For my Jen
Perturbations of Black holes in Einstein-Cartan theory

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Abstract

Torsion is a property of space-time which is not incorporated into the standard formulation of general relativity but which appears as a consequence of unification schemes for fundamental forces. It is, therefore, important to understand its physical consequence. This thesis begins with an introduction to a non-propagating version of torsion theory as an extension to general relativity. The theory can be described in terms of a pair coupled field equations with torsion algebraically linked to elementary particle spin. In order to develop the theory it is necessary to postulate a form for the energy-momentum tensor of spinning matter which is not prescribed in the classical domain. The two main candidates that have been proposed for a spinning fluid are considered. Chapter two contains an independent reworking of Zerilli’s [1] perturbation calculation of a particle falling into a Schwarzschild black hole. The perturbation equations are found and the resulting wave equations are derived. The special case of a particle falling radially is considered in detail. Chapter three contains new work which employs the method of Zerilli in torsion theory to consider a particle with spin falling radially into a black hole. The changes to the black hole are found for each of the two energy-momentum tensors of Chapter one. This enables us to discount one of these as unphysical. The differential equations describing the gravitational radiation released by this system are derived. Finally in Chapter four these equations are solved to find the gravitational radiation from a spinning particle falling radially. These may be significant for observational assessments of torsion theory.
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For I am convinced that neither death nor life, neither angels nor demons, neither the present nor the future, nor any powers, neither height nor depth, nor anything else in all creation, will be able to separate us from the love of God that is in Christ Jesus our Lord.

Romans 8 v 38,39
## Contents

1 Contact Torsion Theory  
1.1 Introduction .......................... 1  
1.2 Definition of Torsion Tensors ........................................................... 3  
1.3 The Einstein, Riemann and Ricci Tensors . . . . . . . . . . . . . . . . . 5  
1.4 The Einstein-Cartan Field Equations ............................................. 6  
1.5 Calculations Using Torsion ............................................................ 8  
  1.5.1 The Symmetric part of the Contortion ............................... 8  
  1.5.2 Finding Contortion in Terms of Spin ............................... 9  
  1.5.3 Combining the Field Equations .................. 10  
1.6 Energy-Momentum Tensors ......................................................... 11  
  1.6.1 Hehl's Spinning Fluid Energy-Momentum Tensor .......................... 11  
  1.6.2 Ray and Smalley's Spinning Fluid Energy-Momentum Tensor .... 12  
1.7 Organisation of the thesis ......................................................... 15  

2 The Zerilli Wave Equation .................................................. 17  
2.1 Introduction ................................................. 17  
2.2 Zerilli's Method .............................................. 19  
2.3 Tensor Harmonics ............................................. 20  
2.4 The Perturbation Equations ............................................. 24
## CONTENTS

2.4.1 The Initial Perturbation ................................................... 24
2.4.2 Combining the Perturbation and Tensor Harmonics ........ 26
2.4.3 Fourier Tranformation of the Perturbation Equations .... 29

2.5 The Energy-Momentum Tensor .............................................. 31
2.5.1 Definition ........................................................................ 31
2.5.2 Harmonic Components ..................................................... 32
2.5.3 Fourier Transformation of the Energy-Momentum Tensor . 34

2.6 Combining the Perturbation Equations to form a Wave Equation 35
2.6.1 The Magnetic Parity Equations ....................................... 35
2.6.2 The Electric Parity Equations ......................................... 36

2.7 The Source for a Radially Infalling Particle ......................... 43

2.8 Gauge Transformations .......................................................... 44

2.9 Non-Radiative Cases ............................................................... 46
2.9.1 Change in Mass ............................................................... 47

2.10 Summary and Conclusion ..................................................... 49

3 A Torsion Wave Equation .......................................................... 50
3.1 Introduction .......................................................... 50

3.2 Hehl's Energy-Momentum Tensor ...................................... 51
3.2.1 Geodesic Motion ............................................................ 51
3.2.2 The Energy-Momentum Tensor ...................................... 55

3.3 Ray and Smalley's Energy-Momentum Tensor ................. 58
3.3.1 Geodesic Motion ............................................................ 58
3.3.2 The Energy-Momentum Tensor ...................................... 59

3.4 Integrating over all time .................................................... 60
3.5 Finding the Harmonic Coefficients .................................... 61
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.6</td>
<td>Solving the Perturbation Equations</td>
<td>63</td>
</tr>
<tr>
<td>3.7</td>
<td>Fourier Transform of Torsion Coefficients</td>
<td>65</td>
</tr>
<tr>
<td>3.8</td>
<td>Torsion Wave Equations</td>
<td>67</td>
</tr>
<tr>
<td>3.9</td>
<td>Summary and Conclusions</td>
<td>68</td>
</tr>
<tr>
<td>4</td>
<td>Gravitational Radiation</td>
<td>69</td>
</tr>
<tr>
<td>4.1</td>
<td>Introduction</td>
<td>69</td>
</tr>
<tr>
<td>4.2</td>
<td>Radiation from a mass falling radially</td>
<td>70</td>
</tr>
<tr>
<td>4.3</td>
<td>The Downhill Simplex Method</td>
<td>73</td>
</tr>
<tr>
<td>4.4</td>
<td>Radiation from a Spinning Particle</td>
<td>78</td>
</tr>
<tr>
<td>4.4.1</td>
<td>The Electric Source Term</td>
<td>79</td>
</tr>
<tr>
<td>4.4.2</td>
<td>The Magnetic Source Term</td>
<td>82</td>
</tr>
<tr>
<td>4.5</td>
<td>Applications</td>
<td>82</td>
</tr>
<tr>
<td>A</td>
<td>The Maple V Code</td>
<td>86</td>
</tr>
<tr>
<td>B</td>
<td>The Matlab Code</td>
<td>94</td>
</tr>
<tr>
<td>B.1</td>
<td>searcher.m</td>
<td>94</td>
</tr>
<tr>
<td>B.2</td>
<td>wave function.m</td>
<td>99</td>
</tr>
<tr>
<td>B.3</td>
<td>imagSecondOrder.m</td>
<td>101</td>
</tr>
<tr>
<td>B.4</td>
<td>realSecondOrder.m</td>
<td>101</td>
</tr>
<tr>
<td>Bibliography</td>
<td></td>
<td>102</td>
</tr>
</tbody>
</table>
# List of Figures

4.1 The real (solid) and imaginary (dashed) parts of the source term coming from a particle falling radially in a Schwarzschild background. 72

4.2 The wave function for $\omega m = 0.3, L = 2$ where the radiation function is $\tilde{R}_{LM}$ (equation 4.3). The real part is the solid line while the imaginary is the dashed line. 73

4.3 The wave function for $\omega m = 0.35, L = 2$ where the radiation function is $\tilde{R}_{LM}$ (equation 4.3). Using the convergence condition of Davis et al[26] causes the algorithm to converge to a solution with the wrong boundary conditions. 74

4.4 Different ways in the simplex is changed using the DSM. a) A 'reflection' away from the worst point. b) A 'reflection' and elongation to increase the area of the simplex, making it converge quicker. c) A contraction of the simplex away from the worse point. d) A contraction of the simplex towards the best point. 76

4.5 The energy $(1/m_0^2)dE/d\omega$ released by the oscillation of the black hole event horizon for a particle falling in radially. The $L = 2$ multipole only is shown. 79
LIST OF FIGURES

4.6 The energy \((1/s^2)\frac{dE}{d\omega}\) released, calculated from the differential equation for electric parity harmonics. The \(L = 2\) multipole only is shown. .......................................................... 80

4.7 The differential equation for magnetic parity harmonics are used to calculate the energy released. Total energy is the sum of the electric and magnetic components. The \(L = 2\) multipole only is shown. .......................................................... 84
Chapter 1

Contact Torsion Theory

1.1 Introduction

General relativity is a metric theory of gravity based on the assumption that structureless particles follow geodesic paths and that the symmetric energy-momentum tensor is the only source of curvature. Einstein, Schrödinger, Eddington and others considered the possibility of an asymmetric metric, or an asymmetric connection, but usually in the context of a unified field theory relating the non-symmetric part to the electromagnetic field. It was Cartan[2] who first introduced the notion of torsion as the antisymmetric part of the connection and suggested a relation to the intrinsic spin of particles. The work of Cartan was largely forgotten until the 1940s when Papapetrou[3] and Weyssenhoff and Raabe[4] showed that the energy-momentum tensor of a spinning particle is asymmetric. Since the source of gravity in Einstein's theory is a symmetric energy-momentum tensor, it is clear that an extension of the theory is necessary for bodies with intrinsic spin.

The systematic development of the extended theory as, in effect, a gauge theory of gravity was continued by Weyl[5] and later by Kibble[6], Sciama[7, 8]
and others. From the work of Hehl[9, 10], the torsion theory emerges as a Poincaré invariant gauge theory.

The work of Hehl in the 1970s provided a strong basis for generalising relativity by means of an asymmetric connection. Torsion had long been thought to be a sensible extension to general relativity, but, until the work of Hehl it did not have a strong physical interpretation and was considered more as a mathematical extension. Since then work on torsion has developed considerably with the attempts towards a Unified Theory of quantum mechanics and relativity in string theory. The majority of modern theories of gravity include torsion as a consequence of elementary particle spin.

The torsion field equation is unusual as the relationship between spin and torsion is algebraic and not differential. This means the torsion field does not propagate through space-time as some other fields do. This form of torsion theory is referred to as ‘contact torsion’ because there is no spin-spin interaction unless two spinning particles are in contact. In this form the theory lends itself to straightforward comparison with general relativity. The asymmetric gravitational field equation can be split into Riemannian and non-Riemannian parts. Substitution for the antisymmetric parts using the torsion field equation yields an Einstein tensor and spin terms which can be considered as additions to the energy-momentum tensor. Thus a direct comparison with classical relativity can be established. For a review of contact torsion see Hehl et al[11].

As an extension of general relativity, contact torsion has been considered, by some, as unphysical. As a consequence propagating forms of the theory have been proposed. In fact a Lagrangian quadratic in torsion leads to a propagating theory. There have been many proposals in the last twenty years along these lines (for an example see Hojman et al[12] and Hojman, Rosenbaum and Ryan[13]).
In an extension to the analytical tools of general relativity Singh and Griffiths[14] have generalised the Newman-Penrose[15] formalism to include torsion theories. More recently Carroll and Field[16] have shown that a propagating torsion will decay rapidly outside a spinning matter distribution. This has consequences for the observability of torsion in future experiments. Hammond has reviewed propagating torsion in [17].

In this thesis we shall be concerned only with contact torsion, although the methods can be extended to the propagating case.

1.2 Definition of Torsion Tensors

It was not until the development of gauge field theory that torsion was placed on a sound theoretical footing. In 1976 Hehl et al [11] showed that torsion can be considered as a byproduct of gauge transformations of the Poincare group. Nevertheless, in this theory the form of the energy-momentum tensor for a perfect spinning fluid is postulated rather than derived from first principles.

More recently Ray and Smalley[18, 19, 20, 21] have developed a Lagrangian theory of a spinning fluid. This theory is set up as applying to a locally rotating fluid rather than intrinsic spin. Nevertheless, it can be used to derive a set of field equations for a perfect spinning fluid. We shall find that these equations differ from those of Hehl by an important factor of 1/2 in one term. In Chapter three we shall resolve this difference by computing explicitly a model problem for a spinning particle. We begin by setting up the machinery of the contact torsion theory.

Any tensor can be split into symmetric and antisymmetric parts. For a general
tensor \( P \),

\[
P_{ij} = P_{(ij)} + P_{[ij]},
\]

where \((\cdot)\) and \([\cdot]\) denote symmetry and antisymmetry respectively and \(i, j, k = 0...3\). Conversely the symmetric and antisymmetric parts can be found by,

\[
P_{(ij)} = \frac{1}{2} (P_{ij} + P_{ji}),
\]

\[
P_{[ij]} = \frac{1}{2} (P_{ij} - P_{ji}).
\]

For the development of torsion theory, the torsion tensor, \( S_{ij}^k \), is set equal to the antisymmetric part of the connection.

\[
S_{ij}^k = \Gamma_{[ij]}^k.
\]

The modified torsion tensor which differs from torsion, is also useful in relating torsion to spin density,

\[
T_{ij}^k = S_{ij}^k + 2\delta_i^k S_{j[l]}^l.
\] (1.1)

Although the torsion tensor has been defined as the antisymmetric part of the connection, the connection also contains a further symmetric, non-Riemannian part. Therefore it is incorrect to define the full connection as the addition of the Christoffel symbol and torsion. Instead an asymmetric tensor called the contortion, \( K_{ij}^k \), is defined by

\[
\Gamma_{ij}^k = \{^k_{ij}\} - K_{ij}^k.
\] (1.2)

The contortion can be written in terms of the torsion as

\[
K_{ij}^k = -S_{ij}^k + S_{j}^k i - S_{ij}^k = -K^{i} j^k.
\] (1.3)
In a torsion theory the covariant derivative is taken with respect to the full asymmetric connection. However, there are occasions when the Christoffel covariant derivative of general relativity is also necessary. Therefore additional notation to distinguish between these derivatives is introduced.

\[ \nabla = \hat{\nabla} \Rightarrow \text{with respect to the full asymmetric connection,} \]

and

\[ \hat{\nabla} \Rightarrow \text{with respect to the second Christoffel symbol.} \]

The definition (1.3) of the contortion is forced upon us by taking the metric to be covariantly constant. Therefore in a space-time with torsion, \( \nabla_i g_{jk} = 0 \) specifies the definition of the connection, where \( \nabla \) is the covariant derivative with respect to \( \Gamma \).

Setting the torsion to zero reduces the Einstein-Cartan theory to classical general relativity.

### 1.3 The Einstein, Riemann and Ricci Tensors

The generalised Riemann tensor is defined in a similar way to general relativity namely,

\[ R_{ijkl} = 2 \partial_{[i} \Gamma_{j]k}^l + 2 \Gamma_{[im}^l \Gamma_{j]k}^m \] \hspace{1cm} (1.4)

The Riemann tensor is antisymmetric on the first two and last two indices and so the Ricci tensor remains the only non-zero contraction of the Riemann tensor. However, the Ricci tensor defined as,

\[ R_{ij} = R_{kij}^k, \]
is asymmetric in general. The Einstein tensor $G_{ij}$ is defined formally in the same way as in general relativity although with torsion it is asymmetric. Thus,

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R .$$  \hspace{1cm} (1.5)$$

The Einstein tensor can also be split into its Riemannian part and a non-Riemannian part

$$G_{ij} = G_{ij}^{\text{R}} + G_{ij}^{\text{NR}} ,$$  \hspace{1cm} (1.6)$$

where $G_{ij}^{\text{NR}}$ denotes the non-Riemannian part.

### 1.4 The Einstein-Cartan Field Equations

The energy-momentum tensor for a Lagrangian dependent on torsion is more complicated than in general relativity. Let

$$\mathcal{L} = \mathcal{L}(\psi, \partial \psi, g, \partial g, S) ,$$  \hspace{1cm} (1.7)$$

where $\psi$ is some matter field. Varying this Lagrangian with respect to the metric and torsion will lead to the metric part of the energy-momentum tensor and the spin energy potential,

$$\sqrt{-g} \sigma^{ij} = \frac{2 \delta \mathcal{L}}{\delta g_{ij}} ,$$  \hspace{1cm} (1.8)$$

$$\sqrt{-g} \mu^{ij} = \frac{\delta \mathcal{L}}{\delta S_{ij}^k} .$$  \hspace{1cm} (1.9)$$

However, because spin couples to contortion and not to torsion,

$$\sqrt{-g} \tau^{ij} = \frac{\delta \mathcal{L}}{\delta K_{ij}^k} .$$  \hspace{1cm} (1.10)$$

These are related by,
\[
\sqrt{-g} \mu^i_{jk} = \frac{\delta \mathcal{L}}{\delta S_{ij}^k} = \frac{\delta \mathcal{L}}{\delta K_{ab}^c} \frac{\delta K_{ab}^c}{\delta S_{ij}^k} = \sqrt{-g} \gamma^{ba}_{c} \frac{\delta K_{ab}^c}{\delta S_{ij}^k} = \sqrt{-g} \gamma^{ba}_{c} \left( -\delta^i_a \delta^j_b \delta^c_k + \delta^i_a \delta^j_c g_{ak} - \delta^i_a \delta^j_c g_{kb} \right) = \sqrt{-g} \left( -\tau_k^{ji} + \tau^{ji}_k - \tau^{ji} \right), \tag{1.11}\]

and therefore,

\[
\mu^i_{jk} = -\tau^{i}_{jk} + \tau^{jki} - \tau^{kij}.\]

The total energy of the system comes not only from the variation of the Lagrangian with respect to the metric, but also with respect to the torsion. Hehl derives an energy-momentum tensor of the form,

\[
E^i_{jk} = \sigma_{ij} - \nabla_k \mu_{ij} = \sigma_{ij} + \nabla_k \left( \tau^{ijk} - \tau^{jki} + \tau^{kij} \right), \tag{1.12}\]

where \(\nabla_k = \nabla_k + S_{kl}^l\).

Variation of the Riemann scalar density in general relativity leads to the Einstein tensor. However, in Einstein-Cartan theory variation of the Riemann scalar, with respect to the metric, leads instead to \(G_{ij}^{(ij)} + \nabla_k (\tau^{kij} + \tau^{kji})\). When combined with equation (1.8) this results in the full variation of the Lagrangian being

\[
G_{ij}^{(ij)} + k \nabla_k (\tau^{kij} + \tau^{kji}) = k \sigma_{ij}, \tag{1.13}\]

where \(k = 8\pi G/c^4\) is a constant. Using the second algebraic identity of the curvature tensor, equation (1.5) can be written as,

\[
G_{ij}^{[ij]} = k \nabla_{\tau_{ijk}}. \tag{1.14}\]

Using the definition of the energy-momentum tensor (equation 1.12), the variation of the Lagrangian with respect to the metric (equation 1.13) and the equation
relating that antisymmetric part of the Einstein tensor (equation 1.14), the field equations of Einstein-Cartan theory are,

\[ G^{ij} = k\Sigma^{ij} \]  \hspace{1cm} (1.15)

\[ T^{ijk} = k\tau^{ijk} . \]  \hspace{1cm} (1.16)

These equations show that the relationship between the modified torsion tensor and the spin density \( \tau \) is algebraic. This implies that torsion does not propagate through space-time.

1.5 Calculations Using Torsion

1.5.1 The Symmetric part of the Contortion

In equation (1.2) the contortion is the non-Riemannian part of the space-time connection but this does not mean that the contortion is antisymmetric. In fact the contortion does have a symmetric part.

The symmetric part of the connection can be calculated from the asymmetric connection. Of course the Christoffel symbol is already symmetric so,

\[ \Gamma_{(ij)}^k = \{^k_{ij}\} - K_{(ij)}^k . \]

Substituting in equation (1.3),

\[ \Gamma_{(ij)}^k = \{^k_{ij}\} + S_{(ij)}^k - S_{(j\ i)}^k + S^k_{(ij)} . \]

Because the torsion tensor is antisymmetric on the first two indices the first torsion term cancels, leaving,

\[ \Gamma_{(ij)}^k = \{^k_{ij}\} - S_{(j\ i)}^k + S^k_{(ij)} . \]
Lowering the \( k \) index, this equation can be written as,

\[
\Gamma_{(ij)}^k = \{_{ij}^k\} + \frac{g^{kl}}{2} (S_{lij} + S_{lij} - S_{jli} - S_{lij}) .
\]

Then using the antisymmetry property of the torsion tensor this can be written as,

\[
\Gamma_{(ij)}^k = \{_{ij}^k\} + g^{kl} (S_{lij} + S_{lij}) ,
\]

and therefore,

\[
\Gamma_{(ij)}^k = \{_{ij}^k\} + 2S_{(ij)}^k ,
\]

gives the symmetric part of the connection.

### 1.5.2 Finding Contortion in Terms of Spin

It is necessary to find the contortion in terms of the spin density, \( r \), to enable the field equations (1.15,1.16) to be combined in section 1.5.3.

Rearranging the definition of the modified torsion, (1.1),

\[
S_{ij}^k = T_{ij}^k - \delta_i^k S_{jl}^l + \delta_j^k S_{il}^l .
\]

Using the definition above, a relationship between the trace of the torsion and contortion can be established:

\[
S_{jl}^l = T_{jl}^l - \delta_j^l S_{lm}^m + \delta_l^m S_{jm}^m
= T_{jl}^l - S_{jm}^m + 4S_{jm}^m .
\]

Therefore,

\[-2S_{jl}^l = T_{jl}^l .\]
This relationship is used to define the torsion tensor in terms of the modified torsion tensor,

\[ S_{ij}^k = T_{ij}^k + \frac{1}{2} \delta_j^k T_{il}^l - \frac{1}{2} \delta_i^k T_{ul}^l. \]  

Equation (1.17) above enables the contortion (1.3) to be defined in terms of the modified torsion rather than the torsion:

\[
K_{ij}^k = -S_{ij}^k + S_j^k i - S_i^k j
= -T_{ij}^k - \frac{1}{2} \delta_i^k T_{jl}^l + \frac{1}{2} \delta_j^k T_{ul}^l + g^{ak} g_{bl} \left( T_{a}^b + \frac{1}{2} \delta^b_a T_{il}^l - \frac{1}{2} \delta_i^b T_{jl}^l \right)
- g^{ak} g_{bj} \left( T_{ai}^b + \frac{1}{2} \delta^b_a T_{il}^l - \frac{1}{2} \delta_i^b T_{al}^l \right)
= -T_{ij}^k + T_j^k i - T_i^k j - \frac{1}{2} \delta_i^k T_{jl}^l + \frac{1}{2} \delta_j^k T_{ul}^l + \frac{1}{2} g_{ij} T^{kl}^l
- \frac{1}{2} \delta_i^k T_{jl}^l - \frac{1}{2} \delta_j^k T_{il}^l + \frac{1}{2} g_{ij} T^{kl}^l
= -T_{ij}^k + T_j^k i - T_i^k j - \delta_i^k T_{jl}^l + g_{ij} T^{kl}^l. \]  

The contortion is easily expressed in terms of the spin density by using (1.18) and the second field equation (1.16). Therefore,

\[
K_{ij}^k = k \left[ -\tau_{ij}^k + \tau_j^k i - \tau_i^k j - \delta_i^k \tau_{jl}^l + g_{ij} \tau^{kl}^l \right]. \]  

This equation is essential to be able to separate the Riemann and non-Riemann terms in the field equations, to enable a comparison with general relativity.

