ON A CONSTRUCTION OF YOUNG MODULES

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On a Construction of Young Modules

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Abstract

Let $n$ be a natural number and $E$ an $n$-dimensional vector space over a field $K$. The symmetric group acts by place permutation on the tensor space $E^\otimes r$. The $\Sigma_r$-module $E^\otimes r$ can be decomposed into a direct sum of permutation modules $M^\lambda$ where $\lambda$ is a composition of $r$ into at most $n$ parts.

Each permutation module labelled by such a composition is isomorphic to one labelled be a partition of $r$ into at most $n$ parts, and therefore we assume that $\lambda$ is such a partition. The indecomposable direct summands of the permutation module $M^\lambda$ are called Young modules, and they are labelled by partitions of $r$ into at most $n$ parts.

Throughout this thesis we consider the case where $E$ has dimension two. For $\lambda$ a two-part partition of $r$, we explicitly decompose the module $M^\lambda$ into a direct sum of Young modules by providing spanning sets for the Young modules.

Moreover, we consider the problem of finding a basis or an algorithm for a basis for the Young modules in this case and, although we have not been able to solve this in general, we give some conjectures and examples showing in which cases we can find a basis.
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Introduction

This thesis is concerned with Young modules for symmetric groups. Young modules are the indecomposable direct summands of certain permutation modules for symmetric groups. As well as being important to the representation theory of symmetric groups, they are also useful in the study of Schur algebras and therefore in the representation theory of general linear groups. Young modules were first investigated by Klyachko [26] then by James [24] who parametrized the Young modules by partitions. James' proof uses Schur algebras, whereas an alternative proof given by Erdmann [9] only uses the representation theory of symmetric groups. Other authors who have studied Young modules include Donkin [2], Grabmeier [11], Green [13], Hemmer and Nakano [16], and Henke [17].

The aim of this thesis is to explicitly construct Young modules. An explicit definition for Young modules does not exist yet, and finding one would give us a different description of Young modules and help us to understand them better. Young modules, denoted by $Y^\lambda$, are parametrized by partitions and we will concentrate on the case of two-part partitions. Understanding this case could provide some ideas of how to approach the case of partitions in three or even more parts, which is still not well understood.

In this thesis we have provided an explicit construction for Young modules corresponding to two-part partitions in the case where $K$ is a field of characteristic two. This is given in Theorem 4.1.4 and Theorem 4.2.1, where the Young modules have been constructed by providing a spanning set. We will now explain in more detail the aims and structure of this thesis.

Let $r$ and $n$ be natural numbers, and denote the symmetric group on $r$ symbols by
Σ_r. If not otherwise stated, $K$ is any field and $K\Sigma_r$ is the group algebra of the symmetric group $\Sigma_r$ over $K$. Modules are taken to be right modules unless stated otherwise. Let $E$ be an $n$-dimensional vector space over $K$. In Section 1.6 we give the definition of the $K\Sigma_r$-module $E^{\otimes r}$, and our problem is to decompose this module into a direct sum of indecomposable $K\Sigma_r$-modules. We start Chapter 1 by introducing compositions and partitions, then define the permutation module $M^\lambda$, where $\lambda$ is a composition of $r$. It is known that the module $E^{\otimes r}$ can be decomposed into a direct sum of permutation modules, and this is described in Proposition 1.6.1.

Our problem of decomposing $E^{\otimes r}$ into a direct sum of indecomposable modules is reduced to that of decomposing permutation modules $M^\lambda$ into a direct sum of indecomposable modules for all partitions $\lambda$ of $r$ (using Proposition 1.6.1). In Section 2.2 we show that there is, up to isomorphism and order, a unique way of doing this. Moreover, in Remark 2.4.8, we see that if $\lambda$ is a two-part partition then the decomposition of $M^\lambda$ into a direct sum of indecomposable modules is actually unique (up to order). The indecomposable direct summands of $M^\lambda$ are called Young modules (see Definition 1.7.1).

In particular, we consider the case where $E$ is two-dimensional and $K$ is a field of prime characteristic $p$. Then $E^{\otimes r}$ can be decomposed into a direct sum of modules $M^\lambda$, where $\lambda = (r - k, k)$ for $0 \leq k \leq r$. We want to decompose the module $M^{(r-k,k)}$ into a direct sum of Young modules, and it is already known which Young modules occur in this decomposition (see Theorem 1.7.4). We would like to find an explicit way of defining these Young modules, as an explicit definition does not yet exist.

For $K$ a field of characteristic two we construct the Young module $Y^{(r-k,k)}$ in $M^{(r-k,k)}$ in Theorem 4.1.4, by giving a spanning set for $Y^{(r-k,k)}$. We then, in Theorem 4.2.1, go on to give a spanning set for the Young module $Y^{(r-s,s)}$ when it occurs as a direct summand of the permutation module $M^{(r-k,k)}$. These results are proved using the fact that the primitive orthogonal idempotents of $\text{End}_{K\Sigma_r}(M^{(r-k,k)})$ are known, and these are described in Section 2.4.

We have given a construction of the Young modules $Y^\lambda$ where $\lambda$ is a two-part partition and $K$ has characteristic two. We have done this by providing a spanning set for the Young module, but we would like to actually give a basis for these Young modules,
or some algorithm that would enable us to construct a basis. In Chapter 5, we use Theorem 4.1.4 to consider this problem in some specific cases, although we have not been able to solve it in general. We also discuss more generally the case where $K$ has characteristic $p$, where $p$ is prime, and look at some examples in Section 5.1.

Throughout, let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ be the set of all non-negative integers and $\mathbb{Z}$ be the set of integers. For a natural number $n$ we define $n := \{1, 2, \ldots, n\}$.

All of our maps will be written on the left, and we use the convention that if $f$ and $g$ are maps, then $g \circ f$ means first apply $f$ then apply $g$. In this way we have, for example, that $(12)(23) = (123)$. The convention that we choose here is not consistent with all of the references that we will use, however most of the results will be the same.
Chapter 1

Representation Theory of Symmetric Groups

We start this chapter with some combinatorics related to symmetric groups, and then describe permutation modules and Specht modules over the symmetric group. This is given in Sections 1.1, 1.4 and 1.5, most of which can be found in [23], but in [23] maps are written on the right. For $n$ and $r$ natural numbers we define the set $I(n,r)$ of multi-indices, and a $\Sigma_r$-action on this set in Section 1.2. We use this in Section 1.6 to define a $K\Sigma_r$-module $E^{\otimes r}$ (as in [12] Sections 2.1 and 2.6). We end this chapter by introducing Young modules.

1.1 Compositions and Partitions

For a natural number $r$, a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers is called a composition of $r$ if $\sum_{i=0}^{\infty} \lambda_i = r$. The elements $\lambda_i$ are called parts of $\lambda$ and the set of all compositions of $r$ is denoted by $\Lambda(r)$. If for some natural number $n$ we have $\lambda_i = 0$ for all $i > n$, we write $\lambda = (\lambda_1, \ldots, \lambda_n)$ and call it a composition of $r$ in $n$ parts. The set of all compositions of $r$ with at most $n$ parts is denoted by $\Lambda(n,r)$. A composition $\lambda$ is called a partition of $r$ if its parts are decreasing, so $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq 0$. The set of partitions of $r$ is denoted by $\Lambda^+(r)$ and that of partitions of $r$ with at most $n$ parts by $\Lambda^+(n,r)$. Given a composition $\lambda \in \Lambda(n,r)$ we can arrange the parts of $\lambda$ in descending order to get a partition $\overline{\lambda} \in \Lambda^+(n,r)$ called the partition associated to $\lambda$. 
If \( \lambda \) is a partition of \( r \) then the \textit{diagram} \([\lambda]\) is the set

\[
\{ (i, j) : i, j \in \mathbb{Z}, 1 \leq i, 1 \leq j \leq \lambda_i \}.
\]

If \((i, j) \in [\lambda]\) then \((i, j)\) is called a \textit{node} of \([\lambda]\). For \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^+(n, r) \) we can denote \([\lambda]\) by drawing a box for each node, with row \( k \) of \([\lambda]\) containing \( \lambda_k \) boxes, for \( 1 \leq k \leq n \). We will use the convention of drawing the diagrams with the first coordinate axis to the right and the second one downwards as in the following example: if \( \lambda = (5, 2, 2, 1) \in \Lambda^+(4, 10) \) then

\[
[\lambda] = \begin{array}{cccc}
\boxed{} & \boxed{} & \boxed{} & \boxed{} \\
\boxed{} & \boxed{} & \boxed{} & \boxed{} \\
\boxed{} & \boxed{} & \boxed{} & \boxed{} \\
\boxed{} & \boxed{} & \boxed{} & \boxed{}
\end{array}
\]

For a prime \( p \), a diagram \([\lambda]\) or a partition \( \lambda \) is called \textit{\( p \)-regular} if no \( p \) rows of \([\lambda]\) have the same length, otherwise \([\lambda]\) or \( \lambda \) is called \textit{\( p \)-singular}.

We now define a partial ordering, the \textit{dominance ordering}, on the set \( \Lambda^+(r) \) of all partitions of \( r \). If \( \lambda \) and \( \mu \) are partitions of \( r \), we say that \( \lambda \) \textit{dominates} \( \mu \), and write \( \lambda \geq \mu \), if for all \( j \in \mathbb{N} \)

\[
\sum_{i=1}^{j} \text{length of the } i^{th} \text{ row of } \lambda \geq \sum_{i=1}^{j} \text{length of the } i^{th} \text{ row of } \mu.
\]

If \( \lambda \geq \mu \) and \( \lambda \neq \mu \), we write \( \lambda > \mu \).

### 1.2 Multi-Indices

For natural numbers \( r \) and \( n \), we write \( I(n, r) \) for the set of all functions \( i : r \to n \). A function of this form is usually written as a \textit{multi-index} \( i = (i_1, i_2, \ldots, i_r) \) with values \( i_j \in \mathbb{N} \). So \( I(n, r) = \{ i = (i_1, i_2, \ldots, i_r) : i_j \in \mathbb{N} \text{ for } j \in r \} \). The symmetric group \( \Sigma_r \) acts on the right on \( I(n, r) \) by place permutation, so \( i\sigma = (i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(r)}) \) for all \( \sigma \in \Sigma_r \). Define a map

\[
\lambda : I(n, r) \to \Lambda(n, r) \quad \begin{array}{ccc}
i = (i_1, \ldots, i_r) & \mapsto & \mu = (\mu_1, \ldots, \mu_n)
\end{array}
\]
where, for \( j \in \mathbb{n} \), \( \mu_j \) is the number of \( \rho \in r \) such that \( i_\rho = j \). If \( \lambda(i) = \mu \) then \( \mu \) is called the content of the multi-index \( i \). As \( \sigma \in \Sigma_r \) acts on \( i \in I(n,r) \) by place permutation, we have that \( \lambda(i) = \lambda(i\sigma) \) for all \( \sigma \in \Sigma_r, i \in I(n,r) \).

**Example 1.2.1** Let \( i = (1,1,2,1,4) = (i_1,i_2,i_3,i_4,i_5) \in I(4,5) \), and \( \sigma = (25)(34) \in \Sigma_5 \). Then \( i\sigma = (i_\sigma(1),i_\sigma(2),i_\sigma(3),i_\sigma(4),i_\sigma(5)) = (i_1,i_5,i_4,i_3,i_2) = (1,4,1,2,1) \), and we have \( \lambda(i) = \lambda(i\sigma) = (3,1,0,1) \in \Lambda(4,5) \).

### 1.3 Binomial Coefficients and \( p \)-adic Decompositions

For integers \( n \) and \( k \) we define the binomial coefficient

\[
\binom{n}{k} := \begin{cases} 
\frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots(1)} & \text{if } k > 0, \\
1 & \text{if } k = 0, \\
0 & \text{if } k < 0.
\end{cases}
\]

In the following lemma we consider some relations that will be needed later (see for example [27], Section 1.2.6).

**Lemma 1.3.1** For integers \( n, k, r, m \) and \( s \) we have

1. for \( n \geq 0 \) we have \( \binom{n}{k} = \binom{n}{n-k} \), so \( \binom{n}{k} = 0 \) if \( 0 \leq n < k \),
2. \( \binom{r}{m} \binom{m}{n} = \binom{r}{n} \binom{r-n}{m-n} \),
3. \( \sum_k \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n} \),
4. for \( r \in \mathbb{N}_0 \), \( \sum_k \binom{r}{k} \binom{s+k}{n} (-1)^k = (-1)^r \binom{s}{n-r} \),
   in particular \( \sum_k \binom{r}{k} \binom{s+k}{n} \equiv \binom{s}{n-r} \mod 2 \),
5. for \( r, t, m \in \mathbb{N}_0 \), \( \sum_{k=0}^{r} \binom{r-k}{m} \binom{s}{k-t} (-1)^k = (-1)^t \binom{r-t-s}{r-t-m} \),
   in particular \( \sum_{k=0}^{r} \binom{r-k}{m} \binom{s}{k-t} \equiv \binom{r-t-s}{r-t-m} \mod 2 \).
Note that we have written \( \sum_k \) in (3) above rather than writing \( \sum_{k=0}^{n} \). If no restriction is placed on \( k \), we are summing over all integers \(-\infty < k < +\infty\). We see that when \( k < 0 \) or \( k > n \) we have \( \binom{n}{k}\binom{n-k}{m-k} = 0 \), so the two notions are equivalent in this case. We will use the form \( \sum_k \) where possible, leaving the limits as infinity, as it saves the effort of having to keep track of the lower and/or upper limits of summation, so making manipulation with summations simpler. This is also the case with part (4) of Lemma 1.3.1, but in part (5) we need the limits on the summation as given above.

Let \( p \) be a prime number, then given a non-negative integer \( n \) there is a unique decomposition of \( n \) as:

\[
n = \sum_{i \in \mathbb{N}_0} n_i p^i
\]

with \( 0 \leq n_i \leq p - 1 \). This is called the \( p \)-adic decomposition of \( n \). For non-negative integers \( n \) and \( m \) with \( p \)-adic decompositions \( n = \sum_{i \in \mathbb{N}_0} n_i p^i \) and \( m = \sum_{i \in \mathbb{N}_0} m_i p^i \), we say that \( m \) is \( p \)-contained in \( n \) if \( m_i \leq n_i \) for all indices \( i \in \mathbb{N}_0 \), and write \( m \subseteq_p n \).

Then we have:

**Lemma 1.3.2 (see for example [23], Lemma 22.4)** Let \( p \) be a prime and let \( n, m \in \mathbb{N}_0 \) with \( p \)-adic decompositions \( n = \sum_{i \in \mathbb{N}_0} n_i p^i \) and \( m = \sum_{i \in \mathbb{N}_0} m_i p^i \). Then \( \binom{n}{m} \equiv \prod_{i \in \mathbb{N}_0} \binom{n_i}{m_i} \mod p \). In particular, \( m \) is \( p \)-contained in \( n \) if and only if \( \binom{n}{m} \neq 0 \mod p \).

### 1.4 Tableaux, Tabloids and Polytabloids

For \( r \in \mathbb{N} \) and \( \lambda \) a partition of \( r \), a \( \lambda \)-tableau is a bijection \( T : [\lambda] \rightarrow r = \{1, \ldots, r\} \). If the image of \((i,j)\) is \( x_{i,j} \) we may denote \( T \) by

\[
T = \begin{array}{cccc}
x_{1,1} & x_{1,2} & \ldots & x_{1,\lambda_1} \\
x_{2,1} & x_{2,2} & \ldots & x_{2,\lambda_2} \\
\vdots & \vdots & \ddots & \vdots \\
x_{\lambda_1,1} & \ldots & \ldots & x_{\lambda_1,\lambda_1}
\end{array}
\]  

(1.1)

where each of the \( x_{i,j} \) are distinct numbers in \( r \). So a \( \lambda \)-tableau is denoted by replacing the nodes of \([\lambda]\) by numbers in the set \( r \) with each number only being used once. For \( \lambda \) a partition of \( r \), the symmetric group \( \Sigma_r \) operates on the set of \( \lambda \)-tableaux as follows:
For \( \pi \in \Sigma_r \) and \( T \) a \( \lambda \)-tableau we define \( T \cdot \pi := \pi^{-1} \circ T \), the composition of the bijection \( T : [\lambda] \to \tau \) and the permutation \( \pi^{-1} : \tau \to \tau \). So if \( T \) maps \((i, j)\) to \( x_{i,j} \) (as in (1.1)), then \( T \cdot \pi \) maps \((i, j)\) to \( \pi^{-1}(x_{i,j}) \). Hence

\[
T \cdot \pi = \begin{pmatrix}
\pi^{-1}(x_{1,1}) & \pi^{-1}(x_{1,2}) & \ldots & \ldots & \pi^{-1}(x_{1,\lambda_1}) \\
\pi^{-1}(x_{2,1}) & \pi^{-1}(x_{2,2}) & \ldots & \ldots & \pi^{-1}(x_{2,\lambda_2}) \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\]

The row stabilizer \( R_T \) of a tableau \( T \) is the subgroup of \( \Sigma_r \) consisting of all permutations which keep the rows of \( T \) fixed as a set. The column stabilizer \( C_T \) is defined similarly. We define an equivalence relation \( \sim \) on the set of all \( \lambda \)-tableaux by \( T \sim T' \) if and only if there exists a permutation \( \pi \in R_T \) such that \( T = T' \cdot \pi \). The equivalence class of \( T \) under \( \sim \) is called a tabloid or \( \lambda \)-tabloid and is denoted by \( \{T\} \). A tabloid can be regarded as a “tableau with unordered row entries”, and in examples we will denote the tabloid \( \{T\} \) by drawing horizontal lines between the rows of \( T \). We define a \( \Sigma_r \)-action on a \( \lambda \)-tabloid \( \{T\} \) by \( \{T\} \cdot \pi = \{T \cdot \pi\} \) for all \( \pi \in \Sigma_r \), and this action is well-defined.

