & \text{if } x > b/a.
\end{cases} \]

(Mittnik and Rachev, 1993).

If \( X, X_1, X_2, \ldots \) are i.i.d. random variables with common max-stable distribution \( M \), then

1) \( \max(X_1, X_2, \ldots, X_n) - (1/a) \ln n = X, \) if \( M \) is of Type I,

2) \( n^{-1/a} \max(X_1, X_2, \ldots, X_n) + (b/a)(1 - n^{-1/a}) = X, \) if \( M \) is of Type II,

3) \( n^{1/a} \max(X_1, X_2, \ldots, X_n) + (b/a)(1 - n^{1/a}) = X, \) if \( M \) is of Type III,

(Mittnik and Rachev, 1993).

Because the \( m_n = \min(Y_1, Y_2, \ldots, Y_n) \) is equivalent to \( m_n = -\max(-Y_1, -Y_2, \ldots, -Y_n) \), all the above results can be easily extended to the \( m_n \) case. However, first the definition of the min-stable variable is presented.

A random variable \( X \) is called min-stable with distribution \( m \) if

\[ c_n m_n + d_n \Rightarrow X, \]

where \( c_n > 0 \) and \( d_n \in R \) are suitable normalising constants. The set of all distribution functions for which the above holds is called the min-domain of attraction of the min-stable distribution \( m \) (Mittnik and Rachev, 1993).

Similarly as before, there exists a location parameter \( b \in R \) and a scale parameter \( a > 0 \) such that the min-stable distribution admits one of the following three parametric forms:

1) min-stable distribution of Type I

\[ m(x) = 1 - \exp\{-e^{ax+b}\}, \quad x \in R, \]

2) min-stable distribution of Type II

\[ m(x) = \begin{cases} 
0, & \text{if } x \leq b/a \\
1 - \exp\{-(ax - b)^{\alpha}\}, & \text{if } x > b/a
\end{cases} \]

3) min-stable distribution of Type III

\[ m(x) = \begin{cases} 
1 - \exp\{-(ax + b)^{-\alpha}\}, & \text{if } x \leq b/a \\
1, & \text{if } x > b/a
\end{cases} \]

(Mittnik and Rachev, 1993).
The multiplicative stable variables are introduced in slightly different way. First, let random variable \( X = \exp(S) \), where \( S \) is a stable distributed random variable with p.d.f. \( g \), have a p.d.f. function of the following form:

\[
m_c(x) = \begin{cases} 
g(\ln x), & x > 0 \\ 
\frac{x}{0}, & x \leq 0 \end{cases}.
\]

Each alpha-stable distribution determines three (\( \alpha \), multiplication)-stable distributions with the respective densities:

1) log-stable density

\[
m_{g,1} = m_g(\left| x \right|) \mathbb{1}(x > 0), \quad x \in \mathbb{R},
\]

2) negative log-stable density

\[
m_{g,2} = m_g(\left| x \right|) \mathbb{1}(x < 0), \quad x \in \mathbb{R},
\]

2) symmetric log-stable density

\[
m_{g,3} = 0.5 m_g(\left| x \right|), \quad x \in \mathbb{R},
\]

(Mittnik and Rachev, 1993).

(\( \alpha \), multiplication)-stable distributions are weak limits of the distributions of the normalized product \( Y_n = X_1X_2...X_n \) of the i.i.d. random variables. The stability property can be derived separately for each functional representation of the distribution function. For example for \( m_{g,1} \) it states that if \( X, X_1, X_2,... \) are i.i.d. non-negative random variables with common (\( \alpha \), multiplication)-stable p.d.f. \( m_{g,1} \), then there exist constants \( A_n > 0 \) and \( B_n > 0 \) such that for any \( n \):

\[
A_n(X_1, X_2,... X_n)^{B_n} \overset{d}{=} X_1,
\]

(Mittnik and Rachev, 1993).

The motivation underlying the geometric analogues to the already presented stable laws is to model process that with some small probability may substantially change in each period.

Let \( X_t \) be a random variable at period \( t \). Let all \( X_t \) be i.i.d. with distribution function:

\[
H(u) = P(X_t < u), \quad u \in \mathbb{R}.
\]

Where \( P(A) \) denoted probability of an event \( A \). In each period, the occurrence of an event altering the characteristics of the underlying process is expected with probability
$p \in (0,1)$. $T(p)$ denotes the period in which such an event is expected to occur. Mittnik and Rachev (1993) assume that $T(p)$ is independent of $\{X_t\}$ and has a geometric distribution:

$$P\{T(p) = k\} = (1 - p)^{k-1}p, \ k = 1, 2, 3,...$$

In the geometric stable distributions the deterministic component $n$ is replaced by the geometric random variable $T(p)$. Such approach allows for varying lengths of investment horizons $T(p)$ at different points of time. Let the geometric sum

$$G_p = \sum_{i=1}^{T(p)} X_i,$$

represent the accumulation of the $X_i$ up to the event at time $t_0 + T(p)$. According to Mittnik and Rachev (1993), the distribution function $H$ describes variable, which is strictly geometric stable with respect to the summation scheme ((geo,sum)-stable) if for any $p \in (0,1)$ there exist constants $a = a(p) > 0$, such that:

$$aG_p \overset{d}{=} X_1.$$

Alternatively, a non-degenerate distribution function is strictly (geo,sum)-stable if and only if its characteristic function has the form:

$$f(t) = \left[1 + \frac{\lambda}{t} \exp\left\{-i\frac{\pi}{2\alpha} \theta \text{sgn}(t)\right\}\right]^{-1},$$

where $0 < \alpha \leq 2$, $|\theta| \leq \min(1, 2/\alpha - 1)$ and $\lambda > 0$.

This Weibull distribution is the limiting distribution for geometric sums of i.i.d. random variables. It proves to be a very good tool in modelling distributions of financial market returns (see e.g. Mittnik and Rachev 1993) and may be an interesting alternative for the traditional stable law.

In a similar way to the (geo,sum)-stable distribution, the geometric min-stable, geometric max-stable and geometric multiplicative stable distributions may be introduced (Mittnik and Rachev, 1993).
II.6.3. Asymptotic stable distribution

Another generalisation of the $\alpha$-stable law introduces the family of heavy-tailed distributions that constitute the normal domain of attraction of a stable Paretian distribution. This family contains distributions whose tails are asymptotically of the $\alpha$-stable type. It is wider than the family of stable distributions, permitting values of $\alpha$ bigger than 2. This allows greater flexibility in modelling the tails of distributions. The rest of the section provides a definition for and main properties of asymptotic stable distributions and discusses their possible applications.

Let $X, X_1, X_2,...$ be i.i.d. random variables with tail behaviour of the asymptotic Pareto-Levy form:

\[
P( X > x ) = p C^a x^{-a(1+\alpha_1(x))}, x>0, \\
P( X > -x ) = q C^a x^{-a(1+\alpha_2(x))}, x>0,
\]

where $\alpha_1(x) \to 0$ and $\alpha_2(x) \to 0$ as $x \to \infty$. The symmetry parameters $p + q = 1$ and $p, q \geq 0$. The parameter $C > 0$ is a scale parameter. The most important parameter in determining the tail shape of the distribution is $\alpha$, which is called the maximal moment exponent of the distribution. Absolute moments of $X$ of order less than $\alpha$ are finite. All higher moments are infinite. For $0 < \alpha < 2$, $X$ lies in the domain of attraction of a stable law with characteristic exponent $\alpha$. When $\alpha > 2$, $X$ is in the domain of attraction of a normal distribution. It means that for $\alpha > 2$ standarized partial sums of $X$ converge in distribution to a normal distribution. When $2 < \alpha < 4$, $X^2$ lies in the domain of attraction of a stable law with characteristic exponent $\alpha/2$.

The asymptotic stable distribution family is applied by Loretan and Philips (1994) to test the covariance stationarity of heavy-tailed time series. Although it has not attracted much attention since then, it is one of the most obvious ways to generalise the stable law. It should be noticed that some of the estimation techniques (e.g. tail estimates) already permit the values of the characteristic exponent $\alpha$ above two. It is, however, not clear how to estimate other parameters and especially, how to specify the terms $\alpha_1(x)$ and $\alpha_2(x)$.
II.7. Conclusions

The stable laws are a natural and important generalisation of the normal distribution. They provide a flexible framework to model heavy tailed data. Four parameters allow quantification of dispersion, skewness, thickness of tails and location of the empirical distributions.

Among various estimation techniques, the maximum likelihood based methods seem to be the most powerful tool to estimate the parameters of stable distributions. However, despite great efficiency of some of them, they are usually relatively complicated, require long calculation and advanced computer programs. Additional problems may occur because of lack of convergence and the necessity of approximations in computation of numerical integrals. All these reasons make them difficult to apply in practical analysis. Consequently, the Fama and Roll (1968) and McCulloch (1986) estimations seem to be the most convenient for practical purposes. Obviously, the best idea is to find an easy to compute and relatively accurate estimator that would employ the maximum likelihood methodology. This goal can be achieved using the minimum $\chi^2$ estimation.

The minimum $\chi^2$ estimation can successfully cope with problems of clustered and censored data. Series of Monte Carlo experiments, comparing the characteristics of the McCulloch (1986) quantile algorithms, show evidence in favour of the minimum $\chi^2$ methodology in cases of grouped data and ungrouped series with more than 400 observations. Clustered series are common on the emerging financial markets and hence the minimum $\chi^2$ estimation technique is recommended when analysing data from these markets.

Analyses of Warsaw and Budapest Stock Exchanges show that the fit of stable distribution to Polish data is much better than to the Hungarian series. The Hungarian market has slightly higher degree of non-normality but distributions of returns on both markets are shown heavy tailed. Excessive numbers of extreme returns and high concentration of returns around zero clearly violate the normality assumption. Examples of Warsaw and Budapest stock markets show that financial time series tend
to be heavy tailed, and stable distribution seem to provide much better description of reality than the normal ones.

To strengthen this conclusion, more extensive empirical research should be undertaken. Parameters of stable distributions of returns to indices and companies from mature and emerging markets should be estimated. Formal tests of the accuracy of fit of the normal and stable distributions should be carried. The fit of alternative distributions should be tested.
Chapter III

SPECULATION ON FINANCIAL MARKETS

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III.5. Conclusions
III.1 Introduction

All investment returns depend to a certain extent on future events. Therefore, the ability to predict the coming news is a crucial component of an investment success. All over the world investors use different asset valuation schemes to make this prediction as successful as possible. The applied asset pricing theories are split into two general groups: the fundamental theories (known as well as the firm-foundation theories) and the technical theories (known as well as cognitive or the castle-in-the-air theories). Both groups have thousands of fierce supporters and both of them seem to be mutually exclusive (see e.g. Malkiel, 1990).

The fundamental theory states that each investment instrument has an intrinsic (fundamental) value, which can be determined by analysis of present conditions and future prospects. In particular, stock prices are usually approximated by discounted streams of future dividends. Correct identification of the actual price deviations from the intrinsic value can lead to extraordinary investment opportunities. Assumption that the price must finally converge to the intrinsic value guarantees the arbitrageurs' profits. There are many models\(^1\), which help investors to determine the intrinsic value of stocks. However, there is no dominating theory and each model has certain limitations. Part of the unsatisfactory performance of the fundamental theory is caused by several risk factors that affect the financial markets. Determining the fundamental value of an asset is affected by three main problems: estimating returns received over time, estimating terminal value of an asset and deciding upon discount rates to be used for translating future returns into current values. Consequently, unexpected changes in stock valuation, cost of capital, exchange rates, taxes and many other factors cause shifts in values that are not always captured by the fundamental-based asset pricing theories.

An alternative to the well-structured but often complicated and inaccurate models offered by the fundamental theory is provided by the cognitive approach. Credit for popularising this type of psychological based analysis is commonly given to John Keynes (see e.g. Malkiel, 1990). According to Keynes, the firm-foundation

\(^{1}\) The best known are presumably the capital asset pricing model and the asset pricing theory.
theory involves too much work of a doubtful value. Instead of finding intrinsic values of stocks, Keynes concentrates on predicting behaviour and preferences of other investors. Well-known is his analogy between stock markets and beauty contests, where the players try to guess which women will win the game. Keynes claims that investor’s psychology and habits affect markets much more strongly than the fundamental values calculated by highly paid professionals.

Keynes’ investment successes attracted thousands of followers to the technical theory. People started to search for patterns in stock prices that would reflect the rules governing human behaviour and, consequently, tell how to make money on the stock exchange. The popularity of the psychological-based analysis varied in time but its contribution to the asset pricing theories seems to be much larger than it is commonly thought. Examples are provided by the financial bubbles and the recent behavioural finance ideas (reviews are given further in this chapter) which have attracted many supporters among the academics and financial professionals.

The problem, whether the fundamental or the technical analysis is right, is connected not only with the question how to make money. If assets indeed do not reflect their fundamentals well, and if their prices have an important effect on resource allocation, than the confidence in the efficiency of market allocations of investment resources is seriously weakened (Stiglitz, 1990). This opens a way for market intervention and government control of financial allocation process.

In this chapter, main attention is given to the financial bubbles and speculation processes that may lead to departures from fundamental values. Financial bubbles are generally understood as the situation when fundamental price does not seem to justify the actual price (Stiglitz, 1990). Speculation is broadly defined as temporary purchase or sale of goods or assets in the hope of profiting from an intervening price change (Hirshleifer, 1977).

The next sections review the concept of speculation and analyse its relation to market efficiency. The definitions of speculative and rational bubbles are introduced and the most famous speculative manias are described. Then, the chapter concentrates

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2 For extensive description of technical analysis techniques see e.g. Pring, 1991.
on speculative bubbles. Derivation of a rational bubble is provided and the possibility of existence of bubbles is discussed. Different examples of rational bubbles are analysed. The chapter reviews the behavioural finance ideas and gives detailed description of processes generated by some of the models developed in this framework. Finally, the Diba-Grossman speculative processes are derived and their properties and estimation techniques are discussed.

III.2. Speculation

III.2.1. Definition

In this section the concept of speculation is reviewed and two major theories of speculation are discussed. Speculation is commonly understood as the purchase of goods for later resale or temporary sale of goods with the intention of later repurchase in hope to profit from price changes (Hirshleifer, 1977). There are two major theories of market speculation. The first, associated mainly with Keynes (1930) and Hicks (1946), is called the risk transfer theory. The second, due to Working (1953, 1962), is named the knowledgeable forecasting hypothesis.

According to the risk-transfer theory, people generally split into risk tolerant and risk averse individuals. In the simple application of risk-transfer theory, the risk averse agents reduce their risk exposure by purchasing or selling contracts on the future or forward markets. If they anticipate that they will need a commodity on a given date in future, they buy now the future delivery at a known price. Similarly, if they expect that they will need to sell a commodity on a given date in future, they sell now at a known price the contract for future delivery of the commodity. In this way, the risk averse agents reduce risk connected with price changes between transaction on future or forward market and time . The risk tolerant speculators may buy or sell the contracts for future deliveries in any single transaction, but in aggregate their commitments must offset any imbalance in the positions of risk averse agents. Consequently, the speculators accept the price risk that is transferred from the risk averse individuals.
The knowledgeable forecasting theory denies such fundamental difference between the motivations of what are conventionally called speculators and hedgers. It claims that what looks like risk transfer behaviour in risk-transfer theory, is only interaction of traders with more and less optimistic beliefs about approaching developments, which will affect prices. An individual, who expects price to fall, will make speculative sales, whereas agent, who anticipates rise in prices, will make speculative purchases.

Hirshleifer (1975) develops a general equilibrium model of speculative processes. He argues that differences in risk tolerance alone could not motivate speculative trading. If all individuals shared the same beliefs there would be no speculation – even when agents had different degrees of risk aversion. Hirshleifer (1977) states that factors possibly involved in the speculation decision include individual beliefs about new information, utility function (preferences and risk tolerance), scale and composition of agents' endowment and availability of markets. He claims that speculative trading could be undertaken only by individuals, whose opinions about future state of the world diverge from representative beliefs on the market. Consequently, if speculation takes place it must be due to the fact that agents do not know the true model of economy and form various, possibly irrational, expectations regarding its form and properties. Such expectations, if shared by significant part of investors, may drive prices far from their intrinsic values, leading, for example, to speculative bubbles (Hirshleifer, 1977).

III.2.2. Relation to market efficiency

This section addresses an important question about the relationship between the level of speculation and market efficiency. Both these concepts are often mixed and confused. Hence, this discussion should help to form accurate conclusions in the remaining parts of the study and clarify conceptual differences between these two phenomena. The main conclusion from this section is that the forms of market efficiency seems to be impossible to verify and, in particular, the existence of market
speculation and its magnitude does not allow any inference about the level of market inefficiency.

The most important role of the financial markets is to allocate available resources among a number of potential investment projects. On an ideal market, prices provide complete and accurate signals for this allocation. On such a market, firms make their production decisions and investors choose among securities that represent ownership in firms’ activities under the assumption that security prices fully reflect all available information sets. The formal definition of market efficiency states that the informationally efficient market is a market on which prices always fully reflect available information and expectations of all market participants (Samuelson, 1965 and Fama, 1970).

Fama (1970) distinguishes three kinds of market efficiency: the weak efficiency, in which the information set consists of historical prices, the semi-strong efficiency, where the information set includes all information that is obviously publicly available and the strong efficiency, in which the information set consists of all public and private information. A precondition for the traditional version of market efficiency hypothesis is that information and trading costs are always equal to zero (Grossman and Stiglitz, 1980). A weaker and economically more sensible version of the market efficiency hypothesis states that prices reflect the information to the point, where the marginal benefits from acting on new information do not exceed the marginal costs of acquiring this information (Jensen, 1978). Some additional intuition about the meanings of particular forms of efficiency is given by Fama (1991). He suggests that tests for weak form of efficiency should cover the more general area of return predictability, including forecasting returns with variables like dividend yields or interest rates. He proposes to use event studies to test for the adjustment of prices to public information. The strong form tests are to be replaced by the family of tests for private information.

The above definitions suggest that any return predictability would violate even the weak form of market efficiency. This is, however, not the case. For example, the market participants face a variety of risks and, provided they are sufficiently risk averse, they expect a reward for bearing them. In such a world, prices do need to be
perfectly random, even if markets are operating efficiently and rationally. Identifying specific kinds of risk and corresponding rewards demanded by investors may lead to substantial predictability (Fama and French, 1993) but does not violate the efficient market hypothesis. Consequently, despite many advances in the statistical analysis, databases and theoretical models, there is still no consensus among financial economists regarding the validity of the efficient market hypothesis (for discussion see e.g. Farmer and Lo, 1999). One of the reasons of such a situation is that the efficient market hypothesis is not well posed and empirically refutable. To make it operational, one has to specify some additional structures as e.g. investor preferences or information structures. However, then each test of efficient market hypothesis becomes a test of a joint hypothesis. Rejection of such a joint hypothesis tells little about which part of it is inconsistent with the data. Hence, the empirical validity of market efficiency does not seem to be testable and the whole problem has some features of a never ending academic discussion. Fama (1998) suggests that one possibility to end this argument would be to replace market efficiency by a better specific model of price formation, which could be empirically tested. Such a model should specify biases in information processing that delay adjustment to new information and cause some agents to underreact to certain events and overreact to others. Despite some serious attempts (see Chapter III.3.7) models competitive to market efficiency do not seem to be fully developed yet. However, even if they existed it would be likely that some authors could still interpret the mispricings within the efficient market paradigm, e.g. by identifying new risk factors or pointing distortions in the information arriving to the market (see e.g. Garber, 1990).

How does the efficient market literature accommodate the concept of speculation? According to Working (1953, 1962), speculation requires traders with different beliefs and different ideas about future development of prices. Grossman (1977) shows that, apart of the liquidity issues, differences in beliefs are crucial incentives to create markets themselves. However, as long as the number (and volume) of traders with optimistic beliefs matches that of pessimistic traders, the market prices do not need to deviate from their fundamental values and the market may remain fully efficient.
The problem occurs when sufficient numbers of traders share similar optimistic (or pessimistic) beliefs. Their buying (selling) behaviour drives prices up (down). This may motivate some of the short horizon traders to purchase (sell) assets in the hope of further increase (decrease) of the value of an asset. Additional demand (supply) may cause prices to deviate from their fundamental values. If deviated asset prices have an important effect on resource allocation, then the efficiency of market allocation of investment resources is seriously weakened (see e.g. Stiglitz, 1990 and Flood and Hodrick, 1990). However, the market prices may still fully reflect the investment opportunity connected with an asset. In other words, they may reflect all available information and (very optimistic or pessimistic) expectations of the market participants, making the markets informationally efficient in the light of Samuelson-Fama definition. In such a situation, the existing price reflects the value of an underlying firm's activity and the value of a specific 'goodwill' created by the market. The collapse of this non-fundamental 'goodwill' value may strongly affect the price of an asset. Such departures from fundamentals cause allocation inefficiency in the sense of Stiglitz-Flood-Hodrick definition. However, markets may still preserve informational efficiency.

Do the speculation and allocation inefficiency have to coexist? The intuitive answer is yes. The historical examples (see Chapter III.3.2) show that large scale speculations are usually followed by macroeconomic crises. This could be, however, due to both misallocation of resources in the time of large scale speculation and very pessimistic information that arrives after the collapse of the market. A more formal answer to this question depends upon the exact definition of large scale speculation. If large scale speculation is understood as a situation when agents' beliefs lead to rapid changes in prices, then allocation efficiency may still be preserved. If large-scale speculation is defined as differences in beliefs leading to departures from fundamental prices, then the allocation efficiency is violated. However, proving the existence of speculation in the sense of the second definition is exactly the same task as proving the efficient market hypothesis. Additionally, the first definition seems to agree more with the traditional views (e.g. Hirshleifer, 1977).

In the remaining part of this study, such situations are described as large scale speculation.
Consequently, it seems that the existence of speculation does not provide any substantial information about kind and level of market efficiency. This is mainly due to difficulties in testing for the informational efficiency of markets and in identifying the magnitude of differences between observed and fundamental prices. Moreover, it should be stressed that stating that markets are speculative does not have much meaning itself. It is well known that agents on all markets buy and sell assets with hope for short term profits coming from price differences. Hence, all markets are to a certain extent speculative in the sense of the Hirshleifer definition and any conclusion about the level of market speculation requires caution and precision.

In this study, markets are said to be speculative when prices seem to follow closely certain speculative (non-fundamental) processes (for definition see Chapter III.3.1) and the estimated values of the parameters responsible for speculation are significantly different from zero. Such a definition of market speculation does not seem to allow any conclusions about the informational or allocation efficiency and therefore any implications in this area are avoided.

III.3. Speculative bubbles

III.3.1. Definition

The theoretical literature has not yet agreed on a unique definition of speculative bubbles and at least few different descriptions of this phenomenon coexist in economics (The New Palgrave Dictionary of Economics, 1987). One of the first attempts to define speculative bubble is due to Palgrave (1926). He defines a bubble as 'any unsound commercial undertaking accompanied by a high degree of speculation'. The New Palgrave Dictionary of Economics (1987) describes speculative bubble as 'a sharp rise in price of an asset or a range of assets in continuous process, with the initial rise generating expectations of further rises and attracting new buyers'. These new buyers are assumed to be speculators, interested in profits from trading the asset rather than its use or earning capacity. Stiglitz (1990) proposes a much shorter definition. He states that bubbles exist 'if the reason that the price is high today is only because investors believe that the selling price will be high.
tomorrow – when fundamental factors do not seem to justify such a price’. Diba and Grossman (1988) state that speculative bubbles reflect a self-confirming belief that an asset’s price depends on a variable (or combination of variables) that is intrinsically irrelevant or on truly relevant variables in a way that involves parameters that are not part of market fundamentals. On the other hand, Froot and Obstfeld (1991) define bubbles as non-stationary deviations from the present values of future cash flows generated by an asset. Similarly, Santos and Woodford (1997) identify speculative bubble with divergence of an asset price from the present value of the stream of dividends expected over the infinite future.

The above brief review, although incomplete, gives examples of different approaches to define the phenomenon of speculative bubbles. However, the presented definitions appear to have certain common features, which characterise the perception of a speculative bubble. Speculative bubble seems to be connected with (a) sharp changes in asset price, (b) departures of price from the fundamental value, defined by the present values of future expected cash flows generated by an asset and (c) high level of speculation on an asset. Therefore, in this study, any situation when financial market are described simultaneously by conditions (a), (b) and (c) is identified with speculative bubble. The terms: ‘speculative mania’, ‘speculative boom’, ‘speculative run’ and ‘bubble’ are treated as equivalent to speculative bubble.

Two more important concepts should be clarified before further analysis. In this study, the name ‘speculative process’ refers to any price generating process that is capable of generating speculative bubbles. The term ‘rational bubble’ describes a speculative process developed under the rational expectation assumption.

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4 The accepted terminology slightly differs from some of the definitions known in the literature (for review see Adam and Szafarz, 1992) but its acceptance helps to avoid confusions and in the remainder of this study.
III.3.2. Real life bubbles

The theoretical literature has not yet agreed on whether speculative bubbles are possible and observable (The New Palgrave Dictionary of Economics, 1987). Hence, it is very difficult to find price paths, which are commonly described as speculative manias. Presumably, the least controversial examples are the Dutch Tulipmania, the British South Sea Bubble and the French Mississippi Company Boom (see e.g. Malkiel, 1990). However, some authors (Garber, 1990) argue that, even during these speculative booms, prices might have been consistent with market fundamentals from the viewpoint of those who had to make investment decisions at these markets at the time. This dichotomy reflects to some extent the conflict between the followers of fundamental and psychological theories, which has been briefly discussed in Chapter III.1.

This section describes the above historical bubbles and provides examples of three recent price paths that seem to have features of speculative bubbles. This includes the stock prices of the US internet provider America Online, the Korean automobile company Hyndai Motor and the Polish construction firm Exbud\(^5\).

The description of Tulipmania follows Malkiel (1990) and Garber (1990). The essence of Tulipmania was connected with the popularity of tulip bulbs in the Netherlands. Tulip bulbs were imported to Holland in 1593 and soon became common in Dutch gardens. Special attention was given to flowers affected by mosaic virus, which developed contrasting coloured stripes. The prices of tulips slowly increased, erupting in a speculative boom between 1634 and 1637. Investors, watching increasing prices, wanted to profit from the ongoing prosperity and started to buy tulips only with the hope of later resale at a better price. The introduction of simple contracts for future delivery allowed huge returns but caused high financial leverage of investors. Investors bartered personal belongings, houses and jewels to get bulbs that were expected to make them even wealthier. Only in the culmination month of the boom, January 1637, prices increased 20 times. The bubble burst in February 1637 reducing the prices by more than 90%. Hundreds of investors lost their fortunes and

\(^5\) Respective series are downloaded from Datastream.
the Dutch economy went into crisis for several subsequent years.

The descriptions of the South Sea Bubble and the Mississippi Bubble follow Malkiel (1990) and Garber (1990). The South Sea Company was formed in the 1711 and obtained the monopoly over all trade to South America. Trade was one of the main sources of British power, so a new trade company easily won trust among investors. The prices of stocks increased, reaching in 1720 value 10 times higher than the initial price. The problem was that the Company did not have any profitable businesses and created only costs by hiring expensive apartments in London. Still, few people could resist the opportunity. Among the unfortunate investors were members of the House of Commons, House of Lords, the King and Isaac Newton. Figure 3.1 shows approximate development of stock prices of the South Sea Company. Strong increase of prices in May 1720 is followed by a rapid fall in September 1720. In November 1720 stocks of the South Sea Company were already five times cheaper than two months earlier.

**Figure 3.1: The South Sea Bubble**

Figure shows the approximate path of stock prices (in British Pounds) of the South Sea Company in 1720.

source: Garber (1990)

At the same time in France, John Law, an exiled Englishman, came with a very interesting idea of replacing gold by more liquid paper national currency. He built a conglomerate that became one of the largest capital enterprises in the financial history.
Figure 3.2: The Mississippi Bubble

Figure shows the approximate path of stock prices (in Livres Tournois) of the Compagnie des Indes in 1719 and 1720.

source: Garber (1990)

Figure 3.2 shows path of the stock prices of Law’s *Compagnie des Indes*. The prices of its stocks increased 20 times in about one year and their total value exceeded 80 times that of all the gold and silver in France. The bubble burst in 1720 with a further fall in price in subsequent years.

Similar stories can be told about the majority of 20th century stock market booms and subsequent crises (Malkiel, 1990). People commonly trade for speculation purposes and their actions prove different opinions about market fundamentals and possible future price changes. The prices are often increasing just because everyone believes they should. Thousands of investors buy stocks with hope for a gain from price changes. The magnitude of the recent years price shifts often exceeds this of the first famous bubbles discussed above. Obviously, it is difficult to evaluate to what extent the recent fast changes in the stock prices can be explained by more volatile fundamentals but it seems that at least some part of their variability is due to the speculative processes.

Figures 3.3 – 3.5 shows rapid changes in stock prices of three companies from US, Korean and Polish markets.
Figure 3.3: America Online Stock Prices
Figure shows the daily stock prices (in USD) of America Online from 01.01.1997 to 24.05.2000.

Figure 3.3 presents stock prices of America Online internet provider quoted at NASDAQ Stock Exchange in New York. The fast and stable growth in 1997 and 1998 increased the value of stocks by about 5 times. During first half of 1999, prices gained almost 900% of the December 1998 value. Then, the bubble seemed to burst, reducing prices almost by half. However, late 1999 brought new sharp growth followed by subsequent collapse in prices.

Figure 3.4: Hyundai Motor Stock Prices
Figure shows the daily stock prices of Hyundai Motor (in South Korean Wons) from 07.03.1984 to 24.05.2000.
Figure 3.4 shows the variability of *Hyundai Motor* stock prices in the last 16 years. *Hyundai Motor* is an automobile company quoted at the Korean stock exchange. Periods of rapid growth are followed by sharp decreases in the value of the company. The most evident bubble erupted in 1999, increasing the stock prices by 3.5 times. It burst few months later, reducing the stock value to the pre-bubble level.

**Figure 3.5: Exbud Stock Prices**

Figure shows the daily stock prices of Exbud (in Polish Zloty) from 07.03.1984 to 24.05.2000.

Figure 3.5 presents stock prices of Polish engineering and construction company *Exbud*. In the analysed period, the prices varied from less than 5 PLN to almost 50 PLN. The sharpest speculative bubble appeared in 1994, when stock prices reached 700% of their initial value. The subsequent fall reduced stock prices to about 33% of the peak value.

Rapid changes are present on many mature and on emerging markets. They occur on assets from almost all industry branches. Although the exact reason of this situation is not clear, it seems that speculative bubbles might provide relatively accurate and reliable explanation of the observed reality.
III.3.3. Derivation of rational bubbles

The rational bubbles are usually derived as non-linear solutions to the linear asset pricing model (Diba and Grossman, 1988). The model is based on the relationship between real stock prices and real dividend payments under the assumption of constant expected rate of return. If \( P_t \) denotes the real price of share at the beginning of period \( t \), \( D_t \) are real dividends per share paid over period \( t \) and \( r \) is the constant discount factor, the asset pricing relationship can be summarised in the following way:

\[
P_t = e^{-r} E_t (D_t + P_{t+1}),
\]

where \( E_t \) is the market expectation conditional on the information available at the beginning of period \( t \).

Note that equation (3.1) is a special case of the general expectation difference equation:

\[
y_t = a E_t (y_{t+1}) + c x_t,
\]

where \( y_t \) depends on the current expectations of its value next period as well as on variable \( x \). All results obtained for equation (3.1) can be easily generalised to the case of the equation (3.2) (see e.g. Blanchard and Fisher, 1989: Chapter 5.1). However, due to the scope of this study, the analysis concentrates on the stock market framework.

The well-known forward-looking solution of the stochastic equation (3.1) is of the form:

\[
P^*_t = \sum_{x=t}^{\infty} e^{-r(t-x+1)} E_t (D_x).
\]

Equation (3.3) states that a stock price is equal to the present discounted value of the future dividend payments. It is commonly assumed that \( P^*_t \), called the fundamental or present value solution to (3.1), always exists\(^6\). It implies that the continuously

---

\(^6\) This description closely follows that by Froot and Obstfeld (1991). It concentrates on the stock prices rational bubbles and presents only the most important results. For general derivation of rational bubbles see Blanchard and Fisher (1989, Chapter 5.1) or Adam and Szafarz (1992). Details about solving linear expectational difference equations (e.g. equation (3.1)) can be found in Blanchard and Fisher (1989, Chapter 5: Appendix).
compounded growth rate of expected dividends is less than \( r \). In order to obtain the present value solution (3.3), the transversality condition must be imposed:

\[
\lim_{T \to \infty} e^{-rT} E_t(P_T) = 0. \tag{3.4}
\]

If (3.4) holds, then the successive forward substitutions of (3.1) converge to (3.3). Relaxing the assumption (3.4) leads to new set of solutions to equation (3.1). These alternative price paths satisfy the period-by-period efficiency but they do not satisfy the transversality condition. The general family of solutions to (3.1) in the absence of (3.4) is of the following form:

\[
P_t = P_t^* + B_t. \tag{3.5}
\]

However, certain restrictions must be imposed on \( B_t \) in order for (3.5) to be a solution to (3.1). If \( P_t = P_t^* + B_t \), then \( E_t(P_t) = E_t(P_t^*) + E_t(B_t) \). Hence, replacing \( P_t \) in (3.1) implies:

\[
P_t^* + B_t = e^{-rT} E_t(D_t + P_{t+1}^*) + e^{-rT} E_t(B_{t+1}).
\]

\( P_t^* \) is a solution to (3.1) and consequently:

\[
P_t^* = e^{-rT} E_t(D_t + P_{t+1}^*). \tag{3.6}
\]

Finally:

\[
B_t = e^{-rT} E_t(B_{t+1}). \tag{3.6}
\]

In other words \( B_t \) is any random variable satisfying the martingale property. For \( B \neq 0 \) the transversality condition (3.4) is clearly violated. Equation (3.5) states that within the rational expectation framework, a bubble is defined as the deviation between any solution to (3.1) and the fundamental solution \( P^* \).

The rational bubble \( B_t \) in (3.5) can be decomposed into stochastic component \( \Theta_t \) and deterministic component \( B_t' \) (Adam and Szafarz, 1992):

\[
B_t = B_t' + \Theta_t \tag{3.7}
\]

All rational bubbles have deterministic and stochastic components. In the most trivial cases, the deterministic component is a starting value or a constant term in the price generating process, whereas the stochastic component is a stationary i.i.d. variable. According to the terminology proposed by Blanchard and Watson (1982), rational bubbles can be divided into two major groups: the deterministic and the stochastic bubbles. The deterministic rational bubbles are of the form (3.7) with a trivial
stochastic component. Similarly, the stochastic rational bubbles have the form (3.7) with trivial deterministic component. Obviously, there may exist mixed rational bubbles, with both non-trivial components.

III.3.4. Examples of rational bubbles

This section presents examples of speculative processes developed in a rational expectation framework. Series generated by such processes seem to resemble price paths described in Chapter III.3.2. The processes are divided into those which generate deterministic, stochastic and mixed bubbles.

Exemplary deterministic bubble

The pioneering article by Flood and Garber (1980) focused on the deterministic component of (3.7) and proposed a rational bubble solution to (3.1) of the form:

\[ B_t = -\alpha A_0 \psi' + \epsilon_t, \]

(3.8)

where \( \psi=(\alpha-1)/\alpha \gg 1 \) and \( \epsilon_t \) is a normally distributed independent random component. This bubble develops in a deterministic way and grows at exponential rate. The rate of its growth depends on parameters \( \alpha \) and \( A \). The relationship between \( A \) and \( B_t \) is linear whereas the relationship between \( \alpha \) and \( B_t \) takes more complicated non-linear form. Figure 3.6 presents behaviour of exemplary bubbles of type (3.8). Increase in \( \alpha \) is connected with more rapid explosion of the bubble. The differences between the simulated processes start to be evident at about seventieth observation. Before this point any discrepancies are offset by the error term. All three bubbles grow without bounds and restrictions. The process with \( \alpha=-18 \) increases almost 5 times within the first 100 observations.
Figure 3.6: Flood and Garber deterministic bubbles

Figure shows deterministic bubbles simulated according to equation (3.8) for \( A_0 = 0.001 \), standard deviation of \( \epsilon \) equal to 0.5 and alphas equal to -18, -20 and -30 respectively.

![Figure 3.6: Flood and Garber deterministic bubbles](image)

Bubbles presented on Figure 3.6 never burst and hence are counterintuitive (Blanchard, 1979). Bubbles are empirically plausible only if, despite explosive conditional expectations, the probability that they become arbitrarily large is moderately small. Such properties may be captured by the stochastic component of (3.7).

Exemplary stochastic bubbles

One of the simplest examples of the stochastic bubbles is suggested by Blanchard (1979). He proposes a solution to equation (3.1) described by the following transition matrix:

\[
\begin{array}{cc}
B_{t+1} - \overline{B} &= (a\pi)^{-1}(B_t - \overline{B}) + \epsilon_t \\
B_{t+1} - \overline{B} &= 0
\end{array}
\]

where \( B_t \) is bubble component of share price at time \( t \), \( \pi \) is the probability that \( B_{t+1} \) will be different from \( \overline{B} \) if \( B_t \) is different from \( \overline{B} \) and \( a = 1/(1+r) \). Similarly as in (3.8), \( \epsilon_t \) denotes normally distributed independent random variable. In each period, the probability that the bubble persists is equal to \( \pi \) and the probability that the market will crash and the price will return to its fundamental value is equal to \( (1-\pi) \). For \( \pi < 1 \),
the bubble is certain to burst and its expected duration is \((1-\pi)^{-1}\). For \(\pi=1\), the bubble behaves like the Flood and Garber (1980) bubble.

Figure 3.7 shows behaviour of exemplary bubbles of the type (3.9). Growth of the probability of collapse \((1-\pi)\) increases the explosive character of the process and reduces its lifetime. Faster growth of the processes with smaller \(\pi\) accounts for higher risk of collapse. The expected duration of the shortest bubble on the Figure 3.7 is 20, of the middle one 50 and the expected lifetime of the longest bubble is equal to 100. The differences in the behaviour of individual processes in the first periods are not evident because of the error term in equation (3.9).

**Figure 3.7: Blanchard stochastic bubble**

Figure shows simple stochastic bubble described by equation (3.9). Parameter \(\alpha=1/1.00025\), the standard deviation of \(\epsilon\), equal to 0.1 and probability of collapse at each time period \((1-\pi)\) is equal to 0.05, 0.02 and 0.01 respectively.