### 1.5.3 Combining the Field Equations

To compare torsion theory and general relativity we need to combine the field equations (1.15) and (1.16). Firstly equation (1.15) is split into its Riemann and non-Riemann parts. Secondly, using the definition of contortion above (1.19), all terms derived from the torsion are substituted with the spin density. This leads
to the result that
\[ G^{ij} = k\Sigma^{ij} - k\nabla_i \tau^{ijl} + k\nabla_i \tau^{jli} - k\nabla_i \tau^{lij}. \]

Using the definition of the energy-momentum tensor, equation (1.12), this can be written as
\[ G^{ij} = k\tilde{\sigma}^{ij}, \]

where,
\[ \tilde{\sigma}^{ij} = \sigma^{ij} + k \left[ -4\tau^{ik}[\tau^{jl}]_k - 2\tau^{ikl}\tau^{jl}_k + \tau^{kli}\tau^{jl}_k \right. \\
\left. \quad + \frac{1}{2}g^{ij} \left( 4\tau^{k}_{m}[\tau^{ml}]_k + \tau^{mkl}\tau^{mkl} \right) \right]. \] (1.20)

All the spin derivative terms cancel in the combined energy-momentum tensor. This energy-momentum tensor gives a basis for a complete comparison with general relativity.

1.6 Energy-Momentum Tensors

1.6.1 Hehl’s Spinning Fluid Energy-Momentum Tensor

As part of the 1976 paper Hehl[11] took the calculation of the energy-momentum tensor to another stage by postulating a convective form for a spinning fluid. Weyssenhoff’s condition is
\[ \tau^{ij}_k = \tau^{ij} u^k. \] (1.21)

This condition together with,
\[ \Sigma^{ij} = \rho u^i u^j + P(u^i u^j + g^{ij}), \] (1.22)
leads to an energy-momentum tensor for a spinning fluid,

$$
\sigma_{ij} = (\rho + P - 2k^2)u^iu^j + (P - k^2)g_{ij} + 2(u_mu^k - \delta_m^k)\nabla_k(\tau^{m(i}u^{j)}) + \frac{1}{2}(2\delta_m^k(u^k - u^j)\delta^j_m)\nabla_ku^m, \quad (1.23)
$$

where $\nabla$ is covariant differentiation with respect to the Christoffel symbol, $\rho$ is the fluid density, $P$ is the pressure and $u^k$ is the velocity.

Hehl has laid the foundation for specific calculations in Einstein-Cartan theory for spinning fluids. This energy-momentum tensor describes a spinning fluid in a general relativistic framework and therefore allows calculations of the effects of spin without the complication of an asymmetric energy-momentum tensor.

### 1.6.2 Ray and Smalley’s Spinning Fluid Energy-Momentum Tensor

In a series of publications during the 1980s Ray and Smalley [18, 19, 20, 21] suggested an alternative energy-momentum tensor for a spinning fluid. Their calculations allowed for the spin to be that of an extended body. However, by ignoring terms which come from considering the spin as a thermodynamic property we obtain a prescription for elementary particle spin only.

Ray and Smalley used the variational approach to guarantee a consistent set of equations. A mild generalisation of their Lagrangian is

$$
\mathcal{L} = c_1\sqrt{-g}R + \frac{1}{c}\sqrt{-g}F(\rho, s, s_{ij}) + \sqrt{-g}\lambda_1(g_{ij}u^iu^j + c^2) + \sqrt{-g}\lambda_2(\rho u^i)_{;i} + \sqrt{-g}\lambda_3X_iu^i + \sqrt{-g}\lambda_4s_{;i}u^i - 2A\sqrt{-g}\frac{\partial k}{c}g_{ij}a^{0i}a^{1j} + \sqrt{-g}\lambda^{00}(g_{ij}a^{0i}a^{0j} - 1) + \sqrt{-g}\lambda^{11}(g_{ij}a^{1i}a^{1j} - 1) + \sqrt{-g}\lambda^{01}(g_{ij}a^{0i}a^{1j} + \sqrt{-g}\frac{2}{c}\lambda^{03}g_{ij}a^{0i}u^j
$$

+ $\sqrt{-g}\lambda^{13}g_{ij}a^{1i}u^j, \quad (1.24)$
CHAPTER 1. CONTACT TORSION THEORY

in which we have allowed for an arbitrary constant $A$. Here $F$ is the energy
density of the fluid, $\rho$ is the fluid density, $s$ is the entropy, $X$ is the Lin particle
identity variable and $\alpha^\mu_i$ is an orthonormal tetrad basis which satisfies,

$$a^\mu_ia_{\mu j} = g_{ij},$$

$$a_{\mu i}a_{\nu}^i = \eta_{\mu \nu},$$

$$a_i^3 = \frac{u_i}{c},$$

$$S_{ij} = \rho s_{ij} = \rho \kappa (a^0_i a^1_j - a^0_j a^1_i).$$

The tetrad vectors are labelled by $\mu, \nu, \ldots = 0, 1, 2, 3$ and the component indices
are $i, j, \ldots = 0, 1, 2, 3$. The notation $\dot{a}^\mu_i = a^\mu_{i,j}u^j$ means differentiation along the
fluid flow.

Variation of this Lagrangian is taken with respect to the variables $\rho, X, s, u^i$,
$a^{0i}, a^{1i}, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda^{00}, \lambda^{11}, \lambda^{01}, \lambda^{03}, \lambda^{13}$ and of course $g_{ij}, S_{ij}$. We have recalculated the equations which result including the constant $A$,

$$\delta \rho : \quad \frac{F_{\rho}}{c} = (\lambda_{2,k} + 2\lambda_2 S_{kl})u^k + \frac{2 A \kappa}{c} a^0_i \dot{a}^{1i}$$

$$\delta X : \quad (\sqrt{-g} \lambda_3 u^i),i = 0$$

$$\delta s : \quad (\sqrt{-g} \lambda_4 u^i),i = \sqrt{-g} F_s$$

$$\delta u^i : \quad \lambda_1 = -\frac{\rho F_{\rho}}{2c^2}$$

$$\quad u_i = \frac{c^3}{\rho F_{\rho}} \left[ \lambda_3 X_{,i} + \lambda_4 s_{,i} - \frac{2 A \rho \kappa}{c} a^0_k a^{1k};i - \rho (\lambda_{2,i} + 2 S_{ik}) \right]$$

$$\quad + \frac{\rho}{c^3} S_{ik} \dot{u}^k$$

$$\delta a^{0i} : \quad -\frac{2 A \rho \kappa}{c} \dot{a}^{1i} + 2 \lambda^{00} a^0_i + \lambda^{01} a^1_i + \lambda^{03} \frac{c}{u_i} = 0$$

$$\delta a^{1i} : \quad \frac{2 A \rho \kappa}{c} a^0_i + \frac{2 A \rho \kappa}{c} a^0_i + 2 \lambda^{11} a^1_i + 2 \lambda^{10} a^0_i + \frac{\lambda^{13}}{c} u_i = 0$$

$$\delta \lambda_1 : \quad g_{ij} u^i u^j + c^2 = 0$$

$$\delta \lambda_2 : \quad (\rho u^i),i = 0$$
\[ \delta \lambda_3 : \quad X_s u^i = 0 \]  
\[ \delta \lambda_4 : \quad s_i u^i = 0 \]  
\[ \delta \lambda^{00} : \quad a_0^0 a_{0i} = 1 \]  
\[ \delta \lambda^{11} : \quad a_1^i a_{1i} = 1 \]  
\[ \delta \lambda^{01} : \quad a_0^0 a_{1i} = 0 \]  
\[ \delta \lambda^{03} : \quad a_0^0 u^i = 0 \]  
\[ \delta \lambda^{13} : \quad a_1^i u^i = 0 \]

and of course the variations,

\[ \delta S_{ij}^k : \quad S_{ijk} = \frac{A_\rho}{2c^2} s_{ij} u^k \]  
\[ \delta g_{ij} : \quad G^{ij} - \hat{\nabla}_k (T^{kij} + T^{kji}) = \frac{1}{2c^2} T^{ij} , \]

where

\[ T^{ij} = -\frac{\rho F_\rho}{c^2} u^i u^j + g^{ij} (F - \rho F_\rho) + 2A \left( \rho u^{(i} s^{j)k} \right) u^k + \frac{2A}{c^2} \rho u^{(i} s^{j)k} \omega_k u^l. \]

These equations can be simplified into an equation comparable with general relativity. In exactly the same way that the Einstein-Cartan field equations were split into Riemannian and non-Riemannian parts the curvature equation can be manipulated to,

\[ \textstyle{\hat{G}}^{ij} = \frac{1}{2c^2} \left\{ -\frac{\rho F_\rho}{c^2} u^i u^j + g^{ij} (F - \rho F_\rho) + 2A \left( \rho u^{(i} s^{j)k} \right) u^k + \frac{2A}{c^2} \rho u^{(i} s^{j)k} \omega_k u^l \right\} , \]

to first order in \( s^{ij} \).

There is now disagreement between the energy-momentum tensors for a spinning fluid given by Hehl (equation 1.23) and that of Ray and Smalley (equation...
1. There are several terms which differ between the two energy-momentum tensors. Inclusion of a constant \( A \) adds additional freedom. For the model problem we study below, the two theories can be brought into agreement by the choice \( A = 1 \).

### 1.7 Organisation of the thesis

Contact torsion theory has been introduced and the differences between two energy-momentum tensors for perfect spinning fluids have been highlighted (see White et al[22]). In Chapter two we introduce a perturbation calculation in the Schwarzschild metric. We consider the changes to the black hole as a particle falls in and we derive a differential equation which describes the gravitational radiation released by this system.

In Chapter three the theories of Chapter one and two are combined. A calculation for a spinning particle radially falling into a Schwarzschild black hole demonstrates the differences between a non-spinning particle and a spinning particle in this system. The changes to the angular momentum of the black hole are calculated (see White et al[23]). It is shown that the energy-momentum tensor of Ray and Smalley (1.29) correctly describes the angular momentum added to the black hole when \( A = 1/2 \). Consequently, Hehl's energy-momentum tensor overestimates the effects of elementary particle spin in this system.

Chapter four contains two gravitational radiation calculations. The first is a recalculation of the radiation expected from a non-spinning particle falling radially. The results show excellent agreement with the published results of Davis et al[26]. The second calculation is of a particular polarisation component of the radiation generated by a spinning particle which is normally absent in the non-
spinning case.
Chapter 2

The Zerilli Wave Equation

2.1 Introduction

There have been many studies of perturbations of a Schwarzschild black hole in general relativity. The first, by Regge and Wheeler[24] in 1957, tested the stability of a Schwarzschild black hole to a metric perturbation. They showed that the black hole remains spherically symmetric by radiating away any asymmetry in the form of gravitational radiation. The analysis was accomplished using tensor harmonics which are a generalisation of the usual spherical harmonics and which provide an orthonormal basis for spherically symmetric tensor fields.

Several years later Zerilli[1] generalised the work of Regge and Wheeler[24] to allow the perturbation to be generated by a source. One such source is a small, non-zero mass particle moving on a geodesic into a Schwarzschild hole. The particle perturbs the background spacetime as it falls and causes a damped oscillation of the event horizon as it approaches $r = 2m$. This oscillation is similar to the perturbation of Regge and Wheeler and is damped by the loss of energy in the form of gravitational radiation. Some of this radiation falls into the black
hole while some escapes and is potentially observable.

Zerilli’s theoretical calculation coincided with the early reports of the detection of gravitational radiation by Weber[25]. Consequently there was considerable interest in calculating the expected energy and form of the radiation emitted. Initially this research was restricted to particles falling radially infalling (see Davis et al[26, 27]). Extensions to these calculations of gravitational waves include the projection of a mass into a hole (Ruffini[28], Oohara and Nakamura[29]), and particles in circular orbits around Schwarzschild black holes have also been considered (Fitchett and Detweiler[30] and Tanaka et al[31]). Later Chandrasekhar[32] also derived the Zerilli equations by using the Newman-Penrose formalism.

Black hole perturbations were further extended by Zerilli[33] to include a charged test particle moving in the Reissner-Nordström solution. Perturbations of rotating black holes were considered by Teukolsky[34, 35], Press and Teukolsky[36] and by Chandrasekhar and Friedman[37]. The relationships between, and unification of the different perturbation techniques of Zerilli and Teukolsky was conducted by Chandrasekhar[38] and Sasaki and Nakamura[39].

In related research Price[40, 41] used the wave equations of Regge and Wheeler and Zerilli to calculate the effects of a nonspherical perturbation on a collapsing spherically symmetric star. His results showed that gravitational radiation carried away energy so that the system collapsed to a Schwarzschild black hole.

Research on black hole perturbations has gone through something of a revival recently. There has been extensive research into the collision of two black holes because of the likelihood of large amounts of gravitational radiation being released. Initially this research was computational but it is now known that in the final stages of the collision the radiation can be accurately described by Zerilli’s equations (see Price and Pullin [42]). As a consequence there have been second
order perturbations of the Schwarzschild solution in an attempt to gain more accuracy in the predictions of the gravitational radiation. For a review see Gleiser et al[43].

2.2 Zerilli’s Method

Both the Regge and Wheeler sourceless perturbation and Zerilli’s perturbation were analysed in tensor harmonics. The background metric in both cases is Schwarzschild which is spherically symmetric.

Tensor harmonics form an orthonormal basis and are invariant under rotations. They are, therefore, perfectly suited to use in a spherically symmetric system. Tensor harmonics allow any tensor to be split into orthogonal parts. Each tensor harmonic, dependent on the angular variables, has a coefficient which is dependent on all other variables. Zerilli uses this technique to split the energy-momentum tensor and the perturbed Einstein tensor into ten perturbation equations in \( r \) and \( t \). These can be solved to calculate the change in the black hole’s mass and angular momentum as a particle falls through the event horizon.

The perturbation equations can also be Fourier transformed and combined into two Schrödinger type wave equations. The wave equations describe the gravitational radiation released when a particle falls into a black hole. They can be solved by use of the WKB asymptotic expansion method or directly using a numerical method (see chapter 4). The different parities of the harmonics results in two wave equations. The magnetic parity wave equation comes from the three magnetic parity perturbation equations while the remaining seven electric parity perturbation equations can be combined to form an electric parity wave equation.

Zerilli has, therefore, laid the foundations for further work in this area by
demonstrating how these spherically symmetric systems can be solved. His work is also suited to generalisation for different energy-momentum tensors. Since the background geometry is the same, only the harmonic contributions from the energy-momentum tensor need to be recalculated.

In the remainder of this chapter the details of Zerilli’s calculations are reviewed.

2.3 Tensor Harmonics

Consider the scalar quantity \( f(\theta, \phi) \), a function of the spherical polar angles \( \theta \) and \( \phi \). This may be expanded in a complete orthonormal set,

\[
f(\theta, \phi) = \sum_{L=0}^{\infty} \sum_{M=-L}^{L} f_{LM} Y_{LM}(\theta, \phi),
\]

where the \( Y_{LM} \) are spherical harmonics defined as follows,

\[
Y_{LM} = (-1)^M \sqrt{\frac{(2L+1)(L-M)!}{4\pi (L+M)!}} P_{LM}(\cos \theta) e^{iM\phi},
\]

(2.1)

where \( P_{LM} \) is the associated Legendre function,

\[
P_{LM}(\cos \theta) = \frac{1}{2^L L!} (1-x^2)^{M/2} \frac{d^{M+L}}{dx^{M+L}} (x^2-1)^L.
\]

(2.2)

These functions provide a separable solution of the scalar wave equation

\[
L^2 Y_{LM} = L(L + 1) Y_{LM}, \quad L^2 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2},
\]

in spherical polar coordinates.

There is a similar expansion for tensor quantities relevant to the tensor wave equation

\[
g^{\mu \nu} \nabla_\mu \nabla_\nu T_{\rho \sigma} = 0,
\]
which enables the angular dependence to be separated if $T$ is in a spherically symmetric background metric $g_{\mu\nu}$. Tensor harmonics, like usual scalar harmonics, transform under a representation of the rotation group and make it possible to decompose a tensor into a sum of orthogonal harmonics.

The ten 2nd rank tensor harmonics are defined by

$$a_{LM}^{(0)} = \begin{pmatrix} Y_{LM} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ (2.3a)

$$a_{LM}^{(1)} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & Y_{LM} & 0 \\ Y_{LM} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ (2.3b)

$$a_{LM} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Y_{LM} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ (2.3c)

$$b_{LM}^{(0)} = \frac{i r^{2L(L+1)}}{2}[2L(L+1)]^{-1/2}$$

$$\begin{pmatrix} 0 & 0 & (\partial/\partial\theta)Y_{LM} & (\partial/\partial\phi)Y_{LM} \\ 0 & 0 & 0 & 0 \\ (\partial/\partial\theta)Y_{LM} & 0 & 0 & 0 \\ (\partial/\partial\phi)Y_{LM} & 0 & 0 & 0 \end{pmatrix}$$ (2.3d)
\[ b_{LM} = r^{2L(L + 1)^{-1/2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (\partial/\partial\theta) Y_{LM} & (\partial/\partial\phi) Y_{LM} \\ 0 & (\partial/\partial\theta) Y_{LM} & 0 & 0 \\ 0 & (\partial/\partial\phi) Y_{LM} & 0 & 0 \end{pmatrix} \] (2.3e)

\[ c_{LM}(0) = r^{2L(L + 1)^{-1/2}} \begin{pmatrix} 0 & 0 & (1/\sin\theta)(\partial/\partial\phi) Y_{LM} & -\sin\theta(\partial/\partial\theta) Y_{LM} \\ 0 & 0 & 0 & 0 \\ (1/\sin\theta)(\partial/\partial\phi) Y_{LM} & 0 & 0 & 0 \\ -\sin\theta(\partial/\partial\theta) Y_{LM} & 0 & 0 & 0 \end{pmatrix} \] (2.3f)

\[ c_{LM} = ir^{2L(L + 1)^{-1/2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (1/\sin\theta)(\partial/\partial\phi) Y_{LM} & -\sin\theta(\partial/\partial\theta) Y_{LM} \\ 0 & (1/\sin\theta)(\partial/\partial\phi) Y_{LM} & 0 & 0 \\ 0 & -\sin\theta(\partial/\partial\theta) Y_{LM} & 0 & 0 \end{pmatrix} \] (2.3g)

\[ d_{LM} = -ir^{2L(L + 1)(L - 1)(L + 2)^{-1/2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(1/\sin\theta) X_{LM} & \sin\theta W_{LM} \\ 0 & 0 & \sin\theta W_{LM} & \sin\theta X_{LM} \end{pmatrix} \] (2.3h)
\[ g_{LM} = \left( \frac{r^2}{\sqrt{2}} \right) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Y_{LM} & 0 \\ 0 & 0 & 0 & \sin^2 \theta \ Y_{LM} \end{pmatrix} \] \hspace{1cm} (2.3i)

\[ f_{LM} = r^2 [2L(L + 1)(L - 1)(L + 2)]^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -W_{LM} & X_{LM} \\ 0 & 0 & X_{LM} & \sin^2 \theta \ W_{LM} \end{pmatrix} \] \hspace{1cm} (2.3j)

where

\[ X_{LM} = 2 \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial \theta} - \cot \theta \right) Y_{LM}, \]

\[ W_{LM} = \left( \frac{\partial^2}{\partial \theta^2} - \cot \theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) Y_{LM}. \]

The tensors are orthonormal with respect to the inner product,

\[ (S, T) \equiv \int \int S^* : T \ d\Omega, \] \hspace{1cm} (2.4)

where,

\[ S : T \equiv \eta^{\mu \lambda} \eta^{\nu \kappa} S_{\mu \nu} T_{\lambda \kappa}, \] \hspace{1cm} (2.5)

\( \eta_{\mu \lambda} \) is the Minkowski metric and * is the complex conjugate.

Using this inner product any rank two tensor can be split into its harmonic parts. Every tensor harmonic, dependent on angular variables, is multiplied by a scalar function, dependent on all other variables and the summation over all \( L \) and \( M \) reproduces the original tensor.
For example, a tensor $T$ could be separated as follows,

\[ T = \sum_{LM} \left\{ A_{LM}^{(0)} a_{LM}^{(0)} + A_{LM}^{(1)} a_{LM}^{(1)} + A_{LM} a_{LM} + B_{LM}^{(0)} b_{LM}^{(0)} \\
+ B_{LM} b_{LM} + Q_{LM}^{(0)} c_{LM}^{(0)} + Q_{LM} c_{LM} + G_{LM} g_{LM} + D_{LM} d_{LM} + F_{LM} f_{LM} \right\}. \]

where $A, B, D, F, G$ and $Q$ are independent of the angular variables $\theta$ and $\phi$.

2.4 The Perturbation Equations

2.4.1 The Initial Perturbation

A perturbation of Einstein’s equations is a perturbation of the total energy of the system, which gives rise to a perturbation of space-time. However in the first example of this type of calculation, Regge and Wheeler [24] assumed a vacuum spacetime and therefore the perturbation had no source. This is effectively a stability calculation of the Schwarzschild solution. By considering late times, they showed that the perturbed Schwarzschild solution settles back down to a spherically symmetric solution thus proving that the Schwarzschild solution is indeed stable. Zerilli generalised the work to include a source of the perturbation which he took to be a point particle.

In this section Einstein’s equations are perturbed, split into spherical harmonics and ten perturbation equations are derived. The energy-momentum tensor is also separated using spherical harmonics. However each term is not expanded explicitly until section 2.5.

Following the scheme of Peters[44], Einstein’s equations are perturbed by considering a small change $\delta$ of the field equations;

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \]
giving,

\[ \delta R_{\mu\nu} - \frac{1}{2} \delta g_{\mu\nu} R^{(0)} - \frac{1}{2} g_{\mu\nu}^{(0)} \delta R = 8\pi G \delta T_{\mu\nu}, \]

where superscript (0) denotes an unperturbed quantity, i.e. the background space-time. The term \( \delta R = \delta (g^{\alpha\beta} R_{\alpha\beta}) \) can be expanded further:

\[ \delta R_{\mu\nu} - \frac{1}{2} \delta g_{\mu\nu} R^{(0)} - \frac{1}{2} g_{\mu\nu}^{(0)} R^{(0)} - \frac{1}{2} \delta R_{\alpha\beta} g_{\mu\nu}^{(0)} g^{\alpha\beta} = 8\pi G \delta T_{\mu\nu}. \]

In a coordinate system in which the \( \Gamma \)'s are zero (i.e. normal coordinates) the Ricci tensor is,

\[ R_{\mu\nu} = \{^\alpha_{\mu\nu}\}_\alpha - \{^\alpha_{\mu\alpha}\}_\nu. \]

The perturbation terms involving the Ricci tensor are now easily calculated. The derivatives can be changed into covariant derivatives through the properties of normal coordinates. Therefore the following equation is valid for all reference frames

\[ \delta R_{\mu\nu} = \delta \{^\alpha_{\mu\nu}\}_\alpha - \delta \{^\alpha_{\mu\alpha}\}_\nu. \quad (2.6) \]

Using the definition of the Christoffel symbol in terms of \( g \)

\[ \{^\alpha_{\mu\nu}\} = \frac{1}{2} g^{\alpha\beta} (g_{\beta\nu,\mu} + g_{\beta\mu,\nu} - g_{\mu\nu,\beta}), \]

perturbing it, taking the covariant derivative and using normal coordinates gives

\[ \delta \{^\alpha_{\mu\nu}\}_\alpha = \frac{1}{2} g^{\alpha\sigma} (\delta g_{\mu\sigma,\nu;\alpha} + \delta g_{\nu\sigma,\mu;\alpha} - \delta g_{\mu\nu,\sigma;\alpha}) \quad (2.7) \]

\[ \delta \{^\alpha_{\mu\alpha}\}_\nu = \frac{1}{2} g^{\alpha\sigma} (\delta g_{\mu\sigma,\alpha;\nu} + \delta g_{\alpha\sigma,\mu;\nu} - \delta g_{\mu\alpha,\sigma;\nu}) \quad (2.8) \]

where \( g^{\alpha\nu}_{\sigma,\nu} = 0 \) and \( \frac{1}{2} g^{\alpha\sigma} (g_{\mu\sigma,\alpha;\nu} + g_{\alpha\sigma,\mu;\nu} - g_{\mu\alpha,\sigma;\nu}) = 0 \) have been used.