For a tableau \( T \), let \( \kappa_T := \sum_{\pi \in C_T} \text{sgn}(\pi) \pi \) be the signed column sum. A polytabloid or \( \lambda \)-polytabloid, \( e_T \), associated with the tableau \( T \) is \( e_T := \{T\} \cdot \kappa_T \). In the next example we will see that \( e_T \) really depends on the tableau \( T \), and not just the tabloid \( \{T\} \).

**Example 1.4.1** Let \( \lambda = (2, 1) \), then \( T_1 = \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix} \) and \( T_2 = \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix} \) are \( \lambda \)-tableaux with

\[
\{T_1\} = \{T_2\} = \begin{pmatrix} \frac{1}{3} \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}.
\]

Now \( C_{T_1} = \{\text{id}, (13)\} \) and \( C_{T_2} = \{\text{id}, (23)\} \), so

\[
e_{T_1} = \{T_1\} - \{T_1 \cdot (13)\} = \begin{pmatrix} \frac{1}{3} \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \quad \text{and} \quad e_{T_2} = \{T_2\} - \{T_2 \cdot (23)\} = \begin{pmatrix} \frac{1}{3} \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{1}{3} \\ 2 \end{pmatrix}.
\]

So we see that \( e_{T_1} \neq e_{T_2} \).

A tableau \( T \) is *standard* if the numbers in \( T \) increase along the rows and down the columns. A tabloid \( \{T\} \) is *standard* if there is a standard tableau in the equivalence class, a polytabloid \( e_T \) is *standard* if \( T \) is standard.
1.5 Permutation Modules and Specht Modules

For $\lambda$ a partition of $r$ we define a $K$-vector space $M^\lambda$ with basis the set of all $\lambda$-tabloids. The action of $\Sigma_r$ on $\lambda$-tabloids has been defined in Section 1.4 by $\{T\} \cdot \pi := \{T \cdot \pi\}$ for all $\pi \in \Sigma_r$. Extending this action to be linear on $M^\lambda$ makes $M^\lambda$ into a $K\Sigma_r$-module, which is cyclic and generated by any one $\lambda$-tabloid. $M^\lambda$ is called the \textit{permutation module corresponding to} $\lambda$. The \textit{Specht module} $S^\lambda$ is then defined to be the subspace of $M^\lambda$ spanned by all the $\lambda$-polytabloids.

\textbf{Theorem 1.5.1} \textit{(see [23] (4.5), [30] Theorem 1.1)} \textit{The Specht module $S^\lambda$ is a cyclic module, generated by any one polytablroid. Moreover, $S^\lambda$ has $K$-basis}

$$\{e_T : T \text{ is a standard } \lambda - \text{tableau}\}.$$

The first part of this holds because, for $T$ a $\lambda$-tableau and $\pi \in \Sigma_r$ we have $e_T \cdot \pi = e_{T \cdot \pi}$.

\textbf{Example 1.5.2} \textit{(see [31], Examples 2.3.6 and 2.3.7)} These examples use Theorem 1.5.1 for the basis of $S^\lambda$.

1. If $\lambda = (r)$, then the only standard $\lambda$-tableau is $T = 1 \ 2 \ \ldots \ r$. Here $C_T = \{id\}$, so $\kappa_T = id$, and hence $e_T = \frac{1 \ 2 \ \ldots \ r}{1 \ 2 \ \ldots \ r}$ is the basis element of $S^{(r)}$. For all $\pi \in \Sigma_r$ we have $e_T \cdot \pi = e_T$, and hence we see that $S^{(r)}$ is the trivial representation.

2. If $\lambda = (1^r)$, then

$$T = \begin{array}{c}
1 \\
2 \\
\vdots \\
r
\end{array}$$

is the only standard tableau. Here $C_T = \Sigma_r$ and so

$$e_T = \sum_{\sigma \in \Sigma_r} \text{sgn}(\sigma) \{T \cdot \sigma\}.$$

It can be shown that $e_T \cdot \pi = \text{sgn}(\pi)e_T$ for all $\pi \in \Sigma_r$, and so $S^{(1^r)}$ is the sign representation.
For \( \lambda \in \Lambda^+(r) \), let \( \langle , \rangle \) be the bilinear form on \( M^\lambda \) given by
\[
\langle , \rangle : M^\lambda \times M^\lambda \to K
\]
\[
\langle \{T_1\}, \{T_2\} \rangle = \begin{cases} 1 & \text{if } \{T_1\} = \{T_2\}, \\ 0 & \text{otherwise} \end{cases}
\]
and extend bilinearly. Clearly, for \( x, y \in M^\lambda \), \( \langle x, y \rangle = \langle y, x \rangle \). Also for \( \pi \in \Sigma_r \),
\[
\langle \{T_1\} \cdot \pi, \{T_2\} \cdot \pi \rangle = \langle \{T_1 \cdot \pi\}, \{T_2 \cdot \pi\} \rangle = \langle \{T_1\}, \{T_2\} \rangle \) (as \( \{T_1 \cdot \pi\} = \{T_2 \cdot \pi\} \) if and only if \( \{T_1\} = \{T_2\} \)). So, extending bilinearly, for all \( x, y \in M^\lambda \) and \( \pi \in \Sigma_r \) we have
\[\langle x \cdot \pi, y \cdot \pi \rangle = \langle x, y \rangle .\]
Hence we have that \( \langle , \rangle \) is a symmetric \( \Sigma_r \)-invariant bilinear form on \( M^\lambda \).

For a \( K\Sigma_r \)-submodule \( U \) of \( M^\lambda \), define
\[
U^\perp := \{ x \in M^\lambda : \langle x, u \rangle = 0 \text{ for all } u \in U \}.
\]
To show that \( U^\perp \) is a \( K\Sigma_r \)-submodule of \( M^\lambda \), let \( v \in U^\perp \) and \( \pi \in \Sigma_r \). As \( \langle , \rangle \) is \( \Sigma_r \)-invariant we have \( \langle v \cdot \pi, u \rangle = \langle v \cdot \pi \cdot \pi^{-1}, u \cdot \pi^{-1} \rangle = \langle v, u \cdot \pi^{-1} \rangle \) for all \( u \in U \). Therefore, for all \( u \in U \), as \( u \cdot \pi^{-1} \in U \) and \( v \in U^\perp \) we have \( \langle v, u \cdot \pi^{-1} \rangle = 0 \). Therefore, for \( v \in U^\perp \) and \( \pi \in \Sigma_r \) we have \( v \cdot \pi \in U^\perp \), and so \( U^\perp \) is a submodule of \( M^\lambda \).

**Theorem 1.5.3 ([22], Theorem 1)** If \( U \) is a \( K\Sigma_r \)-submodule of \( M^\lambda \) then either \( U \supseteq S^\lambda \) or \( U \subseteq (S^\lambda)^\perp \).

Hence, if \( U \) is a \( K\Sigma_r \)-submodule of \( S^\lambda \), we have that either \( U = S^\lambda \) or \( U \subseteq S^\lambda \cap (S^\lambda)^\perp \). So we have that either \( S^\lambda \cap (S^\lambda)^\perp = S^\lambda \), or \( S^\lambda \cap (S^\lambda)^\perp \) is the unique maximal submodule of \( S^\lambda \), and therefore we get the following theorem.

**Theorem 1.5.4 ([22], Theorem 2)** Either \( S^\lambda \subseteq (S^\lambda)^\perp \) or \( S^\lambda /(S^\lambda \cap (S^\lambda)^\perp) \) is simple and \( S^\lambda \cap (S^\lambda)^\perp \) is the unique maximal submodule of \( S^\lambda \).

For a partition \( \lambda \) or \( r \), the module \( S^\lambda \) is defined over any field \( K \). If the characteristic of \( K \) is zero then \( S^\lambda \) is simple. Now consider the case of a field \( K \) with characteristic \( p \), where \( p \) is prime. Here we do not necessarily have that \( S^\lambda \) is simple. If \( S^\lambda \) is not simple, then either \( S^\lambda = S^\lambda \cap (S^\lambda)^\perp \) or \( S^\lambda \) has a unique maximal submodule, which by Theorem 1.5.4 is \( S^\lambda \cap (S^\lambda)^\perp \). The factor module \( S^\lambda /(S^\lambda \cap (S^\lambda)^\perp) = 0 \) if and only if \( \lambda \) is \( p \)-singular (see for example [23] Theorem 11.1). If \( \lambda \) is \( p \)-regular we define
\[
D^\lambda := S^\lambda /(S^\lambda \cap (S^\lambda)^\perp)
\]
which is nonzero, and is the simple head of the module $S^\lambda$. We then have the following theorem.

**Theorem 1.5.5** ([22], Theorems 3 and 6) *If* $\text{char}K = 0$ *then* $\{S^\lambda : \lambda \in \Lambda_p^+(r)\}$ *is a complete set of representatives of simple $KS_r$-modules.* *If* $\text{char}K = p$ *then* $\{D^\lambda : \lambda \text{ is a } p\text{-regular partition of } r\}$ *is a complete set of representatives of simple $K\Sigma_r$-modules.*

Let $K$ be a field of characteristic $p$. For $\lambda$ a partition of $r$, the composition factors of $S^\lambda$ will be $D^\mu$ for some $p$-regular partitions $\mu$ of $r$. For $\lambda$ a partition of $r$ and $\mu$ a $p$-regular partition of $r$ we would like to know how many times each simple module $D^\mu$ occurs as a composition factor of the Specht module $S^\lambda$. These are called the decomposition numbers of $K\Sigma_r$, and are denoted by $[S^\lambda : D^\mu]$. In general these decomposition numbers are unknown, but some results are known. In particular, if $\lambda$ and $\mu$ are two-part partitions, then $[S^\lambda : D^\mu]$ is known (see [23] Theorem 24.15, which uses [20] and [21]).

### 1.6 Permutation Modules as Submodules of $E^{\otimes r}$

Let $n$ and $r$ be natural numbers, $\lambda \in \Lambda^+(n, r)$, and $K$ a field. The $K\Sigma_r$-module $M^\lambda$ was defined in the previous section, but here we will explain another way of defining these permutation modules. Let $E$ be an $n$-dimensional vector space over $K$, with basis $\{e_1, e_2, \ldots, e_n\}$. Then

$$E^{\otimes r} := \underbrace{E \otimes_K E \otimes_K \ldots \otimes_K E}_{\text{$r$ times}}$$

has basis $\{e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r} : i_j \in \mathbb{Z} \text{ for } j \in [r] \}$ over $K$, and $\Sigma_r$ acts on $E^{\otimes r}$ by place permutation:

$$(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r}) \cdot \sigma := e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}} \otimes \ldots \otimes e_{i_{\sigma(r)}}.$$

For $i = (i_1, i_2, \ldots, i_r) \in I(n, r)$ (see Section 1.2) we write $e_i$ for $e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r}$. Then $E^{\otimes r}$ has basis $\{e_i : i \in I(n, r)\}$ over $K$, and $\Sigma_r$ acts on $E^{\otimes r}$ by $e_i \cdot \sigma := e_{i\sigma}$. This action of $\Sigma_r$ on the basis element of $E^{\otimes r}$ can be extended linearly to make $E^{\otimes r}$ into a $K\Sigma_r$-module.
In Section 1.2 we defined a map \( \lambda \) from \( I(n,r) \) to \( \Lambda(n,r) \) such that \( \lambda(i) = \lambda(i\sigma) \) for all \( \sigma \in \Sigma_r \) and \( i \in I(n,r) \). For \( \lambda \in \Lambda(n,r) \) we define a subspace \( \widehat{M}^\lambda \) of \( E^{\otimes r} \) by having \( K \)-basis
\[
\{ e_i : i \in I(n,r), \lambda(i) = \lambda \}.
\]
For \( \sigma \) in \( \Sigma_r \) and \( e_i \in \widehat{M}^\lambda \) we have \( \lambda(i) = \lambda \) now \( e_i \cdot \sigma = e_{i\sigma} \) and \( \lambda(i\sigma) = \lambda(i) = \lambda \) so \( e_i \cdot \sigma \in \widehat{M}^\lambda \). Therefore, for all \( \lambda \in \Lambda(n,r) \), \( \widehat{M}^\lambda \) is a submodule of \( E^{\otimes r} \).

**Proposition 1.6.1** \( E^{\otimes r} \) decomposes as a \( K\Sigma_r \)-module as
\[
E^{\otimes r} = \bigoplus_{\lambda \in \Lambda(n,r)} \widehat{M}^\lambda,
\]
and for all \( \lambda \in \Lambda(n,r) \), \( \widehat{M}^\lambda \cong \widehat{M}^\overline{\lambda} \), where \( \overline{\lambda} \) is the partition associated to \( \lambda \) (see Section 1.1).

This decomposition of \( E^{\otimes r} \) follows from the fact that \( E^{\otimes r} \) has \( K \)-basis \( \{ e_i : i \in I(n,r) \} \), \( \widehat{M}^\lambda \) has \( K \)-basis \( \{ e_i : i \in I(n,r), \lambda(i) = \lambda \} \) and \( I(n,r) = \bigcup_{\lambda \in \Lambda(n,r)} \{ i : \lambda(i) = \lambda \} \) (where the union is disjoint).

The module \( \widehat{M}^\lambda \) has been defined for \( \lambda \) a composition of \( r \), and Proposition 1.6.1 shows that we actually only need to consider the modules \( \widehat{M}^\lambda \) for \( \lambda \) a partition of \( r \). Let \( \lambda \in \Lambda^+(n,r) \), then the following proposition shows that \( \widehat{M}^\lambda \) is actually the same module as \( M^\lambda \) (which was defined in Section 1.5).

**Proposition 1.6.2** For \( \lambda \in \Lambda^+(n,r) \) we have \( M^\lambda \cong \widehat{M}^\lambda \).

**Proof.** We define a map
\[
\phi : M^\lambda \to \widehat{M}^\lambda,
\]
where \( i = (i_1, i_2, \ldots, i_r) \) and for each \( k, i_k \) is the number of the row in \( T \) where the entry \( k \) occurs. We extend \( \phi \) linearly, then \( \phi \) is a well-defined linear map. It can be shown that \( \phi(T \cdot \sigma) = \phi(T) \cdot \sigma \) for all tabloids \( \{ T \} \) and all \( \sigma \in \Sigma_r \). Therefore \( \phi \) is a \( K\Sigma_r \)-module homomorphism, and we also have that \( \phi \) sends a basis element of \( M^\lambda \) to a basis element of \( \widehat{M}^\lambda \), so \( \phi \) is bijective. Hence \( \phi \) is an isomorphism, as required. \( \square \)
Therefore, for $\lambda$ a partition of $r$, instead of $\tilde{M}^\lambda$ we will just call these modules $M^\lambda$ (they are submodules of $E^\otimes r$). Recall from Proposition 1.5.1 that, for $\lambda \in \Lambda^+(n,r)$, $S^\lambda \subseteq M^\lambda$ is a cyclic module generated by any one polytabloid. Let

$$T = \begin{array}{cccc}
1 & 2 & \ldots & \lambda_1 \\
\lambda_1 + 1 & \ldots & \lambda_1 + \lambda_2 \\
\vdots \\
\ldots & r
\end{array}$$

Then $S^\lambda$ is the cyclic $K\Sigma_r$-module generated by $e_T$, where

$$e_T = \{T\} \cdot \kappa_T = \sum_{\pi \in C_T} \text{sgn}(\pi)\{T\} \cdot \pi.$$ 

We have defined $M^\lambda$ as a submodule of $E^\otimes r$, and now we want to define $S^\lambda$ such that $S^\lambda \subseteq M^\lambda \subseteq E^\otimes r$. Let $\phi$ be as in Proposition 1.6.2. Then $\phi(\{T\}) = e_l$, where

$$l = (\underbrace{1,1,\ldots,1}_{\lambda_1}, 2,\ldots,\underbrace{2,\ldots,2}_{\lambda_2}, \ldots, n,\ldots,n).$$

So $S^\lambda$ is the cyclic $K\Sigma_r$-module generated by

$$\sum_{\pi \in C_T} \text{sgn}(\pi)e_l \cdot \pi = \sum_{\pi \in C_T} \text{sgn}(\pi)e_{l\pi},$$

where $C_T$ is the column stabilizer of the tableau $T$.

**Example 1.6.3** Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $r \geq 2k$. Let $\lambda = (r-k,k) \in \Lambda^+(2,r)$ and

$$T = \begin{array}{cccc}
1 & 2 & \ldots & r-k \\
r-k+1 & r-k+2 & \ldots & r
\end{array}$$

Then $C_T = \{(1,r-k+1),(2,r-k+2),\ldots,(k,r)\}$. Let

$$l = \underbrace{(1,1,\ldots,1}_{r-k\choose k}, 2,\ldots,2) \in I(2,r).$$

Then $S^\lambda$ is the cyclic $K\Sigma_r$-module generated by

$$\sum_{\pi \in C_T} \text{sgn}(\pi)e_{l\pi}.$$
Remark 1.6.4 For a partition $\lambda \in \Lambda^+(n, r)$ we have defined the permutation module $M^\lambda$ in two ways. Firstly with basis the set of all $\lambda$-taboids and secondly with basis all $e_i$ with the multi-index $i \in I(n, r)$ having content $\lambda$. In general, for any group, we can define permutation modules as follows (see for example [28] Chapter II, Section 12). Let $G$ and $H$ be groups with $H \leq G$, and let $I_H$ be the trivial $KH$-module. Then a transitive permutation module is a module isomorphic to the induced module $I_H \uparrow^G := I_H \otimes_{KH} KG$ for some subgroup $H$. A permutation module is then a module isomorphic to a direct sum of transitive permutation modules. This is more general than our previous definitions of the permutation module $M^\lambda$, and for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^+(n, r)$ we have $M^\lambda \cong I_{\Sigma^\lambda} \uparrow^G$, where $\Sigma^\lambda := \Sigma_{\{1, 2, \ldots, \lambda_1\}} \times \Sigma_{\{\lambda_1+1, \ldots, \lambda_1+\lambda_2\}} \times \ldots \times \Sigma_{\{\lambda_1+\ldots+\lambda_{n-1}+1, \ldots, r\}}$.