As discussed in Blanchard and Watson (1982), the bubble described in (3.9) has many simple extensions. The probability that a bubble ends may depend on how long the bubble has lasted or how far the price is from the fundamental solution (3.3). Similarly, although in (3.8) and (3.9) the bubbles proceed independently from the fundamental values, this does not always need to be the case. The examples of stochastic bubbles dependant on the fundamental information, as e.g. the probability of the war or the probability of the House of Commons changing the law, are given in Blanchard and Watson (1982) and Hamilton (1986).
An important property of a bubble (3.9) is the finite lifetime. This is relaxed by Hamilton and Whiteman (1985) and Hamilton (1986), who propose the continuously regenerating bubble of the form:

\[ B_t = (1 + r)B_{t-1} + \varepsilon_t/(1 + r)^t, \]  

(3.10)

where \( \varepsilon_t \) is an arbitrary white noise process with constant variance. The simplest interpretation of (3.10) is that \( B_t \) corresponds to a bubble newly set off each period in response to totally irrelevant random events. For example, it might be rational to look at the weather forecast before purchasing stock, because that is what everyone else does. Figure 3.8 shows behaviour of bubbles defined by (3.10). When standard deviation of \( \varepsilon_t \), denoted as \( \sigma(\varepsilon_t) \), is equal to zero, bubble grows with the discount factor without any random shocks. Increase in \( \sigma(\varepsilon_t) \) causes the variability of the process to grow. The process may be both above and below its non-random realisation for \( \sigma(\varepsilon_t)=0 \). It is worth noting that process (3.10) can take both negative and positive values.

**Figure 3.8: Hamilton continuously regenerating bubble**

Figure presents three bubble paths simulated according to equation (3.10). Interest rate \( r \) is set to 0.00025 and standard deviations of normal random variable \( u_t \) (std(u)) are set to 0, 0.15 and 0.3 respectively.

Diba and Grossman (1988) introduced a stochastic bubble with both multiplicative and additive disturbances. They proposed a process of the form:

\[ B_{t-1} = \theta_t B_t + \varepsilon_t, \]  

(3.11)
where \( \theta_t \) and \( \varepsilon_t \) are mutually and serially independent random variables, expected value of \( \varepsilon_t \) is equal to 0 and expected value of \( \theta_t \) is equal to \((1+r)\). They imposed restriction on (3.11) to guarantee the positive values of \( B_t \). If the moment when \( \theta_t=0 \) coincides with positive realisation of \( \varepsilon_t \), then as an existing rational bubble component bursts, a new completely independent rational bubble would simultaneously start.

**Figure 3.9: Diba and Grossman stochastic bubble**

Figure shows the path of a stochastic bubble generated by equation (3.11). Rate of return \( r \) is set to 0.00025. Variable \( \theta_t \) has normal distribution truncated at zero, with standard deviations (std(theta)) equal to \(0.\), \(0.15\) and \(0.2\) respectively. Variable \( \varepsilon_t \) is a standard normal variable with mean zero and variance 0.1.

Figure 3.9 shows path of an exemplary Diba-Grossman bubble for three different distributions of variable \( \theta_t \). Bubble with standard deviation \( \theta_t \) equal to 0.2 collapses at the 92\textsuperscript{nd} observation. This is because the zero realisation of \( \theta_t \) did not coincide with positive realisation of the random term \( \varepsilon_t \).

Another interesting bubble of type (3.11) is proposed by Evans (1991). His positive periodically collapsing bubble is defined by the equations:

\[
\begin{align*}
B_{t+1} &= (1 + r)B_t \varepsilon_{t+1} & \text{if } B_t \leq \alpha \\
B_{t+1} &= \left[ \delta + \pi^{-1}(1 + r)\theta_{t+1}(B_t - (1 + r)^{-1}\delta) \right] \varepsilon_{t+1} & \text{if } B_t > \alpha
\end{align*}
\]  

(3.12)

where \( \alpha \) and \( \delta \) are positive parameters such that \(0<\delta<(1+r)\alpha\), \( \varepsilon_{t+1} \) is an exogenous i.i.d. positive random variable with expected value 1, and \( \theta_t \) is an exogenous i.i.d.
random Bernoulli process, which is independent of $\varepsilon_{t+1}$ and takes the value of 1 with probability $\pi$ and the value of 0 with probability $1-\pi$.

**Figure 3.10: Evans periodically collapsing bubble**

Figure shows exemplary bubbles generated according to equation (3.12). The interest rate $r$ is equal to 0.00025, $\alpha=1.1$, $\delta=1$, and $\pi (\pi_{i})$ are equal to 0.75, 0.9 and 0.95. Standard deviation of $\nu_{t+1}$ is set to 0.1.

![Graph showing exemplary periodically collapsing bubbles](image)

Figure 3.10 shows exemplary periodically collapsing bubbles. When $B_{t} \leq \alpha$, the bubble grows at a mean rate $1+r$. When $B_{t} > \alpha$, the bubble erupts and grows at a mean rate of $(1+r)\pi_{t}$. At this stage, the bubble collapses with probability $1-\pi$ per period. When the bubble collapses, it falls to a mean value of $\delta$ and the process starts again. The smaller the value of parameter $\pi$, the faster the rate of growth of the bubble. This is mainly to provide investors with reward for the risk they are facing because of the probability of collapse in the price of an asset.

An other modification of (3.11) is proposed by Charemza and Deadman (1995) who suggest a non-linear bubble of the form:

$$B_{t} = \theta_{t}B_{t-1} + \varepsilon_{t},$$

(3.13)

where $\theta_{t}$ are random variables exogenous from $B_{t}$ and independent from $\varepsilon_{t}$ and $\varepsilon_{t}$ are mutually and serially identically distributed random disturbances. The expected values of $\theta_{t}$ are chosen so that they guarantee (3.6). In the case of the Charemza and Deadman (1995) bubble it means that $\theta_{t}=\exp(\Theta_{t})$ and $\varepsilon_{t}=\exp(\varepsilon_{t})$, where $E_{t}$ and $\Theta_{t}$ are
i.i.d. random normal variables with variances $\sigma(\theta)^2$, $\sigma(\epsilon)^2$ and means $\ln(1+r) - \sigma(\theta)^2/2$ and $-\sigma(\epsilon)^2/2$ respectively.

**Figure 3.11: Charemza and Deadman non-linear stochastic bubble**

Figure shows paths of three stochastic bubbles generated according to equation (3.13). The interest rate $r=0.00025$ $\sigma(\epsilon) = 0.01$ and $\sigma(\theta)$ (std(theta)) takes the values of 0, 0.025 and 0.05.

![Figure 3.11](image)

Figure 3.11 presents behaviour of non-linear stochastic bubble (3.13). Increase in the variance of $\theta$ is positively correlated with the increase in the variability of the series. For $\sigma(\theta)=0.05$, the series produces bursting and restarting bubbles of different magnitude and maturity. Similar paths, although with smaller absolute shifts characterise other two series.

**Exemplary mixed bubble**

The bubbles (3.8) - (3.13) depend either on time or on variables omitted in the fundamental value equation. There is, however, no particular reason why the rational bubble could not depend on the fundamental variables already taken to account in the equation (3.3). Froot and Obstfeld (1991) propose so called intrinsic bubbles, where the level of bubble depends exclusively on the fundamentals from (3.3):

$$B_t(D_t) = cD_t^\lambda,$$

(3.14)

where $\lambda$ is the positive root of the quadratic equation $\lambda^2\sigma^2/2 + \lambda\mu - r = 0$, $\mu$ is the trend growth in dividends and $\sigma^2$ is the variance of the random term $\eta$, in the log
dividend equation: \( D_{t+1} = \mu + D_t + u_{t+1} \). The functional form (3.14) is chosen for simplicity and can be replaced by any function of \( D_t \), which satisfies (3.6).

**Figure 3.12: Froot and Obstfeld intrinsic bubble**

Figure presents the price path simulated according to equation (3.5) with \( P^* = \kappa D_t, D_t = \mu + D_t + u_t \) and \( \kappa = 12 \), where \( u_t \) is i.i.d. normal variable with conditional mean zero and variance \( \sigma(u)^2 \). The bubble component is simulated according to (3.14). Interest rate \( r \) is set to 0.00025, trend growth in dividends \( \mu \) is equal to \( 10^{-7} \), \( \sigma \) is set to 0.1, \( D_t = 1 \) and \( c = 8 \) and \( \lambda = 0.63145632 \).

Figure 3.12 shows the path of an intrinsic bubble together with the underlying present value and the total price of an asset. It can be seen that behaviour of the bubble strongly resembles variability of the present value price. In this example, the variability of the total price is larger than this of the fundamental value. This, however, does not need to be the case. Movements of bubble in the opposite direction than changes in fundamentals may decrease the variability of prices. This will happen for \( c < 0 \) in equation (3.14). Consequently, intrinsic bubble may be very difficult to identify.

**III.3.5. Tests for speculative bubbles**

The stochastic bubbles of types (3.8) - (3.14) may take almost all kinds of shapes. They may be similar to price paths driven by fundamentals or they may mimic.
behaviour of variables not included in the equation (3.3). Additional complications may be caused by other factors, as e.g. regime switching of the dividend generating process (Driffill and Sola, 1998). Therefore, detecting the presence of bubbles or rejecting their existence may prove to be very hard. One way to verify the existence of bubbles is to construct and apply appropriate empirical tests. The empirical tests designed to detect the rational bubbles may be divided into the main groups: tests based on distributions of excess returns, the parametric tests and the cointegration tests.

Distribution-based tests

The description of distribution-based tests follows Blanchard and Watson (1982). These tests concentrate on the distribution of excess returns to stock prices. The excess returns are defined as differences between the actual rate of returns and the risk free rates of returns. The key idea is that the excess returns generated by speculative bubbles are likely to have both runs and distributions with fat tails. The innovations in the excess returns to a bubble would tend to be of the same sign when bubble grows and of the reverse sign when a crash occurs. Therefore, the runs in the bubble excess return innovations will tend to be longer than for random series, making the total number of runs over the sample smaller. Similarly, crashes produce large outliers so that the distribution of bubble innovations should be leptocurtic (Blanchard and Watson, 1982).

The problem is that these characteristics of bubble processes are unobservable. The actual excess returns are sums of the innovations in the market fundamental and innovations in the bubble process. Consequently, the distribution-based tests require strong assumptions about the behaviour of the innovation in the market fundamentals. The most crucial assumptions state that the distribution of innovations in market fundamentals is symmetric and does not have thick tails. However, if for example the information arrives in clusters, then the innovations in the fundamental value of a stock tend to be leptocurtic independently of the possible bubble component. Moreover, bubbles themselves need not generate long runs and heavy tailed distribution. All of these factors cause serious problems in the application of the distribution-based tests. Indeed, such tests attracted a lot of criticism due to very...
strong underlying assumptions (see Blanchard and Watson, 1982) and never gained substantial interest among researchers, who generally prefer cointegration and parametric tests.

**Parametric tests**

The first parametric test is proposed by Flood and Garber (1980). They reduce testing for the no-bubble hypothesis to test whether the coefficient $A_0$ in the equation (3.8) is significantly different from zero. It is worth noting that similar procedures can be applied to test for any fully specified bubble process. The tests of the no-bubble hypothesis are always reduced to checking whether the coefficients involved in the bubble term are significantly different from zero. A classical example of such methodology is due to Froot and Obstfeld (1991). They estimated the parameters of the regression:

$$P_t = c_0D_t + cD_t^x + \varepsilon_t ,$$

where on the empirical basis $c_0$ is set to 14 and $\varepsilon_t$ is the present value of the errors from equation;

$$P_t = e^{-r}E_0(D_t + P_{t+1}) + e^{-r}u_t ,$$

where $u_t$ is predictable single period excess return. They applied an $F$ test for the hypothesis $c = 0, \lambda = \lambda'$, where $\lambda'$ is the unrestricted estimate of $\lambda$ from regression:

$$\frac{P_t}{D_t} = c_0 + cD_t^{\lambda-1} + \eta_t ,$$

and $\eta_t = \varepsilon_t / D_t$

This test performs well in detecting bubbles of the form (3.14). However, similarly to other parametric tests, it fails when the prices are generated by slightly different model. Indeed, Driffill and Sola (1998) suggest that bubbles (3.14) may behave very similar to the non-bubble price generating processes with regime switching in the underlying dividend generating process. It is not clear whether the Froot and Obstfeld (1991) test can distinguish between the bubble and non-bubble regime switching alternatives. Additionally, it must be remembered that each parametric test is only suitable for particular underlying specification of the bubble process, and rejecting one bubble does not mean that a bubble of different type does
not exist. Consequently, errors of both types can be committed. One can wrongly reject the null about lack of bubbles because some other non-bubble process behaves in a similar way to the bubble under consideration. Similarly, one can wrongly accept the non-bubble hypothesis because the bubble generating the analysed series has completely different specification to that assumed at the beginning of the analysis. Both these facts limit the applicability of parametric tests and divert the researchers' attention to other ways of verifying the existence of financial bubbles.

**Cointegration tests**

The idea of cointegration tests, first proposed by Diba and Grossman (1988), seems to be relatively simple from the econometric point of view. If the following equation holds for dividends $D_t$:

$$E(D_{t+j}) = E_t (cD_{t+j} + u_{t+j}) ,$$

then putting it to the equation (3.3), rearranging the terms and combining the results with (3.5) leads to:

$$P_t - cr^{-1}D_t = B_t + a r^{-1}\left[\sum_{j=1}^{\infty}(1 + r)^{-j} E_t(D_{t+j} - D_{t+j-1})\right] + \sum_{j=1}^{\infty}(1 + r)^{-j} E_tu_{t+j} .$$

(3.15)

If the unobservable component of market fundamentals $u_t$ is stationary in levels, if the dividends are stationary in first differences, and if rational bubbles do not exist, then the right hand side of (3.14) must be stationary. Thus, although prices and dividends are non-stationary in levels, their linear combination given by the left side of (3.14) should be stationary in the absence of bubbles (Diba and Grossman, 1988). Consequently, testing the no bubble hypothesis is equivalent to testing the hypothesis about the cointegration of prices and dividends, as defined by Granger and Engle (1987).

As in the earlier cases, there are certain pitfalls connected with the cointegration tests. For example, Evans (1991) noted that the cointegration tests are misleading in the case of periodically collapsing bubbles. Due to the bursting nature of such bubbles, these tests are biased towards cointegration and have a tendency to
reject the null hypothesis about the non-stationarity of the left side of (3.15) much too often. Charemza and Deadman (1995) showed that this applies as well to the non-linear stochastic bubbles. Taylor and Peel (1998) reported a more robust statistical test for non-cointegration, which overcomes some of these difficulties. Their test is based on the modification of the least squares estimator designed to be robust to the presence of error terms, which may exhibit strong skew and kurtosis. Although it detects periodically collapsing bubbles of the form (3.12), it is not clear how it performs when the stochastic bubbles (3.13) or intrinsic bubbles (3.14) are incorporated in the data generating process. As bubbles of these types do not need to be explosive or significantly more volatile than the fundamental values, it seems to be rather unlikely for any cointegration test to detect their existence. Another line of criticism of the tests based on (3.15) is connected with the constant discount factor assumption implicitly incorporated in the regression. The cointegration tests verify the joint hypothesis about the lack of bubbles and constant discount factor. It seems to be clearly understood now that a rejection of such joint hypothesis is not the same as rejection of the no bubble hypothesis. This is especially obvious after successful introduction of stochastic discount factor models (see Blanchard and Fisher, 1989: Chapter 5.4 and Campbell et al, 1997: Chapter 7.2.2). All these make the conclusions drawn on the basis of cointegration tests relatively questionable and prevent any strong inference about the existence of bubbles.

Some authors (Blanchard and Fisher, 1989: Chapter 5) refer additionally to volatility tests as tests of the rational bubble hypothesis. The volatility tests are investigating whether the fluctuations in asset prices are too large to be explained purely by changing views about fundamentals. In the absence of bubbles, the price path of an asset should be given by (3.3) and hence the easiest approach to test for the excessive volatility of asset prices is to compute the right side of (3.3) and compare it to the real prices. The problem is that neither expectation nor the market participants' information sets are observable. The first tests that do not require full specification of the information set are proposed by Shiller (1981) and LeRoy and Porter (1981). The volatility tests gained substantially in popularity in the last two decades (see Blanchard and Fisher, 1989: Chapter 5) but the detailed evaluation of these tests goes beyond the scope of this study.
III.3.6. Reflection on the idea of speculative bubbles

None of the described empirical tests provides satisfactory tools to verify the hypothesis about the existence of bubbles. The inference about speculative bubbles depends upon many aspects and their existence is difficult to verify with finite data samples. Consequently, clarification of the conditions under which such phenomena are theoretically possible plays an important role in judgements of whether bubbles can be observed in the real data. This section provides a brief review of equilibrium models dealing with the existence of rational bubbles. It is shown that the conclusions of these models are very sensitive even to slight changes in the assumptions.

Diba and Grossman (1988) show that a rational bubble can start only on the first day of trade of an asset and cannot restart after it bursts. The existence of a negative bubble would imply a negative expected asset price in some day in the future, which is inconsistent with the limited liability of agents. Hence, a negative bubble cannot exist on an asset with free disposal. Moreover, if the bubble has a zero value at any time, then due to (3.6), its expected value in the next periods must be equal to zero. As the bubble cannot be negative, its value after collapse to zero must remain on this level. Therefore, bubble can start only at the first day of the trade of a given stock. Moreover, a bubble can not exist if there is any upper limit on the price of an asset. Thus, stock price bubbles are ruled out if firms issue stock in response to price increases. However, these arguments do not cover all possible bubbles. For example, stochastic bubbles that start on the first day, never collapse to zero, and do not exceed certain upper limits may still exist on the stock assets.

General equilibrium analysis also limits the possibility of rational bubbles. Bubbles appear on the market when assets are bought on the anticipation that they can be resold at a higher price. Therefore, it is not surprising that they can not arise when there is a finite number of rational individuals who have infinite horizons and who know the true model governing the economy (Tirole, 1982). Assumptions of this model imply that every asset has the same fundamental value for all market participants. Consequently, if there is a negative bubble, then all individuals will want to buy an asset and, due to infinite horizons, keep it forever. There would be an excess demand and therefore the negative bubble price can not be the equilibrium.
price (market clearing is assumed). If there is a positive bubble and short selling is allowed, a symmetrical argument implies excess supply and disequilibrium. If there is a positive bubble and short sale is not allowed, then an individual who is buying an asset does so only in anticipation of reselling it in the future. If, however, all individuals plan to resell the asset before certain time $T$, no one is going to hold it after $T$ and this can not be the equilibrium. Generally, it can be seen that infinite horizons and short sale rule out bubbles in all cases. When there is no short sale on the market, the additional assumption of a finite number of traders is necessary.

However, in reality new traders come to the market and their horizons are usually finite (sometimes even very short). Therefore, the overlapping generation framework may provide important information on the possibility of existence of rational bubbles. Such analysis is done by Tirole (1985), who disproves deterministic bubbles in the general equilibrium with finite horizon individuals. Following equation (3.6), a bubble must grow at the interest rate. This suggests that in some point the value of the bubble will be too large relative to the economy and the bubbles must be ruled out. However, it may happen that the economy itself is growing and the interest rate is less than this growth rate (economy is dynamically inefficient). In such a situation the above argument does not hold true.

Weil (1987) extends the Tirole (1985) reasoning to the stochastic bubbles of the form proposed by Blanchard (1979). If a stationary bubble (3.9) is to be held by risk-averse agents, its expected rate of return $\pi(1+n)$, where $n$ is the growth rate of the economy and $\pi$ is the probability that bubble will not burst in the next period, must exceed the no trade interest rate. As $\pi$ must be from the interval $[0,1]$, stochastic bubbles may only exist in dynamically inefficient economies. The case when $\pi=1$ coincides with the Tirole (1985) result. Consequently, it has been shown that the deterministic bubbles can not exist in the overlapping generation economy with interest rates higher than the growth rates.

All of the above models are based on quite strong assumptions. For example, both Weil (1987) and Tirole (1985) assume that all dated and contingent goods were traded on a single market. In reality, however, the markets tend to be separated. Santos and Woodford (1997) develop an analysis of bubbles in the intertemporal
general equilibrium model involving spot markets for goods and securities at each of countable infinite sequence of dates. The reasoning disproving bubbles in such a framework is as follows. If there is a bubble on a security in a positive net supply (total endowment of this security is positive), then the net present value of aggregate wealth at date $t$ does not approach zero as $t$ is made unboundedly large. On the other hand, if there is a finite upper bound on the aggregate consumption, defined by the finite bound on the aggregate endowment, the present value of consumption from date $t$ onwards must tend to zero as $t$ increases. Consequently, if there is a bubble, for a date $t$, sufficiently far in the future, the present value of some household’s wealth exceeds that of its consumption, which is inconsistent with optimisation by that household. This holds true even with incomplete markets\textsuperscript{7} and borrowings limits. The presented reasoning is not as intuitive as the before ones but it disproves rational bubbles under much more relaxed conditions. However, lifting some further assumptions clearly leads to the possibility of rational bubbles. Santos and Woodford (1997) provide 5 examples of such situations. Bubbles may exist if no household is endowed with a positive fraction of the aggregate endowment. The other possibility occurs when the borrowing limits are defined other than by one’s ability to repay the debt from the future endowment. Bubbles may appear as well for securities with infinite maturity and zero net supply and, because of the non-stationary stochastic discount factor and lack of well defined fundamental value, on the incomplete markets (for a detailed justification for why, in these cases, rational bubbles can arise, see Santos and Woodford, 1997).

The problem of bubbles and incomplete markets attracted substantial research. Such a situation clearly allows multiple equilibria, where the dynamics depend on extraneous variables only because agents believe it to be so (see Mas-Colell et al, 1995: Chapter 19F or Blanchard and Fisher, 1989: Chapter 5.4). This does not have a close analytical connection to the bubbles, as defined by Santos and Woodford (1997). The price paths generated in such equilibria should not generally be explosive, but in some cases, they may be very similar to those generated by speculative bubbles. Additionally, it should be noted that such price paths can be classified as bubbles in

\textsuperscript{7} Incomplete markets are identified with situation when the number of assets is smaller than the number of possible states of nature.
the light of some other definitions (see e.g. Stiglitz, 1990). Hence, the possibility of
the existence of bubbles will depend upon the applied definition of bubbles and beliefs
whether complete or incomplete markets are a better approximation of financial
markets.

One of the other crucial implicit assumptions of the above analyses is that the
individuals know the model. A weaker assumption states that individuals do not know
the model, but have enough information to form optimal forecasts of the variables
they are interested in. This learning assumption is sufficient to obtain many, but not
all, of the above results (see Blanchard and Fisher, 1989 Chapter 5.5). Grandmont
(1998) states that when agents are uncertain about the dynamics of the economics
system, learning is bound to generate local instability and self-fulfilling expectations.
Under the uncertainty, individuals tend to extrapolate a wide range of regularities out
of past deviations from equilibrium. Basing on past observations they form
expectations of future dynamics, which in turn affect future equilibria. This is
especially evident when the influence of expectations on the system is significant.
Then, even in absence of shocks to the fundamentals or to expectations, learning by
itself may generate convergence to complex nonlinear attractors and self-perpetuating
endogenous fluctuations. These may lead to departures from the fundamental value
(3.3), which is equivalent to the bubble paths.

A simple example of such divergence from fundamental value $P^*$ is provided
by the informational cascade theory (for review see Bikhchandani et al, 1998, for
recent developments and applications to financial markets see Grenadier, 1999 and
Graham, 1999). Following the standard learning framework, it is assumed that the
agents do not know the model. They are not sure about their private information and
they tend to infer from other people behaviour. Additionally, agents are assumed to
trade sequentially, so in the moment of trading they can observe the outcomes of the
decisions of earlier players. Individuals' decisions are based both on the private
information and on the information inferred from other agents' behaviour. Players not
sure about the private impulse assume that other agents act basing on superior
information and tend to put relatively large weight to their behaviour. In such a
context, with relatively inaccurate private information and few self-confident
individuals, people will generally mimic actions of earlier players. New players will not
know whether the actions of the earlier ones are based on the private information or on the observation of other people’s behaviour. Consequently, informational cascades occur. Obviously, a group of informed individuals may act against the majority and then faith in the cascade is broken. Each cascade eventually collapses. However, sooner or later, when a sufficient number of agents decide to mimic each other’s behaviour, a new cascade can occur. The cascades may be interpreted as speculative bubbles formed by the actions of rationally behaving agents under incomplete information.

Obviously, there may be a wide range of learning schemes, among which the perfect knowledge rational expectation framework and the informational cascades are just specific examples. There are no general results about learning yet. However, the above paragraphs show clearly that, although bubbles are generally ruled out in the intertemporal competitive equilibrium model, the non-deterministic divergences from the fundamental price may occur, when the assumption that agents know the true model is lifted and different learning schemes are assumed.

Consequently, neither the empirical tests nor the theoretical models can definitely rule out or prove the existence of financial bubbles. The opinion about the possibility of bubbles depends upon exact definition of the speculative bubbles and particular model applied to approximate the reality. Hence, the question whether bubbles exist or not seems to belong to the same family as the question about market efficiency, and researchers will probably disagree on this topic for many years.

III.3.7. Further developments in the speculative processes: the behavioural finance theory

Processes described in Chapter III.3.4 generated series similar to the real life speculative bubbles discussed in Chapter III.3.2. All of them are based on the rational expectation assumption. However, speculative bubbles may arise as well in the non-rational framework. Such a framework may provide interesting information about the possibly non-rational behaviour of market participants, which results in rapid price changes. This section concentrates on speculative processes recently developed within
behavioural finance framework. It reviews the major motivation of the development of the behavioural finance approach, provides some critic of this type of concepts and analyses three of the asset pricing models developed within this framework.

The behavioural approach to finance rests on two major assumptions. First, some investors are not necessarily fully rational and their demand for risky assets is affected by their beliefs that do not need to be fully justified by market fundamentals. Second, arbitrage, defined as trading by fully rational investors not subject to such sentiments, is risky and limited (Shleifer and Summers, 1990). These two assumptions imply that changes in investor sentiment can not be fully countered by arbitrageurs and affect the security returns. Consequently, the behavioural finance theory claims that the sentiments and habits of investors play an important role on financial markets, and should be incorporated in the asset pricing models.

There are two main arguments that motivate the development of the behavioural finance. The first one is that the security returns present a sharp challenge to the traditional view that stocks are rationally priced to reflect all publicly available information. The second one is based on the observations that agents permanently violate the rational behaviour assumption and seem to take an imperfect rationality approach.

The statistical evidence of imperfect rationality includes event-based returns predictability, short-term momentum, long-term reversal, high volatility of asset prices relative to fundamentals and post earnings announcement price drifts (for literature review see Daniel et al, 1998). The event-based return predictability is connected with the fact that prices underreact to public news in short run. Market does not absorb the information instantaneously but adjusts gradually. As a result, the event date average stock returns are of the same sign as the average subsequent long-run abnormal performance. This phenomenon applies to a large variety of events, including stock splits, earning surprises and analyst recommendations. The short-term momentum effect means that there is a positive short-term autocorrelation of returns on individual stock and whole markets. It is especially visible for small companies. The long-term reversals are connected with negative autocorrelation of short-term returns separated by long lags. This phenomenon is connected with the long-term overreaction of the
stock prices. A market, gradually absorbing information from time $t$, usually ignores the subsequent portions of news, which disagree with the initial signal. Consequently, after some additional time, prices are driven too high or too low comparing to the initial signal and an adjustment is necessary. When agents recognise their mistake, they change positions, which usually initiate a reverse trend in prices. The next consequence of imperfect rationality is the unconditional excess volatility of asset prices relative to fundamentals, presumably connected with the non-linear relationship between asset prices and fundamentals (validity of this argument as anomaly challenging the rational present value models is discussed in Chapter III.3.5). Finally, the post earnings announcement price drifts are caused by the negative correlation between long horizon returns and recently published financial performance measures. This might be partially due to underreaction to the public information or expected lower returns in the future.

The psychological evidence of imperfect rationality provides number of stylised facts and justifications for commonly observed deviations from rational behaviour (for review of behavioural theories in finance see Shiller, 1999). The best known deviations include overconfidence, unjustified optimism, hindsight, overreaction to chance events, errors of preference and narrow framing (see Kahneman and Riepe, 1998). Overconfidence means that people tend to put too much weight to information based on their private signals or intuition and too little weight to public news or price history. Unjustified optimism is connected with the fact that people tend to find themselves lucky and not acknowledge the possibility of extremely negative outcomes. Hindsight means that agents attribute successes to their personal abilities, whereas failures are usually seen as consequences of random and unpredictable events. Overreaction to chance events is connected with human tendency to find patterns even in purely random data. This phenomenon, together with overconfidence, makes agents apply questionable patterns in the analysis of future returns (e.g. trend chasing). Errors of preference are due to undervaluing the probabilities of extremely unlikely events and overvaluing the probabilities of extremely likely events. Additionally, agents put too much weight to very unlikely events and too less weight to very likely events (see prospect theory in Shiller,
Finally, narrow framing means that people tend to analyse investment decisions separately, instead of taking decisions in broad terms.

The behavioural finance has been one of the most active areas in asset pricing during the 1990s (Campbell, 2000). However, the proposed models can usually neither be tested using data on the aggregate consumption and market portfolio nor be verified in the general equilibrium framework. Fama (1998) argues that such an approach is unacceptable. The standard scientific rules imply that the traditional asset pricing models can only be replaced by a better specific model of price formation, itself potentially rejectable in empirical tests. However, recently at least three such models are proposed. All of them concentrate on irrational expectations rather than limits in arbitrage and all are able to generate price paths that are far from fundamentals as defined in the equation (3.3).

Barberis et al (1998) propose a model with one representative investor and one asset. The earnings of the asset follow a random walk but the investor does not know that and assumes two different models of dividends:

<table>
<thead>
<tr>
<th>Model 1</th>
<th>( B_{t+1} = B )</th>
<th>( B_{t+1} = -B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_t = B )</td>
<td>( \pi_l )</td>
<td>1-( \pi_l )</td>
</tr>
<tr>
<td>( B_t = -B )</td>
<td>1-( \pi_l )</td>
<td>( \pi_l )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Model 2</th>
<th>( B_{t+1} = B )</th>
<th>( B_{t+1} = -B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_t = B )</td>
<td>( \pi_H )</td>
<td>1-( \pi_H )</td>
</tr>
<tr>
<td>( B_t = -B )</td>
<td>1-( \pi_H )</td>
<td>( \pi_H )</td>
</tr>
</tbody>
</table>

Models 1 and 2 are Markov processes in the sense that the shock to earnings at time \((t+1)\), \( B_{t+1} \), depends only on the value taken by \( B_t \). The crucial difference between the models lies in the transition probabilities; namely \( \pi_l \in (0, 0.5) \) and \( \pi_H \in (0.5,1) \). In Model 1, a positive shock is likely to be reversed, whereas in Model 2 positive shock is likely to be followed by another positive shock. The investor believes that she knows \( \pi_l, \pi_H \) and the transition matrix governing switching between Model 1\((m_t=1)\) and Model 2 \((m_t=2)\):

<table>
<thead>
<tr>
<th></th>
<th>( m_{t+1}=1 )</th>
<th>( m_{t+1}=2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_t=1 )</td>
<td>1-( \lambda_1 )</td>
<td>( \lambda_1 )</td>
</tr>
<tr>
<td>( m_t=2 )</td>
<td>( \lambda_2 )</td>
<td>1-( \lambda_2 )</td>
</tr>
</tbody>
</table>

If \( \lambda_1 \) and \( \lambda_2 \) are small, than the transition from one model to the other occurs rarely. In this framework, a string of good or bad news can make investors switch from the mean reverting Model 1 to the trending Model 2, so the model can produce
both underreaction and overreaction, depending on recent news. Consequently, if the investor believes that earnings are generated by the above regime-switching model, the asset price is equal to:

\[ P_t = \frac{D_t}{\mu} + B_t(p_1 - p_2q_t) , \]

where \( D_t \) are dividends, \( \mu \) is the dividend growth rate,

\[
q_{t+1} = \frac{((1 - \lambda_1)q_t + \lambda_2(1 - q_t))\pi_L}{((1 - \lambda_1)q_t + \lambda_2(1 - q_t))\pi_L + (\lambda_1q_t + (1 - \lambda_2)(1 - q_t))\pi_H} \quad \text{for } B_{t+1} = B_t ,
\]

\[
q_{t+1} = \frac{((1 - \lambda_1)q_t + \lambda_2(1 - q_t))(1 - \pi_L)}{((1 - \lambda_1)q_t + \lambda_2(1 - q_t))(1 - \pi_L) + (\lambda_1q_t + (1 - \lambda_2)(1 - q_t))(1 - \pi_H)} \quad \text{for } B_{t+1} \neq B_t
\]

and \( p_1 \) and \( p_2 \) are functions of \( \mu, \lambda_1, \lambda_2, \pi_L \) and \( \pi_H \) (see Barberis et al, 1998: Appendix).

**Figure 3.13. Barberis-Shleifer-Vishny price generating process**

Figure shows realisation of price generating process (3.16) for \( \mu = 0.00005, \pi_L = 0.05, \) and \( \pi_H = 0.95 \). For Series 1 \( \lambda_1 = \lambda_2 = 0.5 \), for Series 2 \( \lambda_1 = \lambda_2 = 0.05 \) and Series 3 \( \lambda_1 = \lambda_2 = 0.95 \).

Figure 3.13 shows behaviour of exemplary processes of the type (3.16). The increase in the agents' beliefs in the two-regime model with the domination of price reversals, represented by higher values of \( \lambda_1 \) and \( \lambda_2 \), causes the variability of the process to increase. Similarly, low values of \( \lambda_1 \) and \( \lambda_2 \) make the series smoother and less volatile than the neutral Series 1. Processes of type (3.16) can generate both underreacting and overreacting series. Hence, the variability of prices may be lower than the variability of fundamental values. This resembles the behaviour of intrinsic
bubbles, which could both underreact and overreact to the coming news (see Froot and Obstfeld, 1991).

Daniel et al (1998) explore the fact that stock prices overreact to private information and underreact to public signals. The agents are overconfident in the sense, that they overestimate the precision of private signals. The investors split into those who obtain private signals (the informed ones) and those who do not have any non-public signals (uninformed ones). The informed investors are risk neutral and the uninformed investors are risk averse. Each informed investor has a prior on the precision of her private signal and uses an updating rule that reflects self-attribution bias. If the public signal confirms private information, their self-confidence increases. When the public signal contradicts private information, their self-confidence remains unchanged or decreases only slightly. The unobservable real value of the firm stock is denoted by $s$ and has normal distribution with mean 0 and standard deviation $\sigma_s$. At time 1, each informed investor receives a private signal $s_i = s + \varepsilon$ where $\varepsilon \sim \text{N}(0, \sigma^2)$. At times 2 through $T$, public signals $\phi = \theta + \eta_t$ are released, where $\eta_t \sim \text{N}(0, \sigma^2)$ is i.i.d. The values of $s$ and $\sigma_s$ are common knowledge. It is assumed that, except for the perception of private information precision, investors form expectations about the stock value rationally, using Bayesian updating. Consequently, the price of a security at time $t$ is given by:

$$P_t = \frac{(t-1)v^*_n \Phi_t + v_{C,t} s_t}{v^*_n + v_n + v_{C,t}},$$

where $v_n = 1/\sigma_n^2$, $v_i = 1/\sigma_i^2$ and

$$\Phi_t = s + \frac{1}{(t-1)} \sum_{k=2}^{t} \eta_k,$$

$$v_{C,t} = (1 + k_1)v_{C,t-1} \quad \text{if} \quad \text{sign}(s_t - \Phi_{t-1}) = \text{sign}(\phi_t - \Phi_{t-1}) \quad \text{and} \quad |s_t - \Phi_{t-1}| < 2\sqrt{\sigma_n^2 / (t-1)}$$

Otherwise: $v_{C,t} = (1 - k_2)v_{C,t-1}$.

The ratio $(1+k_1)/(1-k_2)$ is interpreted as an index of the investor's attribution bias.
Figure 3.14 Daniel-Hirshleifer-Subrahmanyam price generating process

Figure shows a path of price generating process (3.17). The values of parameters have been set as follows: \( \sigma_0 = \sigma_1 = 1, \sigma_2 = 7.5 \). Investors initial estimate of her precision is equal to the true precision of the private signal: \( v_{C1} = 1/\sigma_0 = 1 \). Series 1 does not represent any investor's attribution bias \((k_1 = k_2 = 0)\). In Series 2 and 3 \( k_1 = 0.2, k_2 = 0.05 \) and \( k_1 = 0.5, k_2 = 0.2 \) respectively.

Figure 3.9 shows behaviour of three processes generated by (3.17). Generally, increase in the investor's attribution bias causes prices to deviate further from unbiased values represented by Series 1. This relationship is, however, more complicated. In some cases, series less variable than the unbiased value can be obtained. Realisations of process (3.17) are similar to stochastic and intrinsic bubbles defined by (3.13) and (3.14). Rapid movements in prices, implied by some of the Barberis-Shleifer-Vishny models, are generally replaced by smoother shifts.

Barberis et al (1998) and Daniel et al (1998) assume that prices are driven by a single representative agent and concentrate on cognitive biases that this agent may have. Hong and Stein (1999) take a slightly different approach. In their model less action comes from cognitive biases of traders and more from the way these traders interact with one another. They divide agents into ‘newswatchers’ and ‘momentum traders’. The newswatchers make forecasts based on privately observed signals, but they do not analyse the current or past prices. Private information diffuses gradually across the newswatcher population. Momentum traders condition on past prices, but do not look at fundamentals. Their forecasts are simple univariate functions of the history of past prices. If only newswatchers act on the market, the price is given by:

\[
P_t = D_t + (z - 1)e_{z,1} + (z - 2)e_{z,2} + \ldots + e_{t+1,1}) / z - Q.
\]
where $\varepsilon_t$ is the news at time $t$. $D_t = D_0 + \sum_{j=0}^{t} \varepsilon_j$ is the fundamental value and $Q$ is the constant supply of assets. Unlike newswatchers, momentum traders have finite horizons. They enter the market at time $t$ and hold their position until $t+j$. They analyse the cumulative change in price over the last $k$ periods. For $k=1$, the order flow from generation $t$ momentum traders is of the form:

$$F_t = A + \phi \Delta P_{t-1},$$

where $A$ and $\phi$ vary among individual momentum traders and $\Delta P_{t-1} = P_{t-1} - P_{t-2}$ is the last session price increase. Consequently, when newswatcher and momentum trader interact on the market, the prices are generated by the following process:

$$P_t = D_t + \left( (z-1)\varepsilon_{r+1} + (z-2)\varepsilon_{r+2} + ... + \varepsilon_{r+j-1} \right) / z - Q + jA + \sum_{i=1}^{j} \phi \Delta P_{t-i}, \quad (3.18)$$

where $j$ determines the time horizon of momentum traders.