Substituting (2.7) and (2.8) into equation (2.6) now leaves

\[ \delta R_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} (\delta g_{\mu\sigma,\nu;\alpha} + \delta g_{\nu\sigma,\mu;\alpha} - \delta g_{\mu\nu,\sigma;\alpha} - \delta g_{\mu\sigma,\alpha;\nu} - \delta g_{\alpha\sigma,\mu;\nu} + \delta g_{\mu\alpha,\sigma;\nu}). \quad (2.9) \]
The summation over $\alpha$ and $\sigma$ results in two terms cancelling because,

$$g^{\alpha\sigma} \delta g_{\mu\sigma;\alpha;\nu} = g^{\alpha\sigma} \delta g_{\mu\alpha;\sigma;\nu} .$$

The four terms remaining make up the perturbed Ricci tensor

$$\delta R_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} ( \delta g_{\mu\sigma;\nu;\alpha} + \delta g_{\nu\sigma;\mu;\alpha} - \delta g_{\mu\nu;\sigma;\alpha} - \delta g_{\sigma\mu;\nu;\alpha}) .$$

Let $\delta g_{\mu\nu} \equiv h_{\mu\nu}$, $h_{\mu}^{\alpha} \equiv g^{\alpha\nu(0)} \delta g_{\mu\nu}$. After dropping the $^{(0)}$ notation, equation (2.9) becomes

$$\delta R_{\mu\nu} = -\frac{1}{2} ( h_{\mu;\alpha}^{\alpha} - h_{\mu;\alpha}^{\alpha} - h_{\nu;\alpha}^{\alpha} + h_{\alpha}^{\alpha;\mu;\nu}) .$$

The full perturbation of Einstein's equations can be written as,

$$h_{\mu;\nu}^{\alpha} - h_{\mu;\nu}^{\alpha} - h_{\nu;\alpha}^{\alpha} + h_{\alpha}^{\alpha;\mu;\nu} + g_{\mu\nu} \left[ h_{\alpha;\lambda;\alpha;\lambda} - h_{\alpha;\lambda;\alpha;\lambda} \right] + h_{\mu;\nu} R + g_{\alpha\beta} R^{\alpha\beta} = -16\pi G \delta T_{\mu\nu} . \quad (2.10)$$

The background space time is Ricci flat, therefore, $R = 0$ and $R_{\alpha\beta} = 0$

$$h_{\mu;\nu}^{\alpha} - h_{\mu;\nu}^{\alpha} - h_{\nu;\alpha}^{\alpha} + h_{\alpha}^{\alpha;\mu;\nu} + g_{\mu\nu} \left[ h_{\alpha;\lambda;\alpha;\lambda} - h_{\alpha;\lambda;\alpha;\lambda} \right] = -16\pi G \delta T_{\mu\nu} .$$

### 2.4.2 Combining the Perturbation and Tensor Harmonics

A linear perturbation of Einstein's equations is defined by equation (2.10). As the perturbation equations are calculated each $h_{\mu\nu}$ corresponds to a term in a tensor harmonic. Table 2.1 can be used to read off the $r, t$ coefficient of each harmonic and insert it into the perturbation equations. For example, if $\mu = 3, \nu = 0$ then the first term in the perturbation equation (2.10), namely $h_{\mu;\nu}^{\alpha}$, corresponds to the harmonic $c_{LM}^{(0)}$ because this harmonic has a non-zero value in the 03 position. Hence, from table 2.1, $h_{0LM}$ corresponds to this part of the perturbation equation. The second term will have parts such as $h_{31;0}^{(1)}$ which
correspond to the harmonic $c_{LM}$ and therefore $h_{1LM}$ will also be introduced into this perturbation equation. Keeping the harmonics of different parities separate and repeating for all values of $\mu$ and $\nu$ results in a full set of perturbation equations. The energy-momentum tensor is also split into tensor harmonics and the appropriate part of the energy-momentum tensor inserted into the perturbation equations.

Table 2.1: The tensor harmonics and the corresponding coefficients.

<table>
<thead>
<tr>
<th>Tensor Harmonic</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{LM}^{(0)}$</td>
<td>$(-1/r)[2L(L + 1)]^{1/2}h_{0LM}$</td>
</tr>
<tr>
<td>$c_{LM}$</td>
<td>$(i/r)[2L(L + 1)]^{1/2}h_{1LM}$</td>
</tr>
<tr>
<td>$d_{LM}$</td>
<td>$(r^{-2}/2)[2L(L + 1)(L - 1)(L + 2)]^{1/2}h_{2LM}$</td>
</tr>
<tr>
<td>$a_{LM}^{(0)}$</td>
<td>$(1 - 2m/r)H_{0LM}$</td>
</tr>
<tr>
<td>$a_{LM}^{(1)}$</td>
<td>$i\sqrt{2}H_{1LM}$</td>
</tr>
<tr>
<td>$a_{LM}$</td>
<td>$(1 - 2m/r)^{-1}H_{2LM}$</td>
</tr>
<tr>
<td>$b_{LM}^{(0)}$</td>
<td>$(-i/r)[2L(L + 1)]^{1/2}h_{0LM}^{(m)}$</td>
</tr>
<tr>
<td>$b_{LM}$</td>
<td>$(1/r)[2L(L + 1)]^{1/2}h_{1LM}^{(m)}$</td>
</tr>
<tr>
<td>$f_{LM}$</td>
<td>$[\frac{1}{2}L(L + 1)(L - 1)(L + 2)]^{1/2}G_{LM}$</td>
</tr>
<tr>
<td>$g_{LM}$</td>
<td>$\sqrt{2}K_{LM} - (1/\sqrt{2})L(L + 1)G_{LM}$</td>
</tr>
</tbody>
</table>

There is a minor mistake in the published form of these equations in Zerilli's[1] paper (see Rees et al[48] for a list of Errata). In appendix A we give our Maple V r5 code which we used to generate these equations. Each of the coefficient functions in table 2.1 are summed over $L$ and $M$. However, in the perturbation equations below $L$ and $M$ have been dropped for notational convenience. The
correct equations, for the magnetic harmonics are,
\[
\frac{\partial^2 h_0}{\partial r^2} - \frac{\partial^2 h_1}{\partial r \partial t} - \frac{2}{r} \frac{\partial h_1}{\partial t} + \left[ \frac{4m}{r^2} - \frac{L(L+1)}{r} \right] \frac{h_0}{r-2m}
= -\frac{8\pi r^2 Q_{LM}^{(0)}}{(r-2m) \left[ \frac{1}{2} L(L+1) \right]^{1/2}} \tag{2.11a}
\]
\[
\frac{\partial^2 h_1}{\partial t^2} - \frac{\partial^2 h_0}{\partial r \partial t} + \frac{2}{r} \frac{\partial h_0}{\partial t} + (L-1)(L+2)(r-2m) \frac{h_1}{r^3}
= \frac{8\pi i (r-2m) Q_{LM}}{\left[ \frac{1}{2} L(L+1) \right]^{1/2}} \tag{2.11b}
\]
\[
\left( 1 - \frac{2m}{r} \right) \frac{\partial h_1}{\partial r} - \left( 1 - \frac{2m}{r} \right)^{-1} \frac{\partial h_0}{\partial t} + \frac{2m}{r^2} h_1
= \frac{8\pi i r^2 D_{LM}}{\left[ \frac{1}{2} L(L+1)(L-1)(L+2) \right]^{1/2}} \tag{2.11c}
\]

The electric harmonic equations are,
\[
\left( 1 - \frac{2m}{r} \right)^2 \frac{\partial^2 K}{\partial r^2} + \left( 1 - \frac{2m}{r} \right) \left( 3 - \frac{5m}{r} \right) \frac{1}{r} \frac{\partial K}{\partial r} - \left( 1 - \frac{2m}{r} \right)^2 \frac{1}{r} \frac{\partial H_2}{\partial r}
- \left( 1 - \frac{2m}{r} \right) \frac{1}{r^2} (H_2 - K) - \left( 1 - \frac{2m}{r} \right) \frac{1}{2r^2} L(L+1)(H_2 + K)
= 8\pi A_{LM}^{(0)} \tag{2.12a}
\]
\[
\frac{\partial}{\partial t} \left( \frac{\partial K}{\partial r} + \frac{1}{r} (K - H_2) - \frac{m}{r(r-2m)} K \right) - \frac{L(L+1)}{2r^2} H_1
= \frac{8\pi i}{\sqrt{2}} A_{LM}^{(1)} \tag{2.12b}
\]
\[
\left( 1 - \frac{2m}{r} \right)^{-2} \frac{\partial^2 K}{\partial t^2} - \frac{r-m}{r(r-2m)} \frac{\partial K}{\partial r} - \frac{2}{r-2m} \frac{\partial H_1}{\partial t} + \frac{1}{r} \frac{\partial H_0}{\partial r}
+ \frac{1}{r(r-2m)} (H_2 - K) + \frac{L(L+1)}{2r(r-2m)} (K - H_0)
= 8\pi A_{LM} \tag{2.12c}
\]
\[
\frac{\partial}{\partial r} \left[ \left( 1 - \frac{2m}{r} \right) H_1 \right] - \frac{\partial}{\partial t} (H_2 + K)
= \frac{8\pi i r B_{LM}^{(0)}}{\left[ \frac{1}{2} L(L+1) \right]^{1/2}} \tag{2.12d}
\]
These ten perturbation equations must be solved to describe the changes to the black hole as the particle falls in. A Fourier transformation of the perturbation equations can be used to describe the gravitational radiation released.

2.4.3 Fourier Tranformation of the Perturbation Equations

The definition of the Fourier transformation in time,

\[
\hat{f}(\omega, r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t, r) e^{i\omega t} dt
\]

Writing the perturbation equations in a Fourier form removes the explicit time dependence from the system and allows the equations to be combined to form an ordinary differential equation in \( r \) which describes the gravitational radiation emitted.
CHAPTER 2. THE ZERILLI WAVE EQUATION

After transformation, the magnetic parity equations become,

\[
\frac{d^2 \hat{h}_0}{dr^2} + i\omega \frac{d\hat{h}_1}{dr} + 2i\omega \frac{\hat{h}_0}{r} + \left(4m \frac{L(L+1)}{r^2} - \frac{L(L+1)}{r} \right) \frac{\hat{h}_0}{r-2m} = \frac{-8\pi r^2 \hat{Q}_{LM}^{(0)}(\omega, r)}{(r-2m)\left[\frac{1}{2}L(L+1)\right]^{-1/2}} \quad (2.14a)
\]

\[
-\omega^2 \hat{h}_1 + i\omega \frac{d\hat{h}_0}{dr} - \frac{2i\omega}{r} \hat{h}_0 + (L-1)(L+2)(r-2m) \frac{\hat{h}_1}{r^3} = \frac{8\pi i(r-2m)\hat{Q}_{LM}(\omega, r)}{\left[\frac{1}{2}L(L+1)\right]^{-1/2}} \quad (2.14b)
\]

\[
\left(1 - \frac{2m}{r}\right) \frac{d\hat{h}_1}{dr} + i\omega \left(1 - \frac{2m}{r}\right)^{-1} \hat{h}_0 + \frac{2m}{r^2} \hat{h}_1 = \frac{8\pi i\gamma^2 \hat{D}_{LM}(\omega, r)}{\left[\frac{1}{2}L(L+1)(L-1)(L+2)\right]^{-1/2}}. \quad (2.14c)
\]

The electric parity perturbation equations are also Fourier transformed:

\[
\left(1 - \frac{2m}{r}\right)^2 \frac{d^2 \hat{K}}{dr^2} + \left(1 - \frac{2m}{r}\right) \left(3 - \frac{5m}{r}\right) \frac{1}{r^2} \frac{d\hat{K}}{dr} - \left(1 - \frac{2m}{r}\right)^2 \frac{1}{r^2} \frac{d\hat{H}_2}{dr} - \left(1 - \frac{2m}{r}\right) \frac{1}{r^2} L(L+1) \left(\hat{H}_2 + \hat{K}\right) = 8\pi \hat{A}_{LM}^{(0)}(\omega, r) \quad (2.15a)
\]

\[
-i\omega \left(\frac{d\hat{K}}{dr} + \frac{1}{r} \left(\hat{K} - \hat{H}_2\right) - \frac{m}{r(r-2m)} \hat{K}\right) - \frac{L(L+1)}{2r^2} \hat{H}_1 = \frac{8\pi i}{\sqrt{2}} \hat{A}_{LM}^{(1)}(\omega, r) \quad (2.15b)
\]

\[
-\omega^2 \left(1 - \frac{2m}{r}\right)^{-2} \hat{K} - \frac{r - m}{r(r-2m)} \frac{d\hat{K}}{dr} + i\omega \frac{2}{r-2m} \hat{H}_1 + \frac{1}{r} \frac{d\hat{H}_0}{dr} + \frac{1}{r(r-2m)} \left(\hat{H}_2 - \hat{K}\right) + \frac{L(L+1)}{2r(r-2m)} \left(\hat{K} - \hat{H}_0\right) = 8\pi \hat{A}_{LM}(\omega, r) \quad (2.15c)
\]

\[
\frac{d}{dr} \left[\left(1 - \frac{2m}{r}\right) \hat{H}_1\right] + i\omega (\hat{H}_2 + \hat{K}) = \frac{-8\pi i\gamma \hat{B}_{LM}^{(0)}(\omega, r)}{\left[\frac{1}{2}L(L+1)\right]^{-1/2}} \quad (2.15d)
\]
\[ i \omega \dot{H}_1 + \left(1 - \frac{2m}{r}\right) \frac{d}{dr} (\dot{H}_0 - \dot{K}) + \frac{2m}{r^2} \dot{H}_0 + \frac{r - m}{r^2} (\dot{H}_2 - \dot{H}_0) = \frac{-8\pi (r - 2m) \dot{B}_{LM}(\omega, r)}{\left[\frac{1}{2}L(L + 1)\right]^{-1/2}} \] (2.15e)

\[ \omega^2 \left(1 - \frac{2m}{r}\right)^{-1} \ddot{K} + \left(1 - \frac{2m}{r}\right) \frac{d^2 \dot{K}}{dr^2} + \frac{2}{r} \left(1 - \frac{m}{r}\right) \frac{d \ddot{K}}{dr} + \omega^2 \left(1 - \frac{2m}{r}\right)^{-1} \dot{H}_2 
- 2i \omega \frac{d \dot{H}_1}{dr} - \left(1 - \frac{2m}{r}\right) \frac{d^2 \dot{H}_1}{dr^2} - 2i \omega \left(1 - \frac{m}{r}\right) \dot{H}_1 = \frac{(r - m)}{r^2} \frac{d \dot{H}_2}{dr} - \frac{1}{r} \left(1 + \frac{m}{r}\right) \frac{d \dot{H}_0}{dr}
+ \frac{L(L + 1)}{2r^2} (\dot{H}_0 - \dot{H}_2) = -8\sqrt{2} \pi \ddot{G}_{LM}(\omega, r) \] (2.15f)

\[ \frac{1}{2} (\ddot{H}_0 - \ddot{H}_2) = \frac{-8\pi r^2 \ddot{F}_{LM}(\omega, r)}{\left[\frac{1}{2}L(L + 1)(L - 1)(L + 2)\right]^{1/2}} \] (2.15g)

The perturbation equations describe the space-time changes for a general energy-momentum tensor. To solve the system the energy-momentum tensor must be specified.

### 2.5 The Energy-Momentum Tensor

#### 2.5.1 Definition

A test particle moving along a geodesic can be described by an energy-momentum tensor \( T^{\mu\nu} \) integrated over all time, with a \( \delta \) function to pick out the point at which the particle is located. Let \( z^\mu \) define the particle path,

\[ z^\mu = z^\mu(\tau) = (T(\tau), \vec{R}(\tau), \vec{\Theta}(\tau), \vec{\Phi}(\tau)) \]

then,

\[ T^{\mu\nu} = m_0 \int_{-\infty}^{\infty} \delta^{(4)} (x - z(\tau)) \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} d\tau \]
CHAPTER 2. THE ZERILLI WAVE EQUATION

\[ f(x) = m_0 \int_{-\infty}^{\infty} \delta^{(2)}(\Omega - \Omega(\tau)) \frac{\delta \left( r - \bar{R}(\tau) \right)}{r^2} \delta \left( t - T(\tau) \right) \frac{dz^\mu}{dT} \frac{dz^\nu}{dT} \left( \frac{dT}{d\tau} \right)^2 d\tau. \]

Changing the integration variable to \( T \) results in

\[ T^{\mu\nu} = m_0 \int_{-\infty}^{\infty} \delta^{(2)}(\Omega - \Omega(T)) \frac{\delta \left( r - \bar{R}(T) \right)}{r^2} \delta \left( t - T \right) \frac{dz^\mu}{dT} \frac{dz^\nu}{dT} dT dT d\tau, \]

where \( r(t) = \bar{R}(T(t)) \) and \( \Omega(t) = \bar{\Omega}(T^{-1}(t)) \). Here

\[ \delta^{(4)}(x - z(\tau)) = \delta(t - T(\tau)) \frac{\delta(r - \bar{R}(\tau))}{r^2} \delta(\Omega - \bar{\Omega}(\tau)), \]

with

\[ \delta^{(2)}(\Omega) = \delta(\cos \theta) \delta(\phi). \]

The \( \delta \) function is normalised according to

\[ \int \int \int \delta^{(4)}(-g)^{1/2} d^4x = 1. \]

In the equations above \((T, \bar{R}, \bar{\Theta}, \bar{\Phi})\) describes the particle position, \((t, r, \theta, \phi)\) is a general co-ordinate position and \( \tau \) is the proper time. The energy-momentum tensor is non-zero at the position of the particle.

2.5.2 Harmonic Components

The energy-momentum tensor, described by equation (2.16), can be decomposed into spherical harmonics, with the inner product (equations 2.4 and 2.5) being used to find the coefficient functions. For example, for the first of the harmonics \( a_{LM} \) (equation 2.3a) the coefficient \( A_{LM} \) is obtained. In a Schwarzschild background \( A_{LM} \) is

\[ A_{LM} = \int \int a_{LM}^* : T d\Omega \]
where $\gamma = dT/d\tau$.

Similarly the other $r$ and $t$ coefficients of the spherical harmonics are:

\begin{align*}
A_{LM}^{(0)} & = m_0 \gamma r^{-2} \left(1 - \frac{2m}{r}\right)^2 \delta(r - R(t)) Y_{LM}^*(\Omega(t)) \quad \text{(2.18b)} \\
A_{LM}^{(1)} & = \sqrt{2} i m_0 \gamma \frac{dR}{dt} r^{-2} \delta(r - R(t)) Y_{LM}^*(\Omega(t)) \quad \text{(2.18c)} \\
B_{LM}^{(0)} & = \left[\frac{1}{2} L(L + 1)\right]^{-1/2} i m_0 \gamma \left(1 - \frac{2m}{r}\right) r^{-1} \delta(r - R(t)) Y_{LM}^*(\Omega(t)) \quad \text{(2.18d)} \\
B_{LM} & = \left[\frac{1}{2} L(L + 1)\right]^{-1/2} m_0 \gamma (r - 2m)^{-1} \frac{dR}{dt} \delta(r - R(t)) Y_{LM}^*(\Omega(t)) \quad \text{(2.18e)} \\
Q_{LM}^{(0)} & = \left[\frac{1}{2} L(L + 1)\right]^{-1/2} i m_0 \gamma \left(1 - \frac{2m}{r}\right) r^{-1} \delta(r - R(t)) \left[\frac{1}{\sin \Theta} \frac{dY_{LM}^*}{d\Theta} \frac{d\Theta}{dt} - \sin \Theta \frac{dY_{LM}^*}{d\Theta} \frac{d\Phi}{dt}\right] \quad \text{(2.18f)} \\
Q_{LM} & = \left[\frac{1}{2} L(L + 1)\right]^{-1/2} m_0 \gamma (r - 2m)^{-1} \frac{dR}{dt} \delta(r - R(t)) \left[\frac{1}{\sin \Theta} \frac{dY_{LM}^*}{d\Theta} \frac{d\Theta}{dt} - \sin \Theta \frac{dY_{LM}^*}{d\Theta} \frac{d\Phi}{dt}\right] \quad \text{(2.18g)}
\end{align*}
\[ D_{LM} = \left[ \frac{1}{2} L(L+1)(L-1)(L+2) \right]^{-1/2} i m_0 \gamma \delta(r - R(t)) \left\{ \frac{1}{2} \left( \frac{d\theta}{dt} \right)^2 - \sin^2 \Theta \left( \frac{d\Phi}{dt} \right)^2 \right\} \]

\[ F_{LM} = \left[ \frac{1}{2} L(L+1)(L-1)(L+2) \right]^{-1/2} i m_0 \gamma \delta(r - R(t)) \left\{ \frac{d\theta}{dt} \frac{d\Phi}{dt} X_{LM}^*[\Omega(t)] + \frac{1}{2} \left[ \left( \frac{d\theta}{dt} \right)^2 - \sin^2 \Theta \left( \frac{d\Phi}{dt} \right)^2 \right] W_{LM}^*[\Omega(t)] \right\} \]

\[ G_{LM} = \frac{m_0 \gamma}{\sqrt{2}} \left[ \left( \frac{d\theta}{dt} \right)^2 + \sin^2 \Theta \left( \frac{d\Phi}{dt} \right)^2 \right] \delta(r - R(t)) Y_{LM}^*[\Omega(t)]. \]

These energy-momentum tensor terms can be used together with the perturbation equations (2.11a-2.12g) to solve for the changes to the black hole as the particle falls in.

### 2.5.3 Fourier Transformation of the Energy-Momentum Tensor

Using the definition of the Fourier transform in (2.13), the components of the energy-momentum tensor are obtained. This additional integration removes the last remaining \( \delta(r - R(t)) \) function in the energy-momentum terms. For example, the first component is,

\[ \hat{A}_{LM}^{(1)}(\omega, r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_{LM}^{(1)}(t, r) e^{i\omega t} dt \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2im_0 \gamma} \frac{dR}{dt} r^{-2} \delta(r - R(t)) Y_{LM}^*[\Omega(t)] e^{i\omega t} dt \]

\[ = \frac{im_0 \gamma}{\sqrt{\pi}} \int_{-\infty}^{\infty} r^{-2} \delta(r - R) Y_{LM}^* \left( \tilde{\Omega}(R) \right) e^{i\omega T(R)} dR \]
\[ y = \frac{\text{im}_0 \gamma}{\sqrt{\pi}} r^{-2} Y_{LM}^* (\tilde{\Omega}(r)) e^{\omega T(r)}, \quad (2.19) \]

where \( T(r) = R^{-1}(r) \) and \( \tilde{\Omega}(r) = \Omega(R^{-1}(r)) \).