1.7 Young Modules and $p$-Kostka Numbers

Let $r$ be a fixed natural number, $p$ a prime and $K$ a field of characteristic $p$. Let $\lambda$ be a partition of $r$ and let $M^\lambda$ be the corresponding permutation module.

Definition 1.7.1 The indecomposable direct summands of $M^\lambda$ are called Young modules.

Given a decomposition $M^\lambda = \bigoplus_{i=1}^m Y_i$ into a direct sum of indecomposable modules $Y_i$, there exists a unique index $i \in \mathbb{N}$ such that the Specht module $S^\lambda$ is a submodule of $Y_i$, see [29](4.6.1). This Young module $Y_i$ is denoted by $Y^\lambda$. So we have a Young module $Y^\mu$ for each partition $\mu$ of $r$, and we have the following theorem.

Theorem 1.7.2 ([24], Theorem 3.1) The permutation module $M^\lambda$ is a direct sum of $K\Sigma_r$-modules, each of which is isomorphic to some $Y^\mu$ where $\mu$ is a partition of $r$ with $\mu \geq \lambda$, and precisely one summand is isomorphic to $Y^\lambda$. Moreover, if $Y^\lambda \cong Y^\mu$ then $\lambda = \mu$.

Recall that $\geq$ is the dominance ordering, as defined in Section 1.1. The proof of this theorem given in [24] uses Schur algebras, alternatively [9] gives a proof using only the representation theory of symmetric groups.
We can decompose a permutation module $M^\lambda$ into a direct sum of indecomposable modules. The multiplicity of the Young module $Y^\mu$ as direct summand of $M^\lambda$ is well-defined and is called the $p$-Kostka number. So, using Theorem 1.7.2, we have

$$M^\lambda \cong Y^\lambda \oplus \bigoplus_{\mu \succ \lambda} K_{\lambda\mu} Y^\mu,$$

where $K_{\lambda\mu}$ are the $p$-Kostka numbers, which in general are not known.

The problem of determining the $p$-Kostka numbers is equivalent to finding the decomposition numbers of symmetric groups. We will do something slightly different, and study the decomposition of the permutation modules $M^\lambda$ into indecomposable direct summands, which are the Young modules. We aim to construct the Young modules explicitly, and knowing this could help to find new, still unknown $p$-Kostka numbers.

**Theorem 1.7.3** ([2], 2.6) *The Young module $Y^\lambda$ has a Specht filtration, in the sense that there is a filtration, $Y^\lambda = Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_t = \{0\}$, for some $t \in \mathbb{N}$, with each $Y_i/Y_{i+1}$ isomorphic to a Specht module $S^{\lambda_i}$ for some partition $\lambda_i$ of $r$.*

Recall that $K$ is a field of characteristic $p$. We now consider the case where $\lambda$ is a two-part partition. In this case, the multiplicity of the Young module $Y^{(r-s,s)}$ as a direct summand of the permutation module $M^{(r-k,k)}$ is at most one. So all the $p$-Kostka numbers are either zero or one, and the next theorem tells us precisely what the $p$-Kostka numbers are in this case.

**Theorem 1.7.4** ([19], Theorem 3.3) *Let $r \in \mathbb{N}$ and $s, k \in \mathbb{N}_0$ be such that $2s \leq 2k \leq r$. Then the Young module $Y^{(r-s,s)}$ is a direct summand of the permutation module $M^{(r-k,k)}$ if and only if $k - s$ is $p$-contained in $r - 2s$.***
Chapter 2

Idempotents for Schur Algebras

Throughout this chapter, let $n$ and $r$ be natural numbers, $k$ be a non-negative integer with $2k \leq r$, and $K$ a field. In this chapter we start by recalling some general theory about idempotents, in particular aiming to give the idempotents of $\text{End}_{K^r}(M^{(r-k,k)})$ in the case where $K$ has characteristic two. This provides the strategy for the construction of the Young module $Y^{(r-s,s)}$ in $M^{(r-k,k)}$, as used in Chapter 4. Also, in Section 2.3, we will define the Schur algebra $S_K(n,r)$.

2.1 Primitive Idempotents

An element $e$ of an algebra $A$ is said to be an idempotent if $e^2 = e$. Two idempotents $e$ and $f$ such that $ef = fe = 0$ are called orthogonal. The equality $1 = e_1 + e_2 + ... + e_s$, where $e_1, e_2, ..., e_s$ are pairwise orthogonal idempotents will be called a decomposition of the identity of the algebra $A$.

**Theorem 2.1.1** (see for example [7], Theorem 1.7.2) There is a bijective correspondence between the decompositions of an $A$-module $M$ into a direct sum of submodules and the decompositions of the identity of the algebra $\text{End}_A(M)$.

This correspondence is given as follows: Let $M$ be an $A$-module, and $M = M_1 \oplus M_2 \oplus ... \oplus M_s$ a decomposition of $M$ as a direct sum of submodules, for some natural number $s$. Then for $m \in M$, we have a unique way of writing $m = m_1 + m_2 + ... + m_s$.
with \( m_i \in M_i \). We then define a map \( e_i : M \to M \) by \( e_im = m_i \). It can be shown that \( e_i \in \text{End}_A(M) \), \( e_i^2 = e_i \) and \( e_ie_j = 0 \) if \( i \neq j \). Also \( m = e_1m + e_2m + \ldots + e_sm \), so \( e_1 + e_2 + \ldots + e_s = 1 \). So we have a decomposition of the identity of \( \text{End}_A(M) \).

If \( 1 = e_1 + e_2 + \ldots + e_s \) is a decomposition of the identity of the algebra \( \text{End}_A(M) \), then we can define \( M_i := \text{Im} e_i \). This gives \( M = M_1 \oplus M_2 \oplus \ldots \oplus M_s \), a decomposition of \( M \) as a direct sum of submodules. We now have the following corollary:

**Corollary 2.1.2 (see for example [7], Corollary 1.7.3)** A module \( M \) is indecomposable if and only if there are no non-trivial (i.e. different from 0 and 1) idempotents in the algebra \( \text{End}_A(M) \).

An idempotent \( e \) is called a *primitive idempotent* if it cannot be represented in the form \( e = e' + e'' \), where \( e' \) and \( e'' \) are non-zero orthogonal idempotents.

Our aim throughout this chapter is to find a decomposition of the identity of \( \text{End}_{K\Sigma_r}(M^\lambda) \), where \( \lambda \) is a two-part partition of \( r \), into primitive orthogonal idempotents. This then gives a decomposition of \( M^\lambda \) as a direct sum of indecomposable submodules which, by Definition 1.7.1, are Young modules.

### 2.2 Uniqueness of the Decomposition

Let \( A \) be an algebra and \( M \) an \( A \)-module. We can decompose \( M \) as a direct sum of indecomposable \( A \)-modules, and in this section we will see that this decomposition is unique up to isomorphism and order. In particular, when we look at the \( K\Sigma_r \)-module \( M^{(r-k,k)} \) we will see that the decomposition of \( M^{(r-k,k)} \) as a direct sum of Young modules is unique up to order.

**Theorem 2.2.1 (see for example [7], Theorems 3.4.1 and 3.4.2)** Let \( 1 = e_1 + e_2 + \ldots + e_n = f_1 + f_2 + \ldots + f_m \) be two decompositions of the identity of an algebra \( A \) with primitive idempotents \( e_i \) and \( f_j \). Then \( n = m \) and there is an invertible element \( a \) in the algebra \( A \) such that, up to a suitable reindexing, \( f_i = ae_i a^{-1} \) for all \( i \). Moreover, if \( M = M_1 \oplus M_2 \oplus \ldots \oplus M_n = N_1 \oplus N_2 \oplus \ldots \oplus N_m \) are two decompositions of the module \( M \) into a direct sum of indecomposable modules, then \( n = m \) and, after
a suitable reindexing, \( M_i \cong N_i \) for all \( i \).

So the decomposition of an \( A \)-module \( M \) into a direct sum of indecomposable modules is unique up to isomorphism and order, but in general the decomposition is not actually unique. The following corollary gives a particular case where this decomposition is in fact unique (up to order).

**Corollary 2.2.2** If \( \text{End}_A(M) \) is commutative and \( M = M_1 \oplus M_2 \oplus \ldots \oplus M_n = N_1 \oplus N_2 \oplus \ldots \oplus N_m \) are two decompositions of the module \( M \) into a direct sum of indecomposable modules, then \( n = m \) and, after a suitable reindexing, \( M_i = N_i \) for all \( i \).

*Proof.* By Theorem 2.1.1 we have two decompositions of the identity of \( \text{End}_A(M) \) corresponding to the two decompositions of \( M \) into a direct sum of indecomposable modules. Let these be \( 1 = e_1 + \ldots + e_n = f_1 + \ldots + f_m \) where \( e_i, f_j \) are primitive idempotents, \( M_i = \text{Im} \ e_i, N_j = \text{Im} \ f_j \). Then, by Theorem 2.2.1, \( n = m \) and there is an invertible element \( a \in \text{End}_A(M) \) such that, up to a suitable reindexing, \( f_i = a e_i a^{-1} \). Now, as \( \text{End}_A(M) \) is commutative, \( a e_i a^{-1} = e_i \), so up to a suitable reindexing we have \( f_i = e_i \), so \( M_i = \text{Im} \ e_i = \text{Im} \ f_i = N_i \) for all \( i \). \( \square \)

For \( \lambda \) a partition of \( r \), we can write \( M^\lambda \) as a direct sum of Young modules, and our problem is to construct these Young modules. By Theorem 2.2.1 the decomposition of \( M^\lambda \) as a direct sum of Young modules is unique up to isomorphism and order. In particular using Corollary 2.2.2 we will later show that this decomposition is physically unique for \( \lambda \) a two-part partition (see Remark 2.4.8).

### 2.3 The Schur Algebra \( S_K(n, r) \)

We start by defining the Schur algebra \( S_K(n, r) \) as in [12], for \( K \) any field. In [12], Chapter 2, the Schur algebra is defined via regular functions. For these functions to be algebraically independent we need to work over a field \( K \) with infinitely many elements. In the context used in this thesis however, we never work with regular functions and hence we can assume that \( K \) is any (finite or infinite) field.
Recall from Section 1.2 that the symmetric group $\Sigma_r$ acts on $I(n,r)$ by place permutation. We define an equivalence relation $\sim$ on $I(n,r)$ by $i \sim j$ if $j = i\sigma$ for some $\sigma \in \Sigma_r$. Similarly, we define an equivalence relation on $I(n,r) \times I(n,r)$ by $(i,j) \sim (k,l)$ if $k = i\sigma$ and $l = j\sigma$ for some $\sigma \in \Sigma_r$. Let $\Omega(n,r)$ be the set of equivalence classes of $\sim$ on $I(n,r) \times I(n,r)$. The Schur algebra $S_K(n,r)$ is then defined to be the vector space over $K$ with basis $\{\xi_{i,j} : (i,j) \in \Omega(n,r)\}$. We write $\xi_{i,j} = \xi_{k,l}$ if and only if $(i,j) \sim (k,l)$. The multiplication in $S_K(n,r)$ is defined on basis elements by:

$$\xi_{i,j} \xi_{k,l} := \sum_{(p,q) \in \Omega(n,r)} Z(i,j,k,l,p,q) \xi_{p,q}$$

where $Z(i,j,k,l,p,q) = \text{Card}\{s \in I(n,r) : (i,j) \sim (p,s), (k,l) \sim (s,q)\}$, and extended linearly. For a set $X$, $\text{Card}X$ means the cardinality of $X$, which is the number of elements in $X$.

Note that in [12] the Schur algebra $S_K(n,r)$ is actually defined in a different way, as we will now describe. First for $i = (i_1,i_2,\ldots,i_r)$ and $j = (j_1,j_2,\ldots,j_r) \in I(n,r)$ we define $c_{i,j} := c_{i_1 j_1} c_{i_2 j_2} \cdots c_{i_r j_r}$, where the $c_{i_r j_r}$ are commuting variables. We have that $c_{i,j} = c_{k,l}$ if and only if $(i,j) \sim (k,l)$. Then $A_K(n,r)$ is defined to be the vector space over $K$ with basis the monomials $c_{i,j}$. The Schur algebra $S_K(n,r)$ is then defined to be the dual space of $A_K(n,r)$, so $S_K(n,r) = A_K(n,r)^* = \text{Hom}_K(A_K(n,r),K)$, which has basis $\{\xi_{i,j} : (i,j) \in \Omega(n,r)\}$ of $A_K(n,r)$. Therefore for $i,j \in I(n,r)$ the element $\xi_{i,j}$ of $S_K(n,r)$ is given by

$$\xi_{i,j}(c_{p,q}) = \begin{cases} 1 & \text{if } (i,j) \sim (p,q) \\ 0 & \text{if } (i,j) \not\sim (p,q) \end{cases} \tag{2.1}$$

for all $p,q \in I(n,r)$. This is well-defined, and can be extended linearly to $S_K(n,r)$. In (2.3b) of [12] the multiplication rule for $S_K(n,r)$, as we have defined above, is then deduced. Therefore this fits with our approach above of defining the Schur algebra via the basis $\{\xi_{i,j} : (i,j) \in \Omega(n,r)\}$.

Now, recall that the map $\lambda : I(n,r) \rightarrow \Lambda(n,r)$ was defined in Section 1.2. For $i \in I(n,r)$ with $\lambda(i) = \lambda$ we write $\xi_{i,i} = \xi_{\lambda}$. Then $\{\xi_{\lambda} : \lambda \in \Lambda(n,r)\}$ is a set of mutually orthogonal idempotents in $S_K(n,r)$, and the identity of $S_K(n,r)$ is $\epsilon = \sum_{\lambda \in \Lambda(n,r)} \xi_{\lambda}$ (by [12] (2.3c) and (2.3d)).

In the following we provide two further equivalent definitions of the Schur algebra,
one as the endomorphism ring of a module of the symmetric group, the other coming from the fact that the Schur algebra is a quotient of the universal enveloping algebra of $gl_n$.

First recall that $E$ is an $n$-dimensional vector space over $K$ with basis $\{e_1, \ldots, e_n\}$ and that $E^{\otimes r}$ is a $K\Sigma_r$-module with basis $\{e_j : j \in I(n, r)\}$. We can now define an $S_K(n, r)$-action on a basis element $e_j$ of $E^{\otimes r}$ by,

$$\xi \cdot e_j := \sum_{i \in I(n, r)} \xi(c_{i,j})e_i$$

for all $\xi \in S_K(n, r), j \in I(n, r)$ (recall that, for $i, j \in I(n, r)$, $\xi(c_{i,j})$ is given by extending (2.1) linearly to $S_K(n, r)$). This can be extended linearly to $E^{\otimes r}$, and makes $E^{\otimes r}$ into a left $S_K(n, r)$-module. This action commutes with that of $KF(n, r)$, as $(\xi \cdot x) \cdot \pi = \xi \cdot (x \cdot \pi)$ for all $\xi \in S_K(n, r), x \in E^{\otimes r}$ and $\pi \in \Sigma_r$. This can be used to show that

**Proposition 2.3.1** ([12], (2.6c)) $S_K(n, r) \cong \text{End}_{K\Sigma_r}(E^{\otimes r})$.

**Remark 2.3.2** In the proof of this Proposition, given in [12] (2.6c), the isomorphism is given by

$$S_K(n, r) \rightarrow \text{End}_{K\Sigma_r}(E^{\otimes r})$$

$$\xi \mapsto (e_i \mapsto \xi \cdot e_i).$$

In particular, the element $\xi_\lambda \in S_K(n, r)$ corresponds to the endomorphism of $E^{\otimes r}$ given by $e_i \mapsto \xi_\lambda \cdot e_i$ for all $i \in I(n, r)$. Now for $i \in I(n, r)$,

$$\xi_\lambda \cdot e_i = \sum_{j \in I(n, r)} \xi_\lambda(c_{j,i})e_j = \begin{cases} e_i & \text{if } \lambda(i) = \lambda, \\ 0 & \text{otherwise}. \end{cases}$$

Now $\lambda(i) = \lambda$ if and only if $e_i \in M^\lambda$, so the map $e_i \mapsto \xi_\lambda \cdot e_i$ is the projection of $E^{\otimes r}$ onto $M^\lambda$.