**Figure 3.15: Hong-Stein price generating process**

Figure shows price paths generated according to (3.18). The parameters have been fixed as follows: $A=0$, $Q=0.5$, $\text{std}(\varepsilon)=0.5$, $D_0=1$ and $\phi=0.1293$. For the first series $z=j=3$. In Series 2 and 3 values of these parameters increase respectively to $z=j=6$ and $z=j=9$.

Figure 3.15 shows price paths of three processes generated according to (3.18). Higher values of the information diffusion parameter $z$ and the momentum traders’ horizon $j$ cause the variability of series to increase. The price paths form characteristic ups and downs, which resemble the behaviour of stochastic bubbles.

The presented models show that substantial movements in stock prices may be caused by various factors connected with investors’ behaviour. The possible reasons
are strong beliefs in the false model of financial markets, overvaluation of private signals and intensive momentum trading.

The main limitation of the above models is that, in contradiction to rational bubbles, they do not consider equilibrium issues (Campbell, 2000). If they are applicable only to individual assets, then the misvaluations they produce are diversifiable and can be arbitraged by rational agents. On the other hand, if the models apply to the whole market, then it is important to consider implications for consumption. However, it is relatively hard to build a general equilibrium model with irrational expectation. Fama (1998) argued additionally that these models work well only on the anomalies they are designed to explain, failing badly in other situations.

However, to their credit, it must be mentioned that they provide at least some explanation of the deviations from present value models. This explanation seems to be much more convincing than pure chance or commonly used time-varying risk premium. In the latter case, Campbell and Cochrane (1999) showed that a utility function with extreme habit persistence is required to explain the predictable variations in market returns using stochastic risk premium.

The behavioural theory offers an alternative justification of stock price variability, which should be taken in account when interpreting rapid changes in asset prices. Speculative processes developed in this framework form an interesting extension of the rational bubbles presented in Chapter III.3.4.

III.4. Diba-Grossman and related processes

III.4.1. Derivation

This section discusses the price generating processes, as proposed by Diba and Grossman (1988). These processes assume speculative behaviour of investors and neglect the impact of fundamentals. This is a very strong assumption, being in opposition to those of many classical asset pricing models. Nevertheless, it is believed that such processes provide a reasonable approximation to reality and, in particular, allow conclusions to be drawn about the speculative character of analysed assets and
markets. First, the derivation of the Diba-Grossman process is presented. Its description is partially based on Charemza and Kominek (1999). Then, the possible interpretations of its parameters are discussed. Finally, some estimation techniques are suggested.

Following Diba and Grossman (1988), the representative household’s utility function is:

\[ V = E_t \left[ \sum_{t=0}^{\infty} (\beta)^{t} u(cons_{t}) \right] , \]  

(3.19)

where \( \beta \) is the discount factor, \( cons_{t} \) is the logarithm of stochastic consumption, \( u(\cdot) \) is the conventionally defined utility function for a normal (non-Giffen) good and \( E_t(\cdot) \) denotes expectations taken at time \( t \) based on all information available at that time^{8}. The budget constraint for each period \( \tau \) is:

\[ cons_{t} + p_{\tau}(s_{\tau+1} - s_{\tau}) \leq y_{\tau} , \]  

(3.20)

where \( y_{\tau} \) is the fixed endowment of income, \( s_{\tau} \) is the logarithm of shares acquired by the household, and \( p_{\tau} \) is the dividend-adjusted logarithm of share prices. The decision of the household is aimed at smoothing consumption by buying shares \( (s_{\tau}) \) in order to maximise the utility function.

The Lagrangian for the model (3.19) - (3.20) is:

\[ L = E_t \left[ \sum_{t=0}^{\infty} (\beta)^{t} u(cons_{t}) \right] + \lambda_{1}[cons_{t} + p_{t}(s_{t+1} - s_{t}) - y_{t}] + \lambda_{2}[cons_{t+1} + p_{t+1}(s_{t+2} - s_{t+1}) - y_{t+1}] \]

the first-order conditions are:

\[ u'(cons_{t}) + \lambda_{1} = 0 \Rightarrow \lambda_{1} = -u'(cons_{t}) , \]

\[ \beta u'(cons_{t+1}) + \lambda_{2} = 0 \Rightarrow -\lambda_{2} = -\beta u'(cons_{t+1}) , \]

and

\[ \lambda_{1} p_{t} - \lambda_{2} p_{t+1} = 0 . \]

Substituting and applying the expected values for \( p_{t+1} \) and \( cons_{t+1} \) leads to:

\[ -p_{t} u'(cons_{t}) + \beta E_t[p_{t+1} u'(cons_{t+1})] = 0 , \]

or, provided that the distributions of \( p_{t} \) and \( cons_{t} \) are independent:

\[ p_{t} u'(cons_{t}) = \beta E_t[p_{t+1} u'(E_t(cons_{t+1}))]. \]  

(3.21)

^{8} Note that the notation accepted in this section slightly differs from the earlier analyses.
The left-hand side of (3.21) can be interpreted as the marginal utility from selling (buying) a share in the current period, while the right-hand side represents the marginal utility from selling (buying) a share next period. It is clear that if in (3.21) the discounted ratio of the next period expected marginal utility of consumption is equal to the current marginal utility of consumption, that is, if:

$$u'(\text{cons}_t) = \beta u'[E_t(\text{cons}_{t+1})]$$

then the process of price formation can be regarded as the martingale, usually leading either to a random walk model of pricing behaviour or, under the additional assumption of risk neutrality, to an ARCH-type model. Both are consistent with the market efficiency hypothesis (see LeRoy, 1989). This would be the case where (a) the consumers develop expectations regarding marginal utilities, (b) distributions of prices and marginal utilities are independent and (c) current and discounted expected future marginal utilities are identical. These are generally rather tight conditions; in particular it is arguable whether the consumers can develop rational expectations regarding marginal utilities. It seems to be more realistic to assume that the ratio of the discounted future marginal utility to current marginal utility is described by a random variable with the expected value equal to one. In the simplest case this is given by a normal distribution, that is:

$$u'(\text{cons}_t) / \beta u'[E_t(\text{cons}_{t+1})] = \theta_0(\sigma_0^2) \sim IIDN(1, \sigma_0^2)$$

where $IIDN(1, \sigma_0^2)$ stands for a series of identically and independently distributed random normal variables with a unitary mean and variance $\sigma_0^2$. This gives:

$$E_t(p_{t+1}) = \theta_0(\sigma_0^2) \cdot p_t$$

and finally the price formation process:

$$p_{t+1} = \theta_0(\sigma_0^2) \cdot p_t + r_{t+1}^\epsilon$$

where it is convenient to assume that $r_{t+1}^\epsilon \sim IIDN(0, \sigma_\epsilon^2)$. Formally, equation (3.23) is a stochastic root process, where the stochastic root is equal to $\theta_0(\sigma_0^2)$. This process is called herein the Diba-Grossman process (for generalisations see Charemza and Deadman, 1995 and Granger and Swanson, 1997) and $r_{t+1}^\epsilon$ is called the Diba-Grossman (DG) return. If the variance of the stochastic root is equal to zero, then $\theta_0(0) = 1$ and equation (3.23) becomes a random walk. Observed returns, defined as $r_{t+1} = p_{t+1} - p_t$, become equal to the DG returns $r_{t+1}^\epsilon$ and hence the entire market can
be regarded as non-speculative. If, however, $\sigma^2 > 0$, then evidently $r_{t+1} = p_{t+1} - p_t \neq \theta(\sigma^2) p_t$, where $\neq$ reads 'not equal in distribution'.

Figure 3.16 clearly shows that the variation of prices increases with the variance of the stochastic root. Also, these processes show characteristic fluctuations in the series, which might be interpreted as rises and bursts of a speculative bubble. Differences between the processes are also evident in Figure 3.17, showing distributions of observed returns (that is, first differences of each series). For $\sigma^2 = 0$ the distribution is approximately normal, while for the series with $\sigma_o^2 = 0.025^2$ and $\sigma_o^2 = 0.05^2$ there is a clear heavy tailed effect. Further experiments show that, with an increase in $\sigma^2$, the speculative nature of the series becomes more evident; periods of substantial growth and spectacular falls of the simulated prices are more frequent and of a larger amplitude.

**Figure 3.16: Diba-Grossman processes with normally distributed coefficients**

Figure shows series simulated by (3.23), on the basis on the same series of random numbers. The series are generated for $r_t = \text{IIDN}(0,0.01)$ and $p_0 = 1$. For one of the series the variance of the stochastic root (stdtet) is set at zero, so that the process degenerates to a random walk, and for the others $\sigma^2 = 0.025^2$ and $\sigma^2 = 0.05^2$

source: Charemza and Kominek (1999)
Figure 3.17: Returns to the Diba-Grossman processes with normal variables

Figure shows distributions of returns to processes presented on Figure 3.16. The returns are calculated as the first differences of the series.

III.4.2. Interpretation of $\sigma^2$ and $\sigma_{re}^2$

According to (3.22), the stochastic parameter $\theta_i(\sigma^2)$ has been defined as the ratio of the current marginal utility of consumption to the discounted next period expected marginal utility of consumption. The marginal utility depends on the level of consumption $\text{cons}_i$. Additionally, the form of this function can vary in time. As the daily discount factor seems to be relatively constant and not significantly different from one, it is desirable to concentrate mainly on role of the current and expected marginal utility functions in explaining the variability of $\theta_i(\sigma^2)$.

In this context, there seem to be two alternative interpretations of such variability in the literature; approach grounded within the theory of rational behaviour represented e.g. by Campbell and Cochrane (1999) and the hypothesis of non-rational myopic behaviour, developed from psychological sciences. Campbell and Cochrane develop a model, in which consumption is amended by an external or internal habit. If the habit is external, then it is suggested that the marginal utility, as in (3.22), is given by:

$$u'(\text{cons}_i) = (\text{cons}_i - \varphi_i)^\gamma,$$

where $\varphi_i$ is the level of habit and $\gamma > 0$. Defining the surplus consumption ratio $\delta_i$ as:
\[ \phi_i = (\text{cons}_t - \phi_t)/\text{cons}_i \, . \]

It can be noticed that:
\[ u'(\text{cons}_t) = \phi_i^\gamma \text{cons}_t^\gamma \, , \]
and, after substituting this to (3.22):
\[ \theta_i(\sigma_\theta^2) = [ (\phi_i/ E_i(\phi_{t+1})) \times (\text{cons}_i/E_i(\text{cons}_{t+1})) ]^\gamma / \beta \, . \]

Campbell and Cochrane (1999) argue that the variability of \((\text{cons}_i/E_i(\text{cons}_{t+1})\) is low. Hence, the variability of \(\theta_i(\sigma_\theta^2)\) is caused mainly by changes in \((\phi_i/ E_i(\phi_{t+1}))^\gamma\). This leads to the conclusion that a large part of the variability of prices is connected with shifts in habits coming from changes in the level of aggregate consumption. This seems to be relevant for the analysis of long time series, where changes in habits in aggregate consumption can be observed.

The alternative approach, which does not require the assumption of rationality of investors, is developed in the psychological sciences (for literature review, see e.g. Tversky and Kahneman, 1986, 1991). The theory of myopic behaviour, also called 'narrow framing'\(^9\), implies among others that investment decisions are not based on changes in total possible consumption level, but rather on the evaluation of the variability and possible changes of asset prices. Consequently, investors' utility is connected with certain equity and usually does not depend on total welfare, but rather on the current and initial (purchasing) prices of this security. Adopting the myopic behaviour hypothesis to model (3.21), (3.22) can be replaced by:
\[ \theta_i(\sigma_\theta^2) = u'(p_i) / \beta u'[E_i(p_{t+1})] \, . \]

If, additionally, the valuation of changes instead of states is included in the analysis, then \(\theta_i(\sigma_\theta^2)\) is given by:
\[ \theta_i(\sigma_\theta^2) = u'(p_i - p_p) / \beta u'[E_i(p_{t+1} - p_p)] \, , \]
where \(p_p\) is a reference level of price, usually equal to the purchasing price of the analysed equity. Consequently, (3.23) can be rewritten as:
\[ p_{t+1} - p_p = u'(p_i - p_p) / \beta u'[E_i(p_{t+1} - p_p)] \cdot (p_i - p_p) + r_{t+1}^\varepsilon \, . \quad (3.24) \]

---

The utility function \( u'(p_t - p_p) \) is known in the literature as the value function (see e.g. Markowitz, 1952 and Tversky and Kahneman, 1986). Following Tversky and Kahneman (1986) the loss aversion assumption implies that the value function is generally convex for losses and concave for gains and is steeper for losses than for gains. Assuming \( \beta = 1 \), this gives the following four regimes:

(i) If \( p_t - p_p > 0 \) and \( E_i(p_{t+1} - p_p) > p_t - p_p \):

\[
\text{then } u'[E_i(p_{t+1} - p_p)] < u'(p_t - p_p) \text{ and } \theta_i(\sigma^2) > 1 ,
\]

(ii) If \( p_t - p_p > 0 \) and \( E_i(p_{t+1} - p_p) < p_t - p_p \):

\[
\text{then } u'[E_i(p_{t+1} - p_p)] > u'(p_t - p_p) \text{ and } \theta_i(\sigma^2) < 1 ,
\]

(iii) If \( p_t - p_p < 0 \) and \( E_i(p_{t+1} - p_p) > p_t - p_p \):

\[
\text{then } u'[E_i(p_{t+1} - p_p)] < u'(p_t - p_p) \text{ and } \theta_i(\sigma^2) < 1 ,
\]

(iv) If \( p_t - p_p < 0 \) and \( E_i(p_{t+1} - p_p) < p_t - p_p \):

\[
\text{then } u'[E_i(p_{t+1} - p_p)] > u'(p_t - p_p) \text{ and } \theta_i(\sigma^2) > 1 .
\]

In other words, the expectations of an increase in prices causes them to rise, and the expected decrease of prices causes them to fall. Hence, if prices are given by (3.24), \( \theta_i(\sigma^2) \) is responsible for generating self-fulfilling expectations (e.g. bubbles), while \( r_{t+1}^\varepsilon \) represents stationary random shocks to the system. Consequently, it might be conjectured that the greater the variability of \( \theta_i(\sigma^2) \), the more speculative the market.

To sum up, it seems to be plausible to assume that values of \( \sigma_\theta \) can be used to compare the degrees of speculation on different markets or on the same market in different time periods. The value of \( \sigma_\varepsilon \) provides information about the size of random shocks to the market. It can be treated conventionally as a measure of risk, not connected with self-fulfilling expectations. Following these interpretations, the values of \( \sigma_\theta \) and \( \sigma_\varepsilon \) can be used to determine the reasons of shifts in the asset prices on various markets and in different time periods.
III.4.3. Estimation of parameters

Model (3.23) is an example of doubly stochastic process introduced by Tjostheim (1986), specifically analysed by Brandt (1986) and Pourahmadu (1986,1988) and recently reviewed by Granger and Swanson (1997). Processes of this type can be estimated by a number of methods. The first of them is the approximate maximum likelihood proposed by Granger and Swanson (1997). The second one is the Kalman filter technique, as described e.g. in Hamilton (1994).

Following Granger and Swanson (1997), the stochastic unit root processes of the form:

\[ p_{t+1} = \theta_t p_t + r_{t+1}, \]

where \( \theta_t \sim N(1, \sigma_\theta) \) can be estimated by maximising the following likelihood function:

\[ L' = \sum_{k=1}^{K} \left( \frac{1}{\sqrt{2\pi}} \right)^n \sigma_n^{-1} \exp \left\{ \frac{1}{2\sigma_n^2} \sum_{t=1}^{T} \left( \hat{r}_{t,k} \right)^2 \right\}, \tag{3.25} \]

where

\[ \hat{r}_{t+1,k} = p_{t+1} - \hat{\theta}_{k,t} p_t, \]

and \( \sigma_n^2 \) is variance of \( r_t \), \( n \) is the sample size and \( K \) is the number of realisations of \( \theta_t \) that are used to approximate \( \hat{r}_{t+1,k} \). In this procedure, a \( T \times K \) matrix of random numbers with unit mean and variance \( \sigma_\theta^2 \) is drawn and \( K \) different realisations of \( \theta_t \) are obtained for each set of values of other parameters. The likelihood function \( L' \) weights the results obtained for different realisations of \( \theta_t \), so this random element does not affect the estimates of other parameters (apart from \( \sigma_\theta^2 \)). However, it is not feasible to take logarithms of the function (3.25) and therefore for large sample size \( T \) and small standard deviation \( \sigma_\theta \) serious computation problems occur (values of \( L' \) are not significantly different from zero).

The second possible estimation strategy applies the state space Kalman filter. The Kalman filter is an algorithm that sequentially updates a linear projection for the system. It expresses a dynamic system in a particular form, called the state-space representation. Among others, the Kalman filter provides a way to calculate exact finite-sample forecasts and the exact likelihood function for Gaussian ARMA processes and to estimate vector autoregressions with coefficients that change over
time (Hamilton, 1994: Chapter 13). In this section, the Kalman filter is applied to estimate the parameters of the Diba-Grossman price generating process.

The observation and state equation take the forms:

\[ p_t = p_{t-1} \theta_t + r_t \]

Observation equation

\[ \theta_t = 1 + \eta_t \]

State equation

and

\[
\begin{bmatrix}
\varepsilon_t \\
\eta_t
\end{bmatrix}
\sim
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\begin{pmatrix}
\sigma_{r}^2 & 0 \\
0 & \sigma_\theta^2
\end{pmatrix}.
\]

After simple calculations, the likelihood function can be reduced to:

\[
L' = \prod_{t=1}^{T} \left(2\pi\right)^{-\frac{\nu}{2}} \left|F_{y_{t-1}}\right|^{-\frac{\nu}{2}} \exp\left\{-\frac{1}{2}(y_t - y_{t-1}) F_{y_{t-1}}^{-1}(y_t - y_{t-1})\right\},
\]

where

\[
y_{t-1} = y_{t-1}
\]

\[
F_{y_{t-1}} = y_{t-1}\sigma_\theta^2 y_{t-1} + \sigma_{r}^2.
\]

Consequently, it equates to:

\[
L' = \prod_{t=1}^{T} \left(2\pi\right)^{-\frac{\nu}{2}} \left|y_{t-1}\sigma_\theta^2 y_{t-1} + \sigma_{r}^2\right|^{-\frac{\nu}{2}} \exp\left\{-\frac{1}{2}(y_t - y_{t-1})^2\left(y_{t-1}\sigma_\theta^2 y_{t-1} + \sigma_{r}^2\right)^{-1}\right\}.
\]

Maximising of \( L' = \ln(L') \) leads to unbiased maximum likelihood estimates of \( \sigma_\theta^2 \) and \( \sigma_{r}^2 \).

The problem of this technique is that it does not provide an accurate way to compute the standard deviations of the estimates of \( \sigma_\theta^2 \) and \( \sigma_{r}^2 \). One way would be to use the asymptotic properties of the maximum likelihood estimates and evaluate the covariance matrix of the estimates from Hessian of the log likelihood function. This is a reasonable approach when the analysed series are simulated by the process under consideration. However, if the real data is investigated and the assumed process is affected by additional noise, the covariance matrix obtained in this way is rather inaccurate and, in particular, the standard deviations tend to be biased downwards.
III.5. Conclusions

Speculation, defined as investing to gain on the intervening price changes, seems to be common on the financial markets. If a substantial proportion of agents share the same views, an excessive demand (supply) may occur and prices can shift rapidly. In such situations, the classical asset pricing models, based on the fundamental values of stocks, often fail to provide an accurate explanation of reality.

One of the most popular alternative theories developed in the last two decades is the speculative bubble hypothesis. The theoretical literature has not yet agreed on an exact definition of the speculative bubble but certain features seem to be common for all descriptions of this phenomenon. Speculative bubbles are usually identified with large-scale speculations leading to rapid changes in prices and departures from fundamental values of assets. There is no agreement about the possibility of existence of speculative bubbles and no universal test is developed to verify empirically their presence. Consequently, speculative bubbles tend to remain a controversial issue, dividing both the academics and financial practitioners.

The traditional speculative processes, i.e. processes that are capable of producing speculative bubbles, are developed in the rational expectation framework. The rational bubbles can be described as non-standard solutions to the present value relation for asset prices. They usually result in non-negative series, which can be more or less volatile than the market fundamentals. The rational bubbles produce series that closely resemble those, observed on many financial markets. The estimated parameters of such models may give information on the level of market speculation.

Additional intuition about the nature of speculation can be gained from the analysis of non-rational speculative processes. Such models, proposed e.g. by the behavioural finance theory, concentrate on the possibly non-rational motives of individual investors' behaviour. They explain the excessive volatility, often interpreted as speculative bubbles, by e.g. overconfidence in private information, beliefs in false models of economy and financial markets and substantial share of trading motivated purely by the observation of recent price quotations.
Good economic models are usually characterised by simplicity, intuitive interpretation and accurate fit to the empirical data. The doubly stochastic speculative processes proposed by Diba and Grossman (1988) have only two parameters: the standard deviation of the stochastic root of the process and the standard deviation of the additive error term. The first one can be interpreted as a measure of the market (asset) exposure to speculation, whereas the second one acts as a proxy for the exposure to random shocks. The Diba–Grossman speculative processes produce heavy tailed returns, similar to those observed on the financial markets. They generate series with characteristic ups and downs, resembling the speculative bubbles. Hence, the Diba-Grossman processes should form an attractive tool for economic analysis. Their applicability to the empirical series is limited by technical problems with the estimation of parameters. These problems are mainly due to two stochastic components in the series: the stochastic root and additive error term. Even if their standard deviations can be estimated, the precision of such estimates is difficult to evaluate. Nevertheless, low parametrisation, clear interpretation and simulated series, similar to the real markets’ ones, make the Diba–Grossman processes a useful tool in the analysis of financial data.

Future research should determine the applicability of other rational and non-rational speculative processes to analyse the empirical series. New, more accurate models can be proposed. Clear interpretations of parameters should be developed and series generated by various models have to be analysed. Accurate estimation techniques should be provided to fit speculative models to the empirical data.
Chapter IV

SPECULATIVE PROCESSES AND STABLE DISTRIBUTIONS: A NEW APPROACH

IV.1. Introduction

IV.2. Distribution of returns to Diba-Grossman processes
   IV.2.1. The stable hypothesis
   IV.2.2. The goodness of fit tests
   IV.2.3. Verification of stable hypothesis

IV.3. Parameters of Diba-Grossman process and stable distributions
   IV.3.1. Main concepts
   IV.3.2. Estimation through simulations
   IV.3.3. Bivariate character of the relationship

IV.4. Conclusions
IV.1. Introduction

The entire group of tests for speculative bubbles is connected with the distribution of returns to asset prices (see Chapter III.3.5). The intuition behind these tests suggests that the distribution of returns to processes with speculative bubbles should be heavy tailed. This is mainly because crashes and rapid price growths, which are characteristic for many of the discussed processes, produce large outliers, making the distribution of returns leptocur tic. The main weakness of this methodology is that the innovations to market fundamentals may be heavy tailed themselves. In such a situation, the distribution-based tests could wrongly identify fundamental prices as financial bubbles. Additionally, there are bubbles that do not need to result in heavy tailed distribution themselves and their variability may be even lower than the variability of the fundamental prices (see Chapter III.3.4.).

Despite these inaccuracies, the distribution-based tests imply that the existence and magnitude of speculative bubbles might be connected with heavy tailed distribution of returns. The character of this relationship depends on the specification of particular processes and may be different for each of the models presented in Chapter III. Therefore, the possible relationships between price generating processes and distributions of returns should be investigated separately for each model and general conclusions should be avoided before a comprehensive study is completed.

In this chapter, the main attention is given to the speculative processes of the Diba-Grossman type (see Chapter III.4) and to the distributions of returns they produce. First, the hypothesis that Diba-Grossman price generating process result in stable distributed returns is verified and the accuracy of fit of other possible distributions of returns (normal and t-Student) is tested. Then, the relationship between characteristic exponent and scale parameter of the stable distribution from one side and parameters of Diba-Grossman process from the other side is established and appropriate conversion tables are produced. This section extends Charemza and Kominex’s (1999) results.
IV.2. Distribution of returns to Diba-Grossman processes

IV.2.1. The stable hypothesis

In Chapter III.4.1, it is stated that, following Diba and Grossman (1988), the process generating the logarithms of prices \( p_t \) on a speculative market is, under certain assumptions, given by:

\[
p_t = \theta_t \cdot p_{t-1} + r_t^e,
\]

where

\[
\theta_t \sim \text{IIDN}(1, \sigma_\theta^2),
\]

\[
r_t^e \sim \text{IIDN}(0, \sigma_{r_t^e}^2).
\]

The term \( \theta_t \) is called the stochastic root of the process and \( r_{t+1}^e \) is called Diba-Grossman (DG) return. The observed returns \( r_t \) are defined as:

\[
r_t = p_t - p_{t-1},
\]

As it has been stated in the previous chapter if \( \sigma_\theta^2 = 0 \), then the process \eqref{eq:4.1} becomes a random walk:

\[
p_t = p_{t-1} + r_t^e.
\]

By \eqref{eq:4.3}, this implies:

\[
r_t = r_t^e,
\]

and consequently, applying \eqref{eq:4.2b}, the observed returns \( r_{t+1} \) are normally distributed:

\[
r_t \sim \text{IIDN}(0, \sigma_{r_t^e}^2),
\]

and the process can be regarded as non-speculative.

However, if \( \sigma_\theta^2 > 0 \), then \eqref{eq:4.3} implies:

\[
r_t = (\theta_t - 1) \cdot p_{t-1} + r_t^e,
\]

and consequently:

\[
r_t \not\sim r_t^e,
\]

where \( \not\sim \) reads 'not equal in distribution'. Hence, the distribution of \( r_t \) does not need to be normal.

Indeed, for \( t \geq 2 \), \( r_t \) can be represented as a sum of products of independent normal random variables. Backwards substituting of \eqref{eq:4.1} in \eqref{eq:4.5} produces:

\[
r_t = (\theta_t - 1) \cdot (\theta_{t-1} \cdot p_{t-2} + r_{t-1}^e) + r_t^e.
\]
\[
= (\theta_i - 1) \cdot \theta_{i,1} \cdot p_{i,2} + (\theta_i - 1) \cdot r_i^e + r_i^e \\
= (\theta_i - 1) \cdot \theta_{i,1} \cdot \theta_{i,2} \cdot p_{i,3} + (\theta_i - 1) \cdot r_{i,2}^e + r_i^e \\
= (\theta_i - 1) \cdot \theta_{i,1} \cdot \theta_{i,2} \cdot p_{i,3} + (\theta_i - 1) \cdot \theta_{i,1} \cdot r_{i,2} + (\theta_i - 1) \cdot r_i^e + r_i^e \\
= (\theta_i - 1) \cdot \theta_{i,1} \cdot \theta_{i,2} \cdot p_{i,3} + \ldots + (\theta_i - 1) \cdot \theta_{i,k} + (\theta_i - 1) \cdot r_{i,k} + \ldots + (\theta_i - 1) \cdot r_i^e + r_i^e.
\]

Stuart and Ord (1994: p.401) state that variable \( r_n \), defined in such way, must have a heavy tailed distribution. However, the exact shape and probability distribution function remain unknown.

The crucial assumption of this study states that the observed returns \( r_n \), generated by a Diba – Grossman model (4.1), have a symmetric stable distribution \( S(\alpha, 0, c, \delta) \), denoted further by \( SS(\alpha, c, \delta) \) (Charemza and Kominek, 1999):

\[ r_t \sim SS(\alpha, c, \delta). \]

As it is stated in Chapter II, the symmetric stable distributions provide a flexible framework to model financial markets data. Their three parameters allow for separate control of location, dispersion and thickness of tails. The stable laws are a natural generalisation of the normal distribution and hence the conditions (4.4) and (4.5) can be rewritten as:

a) If \( \sigma^2 = 0 \), then \( r_t \sim SS(\alpha = 2, c, \delta) \).

b) If \( \sigma^2 > 0 \), then \( r_t \sim SS(\alpha < 2, c, \delta) \).

The above agrees with common intuition that heavy tailed returns are connected with speculative character of the underlying price generating processes (Peters, 1991).

Hypothesis (4.6) applies to all observed returns generated by Diba – Grossman processes. It imposes a parametric form on the distribution of returns to such processes and hence enables structured analysis of relationship between parameters of speculative processes and shape of the distribution of returns.
IV.2.2. The goodness of fit tests

The essential question, which should be answered before further analysis, is whether the hypothesis (4.6) is confirmed by the empirical distributions of returns to Diba – Grossman processes. In order to verify the null hypothesis that observed returns to (4.1) have symmetric stable distribution, the $\chi^2$ goodness of fit test is applied.

The theory of the $\chi^2$ tests is described in Chapter II.3.2. In this section, a $\chi^2$ test with equiprobable cells is applied. The range of observed returns is arbitrarily divided into $k = 30$ mutually exclusive cells. The cells are chosen to be equiprobable under the hypothesised symmetric stable distribution with estimated parameters. Hence, the probability $p_{oi}$ of an observation falling in each class is equal to $1/30$. The borders of the cells are evaluated by approximating the probability distribution function of stable law in 10,000 points between maximal and minimal values of $r_t$. This is done using McCulloch (1994) code, which applies Zolotarev’s (1983) integral representation of stable probability distribution function. Then, the observed frequencies $n_i$ in each cell are calculated and the $\chi^2$ statistic is computed:

$$\chi^2 = \sum_{i=1}^{k} \frac{(n_i - np_{oi})^2}{np_{oi}},$$

where $n$ denotes the sample size. The obtained numbers are compared with appropriate critical values and conclusions about the null hypothesis are formulated.

Before applying the above test, its main properties, i.e. power and size, should be established. The size of the test is equal to the probability of rejecting the true null hypothesis, whereas the power of the test is equal to the probability of not rejecting the false null hypothesis.

Theoretically, the size of $\chi^2$ tests is well known and should be equal to the respective significance levels. However, the critical values connected with the significance levels are derived basing on the asymptotic properties of analysed statistics. The small sample properties of such tests are generally not known and may differ from the asymptotic ones. Additionally, the probability of rejecting the true null hypothesis for $\chi^2$ tests may depend upon the number and character of cells used to determine the value of the test statistic. Hence, series of symmetric stable
distributed random numbers should be simulated and the size of the applied test should be established.

It is much more difficult to evaluate the power of a statistical test. The possibility of accepting the false null hypothesis depends upon the type of data generating process. If the data is generated by a model, which is very similar to the stable law, it is very likely that the \( \chi^2 \) test incorrectly identifies such a process with symmetric stable disturbances. However, in this study it is not argued that the returns to Diba-Grossman process follow an exact stable distribution. It is rather claimed that a symmetric stable distribution is an accurate approximation of the first differences of a speculation process of type (4.1). Hence, it is worth testing whether some other well known distributions do not approximate the returns to Diba – Grossman process in a better way.

To evaluate the size of the applied \( \chi^2 \) test, 1000 series of symmetric stable distributed variables are simulated for different values of characteristic exponent \( \alpha \) and dispersion parameter \( c \), according to the algorithm described in Chapter II.3.3. Each series has 1000 observations. The parameters of symmetric stable distributions of returns are estimated by McCulloch (1986) quantile method. The \( \chi^2 \) goodness of fit test is applied to verify the null hypothesis that the distributions of the variables is symmetric stable.

Table 4.1 shows the average fraction of non-rejected null hypotheses at 1%, 5% and 10% significance levels. The \( \chi^2 \) test seems to perform fairly well, with just slightly too many rejections for high values of characteristic exponent \( \alpha \). For the nominal significance level of 1%, the real size of the test varies from 0.5% to 2.8%. The smallest size is noted for heavier tailed series with low value of the dispersions parameter. The highest one is observed for highly dispersed returns with distribution close to the normal one. Similar results are obtained for 5% and 10% significance levels. Generally, the \( \chi^2 \) test proves to be an appropriate test to verify the null about symmetric stable distribution and the choice of 30 equiprobable cells seems to be good decision.
Table 4.1: The $\chi^2$ test for the symmetric stable distribution.

The table shows the percentages of the non-rejected true null hypothesis about the symmetric stable distribution of the data. The null hypothesis is verified by $\chi^2$ test with 30 equiprobable cells. Data is simulated for different characteristic exponents (first column) and scale parameters (second row) of the symmetric stable distributions. Results for different level of significance (first row) are reported.

<table>
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<th>$\alpha \backslash c$</th>
<th>Significance level 1%</th>
<th>Significance level 5%</th>
<th>Significance level 10%</th>
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<td>0.03</td>
</tr>
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<td>99.2%</td>
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</tr>
<tr>
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<td>98.1%</td>
<td>98.0%</td>
<td>97.2%</td>
</tr>
</tbody>
</table>

IV.2.3. Verification of stable hypothesis

To test the validity of the null hypothesis about symmetric stable distribution of returns to (4.1), the $\chi^2$ test described in the previous section is applied. To verify (4.6), the following Monte Carlo experiment is designed. About 940,000 'daily' price series of the length of 1,000 each are simulated from (4.1) for different values of $\sigma_\theta$ and $\sigma_{re}$. The values of $\sigma_\theta$ have increase from 0 with step 0.01 and values of $\sigma_{re}$ increase from 0.001 with step 0.001, unless the variability of series is too excessive. That is unless, assuming that the simulated series represent session-by-session returns, 25% or more of the simulated processes exhibit average 'annual' growth higher than 100% or fall by more than 50%. Returns are computed as the first differences of $p_t$. The parameters of symmetric stable distributions of returns are estimated using the McCulloch (1986) quantile method (the use of more accurate methods like Nolan's (1999) maximum likelihood proved to be computationally too expensive for such Monte Carlo study). Then, the $\chi^2$ test is applied in order to verify the null hypothesis that the distributions of returns are symmetrically stable distributed. The obtained numbers are compared with the critical values for 1%, 5% and 10% significance levels and the empirical frequencies of non-rejection of the
null hypothesis about stable distribution of returns are computed. This is done for about 940 different combinations of \( \sigma_\theta \) and \( \sigma_{re} \). The results are grouped, clustering together 25 different combinations of parameters \( \sigma_\theta \) and \( \sigma_{re} \). Each cell in Tables 4.2-4.4 show the average fraction of non-rejected null hypotheses at respectively 1%, 5% and 10% significance levels for non-excessive Diba-Grossman processes and parameters \( \sigma_\theta \) and \( \sigma_{re} \) for individual groups (the first row and column of the table). Each presented fraction is based on 1,000-25,000 generated processes.

<table>
<thead>
<tr>
<th>( \sigma_{re} )</th>
<th>( \sigma_\theta )</th>
<th>0.00-04</th>
<th>0.05-09</th>
<th>0.10-14</th>
<th>0.15-19</th>
<th>0.20-24</th>
<th>0.25-29</th>
<th>0.30-34</th>
<th>0.35-39</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001-0.005</td>
<td>98.3%</td>
<td>97.8%</td>
<td>96.2%</td>
<td>95.1%</td>
<td>94.3%</td>
<td>94.0%</td>
<td>94.2%</td>
<td>94.3%</td>
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</tr>
<tr>
<td>0.006-0.010</td>
<td>98.3%</td>
<td>98.1%</td>
<td>97.5%</td>
<td>97.3%</td>
<td>97.3%</td>
<td>97.5%</td>
<td>97.5%</td>
<td>97.6%</td>
<td>97.4%</td>
</tr>
<tr>
<td>0.010-0.015</td>
<td>98.3%</td>
<td>98.3%</td>
<td>98.0%</td>
<td>98.4%</td>
<td>98.5%</td>
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<tr>
<td>0.016-0.020</td>
<td>98.2%</td>
<td>98.4%</td>
<td>98.6%</td>
<td>98.9%</td>
<td>99.1%</td>
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</tr>
<tr>
<td>0.021-0.025</td>
<td>98.2%</td>
<td>98.5%</td>
<td>98.8%</td>
<td>99.1%</td>
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<tr>
<td>0.026-0.030</td>
<td>98.1%</td>
<td>98.6%</td>
<td>98.8%</td>
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<tr>
<td>0.031-0.035</td>
<td>98.1%</td>
<td>98.6%</td>
<td>99.2%</td>
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<tr>
<td>0.036-0.040</td>
<td>98.3%</td>
<td>98.6%</td>
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<tr>
<td>0.041-0.045</td>
<td>98.1%</td>
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<tr>
<td>0.046-0.050</td>
<td>98.1%</td>
<td>98.6%</td>
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<tr>
<td>0.051-0.055</td>
<td>98.3%</td>
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<td>0.056-0.060</td>
<td>98.1%</td>
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<td>0.061-0.065</td>
<td>98.2%</td>
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<tr>
<td>0.066-0.070</td>
<td>98.2%</td>
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<tr>
<td>0.071-0.075</td>
<td>98.3%</td>
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<tr>
<td>0.075-0.080</td>
<td>98.1%</td>
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<tr>
<td>0.080-0.085</td>
<td>98.1%</td>
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<tr>
<td>0.086-0.087</td>
<td>97.8%</td>
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</tr>
</tbody>
</table>

Tables 4.2-4.4 suggest that the percentages of non-rejection of the null hypothesis are high enough to allow at least for the non-rejection of the hypothesis (4.6). This seems to be particularly true for fairly non-degenerated processes, with substantial variability in both \( r' \) and \( \theta_n \), for which the empirical frequencies are given in the middle of the tables. Hence, the hypothesis that returns to the Diba – Grossman process are symmetric stable distributed cannot be rejected.
Table 4.3: Stable distribution of Diba-Grossman returns 2
Table presents fractions of simulated Diba-Grossman returns for which the null hypothesis of the stable distribution is not rejected at 5% significance level.