### 2.6 Combining the Perturbation Equations to form a Wave Equation

#### 2.6.1 The Magnetic Parity Equations

There are three Fourier transforms of the perturbation equations with magnetic parity and therefore simplification of this parity is relatively straightforward. Clearly the terms \( dh_1/dr \) and \( h_1 \) in equation (2.14c) can be combined. Redefining \( h_1 \) in terms of a new radial factor \( R^{(m)} \),

\[ h_1 = \frac{r^2 R^{(m)}_{LM}}{r - 2m}, \quad (2.20) \]

we have

\[ \begin{align*}
\hat{h}_0 &= \frac{i}{\omega} \frac{d}{dr^*} \left( r R^{(m)}_{LM} \right) + \frac{8\pi r(2m)}{\omega \left[ \frac{1}{2} L(L + 1)(L - 1)(L + 2) \right]^{1/2}} \hat{D}_{LM}(\omega, r), \\
\hat{h}_1 &= \frac{r^2 R^{(m)}_{LM}}{r - 2m},
\end{align*} \quad (2.21) \]

where \( r^* = r + \ln(r/2m - 1) \).

Substituting equations (2.20, 2.21) into (2.14b), results in the following equation.

\[ \begin{align*}
\frac{d^2 R^{(m)}_{LM}}{dr^*^2} + \left[ \omega^2 - V_L(r) \right] R^{(m)}_{LM} \\
&= \frac{8\pi i}{\left[ \frac{1}{2} L(L + 1)(L - 1)(L + 2) \right]^{1/2} r^2} \frac{r - 2m}{r} \left\{ \frac{d}{dr} \left[ r(r - 2m) \hat{D}_{LM}(\omega, r) \right] \\
&- 2(r - 2m) \hat{D}_{LM}(\omega, r) - (r - 2m) \left[ (L - 1)(L + 2) \right]^{1/2} \hat{Q}_{LM}(\omega, r) \right\}, \\
&\quad (2.22)
\end{align*} \]

where

\[ V_L(r) = \frac{r - 2m}{r^2} \left( \frac{L(L + 1)}{r} - \frac{6m}{r^2} \right). \]
Thus, the magnetic parity equations have been simplified into a single Schrödinger type equation. From this equation we can derive some of the gravitational radiation of an infalling particle. However, for a full account, the electric parity equations are also required.

2.6.2 The Electric Parity Equations

Simplification of the seven electric parity equations is considerably more complex than the magnetic case. The process begins by using equations (2.15b, 2.15c, 2.15e and 2.15g) to find an algebraic identity for $\hat{H}_2$, $\hat{H}_1$ and $\hat{K}$. Then, using the outline given by Zerilli[45], the equations are reduced to two coupled first order differential equations. Making a transformation and demanding that the resulting first order equations are of a certain form means that their combination into a single second order equation is straightforward. This results in another Schrödinger type equation for the electric parity perturbations.

To find the algebraic identity we begin by rearranging equation (2.15b) for $dK/dr$,

$$
\frac{dK}{dr} = -\frac{1}{r} (\hat{K} - \hat{H}_2) + \frac{m\hat{K}}{r(r-2m)} + \frac{iL(L+1)H_1}{2\omega r^2} - \frac{8\pi \hat{A}_{LM}^{(1)}}{\omega \sqrt{2}},
$$

and equation (2.15e) for $d\hat{H}_0/dr$,

$$
\frac{d\hat{H}_0}{dr} = \frac{d\hat{K}}{dr} - i\omega \left( 1 - \frac{2m}{r} \right)^{-1} \hat{H}_1 - \frac{2m}{r^2 \left( 1 - \frac{2m}{r} \right)^{-1}} \hat{H}_2 - \frac{16\pi \hat{F}_{LM}}{16r(r-3m)} - \frac{16\pi \hat{B}_{LM}}{\left[ \frac{1}{2}L(L+1)(L-1)(L+2) \right]^{1/2}} - \frac{8\pi \hat{B}_{LM}}{\left[ \frac{1}{2}L(L+1) \right]^{-1/2}},
$$

and then removing $\hat{H}_0$ using (2.15g).

Substituting the two equations above into equation (2.15c) leads to,
\[
\left\{ \frac{-2\omega^2 r^3}{(r-2m)} + (L-1)(L+2) + \frac{2m(r-3m)}{r(r-2m)} \right\} \hat{K} - \left\{ 2 - \frac{6m}{r} \right\} \hat{H}_2
\]

\[
-L(L+1)\hat{H}_0 + \left\{ 2i\omega r - \frac{ml(L+1)}{\omega r^2} \right\} \hat{H}_1 = + 16\pi r(r-2m) \hat{A}_{LM}
\]

\[
-\frac{16\pi m \hat{A}_{LM}^{(1)}}{r \sqrt{2}} + \frac{16\pi r(r-2m) \hat{B}_{LM}}{\left[ \frac{1}{2}L(L+1) \right]^{-1/2}} + \frac{32\pi r(r-3m) \hat{F}_{LM}}{\left[ \frac{1}{2}L(L+1)(L-1)(L+2) \right]^{1/2}}.
\]

Again removing the \( \hat{H}_0 \) dependence leaves the algebraic relation,

\[
\left\{ (L-1)(L+2) - \frac{2\omega^2 r^3}{r-2m} + \frac{2m(r-3m)}{r(r-2m)} \right\} \hat{K}
\]

\[
-\left\{ (L-1)(L+2) + \frac{6m}{r} \right\} \hat{H}_2 + \left\{ 2i\omega r - \frac{ml(L+1)}{\omega r^2} \right\} \hat{H}_1
\]

\[
= -\frac{16\pi m \hat{A}_{LM}^{(1)}}{r \sqrt{2}} + 16\pi r(r-2m) \hat{A}_{LM} + \frac{16\pi r(r-2m) \hat{B}_{LM}}{\left[ \frac{1}{2}L(L+1) \right]^{-1/2}}
\]

\[
-\frac{16\pi^2 \left[ (L-1)(L+2) + \frac{6m}{r} \right] \hat{F}_{LM}}{\left[ \frac{1}{2}L(L+1)(L-1)(L+2) \right]^{1/2}}. \quad (2.23)
\]

The source terms are denoted by \( P_{LM} \),

\[
P_{LM} = -\frac{16\pi m \hat{A}_{LM}^{(1)}}{r \sqrt{2}} + 16\pi r(r-2m) \hat{A}_{LM} + \frac{16\pi r(r-2m) \hat{B}_{LM}}{\left[ \frac{1}{2}L(L+1) \right]^{-1/2}}
\]

\[
-\frac{16\pi^2 \left[ (L-1)(L+2) + \frac{6m}{r} \right] \hat{F}_{LM}}{\left[ \frac{1}{2}L(L+1)(L-1)(L+2) \right]^{1/2}}. \quad (2.24)
\]

This definition is used to simplify the expression for the source terms.

Manipulation of the equations into two first order differential equations begins with the removal of the \( \hat{H}_2 \) dependence from equations (2.15b, 2.15d). Defining \( R_{LM} = (1/\omega) \hat{H}_1 \) these equations are reduced to,

\[
\frac{d\hat{K}_{LM}}{dr} = [\alpha_0(r) + \alpha_2(r)\omega^2] \hat{K}_{LM} + [\beta_0 + \beta_2\omega^2] R_{LM} + [\text{Source of } \hat{K}] \quad (2.25)
\]

\[
\frac{dR_{LM}}{dr} = [\gamma_0(r) + \gamma_2(r)\omega^2] \hat{K}_{LM} + [\delta_0 + \delta_2\omega^2] R_{LM} + [\text{Source of } R]. \quad (2.26)
\]

The source terms are defined by \( \tilde{C}_{1LM} \) and \( \tilde{C}_{2LM} \).
Source of $K = \tilde{C}_{1LM} = \frac{-4\pi\sqrt{2}}{\omega} \hat{A}_{LM}^{(1)} - \frac{1}{r} \hat{B}_{LM}$

\[ + \frac{16\pi r \hat{F}_{LM}}{r^2} \left( \frac{1}{2} L(L + 1)(L - 1)(L + 2) \right)^{1/2} \quad (2.27) \]

Source of $R = \tilde{C}_{2LM} = \frac{-8\pi r^2 \hat{B}_{LM}^{(0)}}{\omega (r - 2m) \left( \frac{1}{2} L(L + 1) \right)} + \frac{ir}{r - 2m} \hat{B}_{LM}$

\[ - \frac{16\pi r^3 \hat{F}_{LM}}{(r - 2m) \left( \frac{1}{2} L(L + 1)(L - 1)(L + 2) \right)^{1/2}} \quad (2.28) \]

where, to be consistent with the published form of Zerilli[1],

\[ \hat{B}_{LM} = -\frac{8\pi mr}{\omega \sqrt{2}(\lambda r + 3m)} \frac{\hat{A}_{LM}^{(1)}}{(\lambda r + 3m)} + \frac{8\pi r^2 (r - 2m)}{(\lambda r + 3m)} \left( \hat{A}_{LM} + \frac{\hat{B}_{LM}}{\left[ \frac{1}{2} L(L + 1) \right]^{1/2}} \right) \quad (2.29) \]

The quantities $\alpha_0, \alpha_2, \beta_0, \beta_2, \gamma_0, \gamma_2, \delta_0$ and $\delta_2$ are,

\[ \alpha_0 = \frac{rm(\lambda - 2) + 6m^2}{r(r - 2m)(\lambda r + 3m)} \]

\[ \alpha_2 = -\frac{r^3}{(r - 2m)(\lambda r + 3m)} \]

\[ \beta_0 = \frac{i(\lambda + 1)(\lambda r + 2m)}{r^2(\lambda r + 3m)} \]

\[ \beta_2 = \frac{ir}{\lambda r + 3m} \]

\[ \gamma_0 = -\frac{ir(2\lambda r(r - 2m) + 4mr - 9m^2)}{(r - 2m)^2(\lambda r + 3m)} \]

\[ \gamma_2 = \frac{ir^5}{(r - 2m)^2(\lambda r + 3m)} \]

\[ \delta_0 = -\frac{m(r(3\lambda + 1) + 6m)}{r(r - 2m)(\lambda r + 3m)} \]
CHAPTER 2. THE ZERILLI WAVE EQUATION

\[ \delta_2 = \frac{r^3}{(r - 2m)(\lambda r + 3m)} , \]

where \( \lambda = (L - 1)(L + 2)/2 \).

Transforming the differential equations (2.25) and (2.26) into a more convenient form allows them to be combined more simply. Therefore, make the transformations,

\[ K_{LM} = f(r)\tilde{K}_{LM} + g(r)\tilde{R}_{LM} , \]
\[ R_{LM} = h(r)\tilde{K}_{LM} + j(r)\tilde{R}_{LM} , \]

and set the conditions,

\[ \frac{d\tilde{K}_{LM}}{dr^*} = \tilde{R}_{LM} + [\text{Source Terms}] , \]
\[ \frac{d\tilde{R}_{LM}}{dr^*} = [V_{LM}(r) - \omega^2] \tilde{K}_{LM} + [\text{Source Terms}] . \]

The equation for \( \tilde{K} \) is,

\[
\left(1 - \frac{gh}{fj}\right) \frac{d\tilde{K}}{dr} = \frac{dr}{dr^*} \frac{1}{f} \left\{ \left[ \alpha_0 + \alpha_2\omega^2 \right] f + \left[ \beta_0 + \beta_2\omega^2 \right] h - \left[ \gamma_0 + \gamma_2\omega^2 \right] \frac{fg}{j} \right. \\
- \left[ \delta_0 + \delta_2\omega^2 \right] h \frac{g}{j} + \frac{g}{j} \frac{dh}{dr} - \frac{df}{dr} \right\} \tilde{K} + \left( \left[ \alpha_0 + \alpha_2\omega^2 \right] g + \left[ \beta_0 + \beta_2\omega^2 \right] j \right. \\
- \left[ \gamma_0 + \gamma_2\omega^2 \right] \frac{g^2}{j} - \left[ \delta_0 + \delta_2\omega^2 \right] g + \frac{g}{j} \frac{dj}{dr} - \frac{dg}{dr} \right\} \tilde{R} - \left( \frac{g}{j} \tilde{C}_{2LM} - \tilde{C}_{1LM} \right) ,
\]

and for \( \tilde{R} , \)

\[
\left(1 - \frac{gh}{fj}\right) \frac{d\tilde{R}}{dr} = \frac{dr}{dr^*} \frac{1}{f} \left\{ \left[ \gamma_0 + \gamma_2\omega^2 \right] f + \left[ \delta_0 + \delta_2\omega^2 \right] h - \left[ \alpha_0 + \alpha_2\omega^2 \right] h \right. \\
- \left[ \beta_0 + \beta_2\omega^2 \right] \frac{h^2}{f} + \frac{h}{f} \frac{df}{dr} - \frac{dh}{dr} \right\} \tilde{K} + \left( \left[ \gamma_0 + \gamma_2\omega^2 \right] g + [\delta_0 + \delta_2\omega^2] j \right. \\
- \left[ \alpha_0 + \alpha_2\omega^2 \right] \frac{gh}{f} - \left[ \beta_0 + \beta_2\omega^2 \right] \frac{hj}{f} + \frac{h}{f} \frac{dg}{dr} - \frac{dj}{dr} \right\} \tilde{R} - \left( \frac{h}{f} \tilde{C}_{1LM} - \tilde{C}_{2LM} \right) ,
\]

where it is clear that the functions \( f, g, h \) and \( j \) are \( r \) dependent.
From these equations several conditions can be obtained so that the forms of equations (2.30a and 2.30b) can be satisfied. The conditions are,

\[ 0 = \alpha_0 f + \beta_0 h - \gamma_0 \frac{f g}{j} - \delta_0 \frac{h g}{j} + g \frac{d h}{j \, d r} - \frac{d f}{d r} \quad (2.31a) \]

\[ 0 = \alpha_2 f + \beta_2 h - \gamma_2 \frac{f g}{j} - \delta_2 \frac{h g}{j} \quad (2.31b) \]

\[ 1 - \frac{g h}{f j} = \frac{1}{f \, d r \ast} \left( \alpha_0 g + \beta_0 j - \gamma_0 \frac{g^2}{j} - \delta_0 g + g \frac{d j}{j \, d r} - \frac{d g}{d r} \right) \quad (2.31c) \]

\[ 0 = \alpha_2 g + \beta_2 j - \gamma_2 \frac{g^2}{j} - \delta_2 g \quad (2.31d) \]

\[ - \left(1 - \frac{g h}{f j} \right) = \frac{1}{j \, d r \ast} \left( \gamma_2 f + \delta_2 h - \alpha_2 h - \beta_2 \frac{h^2}{f} \right) \quad (2.31e) \]

\[ 0 = \gamma_0 g + \delta_0 j - \alpha_0 \frac{g h}{f} - \beta_0 \frac{j h}{f} + h \frac{d g}{f \, d r} - \frac{d j}{d r} \quad (2.31f) \]

\[ 0 = \gamma_2 g + \delta_2 j - \alpha_2 \frac{g h}{f} - \beta_2 \frac{j h}{f} \quad (2.31g) \]

A relationship between \( g \) and \( j \) can be found by rearranging equation (2.31d),

\[ g = \frac{j (r - 2m)}{r^2} \quad (2.32) \]

This equation identically satisfies equations (2.31b, 2.31g). Any dependence on \( g \) or \( j \) can be removed from the remaining equations. Removing \( g \) results in,

\[ 0 = \alpha_0 f + \beta_0 h - \gamma_0 \frac{i (r - 2m) g}{r^2} f - \delta_0 \frac{i (r - 2m) j}{r^2} h + \frac{i (r - 2m) d h}{r^2 \, d r} - \frac{d f}{d r} \quad (2.33a) \]

\[ 0 = 1 - \frac{h i (r - 2m)}{f \, d r \ast} \frac{j d r \, (\lambda r + 3m)}{r^3} \quad (2.33b) \]

\[ 0 = \frac{1}{j \, d r \ast} \left( \gamma_2 f + \delta_2 h - \alpha_2 h - \beta_2 \frac{h^2}{f} \right) + \left(1 - \frac{h i (r - 2m)}{f \, r^2} \right) \quad (2.33c) \]

\[ 0 = \frac{j (2 \lambda r^2 + r (3 - 7 \lambda) - 15m^2)}{r (r - 2m) (\lambda r + 3m)} - \frac{j h i (\lambda (\lambda + 2) r^2 + r m (3 - \lambda) - 6m^2)}{f \, r^3 (\lambda r + 3m)} - \frac{d j}{d r} \left(1 - \frac{h i (r - 2m)}{f \, r^2} \right) \quad (2.33d) \]
Rearranging (2.33b) to an equation linking \( j, h \) and \( f \) we find,

\[
j = -\frac{ir^3}{(\lambda r + 3m)} \frac{dr}{dr^*} \left( f - h \frac{i(r - 2m)}{r^2} \right).
\] (2.34)

Equation (2.33c) is satisfied identically by using (2.34).

Only two conditions now remain. Substituting equation (2.34) into equation (2.33d) removes any \( j \) dependence and the remaining conditions are combined to form an equation relating \( h \) and \( f \). The new relationship together with the other equations linking \( f, g, h \) and \( j \) satisfy all of the conditions. This final relationship is,

\[
f = h \frac{i(r - 2m)(\lambda(\lambda + 1)r^2 + 3\lambda mr + 6m^2)}{r^2(\lambda r^2 - 3\lambda mr - 3m^2)}.
\] (2.35)

Solving the equations (2.32, 2.34, 2.35) leads to an underdetermined system.

Setting \( g = 1 \) the functions are,

\[
\frac{dr}{dr^*} = \frac{r}{r - 2m}, \quad g = 1
\]

\[
j = -\frac{ir^2}{r - 2m}
\]

\[
h = -\frac{i(\lambda r^2 - 3\lambda mr - 3m^2)}{(r - 2m)(\lambda r + 3m)}
\]

\[
f = \frac{(\lambda(\lambda + 1)r^2 + 3\lambda mr + 6m^2)}{r^2(\lambda r + 3m)}
\] (2.36a)

Therefore, equations (2.30a, 2.30b) are satisfied and can be written as,

\[
\frac{d\tilde{K}_{LM}}{dr^*} = \tilde{R}_{LM} - \left( 1 - \frac{hg}{jf} \right) \frac{1}{f} \frac{dr}{dr^*} \left\{ \frac{g}{j} \tilde{C}_{2LM} - \tilde{C}_{1LM} \right\},
\] (2.36a)

\[
\frac{d\tilde{R}_{LM}}{dr^*} = [V_{LM}(r) - \omega^2] \tilde{K}_{LM} - \left( 1 - \frac{hg}{jf} \right) \frac{1}{j} \frac{dr}{dr^*} \left\{ \frac{h}{f} \tilde{C}_{1LM} - \tilde{C}_{2LM} \right\}. (2.36b)
\]
CHAPTER 2. THE ZERILLI WAVE EQUATION

By substituting \( f, g, h \) and \( j \) into the equations above, differentiating (2.36a) with respect to \( r^* \) and using (2.36b) we arrive at a Schrödinger type differential equation.

\[
\frac{d^2 \tilde{K}_{LM}}{dr^*} + \left[ \omega^2 - V_L(r) \right] \tilde{K}_{LM} = S_{LM}, \tag{2.37}
\]

where

\[
V_L = \left( 1 - \frac{2m}{r} \right) \left( \frac{2\lambda^2(\lambda + 1)r^3 + 6\lambda^2mr^2 + 18\lambda m^2r + 18m^3}{r^3(\lambda r + 3m)^2} \right), \tag{2.38}
\]

and

\[
S_{LM} = -i\frac{(r - 2m)}{r} \frac{d}{dr} \left[ \frac{(r - 2m)^2}{r(\lambda r + 3m)} \left( \frac{ir^2}{r - 2m} \tilde{C}_{1LM} + \tilde{C}_{2LM} \right) \right]
+ \frac{i(r - 2m)^2}{r(\lambda r + 3m)^2} \left[ \frac{\lambda(\lambda + 1)r^2 + 3\lambda mr + 6m^2}{r^2} \tilde{C}_{2LM}
+ \frac{i(\lambda r^2 - 3\lambda mr - 3m^2)}{r - 2m} \tilde{C}_{1LM} \right]. \tag{2.39}
\]

To summarise,

\[
\tilde{C}_{1LM} = \frac{-4\pi \sqrt{2}}{\omega} \hat{A}_{LM}^{(1)} - \frac{1}{r} \tilde{B}_{LM} + \frac{16\pi r \hat{F}_LM}{[\frac{1}{2}L(L+1)(L-1)(L+2)]^{1/2}}
\]

\[
\tilde{C}_{2LM} = \frac{-8\pi i r^2 \hat{B}_{LM}^{(0)}}{\omega(r - 2m)[\frac{1}{2}L(L+1)]} + \frac{ir}{r - 2m} \tilde{B}_{LM}
\]

\[
\tilde{B}_{LM} = \frac{-8\pi mr \hat{A}_{LM}^{(1)}}{\omega \sqrt{2}(\lambda r + 3m)} + \frac{8\pi r^2(r - 2m)}{(\lambda r + 3m)} \left( \hat{A}_{LM} + \frac{\hat{B}_{LM}}{[\frac{1}{2}L(L+1)]^{1/2}} \right).
\]

There are now two differential equations for each of the two parities of harmonic.

If the particle falling into the black hole has no angular momentum the magnetic equation (2.22) gives no contribution to the gravitational radiation release. The
electric equation (2.37), therefore, determines exactly the gravitational radiation for a radially infalling particle. However, the equations given here apply to a general energy-momentum tensor.

2.7 The Source for a Radially Infalling Particle

If a particle falls radially into a Schwarzschild black hole there are simplifications to the wave equations derived above. In this case (zero angular momentum), all the harmonics of the source terms are zero apart from the three,

\[ \hat{A}_L^{(0)}(\omega, r) = -\left(\frac{m_0}{2\pi}\right) \left[\left(\frac{L + 1}{2}\right) \left(\frac{2m}{r}\right)\right]^{1/2} \frac{1}{r^2} e^{i\omega T(r)} \]  

\[ \hat{A}_L^{(1)}(\omega, r) = i \left(\frac{m_0}{2\pi}\right) (2L + 1)^{1/2} \frac{1}{r^2} \left(1 - \frac{2m}{r}\right)^{-1} e^{i\omega T(r)} \]

These terms are found using the definition of the spherical harmonics (see equations 2.1 and 2.2 (for example see Arfken[46] or Schiff[47]). In this case of radial infall the spherical harmonic is,

\[ Y_{LM} = \sqrt{\frac{2L+1}{4\pi}} \cdot \]

Now, during radial infall,

\[ \frac{dT}{d\tau} = \left(1 - \frac{2m}{r}\right)^{-1} \]  

\[ \frac{dR}{dt} = -\sqrt{2m} \left(1 - \frac{2m}{r}\right) \]

which can be found from the Lagrangian for particle orbits in the Schwarzschild metric.
Therefore the source term for the magnetic wave equation is zero and for the
electric wave equation the source term is,

\[ S_L = -\frac{4m_0}{\lambda \tau + 3m} \left( L + \frac{1}{2} \right)^{1/2} \left( 1 - \frac{2m}{r} \right) \left[ \left( \frac{r}{2m} \right)^{1/2} - \frac{2i\lambda}{\omega(\lambda \tau + 3m)} \right] e^{i\omega T(r)} \] (2.43)

where,

\[ T(r) = -\frac{4m}{3} \left( \frac{r}{2m} \right)^{3/2} - 4m \left( \frac{r}{2m} \right)^{1/2} + 2m \ln \left( \left[ \left( \frac{r}{2m} \right)^{1/2} + 1 \right] \left[ \left( \frac{r}{2m} \right)^{1/2} - 1 \right] \right)^{-1} \]

which agrees with Davis et al [26] and can be found by solving equation (2.42).