For a field $K$ we use a superscript $K$ to denote the basis elements $\xi_{i,j}^K$ of $S_K(n, r)$. The integral Schur algebra $S_Z(n, r)$ is then the $\mathbb{Z}$-submodule of $S_Q(n, r)$ which is generated by the $\xi_{i,j}^Q$ for $i, j \in I(n, r)$. Then, for any field $K$, there is an isomorphism of $K$-algebras $S_Z(n, r) \otimes K \cong S_K(n, r)$ which takes each $\xi_{i,j}^Q \otimes 1_K \mapsto \xi_{i,j}^K$ (see [12], Section 2.3).
Let $K$ be any field. We will now define the Schur algebra $S_K(2, r)$ as in [6]. We first describe $S_Q(2, r)$ then $S_Z(2, r)$, then we know that $S_K(2, r) \cong S_Z(2, r) \otimes K$. Write

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

for the canonical basis elements of the Lie algebra $\mathfrak{gl}_2$. The elements $e$, $f$ and $h := H_1 - H_2$ form the canonical basis of the Lie subalgebra $\mathfrak{sl}_2$. Let $E$ be a 2-dimensional vector space over $K$ with basis $e_1, e_2$. An action of $\mathfrak{gl}_2$ (and $\mathfrak{sl}_2$) on $E$ is given by,

$$ee_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

and similarly, $ee_2 = e_1$, $fe_1 = e_2$, $fe_2 = 0$, $H_1 e_1 = e_1$, $H_1 e_2 = 0$, $H_2 e_1 = 0$ and $H_2 e_2 = e_2$, making $E$ into a module over $\mathfrak{gl}_2$ and $\mathfrak{sl}_2$. This action extends diagonally to an action on the tensor space $E^\otimes r$ by

$$e(e_1 \otimes e_2 \otimes \cdots \otimes e_r) = \sum_{1 \leq i \leq r} e_{i_1} \otimes \cdots \otimes e_{i_{i-1}} \otimes ee_{i_2} \otimes e_{i_{i+1}} \otimes \cdots \otimes e_{i_r}$$

for all $i_1, i_2, \ldots, i_r \in \{1, 2\}$, and extending linearly. Similarly one can describe the action of $f$, $H_1$ and $H_2$ on the tensor space $E^\otimes r$. Now, for $1 \leq i \leq k$ we define

$$e^{(i)} = \frac{e^i}{i!} \quad \text{and} \quad f^{(i)} = \frac{f^i}{i!}.$$

The action of these so-called divided powers on the tensor space $E^\otimes r$ is given in Section 3.1. Also, for $g \in \mathfrak{gl}_2(\mathbb{Q})$ and $i$ a natural number, we define

$$\binom{g}{i} = \frac{g(g-1) \cdots (g-i+1)}{i!}.$$ 

We then have the following results.

**Theorem 2.3.3 ([6], Theorem 2.1)** The Schur algebra $S_Q(2, r)$ is isomorphic to the associative algebra (with 1) generated by $e$, $f$ and $h$ subject to the relations

$$he - eh = 2e; \quad ef - fe = h; \quad hf - fh = -2f;$$

$$(h + r)(h + r - 2) \cdots (h - r + 2)(h - r) = 0.$$ 

Moreover, this algebra has a 'truncated PBW' basis over $\mathbb{Q}$ consisting of all $f^a h^b e^c$ such that $a + b + c \leq r$. 

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Theorem 2.3.4 ([6], Theorem 2.4) The integral Schur algebra $S_Z(2,r)$ is isomorphic to the subalgebra of $S_Q(2,r)$ generated by all divided powers $e^{(i)}$ and $f^{(i)}$. Moreover, this algebra has a 'truncated Kostant basis' over $\mathbb{Z}$ consisting of all
\[
f^{(a)} \left( \frac{H_2}{b} \right) e^{(c)} \quad (a + b + c \leq r)
\]
and another such basis consisting of all
\[
e^{(a)} \left( \frac{H_1}{b} \right) f^{(c)} \quad (a + b + c \leq r).
\]
Hence the divided powers $e^{(i)}$ and $f^{(i)}$ generate a $\mathbb{Z}$-form for $S_Q(2,r)$ which is isomorphic to $S_Z(2,r)$. More generally, Theorem 3 in [5] gives generators of $S_Z(n,r)$ for any natural number $n$, in terms of the Lie algebra $\mathfrak{gl}_n$. Then for any field $K$ we have that the Schur algebra $S_K(n,r) \cong S_Z(n,r) \otimes K$, but we will be concentrating on the case where $n = 2$.

## 2.4 Primitive Idempotents for $\text{End}_{K\Sigma_r}(M^{(r-k,k)})$

In Section 2.1 we described the general theory about finding idempotents and how this corresponds to a decomposition of a module. In particular, we want to decompose the module $M^{(r-k,k)}$ into a direct sum of Young modules, and in this section we will describe the primitive orthogonal idempotents of $\text{End}_{K\Sigma_r}(M^{(r-k,k)})$ in the case where $K$ has characteristic two.

**Proposition 2.4.1 (see for example [7], Section 1.7)** Let $M = M_1 \oplus \ldots \oplus M_s$ be a decomposition of the $A$-module $M$ as a direct sum of submodules, and $1 = e_1 + \ldots + e_s$ be the corresponding decomposition of the identity of $\text{End}_A(M)$. Then
\[
e_i \text{End}_A(M) e_j \cong \text{Hom}_A(M_j, M_i).
\]

**Remark 2.4.2** In particular, for $E$ an $n$-dimensional vector space, we have a decomposition of the $K\Sigma_r$-module $E^\otimes r$ as $E^\otimes r = \bigoplus_{\lambda \in \Lambda(n,r)} M^\lambda$. Then $\text{End}_{K\Sigma_r}(E^\otimes r) \cong S_K(n,r)$, and by Remark 2.3.2 for $\lambda \in \Lambda(n,r)$ the element $1_\lambda := \xi_\lambda$ can be considered to be the projection of $E^\otimes r$ onto $M^\lambda$ (so $1_\lambda$ is the identity in
Therefore we have that the corresponding decomposition of the identity of $\text{End}_{K^r}(E^{\otimes r}) \cong S_k(n, r)$ is $1 = \sum_{\lambda \in \Lambda(n, r)} 1_\lambda$. So by the previous proposition we have $1_\lambda \text{End}_{K^r}(E^{\otimes r}) 1_\lambda \cong \text{Hom}_{K^r}(M^\lambda, M^\lambda)$ hence,

$$1_\lambda S_k(n, r) 1_\lambda \cong \text{End}_{K^r}(M^\lambda).$$

First, note that by Remark 2.3.2, $1_\lambda = \xi_\lambda$ is the projection of $E^{\otimes r}$ onto $M^\lambda$, so is the identity on $M^\lambda$. Recall that we defined the divided powers $e^{(i)}$ and $f^{(i)}$ which are elements of the universal enveloping algebra of $\text{gl}_n(Q)$. The following lemma shows how for any $\lambda \in \Lambda(n, r)$ the element $1_\lambda$ translates into the setup of the universal enveloping algebra of $\text{gl}_n(Q)$. This is more general than we need, as we are concentrating on the case where $n = 2$, but in general the element $H_i \in \text{gl}_n(Q)$ is the $n \times n$ matrix whose unique nonzero entry is 1 in the $(i, i)$th position.

**Lemma 2.4.3 ([3], Lemma 5.3)** The element $1_\lambda$ in $S_Q(n, r)$ coincides with the element given by the product $\prod_i \left(\frac{H_i}{\lambda_i}\right)$, for any $\lambda \in \Lambda(n, r)$.

Let $\lambda = (r - k, k) \in \Lambda^+(2, r)$, and let $K$ be a field of characteristic two. The primitive orthogonal idempotents of the centralizer subalgebra $S(\lambda) := 1_\lambda S_k(2, r) 1_\lambda \cong \text{End}_{K^r}(M^\lambda)$ have been determined in [4]. So we have a decomposition of the identity of $\text{End}_{K^r}(M^\lambda)$ as $1_\lambda = \varphi_1 + \varphi_2 + \ldots + \varphi_s$, for some natural number $s$, where the $\varphi_i$ are primitive orthogonal idempotents. Then if we let $Y_i = \text{Im } \varphi_i$, we have that $M^\lambda = \bigoplus_{i=1}^s Y_i$ (by Theorem 2.1.1), and as each idempotent $\varphi_i$ is primitive, we have that each $Y_i$ is indecomposable. In the rest of this section we will explain how to calculate these primitive orthogonal idempotents of $S(\lambda)$, following [4].

Define

$$b(i) := 1_\lambda f^{(i)} e^{(i)} 1_\lambda.$$

Then the set $\{b(0), b(1), \ldots, b(k)\}$ forms a basis for $S(\lambda)$. We will see, in Proposition 3.1.5, that for $l > k$ we have $e^{(l)} = 0$. Therefore, if $l > k$ then $b(l) = 0$. We also have that $b(0) = 1_\lambda$.

**Proposition 2.4.4 ([4], Proposition 3.6)** Define $m := r - 2k$, then the multiplication in $S(\lambda)$ is given by:

$$b(i)b(j) = \sum_{l=0}^i \binom{j + l}{i} \binom{j + l}{l} \binom{m + j + i}{i - l} b(j + l),$$
for non-negative integers \(i\) and \(j\).

Now we will show that, for \(\lambda = (r-k,k) \in \Lambda^+(2,r)\), \(S(\lambda) \cong \text{End}_{K^2}(M^\lambda)\) is commutative. Before we do this, we need to define what it means for a module to be liftable. We follow here [28], Chapter 1, Section 14. Let \((K,R,F)\) be a \(p\)-modular system, where \(R\) is a complete discrete valuation ring with maximal ideal \((\pi)\), \(K = R/(\pi)\) and \(F\) is the quotient field of \(R\) (see [28], Chapter 1, Definition 12.1). Then, for \(A\) a finite dimensional algebra over \(R\) and \(\tilde{A} = A/A\pi\) we have the following definition.

**Definition 2.4.5** (see for example [28], Chapter 1, Definition 14.3) An \(A\)-module \(L\) is called liftable if there exists an \(A\)-module \(\hat{L}\) such that \(L/Ln = L\), We call \(\hat{L}\) a lift of \(L\).

Using [28], Chapter 2, Theorem 12.4, we have that any direct summand of a transitive permutation module is liftable, and the endomorphism ring of a transitive permutation module is liftable. This allows us to translate some characteristic zero results into the characteristic \(p\) situation, as we now demonstrate.

In our situation, let \(p\) be a prime, \(R = Z(p)\) the ring of \(p\)-adic integers, \(F = \mathbb{Q}(p)\) the field of \(p\)-adic numbers and \(K = \mathbb{F}_p\) the field with \(p\) elements. Then \((K,R,F)\) is a \(p\)-modular system, and we will take \(A = Z(p)\Sigma\) and \(\tilde{A} = F\Sigma\).

Note that by above for a partition \(\lambda\) the modules \(M^\lambda\) and \(Y^\lambda\) are liftable. Also every endomorphism of \(M^\lambda\) is liftable, so we consider the module \(\text{End}_{F^2}(M^\lambda)\) over a field \(F\) of characteristic zero. Using Young's Rule (see [23], 14.1) it can be shown that over any field of characteristic zero the composition factors of \(M^{(r-k,k)}\) are the Specht modules \(S^{(r-t,t)}\) with \(0 \leq t \leq k\) and each of these occurs precisely once. So, over characteristic zero we have that \(M^{(r-k,k)} = \bigoplus_{t=0}^{k} S^{(r-t,t)}\), therefore \(\text{End}_{F^2}(M^\lambda) = \text{End}_{F^2}(\bigoplus_{t=0}^{k} S^{(r-t,t)}) = \bigoplus_{t=0}^{k} \text{End}_{F^2}(S^{(r-t,t)})\) (see for example [1], page 10, Lemma 5) using the fact that each \(S^{(r-t,t)}\) is simple over characteristic zero, and \(\text{Hom}_{F^2}(S_1,S_2) = 0\) for non-isomorphic simple modules \(S_1\) and \(S_2\). Since, for a simple module \(S\), \(\text{End}_{F^2}(S)\) is isomorphic to a field this shows that \(\text{End}_{F^2}(M^\lambda)\) is a direct sum of several copies of the field \(F\) and hence is commutative. This implies that the same holds for \(\text{End}_{K^2}(M^\lambda)\). Therefore we have proved the following lemma.

**Lemma 2.4.6** Let \(\lambda = (r-k,k) \in \Lambda^+(2,r)\), then \(\text{End}_{K^2}(M^\lambda)\) is commutative.
Remark 2.4.7 Alternatively, as \( \{b(0), b(1), \ldots, b(k)\} \) is a basis of \( S(\lambda) \) where \( \lambda = (r - k, k) \in \Lambda^+(2, r) \), to prove Lemma 2.4.6 we could show that \( b(i)b(j) = b(j)b(i) \) for all non-negative integers \( i \) and \( j \), which we will now do. Let \( i \) and \( j \) be non-negative integers with \( i \neq j \), then by Proposition 2.4.4 we have

\[
b(i)b(j) = \sum_{l=0}^{i} \binom{j + l}{i} \binom{j}{l} \binom{m + j + i}{i - l} b(j + l)
\]

\[
= \sum_{n=j-i}^{j} \binom{i + n}{i} \binom{i + n}{n + i - j} \binom{m + j + i}{j - n} b(i + n)
\]

\[
= \sum_{n=j-i}^{j} \binom{i + n}{n} \binom{i + n}{j} \binom{m + i + j}{j - n} b(i + n), \tag{2.2}
\]

where \( n = j - i + l \) and using Lemma 1.3.1(1). Now if \( j - i > 0 \) then for \( 0 \leq n < j - i \) we have \( i + n < j \) so \( \binom{i + n}{j} = 0 \), therefore by (2.2) we have \( b(i)b(j) = \sum_{n=0}^{j} \binom{i + n}{i} \binom{i + n}{j} \binom{m + i + j}{j - n} b(i + n) \). If \( j - i < 0 \) then for \( j - i \leq n < 0 \) we have \( \binom{i + n}{n} = 0 \), so \( b(i)b(j) = \sum_{n=0}^{i} \binom{i + n}{i} \binom{i + n}{j} \binom{m + i + j}{j - n} b(i + n) \), by (2.2). So we have,

\[
b(i)b(j) = \sum_{n=0}^{j} \binom{i + n}{i} \binom{i + n}{j} \binom{m + i + j}{j - n} b(i + n)
\]

\[
= b(j)b(i),
\]

by Proposition 2.4.4. Therefore for all non-negative integers \( i \) and \( j \) we have \( b(i)b(j) = b(j)b(i) \). So, by extending linearly, \( S(\lambda) \) which has basis \( \{b(0), b(1), \ldots, b(k)\} \) is commutative.

Remark 2.4.8 By Lemma 2.4.6 we have that \( \text{End}_{K^*}(M^{(r-k,k)}) \) is commutative, so by Corollary 2.2.2 we know that the decomposition of \( M^{(r-k,k)} \) into a direct sum of Young modules is physically unique (up to order).

Now let \( \gamma \geq 0 \) with \( m + 2\gamma \leq r \) and let \( h \) be the integer with \( 2^h \leq m + 2\gamma < 2^{h+1} \). Let \( \alpha_0, \ldots, \alpha_{h-1}, \gamma_0, \ldots, \gamma_h \in \{0,1\} \) such that the 2-adic decompositions are:

\[
m + 2\gamma = \alpha_0 \cdot 2^0 + \ldots + \alpha_{h-1} \cdot 2^{h-1} + 1 \cdot 2^h \text{ and }
\]

\[
\gamma = \gamma_0 \cdot 2^0 + \ldots + \gamma_{h-1} \cdot 2^{h-1} + \gamma_h \cdot 2^h.
\]

We then have, from Lemma 1.3.2, that

\[
\begin{pmatrix} m + 2\gamma \\ \gamma \end{pmatrix} \equiv \prod_{i=0}^{h} \binom{\alpha_i}{\gamma_i} \mod 2.
\]

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Now for \( \binom{m+2\gamma}{\gamma} \neq 0 \mod 2 \) we define

\[
e_{m,\gamma} = \prod_{\gamma_u=1} b(2^u) \prod_{\gamma_u=0} (1 - b(2^u)).
\]

**Theorem 2.4.9 (see [4], Idempotent Theorem)** The \( e_{m,\gamma} \) where \( m = r - 2k \), \( \gamma = k - s \geq 0 \) and \( \binom{m+2\gamma}{\gamma} = \binom{r-2s}{k-s} \neq 0 \mod 2 \), are the primitive orthogonal idempotents of \( \text{End}_{K\Sigma_r}(M^{(r-k,k)}) \). In particular, \( e_{m,\gamma}(M^{(r-k,k)}) = Y^{(r-s,s)} \).

Note that this theorem only holds in the case where \( K \) is a field of characteristic two. We will use this theorem in Chapter 4 to explicitly construct some Young modules.
Chapter 3

Action of the Idempotents of
$\text{End}_{K^{\Sigma_r}}(M^{(r-k,k)})$

Throughout this chapter, let $K$ be a field of characteristic two, let $r$ be a natural number, let $k$ and $s$ be non-negative integers and let $E$ be a two-dimensional vector space over $K$ with basis $\{e_1, e_2\}$. In Theorem 2.4.9 the primitive orthogonal idempotents $e_{m,\gamma}$ of $\text{End}_{K^{\Sigma_r}}(M^{(r-k,k)})$ are described, and in this chapter we will calculate the action of these idempotents on the $K^{\Sigma_r}$-module $E^{\otimes r}$. This will then be applied in Chapter 4 to construct the Young modules corresponding to two-part partitions.