<table>
<thead>
<tr>
<th>σε</th>
<th>0.00-0.04</th>
<th>0.05-0.09</th>
<th>0.10-0.14</th>
<th>0.15-0.19</th>
<th>0.20-0.24</th>
<th>0.25-0.29</th>
<th>0.30-0.34</th>
<th>0.35-0.39</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001-0.005</td>
<td>93.1% 93.0% 91.5% 90.3% 89.1% 88.4% 88.2% 87.9%</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>0.006-0.010</td>
<td>93.1% 93.5% 93.0% 92.8% 92.5% 92.7% 92.7% 92.0%</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>0.010-0.015</td>
<td>93.1% 93.7% 93.8% 94.3% 94.5% 94.4%</td>
<td></td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0.016-0.020</td>
<td>93.3% 94.1% 94.7% 95.3% 95.1%</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0.021-0.025</td>
<td>93.0% 94.2% 94.7% 95.1%</td>
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<td></td>
</tr>
<tr>
<td>0.026-0.030</td>
<td>93.1% 94.3% 94.9%</td>
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</tr>
<tr>
<td>0.031-0.035</td>
<td>92.9% 94.2% 95.6%</td>
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<td></td>
</tr>
<tr>
<td>0.036-0.040</td>
<td>93.1% 94.3%</td>
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</tr>
<tr>
<td>0.041-0.045</td>
<td>92.9% 94.1%</td>
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<tr>
<td>0.046-0.050</td>
<td>93.2% 94.1%</td>
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</tr>
<tr>
<td>0.051-0.055</td>
<td>93.1% 93.9%</td>
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<tr>
<td>0.056-0.060</td>
<td>93.2% 94.5%</td>
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<tr>
<td>0.061-0.065</td>
<td>93.1% 93.3%</td>
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</tr>
<tr>
<td>0.066-0.070</td>
<td>93.2%</td>
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</tr>
<tr>
<td>0.071-0.075</td>
<td>93.0%</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0.075-0.080</td>
<td>92.5%</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0.080-0.085</td>
<td>92.6%</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0.086-0.087</td>
<td>92.1%</td>
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<td></td>
</tr>
</tbody>
</table>

Table 4.4: Stable distribution of Diba-Grossman returns 3
Table presents fractions of simulated Diba-Grossman returns for which the null hypothesis of the stable distribution is not rejected at 10% significance level.

<table>
<thead>
<tr>
<th>σε</th>
<th>0.00-0.04</th>
<th>0.05-0.09</th>
<th>0.10-0.14</th>
<th>0.15-0.19</th>
<th>0.20-0.24</th>
<th>0.25-0.29</th>
<th>0.30-0.34</th>
<th>0.35-0.39</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001-0.005</td>
<td>85.3% 86.2% 84.5% 83.2% 81.7% 81.0% 80.4% 79.7%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.006-0.010</td>
<td>85.0% 86.4% 85.6% 86.1% 85.7% 85.4% 85.4% 84.4%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.010-0.015</td>
<td>84.7% 86.3% 87.1% 87.7% 87.9% 87.6%</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.016-0.020</td>
<td>84.9% 86.9% 87.9% 89.0% 89.3%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.021-0.025</td>
<td>85.0% 86.9% 88.2% 88.6%</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>0.026-0.030</td>
<td>84.9% 87.0% 88.5%</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>0.031-0.035</td>
<td>84.8% 87.1% 89.1%</td>
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<tr>
<td>0.036-0.040</td>
<td>84.9% 87.3%</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0.041-0.045</td>
<td>84.6% 87.0%</td>
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<td></td>
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</tr>
<tr>
<td>0.046-0.050</td>
<td>84.9% 86.5%</td>
<td></td>
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</tr>
<tr>
<td>0.051-0.055</td>
<td>84.8% 86.6%</td>
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<tr>
<td>0.056-0.060</td>
<td>85.1% 87.0%</td>
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<td></td>
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<tr>
<td>0.061-0.065</td>
<td>85.0% 86.0%</td>
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<tr>
<td>0.066-0.070</td>
<td>85.2%</td>
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</tr>
<tr>
<td>0.071-0.075</td>
<td>84.9%</td>
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<tr>
<td>0.075-0.080</td>
<td>84.5%</td>
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<tr>
<td>0.080-0.085</td>
<td>84.0%</td>
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<tr>
<td>0.086-0.087</td>
<td>84.1%</td>
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</tbody>
</table>
However, before final conclusion about the symmetric stable distribution of returns to (4.1), the fit of other distribution families should be tested. It is impossible to analyse the goodness of fit of all possible alternative distributions. Hence, the remaining part of this section concentrates only on two families, which could accurately approximate the distribution of returns to Diba – Grossman processes: the normal distribution and the $t$-Student distribution.

The normal distribution is commonly used both in financial and economic applications (see Campbell et al, 1997: Chapter I). It is convenient and estimation of its only two parameters, mean and standard deviation, is very straightforward. If a normal distribution formed a reasonable approximation of returns to (4.1), fitting any other distribution is rather inefficient.

The $t$-Student distribution is one of the most commonly used families to approximate the heavy tailed properties of data. The $t$-Student assumption can be motivated both empirically and theoretically. It captures certain types heavy tailed behaviour, commonly observed on financial markets (see Chapter II.5). Moreover, the multivariate $t$-Student is a return distribution, for which the mean-variance analysis is consistent with agents' expected utility maximisation (Campbell et al, 1997: Chapter 5). Application of the $t$-Student distribution to financial data are analysed e.g. by Bollerslev (1987). The $t$-Student distribution has only one more parameter than the normal law: number of degrees of freedom $v$. The smaller the number of degrees of freedom, the heavier the tails of the analysed distribution. For $v>2$, the $t$-Student distribution has finite variance. This property makes it quite attractive from the theoretical point of view. Non-existence of variance in case of stable distributions with characteristic exponent less than two makes them difficult to accept for some market practitioners. Estimation of the $t$-Student parameters is simpler than the estimation of stable distribution parameters. However, the $t$-Student distribution has specific shape and its dispersion parameter is connected with number of degrees of freedom. This makes it much less flexible than the stable law. Nevertheless, if its fit is better than that of the stable distribution, it would be plausible and convenient to perform the rest of the analysis assuming $t$-Student rather than symmetric stable distribution of returns to Diba – Grossman process.
To test the fit of the normal and $t$-Student distributions of returns to (4.1), series of the length of 1,000 each are simulated for different values $\sigma_0$ and $\sigma_{re}$. The values of $\sigma_0$ increase from 0 with step 0.05 and values of $\sigma_{re}$ increase from 0.001 with step 0.005, unless the variability of series is too excessive. The parameters of $t$-Student distributions are estimated by maximum likelihood method with $\nu$ restricted to be an integer from the interval $[3,100]$. The $\chi^2$ test of fit with 30 equiprobable cells is applied. Tables 4.5-4.6 present fractions of non-rejected null hypotheses about normal and $t$-Student distributions of returns to (4.1) at 5% significance levels.

### Table 4.5: Normal distribution of Diba-Grossman returns

Table presents fractions of simulated Diba-Grossman returns for which the null hypothesis of the normal distribution is not rejected at 5% significance level.

<table>
<thead>
<tr>
<th>$\sigma_{re}$</th>
<th>0</th>
<th>0.05</th>
<th>0.1</th>
<th>0.15</th>
<th>0.2</th>
<th>0.25</th>
<th>0.3</th>
<th>0.35</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>97.2%</td>
<td>84.1%</td>
<td>28.7%</td>
<td>4.9%</td>
<td>0.3%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td>0.006</td>
<td>97.6%</td>
<td>84.0%</td>
<td>28.1%</td>
<td>6.4%</td>
<td>1.3%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td>0.011</td>
<td>97.2%</td>
<td>82.0%</td>
<td>28.5%</td>
<td>6.0%</td>
<td>0.9%</td>
<td>0.2%</td>
<td>0.0%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.016</td>
<td>97.2%</td>
<td>83.4%</td>
<td>34.0%</td>
<td>6.6%</td>
<td>1.6%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.021</td>
<td>97.1%</td>
<td>83.5%</td>
<td>33.3%</td>
<td>8.3%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.026</td>
<td>96.8%</td>
<td>83.2%</td>
<td>34.7%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.031</td>
<td>97.3%</td>
<td>86.1%</td>
<td>33.8%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.036</td>
<td>96.4%</td>
<td>86.6%</td>
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Table 4.5 shows that the normal law fits well the distributions of returns to non-speculative processes with $\sigma_0 = 0$. This is not surprising, as in such cases the distribution of returns should be normal (see Chapter IV.2.1). However, the size of the $\chi^2$ test for the normal hypothesis is slightly larger than this reported in Table 4.3.

1 The estimation procedure is encoded in GAUSS.
This might be due to the fact that the first column of Table 4.3 combines results for different non-negative values of $\sigma_\theta$. Additionally, lack of closed form probability distribution function may cause some flaws in the quality of estimates of symmetric stable distribution parameters. Hence, their accuracy might be lower than that of normal distribution parameters, causing more rejection of the null hypothesis.

The goodness of fit of the normal law decreases for bigger values of $\sigma_\theta$. For standard deviations of the stochastic root of (4.1) larger than 0.25, the null hypothesis about normal distribution of returns is rejected in all cases at 5% significance level. This differs strongly from Table 4.3, where the highest simulated size is just 2.42 larger than the nominal significance level of 5%.

The power of the applied $\chi^2$ test is clearly increasing along with $\sigma_\theta$. As stated in Chapter IV.2.1, the normal hypothesis is false for any $\sigma_\theta > 0$. However, for $\sigma_\theta = 0.05$ the probability of rejecting the false null hypothesis varies only from 2.9% to 16%. The power of the test is much higher, when the analysed series is generated by process with $\sigma_\theta = 0.1$. For $\sigma_\theta \geq 0.15$, the power of the $\chi^2$ test exceeds 90%.

Generally, it can be said that the normal distribution could be a suitable tool to describe the behaviour of returns to non-speculative processes, however, the existence of speculation in the Diba-Grossman sense, makes it rather an inappropriate tool.

The results presented in Table 4.6 show that the $t$-Student distribution is even less appropriate to model returns to Diba-Grossman processes than the normal law. This is slightly surprising, especially that the degrees of freedom should enable the $t$-Student family to capture the heavy tailed behaviour of the data. However, it must be remembered that the shape of $t$-Student distribution is relatively inflexible and is rather inappropriate to capture large concentrations of returns around zero. Indeed, the estimated numbers of degrees of freedom are usually larger than 30 (these results are not reported in the text), making the fitted $t$-Student distributions very similar to the normal ones. Slight inaccuracies in the estimation procedure, connected presumably with additional parameters, cause a worse fit than that which is noted for the normal distribution. Hence, Table 4.6 reports generally larger percentages of rejected null hypotheses than Table 4.5.
Similarly as for normal laws, the goodness of fit of the $t$-Student distribution decreases with growing $\sigma_0$. It fits well the distributions of returns to processes (4.1) only for $\sigma_0 \leq 0.15$. For larger $\sigma_0 > 0.25$, the $t$-Student distribution is rejected in almost all cases.

Table 4.3, Table 4.5 and Table 4.6 show the percentages of rejected null hypotheses that observed returns to Diba – Grossman processes have respectively symmetric stable, normal and $t$-Student distribution. The significance level accepted for these $\chi^2$ tests is equal to 5% and hence the expected non-rejection probabilities are equal to 95%. The percentages of non-rejected hypotheses that returns to process (4.1) with $\sigma_0 \geq 0.15$ have any of the two latter distributions do not exceed 10%. The corresponding values for the symmetric stable laws are above 87%. Consequently, it seems to be clear that neither the normal nor the $t$-Student law form accurate approximations of the distributions of returns to Diba – Grossman processes.

The symmetric stable distribution seems to be a very good approximation of the observed distributions of returns to Diba – Grossman processes. It means that the
hypothesis (4.6): \( r_t \sim SS(\alpha, c, \delta) \) is positively verified by the simulated series. This result establishes a correspondence between speculative processes and distributions of returns and opens a possibility to evaluate the exact character of this relationship.

IV.3. Indirect estimation of parameters of Diba-Grossman process

IV.3.1. Main concepts

The Diba - Grossman process is a first order autoregressive process with stochastic coefficient. As stated in Chapter III.4.3, its parameters could be estimated by the simulated pseudo-maximum likelihood method (see Granger and Swanson, 1997). Alternatively, the model in the state-space form could be formulated and Kalman filter could be used to evaluate the likelihood function (see e.g. Hamilton, 1994, p. 389). However, the first method causes computational problems for large samples with small standard deviations of the stochastic root of (4.1). This is mainly due to the fact that the likelihood function evaluated by Granger and Swanson (1997) is a sum and hence its logarithm can not be taken. The Kalman filter allows estimation of both parameters of Diba - Grossman process, but precision of these estimates is not evaluated.

The main purpose of this section is not to estimate the parameters of (4.1), but rather to establish a relationship between the distributions of returns generated by (4.1) and parameters of the Diba-Grossman process. Obviously, one of the possible applications of such relationship could be estimation of the parameters of Diba-Grossman processes.

In this section, the relationship between stable distribution parameters and parameters of Diba - Grossman process are investigated. Following Charemza and Kominek (1999), the hypothesis formulated herein is that, for \( \sigma^2 \geq 0 \), there exists a mapping:

\[
g(\sigma^2, \sigma^2) = (\alpha, c) \quad ,
\]

\[
(4.7)
\]

\[\text{This might be a serious problem, as the Kalman filter estimations of Diba - Grossman parameters, which proceeded this study, gave rather unreliable results.}\]
such that: \( g(0, \sigma^e_0^2, \sigma^e_2) = (2, (1/\sqrt{2}) \cdot \sigma_n) \). Hence, \( g(\cdot) \) represents a mapping from the domain \{ \sigma^e_0^2 \geq 0, \sigma^e_2 > 0 \} to the domain \{ 0 < \alpha \leq 2, c > 0 \}. Consequently, \( \alpha(\sigma^e_0^2, \sigma^e_2) \) and \( c(\sigma^e_0^2, \sigma^e_2) \) are coordinate functions for the mapping \( g(\cdot) \) and are both functions of the arguments \( \sigma^e_0^2 \) and \( \sigma^e_2 \). The straightforward suggestion is that \( \partial \alpha(\sigma^e_0^2, \sigma^e_2)/\partial \sigma^e_0^2 < 0 \) and \( \partial c(\sigma^e_0^2, \sigma^e_2)/\partial \sigma^e_2 > 0 \). In practice nothing more about the function \( g(\cdot) \) is known, and even if its values are observed, its arguments are not, since these are population parameters.

It is generally feasible to obtain unbiased and invariant estimators of \( \alpha \) and \( c \), say \( \hat{\alpha} \) and \( \hat{c} \) (see Chapter II.3 and Chapter II.4). Hence, if the hypothesis (4.6) is true, it implies that:

\[
E(\hat{\alpha}, \hat{c}) = g(\sigma^e_0^2, \sigma^e_2)
\]

or, more precisely:

\[
E[f_i(\hat{\alpha}, \hat{c}|\sigma^e_0^2, \sigma^e_2)] = (\alpha, c)
\]

where \( f_i(\cdot) \) is the conditional density function. This density function is not usually known but can be approximated by simulation experiments. This may allow a simple evaluation of the degree of market speculation through the evaluation of the inverse function:

\[
(\sigma^e_0^2, \sigma^e_2) = g^{-1}(\alpha, c)
\]

provided that it exists.

In order to obtain the estimates of \( \sigma^e_0^2 \) and \( \sigma^e_2 \), a simulation procedure is proposed. For a large number of replications and different values of \( \sigma^e_0^2 \) and \( \sigma^e_2 \), indexed by \( i \) and \( j \) respectively, the conditional distributions of \( \hat{\alpha} \) and \( \hat{c} \) are evaluated empirically as:

\[
\hat{f}_i(\hat{\alpha}, \hat{c}|\sigma^e_0^2 = \sigma^e_{0,i}, \sigma^e_2 = \sigma^e_{2,j})
\]

That is, for different admissible values of \( \sigma^e_0^2 \) and \( \sigma^e_2 \) the estimates of \( \alpha \) and \( c \) are computed. Next, the conditional empirical distributions:

\[
\hat{f}_i^{-1}(\hat{\sigma}^e_0^2, \hat{\sigma}^e_2|\hat{\alpha}^h, \hat{c}^h)
\]

(4.8)
are evaluated. Means or median values of these distributions can be regarded as point estimates of $\sigma_\theta^2$ and $\sigma_r^2$, with an easy possibility of computing appropriate confidence intervals. Such a procedure enables computation of convenient conversion tables, which can be used for assessing the degree of speculation of particular markets with the use of $\alpha$ and $c$ parameters.

**IV.3.2. Estimation through simulations**

In a modification of the simulation experiment described in Chapter IV.2.3, the conversion tables to map function (4.7) are obtained. The McCulloch (1986) estimates of parameters of the stable distribution are computed for all simulated processes (4.1). Then, the empirical distributions of $\sigma_\theta$ and $\sigma_r$ conditional on $\hat{\alpha}$ and $\hat{c}$ are evaluated according to (4.8). The obtained estimates are grouped, resulting in empirical distributions of $\sigma_\theta$ and $\sigma_r$ conditional on $\hat{\alpha}$ and $\hat{c}$. The 0.05, 0.5 and 0.95 percentiles of these distributions are given in Tables 4.7 and 4.8.

Table 4.9 shows numbers of simulated series in each group. It is suggested that the results for the empirical distributions with less than 1,000 simulated series should be treated with caution, since in these cases the stopping rule (excess variability) has been encountered frequently.
Table 4.7: Quantiles of conditional distributions of $\hat{\sigma}_\theta$ given $c$ and $\alpha$, sample size = 1 000

Table shows 0.05, 0.5 and 0.95 quantiles of conditional distributions of $\hat{\sigma}_\theta$ given $\alpha$ (first column) and $c$ (first row). The simulations are performed for the series with 1000 observations. Results are grouped in intervals, as presented in the first row and first column.

<p>| $\alpha$ | Quantile | $c$ | &lt;0.002 | 0.002- | 0.004- | 0.006- | 0.008- | 0.01- | 0.012- | 0.014- | 0.016- | 0.018- | 0.02- | 0.022- | 0.024- | 0.026- | 0.028- |
|----------|----------|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| &lt;1       | 0.5      | 0.6300 | 0.4500 | 0.3200 | 0.2800 | 0.2500 | 0.2300 | 0.2000 | 0.1900 | 0.1600 | 0.1600 | 0.0750 | 0.0800 | 0.1100 |
| 1-1.05   | 0.5      | 0.5900 | 0.4900 | 0.3600 | 0.3100 | 0.2700 | 0.2500 | 0.2100 | 0.1900 | 0.1600 | 0.1600 | 0.1600 | 0.1600 | 0.1600 | 0.1500 | 0.1500 |
| 1.05-1.1 | 0.5      | 0.5500 | 0.4800 | 0.3800 | 0.3300 | 0.2900 | 0.2500 | 0.2100 | 0.1900 | 0.1800 | 0.1600 | 0.1600 | 0.1500 | 0.1500 | 0.1500 | 0.1500 |
| 1.1-1.15 | 0.5      | 0.5100 | 0.4700 | 0.3600 | 0.3300 | 0.2900 | 0.2500 | 0.2100 | 0.2000 | 0.1800 | 0.1600 | 0.1600 | 0.1600 | 0.1600 | 0.1600 | 0.1600 |
| 1.15-1.2 | 0.5      | 0.4700 | 0.4500 | 0.3800 | 0.3300 | 0.2900 | 0.2500 | 0.2100 | 0.2000 | 0.1800 | 0.1600 | 0.1600 | 0.1600 | 0.1600 | 0.1600 | 0.1600 |
| 1.25-1.3 | 0.5      | 0.4300 | 0.4200 | 0.3700 | 0.3200 | 0.2900 | 0.2500 | 0.2100 | 0.1900 | 0.1800 | 0.1600 | 0.1600 | 0.1600 | 0.1600 | 0.1600 | 0.1600 |
| 1.3-1.35 | 0.5      | 0.3800 | 0.3900 | 0.3600 | 0.3200 | 0.2700 | 0.2400 | 0.2100 | 0.1900 | 0.1700 | 0.1600 | 0.1600 | 0.1600 | 0.1600 | 0.1600 | 0.1600 |
| 1.35-1.4 | 0.5      | 0.3300 | 0.3200 | 0.3100 | 0.2800 | 0.2600 | 0.2300 | 0.2000 | 0.1800 | 0.1600 | 0.1400 | 0.1400 | 0.1300 | 0.1300 | 0.1300 | 0.1300 |
| 1.4-1.45 | 0.5      | 0.2700 | 0.2700 | 0.2800 | 0.2600 | 0.2500 | 0.2100 | 0.1900 | 0.1800 | 0.1600 | 0.1400 | 0.1400 | 0.1300 | 0.1300 | 0.1300 | 0.1300 |
| 1.45-1.5 | 0.5      | 0.2000 | 0.2000 | 0.2100 | 0.2000 | 0.1900 | 0.1800 | 0.1700 | 0.1600 | 0.1400 | 0.1300 | 0.1200 | 0.1100 | 0.1100 | 0.1100 | 0.1100 |
| 1.5-1.55 | 0.5      | 0.1300 | 0.1300 | 0.1300 | 0.1300 | 0.1300 | 0.1300 | 0.1300 | 0.1300 | 0.1300 | 0.1300 | 0.1300 | 0.1300 | 0.1300 | 0.1300 | 0.1300 |</p>
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Table 4.8: Quantiles of conditional distributions of $\hat{\sigma}_{rx}$ given $\alpha$ and $c$, sample size = 1 000

Table shows 0.05, 0.5 and 0.95 quantiles of conditional distributions of $\hat{\sigma}_{rx}$ given $\alpha$ (first column) and $c$ (first row). The simulations are performed for the series with 1000 observations. Results are grouped in intervals, as presented in the first row and first column.

<p>| $\alpha$ | C quantile | &lt;0.002 | 0.002- | 0.004- | 0.006- | 0.008- | 0.01- | 0.012- | 0.014- | 0.016- | 0.018- | 0.02- | 0.022- | 0.024- | 0.026- | 0.028- | 0.03 |
|--------|-----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| &lt;1     | 0.5       | 0.0010 | 0.0020 | 0.0040 | 0.0060 | 0.0070 | 0.0090 | 0.0100 | 0.0120 | 0.0130 | 0.0140 | 0.0160 | 0.0155 | 0.0190 | 0.0090 | 0.0090 |
| 0.05   | 0.0010 | 0.0020 | 0.0030 | 0.0050 | 0.0060 | 0.0070 | 0.0080 | 0.0080 | 0.0090 | 0.0100 | 0.0120 | 0.0140 | 0.0160 | 0.0180 | 0.0155 | 0.0090 | 0.0090 |
| 0.05   | 0.0010 | 0.0020 | 0.0030 | 0.0050 | 0.0070 | 0.0090 | 0.0100 | 0.0120 | 0.0140 | 0.0160 | 0.0180 | 0.0200 | 0.0220 | 0.0240 | 0.0260 | 0.0280 | 0.0300 |
| 0.05   | 0.0010 | 0.0020 | 0.0030 | 0.0050 | 0.0070 | 0.0090 | 0.0110 | 0.0130 | 0.0150 | 0.0170 | 0.0190 | 0.0210 | 0.0230 | 0.0250 | 0.0270 | 0.0290 | 0.0310 | 0.0330 |
| 0.05   | 0.0010 | 0.0020 | 0.0030 | 0.0050 | 0.0070 | 0.0090 | 0.0110 | 0.0130 | 0.0150 | 0.0170 | 0.0190 | 0.0210 | 0.0230 | 0.0250 | 0.0270 | 0.0290 | 0.0310 | 0.0330 |</p>
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Table 4.9: Number of observations in respective groups

Table shows numbers of realisations of Diba – Grossman processes for which the returns followed symmetric stable distributions with parameters in the intervals specified in the first column and first row. On the basis of these processes Tables 4.7-4.8 are constructed.

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<td>3181</td>
<td>3149</td>
<td>3184</td>
<td>3306</td>
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<td>3130</td>
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<td>1.8-1.85</td>
<td>2471</td>
<td>3025</td>
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<td>3156</td>
<td>3154</td>
<td>3206</td>
<td>3181</td>
<td>3124</td>
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</tr>
<tr>
<td>1.9-1.95</td>
<td>2686</td>
<td>3505</td>
<td>3497</td>
<td>3399</td>
<td>3553</td>
<td>3432</td>
<td>3569</td>
<td>3497</td>
<td>3443</td>
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<td>3483</td>
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<td>1.95-2</td>
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<td>11599</td>
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<td>11492</td>
<td>11822</td>
<td>11500</td>
<td>11480</td>
<td>11309</td>
<td>11363</td>
</tr>
</tbody>
</table>
In order to illustrate the relationships identified in the Tables 4.7-4.8, the function (4.3) is simplified by assuming that the relations between $G^q$ and $a$ and between $\sigma_r^2$ and $c$ are separable, so that:

$$\partial \alpha(\sigma_r^2, \sigma_e^2) / \partial \sigma_e^2 = 0 \text{ and } \partial c(\sigma_r^2, \sigma_e^2) / \partial \sigma_r^2 = 0$$

(4.5)

and, consequently, that the following function exists:

$$g^\alpha(\sigma_r^2) = \alpha$$.

This simplifying assumption implies that:

$$E[f_i^\alpha(\hat{\alpha} \mid \sigma_r^2)] = \alpha$$.

**Figure 4.1: 0.5 quantiles of conditional distributions of $\sigma_\theta$ given $\alpha$**

Figure shows the relationship between estimates of characteristic exponent $\alpha$ (alpha) and 0.5 quantiles of the $\sigma_\theta$ parameter (std(theta)) as fitted with the use of Tables 4.7-4.8. Additionally 90% confidence intervals for $\sigma_\theta$ are presented.

Figure 4.1 shows mean values of $\sigma_\theta$ conditional on the $\alpha$’s, together with the 90% confidence intervals. It represents the estimated relationship $(g^\alpha)^{-1}(\alpha) = \sigma_r^2$ by showing its median and 90% confidence interval. It confirms that $\alpha$’s can be regarded as decreasing functions of $\sigma_\theta$’s. A similar relationship, albeit positive, can be observed for the marginal distribution of $c$’s given $\sigma_r$ (see Figure 4.2).
Figure 4.2: 0.5 quantiles of conditional distributions of $\sigma_\theta$ given $\alpha$

Figure shows the relationship between estimates of scale parameter $c$ and 0.5 quantiles of the $\sigma_\alpha$ parameter (std(re)) as fitted with the use of Tables 4.7-4.8. Additionally 90% confidence intervals for $\sigma_\alpha$ are presented.

However, it must be remembered that condition (4.5) imposes a large simplification and the analysed relationship is much more complicated than the linear dependence presented on Figures 4.1 – 4.2. Basing on Tables 4.7-4.8, it is possible to draw the three-dimensional graphs of the relationship between parameters of stable distribution of returns and parameters of the Diba – Grossman price generating process. This is presented on Figures 4.3 – 4.4.

Figure 4.3 shows the 0.5 percentiles of the distribution of $\sigma_\theta$. The negative relationship between $\alpha$ and $\sigma_\theta$ is more evident for small values of $c$. Additionally, it is evident that an increase in $c$ causes $\sigma_\theta$ to fall. These changes are bigger for smaller values of $\alpha$.

Similar distributions for $\sigma_\alpha$ are presented on Figure 4.4. The strength of the positive relationship between $c$ and $\sigma_\theta$ is only slightly increases with $\alpha$. Generally, positive relationship between $\alpha$ and $\sigma_\theta$ is more evident for large $c$ and almost non-existent for $c$ close to zero.
Figure 4.3: 0.5 quantiles of conditional distributions of $\sigma_0$

Figure shows 0.5 quantiles of distributions of $\sigma_0$ conditional on the estimates of the characteristic exponent and scale parameter of the symmetric stable distribution of returns to speculative processes. The stable distribution parameters are shown on the horizontal axes, whereas the quantiles of $\sigma_0$ are given on the vertical axis.

Figure 4.4: 0.5 quantiles of conditional distributions of $\sigma_{re}$

Figure shows 0.5 quantiles of distributions of $\sigma_0$ conditional on the estimates of the characteristic exponent and scale parameter of the symmetric stable distribution of returns to speculative processes. The stable distribution parameters are shown on the horizontal axes, whereas the quantiles of $\sigma_{re}$ are given on the vertical axis.
The relationships presented on Figure 4.3 and Figure 4.4 imply (see Chapter III) that there is a strong positive dependence between exposure to speculation and thickness of tails of return distributions. Similarly, large random shocks generally results in high dispersion of analysed returns. Additionally, exposure to random shocks slightly increases thickness of tails of return distribution and weakens the relationship between the level of speculation and parameter $\alpha$. Low exposure to speculation increases the dispersion of stocks and slightly strengthens the dependence between dispersion and exposure to random shocks.

**IV.3.3. The bivariate character of the relationship**

One of the main questions regarding the analysis developed in Chapter IV.3.2 is whether the addition of the third dimension, as presented on Figures 4.3 - 4.4, to the two-dimensional relationships between $\alpha$ and $\sigma_\theta$ and $c$ and $\sigma_{re}$, shown on Figures 4.1 - 4.2, is justified by the strength of the dependencies between $\alpha$ and $\sigma_{re}$ and $c$ and $\sigma_\theta$. Two dimensional (univariate) analysis offers relative simplicity and intuitive meaning to the results. Hence, if the three dimension (bivariate) approach does not significantly improve precision of the analysis, the addition of the third dimension may prove to be computationally and theoretically inefficient. In other words, the question is, whether the relationships between exposure to speculation and dispersion and size of random shocks and thickness of tails are significant enough to be incorporated into empirical analysis of speculation and distributions of returns.

Figures 4.5 - 4.6 highlight parts of the variability of $\sigma_\theta$ and $\sigma_{re}$, which are lost if the analysis of (4.7) is restricted to two dimensions only. Figure 4.5 shows the impact of $\alpha$ on $\sigma_{re}$. The diagram presents deviations of $\sigma_{re}$ from its expected values for different $c$'s. It is clear that for large values of $c$ the impact of the characteristic exponent $\alpha$ on $\sigma_\theta$ is bigger than in the case of the small $c$.

Similarly, Figure 4.6 shows that for small $\alpha$'s the impact of $c$ on $\sigma_\theta$ is much larger than in the case of big $\alpha$'s. Both impacts are relatively large and omitting them leads to serious errors. Their size reaches 70% of the respective expected values.
Hence, the bivariate analysis, developed in Chapter IV.3.2, seems to be much more appropriate than simple approach illustrated on Figures 4.1-4.2.

**Figure 4.5: The impact of \( \alpha \) on \( \sigma_{re} \)**

Figure shows the magnitude of the impact of characteristic exponent \( \alpha \) on the standard deviation of Diba–Grossman returns. This is shown separately for different values of scale parameter \( c \). If \( \alpha \) was excluded from the analysis, additional errors of the presented size would occur. Parameters of stable distribution are shown in horizontal axes, whereas the impact of \( \alpha \) on \( \sigma_{re} \) is shown on the vertical axis.

**Figure 4.6: The impact of \( c \) on \( \sigma_{\theta} \)**

Figure shows the magnitude of the impact of scale parameter \( c \) on the standard deviation of stochastic root of Diba–Grossman process. This is shown separately for different values of characteristic exponent \( \alpha \). If \( c \) was excluded from the analysis, additional errors of the presented size would occur. Parameters of stable distribution are shown in horizontal axes, whereas the impact of \( \alpha \) on \( \sigma_{\theta} \) is shown on the vertical axis.
The above conclusion means that the relationship between level of speculation and dispersion of the distribution of returns and the relationship between exposure to random shocks and thickness of tails of the distribution of returns are significant and should not be omitted in empirical and theoretical analyses. The straightforward dependence between tail thickness and speculation, suggested by some authors (Peters, 1991), seems to be unjustified oversimplification of reality. Hence, the relationship between exposures to speculation and random shocks and parameters of symmetric stable distribution of returns (see Chapter IV.3.2) have bivariate rather than univariate character.

IV.4. Conclusions

Thanks to the low parametrisation and clear interpretation, the Diba - Grossman processes are useful tools in analysing financial data. The variance of the stochastic root of these processes measures the level of speculation, whereas the variance of additive random term reflects the exposure to random shocks. It is known that for non-speculative markets, when the variance of stochastic root is equal to zero, the observed returns to Diba - Grossman process are normally distributed. When variance of stochastic root is larger than zero, then the distribution becomes heavy tailed. However, the exact form of this distribution is unknown.

In this chapter, it is verified that observed returns to Diba – Grossman process can be approximated by the symmetric stable distributions. This hypothesis is confirmed in $\chi^2$ tests, where thousands of Diba – Grossman processes with different parameters are analysed. The size of the applied test varies slightly, depending on the exact shape of the distribution but its power seems to be large. The test does not reject the null hypothesis about symmetric stable distribution in only a slightly lower number of cases than is suggested by the size of the test. The alternative hypotheses that Diba – Grossman processes results in normal or $t$-Student distribution of returns are tested and in both cases the proposed approximations are rejected. The obtained results provide sufficient confirmation of the symmetric stable hypothesis and hence a direct correspondence between speculative processes of a certain type and
functional form of the distributions of returns is established. It creates a possibility to evaluate the exact character of such relationship and find out how the level of speculation and the exposure of random shocks, both measured by parameters of Diba – Grossman process, contribute to the shape of the distribution of observed returns.

The relationship between parameters of Diba – Grossman process and parameters of symmetric stable distribution of returns is evaluated in a Monte Carlo experiment. The detected negative correlation between the variance of stochastic root and characteristic exponent of stable distributions can be interpreted as positive dependence between level of speculation and the degree of non-normality of the distribution of returns. The positive correlation between variance of random term in Diba – Grossman process and scale parameter of stable distribution means that there is a positive relationship between exposure to random shocks and dispersion of the distributions of returns.

However, the whole relationship proves more complicated and an appropriate bivariate function is analysed. The cross impact of the characteristic exponent on the variance of random term and scale parameter on the standard deviation of stochastic root is evaluated. In both cases positive dependencies are shown but the magnitude of these relationships is varying. This means that market speculation has a significant impact not only on the degree of non-normality but also on the dispersion of the distribution. Similarly, the non-normality might be caused both by speculation and exposure to random shocks. This shows that the non-normality – speculation relationship is more complicated than it could be expected.

The evaluated relationship opens a possibility of indirect estimation of Diba – Grossman process parameters. The parameters of interest may be obtained through conversion from symmetric stable distribution parameters. As the conditional distributions of Diba – Grossman process parameters on scale and characteristic exponent of return distribution are evaluated, this approach offers an easy way to compute both point estimates (0.5 quantiles) and appropriate confidence intervals.
Chapter V

EMPIRICAL ANALYSIS OF SPECULATION AND DISTRIBUTION OF RETURNS

V.1. Introduction

V.2. Analysis US and UK stock markets
V.2.1. Comparative analysis
V.2.2. Time series analysis of Dow Jones Industrial

V.3. International stock markets analysis
V.3.1. Data
V.3.2. Filtering for GARCH effects: an application of the simulated annealing algorithm
V.3.3. Distributions of returns to international indices
V.3.4. Estimation of parameters of the Diba-Grossman speculative processes
V.3.5. Tests for the presence of outliers

V.4. Conclusions
V.1. Introduction

This chapter concentrates on the empirical analysis of speculation, random shocks and distribution of returns. It consists of two distinctive parts. First, the level of speculation and exposure to random shocks on the US and the UK markets are analysed. Then, the empirical relationship between speculation and degree of non-normality is established for a large number of countries.

The variability of stock prices is one of the biggest puzzles of financial economics. Many researchers attribute it to the speculative bubbles and departures of prices from fundamental values (see Chapter III). Some authors (e.g. Stiglitz, 1999) claim that excessively volatile prices, caused by speculative processes, may lead to allocation inefficiency of financial markets. Hence, decomposition of stock price changes, into shifts caused by random shocks to the economy and variations connected with speculative behaviour of investors, can give an important insight into the price generating mechanisms. In this chapter, such a decomposition is applied to the US and the UK stock markets. The levels of speculation on both markets in the second half of 1990s are compared. Then, the time paths of levels of speculation and random shock exposure are evaluated for the US series. This allows conclusions to be drawn about development of market speculation in the last 30 years.

It is well known (see e.g. Chapter II.5) that returns to stock prices have heavy tailed distributions. Some authors blame it on the speculative character of markets (Peters, 1991), others find explanations in rational behaviour of investors under certain conditions (Campbell et al, 1997: Chapter 5). This study concentrates on the speculative hypothesis. However, even if markets are speculative, it is not known whether the heavy tailed distribution is caused by speculative trading or random shocks. The analysis developed in Chapter IV.3 suggests that contribution of both these components might be important. The second part of this chapter tries to address this issue, by analysing broad market indices from 66 different countries. The series are adjusted for possible effects of risk neutrality. The parameters of Diba-Grossman process are fitted to the data, applying methodology developed in Chapter IV.3.2. Then, an empirical relationship between speculation and non-normality is evaluated. Formal tests of outliers are performed to check whether some countries do
not reveal levels of speculation that are significantly different from these implied by the respective degrees of non-normality.

V.2. Analysis of the US and the UK stock markets

V.2.1. Comparative analysis

In this section, the degrees of speculation for two main stock market indices: New York Dow Jones Industrials and London FTSE 100 are evaluated. The data set covers 1000 session to session (daily) observations for the period from April 1994 to April 1998.

Figure 5.1: Distributions of returns on Dow Jones Industrials and FTSE 100

Figure shows distributions of daily returns to Dow Jones Industrials and FTSE 100. Data is collected in the period from 15 April 1994 to 15 April 1998. Returns are computed as first differences of the price logarithms.

Figure 5.1 compares the distributions of returns to Dow Jones Industrial and FTSE 100 indices, calculated as the first differences of the logarithms of the level

1 Data series have been obtained from Datastream.
series. Both indices show slight heavy tailed character of distributions and relatively large concentrations of returns around zero.

### Table 5.1: Stable distribution parameters and standard deviations of $\theta_i$ and $r_i$ for Dow Jones Industrials and FTSE 100 indices

The Table shows McCulloch (1986) estimates of the parameters of stable distribution of returns to Dow Jones Industrial and FTSE 100. Respective standard deviations are given in brackets. Additionally the estimates of parameters of Diba – Grossman processes are presented. These estimates are obtained through conversion from stable distribution parameters. Standard deviations and 90% confidence intervals are given in brackets.