The perturbation equations can be solved directly for the changes to the black
hole in the case of radial infall but this form of the wave equation (2.37) with
this source is required to calculate the gravitational radiation.

2.8 Gauge Transformations

The overall perturbation is the sum of a magnetic parity perturbation plus an
electric parity perturbation,

\[ h = \sum_{LM} \left( h_{LM}^{(m)} + h_{LM}^{(e)} \right) \]

where each part of the total perturbation is the sum of each spherical harmonic
multiplied by its corresponding coefficient. Thus,

\[ h_{LM}^{(m)} = \left( \frac{i}{r} \right) [2L(L + 1)]^{1/2} \left\{ ih_{0LM} c_{LM}^{(0)} + h_{1LM} c_{LM} \right. \\
\left. - \left( \frac{i}{2r} \right) [(L - 1)(L + 2)]^{1/2} h_{2LM} d_{LM} \right\} , (2.44) \]

and

\[ h_{LM}^{(e)} = \left( 1 - \frac{2m}{r} \right) h_{0LM} a_{LM}^{(0)} - \sqrt{2}i \ h_{1LM} a_{LM}^{(1)} \]

\[ + \left( 1 - \frac{2m}{r} \right)^{-1} h_{2LM} a_{LM} - \left( \frac{1}{r} \right) [2L(L + 1)]^{1/2} \left( ih_{0LM}^{(m)} b_{LM}^{(0)} - h_{1LM}^{(m)} b_{LM} \right) \]

\[ + \left[ \frac{1}{2} L(L + 1)(L - 1)(L + 2) \right]^{1/2} G_{LM} f_{LM} + \sqrt{2} \left[ K_{LM} - \frac{1}{2} L(L + 1) G_{LM} \right] g_{LM} \] (2.45)
CHAPTER 2. THE ZERILLI WAVE EQUATION

Reduction of the number of coefficient functions (eg $h_{0LM}$) to be found is achieved using a gauge transformation. Four linearly independent vector fields $\xi$ can be chosen which reduce the number of coefficient functions from ten to six. The new gauge-transformed perturbation tensor is,

$$h' = h - 2[\nabla\xi]_s,$$

where $[\ ]_s$ denotes symmetry and $\nabla$ denotes covariant differentiation with respect to the background space-time.

There is only one vector harmonic of order $(LM)$ and parity $(-1)^{L+1}$, namely $(0, LY_{LM})$, where $L = -i\tau \wedge \nabla$ is the angular momentum operator. Therefore let

$$\xi_{LM}^m = \frac{i}{r} \Lambda_{LM}(r,t)(0, LY_{LM}).$$

Thus,

$$2[\nabla \xi_{LM}^{(m)}]_s = \frac{i}{r}[2L(L+1)]^{1/2} \left\{ \frac{i}{r} \frac{\partial \Lambda(r,t)}{\partial t} c_{LM}^{(0)} + r^2 \frac{\partial}{\partial r} \left( \frac{\Lambda(r,t)}{r^2} \right)c_{LM} \right. + \left. \frac{1}{r}[(L-1)(L+2)]^{1/2} \Lambda(r,t)d_{LM} \right\}.$$

By a suitable choice of $\Lambda(r,t)$ it is possible to remove one of the three magnetic parity harmonic coefficient functions. Regge and Wheeler and Zerilli choose to remove $h_{2LM}$ by setting $\Lambda(r,t) = -ih_{2LM}/2$ initially although for later considerations of the gravitational radiation this function is re-introduced.

Therefore the magnetic perturbation tensor is written as

$$h_{LM}^{(m)} = \frac{i}{r}[2L(L+1)]^{1/2} \left\{ ih_{0LM}c_{LM}^{(0)} + h_{1LM}c_{LM} \right\}, \quad (2.46)$$

where the primes have been dropped for convenience.

Exactly the same method can be used on the electric perturbation tensor. Here there are three vector harmonics, $e_{\tau}Y_{LM}$, $e_{r}Y_{LM}$ and $(0, \nabla Y_{LM})$, where $e_{\tau}$
and $e_r$ are unit vectors along $t$ and $r$. Therefore

$$\xi_{LM}^{(e)} = M_0(r,t)Y_{LM}e_t + M_1(r,t)Y_{LM}e_r + M_2(r,t)(0, \nabla Y_{LM})$$

and so

$$2[\nabla \xi_{LM}^{(e)}]_s = 2 \left( \frac{\partial M_0}{\partial t} - \frac{m}{r^3}(r - 2m)M_1 \right) a_{LM}^{(0)}$$

$$- \sqrt{2} \left( \frac{\partial M_1}{\partial t} + \frac{\partial M_0}{\partial r} - \frac{2m}{r(r - 2m)} M_0 \right) a_{LM}^{(1)} + 2 \left( \frac{\partial M_1}{\partial r} + \frac{m}{r(r - 2m)} M_1 \right) a_{LM}$$

$$- \frac{i}{r} \left[ 2L(L + 1) \right]^{1/2} \left( \frac{\partial M_2}{\partial t} + M_0 \right) b_{LM}^{(0)} + \frac{[2L(L + 1)]^{1/2}}{r} \left( \frac{\partial M_2}{\partial r} - \frac{2}{r} M_2 + M_1 \right) b_{LM}$$

$$+ \frac{\sqrt{2}}{r^2} (2(r - 2m)M_1 - L(L + 1)M_2) g_{LM} + \frac{1}{r^2} [2L(L + 1)(L - 1)(L + 2)]^{1/2} M_2 \xi_{LM}$$

Functions $M_0$, $M_1$ and $M_2$ can be chosen to remove the magnetic coefficient functions $h_{0LM}^{(m)}$, $h_{1LM}^{(m)}$ and $G_{LM}$ to leave the transformed magnetic perturbation tensor as

$$h_{LM}^e = \left( 1 - \frac{2m}{r} \right) H_{0LM} a_{LM}^{(0)} - \sqrt{2} i H_{1LM} a_{LM}^{(1)}$$

$$+ \left( 1 - \frac{2m}{r} \right)^{-1} H_{2LM} a_{LM} + \sqrt{2} K_{LM} g_{LM} \cdot (2.47)$$

### 2.9 Non-Radiative Cases

The wave equation (2.37) describes the gravitational radiation of the particle as it falls into the black hole. However, there are changes to the black hole itself which are a result of the particle falling in. These changes are found from the $L = 0$ and $L = 1$ modes. However, for $L = 0$ all the magnetic parity harmonics are zero which leaves, therefore, only the three non-radiative modes. The electric parity harmonics describe the changes in mass ($L = 0$) and a change of coordinates to a centre of mass system ($L = 1$). The magnetic parity harmonics ($L = 1$) describe the change in angular momentum of the black hole. For the
calculation of these modes it is not necessary to use the Fourier transformed equations, instead equations (2.11a-2.12g) are used. The gravitational radiation is described by the terms $L \geq 2$ in the two wave equations.

2.9.1 Change in Mass

We expect the mass of the black hole to increase as the particle falls in. This can be shown by use of the electric perturbation equations. When $L = 0$, $b_{LM}^{(0)}$, $b_{LM}$ and $f_{LM}$ are all zero and so from equation (2.47) the general perturbation is

$$h_{00}(e) = \left(1 - \frac{2m}{\kappa}\right) H_0 a_{00}^{(0)} + \sqrt{2} H_1 a_{00}^{(1)} + \left(1 - \frac{2m}{\kappa}\right)^{-1} H_2 a_{00} + \sqrt{2}K g_{00}.$$  

This apparent reduction in terms has occurred without the use of any gauge transformations because all the terms removed are identically zero. Since many of the harmonic terms are zero it is straightforward to calculate the change in mass. The gauge transformation $\xi = M_0 Y_0 e_t + M_1 Y_0 e_r$ is used. $M_0$ and $M_1$ are chosen so that $H_1 = K = 0$. Therefore, starting from equation (2.12a) and using equation (2.18b)

$$\frac{\partial}{\partial r} \left\{(r - 2m)H_2(r, t)\right\} = -\sqrt{16\pi m_0} \gamma \left(1 - \frac{2m}{\kappa}\right) \delta(r - R(t)).$$

This can be integrated to

$$(r - 2m)H_2(r, t) = 0, \quad r < R(t)$$

$$= -4\sqrt{\pi} m_0 \left(1 - \frac{2m}{R(t)}\right) \frac{dT}{d\tau}, \quad r > R(t).$$

However, $(1 - 2m/R(t))dT/d\tau$ is a constant of geodesic motion in a Schwarzschild background which is denoted, $\gamma_0$. If the particle has a zero initial velocity and falls from infinity then $\gamma_0 = 1$ and therefore,

$$H_2(r, t) = 0, \quad r < R(t)$$

$$= -\frac{4\sqrt{\pi} m_0 \gamma_0}{r - 2m}, \quad r > R(t). \quad (2.48)$$
Using equation (2.12c) a similar formula can be obtained:

\[ H_0 = - \int \frac{H_2}{r-2m} dr + 8\pi \int r A_{00} dr \]

\[ = 4\sqrt{\pi} m_0 \gamma_0 \int \frac{1}{(r-2m)^2} dr \]

\[ + 8\pi \int m_0 \gamma_0 r \left( \frac{dR}{dt} \right)^2 (r-2m)^{-2} \delta(r-R(t)) Y_{LM} dr \]

\[ = - \frac{4\sqrt{\pi} m_0 \gamma_0}{r-2m} + 4\sqrt{\pi} m_0 \gamma_0 \frac{\left( \frac{dR}{dt} \right)^2}{(1 - \frac{2m}{R(t)}) R(t)} \]

\[ = - \frac{4\sqrt{\pi} m_0 \gamma_0}{r-2m} + f(t) . \]

A gauge condition is now used to remove the function \( f(t) \). Choosing

\[ M_0(r,t) = \frac{1}{2} \left( 1 - \frac{2m}{r} \right) \int f(t) dt , \]

removes \( f(t) \) and therefore

\[ H_0 = H_2 = - \frac{4\sqrt{\pi} m_0 \gamma_0}{r-2m} . \]

Thus, the total electric perturbation in this gauge is written as

\[ h_{00} = - \frac{4\sqrt{\pi} m_0 \gamma_0}{r} a_{00}^{(0)} - \frac{4\sqrt{\pi} m_0 \gamma_0 r}{(r-2m)^2} a_{00} . \]

The late time metric can now be calculated using this expression for the overall perturbation. Up to the linear approximation of this perturbation,

\[ g_{00} = g_{00} + h_{00} = 1 - \frac{2m}{r} - \frac{2m_0}{r} , \]

\[ g_{11} = g_{11} + h_{11} = \left( 1 - \frac{2m}{r} - \frac{2m_0}{r} \right)^{-1} . \]

Here \( h_{00} \) is the tensor component, not the \( L = M = 0 \) harmonic. This clearly shows that the mass of the black hole will increase by the mass of the particle which has fallen in. If the infall of the particle is radial the black hole remains spherically symmetric. If the particle does not fall in along a radial geodesic then the angular momentum of the black hole will also be altered.
2.10 Summary and Conclusion

In this work, Zerilli gives a method of calculating all the effects of a particle as it falls into a black hole. In the specialisation to a radially infalling particle, Zerilli [1] has proved that there is only a correction to the mass of the black hole and Davis et al [26] have solved the electric wave equation to find the gravitational radiation. Zerilli's work can also be used to calculate the effects of accretion disks around black holes, where many particles fall inwards.

In the next chapter we extend this work to deal with a spinning particle falling radially into a black hole in torsion theory.
Chapter 3

A Torsion Wave Equation

3.1 Introduction

The work of Zerilli provides a general method for calculating the metric perturbations due to any perturbing energy momentum tensor in a Schwarzschild background. Furthermore the combined torsion field equations provide an energy momentum tensor which can be used in the Zerilli wave equation in order to include the effect of torsion.

Most calculations of the effects of contact torsion have considered only radially pointing spins for simplicity (see Prasanna[50], Suh[51] and Bedran and Som[52]). The restriction to particles with radially pointing spins simplifies the equations such that the spin provides only a correction to the particle mass and fluid pressure. However, for different orientations of the initial spin direction, additional terms appear in the energy momentum tensor. We show that these terms have an effect on the final state of the system which, while still small in practical situations, is nevertheless much larger that those considered previously.

There are various points that need to be considered at the outset. First
the choice of energy momentum tensor, particularly as 'spin' is not a classical quantity. Second, the deviation from geodesic motion: spinning particles will not generally fall along geodesics. The third consideration is the spin of the elementary particle which obeys its own equation of motion. This equation shows how the direction of spin changes as the particle falls inwards. Taking into account these points we obtain an approximate wave equation with source terms from a particle with torsion.

We approach the problem of the single particle through the Weyssenhoff fluid\cite{4} for a continuous distribution of matter and consider the approximation to a single particle. We set the spin direction orthogonal to the direction of motion as this results in additional terms not previously considered.

### 3.2 Hehl's Energy-Momentum Tensor

In this section the energy-momentum tensor postulated by Hehl et al\cite{11} is considered. The assumption of geodesic motion is tested and the energy-momentum tensor is calculated.

#### 3.2.1 Geodesic Motion

The assumption that a particle with intrinsic spin falls on a geodesic will greatly simplify the calculations of its motion, the spin precession and the final perturbation equations. In this section we check that this assumption is consistent to first order in the spin $s$.

The Weyssenhoff fluid\cite{4} generalises the perfect fluid of general relativity to one that includes intrinsic spin. Hehl\cite{11} showed how to combine the two sets of field equations of torsion and curvature to obtain a single set of modified Einstein
equations for the metric with the total energy momentum

\[ \tilde{\sigma}^{\mu\nu} = \left( \rho + P - 2ks^2 \right) u^\mu u^\nu + \left( P - ks^2 \right) g^{\mu\nu} \]

\[ + 2 \left( u_\lambda u^\kappa - \delta^\kappa_\lambda \right) \nabla_\kappa \left( \tau^{\lambda(\mu} u^{\nu)} \right) + \tau^{(\mu} \left( 2\delta^{\nu)}_\lambda u^\kappa - u^\nu \delta^\kappa_\lambda \right) \nabla_\kappa u^\lambda \], \quad (3.1)

as the source. Here \( \rho \) and \( P \) are the density and pressure respectively and \( \nabla \) is covariant differentiation with respect to the Christoffel symbol. The spin density is represented by the tensor \( \tau_{\mu\nu} \) and \( s^2 = \tau_{\mu\nu} \tau^{\mu\nu} \). The spin tensor is taken to satisfy the condition

\[ \tau_{\mu\nu} u^\nu = \tau_\mu = 0 \]. \quad (3.2)

Using the condition (3.2) the last term can be removed, leaving

\[ \tilde{\sigma}^{\mu\nu} = \left( \rho + P - 2ks^2 \right) u^\mu u^\nu + \left( P - ks^2 \right) g^{\mu\nu} \]

\[ + 2 \left( u_\lambda u^\kappa - \delta^\kappa_\lambda \right) \nabla_\kappa \left( \tau^{\lambda(\mu} u^{\nu)} \right) \]. \quad (3.3)

We are going to model a spinning particle as a Weyssenhoff fluid with non-zero density only in a small region. Internal pressures in a fluid usually cause deviations from geodesic motion. Here the pressure appears combined with the intrinsic spin. To be consistent with the assumption that the particle moves along a geodesic, we follow Prasanna[50] and model the particle by taking the effective pressure \( P - ks^2 \) to be zero. We can justify this for our purposes because the assumption makes no difference in our final result, which is first order in the spin, \( s \). However, we would also argue that we are free to construct a model of a point particle as appropriate. Setting the effective pressure to zero gives a model in which the internal structure of the particle does not produce deviations from geodesic motion (to lowest order).

We now show that geodesic motion is consistent with \( \nabla_\mu \tilde{\sigma}^{\mu\nu} = 0 \). We have

\[ \nabla_\mu \tilde{\sigma}^{\mu\nu} = u^\nu \nabla_\mu (\rho u^\mu) + 2 \nabla_\mu \left[ (u_\lambda u^\kappa - \delta^\kappa_\lambda) \nabla_\kappa \left( \tau^{\lambda(\mu} u^{\nu)} \right) \right] \]. \quad (3.4)
Using conservation of mass \( \nabla_{\nu}(\rho u^\nu) = 0 \), expanding the symmetric terms and removing terms which are identically zero on a geodesic gives

\[
\nabla_{\mu}\tilde{\gamma}^{\mu\nu} = \nabla_{\mu}\left[u_\lambda u^\nu u^\kappa \nabla_{\kappa}(\tau^{\lambda\mu}) + u_\lambda u^\mu u^\kappa \nabla_{\kappa}(\tau^{\lambda\nu})\right.
\]

\[
\left. - \nabla_{\lambda}(\tau^{\kappa\mu} u^\nu) - \nabla_{\kappa}(\tau^{\kappa\nu} u^\mu) \right].
\]  

(3.5)

It is important to distinguish between the spin density, \( \tau^{\mu\nu} \) appearing in the Weyssenhoff fluid and the rank two tensor associated with the particle nature of the spin, \( \tilde{\tau}^{\mu\nu} \). The two are related by the fluid density

\[
\tau^{\mu\nu} = \rho \tilde{\tau}^{\mu\nu}.
\]

The spin direction of a particle is given by the Fermi-Walker transport equation

\[
u^\mu\nabla_\mu S^\nu = u^\nu u^\lambda u_\lambda S^\sigma,
\]

(3.6)

and therefore, a spinning particle falling on a geodesic satisfies

\[
u^\mu\nabla_\mu S^\nu = 0,
\]

where

\[
S^\nu = \frac{1}{2} \epsilon^{\nu\mu\lambda\kappa} u_\mu \tilde{\tau}_{\lambda\kappa}.
\]

This together with the assumption of geodesic motion shows that

\[
u^\mu\nabla_\mu \tilde{\tau}^{\lambda\sigma} = 0.
\]

(3.7)

Equation (3.7) allows further simplification in equation (3.5). In fact only two terms remain

\[
\nabla_{\mu}\tilde{\gamma}^{\mu\nu} = -\nabla_{\mu}\nabla_{\kappa}(\rho \tilde{\tau}^{\kappa\mu} u^\nu) - \nabla_{\mu}\nabla_{\kappa}(\rho \tilde{\tau}^{\kappa\nu} u^\mu).
\]

(3.8)
CHAPTER 3. A TORSION WAVE EQUATION

We shall show that this expression reduces to curvature terms that are zero in a Schwarzschild background.

Taking the first term in (3.8)

\[
\frac{1}{\hat{\nu}}\frac{1}{\hat{\kappa}} (\rho \tilde{\tau}^{\mu\nu} u^\nu) = \nabla_\kappa \nabla_\mu (\rho \tilde{\tau}^{\mu\nu} u^\nu) + R_\kappa^{\kappa\mu} \tilde{\tau}^{\lambda\mu} u^\nu + R_\lambda^{\kappa\mu} \tilde{\tau}^{\kappa\nu} u^\nu + R_\lambda^{\kappa\nu} \tilde{\tau}^{\kappa\mu} u^\lambda,
\]

and by the antisymmetry of \(\tau^{\kappa\mu}\)

\[
2\nabla_\mu \nabla_\kappa (\rho \tilde{\tau}^{\kappa\mu} u^\nu) = R_\lambda^{\kappa\mu} \tilde{\tau}^{\lambda\mu} u^\nu + R_\lambda^{\kappa\mu} \tilde{\tau}^{\kappa\nu} u^\nu + R_\lambda^{\kappa\nu} \tilde{\tau}^{\kappa\mu} u^\lambda.
\]

The terms \(R_\lambda^{\kappa\mu} \tilde{\tau}^{\lambda\mu} u^\nu\) and \(R_\lambda^{\kappa\mu} \tilde{\tau}^{\kappa\nu} u^\nu\) are both zero in a Schwarzschild background because the components of the Riemann tensor are non-zero only if the indices take two different values (e.g. \(R_3^{3\theta\theta}\)) and so this implies that \(\lambda = \mu\) for the first term and that \(\lambda = \kappa\) for the second term. However, both of these cases give zero for antisymmetric \(\tau\). Thus only one term remains namely,

\[
\frac{1}{\hat{\nu}}\frac{1}{\hat{\kappa}} (\rho \tilde{\tau}^{\kappa\mu} u^\nu) = \frac{1}{2} R_\lambda^{\kappa\mu} \tilde{\tau}^{\kappa\mu} u^\lambda.
\]

Taking the spin to point in the \(\theta\) direction and using \(\tau_{\mu\nu} u^\nu = 0\) it is easy to show that only the \(\tau^{31}\) and \(\tau^{03}\) terms are nonzero. Using this, and the fact that the particle is falling radially, it follows that the right hand side of (3.9) is zero.

Taking the second term in (3.8)

\[
\frac{1}{\hat{\nu}}\frac{1}{\hat{\kappa}} (\rho \tilde{\tau}^{\kappa\nu} u^\mu) = \nabla_\kappa \nabla_\mu (\rho \tilde{\tau}^{\kappa\nu} u^\mu) + R_\lambda^{\kappa\nu} \tilde{\tau}^{\lambda\mu} u^\mu + R_\lambda^{\kappa\nu} \tilde{\tau}^{\kappa\lambda} u^\mu + R_\lambda^{\kappa\nu} \tilde{\tau}^{\kappa\mu} u^\lambda.
\]

The covariant derivative term on the RHS is now zero because when expanded there is a geodesic derivative and a term which was shown to be zero earlier (see equation 3.7). Two of the Riemann terms can be shown to cancel which leaves

\[
\frac{1}{\hat{\nu}}\frac{1}{\hat{\kappa}} (\rho \tilde{\tau}^{\kappa\nu} u^\mu) = R_\lambda^{\kappa\mu} \tilde{\tau}^{\kappa\lambda} u^\mu.
\]

This is identical to the RHS of equation (3.9) and therefore zero.
CHAPTER 3. A TORSION WAVE EQUATION

This shows that the assumptions of geodesic motion, radial infall, conservation of mass and Fermi-Walker transport are self consistent. These assumptions are now carried forward in the remainder of the calculation and the system solved exactly to first order in $s$.