3.1 The Operation of the Lie Generators on $E^{\otimes r}$

The vector space $E$ has $K$-basis $\{e_1, e_2\}$. Recall, from Section 2.3, that $e$ and $f$ act on $E$ by $ee_1 = 0$, $ee_2 = e_1$, $fe_1 = e_2$ and $fe_2 = 0$, which we extend diagonally to an action on $E^{\otimes r}$. In this section we explain the action of the divided powers $e^{(i)}$ and $f^{(i)}$ on the tensor space $E^{\otimes r}$, but first we will demonstrate this action with an example.
Example 3.1.1 Let $r = 5$, then in $S_q(2, r)$ we have

$$e(e_{(1,1,2,2)}) = e (e_1 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_2)$$

$$= (ee_1) \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_2 + e_1 \otimes (ee_1) \otimes e_1 \otimes e_2 \otimes e_2 + e_1 \otimes e_1 \otimes (ee_1) \otimes e_2 \otimes e_2 + e_1 \otimes e_1 \otimes e_1 \otimes (ee_2) \otimes e_2 + e_1 \otimes e_1 \otimes e_1 \otimes e_2 \otimes (ee_2)$$

$$= e_1 \otimes e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_1$$

Similarly, $e(e_{(1,1,1,1,2)}) = e_{(1,1,1,1,1)}$ and $e(e_{(1,1,1,2,1)}) = e_{(1,1,1,1,1)}$. Hence we get

$$e^{(2)}(e_{(1,1,1,2,2)}) = \frac{e^2}{2!} (e_{(1,1,1,1,1)}) = \frac{1}{2!} e(e_{(1,1,1,1,2)} + e_{(1,1,1,2,1)})$$

$$= \frac{1}{2} (ee_{(1,1,1,1,2)} + ee_{(1,1,1,2,1)}) = \frac{1}{2} (e_{(1,1,1,1,1)} + e_{(1,1,1,1,1)})$$

$$= e_{(1,1,1,1,1)}.$$ 

Therefore in $S_K(2, r)$ we have that $e^{(2)}(e_{(1,1,1,2,2)}) = e_{(1,1,1,1,1)}$.

Before we describe in general the action of the divided powers $e^{(i)}$ and $f^{(i)}$ on $E^{\otimes r}$ we need to introduce some more notation. Recall, from Section 1.2, that we have $I(r, k) = \{ i = (i_1, i_2, \ldots, i_k) : i_1, \ldots, i_k \in \mathbb{N} \}$ for $r$ and $k$ natural numbers. Also, for $k = 0$, let $I(r, 0) = \{ () \}$. We now define a set $J(r, k) \subseteq I(r, k)$ by

$$J(r, k) := \{ j = (j_1, j_2, \ldots, j_k) : j_1 < j_2 \ldots < j_k \}.$$ 

For $j = (j_1, j_2, \ldots, j_k) \in J(r, k)$, define $i^j$ to be the element in $I(2, r)$ with twos in positions $j_1, j_2, \ldots, j_k$ and ones in the other $r - k$ positions (for example if $r = 5$ and $j = (1, 2) \in J(5, 2)$ then $i^j = i^{(1,2)} = (2, 2, 1, 1, 1) \in I(2, 5)$). Then $M^{(r-k,k)}$ has basis $\{ e_{i^j} : j \in J(r, k) \}$ over $K$. We now define a $\Sigma_r$-action on $J(r, k)$. For $j = (j_1, j_2, \ldots, j_k) \in J(r, k)$ and $\sigma \in \Sigma_r$, define $j \cdot \sigma$ to be the multi-index in $J(r, k)$ containing $\sigma^{-1}(j_1), \sigma^{-1}(j_2), \ldots, \sigma^{-1}(j_k)$.

Example 3.1.2 Let $r = 5$, $k = 2$ and $j = (1, 2) \in J(5, 2)$. Then for $\sigma = (1 \ 3 \ 4) \in \Sigma_5$, $j \cdot \sigma \in J(5, 2)$ contains $\sigma^{-1}(1) = 4$ and $\sigma^{-1}(2) = 2$. Hence $j \cdot \sigma = (2, 4)$, as elements in $J(5, 2)$ are of the form $(j_1, j_2)$ with $1 \leq j_1 < j_2 \leq 5$. 

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Let \( j = (j_1, \ldots, j_k) \in J(r, k) \) and \( \sigma, \pi \in \Sigma_r \), then \( j \cdot \sigma \in J(r, k) \) contains \( \sigma^{-1}(j_1), \ldots, \sigma^{-1}(j_k) \). So \( (j \cdot \sigma) \cdot \pi \) is the multi-index in \( J(r, k) \) which contains \( \pi^{-1}(\sigma^{-1}(j_1)), \ldots, \pi^{-1}(\sigma^{-1}(j_k)) \), i.e. \( (\sigma \pi)^{-1}(j_1), \ldots, (\sigma \pi)^{-1}(j_k) \), so this is the multi-index \( j \cdot (\sigma \pi) \). So for \( \sigma, \pi \in \Sigma_r \) and \( j \in J(r, k) \), we have shown that \( (j \cdot \sigma) \cdot \pi = j \cdot (\sigma \pi) \).

Let \( \sigma \in \Sigma_r \), then for \( j = (j_1, j_2, \ldots, j_k) \in J(r, k) \), and \( i^j = (i_1, i_2, \ldots, i_r) \in I(2, r) \) we have,

\[
i^j \sigma = (i_1, i_2, \ldots, i_r) \sigma = (i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(r)}),
\]

using the action of \( \Sigma_r \) on \( I(n, r) \) given in Section 1.2. Now, by definition of \( i^j \), we have that \( i_\rho = 2 \) for \( \rho = j_1, j_2, \ldots, j_k \) and \( i_\rho = 1 \) otherwise. Hence \( i_{\sigma(\rho)} = 2 \) for \( \sigma(\rho) = j_1, j_2, \ldots, j_k \), so \( \rho = \sigma^{-1}(j_1), \sigma^{-1}(j_2), \ldots, \sigma^{-1}(j_k) \). Therefore \( i^j \sigma = i^{j^\sigma} \sigma \). Also, by definition, \( \Sigma_r \) acts on a basis element \( e_{ij} \) of \( M^{(r-k, k)} \) by \( e_{ij} \cdot \sigma = e_{i^\sigma j} \). So we have \( e_{ij} \cdot \sigma = e_{i^\sigma j} \). So the module \( M^{(r-k, k)} \) has \( K \)-basis \( \{e_{ij} : j \in J(r, k)\} \), and \( \Sigma_r \)-action given by \( e_{ij} \cdot \sigma = e_{i^\sigma j} \) for all \( \sigma \in \Sigma_r \), \( j \in J(r, k) \).

For \( r \in \mathbb{N} \) and integers \( k \) and \( s \) with \( 0 \leq s \leq k \leq r \), if \( j = (j_1, j_2, \ldots, j_s) \in J(r, s) \) and \( l = (l_1, l_2, \ldots, l_k) \in J(r, k) \) we will write \( j \subseteq l \) to mean that \( \{j_1, j_2, \ldots, j_s\} \subseteq \{l_1, l_2, \ldots, l_k\} \).

**Example 3.1.3** Here we demonstrate this new notation and the next proposition which describes the action of the Lie generators on the tensor space. Let \( r = 5 \), then by Example 3.1.1

\[
\begin{align*}
e(e_{i(4,5)}) &= e_{i(4)} + e_{i(5)}, \\
e(e_{i(4)}) &= e(e_{i(5)}) = e_{i(1)} = e(1, 1, 1, 1, 1).
\end{align*}
\]

Similarly, it can be shown that

\[
\begin{align*}
f(e_{i(5)}) &= e_{i(1, 5)} + e_{i(2, 5)} + e_{i(3, 5)} + e_{i(4, 5)}, \\
f(e_{i(1)}) &= e_{i(1)} + e_{i(2)} + e_{i(3)} + e_{i(4)} + e_{i(5)}.
\end{align*}
\]

**Proposition 3.1.4** For \( j \in J(r, k) \), the action of the Lie generators on the tensor space are given by

\[
\begin{align*}
e(e_{ij}) &= \sum_{l \in J(r, k-1) \atop l \leq j} e_{il} \quad \text{and} \quad f(e_{ij}) = \sum_{l \in J(r, k+1) \atop j \leq l} e_{il}.
\end{align*}
\]

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Proof. For \( j = (j_1, j_2, \ldots, j_k) \in J(r, k) \) we have that \( i^j = (i_1, i_2, \ldots, i_r) \) where for \( 1 \leq \rho \leq r \) we have
\[
i_{\rho} = \begin{cases} 
2 & \text{if } \rho = j_1, j_2, \ldots, j_k, \\
1 & \text{otherwise}.
\end{cases}
\]
Note that for \( 1 \leq x \leq r \) we have
\[
e_{i_1} \otimes \ldots \otimes e_{i_x-1} \otimes e_{i_x} \otimes e_{i_x+1} \otimes \ldots \otimes e_{i_r} = \begin{cases} 
0 & \text{if } i_x = 1, \\
e_{(i_1, \ldots, i_x-1, 1, i_{x+1}, \ldots, i_r)} & \text{if } i_x = 2.
\end{cases}
\]
\[
e_{i_1} \otimes \ldots \otimes e_{i_x-1} \otimes e_{i_x} \otimes e_{i_x+1} \otimes \ldots \otimes e_{i_r} = \begin{cases} 
0 & \text{if } i_x = 1, \\
e_{(i_1, \ldots, i_x-1, 2, i_{x+1}, \ldots, i_r)} & \text{if } i_x = 2.
\end{cases}
\]
Hence
\[
e(e^j) = \sum_{1 \leq x \leq r} e_{i_1} \otimes \ldots \otimes e_{i_x-1} \otimes e_{i_x} \otimes e_{i_{x+1}} \otimes \ldots \otimes e_{i_r}
= \sum_{x=j_1, j_2, \ldots, j_k} e_{(i_1, \ldots, i_x-1, 1, i_{x+1}, \ldots, i_r)}
\]
as \( i_x = 2 \) if and only if \( x = j_1, j_2, \ldots, j_k \). Now for \( x = j_\rho \) for some \( 1 \leq \rho \leq k \) we have \((i_1, \ldots, i_{x-1}, 1, i_{x+1}, \ldots, i_r) = (i_1, \ldots, i_{\rho-1}, 1, i_{\rho+1}, \ldots, i_r)\), which will have twos in positions \( j_1, \ldots, j_{\rho-1}, j_{\rho+1}, \ldots, j_k \). Therefore
\[
e(e^j) = \sum_{1 \leq \rho \leq k} e_{i_1, \ldots, i_{\rho-1}, 1, i_{\rho+1}, \ldots, i_k} = \sum_{l \leq j} e_{i_l}.
\]
Similarly, we calculate
\[
f(e^j) = \sum_{1 \leq x \leq r} e_{i_1} \otimes \ldots \otimes e_{i_x-1} \otimes e_{i_x} \otimes e_{i_{x+1}} \otimes \ldots \otimes e_{i_r}
= \sum_{1 \leq \rho \leq r} e_{(i_1, \ldots, i_{x-1}, 2, i_{x+1}, \ldots, i_r)}
= \sum_{l \leq J(r, k+1) \atop j \leq l} e_{i_l},
\]
as for \( x \neq j_1, \ldots, j_k \) the multi-index \((i_1, \ldots, i_{x-1}, 2, i_{x+1}, \ldots, i_r)\) has twos in positions \( j_1, \ldots, j_k \) and \( x \).

Proposition 3.1.5 For all non-negative integers \( x \) and all \( j \in J(r, k) \),
\[
e(x)(e^j) = \sum_{l \leq J(r, k-x) \atop j \leq l} e_{i_l}, \quad \text{and} \quad f(x)(e^j) = \sum_{l \leq J(r, k+x) \atop j \leq l} e_{i_l}.
\]
Note that if \( x > k \) then \( e(x) = 0 \).

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Proof. Let \( j = (j_1, j_2, \ldots, j_k) \in J(r, k) \), and \( x \) be a non-negative integer. We will prove this result by induction on \( x \). If \( x = 0 \) then \( e^{(x)} = e^0 = 1 \), and then \( e^{(0)}(e_{ij}) = e_{ij} = \sum_{l \in J(r, k), l \subseteq j} e_{il} \). Similarly \( f^{(0)}(e_{ij}) = e_{ij} = \sum_{l \in J(r, k), j \subseteq l} e_{il} \), so this result holds for \( x = 0 \). Also, by Proposition 3.1.4, the result holds for \( x = 1 \). Suppose next that \( y \geq 0 \) and

\[
e^{(y)}(e_{ij}) = \sum_{l \in J(r, k-y), l \subseteq j} e_{il} \quad \text{and} \quad f^{(y)}(e_{ij}) = \sum_{l \in J(r, k+y), j \subseteq l} e_{il}.
\]

Then in \( S_Q(2, r) \) we have that

\[
e^{(y+1)}(e_{ij}) = \frac{e^{y+1}}{(y+1)!} = \frac{e e^y}{(y+1)!} = \frac{e}{(y+1)} e^{(y)}
\]

and hence

\[
e^{(y+1)}(e_{ij}) = \frac{e}{(y+1)} (e^{(y)}(e_{ij})) = \frac{e}{(y+1)} \left( \sum_{l \in J(r, k-y), l \subseteq j} e_{il} \right) = \frac{1}{y+1} \left( \sum_{l \in J(r, k-y), l \subseteq j} e(e_{ij}) \right) = \frac{1}{y+1} \left( \sum_{l \in J(r, k-y), l \subseteq j} \left( \sum_{n \in J(r, k-y-1), n \subseteq l} e_{in} \right) \right),
\]

using Proposition 3.1.4. We now consider

\[
T := \sum_{l \in J(r, k-y), l \subseteq j} \left( \sum_{n \in J(r, k-y-1), n \subseteq l} e_{in} \right), \quad (3.1)
\]

which is a sum of elements of the form \( e_{in} \) for \( n \in J(r, k-y-1) \). Let \( n \in J(r, k-y-1) \), we want to see how many times \( e_{in} \) appears in (3.1). For \( e_{in} \) to appear in (3.1) we need to have that \( n \subseteq l \) where \( l \in J(r, k-y) \) with \( l \subseteq j \), i.e. \( n \subseteq j = (j_1, j_2, \ldots, j_k) \). Suppose \( n = (j_1, j_2, \ldots, j_{k-y-1}) \). Then \( e_{in} \) appears in (3.1) for all \( l \in J(r, k-y) \) with \( n = (j_1, j_2, \ldots, j_{k-y-1}) \subseteq l \subseteq j = (j_1, j_2, \ldots, j_k) \). Therefore \( e_{in} \) appears in (3.1) for all \( l = (j_1, j_2, \ldots, j_{k-y-1}, \rho) \) where \( \rho \in \{j_{k-y}, j_{k-y+1}, \ldots, j_k\} \). So \( e_{in} \) appears in (3.1) precisely \( y+1 \) times. Similarly if \( n \in J(r, k-y-1) \) with \( n \subseteq j \) then \( e_{in} \) appears in (3.1) precisely \( y+1 \) times, otherwise \( e_{in} \) does not appear in (3.1). Therefore,

\[
T = (y+1) \sum_{n \in J(r, k-y-1), n \subseteq j} e_{in}.
\]
So we have that
\[ e^{(y+1)}(e_{ij}) = \sum_{n \in J(r,k-(y+1))} e_{in}, \]
and hence this also holds in \( S_K(2,r) \). Similarly, using Proposition 3.1.4, in \( S_Q(2,r) \) we have that,
\[ f^{(y+1)}(e_{ij}) = \frac{1}{y+1} \left( \sum_{l \in J(r,k+y)} \left( \sum_{j \leq l} \sum_{n \in J(r,k+y+1)} e_{in} \right) \right). \]

We now consider
\[ U := \sum_{l \in J(r,k+y)} \left( \sum_{j \leq l} \sum_{n \in J(r,k+y+1)} e_{in} \right), \quad (3.2) \]

Let \( n = (n_1, n_2, \ldots, n_{k+y+1}) \in J(r, k + y + 1) \). We want to see how many times \( e_{in} \) appears in (3.2). For \( e_{in} \) to appear, we need to have \( l \subseteq n \) where \( l \in J(r, k + y) \) with \( j \subseteq l \). So we have \( j = (j_1, j_2, \ldots, j_k) \subseteq l \subseteq n \). Suppose that \( n = (j_1, j_2, \ldots, j_k, n_{k+1}, \ldots, n_{k+y+1}) \), then \( e_{in} \) appears in (3.2) for all \( l \in J(r, k + y) \) with \( j \subseteq l \subseteq n \), which is all \( l = (j_1, \ldots, j_k, l_{k+1}, \ldots, l_{k+y}) \) with \( (l_{k+1}, \ldots, l_{k+y}) \subseteq (n_{k+1}, \ldots, n_{k+y+1}) \). So, given \( n \) we have \( \binom{y+1}{y} = y + 1 \) choices for \( l \), and so \( e_{in} \) appears in (3.2) precisely \( y + 1 \) times. Similarly, for all \( n \in J(r, k + y + 1) \) with \( j \subseteq n \) we have that \( e_{in} \) appears in (3.2) precisely \( y + 1 \) times, otherwise \( e_{in} \) does not appear. So we have,
\[ U = (y+1) \sum_{n \in J(r,k+y+1)} e_{in}, \]

which gives,
\[ f^{(y+1)}(e_{ij}) = \sum_{n \in J(r,k+y+1)} e_{in}, \]
and hence this also holds in \( S_K(2,r) \). Therefore we have proved this proposition by induction. Also, if \( x > k \) then \( J(r, k - x) \) is not defined, and in this case we have \( e^{(x)} = 0 \).  

\[ 36 \]
3.2 The Action of $e_{m,\gamma}$ on $E^\otimes r$

Throughout this section, let $m = r - 2k \geq 0$, $\gamma = k - s \geq 0$ and $\alpha = r - 2s \geq 0$ with \((\gamma) = \binom{r-2s}{k-s} \equiv 0 \mod 2\). Then by Theorem 2.4.9 we know that

\[
e_{m,\gamma} = \prod_{\gamma_u = 1} b(2^n) \prod_{\gamma_u = 0, \alpha_u = 1} (1 - b(2^n))
\]

is a primitive orthogonal idempotent of $\text{End}_{K\Sigma_n}(M^{(r-k,k)})$, and in particular the Young module $Y^{(r-s,\gamma)} = \text{Im } e_{m,\gamma}$. In this section we will determine the action of $e_{m,\gamma}$ on the $K\Sigma_r$-module $E^\otimes r$, by first for a natural number $x$ determining the action of $b(x) = 1_\lambda f(x)e(x)1_\lambda$, where $\lambda = (r-k,k)$, on $E^\otimes r$.