<table>
<thead>
<tr>
<th>Dow Jones Industrials</th>
<th>Estimates of the stable distribution parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$\beta$</td>
<td>$C$</td>
</tr>
<tr>
<td>1.71</td>
<td>-0.20</td>
<td>0.0049</td>
</tr>
<tr>
<td>(0.086)</td>
<td>(0.20)</td>
<td>(0.00019)</td>
</tr>
<tr>
<td>Estimates of $\sigma_\theta$</td>
<td></td>
<td>Estimates of $\sigma_{re}$</td>
</tr>
<tr>
<td>0.116</td>
<td>(0.046, 0.24)</td>
<td>0.0055</td>
</tr>
<tr>
<td>FTSE 100</td>
<td>Estimates of the stable distribution parameters</td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$\beta$</td>
<td>$C$</td>
</tr>
<tr>
<td>1.75</td>
<td>-0.35</td>
<td>0.0050</td>
</tr>
<tr>
<td>(0.091)</td>
<td>(0.24)</td>
<td>(0.00019)</td>
</tr>
<tr>
<td>Estimates of $\sigma_\theta$</td>
<td></td>
<td>Estimates of $\sigma_{re}$</td>
</tr>
<tr>
<td>0.096</td>
<td>(0.027, 0.23)</td>
<td>0.0060</td>
</tr>
</tbody>
</table>

The parameters of the stable distribution are estimated using the McCulloch (1986) method (see Chapter II.3.5). Next, with the use of estimates of $\alpha$ and $c$, the values of standard deviations of $\sigma_\theta$ and $\sigma_{re}$ are estimated using 0.5 interpolated percentiles read from conversion tables in Chapter IV. Results of that analysis are shown in Table 5.1. It might be noticed that, for both markets, the intervals of $\pm 2$ standard deviations of the estimates of $\alpha$ and $c$ do not cover the value of 2, which indicates that the hypothesis that the observed returns are normally distributed has to be rejected. Comparing the results, it might be concluded that the degree of speculation on both markets is very similar, albeit marginally greater for the Dow Jones. In fact the estimates of $\alpha$ do not allow one to reject the hypothesis that they are identical for both markets. It is, however, interesting to note that the differences
between the corresponding $\sigma_\theta$'s are relatively greater than these between $\alpha$'s. It shows that the non-normality of returns cannot be directly interpreted as a measure of the degree of speculation, to which the scale of the distribution of returns also contributes. In fact, the degree of speculation at the New York market, as measured by $\hat{\sigma}_\theta$ seems to be markedly higher than that at the London market. Table 5.1 also indicates that the exposure to random shocks, given by the estimates of $\sigma_{re}$, is similar on both markets.

**V.2.2. Time series analysis of Dow Jones Industrials**

In this section, the historical evolution of the degree of speculation of the New York market is examined. For the Dow Jones Industrials share index, session to session returns from 1 January 1965 to 15 April 1995 are analysed in ‘windows’ of 1,000 observations each\(^2\). Figure 5.2 shows the development of the index for the period analysed. The values stagnate till the early eighties and then start growing. The series’ variability around long term trend seems to decrease in the second part of the sample but shocks (e.g. 1987) are not rare.

**Figure 5.2: Logarithms of Dow Jones Industrials from 1965 to 1995**

Figure presents logarithms of the daily values of Dow Jones Industrial from 1 January 1965 to 15 April 1995

\(^2\) The data for FTSE 100 are not available for such a long period.
Figures 5.3 and 5.4 show the time paths of the estimated values of $\sigma_\theta$ and $\sigma_{re}$ along with the 90% confidence intervals.

**Figure 5.3: Evolution of $\hat{\sigma}_\theta$ for Dow Jones Industrials**
Figure presents standard deviations of the stochastic root of Diba-Grossman process estimated for the logarithms of daily values of Dow Jones Industrial form 1 January 1965 to 15 April 1995. Along with point estimators from moving 1000 observation centred windows, the 90% confidence intervals are presented.

**Figure 5.4: Evolution of $\hat{\sigma}_{re}$ for Dow Jones Industrials**
Figure presents standard deviations of the additive random term of Diba-Grossman process estimated for the logarithms of daily values of Dow Jones Industrial form 1 January 1965 to 15 April 1995. Along with point estimators from moving 1000 observation centred windows, the 90% confidence intervals are presented.
Initially high values of $\hat{\sigma}_\theta$ can be observed, followed by a rapid fall, which suggests the relative increase in the degree of speculation in the late sixties. It might be associated with the development of the so-called Concept Stock Bubble (Malkiel, 1990), which appeared about 1967 and lasted until early seventies.

The late sixties were times of increased competition among the mutual funds. At the same time new, high-performing equities appeared on the market. They were usually of firms using fashionable concepts of modern management and technology. In some cases only a ‘good and new idea’ was enough for an initial market success. This created a conducive atmosphere for a bubble to develop.

In the early seventies, the speculation on the market started to decrease. It reached such low levels between the 1975 and 1978 that for this period the hypothesis of no speculation cannot be rejected. In the meantime, in 1974, there was a brief appearance of the so-called ‘Nifty Fifty Bubble’ (Malkiel, 1990), claimed to be caused by self-fulfilling expectations in the huge growth of the 50 biggest companies on the market. Moreover, an increase of the exposure to random effects can be observed. This is mainly due to the recession of the first half of the seventies and the first oil shock.

The situation changes in the eighties, when the ‘New Issue Boom’ (Malkiel, 1990) occurred. This was a time of enormous popularity of new stocks, which attracted huge numbers of investors and quickly led to overpricing. This bubble was followed by the ‘Biotechnology Bubble’ of 1984 (enormous popularity of the firms involved in biochemical technology) and a series of other smaller bubbles. All these keep $\sigma_\theta$ on a relatively high level through the eighties and early nineties. The high level of speculation in the eighties might be partially connected with increasing stock prices. Such prosperity often leads to overconfidence of investors, common beliefs in false, too optimistic asset pricing models and short term trading based on the analysis of past trends (see Chapter III.3.7). The early eighties are additionally characterised by decreasing value of $\hat{\sigma}_\nu$ and consequently by falling exposure to random shocks.
Generally, it should be noticed that although the degree of speculation on New York stock market seems to vary through time, its exposure to random shocks is more stable.

V.3. International stock market analysis

V.3.1. Data

The analysis developed in Chapter IV.3 suggests that both speculation and random shocks contribute to thick tails of distributions of returns to financial assets. This section investigates this relationship in the empirical framework, using data from 66 different stock markets. Parameters of Diba – Grossman processes are estimated and the levels of speculation and exposure to random shocks are evaluated. Then, the relationship between speculation and degree of non-normality, measured by the characteristic exponent of stable distribution, is investigated. The existence of outliers is verified to establish the strength of random shocks' impact on the thickness of tails of return distributions.

The international stock market analysis covers broad stock market indices from 66 countries. The sample contains daily observations from 1 September 1995 to 18 November 1999, resulting in series of about 1000 quotations. For the majority of countries the Datastream, the Morgan Stanley Capital International (MSCI) or the Hongkong and Shanghai Banking Corporation (HSBC) Price Indices are used. In a few cases, local series are taken3.

For the 38 largest markets, the indices calculated by Datastream Total Market Indices are applied. Their values are calculated on the basis of representative lists of stocks for each market. The numbers of stocks for individual markets are determined by the size of the market capitalisation. They vary from 50 for small markets (e.g. Greece, Portugal and Poland) to 1000 for the largest markets (e.g. Japan and US). For individual companies, the suitability for inclusion is determined by market value and availability of data. Consequently, the largest value stocks tend

3 All series are obtained from Datastream.
to be included more frequently than the ones with small market capitalisation. This may lead to the underestimation of risk on single markets (prices of small firms stocks are usually more volatile than those of the big ones). However, the magnitude of this underestimation should be similar for all markets and hence the international comparison of Datastream indices should lead to the same conclusions as the analysis of all shares indices. Index constituents for Datastream Total Market Indices are reviewed on the annual basis in January each year. The indices do not include unit trusts, mutual funds, warrants, temporary issues, foreign listings and foreign board stocks. They are fixed history indices, which means that they are not recalculated when the constituents change. Such an approach causes the effect of the dead stocks to be seen in the index (for more details see Datastream Online Help).

For 13 countries, the Morgan Stanley Capital International (MSCI) Price Indices are used. MSCI follows six major rules in constructing price indices. First, it defines all listed companies within each country (up to 99% of market capitalisation). Then the securities are sorted into the industry groups. The stocks selected for MSCI indices should have sufficient liquidity and free float. Cross ownership among stocks in the index is avoided. The full market capitalisation weight is applied to each stock in the index. The 60% market cap of each industry is chosen and included in the index. In this way, the country market cap should be included in the global index. The index variability may be correlated with the variability of the largest stocks on the market. As similar problems are noted in case of the Datastream indices, both indices should be easily comparable and the bias in the risk evaluation should be similar in both cases.

For 4 countries, the Hongkong and Shanghai Banking Corporation (HSBC) Price Index is chosen. The HSBC Indices are the only ones which are not calculated in the local currencies but in US dollars. This may cause some disturbances, as both exchange fluctuation risk and price variability are contributing to changes in the index value. Hence, the HSBC indices were chosen only when other indices were not available for a given market. It is believed that even such imperfect data provide more information than could be obtained if countries were excluded from analysis.

Finally, for 10 countries, the local indices are applied. This group includes: Czech Republic, Croatia, Mauritius, Slovenia, Zimbabwe, Romania, Slovakia,
Bangladesh, Oman and Iceland. Detailed description of the local series applied for these countries can be found on the web pages of the respective stock exchanges. Generally, all applied indices tend to overrepresent big companies and not include small market participants. As it has been noticed, this leads to underestimation of risk for individual markets. However, as the magnitude of underestimation should be similar for each market, the cross-country analysis should provide reasonable results about the relative level of risk on the national markets.

Table 5.2 shows which indices are chosen for the analysed countries. It provides their main characteristics, along with the numbers of available observations. The returns to stock indices are calculated as the first differences of the logarithms of the daily values of stock indices. The returns equal to zero are removed from sample. It is assumed that the zero returns to indices are caused by suspended trading or by bank holidays. Their inclusion would result in excessive concentration of returns around zero.

Skewness and kurtosis are calculated according to the formulas:

\[
\text{skewness}(x) = \frac{\text{mean}(x) - \text{median}(x)}{\text{m}^2_2(x)},
\]

\[
\text{kurtosis}(x) = \frac{m_4(x)}{m_2(x)^2},
\]

where

\[
m_r(x) = \frac{\sum [x - \text{mean}(x)]^r}{T}
\]

and \(T\) is the sample size.

A negative, zero or positive value of skewness shows left skew, symmetry or right skew respectively. Values of kurtosis larger (smaller) than 3 show that the analysed distribution is more (less) peaked than the normal one.

The largest daily average returns are noted for Turkey (0.28%), whereas the lowest ones are observed for Latvia (-0.27%). The vast majority of the mean daily returns are positive. Only 15 countries noted negative values in this column. The average mean daily return for the whole sample is equal to 0.04%. The lowest risk, as measured by the standard deviation of daily returns, is noted for Mexico (0.006). The highest standard deviation is noted for Russia (0.043), whereas the average value for analysed sample is equal to 0.016.
Table 5.2. Main characteristics of the analysed international market indices

Column (1) shows the mean daily returns to the analysed stock indices. Column (2) presents their standard deviations. Columns (3) gives the values of the Pearson’s measure of skewness and (4) provides the values of kurtosis. Column (5) shows the number of observations available for individual indices in the analysed period.

<table>
<thead>
<tr>
<th>Country</th>
<th>Index</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argentina</td>
<td>Datastream Global</td>
<td>0.0344%</td>
<td>0.019</td>
<td>0.006</td>
<td>11.738</td>
<td>1099</td>
</tr>
<tr>
<td>Australia</td>
<td>Datastream Global</td>
<td>0.0447%</td>
<td>0.009</td>
<td>-0.072</td>
<td>9.014</td>
<td>1068</td>
</tr>
<tr>
<td>Austria</td>
<td>Datastream Global</td>
<td>0.0165%</td>
<td>0.009</td>
<td>-0.096</td>
<td>8.146</td>
<td>1044</td>
</tr>
<tr>
<td>Bangladesh</td>
<td>SE All Shares</td>
<td>-0.0531%</td>
<td>0.031</td>
<td>0.035</td>
<td>47.460</td>
<td>825</td>
</tr>
<tr>
<td>Belgium</td>
<td>Datastream Global</td>
<td>0.0839%</td>
<td>0.009</td>
<td>-0.197</td>
<td>5.130</td>
<td>1057</td>
</tr>
<tr>
<td>Brazil</td>
<td>Datastream Global</td>
<td>0.0525%</td>
<td>0.019</td>
<td>-0.137</td>
<td>16.342</td>
<td>1042</td>
</tr>
<tr>
<td>Canada</td>
<td>Datastream Global</td>
<td>0.0685%</td>
<td>0.009</td>
<td>-0.133</td>
<td>9.926</td>
<td>1069</td>
</tr>
<tr>
<td>Chile</td>
<td>MSCI Price Index</td>
<td>-0.0042%</td>
<td>0.012</td>
<td>0.175</td>
<td>7.044</td>
<td>1048</td>
</tr>
<tr>
<td>China</td>
<td>Datastream Global</td>
<td>0.0907%</td>
<td>0.021</td>
<td>0.058</td>
<td>6.602</td>
<td>1037</td>
</tr>
<tr>
<td>Columbia</td>
<td>MSCI Price Index</td>
<td>0.0113%</td>
<td>0.014</td>
<td>0.052</td>
<td>11.984</td>
<td>1020</td>
</tr>
<tr>
<td>Croatia</td>
<td>Crobex Price Index</td>
<td>-0.1564%</td>
<td>0.027</td>
<td>-0.094</td>
<td>8.292</td>
<td>465</td>
</tr>
<tr>
<td>Cyprus</td>
<td>HSBC Price Index</td>
<td>0.2606%</td>
<td>0.018</td>
<td>0.338</td>
<td>50.445</td>
<td>916</td>
</tr>
<tr>
<td>Czech Republic</td>
<td>PX 50 Price Index</td>
<td>0.0113%</td>
<td>0.012</td>
<td>-0.062</td>
<td>6.847</td>
<td>1044</td>
</tr>
<tr>
<td>Denmark</td>
<td>Datastream Global</td>
<td>0.0797%</td>
<td>0.010</td>
<td>-0.112</td>
<td>9.462</td>
<td>1059</td>
</tr>
<tr>
<td>Egypt</td>
<td>MSCI Price Index</td>
<td>0.0706%</td>
<td>0.013</td>
<td>0.143</td>
<td>6.906</td>
<td>847</td>
</tr>
<tr>
<td>Estonia</td>
<td>HSBC Price Index</td>
<td>-0.0522%</td>
<td>0.036</td>
<td>-0.130</td>
<td>11.484</td>
<td>723</td>
</tr>
<tr>
<td>Finland</td>
<td>Datastream Global</td>
<td>0.1575%</td>
<td>0.018</td>
<td>-0.167</td>
<td>6.538</td>
<td>1055</td>
</tr>
<tr>
<td>France</td>
<td>Datastream Global</td>
<td>0.0963%</td>
<td>0.011</td>
<td>-0.012</td>
<td>5.840</td>
<td>1063</td>
</tr>
<tr>
<td>Germany</td>
<td>Datastream Global</td>
<td>0.0774%</td>
<td>0.012</td>
<td>-0.258</td>
<td>6.638</td>
<td>1059</td>
</tr>
<tr>
<td>Greece</td>
<td>Datastream Global</td>
<td>0.1679%</td>
<td>0.018</td>
<td>0.052</td>
<td>6.101</td>
<td>1052</td>
</tr>
<tr>
<td>Hong Kong</td>
<td>Datastream Global</td>
<td>0.0370%</td>
<td>0.020</td>
<td>-0.041</td>
<td>13.301</td>
<td>1050</td>
</tr>
<tr>
<td>Hungary</td>
<td>MSCI Price Index</td>
<td>0.1718%</td>
<td>0.024</td>
<td>-0.009</td>
<td>12.026</td>
<td>1047</td>
</tr>
<tr>
<td>Iceland</td>
<td>ICEX All Shares</td>
<td>0.1058%</td>
<td>0.006</td>
<td>0.166</td>
<td>15.693</td>
<td>662</td>
</tr>
<tr>
<td>India</td>
<td>MSCI Price Index</td>
<td>0.0373%</td>
<td>0.016</td>
<td>0.146</td>
<td>5.261</td>
<td>1015</td>
</tr>
<tr>
<td>Indonesia</td>
<td>Datastream Global</td>
<td>0.0424%</td>
<td>0.024</td>
<td>0.117</td>
<td>9.149</td>
<td>1041</td>
</tr>
<tr>
<td>Ireland</td>
<td>Datastream Global</td>
<td>0.0861%</td>
<td>0.011</td>
<td>-0.022</td>
<td>11.937</td>
<td>1061</td>
</tr>
<tr>
<td>Israel</td>
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<td>0.0557%</td>
<td>0.012</td>
<td>0.015</td>
<td>8.593</td>
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</tr>
<tr>
<td>Italy</td>
<td>Datastream Global</td>
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<td>0.014</td>
<td>0.057</td>
<td>5.487</td>
<td>1067</td>
</tr>
<tr>
<td>Japan</td>
<td>Datastream Global</td>
<td>0.0233%</td>
<td>0.012</td>
<td>0.066</td>
<td>5.395</td>
<td>1043</td>
</tr>
<tr>
<td>Jordan</td>
<td>MSCI Price Index</td>
<td>-0.0472%</td>
<td>0.010</td>
<td>0.175</td>
<td>11.209</td>
<td>662</td>
</tr>
<tr>
<td>Kenya</td>
<td>MSCI Price Index</td>
<td>-0.0321%</td>
<td>0.007</td>
<td>0.011</td>
<td>9.237</td>
<td>964</td>
</tr>
<tr>
<td>Korea</td>
<td>Datastream Global</td>
<td>0.0413%</td>
<td>0.026</td>
<td>0.121</td>
<td>5.368</td>
<td>1034</td>
</tr>
<tr>
<td>Latvia</td>
<td>HSBC Price Index</td>
<td>-0.2718%</td>
<td>0.021</td>
<td>-0.152</td>
<td>9.688</td>
<td>469</td>
</tr>
<tr>
<td>Lithuania</td>
<td>HSBC Price Index</td>
<td>-0.0858%</td>
<td>0.018</td>
<td>-0.041</td>
<td>13.335</td>
<td>470</td>
</tr>
<tr>
<td>Luxembourg</td>
<td>Datastream Global</td>
<td>0.0743%</td>
<td>0.008</td>
<td>0.063</td>
<td>13.778</td>
<td>1041</td>
</tr>
<tr>
<td>Country</td>
<td>Index</td>
<td>Daily Return (%)</td>
<td>Volatility</td>
<td>Sharpe Ratio</td>
<td>Beta</td>
<td></td>
</tr>
<tr>
<td>------------------</td>
<td>------------------------------</td>
<td>------------------</td>
<td>------------</td>
<td>--------------</td>
<td>-------</td>
<td></td>
</tr>
<tr>
<td>Malaysia</td>
<td>Datastream Global</td>
<td>-0.0225%</td>
<td>0.023</td>
<td>0.034</td>
<td>27.012</td>
<td>1048</td>
</tr>
<tr>
<td>Malta</td>
<td>HSBC Price Index</td>
<td>0.0846%</td>
<td>0.011</td>
<td>0.098</td>
<td>12.099</td>
<td>931</td>
</tr>
<tr>
<td>Morocco</td>
<td>MSCI Price Index</td>
<td>0.0262%</td>
<td>0.007</td>
<td>0.015</td>
<td>18.301</td>
<td>789</td>
</tr>
<tr>
<td>Mauritius</td>
<td>SE SEMDEX</td>
<td>0.0720%</td>
<td>0.014</td>
<td>0.139</td>
<td>9.511</td>
<td>1074</td>
</tr>
<tr>
<td>Mexico</td>
<td>Datastream Global</td>
<td>0.0924%</td>
<td>0.006</td>
<td>0.148</td>
<td>6.828</td>
<td>1015</td>
</tr>
<tr>
<td>Netherlands</td>
<td>Datastream Global</td>
<td>0.0957%</td>
<td>0.012</td>
<td>-0.201</td>
<td>5.715</td>
<td>1066</td>
</tr>
<tr>
<td>New Zealand</td>
<td>Datastream Global</td>
<td>0.0093%</td>
<td>0.010</td>
<td>-0.036</td>
<td>36.220</td>
<td>1059</td>
</tr>
<tr>
<td>Norway</td>
<td>Datastream Global</td>
<td>0.0546%</td>
<td>0.012</td>
<td>-0.053</td>
<td>8.707</td>
<td>1058</td>
</tr>
<tr>
<td>Oman</td>
<td>Muscat Securities</td>
<td>0.0468%</td>
<td>0.019</td>
<td>-0.082</td>
<td>63.830</td>
<td>497</td>
</tr>
<tr>
<td>Pakistan</td>
<td>MSCI Price Index</td>
<td>-0.0587%</td>
<td>0.025</td>
<td>-0.035</td>
<td>8.638</td>
<td>929</td>
</tr>
<tr>
<td>Peru</td>
<td>MSCI Price Index</td>
<td>0.0040%</td>
<td>0.014</td>
<td>0.014</td>
<td>7.794</td>
<td>1053</td>
</tr>
<tr>
<td>Philippines</td>
<td>Datastream Global</td>
<td>-0.0163%</td>
<td>0.016</td>
<td>-0.087</td>
<td>7.431</td>
<td>1055</td>
</tr>
<tr>
<td>Poland</td>
<td>Datastream Global</td>
<td>0.0716%</td>
<td>0.019</td>
<td>-0.085</td>
<td>5.763</td>
<td>1052</td>
</tr>
<tr>
<td>Portugal</td>
<td>Datastream Global</td>
<td>0.0889%</td>
<td>0.011</td>
<td>0.067</td>
<td>11.305</td>
<td>1039</td>
</tr>
<tr>
<td>Romania</td>
<td>BET Price Index</td>
<td>-0.1125%</td>
<td>0.023</td>
<td>0.083</td>
<td>6.695</td>
<td>537</td>
</tr>
<tr>
<td>Russia</td>
<td>MSCI Price Index</td>
<td>0.0439%</td>
<td>0.043</td>
<td>-0.019</td>
<td>9.209</td>
<td>1065</td>
</tr>
<tr>
<td>Singapore</td>
<td>Datastream Global</td>
<td>0.0208%</td>
<td>0.014</td>
<td>-0.007</td>
<td>8.459</td>
<td>1060</td>
</tr>
<tr>
<td>Slovakia</td>
<td>SIX All Price Index</td>
<td>-0.0300%</td>
<td>0.009</td>
<td>-0.057</td>
<td>4.365</td>
<td>985</td>
</tr>
<tr>
<td>Slovenia</td>
<td>PIX Price Index</td>
<td>0.1710%</td>
<td>0.014</td>
<td>0.406</td>
<td>6.619</td>
<td>184</td>
</tr>
<tr>
<td>South Africa</td>
<td>Datastream Global</td>
<td>0.0473%</td>
<td>0.013</td>
<td>-0.084</td>
<td>18.615</td>
<td>1072</td>
</tr>
<tr>
<td>Spain</td>
<td>Datastream Global</td>
<td>0.1033%</td>
<td>0.013</td>
<td>-0.090</td>
<td>7.421</td>
<td>1053</td>
</tr>
<tr>
<td>Sri Lanka</td>
<td>MSCI Price Index</td>
<td>-0.0367%</td>
<td>0.012</td>
<td>-0.091</td>
<td>7.750</td>
<td>1009</td>
</tr>
<tr>
<td>Sweden</td>
<td>Datastream Global</td>
<td>0.1043%</td>
<td>0.013</td>
<td>-0.052</td>
<td>9.258</td>
<td>1062</td>
</tr>
<tr>
<td>Switzerland</td>
<td>Datastream Global</td>
<td>0.0807%</td>
<td>0.011</td>
<td>-0.153</td>
<td>6.481</td>
<td>1061</td>
</tr>
<tr>
<td>Taiwan</td>
<td>Datastream Global</td>
<td>0.0705%</td>
<td>0.017</td>
<td>0.093</td>
<td>5.154</td>
<td>1023</td>
</tr>
<tr>
<td>Thailand</td>
<td>Datastream Global</td>
<td>-0.0911%</td>
<td>0.025</td>
<td>0.245</td>
<td>6.566</td>
<td>1039</td>
</tr>
<tr>
<td>Turkey</td>
<td>Datastream Global</td>
<td>0.2821%</td>
<td>0.032</td>
<td>0.222</td>
<td>5.676</td>
<td>1097</td>
</tr>
<tr>
<td>UK</td>
<td>Datastream Global</td>
<td>0.0567%</td>
<td>0.009</td>
<td>-0.100</td>
<td>4.753</td>
<td>1079</td>
</tr>
<tr>
<td>US</td>
<td>Datastream Global</td>
<td>0.0904%</td>
<td>0.010</td>
<td>-0.156</td>
<td>7.780</td>
<td>1066</td>
</tr>
<tr>
<td>Venezuela</td>
<td>Datastream Global</td>
<td>0.1036%</td>
<td>0.022</td>
<td>0.072</td>
<td>14.268</td>
<td>1049</td>
</tr>
<tr>
<td>Zimbabwe</td>
<td>SE Industrials</td>
<td>0.1259%</td>
<td>0.012</td>
<td>0.039</td>
<td>15.439</td>
<td>999</td>
</tr>
</tbody>
</table>

The largest reward for unit of carried risk is noted for Iceland (0.176333), Mexico (0.154) and Cyprus (0.144778). It is measured by the simplified Sharpe ratio (not presented in the table), which is equal to the mean daily return divided by the standard deviation of daily returns. The lowest simplified Sharpe ratios are noted for Latvia (-0.12943) and Croatia (-0.05793). The average for all 66 countries is equal to 0.037. There is no clear tendency in the data to be positively or negatively skewed. The largest left skew is noted for Germany (-0.258), the largest right skew is observed for Slovenia (0.406) and the most symmetric distributions are these of the

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daily returns in Argentina (0.006) and Singapore (-0.007). Generally, the distributions tend to be symmetric. The absolute values of skewness parameter are less than 0.22 for 60 and less than 0.1 for 42 countries. The fourth column of the Table 5.2 shows that the analysed distributions are rather heavy tailed, with kurtosis ranging from 4.365 for Slovakia to 63.83 for Oman (the average kurtosis is equal to 11.743). This would suggest that the real data is leptocurtic and heavy tailed distribution should be a good approximation of the observed returns.

V.3.2. Filtering for GARCH effects: an application of the simulated annealing algorithm

The kurtosis presented in the Table 5.1 shows that distributions of daily returns to the 66 analysed markets are heavy tailed and hence do not seem to be generated by normal random innovations. As stated earlier, such a situation may be caused by speculative character of the price generating process. However, it is also possible that the heavy tailed behaviour of the distributions of returns is simply due to risk neutrality of investors. In such a situation, the prices follow an ARCH-type process, which results in a heavy-tailed distribution without violating the no-speculation hypothesis. In order to avoid confusion of market speculation and risk neutrality and to make the interpretation of results more transparent, the ARCH-caused heteroscedascity of the data should be removed before estimating the heavy tailed distribution parameters.

The GARCH(1,1) process can be described by the following equations (see e.g. Hamilton, 1994: Chapter 21):

\[ x_t = \mu + h_t \varepsilon_t, \]  
\[ \text{where:} \]
\[ h_t = \kappa + \alpha (x-\mu)^2_{t-1} + \beta h_{t-1}, \]
\[ \varepsilon_t \sim N(0,1). \]

The condition (5.1c) means that independent identically distributed random variable \( \varepsilon_t \) follows standard normal distribution with mean 0 and variance 1. Application of models described by equations (5.1a) - (5.1c) requires estimation of only four parameters. This is a relatively simply task and the analysed data set offers
sufficient information for such estimation. More advanced analysis of the GARCH-type processes, although very interesting and promising, seems to go beyond the scope of this study.

A complete analysis of possible risk neutral GARCH effects in a series of returns should include estimation of a number of different GARCH models and the one that fits the sample the most closely should be chosen. However, there are some difficulties in appropriate selection of the GARCH models. First, the decision about the maximum possible lags in the GARCH process must be taken in an arbitrary way. Second, due to small absolute values of the daily returns, the likelihood function takes values close to zero, making the standard criteria (e.g. Akaike or Schwarz) not fully applicable. The solution would be to test all the possible models against each other but this is rather time-consuming. Consequently, it seems to be plausible (and convenient) to assume that the majority of the ARCH-type heteroscedasticity can be captured by a GARCH(1,1) model. Even if returns were generated by a more complicated GARCH process, the GARCH(1,1) model should capture the majority of the variability of conditional variance. The remaining heteroscedasticity should not be significant for further analysis.

The parameters of the GARCH(1,1) model, as described in (5.1a) – (5.1c), can be estimated using the maximum likelihood method. For normally distributed disturbances, the log likelihood function for a sample of T observations is:

\[ \ln L = \sum_t -0.5 \cdot \left[ \ln(2\pi) + \ln h_t + \frac{(x_t-\mu)^2}{h_t} \right] \]

The possibly multimodal character of the likelihood function causes difficulties in convergence when standard optimisation procedures (see e.g. American University GAUSS Software Archive) are applied. In order to find the global maximum of this function, the simulated annealing optimisation algorithm is applied (see e.g. Corana et al, 1987 and Goffe et al, 1994).

The simulated annealing algorithm not only deals with the problem of multiple local optima but also sorts out the difficulties connected with lack of differentiability of certain parameters. It is based on an analogy to the

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4 GARCH models of higher orders have been fitted but produced very similar results.
thermodynamics systems, in particular to the way the liquid metals solidify when the temperature falls. If the drop in temperature is slow, the molecules in material become settled in a highly ordered way, forming a native crystal, which is the global minimum energetic state of the material. However, when the drop in temperature is too fast, the material does not solidify in such an ordered way and its structure presents defects, resulting in the local energy minimum, with higher energy than the native crystal, being achieved. Maximising a function with conventional algorithms, which only accept points leading to higher values of the function under consideration (uphill moves), is similar to cooling a liquid rapidly. This means that such algorithms are very likely to reach a local maximum instead of the global one. The simulated annealing allows not only uphill but also downhill moves. Their frequency is controlled by the parameter called temperature. At high temperature the algorithm overviews the function domain and forms a general picture. As the temperature drops, the algorithm concentrates on the most promising parts of the domain (Corana et al, 1987 and Goffe et al, 1994).

The optimisation starts by drawing a sequence of points around the initial parameters. The new candidate points are generated around the current points, applying random moves along each axis. The candidate points are uniformly distributed in intervals centred about the corresponding element of the parameter vector. If the point falls outside the domain of the function, a new point is randomly generated in its place. If the value of the function in a new point is higher than this in the old one, the point is accepted and the parameter vector is updated. If the function at the new point takes lower value, the point is accepted or rejected according to the Metropolis criterion (Metropolis et al, 1953). The point is accepted with probability $p$ when the move causes the maximised function to decrease. Probability $p$ is equal to exponent of the ratio of the decease in the maximised function and the value of the temperature parameter $T$. In practice, a pseudo-random number is generated in the range $[0,1]$ and compared with $p$. If it is larger than $p$, the point is rejected, otherwise it is accepted. The initial value of $T$ is defined by the researcher and $T$ decreases when the estimation proceeds. If the value of $T$ is large, almost every new point is accepted and the succession is a random sampling from the domain. When $T$ increases, the probability of accepting a candidate point decreases and eventually almost only points connected with increase of the maximised function are accepted.
Careful choice of the initial $T$ guarantees a relatively short time of estimation and searching through the whole domain of the function. The whole procedure is relatively fast. For unimodal functions, it gives similar results to the traditional optimisations. In contradiction to traditional methods, it does work with all analysed series (Corana et al, 1987 and Goffe et al, 1994).

Table 5.3 presented the estimated parameters of $\text{GARCH}(1,1)$ process for the 66 stock market indices. The estimates of series means and constant terms in the conditional variance equation tend to be close to zero. Additionally, sums of the estimates of $\alpha$ and $\beta$ tend to be close to the imposed limits (to preserve stationarity and invertability of the process, $\alpha+\beta$ must be less than 1). The simulated annealing algorithm does not provide the standard deviations of the estimates and hence their significance cannot be tested.

It is believed that filtering the data by estimated $\text{GARCH}(1,1)$ processes can help to separate the heavy tail effect connected with the risk neutrality of agents. In other words, it is claimed that the filtered series should provide more accurate information about the level of speculation than the original ones. Therefore, after estimating the parameters of $\text{GARCH}(1,1)$ processes, the theoretical values of the conditional variances are calculated. Then, the observed returns are divided by the respective theoretical conditional standard deviations and, finally, they are multiplied by the average conditional standard deviation for a given series.

---

5 The parameters of $\text{GARCH}(1,1)$ model are estimated using software developed in GAUSS. The values of the maximum likelihood function of $\text{GARCH}(1,1)$ processes are computed using the Ronald Schoenberg procedure, which can be downloaded from http://faculty.washington.edu/rons/garch.html. The simulated annealing maximisation procedure has been written by E.G. Tsionans on the basis of Corana et al (1987) and Goffe et al (1994) papers and downloaded from: http://netec.wustl.edu/CodEe/GaussAtAmericanU/OPTIMIZE/OPTIMIZE.HTM. The procedure is modified to incorporate the necessary restrictions.
Table 5.3. Parameters of GARCH(1,1) model for international market indices

Column (1) gives the estimates of mean of the series, column (2) shows the estimates of constant term in conditional variance equation. Columns (3) and (4) present the estimates of coefficients at the lagged conditional variance and lagged square observation in the conditional variance equation respectively. Column (5) gives the maximised values of the likelihood function.

<table>
<thead>
<tr>
<th>Country</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>μ</td>
<td>κ</td>
<td>α</td>
<td>β</td>
<td>Max. Lik.</td>
</tr>
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<td>Argentina</td>
<td>0.00102</td>
<td>0.00001850</td>
<td>0.77820</td>
<td>0.17044</td>
<td>2.73160</td>
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<tr>
<td>Australia</td>
<td>0.00055</td>
<td>0.00000703</td>
<td>0.81344</td>
<td>0.09556</td>
<td>3.35138</td>
</tr>
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The filtered series is of the form:

$$r_t = \frac{x_t}{\sqrt{h_t}} \cdot \sqrt{\bar{h}_{t}}$$

where $x_t$ is the original series of returns and $h_t$ is the conditional variance evaluated on the basis of the parameters of an appropriate GARCH(1,1) model and $\bar{h}_t$ is the average $h_t$ for a given series. Such an approach should remove the GARCH-type heteroskedascity without affecting the average dispersion of returns.
V.3.3. Distribution of returns to filtered international indices

It is assumed that, after removing the effects of risk neutrality, the thickness of tails and dispersion of distribution of returns should reflect market’s exposure to speculation and random shocks. Hence, in this section, parameters of the stable distribution are estimated for the filtered series of returns to the 66 analysed international indices. The hypothesis, that the stable law is an accurate approximation of the actual distributions of returns, is tested.

In order to check the heavy tailed character of the filtered distributions of returns, for each series the parameters of stable distributions are estimated using Nolan (1999) maximum likelihood method. Nolan’s (1999) program maximises the log likelihood function for an independent identically distributed stable sample $X_1, X_2, \ldots X_n$:

$$l(\theta) = \prod_{i=1}^{n} \log f(X_i | \theta).$$

The maximisation is done over the parameter space $\Theta = (0,2] \times [-1,1] \times (0,\infty) \times (-\infty,\infty)$. Parameter vector $\theta$ consists of characteristic exponent $\alpha$, skewness parameter $\beta$, scale parameter $c$ and location parameter $\delta$.

The program to estimate parameters of stable distribution using the maximum likelihood approach can be downloaded from the John Nolan web page at American University (http://www.cas.american.edu/~jpnolan). Not many details about the computational part of the procedure and properties of the finite sample estimates are known. The available program does not compute the standard deviations of the estimated parameters and hence it is impossible to formally test for heavy-tailed or character of the data and for symmetric form of the distribution of returns. Despite these limitations, Nolan (1999) maximum likelihood seems to be the most accurate available method to estimate the stable distribution parameters and for conventional series it tends to out-perform other approaches (for discussion see Chapter II.2.).

---

* The McCulloch (1986) quantile algorithm applied to the analysed data set gave very similar results to those obtained in Nolan (1999) maximum likelihood.
Table 5.4: Estimates of stable distribution parameters for international indices

Column (1) gives the estimates of characteristic exponents, column (2) provides the estimates of skewness parameter and columns (3) and (4) of the scale and location parameters respectively.

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</table>

The estimates of the characteristic exponent $\alpha$ vary from 1.1606 for Bangladesh to 1.8665 for Australia with the average $\alpha$ being equal to 1.5806. The skewness parameter $\beta$ lies in the range from –0.4845 (Germany) to 0.3881 (Cyprus). The most symmetric distributions are noted for Lithuania and Bangladesh. This is similar to results presented in Table 5.2. The scale parameter $c$ varies from 0.0289 for Mexico to 0.02021 for Russia. Again, the ranking is not very different from classification based on the standard deviations of raw returns in Table 5.1. The estimates of location parameters are generally close to zero, with the biggest value for Germany (0.001189) and the lowest value for Thailand (-0.0031).
In Chapter IV.2, it is shown that the distribution of returns to Diba-Grossman type processes can be approximated by the symmetric stable distribution. Hence, it is desirable to test whether the symmetric stable distributions, described by parameters $\alpha$, $c$ and $\delta$ from Table 5.4, form acceptable approximations of the actual returns. Only if this is true, further analysis, including conversion to the parameters of the Diba-Grossman process, is plausible. In order to test the goodness of fit of the above stable distributions, four different statistical tests of fit are applied: the $\chi^2$ test, the Anderson-Darling test (1952, 1954), the Cramer-von Mises test (see e.g. Anderson and Darling, 1954) the Kolmogorov-Smirnov test (see e.g. Stuart and Ord, 1991: p. 1187).

The general properties of the $\chi^2$ test are described in Chapter II.3.2. In this section, a $\chi^2$ test with 30 equiprobable cells is applied. Its applications to test for symmetric stable distribution of returns to Diba-Grossman processes are discussed in Chapter IV.2.2.