3.2.2 The Energy-Momentum Tensor

The spin of the particle is modelled by a Weyssenhoff fluid with non-zero density only in a small region. Following the method of the previous section the geodesic terms and other zero terms are removed to give

$$
\tilde{\sigma}^{\mu \nu} = (\rho - ks^2)u^\mu u^\nu + 2u_\lambda u^\kappa \left( \nabla_\kappa (\tau^{(\lambda)}(\nu)) u^\mu \right) - 2\tau^{(\mu \nu)} - 2u^{(i \nu} \nabla_k \tau^{k \mu}) .
$$

(3.10)

To calculate the non-zero component of the spin tensor $\tau^{\mu \nu}$ we consider an element of fluid with spin in the $\hat{\theta}$-direction moving radially. At rest the spin tensor has $\tau_{13} = -\tau_{31}$ as the only non-zero component (see Aharoni[53]),

$$
\tau^{\kappa \xi} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \tau_{12} & \tau_{13} \\
0 & -\tau_{12} & 0 & \tau_{23} \\
0 & -\tau_{13} & -\tau_{23} & 0
\end{pmatrix} .
$$

(3.11)

As the fluid moves the condition $\tau^{\mu \nu}u_\nu = 0$ shows that the $\tau_{03}$ component is also non-zero. The components $\tau_{13}$ and $\tau_{03}$ are related by

$$
\tau_{03} = -\frac{\tau_{13}u^1}{u^0} .
$$

(3.12)

As the fluid is falling radially along a geodesic, $S^2$ is the only non-zero component of $S^\nu$. Using equation (3.12) and the Fermi Walker equation for particle motion
CHAPTER 3. A TORSION WAVE EQUATION

(3.6)

\[ S^2 = -(u_0 \tilde{\tau}_{13} - u_1 \tilde{\tau}_{03}) (-g)^{1/2} \]

\[ = \frac{\tilde{\tau}_{13}}{u_0} (-g)^{1/2} . \]  (3.13)

Equations (3.6) and (3.13) show that only the \( S^2 \) component of \( S \) changes as the fluid falls in. Using the definition of the covariant derivative with respect to the normal symmetric connection, equation (3.6) can be expanded to,

\[ \frac{dS^3}{d\tau} = -\Gamma^3_{13}S^3u^1 = 0 , \]  (3.14)

\[ \frac{dS^2}{d\tau} = -\Gamma^2_{12}S^2u^1 , \]  (3.15)

\[ \frac{dS^1}{d\tau} = -\Gamma^1_{00}S^0u^0 - \Gamma^1_{11}S^1u^1 = 0 , \]  (3.16)

\[ \frac{dS^0}{d\tau} = -\Gamma^0_{01}S^0u^1 - \Gamma^0_{10}S^1u^0 = 0 , \]  (3.17)

where \( \tau \) is the proper time. Due to the definition of \( S^\nu \) all these equations have zero right hand sides except equation (3.15) and the spin direction remains unchanged. Equation (3.15) can be integrated to

\[ S^2 = \frac{D}{r} , \]

where \( D \) is a constant (agreeing with Apostolatos[55]). Therefore, equation (3.13) gives

\[ \tilde{\tau}_{13} = \frac{Du^0(-g)^{1/2}}{r} , \]  (3.18)

and

\[ \tilde{\tau}_{03} = -\frac{Du^1(-g)^{1/2}}{r} . \]  (3.19)

Using equations (3.18) and (3.19) it now also possible to calculate the magnitude of the spin throughout the motion. According to Hehl \( s^2 = \tau_{ij}\tau^{ij} \). However, the
definition used here is \( s^2 = \frac{\tau_{ij}\tau^{ij}}{2} \), which is more common. We have

\[
s^2 = \frac{\tau_{ij}\tau^{ij}}{2} = \tau_{13}\tau_{13} + \tau_{03}\tau_{03} = g^{11}g^{33}\tau_{13}\tau_{13} + g^{00}g^{33}\tau_{03}\tau_{03} = \rho^2D^2\left(\frac{g^{11}g^{33}(u^0)^2}{r^2(\epsilon^{2103})^2} + \frac{g^{00}g^{33}(u^1)^2}{r^2(\epsilon^{2103})^2}\right) = \rho^2D^2\left(\frac{g^{33}}{r^2(r^4\sin^2\theta)^{-1}} - g^{00}u^0u^0 + g^{00}u^1u^1\right) = \rho^2D^2(-g^{00}u^0u^0 - g^{11}u^1u^1)
\]

because \( u^\mu u_\mu = -1 \). Thus the magnitude of the spin remains constant throughout the motion, as expected.

The additional terms required to calculate the energy-momentum tensor are

\[
\begin{align*}
\nabla_0 u^0 &= \Gamma^0_{10} u^1 \\
\nabla_0 u^2 &= 0 \\
\nabla_1 u^0 &= \frac{\partial u^0}{\partial \tau} + \Gamma^0_{10} u^0 \\
\nabla_1 u^2 &= 0 \\
\nabla_2 u^0 &= 0 \\
\nabla_2 u^2 &= \Gamma^2_{12} u^1 \\
\nabla_3 u^0 &= 0 \\
\nabla_3 u^2 &= 0
\end{align*}
\]

Calculation of the energy-momentum tensor can now be completed so that,

\[
\tilde{\sigma}^{ij} = \int_{-\infty}^{\infty} \gamma^{ij} d\tau,
\]
where,
\[
\gamma^{ij} = \begin{pmatrix}
\frac{r^2 m_0}{(r-2m)^2} & -\sqrt{\frac{2m}{r}} \frac{r m_0}{(r-2m)} & 0 & \frac{s(r+6m)}{2r^2(r-2m)} \\
-\sqrt{\frac{2m}{r}} \frac{r m_0}{(r-2m)} & \frac{2nm_0}{r} & 0 & \frac{s(r+6m)}{\sqrt{2} r^{3/2}} \\
0 & 0 & 0 & 0 \\
\frac{s(r+6m)}{2r^2(r-2m)} & \frac{s m^{1/2}}{\sqrt{2} r^{3/2}} & 0 & 0
\end{pmatrix} \delta^{(4)}(x - z(\tau)) .
\]

The coordinate \(x^\mu\) is a general event, \(z(\tau) = (T(\tau), \bar{R}(\tau), \bar{\Theta}(\tau), \bar{\Phi}(\tau))\) is the particle position at proper time \(\tau\) and we have put \(G = c = 1\).

## 3.3 Ray and Smalley’s Energy-Momentum Tensor

The energy-momentum tensor of Ray and Smalley\cite{18}-\cite{21} has been discussed in chapter one. There are several differences between the work of Hehl and Ray and Smalley. Here we show that the assumption of geodesic motion is consistent with the energy-momentum tensor of Ray and Smalley and that it reduces to a form similar to that of Hehl.

### 3.3.1 Geodesic Motion

The variation of the Lagrangian given by Ray and Smalley in chapter one leads to a gravitational field equation of
\[
G^{(ij)} - \nabla_k (T^{kij} + T^{kji}) = \frac{1}{2c_1} T^{ij} .
\]  
\hspace{1cm} (3.20)

Splitting the curvature tensor into an Einstein tensor and removing all the \(s^2\) terms reduces 3.20 to
\[
G^{ij} = \frac{1}{2c_1} T^{ij} ,
\]  
\hspace{1cm} (3.21)
where $G^{ij}$ is the normal Einstein tensor. Therefore the assumption of geodesic motion must be tested using the divergence of the energy-momentum tensor,

$$T^{ij} = -\frac{\rho F^i}{c^2} u^i u^j + g^{ij} (F - \rho F^i) + 2A \left( \rho u^{(i} \tau^{j)} k \right)_{;k} + \frac{2A}{c^2} \rho u^{(i} \tau^{j)} k \dot{u}_k$$

$$- 2A \rho \omega^{k(i} \tau^{j)} _k + 2A \rho u^{(i} \tau^{j)} k \omega_{kl} u^l . \quad (3.22)$$

Therefore,

$$\nabla_k T^{ij} = \nabla_k \left\{ -\frac{\rho F^i}{c^2} u^i u^j + g^{ij} (F - \rho F^i) + 2A \left( \rho u^{(i} \tau^{j)} k \right)_{;k} + \frac{2A}{c^2} \rho u^{(i} \tau^{j)} k \dot{u}_k$$

$$- 2A \rho \omega^{k(i} \tau^{j)} _k + 2A \rho u^{(i} \tau^{j)} k \omega_{kl} u^l \right\} . \quad (3.23)$$

The terms $\rho F^i u^i u^j / c^2 + g^{ij} (F - \rho F^i)$ are equivalent to a perfect fluid without spin, and with geodesic motion and $P = 0$, these give zero. For an elementary particle $w^{ij} = 0$ and therefore,

$$\nabla_k T^{ij} = 2A \nabla_k \left\{ \left( \rho u^{(i} \tau^{j)} k \right)_{;k} + \frac{1}{c^2} \rho u^{(i} \tau^{j)} k \dot{u}_k \right\} . \quad (3.24)$$

The geodesic term, $\dot{u}_k = 0$, which leaves only one term which is the same as the Hehl energy-momentum tensor except for a constant factor. Therefore, the assumption of geodesic motion is consistent with the energy-momentum tensor of Ray and Smalley.

3.3.2 The Energy-Momentum Tensor

Calculation of Ray and Smalley's energy-momentum tensor is trivial now that it has been shown to reduce to that of Hehl (except for a constant) to this order. Therefore, the energy-momentum tensor can be written as, $\tilde{\sigma}^{ij} = \int_{-\infty}^{\infty} \gamma^{ij} d\tau$, 
where

\[
\gamma_{ij} = \begin{pmatrix}
\frac{r^2 m_0}{r(r-2m)} & -\sqrt{\frac{2m}{r}} \frac{r m_0}{r(r-2m)} & 0 & \frac{A_s(r+6m)}{2r^2(r-2m)} \\
-\sqrt{\frac{2m}{r}} \frac{r m_0}{r(r-2m)} & \frac{2m m_0}{r} & 0 & -\frac{A_s m_1^{1/2}}{\sqrt{2} r^{7/2}} \\
0 & 0 & 0 & 0 \\
\frac{A_s(r+6m)}{2r^2(r-2m)} & -\frac{A_s m_1^{1/2}}{\sqrt{2} r^{7/2}} & 0 & 0
\end{pmatrix} \delta^{(4)}(x-z(\tau)) \label{3.25}
\]

The important constant \( A_s \) now distinguishes between these two energy-momentum tensors. It is possible to continue the analysis including \( A_s \). Once we have calculated the radial infall of a spinning particle, \( A_s \) will be chosen to give a meaningful physical result.

### 3.4 Integrating over all time

The energy-momentum tensor (3.25) must be integrated over all time as in Zerilli's case. This is done, term by term, using the \( \delta \) functions as before. For every density \( \rho \) or torsion term, \( s^2 \) and \( \tau^{13} \) a rank four normalised delta function is inserted to guarantee a point particle system. Therefore, following the scheme of section 2.5.1, the first term of the energy-momentum tensor can be integrated as follows,

\[
\bar{\sigma}^{00} = \int_{-\infty}^{\infty} m_0 \delta^{(4)}(x-z(\tau))(u^0)^2 d\tau
\]

\[
= \int_{-\infty}^{\infty} m_0 \delta(t-T(\tau)) \frac{\delta(r-R(t))}{r^2} \delta^{(2)}(\Omega-\Omega(t)) \left( \frac{dT}{d\tau} \right)^2 d\tau
\]

\[
= m_0 \frac{\delta(r-R(t))}{r^2} \delta^{(2)}(\Omega-\Omega(t)) \frac{dT}{d\tau} \label{3.26a}
\]

where \( \tau \) denotes the proper time, \( R(t) = \bar{R}(T^{-1}(t)) \) and \( \Omega(t) = \bar{\Omega}(T^{-1}(t)) \). Similarly the other components are integrated and they become,

\[
\bar{\sigma}^{10} = m_0 \frac{\delta(r-R(t))}{r^2} \delta^{(2)}(\Omega-\Omega(t)) \frac{dR}{dt} \frac{dT}{d\tau} \label{3.26b}
\]
CHAPTER 3. A TORSION WAVE EQUATION

\[\tilde{\sigma}^{11} = m_0 \frac{\delta(r - R(t))}{r^2} \delta^{(2)}(\Omega - \Omega(t)) \frac{dR}{dt} \frac{dT}{d\tau}\] (3.26c)

\[\tilde{\sigma}^{03} = sF(r) \frac{\delta(r - R(t))}{r^2} \delta^{(2)}(\Omega - \Omega(t)) \frac{d\tau}{dT}\] (3.26d)

\[\tilde{\sigma}^{13} = sG(r) \frac{\delta(r - R(t))}{r^2} \delta^{(2)}(\Omega - \Omega(t)) \frac{d\tau}{dT}\] (3.26e)

where \(F(r) = A(r + 6m)/(2r^2(r - 2m))\) and \(G(r) = -Am^{1/2}/(\sqrt{2r^{5/2}})\). Here \(dR/dt\) is taken to mean \(dR/dT\) evaluated at a field point \(t\). The next stage is to calculate the components of each of the tensor harmonics ready for substitution into the wave equation.

3.5 Finding the Harmonic Coefficients

To find the coefficients of each of the tensor harmonics the inner product described in section 2.5.2 is used. This amounts to an integration over the spatial variables in the energy-momentum tensor.

\[A_{LM}^{(0)} = \int \int \eta^{00} \eta^{00} (a_{LM})_{00} \tilde{\sigma}_{00} d\Omega\]
\[= \int \int (a_{LM})_{00} (g_{00})^{2} \tilde{\sigma}_{00} d\Omega\]
\[= \frac{(\rho - ks^2)Y_{LM}^*(\Omega(t))\delta(r - R(t))}{r^2} \left(1 - \frac{2m}{r}\right)^2 \frac{dT}{d\tau}.\] (3.27a)

Similarly for the other harmonic coefficients,

\[A_{LM}^{(1)} = \frac{i\sqrt{2}(\rho - ks^2)\delta(r - R(t))}{r^2} \frac{dR}{dt} \frac{dT}{d\tau} Y_{LM}^*(\Omega(t))\] (3.27b)

\[A_{LM} = \frac{(\rho - ks^2)Y_{LM}^*(\Omega(t))\delta(r - R(t))}{r^2} \left(1 - \frac{2m}{r}\right)^{-2} \left(\frac{dR}{dt}\right)^2 \frac{dT}{d\tau}\] (3.27c)
\( B_{LM}^{(0)} = \frac{i \left[ \frac{1}{2} L (L + 1) \right]^{-1/2}}{r} \left( 1 - \frac{2m}{r} \right) sF(r) \delta(r - R(t)) \)

\[ \frac{\partial Y_{LM}^{*}(\Omega(t))}{\partial \phi} \frac{d\tau}{dT} \quad (3.27d) \]

\( B_{LM} = \frac{1}{2} L (L + 1) \right]^{-1/2} \left( 1 - \frac{2m}{r} \right)^{-1} sG(r) \delta(r - R(t)) \)

\[ \frac{\partial Y_{LM}^{*}(\Omega(t))}{\partial \phi} \frac{d\tau}{dT} \quad (3.27e) \]

\( Q_{LM}^{(0)} = \left[ \frac{1}{2} L (L + 1) \right]^{-1/2} \left( 1 - \frac{2m}{r} \right) sF(r) \sin \theta |_{\Omega(t)} \delta(r - R(t)) \)

\[ \frac{\partial Y_{LM}^{*}(\Omega(t))}{\partial \theta} \frac{d\tau}{dT} \quad (3.27f) \]

\( Q_{LM} = - \frac{1}{2} L (L + 1) \right]^{-1/2} \left( 1 - \frac{2m}{r} \right)^{-1} sG(r) \sin \theta |_{\Omega(t)} \delta(r - R(t)) \)

\[ \frac{\partial Y_{LM}^{*}(\Omega(t))}{\partial \theta} \frac{d\tau}{dT} \quad (3.27g) \]

\( D_{LM} = 0 \quad (3.27h) \)

\( F_{LM} = 0 \quad (3.27i) \)

\( G_{LM} = 0 \quad (3.27j) \)

Splitting the energy-momentum tensor into harmonic coefficients allows the perturbation equations to be solved for this particular system.
3.6 Solving the Perturbation Equations

The energy-momentum tensor has been separated into its harmonic components and the perturbation equations are now solved to calculate the changes to the black hole as the particle falls in. An observer near a black hole will experience only the black hole’s mass, charge and angular momentum. The observer can see the effect of these attributes through the curvature of the space. In this case the effect of the elementary particle spin of the particle does not propagate outside the event horizon. Therefore, the expected results will be that the elementary particle spin will manifest itself as an angular momentum once the particle has fallen into the black hole.

For the $L = 0$ there is only an electric component which corresponds to the change in mass of the black hole. The case of a spinning particle and that of Zerilli are the same and therefore,

\[
\text{Change in mass of the black hole} = m_0.
\]

For the $L = 1$ we begin with perturbation equation (2.11a)

\[
\frac{1}{r^2} \frac{\partial^2}{\partial t \partial r} (r^2 h_1) = \frac{8\pi r^2 Q_{1M}^{(0)}}{r - 2m},
\]

where a gauge transformation is used to eliminate the variable, $h_{0LM}$. There is freedom to only eliminate one variable for this parity, for $L = 1$ the $h_{2LM}$ coefficient is identically zero and therefore the gauge transformation can be used to remove $h_{0LM}$.

Substituting the value for $Q_{1M}^{(0)}$ into equation (3.28) above gives

\[
\frac{1}{r^2} \frac{\partial^2}{\partial t \partial r} (r^2 h_1) = -8\pi s F(r) \sin \theta \delta(r - R(t)) \frac{\partial Y_{1M}}{\partial \theta} \frac{d\tau}{dT}.
\]

Because the particle falls in the $\theta = \pi/2$ plane, the only non-zero term, coming from $\partial Y_{1M}/\partial \theta$ evaluated on the path, is $\partial Y_{10}/\partial \theta$. Substituting in $d\tau/dT =$
(r - 2m)/r and integrating with respect to r and then t leaves

\[ h_1 = \begin{cases} 0, & r < R(t) \\ 8\pi sA\sqrt{\frac{3}{4\pi r^3}} p(R), & r > R(t) \end{cases} \]

where,

\[ p(R) = \int^{t(R)} R'(R' - 2m)F(R')dt = \int^{t(R)} \frac{(R' + 6m)}{2R'} dt. \]

It is now possible to remove the integral in \( h_1 \), using a further gauge transformation. The gauge transformed perturbation is of the form,

\[ h' = \frac{2i}{r} \left[ h_1 - r^2 \frac{\partial}{\partial r} \left( \frac{\Lambda_{10}}{r^2} \right) \right] c_{10} + \frac{2}{r} \frac{\partial \Lambda_{10}}{\partial t} c_{10}^{(0)}. \]  

Thus, to remove the integral (i.e. set \( h'_1 = 0 \)) we choose,

\[ h_1 = r^2 \frac{\partial}{\partial r} \left( \frac{\Lambda_{10}}{r^2} \right), \]

which gives,

\[ \Lambda_{10} = -4sA\sqrt{\frac{\pi}{3}} \frac{p(R)}{r}. \]  

Substituting equation (3.30) back into (3.29) gives a final equation describing the perturbation of the background space from the spinning particle.

\[ h = -4sA\sqrt{\frac{\pi}{3}} \frac{1}{r^2} \frac{1}{R} \frac{(R + 6m)}{R} c_{10}^{(0)}. \]

This can be interpreted either by comparison with the angular momentum of a weak field (Landau and Lifschitz[49]) or by comparing the metric with that of Kerr; here we compare the metrics. The harmonic \( c_{10}^{(0)} \) has only a 03 component so, the perturbed metric \( g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu} \), where \( g_{\mu\nu}^{(0)} \) is the background space-time, can be written as,
CHAPTER 3. A TORSION WAVE EQUATION

Here $R(t)$ is the radial position of the particle at co-ordinate time $t$. Therefore as the particle falls through the black hole horizon ($R(t) \to 2m$) the off diagonal contribution tends to a non-zero constant. This leads to a metric

$$g_{\mu\nu} = \begin{pmatrix}
-\left(1 - \frac{2m}{r}\right) & 0 & 0 & \frac{-sA(R(t)+6m)}{r} \\
0 & \left(1 - \frac{2m}{r}\right)^{-1} & 0 & 0 \\
0 & 0 & r^2 & 0 \\
\frac{-sA(R(t)+6m)}{r} & 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}.$$  

This clearly has off diagonal terms identical to that of the Kerr metric in Boyer-Lindquist coordinates to first order in the angular momentum parameter $a$ (putting $2sA = a$).

It is now perfectly clear that the infall of a spinning particle leads to a rotating black hole where the rotation of the black hole is equal to the intrinsic spin of the infalling particle provided that $A = 1/2$. Therefore, we conclude that the energy-momentum tensor of Ray and Smalley correctly describes a spinning fluid while in the energy-momentum tensor of Hehl et al there is a spurious factor 2.

3.7 Fourier Transform of Torsion Coefficients

To calculate the gravitational radiation released by the radial infall of a spinning particle, the sources of the wave equations of section 2.6 must be calculated. To do
this it is convenient to Fourier transform the harmonic components of the energy-
momentum tensor. Using the definition of the Fourier transformation given in
equation (2.13) and following the method of section 2.5.3, the first harmonic
coefficient becomes

\begin{align}
\hat{A}_{LM}^{(0)}(\omega, r) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Y_{LM}(\Omega(t)) \frac{\delta(r - R(t))}{r^2} \left(1 - \frac{2m}{r}\right)^2 m_0 \frac{dT}{d\tau} \text{e}^{i\omega T} \, dt \\
&= \frac{1}{\sqrt{2\pi}} m_0 Y_{LM}^* (\tilde{\Omega}(r)) \left(1 - \frac{2m}{r}\right)^2 \frac{dT}{d\tau} \frac{dr}{dR} \text{e}^{i\omega T(r)} , \quad (3.32a)
\end{align}

where \( T(r) = R^{-1}(r) \) and \( \tilde{\Omega}(r) = \Omega(R^{-1}(r)) \). The other terms are,

\begin{align}
\hat{A}_{LM}^{(1)}(\omega, r) &= \frac{i}{\sqrt{2\pi}} m_0 \frac{dT}{d\tau} Y_{LM}^* (\tilde{\Omega}(r)) \text{e}^{i\omega T(r)} \quad (3.32b) \\
\hat{A}_{LM}(\omega, r) &= \frac{1}{\sqrt{2\pi}} m_0 Y_{LM}^* (\tilde{\Omega}(r)) \frac{1}{r^2} \left(1 - \frac{2m}{r}\right)^2 \frac{dR}{dt} \frac{dT}{d\tau} \text{e}^{i\omega T(r)} \quad (3.32c) \\
\hat{B}_{LM}^{(0)}(\omega, r) &= \frac{i}{\sqrt{2\pi}} \left[ \frac{1}{2} L(L + 1) \right]^{-1/2} \frac{1}{r} \left(1 - \frac{2m}{r}\right)^{-1} sF(r) \frac{\partial Y_{LM}^* (\tilde{\Omega}(r))}{\partial \phi} \\
&\quad \frac{dt}{dR} \frac{d\tau}{dT} \text{e}^{i\omega T(r)} \quad (3.32d) \\
\hat{B}_{LM}(\omega, r) &= \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{2} L(L + 1) \right]^{-1/2} \left(1 - \frac{2m}{r}\right)^{-1} sG(r) \frac{\partial Y_{LM}^* (\tilde{\Omega}(r))}{\partial \phi} \\
&\quad \frac{dt}{dR} \frac{d\tau}{dT} \text{e}^{i\omega T(r)} \quad (3.32e) \\
\hat{Q}_{LM}^{(0)}(\omega, r) &= -\frac{1}{\sqrt{2\pi}} \left[ \frac{1}{2} L(L + 1) \right]^{-1/2} \frac{1}{r} \left(1 - \frac{2m}{r}\right)^{-1} sF(r) \sin \theta \frac{\partial Y_{LM}^* (\tilde{\Omega}(r))}{\partial \theta} \\
&\quad \frac{dt}{dR} \frac{d\tau}{dT} \text{e}^{i\omega T(r)} \quad (3.32f) \\
\hat{Q}_{LM}(\omega, r) &= \frac{i}{\sqrt{2\pi}} \left[ \frac{1}{2} L(L + 1) \right]^{-1/2} \frac{1}{r} \left(1 - \frac{2m}{r}\right)^{-1} sG(r) \sin \theta \frac{\partial Y_{LM}^* (\tilde{\Omega}(r))}{\partial \theta} \\
&\quad \frac{dt}{dR} \frac{d\tau}{dT} \text{e}^{i\omega T(r)} \quad (3.32g)
\end{align}
3.8 Torsion Wave Equations

The background space-time is Schwarzschild and so the curvature part of Zerilli’s wave equation will be unchanged and the substitution of a torsion energy-momentum tensor will lead to a pair of wave equations for each parity of harmonic.