For natural numbers $s$ and $k$ we define a map

$$\psi_{s,k} : J(r,s) \times J(r,k) \rightarrow \{0,1,\ldots,\min\{s,k\}\}$$

by

$$\psi_{s,k}((j_1, j_2, \ldots, j_s), (l_1, l_2, \ldots, l_k)) = \text{Card}((\{j_1, j_2, \ldots, j_s\} \cap \{l_1, l_2, \ldots, l_k\})).$$

For all $j \in J(r,s)$, $l \in J(r,k)$ note that $\psi_{s,k}(j,l) = \psi_{k,s}(l,j)$, and from now on let $\psi := \psi_{s,k}$.

**Lemma 3.2.1** For $j \in J(r,k)$, and $x$ a natural number we have,

$$b(x)(e_{ij}) = \sum_{l \in J(r,k)} \binom{\psi(j,l)}{k-x} e_{il}.$$

**Proof.** Let $j = (j_1, j_2, \ldots, j_k) \in J(r,k)$. If $x > k$ then for all $l \in J(r,k)$ we have $\binom{\psi(j,l)}{k-x} = 0$, so the result follows from the fact that $e(x) = 0$ for $x > k$ (see Proposition...
3.1.5). Now suppose \( x \leq k \). We have that
\[
\begin{align*}
  b(x)(e_i) &= 1_{\lambda} \mathbf{f}^{(x)}(e^{(x)}) 1_{\lambda}(e_i) \\
  &= \mathbf{f}^{(x)}(e^{(x)}(e_i)) \\
  &= \mathbf{f}^{(x)}(\sum_{n \in J(\mathbf{r},k-x)} e_i^{(n)}) \\
  &= \sum_{n \in J(\mathbf{r},k-x)} \mathbf{f}^{(x)}(e_i^{(n)}) \\
  &= \sum_{n \in J(\mathbf{r},k-x)} \left( \sum_{\ell \in J(\mathbf{r},k)} e_i^{(\ell)} \right),
\end{align*}
\]  

using Proposition 3.1.5.

If \( l = (l_1, l_2, \ldots, l_k) \in J(\mathbf{r}, k) \), then \( e_{ij} \) appears in (3.3) for any \( n \in J(\mathbf{r}, k-x) \) with \( n \subseteq j = (j_1, j_2, \ldots, j_k) \) and \( n \subseteq l = (l_1, l_2, \ldots, l_k) \). Therefore, \( e_{ij} \) is in (3.3) for any \( n = (n_1, n_2, \ldots, n_{k-x}) \in J(\mathbf{r}, k-x) \) with each \( n_\rho \in \{j_1, j_2, \ldots, j_k\} \cap \{l_1, l_2, \ldots, l_k\} \), for \( 1 \leq \rho \leq k-x \). So, for each \( l \in J(\mathbf{r}, k) \), \( e_{ij} \) appears in this sum \( \binom{\psi(j,l)}{k-x} \) times, and the result follows.

**Lemma 3.2.2** Let \( x \) and \( u \) be non-negative integers with \( 2^u > x \), then \( b(x)b(2^u) = b(x + 2^u) \). In particular if \( \gamma \) is a natural number with 2-adic decomposition \( \gamma = \sum_{u \in \mathbb{N}_0} \gamma_u 2^u \), then
\[
\prod_{\gamma_u = 1} b(2^u) = b(\sum_{\gamma_u = 1} 2^u) = b(\gamma).
\]

**Proof.** By Proposition 2.4.4 we have
\[
\begin{align*}
  b(x)b(2^u) &= \sum_{l=0}^{x} \binom{2^u + l}{x} \binom{2^u + l}{l} \binom{m + x + 2^u}{x - l} b(2^u + l).
\end{align*}
\]

As \( x < 2^u \) we have a 2-adic decomposition of \( x \) as \( x = \sum_{i=0}^{u-1} x_i 2^i \). Now let \( l \) be an integer with \( 0 \leq l \leq x \), then we have a 2-adic decomposition \( l = \sum_{i=0}^{u-1} l_i 2^i \) of \( l \). By
Lemma 1.3.2 we have
\[
\binom{2^u + l}{x} = \binom{\binom{l_0}{x_0} \binom{l_1}{x_1} \ldots \binom{l_{u-1}}{x_{u-1}}}{1} \mod 2
\]
\[
= \binom{\binom{l_0}{x_0} \binom{l_1}{x_1} \ldots \binom{l_{u-1}}{x_{u-1}}}{x_{u-1}} \mod 2
\]
\[
= \binom{l}{x} \mod 2.
\]
So, as \( K \) has characteristic two, for \( 0 \leq l \leq x \) we have \( \binom{2^u + l}{x} = \begin{cases} 0 & \text{if } 0 \leq l < x \\ 1 & \text{if } l = x \end{cases} \).
So by (3.4) we have
\[
b(x)\gamma(2^u) = \binom{2^u + x}{x} \binom{2^u + x}{x} \binom{m + x + 2^u}{x - x} \gamma(2^u + x) = \gamma(2^u + x),
\]
and the result follows. \( \square \)

Recall, from Section 1.3, for non-negative integers \( m \) and \( n \) we write \( m \subseteq 2 \) if \( m \) is 2-contained in \( n \).

**Lemma 3.2.3** Let \( x \) be a natural number with 2-adic decomposition \( x = \sum_{i \in N_0} x_i 2^i \).
Then we have
\[
\prod_{x_i=1} (1 - \gamma(2^u)) = \sum_{y \in N_0 \atop y \leq 2^x} \gamma(y)
\]

*Proof*. Let \( x \in N \) and \( 0 \leq u_1 < u_2 < \ldots < u_t \) be integers such that \( x_{u_1} = x_{u_2} = \ldots = x_{u_t} = 1 \) and \( x_\rho = 0 \) for \( \rho \not\in \{u_1, u_2, \ldots, u_t\} \). Then we have that \( x = 2^{u_1} + 2^{u_2} + \ldots + 2^{u_t} \), and we will prove this lemma by induction on \( t \).

For \( t = 1 \) we have that \( x = 2^{u_1} \) for some \( u_1 \geq 0 \) and \( \prod_{x_i=1} (1 - \gamma(2^u)) = 1 - \gamma(2^{u_1}) = 1 + \gamma(2^{u_1}), \) as \( K \) has characteristic two. Now a non-negative integer \( y \) is 2-contained in \( 2^{u_1} \) if and only if \( y = 0 \) or \( y = 2^{u_1} \). So
\[
\sum_{y \in N_0 \atop y \leq 2^x} \gamma(y) = \gamma(0) + \gamma(2^{u_1}) = 1 + \gamma(2^{u_1}),
\]
as \( \gamma(0) = 1 \) by definition. So the result holds for \( t = 1 \).
Let $t = s$, and suppose that for $x$ of the form $x = 2^{u_1} + 2^{u_2} + \ldots + 2^{u_s}$ we have

$$\prod_{i=1}^{s}(1 - b(2^{u_i})) = \sum_{y \in \mathbb{N}_0 \atop y \leq 2^x} b(y).$$

Now, let $t = s + 1$ and $x' = 2^{u_1} + 2^{u_2} + \ldots + 2^{u_s} + 2^{u_{s+1}}$ for some integers $0 \leq u_1 < u_2 < \ldots < u_{s+1}$. Let $x = 2^{u_1} + 2^{u_2} + \ldots + 2^{u_s}$, then $x' = x + 2^{u_{s+1}}$ and $2^{u_{s+1}} > x$. Then

$$\prod_{i=1}^{s+1}(1 - b(2^{u_i})) = (1 - b(2^{u_{s+1}})) \prod_{i=1}^{s}(1 - b(2^{u_i}))$$

$$= (1 - b(2^{u_{s+1}})) \sum_{y \in \mathbb{N}_0 \atop y \leq 2^x} b(y)$$

$$= \sum_{y \in \mathbb{N}_0 \atop y \leq 2^x} b(y) + \sum_{y \in \mathbb{N}_0 \atop y \leq 2^{u_{s+1}}} b(y),$$

using the induction hypothesis and the fact that $K$ has characteristic two. Now we know that $x < 2^{u_{s+1}}$ and if $y \subseteq_2 x$ then $y \leq x$ so $y < 2^{u_{s+1}}$. Therefore by Lemma 3.2.2 and using the fact that $\text{End}_{K \Sigma_r}(M^\lambda)$ is commutative (see Lemma 2.4.6) we have $b(2^{u_{s+1}})b(y) = b(y + 2^{u_{s+1}})$. Now, for $y' \in \mathbb{N}_0$ we have that $y' \subseteq_2 x' = x + 2^{u_{s+1}}$ if and only if $y' \subseteq_2 x$ or $y' = 2^{u_{s+1}} + y$ for some $y \in \mathbb{N}_0$ with $y \subseteq_2 x$. Therefore

$$\prod_{i=1}^{s+1}(1 - b(2^{u_i})) = \sum_{y \in \mathbb{N}_0 \atop y \leq 2^x} b(y) + \sum_{y \in \mathbb{N}_0 \atop y \leq 2^{u_{s+1}}} b(y + 2^{u_{s+1}})$$

$$= \sum_{y' \in \mathbb{N}_0 \atop y' \subseteq_2 x'} b(y'),$$

and so the result is proved by induction on $t$. \hfill $\square$

In particular, as $K$ has characteristic two and $y \subseteq_2 x$ if and only if \(\binom{x}{y} \equiv 1 \mod 2\) (by Lemma 1.3.2) we have

**Corollary 3.2.4** For a natural number $x$,

$$\prod_{x_u=1} (1 - b(2^u)) = \sum_{y \in \mathbb{N}_0} \binom{x}{y} b(y).$$
**Proposition 3.2.5** Let \( j \in J(r, k) \), then for any natural number \( x \) with 2-adic decomposition given by \( x = \sum_{i \in \mathbb{N}_0} x_i 2^i \) we have,

\[
\prod_{x_i = 1} (1 - b(2^u))(e_{i,i}) = \sum_{l \in J(r,k)} \left( x + \psi(j, l) \right) k \right) e_{i,i}.
\]

**Proof.** Let \( j \in J(r, k) \), then using Corollary 3.2.4 we have

\[
\prod_{x_i = 1} (1 - b(2^u))(e_{i,i}) = \sum_{y \in \mathbb{N}_0} \left( \frac{x}{y} \right) b(y)(e_{i,i})
= \sum_{y \in \mathbb{N}_0} \left( \frac{x}{y} \right) \sum_{l \in J(r,k)} \left( \psi(j, l) \right) k - y e_{i,i} \quad \text{(by Lemma 3.2.1)}
= \sum_{l \in J(r,k)} \sum_{y \in \mathbb{N}_0} \left( \frac{x}{y} \right) \left( \psi(j, l) \right) k - y e_{i,i}.
\]

Now, for \( l \in J(r, k) \), by Lemma 1.3.1 (3) we have that \( \sum_{y \in \mathbb{N}_0} \left( \frac{x}{y} \right) \left( \psi(j, l) \right) = \left( \frac{x + \psi(j, l)}{k} \right) \), so the result follows. \( \square \)

In particular, if we take \( \gamma = 0 \) (so \( k = s \)), then \( \alpha = r - 2s = r - 2k = m \), so \( e_{m,0} = \prod_{x_i = 1} (1 - b(2^u)) \), so we have the following corollary.

**Corollary 3.2.6** For \( j \in J(r, k) \),

\[
e_{m,0}(e_{i,i}) = \sum_{l \in J(r,k)} \left( m + \psi(j, l) \right) k e_{i,i}.
\]

Before we calculate the action of \( e_{m,\gamma} \) on \( M(r-k,k) \) in general we need the following lemma. Recall that, throughout this section, \((\cdot) = (r f c E2 /) \equiv 1 \) modulo two.

**Lemma 3.2.7** For all \( n \in \mathbb{N}_0 \),

\[
\left( \begin{array}{c} k - s + n \\ n \\ \end{array} \right) \sum_{y \in \mathbb{N}_0} \left( \begin{array}{c} r - k - s \\ y \\ \end{array} \right) \left( \begin{array}{c} k - s + n \\ y \\ \end{array} \right) \left( \begin{array}{c} r - k - s + y \\ y - n \\ \end{array} \right) 
\equiv \left( \begin{array}{c} k - s + n \\ n \\ \end{array} \right) \left( \begin{array}{c} r - k - s \\ n \\ \end{array} \right) \text{ mod } 2.
\]

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Proof. If \( \binom{k-s+n}{n} \equiv 0 \mod 2 \) then this lemma is clearly true. Assume that \( \binom{k-s+n}{n} \equiv 1 \mod 2 \). Let \( t \in \mathbb{N} \) be such that the 2-adic decompositions of \( r-2s \) and \( k-s \) are as follows:

\[
\begin{align*}
  r-2s &= \alpha = \alpha_0 2^0 + \alpha_1 2^1 + \ldots + \alpha_t 2^t, \\
  k-s &= \gamma = \gamma_0 2^0 + \gamma_1 2^1 + \ldots + \gamma_t 2^t,
\end{align*}
\]

where \( \alpha_0, \ldots, \alpha_t, \gamma_0, \ldots, \gamma_t \in \{0,1\} \). Let \( u_1, \ldots, u_w \in \{0,1, \ldots, t\} \) be such that \( \gamma_{u_1} = \gamma_{u_2} = \ldots = \gamma_{u_w} = 1 \) and \( \gamma_\rho = 0 \) for all \( \rho \not\in \{u_1, \ldots, u_w\} \). Then, as \( \binom{\alpha}{\gamma} \equiv \binom{\alpha_{u_1}}{\gamma_{u_1}} \binom{\alpha_{u_2}}{\gamma_{u_2}} \ldots \binom{\alpha_{u_w}}{\gamma_{u_w}} \equiv 1 \mod 2 \) we have \( \alpha_{u_1} = \ldots = \alpha_{u_w} = 1 \). Let \( \beta_0, \beta_1, \ldots, \beta_t \in \{0,1\} \) be such that

\[
r - k - s = \beta_0 2^0 + \beta_1 2^1 + \ldots + \beta_t 2^t.
\]

Then, as \( r - k - s = (r - 2s) - (k - s) \) we have that \( \beta_{u_1} = \ldots = \beta_{u_w} = 0 \) and \( \beta_\rho = \alpha_\rho \) for all \( \rho \not\in \{u_1, \ldots, u_w\} \). Let \( z, x \in \mathbb{N} \) and \( v_1, v_2, \ldots, v_x \in \{0,1, \ldots, z\} \) be such that

\[
n = n_0 2^0 + n_1 2^1 + \ldots + n_x 2^x
\]

with \( n_{v_1} = \ldots = n_{v_x} = 1 \) and \( n_\rho = 0 \) for all \( \rho \not\in \{v_1, v_2, \ldots, v_x\} \). As \( \binom{k-s+n}{n} \equiv 1 \mod 2 \), we have that \( (k-s+n)_{v_1} = \ldots = (k-s+n)_{v_x} = 1 \). Now \( \gamma = k-s = (k-s+n)-n, \) so \( \gamma_{v_1} = \ldots = \gamma_{v_x} = 0, \) hence \( \{v_1, \ldots, v_x\} \cap \{u_1, \ldots, u_w\} = \emptyset \). So as \( k-s+n = \gamma + n \) we have \( (k-s+n)_{u_1} = \ldots = (k-s+n)_{u_w} = (k-s+n)_{v_1} = \ldots = (k-s+n)_{v_x} = 1 \) and \( (k-s+n)_\rho = 0 \) for \( \rho \not\in \{u_1, \ldots, u_w, v_1, \ldots, v_x\} \).