Anderson and Darling (1952, 1954) propose the following procedure to test for goodness of fit of fully specified cumulative distribution functions to the sample data. If $X_1 \leq X_2 \leq \ldots \leq X_n$ are the $n$ ordered observations in the sample and $w = F(X_i)$, then the suggested test statistic is of the form:

$$W = -n - \frac{1}{n} \sum_{j=1}^{n} (2j - 1) \left[ \ln(w_j) + \ln(1 - w_{n-j+1}) \right].$$

Lewis (1961) provides comprehensive tables of this statistic.

Cramer and von Mises suggest a different statistic to test the goodness of fit (Anderson and Darling, 1954):

$$W = \frac{1}{12n} + \sum_{j=1}^{n} \left( u_j - \frac{2j-1}{2n} \right)^2.$$

Kolmogorov and Smirnov apply the principle of biggest distance (largest divergence) between theoretical and sample distribution functions. The statistic is of the form (Stuart and Ord, 1991: p. 1187):

$$W = \sup_j \left| \frac{j}{n} - u_j \right|.$$
Extensive research has been done to compare the efficiency of the above methods (for summary, see e.g. Stuart and Ord, 1991: Chapter 30). However, it goes beyond the scope of this study to verify the small sample properties and other characteristics of the goodness of fit tests. Hence, all reviewed methods are applied and the final conclusion is drawn after analysing all obtained statistics.

Table 5.5 presents statistics obtained in the above tests, along with the corresponding $p$ values. $P$ values are equal to the maximum significance levels at which the null hypothesis, that the analysed series follow stable distributions, cannot be rejected. The $\chi^2$ test rejects the null about stable distribution of returns to stock market indices for 12 countries at 1% significance level and for 23 and 26 at 5% and 10% significance levels. Respective numbers for the Anderson–Darling test are 3, 7 and 13, for the Cramer–von Mises test 2, 3 and 8 and for the Kolmogorov–Smirnov test 1, 5 and 12. Clearly, the $\chi^2$ test rejects the null hypothesis more often than other applied tests. Additionally, it should be noted that the tests lead to slightly different conclusions, as only for three countries: Cyprus, Thailand and Germany, the null hypothesis is rejected by all tests at a relatively restrictive 10% significance level. As this is only 4.5% of the sample, it seems to be plausible to assume that generally the filtered returns to the 66 analysed international stock price indices follow the symmetric stable distribution with parameters listed in Table 5.3. However, special caution will be taken when interpreting results connected with Cyprus, Thailand and Germany.

7 Testing is performed with the help of a GAUSS code, similar to those applied in Chapter II. For the other three tests, the GAUSS code developed by Baird (1994) is adapted.
Table 5.5: Goodness of fit statistics for international indices: stable distribution

Table shows the values of goodness of fit statistics to test the hypothesis that returns to price indices in the listed countries follow symmetric stable distributions with parameters listed in Columns (1), (3) and (4) of Table 5.3. Column (1) gives the value of the chi-squared statistic. Column (2) provides the Anderson-Darling statistic for each series. Columns (3) and (4) list values of Cramer-von Mises and Kolmogorov-Smirnov statistics. Columns (1a) - (4a) show the respective maximum significance levels at which the hypothesis that the series are stable distributed cannot be rejected.

<table>
<thead>
<tr>
<th>Country</th>
<th>Chi-squared</th>
<th>P value</th>
<th>Anderson-Darling</th>
<th>P value</th>
<th>Cramer-von Mises</th>
<th>P value</th>
<th>Kolmogorov-Smirnov</th>
<th>P value</th>
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<td>0.051</td>
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<td>0.098</td>
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<td>0.598</td>
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<td>0.561</td>
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<td>0.052</td>
<td>0.360</td>
<td>0.089</td>
<td>0.037</td>
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</tr>
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<td>2.899</td>
<td>0.031</td>
<td>0.384</td>
<td>0.077</td>
<td>0.035</td>
<td>0.141</td>
</tr>
<tr>
<td>New Zealand</td>
<td>26.083</td>
<td>0.621</td>
<td>0.389</td>
<td>0.734</td>
<td>0.063</td>
<td>0.567</td>
<td>0.020</td>
<td>1.000</td>
</tr>
<tr>
<td>Norway</td>
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<td>0.122</td>
<td>0.722</td>
<td>0.468</td>
<td>0.107</td>
<td>0.427</td>
<td>0.025</td>
<td>0.734</td>
</tr>
<tr>
<td>Oman</td>
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<td>1.528</td>
<td>0.165</td>
<td>0.255</td>
<td>0.169</td>
<td>0.053</td>
<td>0.114</td>
</tr>
<tr>
<td>Pakistan</td>
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<td>0.395</td>
<td>1.081</td>
<td>0.292</td>
<td>0.194</td>
<td>0.247</td>
<td>0.033</td>
<td>0.273</td>
</tr>
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<td>Peru</td>
<td>20.814</td>
<td>0.866</td>
<td>0.908</td>
<td>0.366</td>
<td>0.138</td>
<td>0.350</td>
<td>0.029</td>
<td>0.394</td>
</tr>
<tr>
<td>Philippines</td>
<td>29.700</td>
<td>0.429</td>
<td>0.821</td>
<td>0.410</td>
<td>0.112</td>
<td>0.413</td>
<td>0.025</td>
<td>0.659</td>
</tr>
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<td>Poland</td>
<td>25.775</td>
<td>0.638</td>
<td>0.756</td>
<td>0.448</td>
<td>0.126</td>
<td>0.377</td>
<td>0.029</td>
<td>0.395</td>
</tr>
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<td>0.873</td>
<td>0.716</td>
<td>0.472</td>
<td>0.107</td>
<td>0.427</td>
<td>0.028</td>
<td>0.484</td>
</tr>
<tr>
<td>Romania</td>
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<td>0.611</td>
<td>0.276</td>
<td>0.857</td>
<td>0.038</td>
<td>0.671</td>
<td>0.026</td>
<td>1.000</td>
</tr>
<tr>
<td>Russia</td>
<td>46.564</td>
<td>0.021</td>
<td>1.238</td>
<td>0.238</td>
<td>0.201</td>
<td>0.236</td>
<td>0.030</td>
<td>0.327</td>
</tr>
<tr>
<td>Singapore</td>
<td>34.966</td>
<td>0.206</td>
<td>0.555</td>
<td>0.586</td>
<td>0.093</td>
<td>0.469</td>
<td>0.022</td>
<td>1.000</td>
</tr>
<tr>
<td>Slovakia</td>
<td>21.854</td>
<td>0.826</td>
<td>0.711</td>
<td>0.475</td>
<td>0.133</td>
<td>0.362</td>
<td>0.030</td>
<td>0.380</td>
</tr>
<tr>
<td>Slovenia</td>
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<td>0.571</td>
<td>0.332</td>
<td>0.793</td>
<td>0.041</td>
<td>0.658</td>
<td>0.015</td>
<td>1.000</td>
</tr>
<tr>
<td>South Africa</td>
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<td>0.113</td>
<td>1.112</td>
<td>0.281</td>
<td>0.168</td>
<td>0.290</td>
<td>0.028</td>
<td>0.423</td>
</tr>
<tr>
<td>Spain</td>
<td>30.510</td>
<td>0.389</td>
<td>1.154</td>
<td>0.266</td>
<td>0.172</td>
<td>0.282</td>
<td>0.029</td>
<td>0.405</td>
</tr>
<tr>
<td>Sri Lanka</td>
<td>29.857</td>
<td>0.421</td>
<td>1.502</td>
<td>0.170</td>
<td>0.225</td>
<td>0.203</td>
<td>0.034</td>
<td>0.210</td>
</tr>
<tr>
<td>Sweden</td>
<td>22.591</td>
<td>0.795</td>
<td>1.224</td>
<td>0.243</td>
<td>0.165</td>
<td>0.296</td>
<td>0.025</td>
<td>0.740</td>
</tr>
<tr>
<td>Switzerland</td>
<td>31.094</td>
<td>0.361</td>
<td>2.165</td>
<td>0.075</td>
<td>0.326</td>
<td>0.110</td>
<td>0.036</td>
<td>0.120</td>
</tr>
<tr>
<td>Taiwan</td>
<td>48.841</td>
<td>0.012</td>
<td>1.059</td>
<td>0.300</td>
<td>0.143</td>
<td>0.339</td>
<td>0.029</td>
<td>0.402</td>
</tr>
<tr>
<td>Thailand</td>
<td>51.538</td>
<td>0.006</td>
<td>5.682</td>
<td>0.002</td>
<td>0.863</td>
<td>0.005</td>
<td>0.051</td>
<td>0.010</td>
</tr>
<tr>
<td>Turkey</td>
<td>120.310</td>
<td>0.000</td>
<td>1.307</td>
<td>0.218</td>
<td>0.272</td>
<td>0.151</td>
<td>0.044</td>
<td>0.026</td>
</tr>
<tr>
<td>UK</td>
<td>31.610</td>
<td>0.337</td>
<td>0.858</td>
<td>0.391</td>
<td>0.122</td>
<td>0.389</td>
<td>0.026</td>
<td>0.558</td>
</tr>
<tr>
<td>US</td>
<td>42.690</td>
<td>0.049</td>
<td>1.274</td>
<td>0.228</td>
<td>0.181</td>
<td>0.267</td>
<td>0.026</td>
<td>0.569</td>
</tr>
<tr>
<td>Venezuela</td>
<td>28.737</td>
<td>0.479</td>
<td>0.459</td>
<td>0.667</td>
<td>0.064</td>
<td>0.566</td>
<td>0.018</td>
<td>1.000</td>
</tr>
<tr>
<td>Zimbabwe</td>
<td>35.226</td>
<td>0.197</td>
<td>0.323</td>
<td>0.803</td>
<td>0.044</td>
<td>0.646</td>
<td>0.019</td>
<td>1.000</td>
</tr>
</tbody>
</table>
Table 5.6: Goodness of fit statistics for international indices: normal distribution

Table shows the values of goodness of fit statistics to test the hypothesis that returns to price indices in the listed countries follow normal distribution. Column (1) gives the value of the chi-squared statistic. Column (2) provides the Anderson- Darling statistic for each series. Columns (3) and (4) list values of Cramer-von Mises and Kolmogorov- Smirnov statistics. Columns (1a) - (4a) show the respective maximum significance levels at which the hypothesis that the series are stable distributed cannot be rejected.

<table>
<thead>
<tr>
<th>Country</th>
<th>(1) Chi-squared</th>
<th>(1a) P value</th>
<th>(2) Anderson-Darling</th>
<th>(2a) P value</th>
<th>(3) Cramer-von Mises</th>
<th>(3a) P value</th>
<th>(4) Kolmogorov-v-Smirnov</th>
<th>(4a) P value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argentina</td>
<td>273.400</td>
<td>0.000%</td>
<td>23.610</td>
<td>0.000%</td>
<td>4.210</td>
<td>0.000%</td>
<td>0.093</td>
<td>0.000%</td>
</tr>
<tr>
<td>Australia</td>
<td>46.260</td>
<td>2.210%</td>
<td>4.017</td>
<td>0.886%</td>
<td>0.609</td>
<td>2.104%</td>
<td>0.038</td>
<td>8.769%</td>
</tr>
<tr>
<td>Austria</td>
<td>116.000</td>
<td>0.000%</td>
<td>10.930</td>
<td>0.001%</td>
<td>1.913</td>
<td>0.003%</td>
<td>0.074</td>
<td>0.001%</td>
</tr>
<tr>
<td>Bangladesh</td>
<td>701.500</td>
<td>0.000%</td>
<td>64.660</td>
<td>0.000%</td>
<td>12.470</td>
<td>0.000%</td>
<td>0.189</td>
<td>0.000%</td>
</tr>
<tr>
<td>Belgium</td>
<td>76.670</td>
<td>0.000%</td>
<td>6.466</td>
<td>0.071%</td>
<td>1.112</td>
<td>0.145%</td>
<td>0.062</td>
<td>0.072%</td>
</tr>
<tr>
<td>Brazil</td>
<td>228.700</td>
<td>0.000%</td>
<td>23.920</td>
<td>0.000%</td>
<td>4.280</td>
<td>0.000%</td>
<td>0.100</td>
<td>0.000%</td>
</tr>
<tr>
<td>Canada</td>
<td>106.200</td>
<td>0.000%</td>
<td>12.010</td>
<td>0.000%</td>
<td>1.976</td>
<td>0.003%</td>
<td>0.072</td>
<td>0.002%</td>
</tr>
<tr>
<td>Chile</td>
<td>96.810</td>
<td>0.000%</td>
<td>8.620</td>
<td>0.009%</td>
<td>1.388</td>
<td>0.038%</td>
<td>0.066</td>
<td>0.019%</td>
</tr>
<tr>
<td>China</td>
<td>167.000</td>
<td>0.000%</td>
<td>15.130</td>
<td>0.000%</td>
<td>2.519</td>
<td>0.000%</td>
<td>0.079</td>
<td>0.000%</td>
</tr>
<tr>
<td>Columbia</td>
<td>229.900</td>
<td>0.000%</td>
<td>25.020</td>
<td>0.000%</td>
<td>4.529</td>
<td>0.000%</td>
<td>0.108</td>
<td>0.000%</td>
</tr>
<tr>
<td>Croatia</td>
<td>86.220</td>
<td>0.000%</td>
<td>6.475</td>
<td>0.069%</td>
<td>1.178</td>
<td>0.103%</td>
<td>0.091</td>
<td>0.088%</td>
</tr>
<tr>
<td>Cyprus</td>
<td>563.800</td>
<td>0.000%</td>
<td>63.080</td>
<td>0.000%</td>
<td>11.560</td>
<td>0.000%</td>
<td>0.179</td>
<td>0.000%</td>
</tr>
<tr>
<td>Czech Republic</td>
<td>118.200</td>
<td>0.000%</td>
<td>11.960</td>
<td>0.001%</td>
<td>2.116</td>
<td>0.001%</td>
<td>0.076</td>
<td>0.000%</td>
</tr>
<tr>
<td>Denmark</td>
<td>133.600</td>
<td>0.000%</td>
<td>15.620</td>
<td>0.000%</td>
<td>2.733</td>
<td>0.000%</td>
<td>0.087</td>
<td>0.000%</td>
</tr>
<tr>
<td>Egypt</td>
<td>221.900</td>
<td>0.000%</td>
<td>20.390</td>
<td>0.000%</td>
<td>3.828</td>
<td>0.000%</td>
<td>0.128</td>
<td>0.000%</td>
</tr>
<tr>
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<td>246.200</td>
<td>0.000%</td>
<td>28.950</td>
<td>0.000%</td>
<td>5.178</td>
<td>0.000%</td>
<td>0.135</td>
<td>0.000%</td>
</tr>
<tr>
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<td>75.180</td>
<td>0.001%</td>
<td>7.197</td>
<td>0.034%</td>
<td>1.194</td>
<td>0.097%</td>
<td>0.054</td>
<td>0.406%</td>
</tr>
<tr>
<td>France</td>
<td>73.370</td>
<td>0.001%</td>
<td>6.215</td>
<td>0.089%</td>
<td>0.943</td>
<td>0.346%</td>
<td>0.059</td>
<td>0.123%</td>
</tr>
<tr>
<td>Germany</td>
<td>88.410</td>
<td>0.000%</td>
<td>9.510</td>
<td>0.004%</td>
<td>1.569</td>
<td>0.016%</td>
<td>0.072</td>
<td>0.002%</td>
</tr>
<tr>
<td>Greece</td>
<td>168.400</td>
<td>0.000%</td>
<td>15.910</td>
<td>0.000%</td>
<td>2.805</td>
<td>0.000%</td>
<td>0.082</td>
<td>0.000%</td>
</tr>
<tr>
<td>Hong Kong</td>
<td>229.200</td>
<td>0.000%</td>
<td>24.390</td>
<td>0.000%</td>
<td>4.352</td>
<td>0.000%</td>
<td>0.100</td>
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</tr>
<tr>
<td>Hungary</td>
<td>142.600</td>
<td>0.000%</td>
<td>19.060</td>
<td>0.000%</td>
<td>3.196</td>
<td>0.000%</td>
<td>0.090</td>
<td>0.000%</td>
</tr>
<tr>
<td>Iceland</td>
<td>108.700</td>
<td>0.000%</td>
<td>10.100</td>
<td>0.002%</td>
<td>1.749</td>
<td>0.007%</td>
<td>0.086</td>
<td>0.009%</td>
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<tr>
<td>India</td>
<td>51.620</td>
<td>0.601%</td>
<td>3.727</td>
<td>1.222%</td>
<td>0.531</td>
<td>3.275%</td>
<td>0.044</td>
<td>3.493%</td>
</tr>
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<td>Indonesia</td>
<td>290.000</td>
<td>0.000%</td>
<td>28.500</td>
<td>0.000%</td>
<td>5.251</td>
<td>0.000%</td>
<td>0.119</td>
<td>0.000%</td>
</tr>
<tr>
<td>Ireland</td>
<td>140.800</td>
<td>0.000%</td>
<td>17.530</td>
<td>0.000%</td>
<td>2.962</td>
<td>0.000%</td>
<td>0.088</td>
<td>0.000%</td>
</tr>
<tr>
<td>Israel</td>
<td>186.300</td>
<td>0.000%</td>
<td>16.760</td>
<td>0.000%</td>
<td>3.098</td>
<td>0.000%</td>
<td>0.090</td>
<td>0.000%</td>
</tr>
<tr>
<td>Italy</td>
<td>61.110</td>
<td>0.045%</td>
<td>5.846</td>
<td>0.129%</td>
<td>0.945</td>
<td>0.342%</td>
<td>0.052</td>
<td>0.647%</td>
</tr>
<tr>
<td>Japan</td>
<td>70.820</td>
<td>0.002%</td>
<td>5.954</td>
<td>0.116%</td>
<td>0.987</td>
<td>0.276%</td>
<td>0.051</td>
<td>0.948%</td>
</tr>
<tr>
<td>Jordan</td>
<td>147.000</td>
<td>0.000%</td>
<td>12.940</td>
<td>0.000%</td>
<td>2.451</td>
<td>0.000%</td>
<td>0.117</td>
<td>0.000%</td>
</tr>
<tr>
<td>Kenya</td>
<td>196.100</td>
<td>0.000%</td>
<td>22.750</td>
<td>0.000%</td>
<td>3.834</td>
<td>0.000%</td>
<td>0.099</td>
<td>0.000%</td>
</tr>
<tr>
<td>Korea</td>
<td>158.100</td>
<td>0.000%</td>
<td>14.760</td>
<td>0.000%</td>
<td>2.584</td>
<td>0.000%</td>
<td>0.092</td>
<td>0.000%</td>
</tr>
</tbody>
</table>
Table 5.6 lists statistics obtained when testing the fit of the normal distribution to returns on the analysed indices. The $\chi^2$ test rejects the null about normal distribution of returns to stock market indices in all but 2 countries at 1% significance level. At the same significance level, the Anderson – Darling that does not reject the null hypothesis in 2 cases, the Cramer – von Mises test in 3 cases and
the Kolmogorov – Smirnov test in 4 cases. For the $\chi^2$ test, all but three $p$ values do not exceed 0.05%. This suggests a high degree of non-normality of the distribution of returns. Such a result can be interpreted as an indirect proof that $\alpha$'s, listed in Table 5.4, are significantly less than two for all countries except Slovakia and Australia. Consequently, the normal distribution proves to be an inaccurate approximation of returns to analysed stock market indices. The symmetric stable alternative, with all but 12 $p$ values for the $\chi^2$ test larger than 1%, seems to perform much better.

V.3.4. Estimation of parameters of the Diba-Grossman speculative processes

In Chapter IV, a general method to estimate parameters of Diba – Grossman processes, using conversion tables from the parameters of the stable distributions of returns is proposed. As the hypothesis that distributions of returns on the analysed stock markets are symmetric stable is not rejected, this methodology can be applied to the analysed data.

The parameters of interest in the Diba and Grossman (1988) price generating process given as (see Chapter III.4):

$$p_t = \theta_t p_{t-1} + r_t^e,$$

where $\theta_t \sim IIDN(1, \sigma_\theta^2)$, $r_t^e \sim IIDN(0, \sigma_e^2)$ are $\sigma_e$ and $\sigma_\theta$. Table 5.7 shows the estimates of 0.05, 0.5 and 0.95 quantiles of the distribution of estimates of $\sigma_e$ and $\sigma_\theta$ for the analysed countries. The 0.5 quantiles are the point estimates of respective parameters.

---

8 Tests of fit of the t-Student distribution to the analysed series produced similar results to tests of the normal fit and hence are not reported in this chapter.
Table 5.7: Parameters of Diba – Grossman process for international indices

Table shows the estimates of 0.05, 0.5 and 0.95 quantiles of the distribution of estimates of $\sigma_\theta$ (Columns (1) - (3) respectively) and $\sigma_\xi$ (Columns (4) - (6) respectively) for 66 analysed countries.

<table>
<thead>
<tr>
<th>Country</th>
<th>(1) $\sigma_\theta (0.5)$</th>
<th>(2) $\sigma_\theta (0.05)$</th>
<th>(3) $\sigma_\theta (0.95)$</th>
<th>(4) $\sigma_\xi (0.5)$</th>
<th>(5) $\sigma_\xi (0.05)$</th>
<th>(6) $\sigma_\xi (0.95)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argentina</td>
<td>0.2119</td>
<td>0.0933</td>
<td>0.3435</td>
<td>0.0086</td>
<td>0.0076</td>
<td>0.0096</td>
</tr>
<tr>
<td>Australia</td>
<td>0.0517</td>
<td>0.0000</td>
<td>0.1439</td>
<td>0.0068</td>
<td>0.0056</td>
<td>0.0088</td>
</tr>
<tr>
<td>Austria</td>
<td>0.1280</td>
<td>0.0572</td>
<td>0.2577</td>
<td>0.0054</td>
<td>0.0042</td>
<td>0.0072</td>
</tr>
<tr>
<td>Bangladesh</td>
<td>0.3200</td>
<td>0.1389</td>
<td>0.4325</td>
<td>0.0065</td>
<td>0.0055</td>
<td>0.0075</td>
</tr>
<tr>
<td>Belgium</td>
<td>0.1066</td>
<td>0.0364</td>
<td>0.2252</td>
<td>0.0064</td>
<td>0.0045</td>
<td>0.0077</td>
</tr>
<tr>
<td>Brazil</td>
<td>0.2229</td>
<td>0.0971</td>
<td>0.3503</td>
<td>0.0087</td>
<td>0.0077</td>
<td>0.0097</td>
</tr>
<tr>
<td>Canada</td>
<td>0.1255</td>
<td>0.0555</td>
<td>0.2541</td>
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<td>0.4531</td>
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<td>0.0038</td>
<td>0.0058</td>
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</table>

The estimates of $\sigma_\theta$ vary from 0.0517 for Australia to 0.3232 for Cyprus with the average value of 0.1745. Consequently, the analysis suggests that the five markets with the lowest levels of speculation are Australia, Slovakia, Indonesia, Sweden and USA. The highest levels of speculation are noted for Cyprus, Egypt,
Malta, Bangladesh and Zimbabwe. It is worth noting that for two countries (Australia and Slovakia) the estimates of 0.05 quantile of $\sigma_0$ are equal to zero. This means that the hypothesis about lack of speculation in the Diba – Grossman sense cannot be rejected for both these markets at 5% significance level.

Table 5.7 shows a general tendency for mature markets to reveal lower levels of speculation than the emerging ones. This could mean (see Chapter III.3.7) that market participants on small and emerging markets tend to overvalue private information more than players on the mature markets. Other explanations could include common beliefs in false models of economy and financial markets and momentum trading among investors in the emerging markets.

The estimates of $\sigma_e$ are in the range from 0.0029 for Morocco, Iceland and Mauritius to 0.02 for Turkey. The average estimate of $\sigma_e$ is equal to 0.081. Consequently, the five countries with relatively large random shocks are Turkey, Russia, Croatia, Pakistan and Thailand, whereas Morocco, Iceland, Mauritius, Kenya and Luxembourg seem to suffer from random shocks in relatively small way.

Results presented in Table 5.7 allow certain conclusions to be drawn about the reasons of price volatility on individual markets. For example, Russia is characterised by large random shocks and medium level speculation. The price volatility in Luxembourg seems to be mainly due to the relatively high level of speculation. Finally, countries like the US and the UK are characterised by rather low levels of speculation and medium exposures to random shocks.

Obviously, the above conclusions are based on the assumptions that speculation and exposure to random shocks are well captured by parameters of the Diba – Grossman model and that financial series variability is well captured by such processes. However, even if the accepted proxies for speculation and random shocks are not perfect, they provide important information concerning reasons for volatility on individual markets, and can be used as a measure of types and reasons for certain kinds of investment risk.
Figure 5.5: Relationship between non-normality and speculation

Figure shows the relationship between degree of non-normality of distribution of returns and level of speculation on 66 financial markets. Non-normality is measured as $\log(\alpha/(2-\alpha))$, where $\alpha$ is an estimate of characteristic exponent of stable distribution. Speculation is measured by the logarithms of estimated values of variances of stochastic root in Diba–Grossman price generating process.

Figure 5.5 shows the relationship between the logarithm of estimated $\sigma_\theta$ and the corresponding logit transformation of $\alpha$ (see Charemza and Kominek, 2000). The logit transformation is chosen to present the existing relationship in a linear form. As the largest admissible value of $\alpha$ is equal to 2, the following logit transformation is applied: $\log\left(\frac{\alpha}{2-\alpha}\right)$. To identify the points the ISO 3166 A3 country codes have been presented (see Appendix A).

For most of the analysed countries, the relationship between the parameters showing deviations from normality and the degree of speculation is clearly negatively linear. That is, a higher logit transformation of $\alpha$ (lower deviation from normality) corresponds to a lower degree of speculation. As it has been noticed earlier, there is a slight tendency for richer countries to exhibit lower deviations from normality and lower degrees of speculation than the poor countries. However, there are some distinctive outliers, most notably Russia, Bangladesh, Estonia, Indonesia and Korea, which show an unusually low degree of speculation relative to the
corresponding degree of non-normality. This would suggest that the correspondence between the non-normality of the distribution of returns and the degree of speculation is more complicated than that derived by a simple relationship between $\alpha$ and $\sigma_0$. However, in order to verify the validity of this hypothesis, a formal test for the presence of outliers should be performed. Outliers, if detected, disprove the linear character of the relationship between speculation and thick tails of the distributions of returns and, consequently, confirm the rationale behind the applied bivariate rather than simple univariate analysis.

V.3.5. Tests for the presence of outliers

This section formally addresses the problem illustrated on Figure 5.5. It tries to establish whether countries not conforming to the general relationship between speculation and non-normality (e.g. Bangladesh, Estonia and Russia) can indeed by classified as outliers. In other words, it tests whether the unusually low degree of speculation relative to the observed level of non-normality for these countries can be regarded as statistically significant. If this is the case, then it can be implied that the relationship between dispersion and speculation, detected in Monte Carlo experiments in Chapter IV.2, does exist on the financial markets. If outliers are not detected, then the bivariate dependance form Chapter IV.2 should be treated as a theoretical result, without significant applicability to the observed financial series.

Outliers are usually defined as observations that do not follow the pattern of the majority of the data (Rousseeuw and van Zomeren, 1990). Detecting such observations may provide analysts with important information about the nature of the data and investigated relationship. Outlier tests are used, among others, to verify whether certain points in the multidimensional space are significantly different from the sample characteristics. They can be applied to filter out the 'odd' observations, e.g. in panel data analysis, where information is often not fully reliable. In this situation, detecting outliers may help to identify the relationship and relieve it from the effect of inaccurate observations.

This section is designed to test for the presence of outliers in the data set containing characteristic exponents of stable distributions and standard deviations of
the stochastic roots in Diba-Grossman process for 66 international stock price indices. First, the main concepts of the outlier detection are discussed. Specific attention is given to tests designed for linear regressions, as this is the closest case to the empirical analysis carried in this Chapter. Then, the empirical results are presented.

If a data set contains a single outlier, the problem of identifying such observation is relatively simple from both an analytical and a computational point of view. However, multiple outliers in a multivariate cloud tend to be very hard to detect especially when the dimensions of the data exceeds two and visual perception can no longer be used. Even if the data has only two dimensions, some formal tests are necessary to confirm that certain suspected points are indeed outliers, violating the general pattern in the sample. For example, in Figure 5.5 there are at least three points that intuitively could be classified as distinctive outliers: Russia, Estonia and Bangladesh. However, it is not clear if this status can be extended to Indonesia, Malaysia and Korea. Similarly, it could be argued that the distances between Russia, Estonia and Bangladesh and the rest of the points are not sufficiently large to call these countries outliers.

A classical method to verify whether a single point \( x \) is an outlier is to calculate the Mahalanobis distance:

\[
A/D, (Cx, Sx) = \frac{1}{(x - Cx)^T Sx^{-1} (x - Cx)},
\]

where \( Cx \) is the arithmetic mean of the data set \( X \) and \( Sx \) is the usual sample covariance matrix (see e.g. Rousseeuw and van Zomeren, 1990). The distance \( MD_i \) should show how far \( x_i \) is from the centre of the cloud, taking into account the shape of the cloud. The cut-off value for the statistic (5.4.) is equal to \( (\chi^2_{s,(1-\alpha)})^{0.5} \), where \( s \) is the vertical dimension of the data set (number of variables) and \( \alpha \) is the significance level. Points, with \( MD_i \) larger than the cut-off value, are classified as outliers, whereas the remaining ones are counted as regular observations. This approach works very well for data sets with one distinctive observation. However, if the data contains more than one outlier or influential observation, it may suffer from masking and from swamping effects. Masking occurs when an outlying subset remains undetected because of the presence of another, usually adjacent, subset. Swamping occurs when regular observations are classified as outliers because of presence of
another remote set of observations. The main reason for these problems is that \( Cx \) and \( Sx \) are not robust. A small cluster of outliers may attract \( Cx \) and inflate \( Sx \) in its direction. Hence, it is natural to replace \( Sx \) and \( Cx \) by their robust estimators.

A number of different robust estimators of \( Sx \) and \( Cx \) are proposed (for literature review see Rousseeuw and van Zomeren, 1990). One of them is the minimum volume ellipsoid estimator (MVE) introduced by Rousseeuw (1985). This approach concentrates on computing \( Cx \) and \( Sx \) on the basis of the regular observations, putting relatively less weight on the possible outliers. The estimates of the mean and covariance matrices are obtained from the formulas:

\[
Cm = \left( \sum_{i=1}^{n} w_i \right)^{-1} \sum_{i=1}^{n} w_i x_i, \quad (5.5)
\]

\[
Sm = \left( \sum_{i=1}^{n} w_i - 1 \right)^{-1} \sum_{i=1}^{n} (x_i - Cx)^T (x_i - Cx), \quad (5.6)
\]

where the weights \( w_i \) depend on the robust distances \( RD_i = MD_i(Cm, Sm) \) and \( n \) is the sample size. The robust distances \( RD_i \) have the same cut-off points as the Mahalanobis distances \( MD_i \) but they take different (usually larger) values. In the simplest case, which is applied in this analysis (see as well Hadi, 1992), the weights \( w_i \) are equal to one for points with \( RD_i \) below the cut-off values and to zero for other elements of the analysed data set. In the recursive procedure, the values of \( Cm, Sm, w_i \) and \( RD_i \) are updated until the new change does not enlarge the set of points classified as outliers. This approach is computationally inexpensive and offers a substantial improvement over the classical Mahalanobis distance (see Rousseeuw and van Zomeren, 1990 and Hadi, 1992).

The above methods are designed to identify outliers among points characterised by \( k \times 1 \) dimensional vectors. However, often the main purpose of this analysis is to test the dependence between two variables and hence the investigation of linear regression of a certain type is involved. In such cases, tests based on formula (5.4) may turn out to be inappropriate. They are based on the distance between the centre of the cloud and candidate points. It means that if the elements of the data set were positioned on a line in \( k \) dimensional space, the most extreme of them would probably be identified as outliers. However, the linear model fitted to the data could still match the sample perfectly. In order to avoid swamping in such cases,
procedures specially designed for the identification of multiple outliers in linear models should be used. For this study, methodology proposed by Hadi and Simonoff (1992) is chosen (for review of other tests and brief Monte Carlo analysis of their efficiency see Hadi and Simonoff, 1992). This method seems to be relatively resistant to masking and swamping effects and, additionally, is relatively suitable for automation and encoding in a computer program.

First, a clean subset of the size of \( h = (n + k - 1)/2 \) is constructed using an approach similar to the robust distances. A group of \( k + l \) observations with lowest \( RD_i \) is chosen to form the basic subset \( B \). Then, the following statistics are computed for all \( x_i \):

\[
\begin{align*}
    f_i &= \frac{|y_i - x_i^T \beta_B|}{\sqrt{1 - x_i^T (X_B^T X_B)^{-1} x_i}} \quad \text{if } i \in B, \\
    f_i &= \frac{|y_i - x_i^T \beta_B|}{\sqrt{1 + x_i^T (X_B^T X_B)^{-1} x_i}} \quad \text{if } i \not\in B,
\end{align*}
\]

where \( X_B \) is the candidate clean subset \( B \) and \( \beta_B \) is the vector of parameter estimates from the regression fitted to the subset \( B \). In the next step, the observations are arranged in ascending order and \( s + 1 \) points with the lowest values of \( f_i \) are chosen to form the new candidate clean subset (\( s \) is the size of the clean subset in the before iteration). If \( s = h \) the procedure is interrupted, clean subset \( M \) is formed and the following statistics are computed for all points:

\[
\begin{align*}
    d_i &= \frac{|y_i - x_i^T \beta_M|}{\sigma_M \sqrt{1 - x_i^T (X_M^T X_M)^{-1} x_i}} \quad \text{if } i \in M, \\
    d_i &= \frac{|y_i - x_i^T \beta_M|}{\sigma_M \sqrt{1 + x_i^T (X_M^T X_M)^{-1} x_i}} \quad \text{if } i \not\in M,
\end{align*}
\]

where \( X_M \) is the clean subset \( M \) and \( \beta_M \) is the vector of parameter estimates and \( \sigma_M \) is the residual mean square from the regression fitted to the subset \( M \). The observations are arranged according to the absolute values of \( d_i \). If \( s \) is the current size of \( M \), then the \( s + 1 \) observation is chosen and the corresponding value of \( d_{i+1} \) is compared with the critical value of t-Student statistic: \( t_{(\alpha/2(s+1), k)} \). If \( d_{i+1} \geq t_{(\alpha/2(s+1), k)} \), then all sorted observations \( x_i \) with \( i > s \) are declared outliers and the procedure is stopped.
Otherwise, a new subset $M$ is constructed from first $s+1$ observations and the procedure is repeated.

Table 5.8 shows the outliers detected by the classical Mahalanobis distance at 10% significance level. This group includes the three most significant outliers from Figure 5.5: Russia, Estonia and Bangladesh. The additional presence of Australia is mainly due to the construction of the Mahalanobis statistic, which takes into account distance between centre of the data set and specific points. Australia is detected as an outlier because of the low estimated value of $\log(\sigma_0)$ and high estimated value of $\log(\alpha/(2-\alpha))$.

<table>
<thead>
<tr>
<th>Country</th>
<th>(1) MD</th>
<th>(2) p – value</th>
<th>Country</th>
<th>(1) MD</th>
<th>(2) p – value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Australia</td>
<td>2.3212</td>
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<td>0.0108</td>
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<td>0.0058</td>
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</table>

Table 5.9: Outliers detected by robust Mahalanobis distance

Table shows values of the robust Mahalanobis distance (Columns (1)) for countries, for which the null non-outlier hypothesis is rejected at 10% significance level. Columns (2) list the largest significance levels at which the null hypothesis can not be rejected.

<table>
<thead>
<tr>
<th>Country</th>
<th>(1) RD</th>
<th>(2) p – value</th>
<th>Country</th>
<th>(1) RD</th>
<th>(2) p – value</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.0010</td>
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<td>India</td>
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<td>Estonia</td>
<td>5.3742</td>
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<td>Slovakia</td>
<td>3.0009</td>
<td>0.0111</td>
<td>Russia</td>
<td>5.4502</td>
<td>0.0000</td>
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<td>Indonesia</td>
<td>3.2790</td>
<td>0.0046</td>
<td>Bangladesh</td>
<td>5.7405</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

This feature is even clearer when looking at results obtained using robust Mahalanobis distance (see Table 5.9). The detected outliers can be divided into two major groups: observations being far from the main line connecting Cyprus and Australia, i.e. Russia, Estonia, Bangladesh, Indonesia and Korea, and observations lying in the south east corner of Figure 5.5, i.e. Australia, Slovakia and India. The second group does not violate the general pattern of the data and is classified as
outliers only due to its long distance from the centre of the cloud. Consequently, it should not be detected by tests designed for linear regressions.

Table 5.10 shows values of Hadi - Simonoff distance for 14 countries, which are detected as outliers at 10% significance level. All of them lie below the line sketched by the majority of the points. It supports earlier expectations that regression designed tests for outliers should not detect all points lying far from the centre of the cloud. They rather concentrate on the impact of single observations on the final estimates of the regression coefficients. It could be argued that the group listed in Table 5.10 is too large for a sample with 66 observations. Changing the significance level to 1%, would limit its size to 8: China, Turkey, Croatia, Indonesia, Korea, Bangladesh, Estonia and Russia. There is no doubt that these countries behave at least slightly different than the rest of the sample.

Table 5.10: Outliers detected by Hadi - Simonoff method

Table shows values of the Hadi - Simonoff distance (Columns (1)) for countries, for which the null non-outlier hypothesis is rejected at 10% significance level. Columns (2) list the largest significance levels at which the null hypothesis can not be rejected.

<table>
<thead>
<tr>
<th>Country</th>
<th>(1) HS</th>
<th>(2) p-value</th>
<th>Country</th>
<th>(1) HS</th>
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<td>4.6278</td>
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<td>0.0015</td>
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<td>0.0008</td>
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<td>0.0259</td>
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<td>14.0033</td>
<td>0.0000</td>
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<td>0.0000</td>
</tr>
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<td>Malaysia</td>
<td>5.5821</td>
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<td>27.0873</td>
<td>0.0000</td>
</tr>
<tr>
<td>China</td>
<td>6.4084</td>
<td>0.0073</td>
<td>Russia</td>
<td>33.8885</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Generally, Tables 5.7-5.9 specify three groups of outliers: the strong outliers, the low speculation outliers and the moderate outliers. The strong outliers group includes countries detected by all tests: Estonia, Russia and Bangladesh. The behaviour of Estonian stock index can be connected with measurement error of the HSBC Estonia Index. This index is denominated in US dollars and hence both currency and stock markets contribute to level of speculation and thick tails of distribution of returns. Bangladesh and Russia have stock markets with relatively low liquidity and few traders. Additionally, as there is no long tradition of data collection in these countries, some measurement errors might be involved. However, these
reasons can only account for a limited part of the distinctive behaviour of these outliers and direction of their impact is uncertain. Hence, the outlying character of the estimated values can be due exclusively to the bivariate character of the relationship between speculation, random shocks and parameters of the distribution of returns.