Only the source term need be given because the rest of the wave equation is given by equations (2.37) and (2.38). The source terms are found for radial infall, in this case,

\[
\frac{dR}{dt} = -\sqrt{\frac{2m}{r}} \left(1 - \frac{2m}{r}\right)
\]

\[
\frac{dT}{dr} = \left(1 - \frac{2m}{r}\right)^{-1}.
\]

Therefore, the source term for the electric parity harmonics is,

\[
S^e_{LM} = -\frac{4m_0 \sqrt{\pi}}{\lambda r + 3m} \left(1 - \frac{2m}{r}\right) \left[\frac{r}{2m} \frac{\partial Y_{LM}}{\partial \phi} \left(1 - \frac{2m}{r}\right) + \frac{1}{\sqrt{2m}} + \frac{ib(r)}{\sqrt{m}} \sqrt{\frac{r}{2m}} \left(\frac{r}{2m}\right)^{3/2} \left(\frac{r}{2m}\right)^{1/2} \right] e^{i\omega T(r)},
\]

where \(b(r) = \lambda r^3(5 + 2\lambda) + m r^2(3 + 38\lambda + 12\lambda^2) + 48 m^2 r (1 - \lambda) - 108 m^3\).

For radial infall there is no magnetic parity harmonic in the Zerilli case. However, with torsion the source term is,

\[
S^m_{LM} = \sqrt{2\pi} \left(1 - \frac{2m}{r}\right) \frac{s \sin \theta}{r^3 (\lambda + 1)} \frac{dY_{LM}}{d\theta} e^{i\omega T(r)},
\]

where

\[
T(r) = -\frac{4m}{3} \left(\frac{r}{2m}\right)^{3/2} - 4m \left(\frac{r}{2m}\right)^{1/2} + 2m \ln \left[\left(\frac{r}{2m}\right)^{1/2} + 1\right] \left[\left(\frac{r}{2m}\right)^{1/2} - 1\right]^{-1}.
\]

In the case that the torsion is zero these equations reduce to those of Zerilli as expected, equation (2.43).
3.9 Summary and Conclusions

In previous work the effect of torsion has been considered negligible because it was thought that the torsion only occurred as a correction to the mass. This is true for a radially pointing spin but here an azimuthal pointing spin has been considered and has led to additional terms in the energy-momentum tensor and finally in the wave equations. Unlike the terms which correct the mass, these additional terms have no coupling term, $k$, which is a small quantity and although the spin density $s$ is also small, these terms are more significant than the $ks^2$ terms previously considered.

We have shown that the Zerilli method produces a wave equation which can be solved to find the effect on the background black hole and the gravitational radiation released. Contact torsion theory has been introduced and Zerilli’s method has been extended to include the case of a radially infalling elementary spinning particle. In the case of zero torsion the equations reduce to those of Davis et al [26] as required.

The discrepancy between the energy-momentum tensors of Hehl[11] and those of Ray and Smalley[18] has been resolved in favour of Ray and Smalley because, for their energy-momentum tensor, the spin-angular momentum is conserved while Hehl predicts twice the black hole rotation expected. Therefore, in the remainder of this thesis we adopt the energy-momentum tensor of Ray and Smalley in order to calculate the gravitational radiation that we would expect from this system.
Chapter 4

Gravitational Radiation

4.1 Introduction

Gravitational radiation occurs throughout the universe. In the cases of chapters 2 and 3, particles fall towards a black hole and this will cause an emission of gravitational radiation.

There are several ways of calculating the radiation released in different situations. The early theoretical predictions of the radiation were carried out by Regge and Wheeler[24] as part of the work on the stability of the Schwarzschild metric. The perturbed Schwarzschild black hole must lose energy so that it can settle back down to a stable symmetry. This energy loss is achieved through the emission of gravitational radiation. As the black hole returns to its symmetric state the event horizon oscillates, which is the source of the gravitational radiation.

In the case of a particle falling into the black hole, it is assumed that there is no radiation reaction and the particle falls on a geodesic. The particle falls into the black hole, the mass of the black hole increases and the event horizon
CHAPTER 4. GRAVITATIONAL RADIATION

radiates away energy.

The early calculations of gravitational radiation where completed using the simplification of non-relativistic motion of a particle in flat space-time (see Kenyon [56]). Despite these restrictions the calculations appeared to give reasonable results. Later, when fully relativistic calculations were made, this proved to be the case.

Zerilli's derivation of a wave equation led the way to solving this problem in a fully relativistic framework. Solution of the wave equation gives the different multipole components of the gravitational radiation. Davis et al [26] solved the wave equations using the Green's function technique, laid out by Zerilli, and also directly using a numerical search routine. This was the first in a series of papers which covered the pulses of gravitational radiation released (Davis et al [27]), the difference in radiation when the particle is projected in (Ruffini[28]) and the radiation from circular orbits (for example see Fitchett and Detweiler[30]).

4.2 Radiation from a mass falling radially

Davis et al [27] use a numerical search routine to solve the wave equation (2.37) directly without use of a Green's function. They choose a phase and amplitude at one boundary (ingoing or outgoing wave) which exactly gives a wave (outgoing or ingoing) at the other boundary. The boundary conditions are equivalent to specifying that the radiation should only come from the particle - black hole system and not from any external object. Using these boundary conditions demands that the radiation is 'outgoing' in the sense that it radiates away from the particle i.e. down the hole and out to infinity.
CHAPTER 4. GRAVITATIONAL RADIATION

The boundary conditions can be written as,

\[ R_L \rightarrow \begin{cases} 
A_L^{\text{out}}(\omega)e^{i\omega r} & \text{as } r^* \to \infty \\
A_L^{\text{in}}(\omega)e^{-i\omega r} & \text{as } r^* \to -\infty 
\end{cases} \]

where \( R_L \) is the solution of the wave equation.

The numerical technique employed requires the wave equation to be split into real and imaginary parts. The real and imaginary parts of the source term (equation 2.43) for a particle falling radially are,

\[
S_{LM}^{\text{real}} = -\frac{4m_0\sqrt{(L + 1/2)(1 - \frac{2m}{r})}}{\lambda r + 3m} \left( \frac{r}{2m} \cos \omega T + \frac{2\lambda \sin \omega T}{\omega(\lambda r + 3m)} \right) \tag{4.1}
\]

\[
S_{LM}^{\text{imag}} = -\frac{4m_0\sqrt{(L + 1/2)(1 - \frac{2m}{r})}}{\lambda r + 3m} \left( \frac{r}{2m} \sin \omega T - \frac{2\lambda \cos \omega T}{\omega(\lambda r + 3m)} \right) \tag{4.2}
\]

where

\[ T(r) = -\frac{4m}{3} \left( \frac{r}{2m} \right)^{3/2} - 4m \left( \frac{r}{2m} \right)^{1/2} + 2m \ln \left( \left[ \left( \frac{r}{2m} \right)^{1/2} + 1 \right] \left[ \left( \frac{r}{2m} \right)^{1/2} - 1 \right]^{-1} \right) \]

and \( \lambda = (L + 2)(L - 1)/2 \).

Splitting the wave equation into real and imaginary parts makes the coupled equations easier to solve, because the boundary conditions are then simpler to check. For an ingoing wave the imaginary part must appear to follow behind the real part, while for an outgoing wave the imaginary part leads the real part, with the difference between the real and imaginary parts being \( \pi/2 \) in each case. This is shown in figure 4.2 for a specific value of omega for which we have completed this search successfully.

Figure 4.2 shows a solution to the wave equation

\[
\frac{d^2 \tilde{R}_{LM}}{dr^*} + \left[ \omega^2 - V_L(r) \right] \tilde{R}_{LM} = S_{LM} \tag{4.3}
\]

where

\[ V_L = \left( 1 - \frac{2m}{r} \right) \frac{(2\lambda^2(\lambda + 1)r^3 + 6\lambda^2mr^2 + 18\lambda m^2r + 18m^3)}{r^3(\lambda r + 3m)^2} \]
Figure 4.1: The real (solid) and imaginary (dashed) parts of the source term coming from a particle falling radially in a Schwarzschild background.

where $\omega m = 0.3, L = 2$. The dimensionless quantity $\omega m$ is used to enable the frequency peak of any black hole mass to be read from the results. As $r^*$ increases the solution takes on a wave-like form with the imaginary part leading the real part, which shows an outgoing wave. For negative $r^*$ the solution is also a wave with the imaginary part following the real part. Thus, for negative $r^*$ the solution to the wave equation is a wave ingoing i.e. down the black hole. This solution was found by choosing the phase and amplitude of the ingoing wave so that the solution satisfies the boundary conditions at large $r^*$. 
Figure 4.2: The wave function for $\omega m = 0.3, L = 2$ where the radiation function is $\tilde{R}_{LM}$ (equation 4.3). The real part is the solid line while the imaginary is the dashed line.

4.3 The Downhill Simplex Method

Davis et al[26] make no comment about their numerical method employed to solve these equations. However, they test that the boundary conditions are satisfied by checking if

$$|A_L(\omega)|^2_{\text{max}} = |A_L(\omega)|^2_{\text{min}} .$$

(4.4)

They state that if the boundary conditions are achieved then equation (4.4) will hold at $r^* \to \pm \infty$. Addition of the real and imaginary parts of the solution leads to the amplitude of the solution. If the amplitude is constant then the solution has a wave form. Therefore provided this condition holds the system has been
solved.

The condition (4.4) is therefore ideal for use with a numerical search technique. A phase and amplitude at one boundary can be chosen and the condition checked for a wave form at the other boundary. Once the numerical method has converged the system is solved. However, Davis et al [26] did not consider that the condition (4.4) will also allow convergence to a wave of the wrong direction. Therefore, while it is possible to guarantee a wave at both boundaries, it is not possible with this method alone to guarantee an ingoing wave at $r^* = -\infty$ and an outgoing wave at $r^* = \infty$ (see figure 4.3).

![Figure 4.3](image)

Figure 4.3: The wave function for $\omega m = 0.35, L = 2$ where the radiation function is $\tilde{R}_{LM}$ (equation 4.3). Using the convergence condition of Davis et al [26] causes the algorithm to converge to a solution with the wrong boundary conditions.

Another approach to checking the boundary conditions is to check the phase
difference of the real and imaginary parts instead of the amplitude

\[
\frac{dR_{\text{real}}}{dr^*} = -\frac{R_{\text{imag}}}{\omega}.
\]  

Using this condition there is no ambiguity between different wave directions at the boundaries. However, an ideal method would use both conditions: the phase condition of equation (4.5) to begin with in order to pick out the right solution to converge to, switching to the amplitude condition used by Davis et al. [26] later for efficiency. This will guarantee that both the phase and amplitude of the outgoing wave satisfies the boundary conditions. These considerations suggest that a numerical method which allows changes in the convergence conditions would be most suited to this problem.

The Downhill Simplex Method (DSM) [57] to be described below has several advantages for this problem. Unlike steepest descent methods it does not require the derivative of the solution with respect to the phase or amplitude. It is also suited to restarting, at which point a different condition can be used. The DSM starts with a number of initial conditions. The number of initial conditions required is the dimension of the solution space plus one. The DSM uses four types of change to move these initial conditions towards a local minimum. In figure 4.4 a two dimensional problem is illustrated and therefore three initial conditions are required to enable a local minimum to be calculated. In the case of gravitational radiation both the phase and amplitude of the ingoing wave need to be given. As the method iterates it moves the phase and amplitude of the ingoing wave until the conditions at the other boundary are met. In this way the method finds an outgoing wave at the other boundary. The second order wave equation is first split into two coupled first order equations. To this a third first order equation
Figure 4.4: Different ways in the simplex is changed using the DSM. a) A 'reflection' away from the worst point. b) A 'reflection' and elongation to increase the area of the simplex, making it converge quicker. c) A contraction of the simplex away from the worse point. d) A contraction of the simplex towards the best point.
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Table 4.1: The results of the DSM when calculating the results from a non-spinning particle falling radially into a Schwarzschild black hole.
is added which specifies the relationship between \( r \) and \( r^* \):

\[
\frac{dR}{dr^*} = Z
\]

\[
\frac{dZ}{dr^*} = -[\omega^2 - V(r)]R + S_{LM}
\]

\[
\frac{dr}{dr^*} = \frac{r - 2m}{r}.
\]

These equations can be integrated using the variable \( r^* \) rather than \( r \). For the full Matlab code used see the appendix B.

Firstly the method was used to recalculate the radiation investigated by Davis et al [26]. The results in obtained are found in table 4.1. The energy is given by

\[
\left( \frac{dE}{d\omega} \right)_{2^L-pole} = \frac{1}{32\pi (L - 2)} \omega^2 |A_{L}^{out}(\omega)|^2,
\]

which has been plotted in figure 4.5 as a function of \( \omega \) for \( L = 2 \).

The results generated by the Matlab code show excellent agreement with the Davis et al[26] and therefore act as validation of the code.

### 4.4 Radiation from a Spinning Particle

Some of the differences between the radial infall of a spinning particle and non­spinning particle are seen by changes to the black hole. We might also expect a difference in the gravitational radiation release from the system. The curvature terms in the wave equations remain the same for both systems because the background space-time is still Schwarzschild. However, the source term will change for each different energy momentum tensor. The terms for the electric and magnetic source for a spinning particle are given below.
Figure 4.5: The energy \( \frac{1}{m_0^2} \frac{dE}{d\omega} \) released by the oscillation of the black hole event horizon for a particle falling in radially. The \( L = 2 \) multipole only is shown.

4.4.1 The Electric Source Term

When a spin component is added to the energy momentum tensor the source for the electric equation changes. The source term is still based on the perfect fluid and so the terms associated with the nonspinning particle remain the same. Setting \( s = 0 \) in any of the following equations reverts them to the non-spinning case.

For a spinning particle radially infalling the electric part of the source is found from equation (3.34),
\[ S_{LM} = \frac{4m_0 \sqrt{2} \pi Y_{LM}}{\lambda r + 3m} \left( 1 - \frac{2m}{r} \right) \left[ \sqrt{\frac{r}{2m}} - \frac{2i \lambda}{\omega (\lambda r + 3m)} \right] e^{i \omega T(r)} \]

\[ + \frac{s \sqrt{\pi}}{r(\lambda r + 3m)(\lambda + 1)} \frac{\partial Y_{LM}}{\partial \phi} \left( 1 - \frac{2m}{r} \right) \left[ \frac{r + 4m}{\sqrt{2m}} \right] e^{i \omega T(r)}. \] (4.7)

As before this can be split into real and imaginary parts and used in the calculation of gravitational radiation. The contribution to the radiation from the nonspinning part of equation (4.7) is already known and was calculated in section 4.2. Therefore, only the contribution from the spin part need be calculated. A solution of the spin parts of this equation has been found using the DSM and the following results have been obtained. Clearly the spin makes a contribution to the overall gravitational radiation for the electric parity.

Figure 4.6: The energy \((1/s^2)dE/d\omega\) released, calculated from the differential equation for electric parity harmonics. The \(L = 2\) multipole only is shown.
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Table 4.2: The spin contribution to the gravitational radiation of the electric parity equations for a spinning particle falling radially into a Schwarzschild black hole.
4.4.2 The Magnetic Source Term

A spinning particle appears as if it has angular momentum even when it is radially falling. The black hole acquires a small rotation (Chapter 3) and this effect is also demonstrated in the gravitational radiation. For a spinless particle falling in radially there is no magnetic source term for the gravitational radiation. For a spinning particle this source term is non-zero. The magnetic source is given by,

\[ S_{LM}^{(m)\text{real}} = \sqrt{2\pi} \left(1 - \frac{2m}{r}\right) \frac{s \sin \theta}{r^3 (\lambda + 1)} \frac{dY_{LM}}{d\theta} \cos \omega T, \]  

\[ S_{LM}^{(m)\text{imag}} = \sqrt{2\pi} \left(1 - \frac{2m}{r}\right) \frac{s \sin \theta}{r^3 (\lambda + 1)} \frac{dY_{LM}}{d\theta} \sin \omega T. \]

The magnetic part of the radiation is therefore particularly interesting as the corresponding components in the non-spinning case are zero. The DSM was used to solve these equations for the real and imaginary parts of the radiation. The results are tabulated in table (4.3). These results show a small but previously uninvestigated source of magnetic multipole gravitational radiation. The energy peaks at a low frequency before exponentially decaying away to zero for higher frequencies.

4.5 Applications

The contribution to gravitational radiation from a single infalling particle is clearly negligible. Nevertheless, there may be some circumstances where radiating spinning particles in a dense medium may contribute a significant energy loss.

The order of magnitude of the spin terms in the electric source equation (4.7) are \( \hbar/L^2 \). The normal terms for the mass of the particle are of the order \( m/L \).
<table>
<thead>
<tr>
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Table 4.3: The results obtained by the DSM for the magnetic parity equations for a spinning particle falling radially into a Schwarzschild black hole.
Figure 4.7: The differential equation for magnetic parity harmonics are used to calculate the energy released. Total energy is the sum of the electric and magnetic components. The $L = 2$ multipole only is shown.

The ratio of these terms is

$$\text{Ratio of spin terms to mass terms} = \frac{\frac{\hbar c}{L^2}}{\frac{m c^2}{L}} = \frac{\frac{\hbar c}{m c}}{L} = \frac{\lambda_c}{L},$$

where $c$ has been re-introduced to exhibit dimensional consistency and $\lambda_c$ is the Compton wavelength. This shows that the spin terms are important only if the length scale over which the particle is accelerated is of the order of its Compton wavelength. This order of magnitude estimate is consistent (within a factor of 10) with the exact ratio found in section 4.4.

In a spherical collapse there is no gravitational radiation in general relativity. Thus, predictions of gravitational radiation from supernovae assume a non-spherical collapse phase. If, however, spinning particles were aligned by a mag-
netic field, the spin moment term could differ from zero, even in spherical col-
lapse. Furthermore, the Compton wavelength of electrons is ~ 1000 times the
mean separation of material at neutron star densities (which is of order of the
neutron Compton wavelength). It is therefore possible that the infall of partially
aligned electron spins in quasi-spherical stellar collapse on to a neutron star core
could produce accelerations large enough that the spin component would domi-
minate the radiation and for sufficiently dense material, be of comparable magnitude
to the spherical general relativity term. Furthermore, this radiation would, in
principle, be identifiable by its magnetic parity. Observations of gravitational
radiation might therefore provide a test of contact torsion theory.
Appendix A

The Maple V Code

This Maple code calculates the perturbation equations for a Schwarzschild background. It shows that the equations published by Zerilli[1] are incorrect.

\[ \text{with(tensor):} \]

Definition of coordinates.

\[ x_0 := t; x_1 := r; x_2 := \theta; x_3 := \phi; \text{coord := [t, r, \theta, \phi]} \]

Definition of the metric tensor.

\[ g := \text{create([-1,-1], eval(g_compts))}; \]
APPENDIX A. THE MAPLE V CODE

\[ g := \text{table}([[ \begin{array}{cccc} -1 + 2 \frac{M}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{r} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin(\theta)^2 \end{array} ] \]) \]

\[ \text{index\_char} = [-1, -1] \]

Metric inverse.
\[ > \text{ginv} := \text{invert}(g, '\text{detg}'); \]

\[ \text{ginv} := \text{table}([[ \begin{array}{cccc} \frac{r}{r - 2M} & 0 & 0 & 0 \\ 0 & \frac{r - 2M}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin(\theta)^2} \end{array} ] \]) \]

\[ \text{index\_char} = [1, 1] \]

Gammas.
\[ > \text{dig} := \text{dimetric}(g, \text{coord}); \]
\[ > \text{cs1} := \text{Christoffel1}(\text{dig}); \]
\[ > \text{cs2} := \text{Christoffel2}(\text{ginv}, \text{cs1}); \]

Metric, metric inverse, gammas are index adjusted.
\[ > \text{read} 'e:/mapleWork/mapleProcedures/indexAdjustments.txt'; \]
\[ > \text{adjust2}(g, g); \]
\[ > \text{adjust2}(\text{ginv}, \text{ginv}); \]
\[ > \text{adjust3}(\text{cs2}, \text{gamma}); \]
The Riemann tensor is calculated and index adjusted.

> read 'e:/mapleWork/mapleProcedures/
> adjustedRiemannTensor.txt';
> Adjusted_Riemann_Tensor(cs2,R):

A Procedure for calculating a double covariant derivate.