Now, let \( y = y_0 2^0 + y_1 2^1 + \ldots + y_t 2^t \) be such that \( \binom{r-k-s}{y} \equiv 1 \mod 2 \) and \( \binom{k-s+n}{y} \equiv 1 \mod 2 \). So \( y_{u_1} = \ldots = y_{u_w} = 0 \) and \( y_i = 0 \) for all \( i \not\in \{u_1, \ldots, u_w, v_1, \ldots, v_x\} \). So the only possible \( y_i \)'s equal to one are \( y_{v_1}, \ldots, y_{v_x} \), so \( y \leq n \). Now \( \binom{r-k-s+y}{y-n} = 0 \) if \( y < n \). So if \( \binom{r-k-s}{y} \binom{k-s+n}{y} \binom{r-k-s+y}{y-n} \equiv 1 \mod 2 \) then we must have \( y \geq n \), so \( y = n \). So we have

\[
\begin{align*}
  \binom{k-s+n}{n} \sum_{y \in \mathbb{N}} \binom{r-k-s}{y} \binom{k-s+n}{y} \binom{r-k-s+y}{y-n} \\
  = \binom{k-s+n}{n} \binom{r-k-s}{n} \binom{k-s+n}{n} \binom{r-k-s+n}{n-n} \\
  = \binom{k-s+n}{n} \binom{r-k-s}{n},
\end{align*}
\]

as required. \( \Box \)
Lemma 3.2.8 We have that
\[ e_{m,\gamma} = \sum_{n=0}^{s} \binom{k-s+n}{n} \binom{r-k-s}{n} b(k-s+n). \]

Proof. Recall that
\[ e_{m,\gamma} = \prod_{\gamma_u=1} b(2^u) \prod_{\alpha_u=1} (1 - b(2^u)). \]

Now, as \( \binom{s}{\gamma} = \binom{r-s}{k-s} \equiv 1 \mod 2 \), we have that \( \gamma_u = 0 \) and \( \alpha_u = 1 \) if and only if \( (\alpha - \gamma)_u = 1 \). So using Corollary 3.2.4 we have
\[ \prod_{\gamma_u=0, \alpha_u=1} (1 - b(2^u)) = \prod_{(\alpha - \gamma)_u=1} (1 - b(2^u)) = \sum_{y \in \mathbb{N}_0} \binom{r-k-s}{y} b(y), \]
as \( \alpha - \gamma = r - k - s \). Also by Lemma 3.2.2 we have \( \prod_{\gamma_u=1} b(2^u) = b(\gamma) = b(k-s) \). Therefore
\[ e_{m,\gamma} = \sum_{y \in \mathbb{N}_0} \binom{r-k-s}{y} b(k-s)b(y). \]

Now, by Proposition 2.4.4 we have,
\[ b(k-s)b(y) = \sum_{l \in \mathbb{N}_0} \binom{y+l}{k-s} \binom{y+l}{l} \binom{m+k-s+y}{k-s-l} b(y+l) \]
\[ = \sum_{n \in \mathbb{N}_0} \binom{k-s+n}{k-s} \binom{k-s+n}{k-s+n-y} \binom{m+k-s+y}{y-n} b(k-s+n) \]
\[ = \sum_{n \in \mathbb{N}_0} \binom{k-s+n}{n} \binom{k-s+n}{y} \binom{r-k-s+y}{y-n} b(k-s+n), \]
where \( n = y+l-k+s \), and using Lemma 1.3.1 (1) and \( m = r-2k \). So,
\[ e_{m,\gamma} = \sum_{y \in \mathbb{N}_0} \binom{r-k-s}{y} \sum_{n \in \mathbb{N}_0} \binom{k-s+n}{n} \binom{k-s+n}{y} \binom{r-k-s+y}{y-n} b(k-s+n) \]
\[ = \sum_{n \in \mathbb{N}_0} \binom{k-s+n}{n} \sum_{y \in \mathbb{N}_0} \binom{r-k-s}{y} \binom{k-s+n}{y} \binom{r-k-s+y}{y-n} b(k-s+n) \]
\[ = \sum_{n \in \mathbb{N}_0} \binom{k-s+n}{n} \binom{r-k-s}{n} b(k-s+n), \]
by Lemma 3.2.7. If \( n > s \) then \( k-s+n > k \) so \( b(k-s+n) = 0 \), and the result follows. \( \square \)
Proposition 3.2.9 For $j \in J(r, k)$,

$$e_{m, r}(e_{ij}) = \sum_{l \in J(r, k)} \binom{r-k-s}{s} \binom{r}{n} \binom{k-s+n}{n} b_{k-s+n}(e_{ij})$$

Proof. For $j \in J(r, k)$, by Lemma 3.2.8 we have:

$$e_{m, r}(e_{ij}) = \sum_{n=0}^{s} \binom{k-s+n}{n} \binom{r-k-s}{n} \sum_{l \in J(r, k)} \binom{\psi(j, l)}{s-n} e_{il} \text{ (by Lemma 3.2.1)}$$

Now to complete this proof we will show that, for $l \in J(r, k)$,

$$\sum_{n=0}^{s} \binom{k-s+n}{n} \binom{r-k-s}{n} \binom{\psi(j, l)}{s-n} \equiv \binom{r-k-s+\psi(j, l)}{s} \mod 2.$$

Let $t \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_t, \gamma_1, \ldots, \gamma_t \in \{0,1\}$ be such that the 2-adic decompositions of $\alpha = r-2s$ and $\gamma = k-s$ are as follows:

$$r-2s = \alpha_0 2^0 + \alpha_1 2^1 + \ldots + \alpha_t 2^t,$$

$$k-s = \gamma_0 2^0 + \gamma_1 2^1 + \ldots + \gamma_t 2^t.$$

Let $u_1, \ldots, u_w \in \{0,1,\ldots,t\}$ be such that $\gamma_{u_1} = \gamma_{u_2} = \ldots = \gamma_{u_w} = 1$ and $\gamma_{\rho} = 0$ for all $\rho \not\in \{u_1, \ldots, u_w\}$. Then, as $\binom{\alpha}{\gamma} \equiv \binom{\alpha_1}{\gamma_1} \ldots \binom{\alpha_t}{\gamma_t} \equiv 1 \mod 2$ we have $\alpha_{u_1} = \ldots = \alpha_{u_w} = 1$. Let $r-k-s$ have 2-adic decomposition

$$r-k-s = \beta_0 2^0 + \beta_1 2^1 + \ldots + \beta_t 2^t.$$

Then, as $r-k-s = (r-2s)-(k-s)$ we have that $\beta_{u_1} = \ldots = \beta_{u_w} = 0$ and $\beta_{\rho} = \alpha_{\rho}$ for all $\rho \not\in \{u_1, \ldots, u_w\}$.

Let $n \in \{0,1,2,\ldots,s\}$ have a 2-adic decomposition $n = n_0 2^0 + n_1 2^1 + \ldots + n_z 2^z$, for some $z \in \mathbb{N}_0$. If $\binom{r-k-s}{n} \equiv 1 \mod 2$ then $n_{u_1} = \ldots = n_{u_w} = 0$. Now $\binom{k-s+n}{n} = \ldots = $
\[(k-s+n) \equiv 1 \mod 2 \text{ if and only if } (k-s+n)_{u_1} = \ldots = (k-s+n)_{u_w} = 1 \text{ if and only if } n_{u_1} = \ldots = n_{u_w} = 0. \] So if \((r-k-s) \equiv 1 \mod 2\) then \((k-s+n) \equiv 1 \mod 2\). Therefore,

\[
\sum_{n=0}^{s} \binom{k-s+n}{n} \binom{r-k-s}{n} \binom{\psi(j,l)}{s-n} \\
\equiv \sum_{n=0}^{s} \binom{r-k-s}{n} \binom{\psi(j,l)}{s-n} \mod 2 \\
\equiv \binom{r-k-s+\psi(j,l)}{s} \mod 2,
\]

by Lemma 1.3.1 (3).

\[\square\]

So we have now calculated the action of \(e_{m,\gamma}\) on the module \(E^\otimes \).

**Remark 3.2.10** If \(k = s\) then Proposition 3.2.9 gives that

\[e_{m,0}(e_{i;j}) = \sum_{l \in \mathcal{J}(r,k)} \binom{r - 2k + \psi(j,l)}{k} e_{i;l},\]

which is what we expected, by Corollary 3.2.6 (as \(m = r - 2k\)).
Chapter 4

Construction of Young Modules

Throughout this chapter let $K$ be a field of characteristic two. We fix a natural number $r$ and non-negative integers $k$ and $s$ such that $m = r - 2k \geq 0$, $\gamma = k - s \geq 0$ and $\alpha = r - 2s \geq 0$. Let $\binom{r}{k} \mod 2 \neq 0 \mod 2$, so that the Young module $Y^{(r-s,s)}$ is a direct summand of the permutation module $M^{(r-k,k)}$ (see Proposition 1.7.4). Then by Theorem 2.4.9 we know that $Y^{(r-s,s)} = \text{Im } e_{m,\gamma}$ where the $e_{m,\gamma}$ are the primitive orthogonal idempotents of $\text{End}_{K^r}(M^{(r-k,k)})$, with action described in Proposition 3.2.9.

In this chapter, we give an explicit construction of the Young modules $Y^{(r-s,s)}$ in $M^{(r-k,k)}$. Firstly, in Section 4.1, we consider the case where $s = k$, and define some vectors $v_a$ for all $a \in J(r,k)$ (see Definition 4.1.2). Throughout the rest of the section we show that the Young module $Y^{(r-k,k)}$ in $M^{(r-k,k)}$ is spanned by these vectors using the fact that $Y^{(r-k,k)} = \text{Im } e_{m,0}$. In Proposition 4.1.8 we show that $\text{span}\{v_a : a \in J(r,k)\} \subseteq \text{Im } e_{m,0}$, then in Proposition 4.1.9 we show that $\text{Im } e_{m,0} \subseteq \text{span}\{v_a : a \in J(r,k)\}$.

In Section 4.2 we consider the module $Y^{(r-s,s)}$ in $M^{(r-k,k)}$ in the case where $s \neq k$. Here (see (4.3)) we define some vectors $w_a$, for all $a \in J(r,s)$. In a similar way to the previous case, we use the fact that $Y^{(r-s,s)} = \text{Im } e_{m,\gamma}$ and our strategy is to show that $\text{Im } e_{m,\gamma} \subseteq \text{span}\{w_a : a \in J(r,s)\}$ (see Proposition 4.2.5) and $\text{span}\{w_a : a \in J(r,s)\} \subseteq \text{Im } e_{m,\gamma}$ (see Proposition 4.2.9). In this case the calculations are more complicated, and Section 4.2 contains lots of calculations with binomial coefficients,
which are needed to prove the results.

### 4.1 Constructing the Young Module $Y^{(r-k,k)}$ in $M^{(r-k,k)}$

Recall that in Section 3.1 we defined $J(r,k) \subseteq I(r,k)$, and for $j = (j_1, j_2, \ldots, j_k) \in J(r,k)$ we defined $i_j$ to be the element in $I(2,r)$ with twos in positions $j_1, j_2, \ldots, j_k$ and ones in the other $r - k$ positions. In this way, we have that $M^{(r-k,k)}$ has basis \{e_{i_j} : j \in J(r,k)\} over $K$. Now, for $j = (j_1, j_2, \ldots, j_k) \in J(r,k)$, we define $j' := (j'_1, j'_2, \ldots, j'_{r-k}) \in J(r, r-k)$ where each $j'_\rho \in \mathbb{Z}\setminus\{j_1, j_2, \ldots, j_r\}$. So for $j \in J(r,k)$, $i_j \in I(2,r)$ has twos in positions $j_1, j_2, \ldots, j_k$ and ones in positions $j'_1, j'_2, \ldots, j'_{r-k}$.

For example, if $r = 5, k = 2$ and $j = (1,2) \in J(5,2)$, then $j' = (3,4,5) \in J(5,3)$ and $i_j = (2,2,1,1,1) \in I(2,5)$.

**Lemma 4.1.1** Let $k$ and $l$ be natural numbers. For $i \in J(r,k)$, $j \in J(r,l)$ and $\sigma \in \Sigma_r$ we have

- $i \subseteq j'$ if and only if $j \subseteq i'$,
- $i \subseteq j$ if and only if $i \cdot \sigma \subseteq j \cdot \sigma$,
- $i' \cdot \sigma = (i \cdot \sigma)'$.

**Proof.** Let $i = (i_1, i_2, \ldots, i_k) \in J(r,k)$ and $j = (j_1, j_2, \ldots, j_l) \in J(r,l)$.

If $i \subseteq j'$ then $k \leq r - l$ and $j' \in J(r,r-l)$ contains $i_1, i_2, \ldots, i_k$, so $j$ does not contain $i_1, i_2, \ldots, i_k$ which means that $j \subseteq i'$. So if $i \subseteq j'$ then $j \subseteq i'$, and conversely if $j \subseteq i'$ then $i \subseteq j'$, as required.

Let $\sigma \in \Sigma_r$ and $i \subseteq j$ so that \{i_1, i_2, \ldots, i_k\} $\subseteq$ \{j_1, j_2, \ldots, j_k\}. Therefore we have that \{\sigma^{-1}(i_1), \sigma^{-1}(i_2), \ldots, \sigma^{-1}(i_k)\} $\subseteq$ \{\sigma^{-1}(j_1), \sigma^{-1}(j_2), \ldots, \sigma^{-1}(j_k)\}, so $i \cdot \sigma \subseteq j \cdot \sigma$. So $i \subseteq j$ implies that $i \cdot \sigma \subseteq j \cdot \sigma$, similarly applying $\sigma^{-1}$ gives that $i \cdot \sigma \subseteq j \cdot \sigma$ implies that $i \cdot \sigma \cdot \sigma^{-1} \subseteq j \cdot \sigma \cdot \sigma^{-1}$, hence $i \subseteq j$. So $i \subseteq j$ if and only if $i \cdot \sigma \subseteq j \cdot \sigma$. 

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Let $i' = (a_1, a_2, \ldots, a_{r-k}) \in J(r, r - k)$, then \{i_1, i_2, \ldots, i_k, a_1, a_2, \ldots, a_{r-k}\} = r, so \{\sigma^{-1}(i_1), \sigma^{-1}(i_2), \ldots, \sigma^{-1}(i_k), \sigma^{-1}(a_1), \sigma^{-1}(a_2), \ldots, \sigma^{-1}(a_{r-k})\} = r$. Now $i \cdot \sigma \in J(r, k)$ contains $\sigma^{-1}(i_1), \sigma^{-1}(i_2), \ldots, \sigma^{-1}(i_k)$, so by definition $(i \cdot \sigma)' \in J(r, r - k)$ contains $\sigma^{-1}(a_1), \sigma^{-1}(a_2), \ldots, \sigma^{-1}(a_{r-k})$. Now, as $i' = (a_1, a_2, \ldots, a_{r-k})$, we know that $i' \cdot \sigma \in J(r, r - k)$ contains $\sigma^{-1}(a_1), \sigma^{-1}(a_2), \ldots, \sigma^{-1}(a_{r-k})$, therefore $(i \cdot \sigma)' = i' \cdot \sigma$.

**Definition 4.1.2** For $a \in J(r, k)$ define

$$v_a := \sum_{j \in J(r, k)} e_{ij}.$$  

So if $a = (a_1, a_2, \ldots, a_k) \in J(r, k)$, then $v_a$ is the sum of all the $e_{ij}$ with $i^j \in I(2, r)$ having ones in positions $a_1, a_2, \ldots, a_k$.

**Example 4.1.3** Let $r = 5$, $k = 2$, and $a = (1, 2) \in J(5, 2)$. Then $a' = (3, 4, 5)$ and

$$v_{(1,2)} = \sum_{j \in J(5,2)} e_{ij} = e_{i(3,4)} + e_{i(3,5)} + e_{i(4,5)} = e_{(1,1,2,2,1)} + e_{(1,1,2,1,2)} + e_{(1,1,1,2,2)}.$$

From Theorem 1.7.2 we know that $Y^{(r-k,k)}$ is a direct summand in $M^{(r-k,k)}$, and we will prove the following theorem:

**Theorem 4.1.4** For $a \in J(r, k)$ let $v_a$ be defined as in Definition 4.1.2. Then

$$Y^{(r-k,k)} = \text{span}\{v_a : a \in J(r, k)\}.$$  

Fix $r$ and $k$ and let $Y := \text{span}\{v_a : a \in J(r, k)\}$. We will first show that $Y$ is a submodule of $M^{(r-k,k)}$ with $S^{(r-k,k)} \subseteq Y$. So if we could also show that $Y$ was an indecomposable direct summand of $M^{(r-k,k)}$ then $Y = Y^{(r-k,k)}$ and so Theorem 4.1.4 would be proved. This latter part of the proof will be completed later in this section.

**Lemma 4.1.5** $Y$ is a $K\Sigma_r$-submodule of $M^{(r-k,k)}$.

**Proof.** Clearly $Y \subseteq M^{(r-k,k)}$. Recall, from Section 3.1, that we defined an action of $\Sigma_r$ on $J(r, k)$ in the following way. For $j = (j_1, j_2, \ldots, j_k) \in J(r, k)$ and $\sigma \in \Sigma_r$, $j \cdot \sigma$ is
the multi-index in $J(r, k)$ containing $\sigma^{-1}(j_1), \sigma^{-1}(j_2), \ldots, \sigma^{-1}(j_k)$. This gave us that $i^j \sigma = i^j \sigma$, and so $e_{ij} \cdot \sigma = e_{ij} \sigma$. Hence for $a = (a_1, a_2, \ldots, a_k) \in J(r, k)$ we have

$$v_a \cdot \sigma = \sum_{j \in J(r, k) \cap a'} e_{ij} \cdot \sigma = \sum_{j \in J(r, k) \cap a'} e_{ij} \sigma = \sum_{l \in J(r, k) \cap a' \sigma} e_{il} = \sum_{l \in J(r, k) \cap (a \sigma)'} e_{il} = v_{a' \sigma},$$

using parts two and three of Lemma 4.1.1. Hence $v_a \cdot \sigma \in Y$ as $a \cdot \sigma \in J(r, k)$. So $Y$ is a $K\Sigma_r$-submodule of $M^{(r-k,k)}$.

\begin{proof}
Let $\lambda = (r-k,k)$. Then, as in Example 1.6.3, $S^\lambda$ is the cyclic $K\Sigma_r$-module generated by

$$\sum_{\sigma \in C_T} \text{sgn}(\sigma) e_{l \sigma},$$

where $C_T = \{(1, r-k+1), (2, r-k+2), \ldots, (k, r)\}$ and $l = (1, 1, \ldots, 1, 2, \ldots, 2)$. Let $a = (1, 2, \ldots, k) \in J(r, k)$ and $b = a' = (k+1, k+2, \ldots, r) \in J(r, r-k)$. Then $Y$ is the cyclic $K\Sigma_r$-module generated by

$$v_a = \sum_{j \in J(r, k) \cap b} e_{ij}.$$

By Theorem 1.5.3 (the Submodule Theorem), as $Y$ is a $K\Sigma_r$-submodule of $M^\lambda$, either $Y \supseteq S^\lambda$ or $Y \subseteq (S^\lambda)^\perp$. Now $(S^\lambda)^\perp := \{x \in M^\lambda : \langle x, u \rangle = 0 \text{ for all } u \in S^\lambda\}$ where $\langle \cdot, \cdot \rangle$ is the bilinear form defined on $M^\lambda$ by

$$\langle e_{ij}, e_{ij} \rangle = \begin{cases} 1 & \text{if } e_{ij} = e_{ij}, \\ 0 & \text{otherwise.} \end{cases}$$