The low speculation outliers group includes Australia, Slovakia and India. Their behaviour does not violate the general relationship between thickness of tails and speculation. Instead, they are distinguishable by low speculation and high $\alpha$ parameters. In order to find why these countries are different from the rest of the sample, a detailed analysis of their financial market structures and legal systems should be undertaken but this seems to go beyond the scope of this study.

The moderate outliers group consists of China, Turkey, Croatia, Indonesia and Korea. There is not much to be said about the reasons for the slightly different behaviour of these countries. The obtained results can be both due to specific country characteristics and the bivariate relationship between speculation, random shocks and shape of the distribution of returns.

Generally, it seems that the number of detected outliers and their significance is more than sufficient to claim the bivariate character of the relationship between the parameters of Diba – Grossman speculation processes and heavy tailed distributions of returns generated by such processes. Hence, this section confirms the rationale behind the analysis developed in Chapter IV.3.

V.4. Conclusions

This chapter concentrates on the empirical analysis of the levels of speculation and exposures to random shocks on different financial markets. First, the US and the UK markets are investigated. The comparative analysis does not show that in the second half of the 1990s there were any any significant differences between these two markets. The London Stock Exchange shows slightly higher exposure to random shocks, whereas the New York Stock Exchange is characterised by a slightly larger level of speculation. For both markets the hypotheses about
normally distributed returns and lack of speculation can be rejected at 5% significance level.

The analysis of the Dow Jones Industrial price index in the last 30 years shows a steady level of random shock exposure but a rather volatile level of speculation. The period of relatively larger exposure to random shocks coincides with oil shocks and high inflation in the US. The established swings in the market speculation correspond to events commonly described as speculative bubbles. These facts suggest that parameters of the Diba – Grossman process act as reasonable proxies for speculation and random shock exposure. Results obtained by this methodology generally agree with common perception of processes on the analysed stock market.

The comparison of the characteristics of the stock market indices from 66 countries confirms that financial returns have heavy tailed distributions. The symmetric stable distribution turns out to be an accurate approximation of the observed series. As the normal distribution is rejected in the vast majority of series, it can be concluded that the characteristic exponents of the stable distributions of market returns are generally significantly less than two. The estimated parameters of Diba – Grossman process show that for all but two markets, the hypothesis about lack of speculation has to be rejected. Moreover, there is a slight tendency for mature markets to reveal lower exposure to random shocks than the emerging markets. This might be, among other things, due to more overconfidence, stronger beliefs in false asset pricing models and more frequent momentum trading in the emerging than mature markets. The performed analysis additionally allows for simple decomposition of individual countries’ price changes into speculation and random shocks components.

The positive relationship between degrees of speculation and non-normality is confirmed in the analysis of the relationship between estimates of characteristic exponents and variances of stochastic roots of Diba – Grossman process. However, some statistically significant outliers are detected. Hence, the bivariate character of the relationship between the parameters of Diba – Grossman processes and heavy tailed distributions of returns generated by such processes is confirmed.
The Diba – Grossman process proves to be a useful tool to analyse the levels of speculation and exposure to random shocks of financial markets and assets. More extensive analysis of financial series should allow conclusions to be drawn about speculation on different kinds of markets and assets. In addition, longer historical time series should enable analysis of the time varying levels of speculation in various markets. The results could be confirmed by applying other processes, which parameters (or their combinations) could act as proxies for degrees of speculation and exposure to random shocks.
Chapter VI

CONCLUSIONS

VI.1. Summary of thesis

VI.2. Main results

VI.3. Suggestions for further research
VI.1. Summary of thesis

Traditionally, economists and financial practitioners favoured the normal and log-normal distributions when analysing returns to financial assets. However, in recent decades the popular assumption about normality of capital markets has been seriously questioned. The frequently detected excess kurtosis suggests rejection of the hypothesis about the normality of returns and its replacement by that of heavy tailed distribution. However, the thick tailed data can be modelled by a number of different distribution functions.

This thesis concentrates on the stable distribution as the alternative to the normal distribution. The stable law is described by four parameters responsible respectively for location, dispersion, skewness and thickness of tails. A large number of estimation methods are designed to fit the stable distributions to observed data. The main problems arise due to lack of straightforward functional representation of the density function. Hence, various approximations are proposed. They usually perform well for developed markets with frequent and regular data. Yet, problems occur when clustered, censored or grouped series are analysed. To overcome this issue, in Chapter II the minimum $\chi^2$ estimation is proposed. Flexible selection of bins (cells) helps to accommodate various irregularities in the data. The Monte Carlo experiments confirm that its efficiency is higher than that of some earlier methods. The minimum $\chi^2$ estimation is especially suggested for series from emerging markets, because of its frequent truncations and irregularities.

The proposed minimum $\chi^2$ estimation is applied for the analysis of the series from two Eastern European emerging markets. Symmetric stable distributions are fitted to the data from Warsaw and Budapest stock exchanges. The estimates of tail parameter are far below two, suggesting a high degree of non-normality of returns on these markets. The fitted characteristic exponents show larger dispersion for the Polish stocks market than for the Hungarian one.

There may exist many reasons for such a situation. Heavy tailed returns might originate, among others, from risk neutrality of investors, information arriving in clusters, or speculative behaviour of market participants. The thesis concentrates on the market speculation possibility. In Chapter III, the general concept of speculation
is reviewed and rational bubbles as a mean of speculation analysis are extensively discussed. Then, the behavioural finance theory is used for interpretation of speculative behaviour of investors.

As a stochastic process describing price formation on a speculative market, the Diba – Grossman process is chosen for further analysis. The Diba – Grossman process is an example of a doubly stochastic autoregressive process. The means of the autoregressive parameter and the additive random term are set to one and zero respectively. Two parameters of interest are the standard deviation of the autoregressive parameter, called herein the stochastic root of the process, and the standard deviation of the additive random term, called the Diba – Grossman return. It is shown that increase in the variance of the stochastic root results in processes with characteristic ups and downs, resembling the behaviour of financial bubbles and behavioural processes. It is claimed that the standard deviation of stochastic root can be interpreted as a measure of level of speculation and standard deviation of Diba – Grossman return can be treated as a measure of exposure to random shocks. The possible estimation techniques of the Diba – Grossman process are critically analysed.

To provide an acceptable approximation to reality, the Diba – Grossman processes should generate heavy tailed returns, similar to these observed on the emerging financial markets of Eastern Europe. In Chapter IV, the distribution of returns to Diba – Grossman process is analysed. It is hypothesised that such process results in stable distributed returns. This hypothesis is not rejected in a series of simulation experiments. Moreover, the normal and the t-Student alternatives are clearly rejected. Consequently, the analysis of the processes’ parameters should provide information about the level and character of market speculation. It is interesting to investigate the general relationship between parameters of Diba – Grossman price generating process and parameters of stable distribution of returns generated by such process.

An extensive Monte Carlo study is designed and above dependence is evaluated. About one million Diba – Grossman processes are simulated for different variances of stochastic root and additive random term. The stable distribution is fitted to returns to each of the series. The results are grouped according to the estimated
parameters of the stable laws and conditional distributions of standard deviations of stochastic root and additive random term are evaluated. Positive relationships between the level of speculation and the tail thickness and between the exposure to random shock and the dispersion parameter are established. The cross impact of the tail thickness parameter on the variance of random term and the scale parameter on the standard deviation of stochastic root is analysed and in both cases positive dependencies are shown. The established bivariate relationship is very interesting from theoretical and empirical points of view.

The evaluated function can be used to estimate parameters of the Diba – Grossman processes through inverted conversion from the parameters of the stable distribution of returns. In Chapter V.2 this methodology is applied to the UK and the US stock indices and levels of speculation on both markets are compared. The time path of speculation and exposure to the random shock of the US stock market is analysed. The obtained results coincide with intuitive perception, supporting the developed methodology.

The character of the relationship between speculation processes and distribution of returns is verified for daily financial data. In Chapter V.3, a sample of 66 stock market indices is analysed. Returns are filtered to remove the heavy tailed effect caused by risk neutrality of some investors. The fit of the stable distribution to the data is confirmed in $\chi^2$ tests and the normal alternative is clearly rejected. The parameters of Diba – Grossman process are estimated through conversion from the stable distribution parameters. The levels of speculation and exposure to random shocks are analysed for different countries. The empirical relationship between level of speculation and variance of stochastic root of Diba – Grossman process is analysed. As it results in significant outliers, the bivariate character of the dependence between speculation and distribution of returns is confirmed.

VI.2. Main results

The main findings of this thesis can be divided into five groups. The theoretical results concern the minimum $\chi^2$ estimation of the stable distribution parameters and the relationship between the stable distributions of market returns and
the parameters of the Diba – Grossman price generating process. The empirical results include conclusions about the distributions of returns on the emerging markets of Eastern Europe, the detailed analysis of speculation and exposure to random shocks on the UK and the US markets and the cross country analysis of the relationship between speculation and non-normality of returns.

First, the minimum $\chi^2$ estimation of the parameters of symmetric stable distributions is proposed. This approach is based on the minimisation of certain $\chi^2$ criteria with respect to the distribution parameters. Careful choice of criterion and appropriate construction of the bins, into which data is divided, guarantee high accuracy of the estimation. Properties of the minimum $\chi^2$ estimation are compared to the characteristics of the quantile algorithm in a series of Monte Carlo experiments. Evidence in favour of the minimum $\chi^2$ methodology is reported for grouped data and ungrouped series with more than 400 observations. The minimum $\chi^2$ approach is shown to be especially suitable for clustered, censored and grouped data. The flexible construction of bins helps to accommodate irregular samples and hence to avoid pitfalls that rule out the applicability of many traditional estimation techniques.

The most important result of the thesis is the evaluation of the relationship between parameters of stable distribution of returns and parameters of Diba – Grossman speculative process. Two main patterns are established: the negative correlation between variance of stochastic root and characteristic exponent and the positive correlation between variance of random term in Diba – Grossman process and scale parameter of stable distribution. The first can be interpreted as the positive relationship between speculation and degree of non-normality, whereas the latter shows the positive dependence between magnitude of random shocks and dispersion of the distribution of returns. Additionally, the positive cross impact of the characteristic exponent on the variance of random term is shown. A similar result is obtained for the scale parameter and the standard deviation of the stochastic root. The strength of these relationships is varying. The established dependence allows the analysis of the level of speculation on individual markets and securities through looking at the respective distributions of returns. It gives an intuition about the impact of speculation on the market risk and dispersion of returns. Finally, it offers a new method to estimate parameters of Diba – Grossman speculative process. This
method is based on the evaluated conditional distribution of the parameters of Diba - Grossman process on the scale and characteristic exponent of the stable distribution. The point estimates are obtained as 0.5 quantiles of these distributions. The computations are very fast and offer an easy way to evaluate the appropriate confidence intervals.

The first empirical result is connected with returns to assets from the Warsaw Stock Exchange and the Budapest Stock Exchange. They prove heavy tailed with slightly thicker tails on the Hungarian market. The values of estimated parameters are more diversified for assets listed on the Budapest Stock Exchange. For the Warsaw Stock Exchange, the differences between standard and the minimum $\chi^2$ estimators are regressed on the number of hits in the $\pm 10\%$ limits. The strong positive dependence shows that minimum $\chi^2$ estimation deals with censored data much better than the traditional, often inflexible, estimation techniques. The empirical analysis proves the non-normality of returns on the emerging markets and presents the flexibility of the developed minimum $\chi^2$ estimation.

The second empirical result concerns the analysis of speculation and random shocks on the UK and the US markets. The UK stock market shows a slightly higher level of random shocks and the US market presents a larger degree of speculation but no significant differences between these two markets are detected in the second half of the 1990s. The analysis of 30 year series of Dow Jones Industrial shows that the degree of speculation on the New York stock market seems to vary through time with relatively higher levels in the late sixties, early seventies and eighties. The US market's exposure to random shocks is more stable, with the only increase being in the late seventies and early eighties, presumably due to oil shocks and high inflation.

The third empirical result is based on the analysis of the behaviour of 66 stock market indices in the late 1990s. Returns on these markets prove symmetric stable distributed with characteristic exponent significantly less than two. The estimated values of parameters of Diba - Grossman processes suggest that there is a slight tendency for mature markets to reveal a lower level of speculation than the emerging ones. The lowest level of speculation is noted for Australia and Slovakia, and the highest one for Bangladesh, Cyprus, Egypt and Malta. The mature markets are usually located among the countries with medium low speculation. The London
Stock Exchange reveals a relatively low level of speculation and medium exposure to random shocks. The relationship between non-normality, measured by the tail thickness of the distributions of returns, and levels of speculation, measured by the standard deviation of stochastic roots of Diba – Grossman process, is analysed. A log linear pattern is detected but the existence of a few distinctive outliers is proven. This means that the dependence between speculation and non-normality is far more complicated than that given by a log linear function. Hence, the bivariate relationship between speculation and distribution of returns, established in the Monte Carlo experiments, proves more appropriate.

VI.3. Suggestions for further research

The presented results seem to be complete, but there are at least some aspects which could be taken into consideration in future research.

The first one is connected with estimation of the stable distribution for censored and irregular data. It seems to be possible to extend the presented analysis by including the maximum likelihood method. A maximum likelihood approach could be directly applied to estimate parameters for non-standard samples. This should result in more accurate estimates and better knowledge of the tail behaviour of return distributions. An additional extensive analysis of international financial data would be desirable.

Second, after analysing distribution of returns to Diba – Grossman process, it would be interesting to look at the distributions of returns to other speculative processes, including these generated by recently developed behavioural models. These models claim to capture certain types of speculative behaviour and hence relationships similar to the presented one should be expected. Moreover, the analysis of distributions of returns to Diba – Grossman should be repeated for series of different lengths.

Third, more extensive empirical analysis should be carried out on the basis of the developed methodology. A comparative analysis of historical evaluation of levels of speculation on mature and emerging markets should provide important
information about the development of financial systems. Comparison of speculations and random shock exposures for different companies and industries should give an insight into the possible motives of stock purchases and help one to understand the mechanisms governing investment behaviour.

Fourth, work on the financial models that could produce distribution of returns possibly identical to those seen on the financial markets is advisable. This problem coincides with the general need for an accurate asset pricing scheme. The recent ideas, how to improve the traditional models, form a broad stream in the financial literature. The results presented in this thesis suggest that one should look closer at processes with stochastic parameters and possibly non-rational expectations. This problem seems to be difficult and not straightforward, but it is certainly challenging and intellectually rewarding.
# APPENDIX A

ISO 3166 A3 Country Codes

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<th>Code</th>
<th>Country</th>
<th>Code</th>
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Appendix B

Exemplary GAUSS programs

1. Program to estimate parameters of symmetric stable distribution using the minimum \( \chi^2 \) methodology (applied in Chapter II)

```gauss
new;
library optmum;
optset;
output file = c:\gauss\zibi\ss3.out reset;
load z[] = c:\gauss\zibi\wigdln.txt;
y=100*z;
n=rows(y);
start=start1;
start={ 1.2397, 1.66, 0 };
_opstmth="newton";

print("Minimizing chi-square value");
[x,f,g,retcode] = optprt(optmum(&chi2,start));
print("Minimizing g-square value");
{x,f,g,retcode} = optprt(optmum(&g2,start));
print("Minimizing ft value");
{x,f,g,retcode} = optprt(optmum(&ft,start));
```

```gauss
@ PROCEDURES@

@ Procedure to calculate chi-squared statistic @

```
proc chi2(param);
local ee, tt, chi, vv, inter;
chi=10000000*(1+abs(param[1,1])+abs(param[2,1]));
if param[1,1]>=0.85;
if param[1,1]<2;
if param[2,1]>=0;
vv=empdens(y,param[2,1],param[3,1]);
inter=vv[.,1:2];
ttt=prob(param[1,1],inter);

tt=ttt[.,4]*n;
ee=vv[.,3];
chi=sumc((tt-e)^2/e);
endif;
endif;
endif;
retchi;
endp;
```

```gauss
@ Procedure to calculate g-squared statistic @

```
proc g2(param);
local eel, tt1, ti, g, vv1, inter1;
g=10000000*(1+abs(param[1,1])+abs(param[2,1]));
if param[1,1]>=0.85;
if param[1,1]<2;
if param[2,1]=0;
vv1=empdens(param[2,1],param[3,1]);
inter1=vv1[.,1:2];

```

```gauss
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@ Procedure to calculate Freeman-Tukey ratio @

proc ft(parain2);
local e e2, ttt2, tt2, ft, vv2, inter2;
ft = 10000000*(1+abs(parain2[1,1])+abs(parain2[2,1]));
if param2[1,1] > = 0.85;
if param2[1,1] < = 2;
if param2[2,1] > = 0;
vv2 = emdpens(param2[2,1],param2[3,1]);
inter2 = vv2[1,2];
ttt2 = prob(param2[1,1],inter2);
tt2 = ttt2[4,1]*n;
e2 = vv2[3,3];
ft = 4*sumc(((ee2 A 0.5)-(tt2 A 0.5)) A 2);
endif;
endif;
endif;
retp(ft);
endp;

@ Procedure to estimate parameters of stable distribution @
@ J. Houston McCulloch, 1986, Simple consistent estimators of stable @
@ distributions parameters, Commun. Statist. - Simula. 15, 1109-1136 @

proc mcculloch(as);
local x, n, q05a, qw05, x05, q25a, qw25, x25, q50a, qw50, x50,
q75a, qw75, x75, q95a, qw95, x95, m50, table, table3, table4,
valfa, valfal, vbeta, alphal, w1, w2, alphasl, vc, c, loc,
beta, rlO;
x = sortc(as.1);
n = rows(x);

@ quantiles @
q05a = 0.05*n;
q25a = 0.25*n;
q50a = 0.5*n;
q75a = 0.75*n;
q95a = 0.95*n;
qw05 = q05a - trunc(q05a);
x05 = (1-qw05)*x[trunc(q05a)] + qw05*x[trunc(q05a)+1];
qw25 = q25a - trunc(q25a);
x25 = (1-qw25)*x[trunc(q25a)] + qw25*x[trunc(q25a)+1];
qw50 = q50a - trunc(q50a);
x50 = (1-qw50)*x[trunc(q50a)] + qw50*x[trunc(q50a)+1];
qw75 = q75a - trunc(q75a);
x75 = (1-qw75)*x[trunc(q75a)] + qw75*x[trunc(q75a)+1];
qw95 = q95a - trunc(q95a);
x95 = (1-qw95)*x[trunc(q95a)] + qw95*x[trunc(q95a)+1];
m50 = x75 - x25;

@ fi functions for alfa and beta @
valfa = (x95 - x05) / m50;
table1 = [ 2.000, 1.916, 1.808, 1.729, 1.664, 1.563, 1.484, 1.391, 1.279, 1.129, 1.029, 0.896, 0.818, 0.698, 0.593 ];
valfal = [ 2.439, 2.5, 2.6, 2.7, 2.8, 3.0, 3.5, 4.0, 5.0, 6.0, 8.0, 10.0, 15.0, 25.0 ];

i = 1; alphal = 0;
w1 = 0; w2 = 0; denl = 0;
do until i > 14;
if valfa > valfal[i,1];
if valfa <= valfal[(i+1),1];
denl = valfal[(i+1),1]-valfal[i,1];
w1 = (valfa - valfal[i,1])/denl;
w2 = (valfal[i+1,1] - valfa)/denl;
alphal = w1 * table1[(i+1),1] + w2 * table1[i,1];
endif;
endif;
i = i + 1;
beta=0;

table3 = { 1.908, 1.914, 1.921, 1.927, 1.933, 1.939, 
   1.946, 1.955, 1.965, 1.980, 2.000, 2.040, 
   2.098, 2.189, 2.337, 2.588 
};

alfasl = { 2, 1.9, 1.8, 1.7, 1.6, 1.5, 1.4, 1.3, 1.2, 1.1, 1.0, 
   0.9, 0.8, 0.7, 0.6, 0.5 
};

i = 1; vc=0;
deni=0;
w l=0; w 2=0;
do until i > 15;
   if alfa < alfasl[i,1];
     if alfa >= alfasl[i+1,1];
       deni = abs(alfasl[i+1,1]-alfasl[i,1]);
       w l = abs(alfa - alfasl[i,1])/deni;
     endif;
   endif;
   w2 = abs(alfasl[i+1,1] - alfa)/deni;
   vc = w l*table3[i+1,1] + w2*table3[i,1];
   endif;
endi;
endo;
c = m50 / vc;
loc = x50;
rl0 = zeros(1,3);
rl0[1,1] = alfa; rl0[1,2] = c; rl0[1,3] = loc;
retp(rl0);
endp;

proc prob(alfa, inter);
local pp, mm, j1, j2, kk1, kk2, kk3, sum, mat;
mm=rows(inter);
pp=zeros(mm,4);
pp[.,1:2]=inter;
jjl=1;
do until jjl>mm;
   suml=0;
   kk1=pp[jj,2]-pp[jj,1]/100;
   jjj=jjl+1;
   mat=zeros(101,2);
   do until jj3>101;
      kk2=pp[jj,1]+(jj3-jjj)*kk1;
      mat[jj3,1]=kk2;
      mat[jj3,2]=symtab(kk2,alfa);
      jjj=jjj+1;
   enddo;
   suml=mean(mat);
   pp[.,3]=suml[2,1];
   pp[.,4]=pp[.,3]*(pp[.,2]-pp[.,1]);
   jjl=jjl+1;
endo;
retp(pp);
endp;

proc empdens(yy11,c,d);
local yy1, yy2, nni, nn2, dd1, dd2, dd3, ii1, ll1, spr, aal, aa2;
yy1=(yy11-d)/c;
yy2=sortc(yy1);
nm=rows(yy2);
nn2=round((t2*(nn1^2))/1.96^2)*0.2);
spr = (nn1/nn2);
pp = zeros(nni,2);
dd1 = zeros(nn2,4);
dd[1,1]=yy2[1,1]-0.00000001;
aal1=floor(spr);
aa2=spr;
dd[1,1]=a2-aal1*yy2[aal1+1,1];
aal1=0; a2=0;
ii1=2;
do until ii1>nn2-1;
aal1=floor(ii1*spr);

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aa2=ii *spr;
dd1[iii,1]=dd1[iii-1,2];
dd1[iii,2]=(aa2-aa1)*yy2[aa1+1,1]+(1-(aa2-aa1))*yy2[aa1,1];
ii=ii+1;
end;

dd1[nnn,1]=dd1[nnn-1,2];
dd1[nnn,2]=yy2[nnn,1];
dd2=zero(nnn+1,1);
dd2[1:nnn,1]=dd1[1:nnn,2];
dd3=counts(yy2,dd2);
dd1[.,3]=dd3[2:nnn+1,.;];
dd1[.,4]=(dd1[.,3]/nnn-1)/dd1[.,2]-dd1[.,1];
retp (dd1);
endp:

June 1993; GAUSS version

x may be a column vector; alpha is a scalar returns sden = symmetric stable density

proc symstbl(x,alpha);
    declare s_, sqpi_, a2_, cpxp0_, gpxp0_, gpxpp0_,
    declare ccpp_, gppp_, c5i_,
    declare a_, alf2_, alfl_, pd_, p_, pdd_,
    declare first_ = 1;
    declare old alf_ = -999.99;
    local r, q, znot, zn4, zn5;
    local i, il, j, jl;
    local pialf, spO, spppO, xpO, xppO, xpppO, sp z p l;
    local rpO, rppO, rpppO, r p l, b, c;
    local x a l, x a la, x2, x2p, cd, gd;
    fn afun(alpha) = 2 A( 1/alpha) - 1;
    fn cden(x) = l/(p i* (l+ x .* x ));
    fn gden(x) = exp (max(-700*ones(l,rows(x)) I  -(x.*x)'/4)) /(2 * sqpi_);
    fn zfun(x, alpha, a) = 1  - (1 + a*x) A  (-alpha);
    fn zpfun(x, alpha, a) = alpha * a * ( l+ a * x ) A  (-alpha -I);
    fn xfun(z, alpha, a) = ( ( l - z ) A(-l/alpha ) - 1) /  a;
    if(not first_); goto starta; endif;
    @ this s is the gij o f  the writeup @
\begin{verbatim}
1.33979992711D4, -3.1246611987D3,
-7.2435775303D3, 4.3545399418D3,
2.3616155949D2, -7.6571653073D2,
-8.7376725439D3, 1.5510852129D4,
-1.3789764158D4, 4.6387417712D3);
\end{verbatim}
x1 = x11 - 1;
x1p = x11 * x11;
x2a1 = 1/sqrt(x1a):
x2 = (x2a1 -1)/a2_x;
x2p = x2a1 / (2*a2_x * x1a);
cd = cden (x1);

gd = gden(x2);
j1 = trunc(20 * z) + 1;

rpz = poly(pd_f., j1, z, 4);

sden = (a1fl_ * cd * x1p + a1fl_ * gd * x2p - rpz) * zp;

retp (sden);

endp;

proc poly(a, x, k);
local sum, j;
sum = a[k+1];
j = k;
do until j == 0;
sum = sum * x + a[j-];
j = j - 1;
endo;
retp(sum);
endp;

proc dgamma(x):
@ Note -- GAUSS 3.2+ has adopted an improved gamma function similar
to this one, making this proc redundant.
@
local t, ser, c, stp;
c = [76.18009172947146, -86.50532032941677, .01409824083091,
-1.231739572450155, .001208650973866179, -5395239384953d-5];

stp = 2.506628746310005;
t = x + 5.5;

t = t * (x+5.5)/exp(t);

ser = 1.000000000190015 + sumc(c./(x+seqa(1,1.6)));
retp( t * stp * ser / x);
endp;

end;
new;
library pgraph;
output file = bub8.out reset;

@Parameters@
tmax=100;
sigma=0.00005;
lam1=0.9;
lam2=0.1;
pil=0.05;
pih=0.95;
Div0=0.00005;
y0=0.00001;

n=tmax+2;
a=rdtn(1,2);
y=zeros(n,1);
y[1]=a[1]/abs(a[1])*ya0;
y[2]=a[2]/abs(a[2])*ya0;
i=3;
do until i>n;
a=rdtn(1,1);
y[i]=a/abs(a)*ya0;
i=i+1;
endo;

P1=bub(0.5,0.5);
P2=bub(0.05,0.05);
P3=bub(0.95,0.95);
b=seql(1,100);
xy(b,P);
output on; P; output off;

proc bubf(lam1,lam2);
local gamma0, gamma1, gamma2, Q, p1, p2, ql, qN, qNN, i, a, P, i, y;
P=zeros(tmax,1);
a=rdtn(1,2);
y=zeros(n,1);
y[1]=a[1]/abs(a[1])*ya0;
y[2]=a[2]/abs(a[2])*ya0;
i=3;
do until i>n;
a=rdtn(1,1);
y[i]=a/abs(a)*ya0;
i=i+1;
endo;

gamma0={1, -1.1, -1};
gamma1={0, 0, 1, 0};
gamma2={1, 0, -1.0};
Q=zeros(4,4);
Q[1,1]=(1-lam1)*pil;
Q[1,2]=(1-lam1)*(1-pil);
Q[1,3]=lam2*pil;
Q[1,4]=lam2*(1-pil);
Q[2,1]=(1-lam1)*(1-pil);
Q[2,2]=(1-lam1)*pil;
Q[2,3]=lam2*(1-pil);
Q[2,4]=lam2*pil;
Q[3,1]=lam1*(1-pil);
Q[3,2]=lam1*(1-pil);
Q[3,3]=(1-lam2)*pil;
Q[3,4]=(1-lam2)*(1-pil);
Q[4,1]=lam1*pil;
Q[4,2]=lam1*(1-pil);
Q[4,3]=(1-lam2)*pil;
Q[4,4]=(1-lam2)*pil;
p1=(gamma0' + sigma)*inv(cyc(4)*(1+sigma)-Q)*Q*gamma1/4;
p2=(gamma0' + sigma)*inv(cyc(4)*(1+sigma)-Q)*Q*gamma2/4;
q1=zeros(n,1);
NN=q1;
NNN=q1;
NNN[1]=Div0+y[1];
q1[2]=0.5;

i=3;
do until i>n;
if y[i-2]==y[i-1];
  q1[i]=((1-lam1)*q1[i-1]+lam2*(1-q1[i-1]))*pil/
  (((1-lam1)*q1[i-1]+lam2*(1-q1[i-1]))*pil+(lam1*q1[i-1]+(1-lam2)*
  (1-q1[i-1]))*ph);
else:
  q1[i]=((1-lam1)*q1[i-1]+lam2*(1-q1[i-1]))*(1-pil)/
  (((1-lam1)*q1[i-1]+lam2*(1-q1[i-1]))*(1-pil)+(lam1*q1[i-1]+(1-lam2)*
  (1-q1[i-1]))*(1-ph));
endif;
a=rand(1,1);
NNN[i]=NNN[i-1]+y[i];
NN[i]=y[i]*(p1-p2*q1[i]);
P[i-2]=NNN[i]/sigma+NN[i];
i=i+1;
end;
retp(P);
endp;
3. Program to test goodness of fit of normal and t - Student distributions to returns to Diba - Grossman processes (applied in Chapter IV)

new;
library optimum:

@ Parameters @
@ General @
Noobs=1000;  @ No of observations @
noise=1000;  @ No of iterations for each std of theta @
X0=0;  @ Initial price @
drift=0;  @ drift in Diba-Grossman process @

@ Thetas @
meantet=1;  @ Mean of theta @
lowestet=0;  @ Lowest analysed std of theta @
higset=0.4;  @ Highest analysed std of theta @
step=0.05;  @ Step in std of theta @

@ Random term @
meanran=0;  @ Mean of random term @
lowstras=0.001;  @ Lowest std of random term @
higstran=0.09;  @ Highest analysed std of random term @
step=0.005;  @ Step in the std of random term @

@ Chi-square test specification @
n=30;  @ Mann and Wald - about 30 @
Crit01=49.588;  @ Critical values for different level of significance @
Crit05=42.557;  @ Must be changed with NCell @
Crit10=39.087;  @ No of degrees of freedom = NCell-1 @

@ Limits for prices @
manret=1;  @ Max average annual return @
minret=-0.5;  @ Min average annual return @
obsyear=250;  @ No of observation per year @
nofail=0.25;  @ Max. percentage of explosive series @

@ T-student @
@ Number of degrees of freedom is restricted to be integer and in the interval (3,102) @

@ Equiprobable cells for normal fit @
cellsn=cdfn(seqa(1,1,n-1)/n);

@ Others @
rnds=196446388;
limit1 = ln(exp(XO)*((1+manret)/(Noobs/obsyear)));
limit2 = ln(exp(XO)*((1+minret)/(Noobs/obsyear)));
theta = (higset-lowset)/step+1;
sigma = (higstran-lowstran)/step+1;

@ Output files @
output file = quantn.out reset;
output file = quantt.out reset;
output file = chit01n.out reset;
output file = chit05n.out reset;
output file = chit01t.out reset;
output file = chit05t.out reset;
output file = chit10t.out reset;

@ Monte Carlo loops @

col=0; i=0;
do until i > thetas-1:
    sss1=(i+1)/theta;
    stdtet = lowestet + i * step ;
    k=0; label1:
do until k > sigmas-1:
    sss2=(k+1)/sigma;

stdran = low sran + k * stepr;
col=col+1;
excvar=0;
chis1 = zeros(noiter,1);
chis2 = chis1;
j=1;
do until j > noiter:
  col, ss1, ss2, j/noiter;
  label2:
  if excvar>nofail*(noiter+excvar):
    "For stdtet = ":stdtet;
    "number of explosive and collapsing time series is too big!";
    k=sigmas;
    col=col-1;
    goto label1;
  endif;
  e1= randn(Noobs,1);
e2= randn(Noobs,1);
  thet = meanet + ((e1-mean(e1))/stdc(e1))*stdtet;
  rand = meanran + ((e2-mean(e2))/stdc(e2))*stdran;
  logprices=zero(Noobs+1,1);
  logreturns=zeros(Noobs,1);
  l=2;
do until l > Noobs+1:
  logprices[l,1]=drift+thet[l-1,1]*logprices[l-1,1]+rand[l-1,1];
  @ Checking for explosive time series @
  if logprices[l,1]>limit1:
    excvar=excvar+1;
    goto label2;
  endif;
  if logprices[l,1]<limit2:
    excvar=excvar+1;
    goto label2;
  endif;
  logreturns[l,1]=logprices[l,1]-logprices[l-1,1];
  l=l+1;
  endo;
@ Testing the fit of the normal and t-Student distributions @
x=(logreturns-meanc(logreturns))/stdc(logreturns);
v=tmlik(x);
cellst=tcells(v);
chst=chitest(cellst,x);
chnt=chitest(cellstn,x);
chis[j,1]=chnt;
  j=j+1;
endo;
chis1=sortc(chis1,1);
chis2=sortc(chis2,1);
No01n=counts(chis1.crit01);
Acc01n=No01n/noiter;
No05n=counts(chis1.crit05);
Acc05n=No05n/noiter;
No10n=counts(chis1.crit10);
Acc10n=No10n/noiter;
No01t=counts(chis2.crit01);
Acc01t=No01t/noiter;
No05t=counts(chis2.crit05);
Acc05t=No05t/noiter;
No10t=counts(chis2.crit10);
Acc10t=No10t/noiter;
format 6,9;
output file = quantn.out;
output on;
aq=noiter;
stdtet=stdran-chis1[0.9*aq,1]-chis1[0.95*aq,1]-chis1[0.99*aq];
output off;
output file = chit01n.out;
@ Procedure to compute chi-squared statistic @
proc chitest(cells, data);
  local d1, d2;
  d1 = counts(data, cells);
  d1 = d1' - (rows(data) - sumc(d1))';
  d2 = rows(data) * ones(n, 1) / n;
  retp(sumc((d1 - d2) * (d1 - d2) / d2));
endp;

@ Procedure to compute borders of equiprobable cells for @
@ t-student distribution with v degrees of freedom @
proc tcells(p);
  retp(cdftci((seq(1, n - 1) / n), p));
endp;

@ Procedure to estimate parameters of t-Student distribution @
@ Maximum likelihood with integer n from (1, 100) @
proc tmlik(xx);
  local i_t, lik_t, k_t;
  lik_t = zeros(100, 2);
  i_t = 3; do until i_t > 102;
    lik_t[i_t-2.1] = i_t;
    k_t = gamma(0.5 * i_t - 0.5) / (sqrt(pi * i_t) * gamma(0.5 * i_t - 1));
    lik_t[i_t-2.2] = sumc(lnk_t[i_t-2.1] + sumc(lnk_t[i_t-2.2] + sumc(lnk_t[i_t-2.2])));  
    i_t = i_t + 1; endo;
  lik_t = sortc(lik_t, 2);
  retp(lik_t[1, 1]);
endp;
4. Programs to Monte Carlo analysis of returns to Diba – Grossman processes (applied in Chapter IV):

a) Simulation and estimation of distribution parameters

@ Efficient returns, thick tails and speculative processes @
@ Simulates \( X(t) = \text{theta}(t) \times X(t-1) + w(t) @
@ Estimates stable distribution parameters of returns @

@ Description @
@ \( X(t) \) - In of prices @
@ \( \text{theta}(t) \) - normal distributed variable @
@ \( w(t) \) - normal distributed random term @

new;

@ Output files @
@ alfasm.out, betas.out, scale.out, locat.out - matrixes of estimates
for respective values of stdtet;

to change names of output files goto line 100 @@

output file = alfasm.out reset;
output file = betas.out reset;
output file = scale.out reset;
output file = locat.out reset;

@ Parameters @
@ General @

Noobs=1000; @ No of observations @
noiter=1000; @ No of iterations for each std of theta @
X0=0; @ Initial price @
drift=0; @ drift in Diba-Grossman process @

@ Thetas @
meantet=1; @ Mean of theta @
lowstet=0; @ Lowest analysed std of theta @
higstet=1; @ Highest analysed std of theta @
step=0.01; @ Step in std of theta @

@ Random term @
meanran=0; @ Mean of random term @
lowsran=0.001; @ Lowest std of random term @
higran=0.2; @ Highest analysed std of random term @
stepr=0.001; @ Step in the std of random term @

@ Limits for prices @
manret=1; @ Max average annual return @
mintet=-0.5; @ Min average annual return @
obyear=250; @ No of observation per year @
nofail=0.25; @ Max. percentage of explosive series @

@ Others @
rndseed 196446388:

limit1 = ln(exp(X0)*((1+manret)*(Noobs/obsyear)));
limit2 = ln(exp(X0)*((1+mintet)*(Noobs/obsyear)));

thetas = (higstet-lowstet)/step+1;
sigmas = (higran-lowsran)/stepr+1;
output on; print(noiter); output off;

@ Monte Carlo loops @
col=0;
is=0;
do until i > thetas-1;
stdet = lowstet + i * step ;
k=0;
label1:
do until k > sigmas-1;
sdtran = lowsran + k * stepr;
cls;
stdet=sdtran;
col=col+1;
exvar=0;
alf = zeros(noiter+3,1);
bet = zeros(noiter+3,1);
sea = zeros(noiter+3,1);
loc = zeros(noiter+3,1);
j=1;
do until j > noiter;
label2:
    if excvar>nofail*(noiter+excvar):
        "For stdtet = ";stdtet,"and stdgdet = ";stdran;
        "number of explosive and collapsing time series is too big!";
        k=sigma;
        col=col-1;
        goto label1;
    endif;
    e1=rdm(Noobs*1.5,1);
e2=rdm(Noobs*1.5,1);
    thet = meanet + ((el-mean(c(e1))/stdtet1))*stdtet;
    rand = meanran + ((c2-mean(c(e2))/stdc(e2))*stdran;
    logprices=zeros(Noobs*1.5+1,1);
    logreturns=zeros(Noobs*1.5,1);
    logprices[1,1]=XO;
    do until 1 > Noobs+1 ;
        logprices[1,1]=drift+thet[I-1,1]*logprices[I-1,1]+rand[I-1,1];
        @ Checking for explosive time series @
        if logprices[I,1]>limit1;
            excvar=excvar+1.
            goto label2;
        endif;
        if logprices[I,1]<limit2;
            excvar=excvar+1.
            goto label2;
        endif;
        if exp(abs(logreturns[I-1]-logreturns[I]))-1>0.5;
            excvar=excvar+1.
            goto label2;
        endif;
        logreturns[I-1,1]=(logprices[I,1]-logprices[I-1,1]);
        ls=ls+1;
        endo;
    logprices=logprices[1:Noobs];
    logreturns=logreturns[1:Noobs];
    parameters=mcu(loc(logreturns));
    alf[4:noiter+3]=parameters[1,1];
    bet[4:noiter+3]=parameters[1,2];
    sca[4:noiter+3]=parameters[1,3];
    loc[4:noiter+3]=parameters[1,4];
    j=j+1;
endo;
alf[4:riter+3]=sortc(alf[4:noiter+3],1);
bet[4:noiter+3]=sortc(bet[4:noiter+3],1);
sca[4:noiter+3]=sortc(sca[4:noiter+3],1);
alf[1,1]=stdtet;
bet[2,1]=stdran;
    alf[3,1]=excvar/(noiter+excvar);
    bet[1,1]=stdtet;
bet[2,1]=stdran;
    bet[3,1]=excvar/(noiter+excvar);
sca[1,1]=stdtet;
sca[2,1]=stdran;
sca[3,1]=excvar/(noiter+excvar);
loc[1,1]=stdtet;
loc[2,1]=stdran;
loc[3,1]=excvar/(noiter+excvar);
output file = alfas.out;
outwidth 256;
output on;
print(alf);
output off;
output file = betas.out;
outwidth 256;
output on;
print(bet);
output off;
output file = scal.out;
outwidth 256;
output on;
print(sca);
output off;
output file = scale.out
@ Procedure to estimate parameters of stable distribution
It follows the McCulloch (1986) method and is based on J. Huston McCulloch, 1986, Simple consistent estimators of stable distributions parameters, Commun. Statist. - Simulat. 15, 1109-1136 suitable for alpha in range [0.6, 2.0] and beta in range [-1.1]
Input - column vector with data
Returns 1 by 4 matrix of parameters:
alpha, beta, scale, location delta (usually used) @

proc mcculloch(yy):
local xx, nn, q05a, q25a, q50a, q75a, q95a, qw5, qw25, qw50, qw, x50, x25, x75, x95, m50, valfa, vbeta, table1, table2, valfa1, vbeta1, ii, alfa, alfa1, alfa2, beta0, beta2, w1, w2, w3, w4, jj, den1, den2, table3, table4, table5, table6, table7, table8, table9, alfas1, betas1, bet, vc, vcl, vcl2, vloc, vloc2, c, loc1, loc, alfas2, betas2, vcl1;

xx = sort(yy,1);
nn = rows(xx);
@ Sample quantiles @
q05a = 0.05*nn;
q25a = 0.25*nn;
q50a = 0.50*nn;
q75a = 0.75*nn;
q95a = 0.95*nn;
qw5 = q50a - trunc(q50a);
x05 = (1-qw5)*xx[(trunc(qw5)+1)];
qw25 = q25a - trunc(q25a);
x25 = (1-qw25)*xx[(trunc(qw25)+1)];
qw50 = q50a - trunc(q50a);
x50 = (1-qw50)*xx[(trunc(qw50)+1)];
qw75 = q75a - trunc(q75a);
x75 = (1-qw75)*xx[(trunc(qw75)+1)];
qw95 = q95a - trunc(q95a);
x95 = (1-qw95)*xx[(trunc(qw95)+1)];
m50 = x75 - x25;

@ fi functions for alpha and beta @
valfa = (x95 - x05) / m50;
valfa1 = valfa + 2*50;
valfa2 = valfa - 2*50;

vbeta = abs(vbeta);
valfa1 = valfa + 2*50;
valfa2 = valfa - 2*50;

Table 1 = table 1;
...
\begin{verbatim}

valfal = { 2.439, 2.5, 2.6, 2.7, 2.8, 3.0, 3.2, 3.5, 4.0, 5.0,
6.0, 8.0, 10.0, 15.0, 25.0 };

vbeta1 = { 0, 0.1, 0.2, 0.3, 0.5, 0.7, 1 };

ii = 1; alfa=0; alfa1=0; alfa2=0;
beta0=0; beta1=0; beta2=0;
w1=0; w2=0; w3=0; w4=0;
do until ii > 14;
if valfa > valfal[ii,1];
if valfa <= valfal[(ii+1),1];
jj = 1:
do until jj > 6;
if vbeta > vbeta1[jj,1];
if vbeta <= vbeta1[(jj+1),1] - vbeta1[jj,1];
deni = valfal[(ii+1),1] - valfal[ii,1];
den2 = vbeta1[(jj+1),1] - vbeta1[jj,1];
w1 = (valfa - valfal[ii,1])/den1;
w2 = (valfa - valfal[ii,1])/den2;
w3 = (vbeta - vbeta1[jj,1])/den1;
w4 = (vbeta - vbeta1[jj,1])/den2;
alfa1 = w1*w3*table1[(ii+1),(jj+1)] + w1*w4*table1[(ii+1),jj];
alfa2 = w2*w3*table2[ii,(jj+1)] + w2*w4*table2[ii,jj];
alfa = alfa1 + alfa2;
beta1 = w1*w3*table2[(ii+1),(jj+1)] + w1*w4*table2[(ii+1),jj];
beta2 = w2*w3*table2[ii,(jj+1)] + w2*w4*table2[ii,jj];
beta0 = (vbeta/vbeta1)*(beta1 + beta2);
if abs(beta0) > 1;
beta = abs(beta0)/beta0;
else;
beta = beta0;
endif;
endif;
jj = jj + 1;
enddo;
endif;
i = i + 1;
enddo;
if valfa<=2.439;
alfa=2;
beta=abs(vbeta)/vbeta;
endif;

@ fi function for scale c and location loc @

table3 = { 0.908 1.908 1.908 1.908 1.908,
1.914 1.915 1.916 1.918 1.921,
1.921 1.922 1.927 1.936 1.947,
1.927 1.930 1.943 1.961 1.987,
1.933 1.940 1.962 1.997 2.043,
1.939 1.952 1.988 2.045 2.116,
1.946 1.967 2.022 2.106 2.211,
1.955 1.984 2.067 2.188 2.333,
1.965 2.007 2.125 2.294 2.491,
1.980 2.040 2.205 2.435 2.696,
2.000 2.085 2.311 2.624 2.973,
2.040 2.149 2.461 2.866 3.356,
2.098 2.244 2.676 3.265 3.912,
1.908 1.908 1.908 1.908 1.908,
1.914 1.915 1.916 1.918 1.921,
1.921 1.922 1.927 1.936 1.947,
1.927 1.930 1.943 1.961 1.987,
1.933 1.940 1.962 1.997 2.043,
1.939 1.952 1.988 2.045 2.116,
1.946 1.967 2.022 2.106 2.211,
1.955 1.984 2.067 2.188 2.333,
1.965 2.007 2.125 2.294 2.491,
1.980 2.040 2.205 2.435 2.696,
2.000 2.085 2.311 2.624 2.973,
2.040 2.149 2.461 2.866 3.356,
2.098 2.244 2.676 3.265 3.912,

\end{verbatim}
\begin{verbatim}
2.189 2.392 3.004 3.844 4.775,
2.337 2.635 3.542 4.808 6.247,
2.588 3.073 4.534 6.636 9.144 ];
table4 = [ 0.000 0.000 0.000 0.000, 0.0-0.017 -0.032 -0.049 -0.064, 0.0-0.030 -0.061 -0.092 -0.123, 0.0-0.043 -0.088 -0.132 -0.179, 0.0-0.056 -0.111 -0.170 -0.232, 0.0-0.066 -0.134 -0.206 -0.283, 0.0-0.075 -0.154 -0.241 -0.335, 0.0-0.084 -0.173 -0.276 -0.390, 0.0-0.090 -0.192 -0.310 -0.447, 0.0-0.095 -0.208 -0.346 -0.508, 0.0-0.098 -0.223 -0.383 -0.576, 0.0-0.099 -0.237 -0.424 -0.652, 0.0-0.096 -0.250 -0.469 -0.742, 0.0-0.089 -0.262 -0.520 -0.853, 0.0-0.078 -0.272 -0.581 -0.997, 0.0-0.061 -0.279 -0.659 -1.198 ];
alphas1 = [ 2.1.9.1.8.1.7.1.6.1.5.1.4.1.3.1.2.1.1.1.0, 0.9.0.8.0.7.0.6.0.5 ];
betas1 = [ 0.0.25.0.5.0.75.1 ];
bet = abs(beta);
ii = 1; vec0 = 0; vec1 = 0; vec2 = 0;
vloc0 = 0; vloc1 = 0; vloc2 = 0;
den1 = 0; den2 = 0;
w1 = 0; w2 = 0; w3 = 0; w4 = 0;
do until ii > 15;
if alfa <= alfas1[ii, 1];
if alfa > alfas1[(ii+1), 1];
jj = 1;
do until jj > 4;
if bet > betas1[jj, 1];
if bet <= betas1[(jj+1), 1];
den1 = abs(alfas1[(ii+1), 1] - alfas1[ii, 1]);
den2 = abs(betas1[(jj+1), 1] - betas1[jj, 1]);
w1 = abs(alfa - alfas1[ii, 1])/den1;
w2 = abs(alfa - alfas1[(ii+1), 1] - alfa)/den1;
w3 = abs(bet - betas1[jj, 1])/den2;
w4 = abs(betas1[(jj+1), 1] - bet)/den2;
vec1 = w1*w3*table3((ii+1), jj) + w1*w4*table3((ii+1), jj);
vec2 = w2*w3*table3(jj, (jj+1)) + w2*w4*table3(jj, jj);
vec = vec1 + vec2;
vloc1 = w1*w3*table4((ii+1), jj) + w1*w4*table4((ii+1), jj);
vloc2 = w2*w3*table4(jj, (jj+1)) + w2*w4*table4(jj, jj);
vec = (bet/beta)*vloc1 + vloc2;
endif;
endif;
jj = jj + 1;
endo;
endif;
ii = ii + 1;
endo;
if vec0 = 0;
c = 0;
else:
c = m50 / vec;
endif;
loc1 = x50 + c*vec;
loc = loc1 - beta*vec*tan(pi*alfa/2);
endo;
@end output @
format 12.6;
vec1 = zeros(1, 4);
vec1[1, 1] = alfa,
vec1[1, 2] = beta;
vec1[1, 3] = c;
vec1[1, 4] = loc;
retp(vec1);
endp;
end:

\end{verbatim}
b) Construction of conversion tables

@ Efficient returns, thick tails and speculative processes @
@ Analysis of output matrixes @

new;
output file = alf10 out reset;
load noiter[] = noiter.out; load y1[] = alfas3.out;

@ Ranges of alpha for which cond std teta distribution are evaluated @
alfas = [ 1 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2 ];

@ Quantiles for alfas distribution for given std teta @
quantum1 = [ 0.05 0.50 0.95 ];

@ Quantiles for std teta distribution for given alfa @
quantum2 = [ 0.05 0.50 0.95 ];

@ Data transformation @
y2=reshape(y1,rows(y1)/(noiter+3),noiter+3)';
row=rows(y2);
col=rows(y2)';
obs=row-2;
y=y2[4:row,:];
y20=y;
i=1;
do until ii>col;
y30=sortc(y20,ii);
y[,]ii=y30[,]ii;
i=ii+1;
end;

@ Alpha conditional distribution analysis @
r=rows(quantum1');
resalfa=zeros(r+5,col+1);
resalfa[1,2:col+1]=y2[1,:];
resalfa[2,2:col+1]=y2[3,:];
r1=3;
do until rl>r+2;
resalfa[r1,1]=quantum1[1,r1-2];
resalfa[r1,2:col+1]=quant(quantum1[1,r1-2]);
resalfa[r1,2:col+1]=quant(quant(quantum1[1,r1-2]));
r1=r1+1;
end;
resalfa[r+3,1]=25.75;
resalfa[r+3,2:col+1]=(quant(0.75)+quant(0.25))/2;
resalfa[r+3,2:col+1]=mean(c(y));
resalfa[r+5,2:col+1]=std(c(y));
format 8,4;
"Distributions of alphas conditional on std teta";
y=
"First column : information about quantiles"
"First row : std of teta"
"Second row : ratio of too explosive series to all drawn series"
"Next rows : respective quantiles"
"Last three rows: quantile mean, mean and std of alpha"
wydruk(resalfa);

@ Std teta conditional distribution analysis @
stdtet=y2[1,:];
noranges=rows(alfas');
v2 = alfas';
v1=sortc(v2,1);
count1=zeros(noranges,col);
j=1;
do until j>col;
count1[,]j=counts(y[,]j,v1);
j=j+1;
end;
count=sumc(count1);
elements=sumc(count1);
\[ v_3 = v_1' \]

\[ \text{tetas} = \text{stdtet}' \]

\[ \text{prob} = \text{count} / (\text{sum(count)})' \]

\[ \text{cumdens} = \text{zeros} (\text{noranges}, \text{col})' \]

\[ k = 1 \]

\[ \text{do until } k > \text{col} : \]

\[ \text{cumdens} (k..) = \text{sum(prob}[1..k,..])' \]

\[ k = k + 1 \]

\[ \text{enddo} \]

\[ \text{table} = \text{zeros} (\text{col} + 1, \text{noranges} + 1) \]

\[ \text{table}[1, 2: \text{noranges} + 1] = v_3 \]

\[ \text{table}[1, \text{col}, 1] = \text{tetas} \]

\[ \text{table}[\text{col} + 1, 1] = \text{table}[\text{col}, 1] \]

\[ \text{table}[2: \text{col} + 1, 2: \text{noranges} + 1] = \text{cumdens} \]

\[ \text{table}[1] = \text{quantile2} (\text{quantum}2, \text{table}) \]

@ Procedures @

@ Procedure to calculate quantiles of std of teta distribution for given values of alpha @

\[ \text{proc quantile(q1, w);} \]

\[ @ \text{q1} - \text{vector with demanded quantiles} @ \]

\[ @ \text{w} - \text{matrix: first column - std teta values} \]

\[ : \text{other columns - cum dis for different alphas} \]

\[ \text{first row} \quad \text{- respective alfa values} \] @

\[ \text{local q, noquant, noalf, notet, a1, a2, a3, quantiles, qq, weight, w1;} \]

\[ q = q1' \]

\[ \text{noquant} = \text{rows(q)} \]

\[ \text{noalf} = \text{rows(w') - 1} \]

\[ \text{notet} = \text{rows(w) - 1} \]

\[ \text{quantiles} = \text{zeros} (\text{noquant} + 1, \text{noalf} + 1) \]

\[ \text{quantiles}[2: \text{noquant} + 1, 1] = q \]

\[ \text{quantiles}[1, 2: \text{noalf} + 1] = w [1, 2: \text{noalf} + 1] \]

\[ w1 = w \]

\[ w1[1, 1] = \text{zeros} (1, \text{noalf} + 1) \]

\[ w1[1, 1] = v1 [\text{noranges}, 1] \]

\[ a1 = 2 \]

\[ \text{do until } a1 > \text{noalf} + 1 ; \]

\[ a2 = 1 \]

\[ \text{do until } a2 > \text{noquant} ; \]

\[ a3 = 1 \]

\[ \text{weight} = 0 ; \text{qq} = 0 ; \]

\[ \text{do until } a3 = \text{notet} ; \]

\[ \text{if } q[a2, 1] > w1[a3, a1] ; \]

\[ \text{if } q[a2, 1] <= w1[a3 + 1, a1] ; \]

\[ \text{weight} = q[a2, 1] - w1[a3, a1] / w1[a3 + 1, a1] - w1[a3, a1] \]

\[ \text{qq} = \text{weight} * w[a3 + 1, 1] + (1 - \text{weight}) * w[a3, 1] ; \]

\[ \text{quantiles}[a2 + 1, a1] = \text{qq} \]

\[ \text{endif} \]

\[ \text{endif} \]

\[ a3 = a3 + 1 \]

\[ \text{enddo} \]

\[ a2 = a2 + 1 \]

\[ \text{enddo} \]

\[ a1 = a1 + 1 \]

\[ \text{enddo} \]

\[ \text{rdtp(quantiles)} \]

endp
 Procedure for printing output 