> DCD:=proc(a,b,c,d) local i,j,a1,a2,a3,
> a4,a5,a6,a7,a8,a9,a10,a11,a12,a13,a14,total_sum;
> total_sum:=P.a.b.c.d;
> for i from 0 to 3 do
> a2:=-simplify(P.i.b*diff(gamma.i.a.c,x.d));
> a3:=-simplify(gamma.i.a.c*P.i.b.d);
> a4:=-simplify(P.a.i*diff(gamma.i.b.c,x.d));
> a5:=-simplify(gamma.i.b.c*P.a.i.d);
> a6:=-simplify(gamma.i.a.d*P.i.b.c);
> a9:=-simplify(gamma.i.a.d*P.i.b.c);
> a12:=-simplify(gamma.i.c.d*P.a.b.i);
> total_sum:=total_sum+a2+a3+a4+a5+a6+a9+a12;
> for j from 0 to 3 do
> a7:=simplify(gamma.i.a.d*gamma.j.i.c*P.j.b);
> a8:=simplify(gamma.i.a.d*gamma.j.b.c*P.i.j);
> a10:=simplify(gamma.i.b.d*gamma.j.a.c*P.j.i);
> a11:=simplify(gamma.i.b.d*gamma.j.i.c*P.a.j);
> a13:=simplify(gamma.i.c.d*gamma.j.a.i*P.j.b);
> a14:=simplify(gamma.i.c.d*gamma.j.b.i*P.a.j);
> total_sum:=total_sum+a7+a8+a10+a11+a13+a14;
> od;
> od;
> end:
> mu:=0;nu:=0;

\[
\mu := 0 \\
\nu := 0
\]

Einstein's equations for the Zerilli equation is now calculated

> Zerilli_Einsteins_Equations:=proc() local alpha,beta,delta,
> lambda,grand_total,gt1,gt2,gt3,gt4,gt5,gt6,gt7;
> grand_total:=0;
APPENDIX A. THE MAPLE V CODE

> for alpha from 0 to 3 do
>   for beta from 0 to 3 do
>     gt1:=simplify(ginv.alpha.beta*DCD(mu,nu,alpha,beta));
>     gt2:=-simplify(ginv.alpha.beta*DCD(mu,alpha,beta,nu));
>     gt3:=-simplify(ginv.alpha.beta*DCD(nu,alpha,beta,nu));
>     gt4:=simplify(ginv.alpha.beta*DCD(beta,alpha,nu,mu));
>     grand_total:=grand_total+gt1+gt2+gt3+gt4;
>   for delta from 0 to 3 do
>     gt5:=-simplify(2*ginv.alpha.beta*R.delta.mu.beta.nu*P.delta.alpha);
>     grand_total:=grand_total+gt5;
>   for lambda from 0 to 3 do
>     gt6:=simplify(g.mu.nu*ginv.lambda.delta*
>       ginv.alpha.beta*DCD(lambda,alpha,beta,lambda));
>     gt7:=-simplify(g.mu.nu*ginv.lambda.delta*
>       ginv.alpha.beta*DCD(beta,alpha,lambda,delta));
>     grand_total:=grand_total+gt6+gt7;
>   od;
> od;
> od;
> grand_total;
> end:
> ZEE1:=Zerilli_Einsteins_Equations();

Introduction of the perturbation tensor.
Defines the Even perturbation tensor.
> P.0.0:=(1-2*M/r)*Y(theta,phi)*H0(t,r);
> P.1.1:=1/(1-2*M/r)*Y(theta,phi)*H2(t,r);
> P.2.2:=r^2*Y(theta,phi)*K(t,r);
> P.3.3:=r^2*Y(theta,phi)*K(t,r);
> P.0.1:=Y(theta,phi)*H1(t,r);
> P.1.0:=P.0.1;

\[ P_{00} := (1 - 2 \frac{M}{r}) Y(\theta, \phi) H0(t, r) \]
\[ P11 := \frac{Y(\theta, \phi) H2(t, r)}{1 - 2 \frac{M}{r}} \]

\[ P22 := r^2 Y(\theta, \phi) K(t, r) \]

\[ P33 := r^2 \sin(\theta)^2 Y(\theta, \phi) K(t, r) \]

\[ P01 := Y(\theta, \phi) H1(t, r) \]

\[ P10 := Y(\theta, \phi) H1(t, r) \]

> `P.1.3:=0; P.3.1:=0; P.3.2:=0; P.2.3:=0; P.3.0:=0; P.0.3:=0;`
> `P.2.0:=0; P.0.2:=0; P.2.1:=0; P.1.2:=0;`
> `P13 := 0`
> `P31 := 0`
> `P32 := 0`
> `P23 := 0`
> `P30 := 0`
> `P03 := 0`
> `P20 := 0`
> `P02 := 0`
> `P21 := 0`
> `P12 := 0`

This calculates the differentials of the P.i.j.

> `for i from 0 to 3 do`
> `for j from 0 to 3 do`
> `for k from 0 to 3 do`
> `for l from 0 to 3 do`
> `P.i.j.k.l:=diff(P.i.j,x.k,x.l);`
> `od;`
> `P.i.j.k:=diff(P.i.j,x.k);`
> `od;`
> `od;`
> `od;`
> `ZEE1:

Simplification of the Zerilli equations leads to:
Substitutes in H.i.j for the differential diff(H.i,x.j).

> `substitute:=proc() local i,j,k; global ZEE2;`
> `ZEE2:=ZEE1;`
> `for i from 0 to 3 do`
APPENDIX A. THE MAPLE V CODE

> for j from 0 to 3 do
>   for k from 0 to 3 do
>     ZEE2 := subs(diff(H.i(t, r), x.j, x.k) = H.i.j.k(t, r), ZEE2);
>   od;
> od;
> ZEE2 := subs(diff(H.i(t, r), x.j) = H.i.j(t, r), ZEE2);
> ZEE2 := subs(diff(K(t, r), x.i, x.j) = K.i.j(t, r), ZEE2);
> ZEE2 := subs(diff(Y(theta, phi), x.i, x.j) = Y.i.j(theta, phi), ZEE2);
> od:
> end:

This line calls the substitute procedure and defines it to be ZEE2
> ZEE2 := substitute():

Collecting all the different terms and defining it to be ZEE3
> ZEE3 := collect(ZEE2, [Y, Y2, Y3, K, H0, H1, H2, H00, H01, H10, H11, H20, H200, H21, H210, H211], distributed):

This is the simpleZEE procedure which takes each co-efficient and simplifies it
> simpleZEE := proc() local a, b, c, d, temp, temp2, total, final;
> total := 0;
> temp := collect(simplify(coeff(ZEE3, K(t, r))), [Y, Y2, Y3, Y22, Y33, Y2.2, Y3.3]);
> temp2 := simplify(coeff(temp, Y(theta, phi)))*Y(theta, phi);
> temp2 := temp2 + simplify(coeff(temp, Y2(theta, phi)))*Y2(theta, phi);
> temp2 := temp2 + simplify(coeff(temp, Y3(theta, phi)))*Y3(theta, phi);
> temp2 := temp2 + simplify(coeff(temp, Y2.2(theta, phi)))*Y2.2(theta, phi);
> temp2 := temp2 + simplify(coeff(temp, Y2.3(theta, phi)))*Y2.3(theta, phi);
> temp2 := temp2 + simplify(coeff(temp, Y3.2(theta, phi)))*Y3.2(theta, phi);
> temp2 := temp2 + simplify(coeff(temp, Y3.3(theta, phi)))*Y3.3(theta, phi);
> total := total + temp2*K(t, r);
> for a from 0 to 2 do
APPENDIX A. THE MAPLE V CODE

> temp:=collect(simplify(coeff(ZE3,H.a(t,r))),[Y,Y.2,Y.3,
   Y.2.2,Y.2.3,Y.3.2,Y.3.3]);
> temp2:=simplify(coeff(temp,Y(theta,phi)))*Y(theta,phi);
> temp2:=temp2+simplify(coeff(temp,Y.2(theta,phi)))*
   Y.2(theta,phi);
> temp2:=temp2+simplify(coeff(temp,Y.3(theta,phi)))*
   Y.3(theta,phi);
> temp2:=temp2+
   simplify(coeff(temp,Y.2.2(theta,phi)))*
   Y.2.2(theta,phi);
> temp2:=temp2+
   simplify(coeff(temp,Y.2.3(theta,phi)))*
   Y.2.3(theta,phi);
> temp2:=temp2+
   simplify(coeff(temp,Y.3.2(theta,phi)))*
   Y.3.2(theta,phi);
> temp2:=temp2+
   simplify(coeff(temp,Y.3.3(theta,phi)))*
   Y.3.3(theta,phi);
> total:=total+temp2*H.a(t,r)
   +simplify(coeff(ZE3,K.a(t,r)))*K.a(t,r);
> for b from 0 to 3 do
   total:=total+simplify(coeff(ZE3,H.a.b(t,r)))*
   H.a.b(t,r)
   +simplify(coeff(ZE3,K.a.b(t,r)))*K.a.b(t,r);
> for c from 0 to 3 do
   total:=total
   +simplify(coeff(ZE3,H.a.b.c(t,r)))*H.a.b.c(t,r);
   od;
> od;
> final:=subs((-1+cos(theta)^2)=-sin(theta)^2,total);
>
end:
The simplification procedure is called and the results
defined to be SZEE and displayed.
> SZEE:=simpleZEE();
\[
SZEE := \left(2 \frac{(r - 2M) Y(\theta, \phi)}{r^3} + \frac{\cos(\theta) (r - 2M) Y2(\theta, \phi)}{\sin(\theta) r^3} + \frac{(r - 2M) Y22(\theta, \phi)}{r^3} \right)
+ \left(\frac{(r - 2M) Y33(\theta, \phi)}{r^3 \sin(\theta)^2}\right) K(t, r) + 2 \frac{Y(\theta, \phi) (-11M r + 10M^2 + 3r^2) K1(t, r)}{r^3}
+ 2 \frac{(r^2 - 4Mr + 4M^2) Y(\theta, \phi) K11(t, r)}{r^2} + (-2) \frac{(r - 2M) Y(\theta, \phi)}{r^3}
+ \frac{\cos(\theta) (r - 2M) Y2(\theta, \phi)}{\sin(\theta) r^3} + \frac{(r - 2M) Y22(\theta, \phi)}{r^3} + \frac{(r - 2M) Y33(\theta, \phi)}{r^3 \sin(\theta)^2}
\]
\[
H2(t, r) - 2 \frac{Y(\theta, \phi) (r^2 - 4Mr + 4M^2) H21(t, r)}{r^3}
\]
Appendix B

The Matlab Code

This matlab code implements the Downhill Simplex Method. It is used to solve the equations describing the gravitational radiation.

B.1 searcher.m

```matlab
function z = searcher()

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% %
% THIS FUNCTION IMPLEMENTS %
% THE SIMPLEX SEARCH ROUTINE %
% FOR FINDING LOCAL EXTREMES %
% %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% For an explanation see Numerical Recipes for FORTRAN p289-293

% Parameters for the search algorithm
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
LASTTOL=0.0005; % the function tolerance which can end the algorithm
FIRSTTOL=0.002; % the function tolerance of the phase shift
NDIM=2; % number of dimensions.
ALPHA=1.0; % three parameters
BETA=0.5; % which describe the
GAMMA=1.5; % expansions and contractions.
ITMAX=500; % max number of iterations.
SAME_ERRORS=25; % max number of times the same error can be produced
RESTART_ERRORS=15; % this many of the same errors restart the system
```
APPENDIX B. THE MATLAB CODE

% this is the loop for different omega values
for loopindex=1:1:1

%omega=loopindex*0.05;
omega=0.4;
L=2;
no_same_errors=0;
restart=0;
endnow=0;
iter=0;  % iteration counter

% Initial Calculations of the Wave Function for this Omega
P(1,1)=12.4;  % initial amplitude 1
P(1,2)=-1.8;  % initial phase 1
P(2,1)=12.5;  % initial amplitude 2
P(2,2)=-1.7;  % initial phase 2
P(3,1)=12.4;  % initial amplitude 3
P(3,2)=-1.8;  % initial phase 3

total_wave_output(1,1:4)=wave_function(P(1,1), P(1,2),L,omega,restart);
total_wave_output(2,1:4)=wave_function(P(2,1), P(2,2),L,omega,restart);
total_wave_output(3,1:4)=wave_function(P(3,1), P(3,2),L,omega,restart);

y(1)=total_wave_output(1,1);
y(2)=total_wave_output(2,1);
y(3)=total_wave_output(3,1);

% display the omega value so far
ILO=1;
while (endnow==0)

last_lowest_error=y(ILO);  % stores lowest error from last calculation

% If the system has got stuck then restart it
if (restart==1)
output='System Restarted'
iterations=iter  % the system time to converge
restart=2;  % this stops the system restarting again

temp_P_amp=P(ILO,1);
temp_P_phase=P(ILO,2);

P(1,1)=temp_P_amp+0.1;  % Reset the points
P(1,2)=temp_P_phase+0.1;
P(2,1)=temp_P_amp+0.11;
P(2,2)=temp_P_phase+0.1;
P(3,1)=temp_P_amp+0.1;
P(3,2)=temp_P_phase+0.11;

% recalculate the results

total_wave_output(1,1:4)=wave_function(P(1,1), P(1,2),L,omega,restart);
total_wave_output(2,1:4)=wave_function(P(2,1), P(2,2),L,omega,restart);

end

endnow=1;
APPENDIX B. THE MATLAB CODE

```

total_wave_output(3,1:4)=wave_function(P(3,1),P(3,2),L,omega,restart);

total_wave_output

y(1)=total_wave_output(1,1);  % recalculate the errors
y(2)=total_wave_output(2,1);
y(3)=total_wave_output(3,1);

no_same_errors=0;
end

% Sort the Values into the Highest (worst), next highest and lowest

IL0=1;  % Initial lowest point

if (y(1) > y(2))
    IHI=1;
    INHI=2;
else
    IHI=2;
    INHI=1;
end

for k = 1:1:NDIM+1
    if (y(k) < y(IL0)) IL0=k;
    end
    if (y(k) > y(IHI))
        INHI=IHI;
        IHI=k;
        elseif (y(k) > y(INHI))
            if (k ~= IHI) INHI=k;
            end
    end
end

% Test the different restart and end conditions

if (no_same_errors==RESTART_ERRORS)
    if (restart ==0)
        restart=1;
    end
end

test_error=abs(y(IHI)-y(IL0))/abs(y(IHI)+y(IL0));
if (test_error < FIRSTTOL)
    if (restart == 0)
        restart=1;
    end
end

no_phase_finished=0;
for k=1:1:NDIM+1
    if (y(k)<FIRSTTOL)
        no_phase_finished=no_phase_finished+1;
    end
end
```
if (no_phase_finished==3) % all three errors passed and so
    restart=1; % this loop ends at the next condition
end

no_finished=0;
for k=1:1:NDIM+1 % are all the errors below a certain value?
    if (y(k)<LASTTOL)
        no_finished=no_finished+1;
    end
end

if (no_finished==3) % all three errors passed and so this loop ends
    output='Required Accuracy Achieved' % at the next condition
    iterations=iter
    endnow=1;
    break;
end

if (iter==ITMAX) % check the number of iterations is below the max
    output='Iteration Level Exceeded'
    endnow=1;
end

if (last_lowest_error==y(ILO)) % Checks if same lowest error as last time
    no_same_errors=no_same_errors+1; % yes so same error counter incremented
else
    no_same_errors=0;
end

if (no_same_errors==SAME_ERRORS) % best error 15 times so loop ends
    output='Same Lowest Error'
    if (restart==2) % provided haven't just restarted calculation can finish
        iterations=iter
        endnow=1;
        break;
    else
        restart=3; % if we have just restarted then ignore this lowest error
        end % and keep going
    end
end

iter=iter+1; % increment the loop counter

pbar(1:NDIM)=0;
for k=1:1:NDIM+1
    if (k ~= IHI)
        for l=1:1:NDIM
            pbar(l)=pbar(l)+P(k,l);
        end
    end
end

% Select a New Point and test it against the current points
for k=1:1:NDIM
    pbar(k)=pbar(k)/NDIM;
    PR(k)=(1+ALPHA)*pbar(k)-ALPHA*P(IHI,k);
APPENDIX B. THE MATLAB CODE

end

% evaluate the result at this new point
PRtotal_wave_output=wave_function(PR(1),PR(2),L,omega,restart);
YPR=PRtotal_wave_output(1); % and store the error here

if (YPR < y(ILO)) % if new point is better than best, try going further
    for k=1:1:NDIM
        pbar(k)=pbar(k)/NDIM;
P(k)=GAMMA*PR(k)+(1-GAMMA)*P(IHI,k);
    end

    % evaluate the result at this extended point
PRRtotal_wave_output=wave_function(PRR(1),PRR(2),L,omega,restart);
YPRR=PRRtotal_wave_output(1); % and store the error here

    if (YPRR < y(ILO)) % the extra extrapolation succeeded
        for k=1:1:NDIM
            P(IHI,k)=PR(k); % this first point replaces the worst one
        end
        total_wave_output(IHI,1:4)=PRRtotal_wave_output;
y(IHI)=YPRR; % and the error is also replaced
    else % the extra extrapolation didn’t work
        for k=1:1:NDIM
            P(IHI,k)=PR(k); % this first point replaces the worst one
        end
total_wave_output(IHI,1:4)=PRRtotal_wave_output;
y(IHI)=YPR;
end

elseif (YPR > y(INHI)) % if the new point is worse than second-highest
    if (YPR < y(IHI)) % but still better than the worst
        for k=1:1:NDIM
            P(IHI,k)=PR(k); % this first point replaces the best one
        end
total_wave_output(IHI,1:4)=PRRtotal_wave_output;
y(IHI)=YPR;
end

    for k=1:1:NDIM % defining inbetween point is a contraction of simplex
        PRR(k)=BETA*P(IHI,k)+(1-BETA)*pbar(1);
    end

    % evaluate the result at the inbetween point
PRRtotal_wave_output=wave_function(PRR(1),PRR(2),L,omega,restart);
YPRR=PRRtotal_wave_output(1); % and store the error here

    if (YPRR < y(IHI)) % now test inbetween point and accept if better
        for k=1:1:NDIM
            P(IHI,k)=PRR(k); % this inbetween point replaces the worst one
        end
total_wave_output(IHI,1:4)=PRRtotal_wave_output;
y(IHI)=YPRR;
else % can’t get rid of the high so contract around the best point
    for k=1:1:NDIM+1
        if (1 ~= ILO)
            for k=1:1:NDIM
                PR(k)=0.5*(P(IHI,k)+P(ILO,k));
            end
        end
    end
end
APPENDIX B. THE MATLAB CODE

```matlab
P(1,k)=PR(k);
end
total_wave_output(1,1:4)=wave_function(PR(1),
                                      PR(2),L,omega,restart);
y(1)=total_wave_output(1,1);
end
end
end

else  % if new point is a middle point then replace high point and loop
    for k=1:1:NDIM
        P(IHI,k)=PR(k);  % this inbetween point replaces the best one
    end
    total_wave_output(IHI,1:4)=PRtotal_wave_output;
y(IHI)=YPR;
end
end  % end the while loop

% take the best result of the three
total_results_store(loopindex,1:5)=[omega total_wave_output(1LO,1:4)]
end  % end of the for loop for different omega
z=total_results_store

B.2 wave function.m

function z=wave_function(amplitude_guess, phase_guess, L, omega, restart)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% %
%%% % ALGORITHM TO CALCULATE %
%%% % THE GRAVITATIONAL RADIATION %
%%% % USED BY THE INTERATIVE RM %
%%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%% Initial Setups %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

n=1000;
start_rstar=-50;
finish_rstar=500;
check_length=50;

M=1;
lambda=(L-1)*(L+2)/2;

%%% Calculations on initial setups %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

total_range=finish_rstar-start_rstar;
h=total_range/n;

%%% Calculations of the Range
```
APPENDIX B. THE MATLAB CODE

for k=0:1:n
    rstar_span(k+1)=start_rstar+h*k;
end

% Boundary Conditions
real_init_in_R=amplitude_guess*cos(omega*start_rstar+phase_guess);
real_init_in_dR=-omega*amplitude_guess*sin(omega*start_rstar+phase_guess);
imaginary_init_in_R=amplitude_guess*sin(omega*start_rstar+phase_guess);
imaginary_init_in_dR=-omega*amplitude_guess*cos(omega*start_rstar+phase_guess);

% The Second Order Equations
[rstar R_real]=ode45('realSecondOrder',rstar_span,
    [real_init_in_R; real_init_in_dR; 2.000000000010218],',M,L,omega);
[rstar R_imag]=ode45('imagSecondOrder',rstar_span,[imaginary_init_in_R;
    imaginary_init_in_dR; 2.000000000010218],',M,L,omega);

% Testing the Amplitudes and Phases
A_max=0;
A_min=Inf;
for k=n:-1:n-check_length
    A_test=abs(R_real(k,1))^2+abs(R_imag(k,1))^2;
    if (A_test>A_max) A_max=A_test;
    end
    if (A_test<A_min) A_min=A_test;
    end
end
A_amp_diff=abs(A_max-A_min);
if (restart==0) % if error has been same for while then must be close.
    A_phase_diff=0; % using the Amplitude as only error indicator
    for k=n:-1:n-check_length % to the correct solution
        A_test=abs(R_real(k,2)+R_imag(k,1)/omega);
        if (A_phase_diff<A_test) A_phase_diff=A_test;
        end
        total_error=A_phase_diff;
    end
else
    total_error=A_amp_diff;
end
store(1,1:4)=[total_error amplitude_guess phase_guess A_max^(1/2)];
z=store(1,1:4);
APPENDIX B. THE MATLAB CODE

B.3 imagSecondOrder.m

function coupledsystem = imagSecondOrder(r,y, flag,M,L,omega)

% A function which sets up the second order
% equation as a coupled first order systems

lambda = (L-1)*(L+2)/2;

% The potential
V = (1-2*M/y(3))*(2*lambda^2*(lambda+1)y(3)^3 + 6*lambda^2*M*y(3)^2
+18*lambda^2*M^2*y(3)+18*M^3)/(y(3)^3*(lambda*y(3)+3*M)^2);

% Term from the time of a radially infalling particle
T = -4/3*(y(3)/(2*M))^(3/2) - 4*(y(3)/(2*M))^(1/2)
+2*log((sqrt(y(3)/(2*M))+1)/(sqrt(y(3)/(2*M))-1));

% Imaginary Part of the Source Term
imaginary.source = 4/(lambda*y(3)+3*M)*sqrt(L+l/2)*(1-2*M/y(3))
*(sqrt(y(3)/(2*M))*sin(omega*T)-2*lambda*cos(omega*T)
/(omega*(lambda*y(3)+3*M)));

coupledsystem = [y(2); -(omega^2-V)*y(1)+imaginary_source; (y(3)-2*M)/y(3)];

B.4 realSecondOrder.m

function coupledsystem = realSecondOrder(r,y, flag,M,L,omega)

% A function which sets up the second order
% equation as a coupled first order systems

lambda = (L-1)*(L+2)/2;

% The potential
V = (1-2*M/y(3))*(2*lambda^2*(lambda+1)y(3)^3 + 6*lambda^2*M*y(3)^2
+18*lambda^2*M^2*y(3)+18*M^3)/(y(3)^3*(lambda*y(3)+3*M)^2);

% Term from the time of a radially infalling particle
T = -4/3*(y(3)/(2*M))^(3/2) - 4*(y(3)/(2*M))^(1/2)
+2*log((sqrt(y(3)/(2*M))+1)/(sqrt(y(3)/(2*M))-1));

% Real Part of the Source Term
real_source=4/(lambda*y(3)+3*M)*sqrt(L+l/2)*(1-2*M/y(3))
*cos(omega*T)+2*lambda*sin(omega*T)/(omega*(lambda*y(3)+3*M)));

coupledsystem = [y(2); -(omega^2-V)*y(1)+real_source; (y(3)-2*M)/y(3)];
Bibliography