```plaintext
proc wydruk(paper);
    local mm1, mm2, mm3;
    mm1=rows(paper);
    mm2=trunc((mm1-1)/6);
    mm3=0;
    do until mm3>mm2+1;
        print(paper[.,I]-paper[.,mm3*6+2:mm3*6+7],'' );
        mm3=mm3+1;
    endo;
    if mm2<mm3/6;
        print(paper[.,I]-paper[.,mm2*6+2:mm3],'' );
    endif;
    retp(0);
endp;
```

 Procedure to calculate quantiles of alpha distribution for given values of std of teta 

```plaintext
proc quant(x);
    local nn1, nn2, nn;
    nn1=x*obs;
    if nn1<1; nn1=1; endif;
    nn2=nn1-trunc(nn1);
    nn=zeros(1,icol);
    nn=(1-nn2)*y[trunc(nn1),.]+nn2*y[trunc(nn1)+1,.];
    retp(nn);
endp;
end;
```
5. Program to estimate parameters of GARCH(1,1) process (applied in Chapter V)

@ Program to filter the data for GARCH[1,1] effect @
@ Program estimates GARCH[1,1] parameters and rescales the data @
@ using estimated values of conditional standard deviations @
@ Series in the output should be homoscedastic @

new;
output file = dataint.out reset;
output file = dataint2.out reset;
output file = intparg.out reset;
output off;
load x[]=uklev4.txt;

@ No of rows and columns of the input matrix of data @
@ Series should be given in columns @
"mrow"=1100;
"ncol"=39;
out1=zeros(ncol,7);
out2=zeros(mrow+2,ncol);
xtot=reshape(x,mrow,ncol);

@ Delete from sample first observations equal to zero @
i1=22;
do until i1>28:
  z=zeros(mrow,1);
  z=xtot[.,i1];
  label la;
  if z[1]=0;
    z=z[2:rows(z)];
goto label la:
  endif;
endo;

@ Compute log returns and check for missing observations @
i2=2;
z1=z;
z=zeros(rows(z1)-counts(z1,0),1);
do until i2>rows(z1)-counts(z1,0);
  if z1[i2]=0;
    z[i2]=ln(z1[i2])-ln(z1[i2-1]);
i2=i2+1;
  else;
    z2=z1; z1=zeros(rows(z2)-1,1);
    z1[1:i2-1]=z2[1:i2-1];
    z1[i2:rows(z1)]=z2[i2+1:rows(z2),1];
  endif;
endo;
z=z[2:rows(z)];

@ Delete from sample zero log returns @
@ It is assumed that zero log return on the price index is @
@ due to holidays or suspended trading @
i2=1; i3=1;
z1=z;
do until i2>rows(z);
  if z1[i2]=0;
    z[i3]=z1[i2];
i3=i3+1;
  endif;
i2=i2+1;
endo;
z=z[1:i3-1];

@ Specification for the GARCH[1,1] model @
@ May be changed upto GARCH[2,2] @
p=1;
q=1;
z=z-ones(rows(z),1);
aw={0.0,1.0,9.0,0.2};

@ Start at local, but not global, optima of the Judge function @

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@ Random search for optimal parameters @
maxl_a1=1;
maxl_start=zeros(p+q+2,1);
maxl_start[1]=mean(z[1:1]);
maxl_m=-1000;
do until maxl_a1>1000;
maxl_start[1]=maxl_start[1]*rndn(1,1)+rndn(1,1)*0.00000001;
maxl_start[2]=rndn(1,1)/100;
sos:
maxl_start[3]=rndn(1,1);
maxl_start[4]=rndn(1,1);
if garch(maxl_start)>maxl_m;
maxl_par=maxl_start;
maxl_m=garch(maxl_par);
cis:
scr(maxl_par,maxl_m,maxl_a1,i1);
endif;
maxl_a1=maxl_a1+1;
endo;
b = maxl_par;
neps = 4;

/* Set input parameters. */
max = 1;
eps = 1.0E-6;
rt = .85;
seed = 123;
nr = 20;
nl = 5;
maxevl = 100000;
iprint = 1;
npar = rows(b);
lb = [-10000,0,0,0,0,0,0,0];
ub = [+10000,10000,1,1];
c = 2.0*ones(npar,1);

/** Set input values of the input/output parameters. **/ 
t = 5.0;
vm = 1.0*ones(npar,1);
rndseed seed;

print "number of parameters=", npar, " max=", max, " t=", t;
print "rt=", rt, " eps=", eps, " ns=", ns;
print "neps=", neps, " maxevl=", maxevl;
print "iprint=", iprint, " seed=", seed;
print "c vector=", c;

/** Call SA **/
tim = hsec;
{xopt, fopt, nacc, nfcnev, nobds, ier, t, vm} = 
sai&garch,b,max,rt,eps,ns,rt,neps,maxevl,lb,ub,c,iprint,t,vm);
tim = hsec-tim;

print "**** results after sa ****";
print "solution" xopt;
print "final step length" vm;
print "optimal function value " fopt;
print "number of function evaluations " nfcnev;
print "number of accepted evaluations " nacc;
print "number of out-of-bounds evaluations " nobds;
print "final temperature " t;
print "elapsed time (minutes) " tim/6000;
print "error" ier.

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out1[1,1]=i1;
out1[1,2]=xopt';
out1[1,6]=fopt;
out1[1,7]=rows(z);
x=xopt';
f=fopt;

@ Compute conditional standard deviations @
"Parameters":
format 4,6;
x:
"Standardizing series":
h=zeros(rows(z),1);
h[1]=x[2]+x[3]*h[1]+x[4]*(z[1,1]-x[1])*2*0.5;
i2=2;
do until i2>rows(z);
h[i2]=x[2]+x[3]*h[i2-1]+x[4]*(z[i2-1,1]-x[1])*2;
h[i2]=h[i2]*0.5;
i2=i2+1;
end;

@ Standarize conditional standard deviations @
h=h/mean(h);

@ Standarize data @
The series should be homoscedastic and have average variability @
on the same level as series in input @
z=(z-x[1])/h;
z=z[2:rows(z),1];

out2[1,1]=i1;
out2[2,1]=rows(z);
out2[3:rows(z)+2,1]=z[..,1];
output file = dataint.out;
output on:
out2[..,1];
output off:
output file = intparg.out;
outwidth ncol:
output on:
out1[..,1];
output off:
i1=i1+1;
end:

output file = dataint2.out;
outwidth ncol:
output on:
out2;
output off:

format 6,9;
out1:

/**************************** procedures ****************************/

proc scr(spar,sfun,siter,st):
format 6,9;
print("GARCH by Zbigniew Kominek");
print("GARCH by Zbigniew Kominek");
print("Constant ":spar[1];
"Kappa ":spar[2];
if p>0; "Delta ":spar[3:2+p]; endif;
if q>0; "Alpha ":spar[3+p:3+p+q+2]; endif;
"Max-lik ":sfun;
"Iteration ":siter;
print("GARCH by Zbigniew Kominek");
"Index No: ":st;
retp("GARCH by Zbigniew Kominek");
endp;

proc garch(b):
local u2,kappa,delta,alpaha,g,gar,suma,
weight,r,hp0,hq0,garq0;
u2 = (z[.,1] - z[.,2] * b[1])^2;
kappa = b[2];
if p>0: delta = b[3:2+p];
else: delta = [0];
endif;
if q>0: alpha = b[p+3:p+q+2];
else: alpha = [0];
endif;
if p*q>0:
h = recserar(kappa + _shape(u2,q,1)*alpha,mean(u2)*ones(p,1),delta);
endif;
if p+q>0:
delta = [0];
h = recserar(kappa + _shape(u2,q,1)*alpha,mean(u2),delta);
endif;
if p+q==0;
delta = [0]; alpha = [0]; h = kappa;
endif:
gar = -0.5*( (u2 / h) + ln(2 * pi) + ln(h) );
r = maxc(plq);
weight = (rows(z))/max(rows(z));
if p+q>0, weight[1:r] = zeros(r,1); endif;
gar = gar * weight;
gar = sumc(gar);
if p+q>0:
if sumc(b[3:p+q]) >= l:
endif;
endif:
retp(gar);
endp:

proc _shape(z,m,r):
local y,n,v;
N = rows(z) - m - 1;
y = zeros(rows(z),m):
if r == 1:
v = seqa(1.1,m)' + seqa(0.1,n+1);
else:
v = seqa(m,-1,m)' -t- seqa(0.1,n+1);
endif:
y[m+1:rows(z),.] = reshape(z[v].n+1,m);
retp(y);
endp:

proc symgarch(n,mean,kappa,sigma1,sigma2,tau1,tau2):
local h,v,u,y,i;
h = zeros(n,1);
h[1,1]=kappa;
h[2,1]=kappa;
v = radn(n,1);
u = zeros(n,1);
u[1,1]=h[1,1]*0.5^*[1,1];
u[2,1]=h[2,1]*0.5^*[2,1];
i=3;
do until i>n;
h[i,1]=kappa + sigma1*h[i-1,1] + sigma2*h[i-2,1] +
tau1*u[i-1,1]^2 + tau2*u[i-2,1]^2;
u[i,1]=h[i,1]*0.5^*[1,1];
i=i+1;
endo:
y=u+mean;
retp(y);
endp:

THE SIMULATED ANNEALING PROGRAM FOLLOWS:

*/
proc (8) = sa(& fcn, x, max.n, eps.nt, neps, maxevl, lb, ub, c, iprint, t, vm);

local n, xopt, xp, nacp, nobds, nfcnev, ier, fstar,
     f, fopt, nup, nrej, nnew, ndown, lnoobs, m, j, h, i,
     fp, p, pp, ratio, quit, fcn.proc;

n = rows(x);

xopt = zeros(n,1);
xp = zeros(n,1);
naep = zeros(n,1);

/* * Set initial values. */

nacp = 0;
nobds = 0;
nfcnev = 0;
ier = 99;
xopt = x;
naep = zeros(n,1);
fstar = 1e20*ones(neps.i);

/* * If the initial temperature is not positive, notify the user and abort. */

if (t <= 0.0);
    print"The initial temperature is not positive. Reset the variable t";
    ier = 3; stop;
endif:

/* * If the initial value is out of bounds, notify the user and abort. */

if (sum c(x,> ub) + sum c(x,< lb) > 0);
    print"initial condition out of bounds";
    ier = 2; stop;
endif;

/** Evaluate the function with input x and return value as f. **/

f = fcn(x);

/**
If the function is to be minimized, switch the sign of the function.
Note that all intermediate and final output switches the sign back
to eliminate any possible confusion for the user.
**/

if (max == 0): f = -f; endif;
nfcnev = nfcnev + 1;
fopt = f;
fstar = f;
if (ierror > 1): call prt2tma.x.f); endif:

/**
Start the main loop. Note that it terminates if (i) the algorithm
successfully optimizes the function or (ii) there are too many
function evaluations more than MAXEVL).
**/

m1: do while m <= nt:
    j1: do while j <= ns:
        h1: do while h <= n:

        /* * Generate xp, the trial value of x. Note use of vm to choose xp. */
        label z:

        i1: do while i <= n:
            if (i == h):
                xp[i] = v[i] = -round(1.1)*2.-1.) * vm[i];
            else:
                xp[i] = v[i];
            endif:

            i1 = i1 + 1;
        endwhile:

        j1 = j1 + 1;
    endwhile:

    m1 = m1 + 1;
endwhile:

ier = 99;
if (fopt > 0):
    print"The algorithm did not converge."
    endif:

ier = ierror;
if (ier == 99):
    print"The algorithm did not converge."
    endif:

ier = ierror;
if (ier == 99):
    print"The algorithm did not converge."
    endif:
/** If xp is out of bounds, select a point in bounds for the trial. **/

if((xp[i] < lb[i]) or (xp[i] > ub[i])):
    xpl[i] = lb[i] + (ub[i] - lb[i]) * random(1,1);
    nobds = nobds + 1;
    inobds = inobds + 1;
    if(iprint >= 3); call prt5(max.xp.x,fp,f); endif;
endif;

i = i+1; endo;

@ Constraint for parameters @
@if p+q >0;@
    if (sum c(xp[3:4]) >= l):
        goto labelzk;
    endif;
@ endif;@

/* * Evaluate the function with the trial point xp and return as fp. **/

fp = fcn(xp);
if (max == 0); fp = -fp; endif;

nfcnev = nfcnev + 1;
if(iprint >= 3); call prtdimax.xp.x.fp,f); endif;

/* * If too many function evaluations occur, terminate the algorithm. **/

if (nfcnev >= maxevl): call prt5();
    if (max == 0); fopt = -fopt; endif;
    ier = 1;
    stop;
endif;

/** Accept the new point if the function value increases. **/

if (fp >= f):
    if[iprint >= 3]; print "point accepted": endif;
    x = xp;
    f = fp;
    nacc = nacc + 1;
    napc[h] = napc[h] + 1;
    nup = nup + 1;

/** If greater than any other point, record as new optimum. **/

if (fp > fopt); if[iprint >= 3]; print "new optimum": endif;
    xopt = xp;
    fopt = fp;
    nnew = nnew + 1;
endif;

/** If the point is lower, use the Metropolis criteria to decide on acceptance or rejection. **/

else;
    p = exp(exp(fp - f))/t);
    pp = random(1,1);
    if (pp < p):
        if[iprint >= 3]; call prtdimax: endif;
        x = xp;
        f = fp;
        nacc = nacc + 1;
        napc[h] = napc[h] + 1;
        ndown = ndown + 1;
        else;
        nrej = nrej + 1;
        if[iprint >= 3]; call prtdimax: endif;
endif:

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endif;

h = h+1; endo;
j = j+1; endo;

/** Adjust vm so that approximately half of all evaluations are accepted. **/
i=1; do while i <= n;
    ratio = nacp[i]/ns;
    if (ratio > .6);
        vm[i] = vm[i] * (1.0 - c[i] * (ratio - .6)/.4);
    elseif (ratio < .4);
        vm[i] = vm[i]/(1.0 - c[i] * (.4 - ratio)/.4));
    endif;
    if (vm[i] > (ub[i] - lb[i]));
        vm[i] = ub[i] - lb[i];
    endif;
    i = i+1; endo;
if(iprint >= 2); call pr8(vm.xopt.x); endif;

nacp = zeros(n,1);
m = m+1; endo;
if(iprint >= 1);
call pr9(max,t.xopt.vm.fopt.nup.down.nrej.inobds.nnew); endif;

/** Check termination criteria. **/
quit = 0;
fs= f;
if (abs(fopt - fstar[1]) <= eps); quit = 1; endif;
if (sum(abs(fabs(fstar) > eps) > 0); quit = 0; endif;

/** Terminate SA if appropriate. **/
if (quit);
x = xopt;
ier = 0;
if (max == 0); fopt = -fopt; endif;
if(iprint >= 1); call pr10; endif;
reps xopt. fopt. nacc. nnev. nobds. ier. t. vm);
endif;

/** If termination criteria are not met, prepare for another loop. **/
t = rt*t;
steps; do while i >= 2;
    (star[i] = fstar[i-1];
    i = i-1; endo;
f = fopt;
x = xopt;

/** Loop again. **/
goto _100;
endp;

proc exprep(x);
/**
This function replaces exp to avoid under- and overflows.
Note that the maximum and minimum values of expre are such that they have no effect on the algorithm.
**/
local e;
if (x > 709); e = 8.2184e+307;
    elseif (x < -708); e = 1;
    else: e = exp(x);
    endif;
repret;
endp;

-----------------------------------------------------------------------------------
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/**
This function replaces exp to avoid under- and overflows.
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**/
local e;
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    elseif (x < -708); e = 1;
    else: e = exp(x);
    endif;
repret;
endp;

-----------------------------------------------------------------------------------
This subroutine prints intermediate output, as does prt2 through prt10. Note that if SA is minimizing the function, the sign of the function value and the directions (up/down) are reversed in all output to correspond with the actual function optimization.

print "The starting value \((x)\) is outside the bounds";
print "\((lb and ub)\). execution terminated without any";
print "optimization. Respecify \(x\), \(ub\) or \(lb\) so that";
print '\(lb(i) < x(i) < ub(i)\). i = 1, n";
endp:

proc (0 ) =  prt2(max,x,f);
print "":
print "initial x" x;
if  (max); print "initial f" f: else: print "initial f" -f: endif;
endp:

proc (0 ) =  prt3(max,xp.x.fp.f):
print "current x" x;
if  (max);
print "current f" f:
else:
print "current f" -f:
endif:
print "trial x" xp;
print "point rejected since out of bounds";
endp:

proc (0 ) =  prt4(max.xp.x.fp.f):
print "current x" x;
if  (max):
print "current f" f:
print "trial x" xp;
print "resulting f" fp:
else:
print "current f" -f:
print "trial x" xp;
print "resulting f" -fp:
endif:
endp:

proc (0 ) =  prt5():
print "Too many function evaluations: consider";
print "increasing maxevl or eps. or decreasing";
print "nt or rt. These results are likely to be poor";
endp:

proc (0 ) =  prt6(max):
if  (max);
print "though lower, point accepted":
else:
print "though higher, point accepted":
endif:
endp:

proc (0 ) =  prt7(max):
if  tm ax);
print "lower point rejected":
else:
print "higher point rejected":
endif:
endp:

proc (0 ) =  prt8(max.xopt.x):

print "Intermediate results after step length adjustment";
print "new step length (vm)" vm;
print "current optimal x" xopt;
print "current x" x;
print "":
endp;

proc (0 ) = prtd(max,t,xopt,vm,fopt,nup,ndown,nrej,nobds,nnew);
local totmov;
totmov = nup + ndown + nrej;

print "Intermediate results before next temperature reduction";
print "current temperature " t;
if (max);
print "max function value so far" fopt;
print "total moves" totmov;
print "uphill" nup;
print "accepted downhill" ndown;
print "rejected downhill" nrej;
print "out of bounds trials" nobds;
print "new maxima this temperature" nnew;
else;
print "min function value so far" -fopt;
print "total moves" totmov;
print "downhill" nup;
print "accepted uphill" ndown;
print "rejected uphill" nrej;
print "trials out of bounds" nobds;
print "new minima this temperature" nnew;
endif;
print "current optimal x" xopt;
print "strength (vm)" vm;
print "":
endp;

proc (0 ) = prtl0Q();
print "SA achieved termination criteria. ier = 0";
endp;

end:
7. Program to test for outliers using Hadi – Simonoff (1993) approach (applied in Chapter V)

@ Program to detect outliers using Hadi - Simonoff approach @

new;
output file = outlier.out reset;

load data[]=datafin.txt;
N=66;
c=3;
data=reshape(data,n,c);
@ Stopping criterion for Step 1 @
h=trunc((n+c-1)/2);
@ Significance level for test for outliers @
alpha=0.05;

@ Choosing the initial basic set @
d1=sortc(data,2);
d2=sortc(data,3);
Cm=[d1[n/2,2]+d1[n/2+1,2]/2-d2[n/2,3]+d2[n/2+1,3]/2]/2;
s=0;
i=1; do until i>n;
s=s1+(data[i,2:3]-Cm)'*(data[i,2:3]-Cm); i=i+1; endo:
Sm=s1/(n+1);
@ First sorting @
rank1=zeros(n,4);
rank1[,2:4]=data;
i=1; do until i>n;
rank1[i,1]=data[i,2:3]-Cm)*inv(Sm)*(data[i,2:3]-Cm);
i=i+1; endo;
rank1=sortc(rank1,1);
@ Second sorting @
weight1=zeros(n,1);
weight1(1:trunc((n+c+1)/2))=ones(trunc((n+c+1)/2),1);
x1=rank1[,3:4];
Cv=sumc(weight1*x1)/sumc(weight1);
s1=zeros(2,2);
i=1; do until i>n;
s1=s1+weight1(i)*(x1(i,1)-Cv)'*(x1(i,1)-Cv);
i=i+1; endo;
Sv=s1/(sumc(weight1)-1);
rank2=zeros(n,4);
rank2[,2:4]=data;
i=1; do until i>n;
rank2[i,1]=(data[i,2:3]-Cv)*inv(Sv)*(data[i,2:3]-Cv);
i=i+1; endo:
rank2=sortc(rank2,1);

@ Updating the basic set @
@ Computing clean subset M of size h @
t=c+2;
do until t>h:
b=rank2[1:t,];
theta=rank2[1+t:n,];
x=ones(n,1)-rank2[,3];
bet=rank2[,1];
theta=y=rank2[1:n,];
(xb*y[1:t])=ols1(xb y[1:t]);
rank3=rank2;
i=1; do until i>n:
if i<=t:
rank3[1,i]=abs(y[i]-x[i]*bet)/sqrt(1-x[i]*inv(xb'*xb)*x[i,]);
else:
rank3[1,i]=abs(y[i]-x[i]*bet)/sqrt(1+x[i]*inv(xb'*xb)*x[i,]);
endif:
i=i+1; endo;
rank2=sortc(rank3,1);
i=t+1; endo;

@ Step 2: Updating set M using internally studentized residuals @
t=h;
do until t->n-1;
b=rank2[1:t,1:n];
x=ones(n,1)-rank2[..3];  \text{ @Alphas@}
xb=x[1:t,1:n];
y=rank2[..,4]; \text{ @Thetas@}
[ bet, sm ] = ols1(xb, y[1:t]);
rank3=rank2;
i=1; do until i=n;
if i<=t;
    rank3[i,1]=abs( (y[i]-xb .*bet)/
        (sm * sqrt(1-xb .*inv(xb *xb) *xb .*xb)) )
else;
    rank3[i,1]=abs( (y[i]-xb .*bet)/
        (sm * sqrt(1+xb .*inv(xb *xb) *xb .*xb)) )
endif;
i=i+1; enddo;
rank2=sortc(rank3,1);
d=rank2[t+1,1:n];
ti=cdftci(alphas/(2*(t-1))+t-c);  
if d>i:
    print("This is the list of outliers:");
    o=rank2[t+1:n,1:n];
    t=n+1;
endif;
t=t+1;
if t==n;
    print("There is no outliers");
endif;
endo;

\text{@ Procedures @}

\text{@ OLS estimation @}
proc(2) = ols1(x2,x1):
local coef, sm, p, e, sse:
    coef=inv(x2'*x2)*x2',x1:
p=x2*inv(x2'*x2)*x2':
e=(eye(rows(x2))+p)*x1:
    sse=e'*e:
    sm=sqrt(sse/(rows(x2)-cols(x2))):
    retp(coef,sm):
endp:
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SPECIAL NOTE

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