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Suppose that $Y \subseteq (S^\lambda)^\perp$. Then $v_a \in (S^\lambda)^\perp$, and so $\langle v_a, u \rangle = 0$ for all $u \in S^\lambda$. In particular

\[
0 = \langle v_a, \sum_{\sigma \in C_T} \text{sgn}(\sigma)e_{l \sigma} \rangle = \left( \sum_{j \notin J(r,k)} \sum_{\sigma \in C_T} e_{ij}, \sum_{\sigma \in C_T} \text{sgn}(\sigma)e_{l \sigma} \right)
\]

\[
= \sum_{\sigma \in C_T} \sum_{j \notin J(r,k)} \text{sgn}(\sigma)\langle e_{ij}, e_{l \sigma} \rangle
\]

\[
= \sum_{j \notin J(r,k)} \langle e_{ij}, e_l \rangle + \sum_{\sigma \in C_T, j \notin J(r,k)} \sum_{\sigma \neq \text{id}} \text{sgn}(\sigma)\langle e_{ij}, e_{l \sigma} \rangle.
\]

Now $\langle e_{ij}, e_l \rangle = \begin{cases} 1 & \text{if } i^j = l \text{ i.e. } j = (r - k + 1, r - k + 2, \ldots, r), \\ 0 & \text{otherwise}. \end{cases}$

Hence

\[
\sum_{j \notin J(r,k)} \langle e_{ij}, e_l \rangle = 1.
\]

Now for $\sigma \neq \text{id} \in C_T$, $\langle e_{ij}, e_{l \sigma} \rangle = \begin{cases} 1 & \text{if } i^j = l \sigma, \\ 0 & \text{otherwise}. \end{cases}$

Now $l \sigma = (1,1,\ldots,1,2,2,\ldots,2)\sigma$ is the multi-index which has ones in positions $\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(r - k)$, and twos in positions $\sigma^{-1}(r - k + 1), \sigma^{-1}(r - k + 2), \ldots, \sigma^{-1}(r)$. Hence $i^j = l \sigma$ if and only if $j$ contains $\sigma^{-1}(r - k + 1), \sigma^{-1}(r - k + 2), \ldots, \sigma^{-1}(r)$, which means $j = (r - k + 1, r - k + 2, \ldots, r) \cdot \sigma$. But for $\sigma \in C_T$, $\sigma \neq \text{id}$, at least one of $\sigma^{-1}(r - k + 1), \sigma^{-1}(r - k + 2), \ldots, \sigma^{-1}(r)$ will be between 1 and $k$. Hence if $j \subseteq b$, then $j$ does not contain 1, 2, $\ldots, k$ and so $\langle e_{ij}, e_{l \sigma} \rangle = 0$. Therefore we have that

\[
\sum_{\sigma \in C_T, j \notin J(r,k)} \sum_{\sigma \neq \text{id}} \text{sgn}(\sigma)\langle e_{ij}, e_{l \sigma} \rangle = 0.
\]

Hence,

\[
\sum_{j \notin J(r,k)} \langle e_{ij}, e_l \rangle + \sum_{\sigma \in C_T, j \notin J(r,k)} \sum_{\sigma \neq \text{id}} \text{sgn}(\sigma)\langle e_{ij}, e_{l \sigma} \rangle = 1 + 0 = 1,
\]

which is a contradiction. Therefore we do not have that $Y \subseteq (S^\lambda)^\perp$, so by the Submodule Theorem (1.5.3) we have that $Y \supseteq S^\lambda$. \qed
Remark 4.1.7 If \( k = 0 \) then \( a \in J(r, k) \) implies that \( a = () \). Then we have (from Definition 4.1.2) that \( v_a = \sum_{j \in J(r,0)} e_{ij} \). Now if \( j \in J(r,0) \) then \( j = () \) so by definition \( i^j \) has twos in no positions, hence \( i^j = (1,1,\ldots,1) \in I(2,r) \). Therefore \( v_a = e_{(1,1,\ldots,1)} \), and so Theorem 4.1.4 gives that \( Y^{(r,0)} = \text{span}\{e_{(1,1,\ldots,1)}\} \), which we know to be correct, as we know that \( Y^{(r,0)} = M^{(r,0)} \) is one-dimensional with basis \( \{e_{(1,1,\ldots,1)}\} \).

We will prove Theorem 4.1.4 in the following two propositions, using Theorem 2.4.9 which tells us that \( Y^{(r-k,k)} = \text{Im} \, e_{m,0} \) where

\[
e_{m,0} = \prod_{m_u = 1} (1 - b(2^u)).
\]

Recall from Section 2.4 that \( b(u) = 1, \lambda e^{(u)} f^{(\lambda)} 1_\lambda \), where \( \lambda = (r-k,k) \). We want to show that \( Y = \text{Im} \, e_{m,0} \), and so prove Theorem 4.1.4. First we will show that \( e_{m,0}(v_a) = v_a \) for all \( a \in J(r,k) \), which shows that \( Y \subseteq \text{Im} \, e_{m,0} \). We then need to show that \( Y \supseteq \text{Im} \, e_{m,0} \) by showing that for all \( j \in J(r,k) \), \( e_{m,0}(e_{ij}) \in Y \), so \( e_{m,0}(e_{ij}) = \sum_{a \in J(r,k)} \lambda_a v_a \), for some \( \lambda_a \in K \).

Proposition 4.1.8 For all \( a \in J(r,k) \), we have \( e_{m,0}(v_a) = v_a \). In particular this says that \( Y \subseteq \text{Im} \, e_{m,0} \).

Proof. To prove this we just need to show that \( e_{m,0}(v_{(1,2,\ldots,k)}) = v_{(1,2,\ldots,k)} \), because then for \( a \in J(r,k) \), \( a = (1,\ldots,k) \cdot \sigma \) for some \( \sigma \in \Sigma_r \), therefore \( e_{m,0}(v_a) = e_{m,0}(v_{(1,\ldots,k)} \cdot \sigma) = e_{m,0}(v_{(1,\ldots,k)}) \cdot \sigma = v_{(1,\ldots,k)} \cdot \sigma = v_a \).

To prove that \( e_{m,0}(v_{(1,2,\ldots,k)}) = v_{(1,2,\ldots,k)} \), we will show that \( e^{(x)}(v_{(1,2,\ldots,k)}) = 0 \) for all \( u \) with \( m_u = 1 \) (recall that \( m = r - 2k \)). Using Proposition 3.1.5, for any natural number \( x \) we have

\[
e^{(x)}(v_{(1,2,\ldots,k)}) = e^{(x)} \left( \sum_{j \in J(r,k)} e_{ij} \right)
= \sum_{j \in J(r,k)} e^{(x)}(e_{ij})
= \sum_{j \in J(r,k)} \left( \sum_{l \in J(r,k-2)} e_{il} \right). \quad (4.1)
\]
Now, for example, \( e_{(k+1,k+2,...,2k-x)} \) appears in (4.1) for \( l = (k + 1, k + 2, \ldots, 2k - x) \), which appears for all \( j \in J(r, k) \) with \( l \subset j \subset (k + 1, k + 2, \ldots, r) \), which is all \( j = (k + 1, \ldots, 2k - x, j_{k-x+1}, \ldots, j_k) \) with \( (j_{k-x+1}, \ldots, j_k) \subset (2k - x + 1, \ldots, r) \). Hence \( e_{i(k+1,k+2,...,2k-x)} \) appears in (4.1) \( \left( \begin{array}{c} r-2k+x \\ x \end{array} \right) = \left( \begin{array}{c} m+x \\ x \end{array} \right) \) times. Similarly, each \( e_{i} \) with \( l = (l_1, l_2, \ldots, l_{k-x}) \in J(r, k - x) \) and \( l_1 \geq k + 1 \) appears in (4.1) \( \left( \begin{array}{c} m+x \\ x \end{array} \right) \) times. If \( l \) is not of this form then \( e_{i} \) does not occur in (4.1). Now let \( x = 2^u \), and \( m = r - 2k \) with \( m_u = 1 \).

\[
m = m_0 \cdot 2^0 + m_1 \cdot 2^1 + \ldots + m_{u-1} \cdot 2^{u-1} + 1 \cdot 2^u + m_{u+1} \cdot 2^{u+1} + \ldots
\]
\[
x = 2^u
\]
\[
m + x = m_0 \cdot 2^0 + m_1 \cdot 2^1 + \ldots + m_{u-1} \cdot 2^{u-1} + 0.2^u + m_{u+1} \cdot 2^{u+1} + \ldots
\]

for some \( \alpha_{u+1}, \alpha_{u+2}, \ldots \in \{0, 1\} \). Now, by Lemma 1.3.2 we have

\[
\left( \begin{array}{c} m+x \\ x \end{array} \right) \equiv \left( \begin{array}{c} m_0 \\ 0 \end{array} \right) \left( \begin{array}{c} m_1 \\ 0 \end{array} \right) \ldots \left( \begin{array}{c} m_{u-1} \\ 0 \end{array} \right) \left( \begin{array}{c} \alpha_{u+1} \\ 0 \end{array} \right) \ldots \mod 2 \equiv 0 \mod 2.
\]

Hence \( e^{(2^u)}(v_{(1,2,\ldots,k)}) = 0 \). So, for \( m_u = 1 \), \( e^{(2^u)}(v_{(1,2,\ldots,k)}) = 0 \), so we have that \( b(2^u)(v_{(1,2,\ldots,k)}) = 1.A^{(2^u)}(v_{(1,2,\ldots,k)}) = 0 \). Therefore \( e_{m,0}(v_{(1,2,\ldots,k)}) = (\prod_{m_u=1}(1-b(2^u)))(v_{(1,2,\ldots,k)}) = v_{(1,2,\ldots,k)} \), and we have proved Proposition 4.1.8. Therefore \( Y \subseteq \text{Im} e_{m,0} \).

In the rest of this section we will show that \( \text{Im} e_{m,0} \subseteq Y \), by proving the following proposition.

**Proposition 4.1.9** For all \( j \in J(r, k) \)

\[
e_{m,0}(e_{i}) = \sum_{a \in J(r, k) \atop a \subseteq j} v_a.
\]

**Proof.** Let \( j = (j_1, \ldots, j_k) \in J(r, k) \). Now, by definition of \( v_a \) for \( a \in J(r, k) \) we have,

\[
\sum_{a \in J(r, k) \atop a \subseteq j'} v_a = \sum_{a \in J(r, k) \atop a \subseteq j'} \sum_{e_{i} \atop a \subseteq j'} e_{i} = \sum_{b \in J(r, r-k) \atop j \subseteq b} \sum_{l \in J(r, k) \atop l \subseteq b} e_{i},
\]

where \( b = a' \in J(r, r - k) \) and using the first property in Lemma 4.1.1. For \( l = (l_1, \ldots, l_k) \in J(r, k) \) we want to see how many times \( e_{i} \) appears in (4.2). We see...
that $e_i$ appears in (4.2) for all $b \in J(r, r - k)$ with $j \subseteq b$ and $l \subseteq b$. Let $b = (b_1, \ldots, b_{r-k}) \in J(r, r - k)$ with $j = (j_1, \ldots, j_k) \subseteq b$ and $l = (l_1, \ldots, l_k) \subseteq b$, then \{j_1, \ldots, j_k\} \cup \{l_1, \ldots, l_k\} \subseteq \{b_1, \ldots, b_{r-k}\}$. Let $x := \text{Card}((j_1, \ldots, j_k) \cup \{l_1, \ldots, l_k\}) = 2k - \psi(j, l)$. Then for $b \in J(r, k)$ with $j \subseteq b$ and $l \subseteq b$ we have left $r - x$ numbers for the $(r - k) - x$ positions in $b$. So the number of times $e_i$ appears is

$$\binom{r - x}{k} = \binom{r - 2k + \psi(j, l)}{k} = \binom{m + \psi(j, l)}{k},$$

as $m = r - 2k$, and using Lemma 1.3.1(1). So we have that

$$\sum_{b \in J(r, r - k)} \sum_{j \subseteq b} e_i = \sum_{l \in J(r, k)} \binom{m + \psi(j, l)}{k} e_i.$$

Therefore,

$$\sum_{a \in J(r, k)} \sum_{l \in J(r, k)} \binom{m + \psi(j, l)}{k} e_i,$$

and the result follows from Corollary 3.2.6

So we have shown that $\text{Im } e_{m,0} \subseteq Y$ and $Y \subseteq \text{Im } e_{m,0}$. Therefore $Y = \text{Im } e_{m,0}$ and so $Y^{(r - k,k)} = Y = \text{span}\{v_a : a \in J(r, k)\}$ and we have proved Theorem 4.1.4.

### 4.2 Constructing the Young Module $Y^{(r-s,s)}$ in $M^{(r-k,k)}$

Recall that $r$ is a natural number and $k$ and $s$ are non-negative integers with $(r - 2s) \not\equiv 0 \mod 2$, so that $Y^{(r-s,s)}$ is a direct summand in $M^{(r-k,k)}$. For $a \in J(r, s)$ we define

$$w_a := \sum_{j \in J(r, k)} \binom{k - \psi(a, j)}{s} e_{ij} \in M^{(r-k,k)}.$$  \hspace{1cm} (4.3)

We then have the following Theorem, which will be proved later in this section.

**Theorem 4.2.1** For $a \in J(r, s)$, the vector $w_a \in M^{(r-k,k)}$ is as defined by (4.3), then we have that $Y^{(r-s,s)} = \text{span}\{w_a : a \in J(r, s)\}$ in $M^{(r-k,k)}$.
Remark 4.2.2 In particular if \( k = s \) then for \( a \in J(r, k) \) we have

\[
w_a = \sum_{j \in J(r, k)} \left( \frac{k - \psi(a, j)}{k} \right) e_{ij}
\]

and for \( j \in J(r, k) \) we have

\[
\left( \frac{k - \psi(a, j)}{k} \right) = \begin{cases} 1 & \text{if } \psi(a, j) = 0 \\ 0 & \text{otherwise} \end{cases}
\]

Now for \( j \in J(r, k) \), \( \psi(a, j) = 0 \) if and only if \( j \subseteq a' \), where for \( a \in J(r, s) \) the element \( a' \in J(r, r - s) \) is as defined as the start of Section 4.1. Therefore,

\[
w_a = \sum_{j \in J(r, k) \atop j \subseteq a'} e_{ij} = v_a
\]

by Definition 4.1.2. So, by Theorem 4.1.4 we know that Theorem 4.2.1 is true in this case.

Example 4.2.3 Let \( r = 5, k = 2 \), then by Theorem 1.7.4 we have that \( M^{(3,2)} = Y^{(3,2)} \oplus Y^{(4,1)} \), and we will now give an explicit construction of these Young modules. First, to construct \( Y^{(4,1)} \) let \( s = 1 \) and we will use Theorem 4.2.1. We have that \((1) \in J(5,1)\) and,

\[
w_{(1)} = \sum_{j \in J(5,2)} \left( 2 - \psi((1), j) \right) e_{ij}. 
\]

Now \( \psi((1), j) \in \{0, 1\} \) and

\[
\left( 2 - \psi((1), j) \right) = \begin{cases} \binom{2}{2} = 0 & \text{if } \psi((1), j) = 0, \\ \binom{2}{1} = 1 & \text{if } \psi((1), j) = 1. \end{cases}
\]

So

\[
w_{(1)} = e_{(1,2)} + e_{(1,3)} + e_{(1,4)} + e_{(1,5)}
\]

\[
= e_{(2,2,1,1,1)} + e_{(2,1,2,1,1)} + e_{(2,1,1,2,1)} + e_{(2,1,1,1,2)}.
\]

Similarly, we have that

\[
w_{(2)} = e_{(2,2,1,1,1)} + e_{(1,2,2,1,1)} + e_{(1,2,1,2,1)} + e_{(1,2,1,1,2)},
\]

\[
w_{(3)} = e_{(2,1,2,1,1)} + e_{(1,2,2,1,1)} + e_{(1,1,2,2,1)} + e_{(1,1,2,1,2)},
\]

\[
w_{(4)} = e_{(2,1,1,2,1)} + e_{(1,2,1,2,1)} + e_{(1,1,2,2,1)} + e_{(1,1,1,2,2)},
\]

\[
w_{(5)} = e_{(2,1,1,1,2)} + e_{(1,2,1,1,2)} + e_{(1,1,2,1,2)} + e_{(1,1,1,2,2)}.
\]
By Theorem 4.2.1 we have that $Y^{(4,1)} = \text{span}\{w_{(1)}, w_{(2)}, w_{(3)}, w_{(4)}, w_{(5)}\}$.

Now, to construct the module $Y^{(3,2)}$ in $M^{(3,2)}$ we will use Theorem 4.1.4 which is a specific case of Theorem 4.2.1. Using Theorem 4.1.4 we have that $Y^{(3,2)} = \text{span}\{v_a : a \in J(5,2)\}$, where by Definition 4.1.2,

\begin{align*}
v_{(1,2)} &= e_{i(3,4)} + e_{i(3,5)} + e_{i(4,5)} = e_{(1,1,2,2,1)} + e_{(1,1,2,1,2)} + e_{(1,1,1,2,2)}, \\
v_{(1,3)} &= e_{i(2,4)} + e_{i(2,5)} + e_{i(4,5)} = e_{(1,2,1,2,1)} + e_{(1,2,1,1,2)} + e_{(1,1,1,2,2)}, \\
v_{(1,4)} &= e_{i(2,3)} + e_{i(2,5)} + e_{i(3,5)} = e_{(1,2,2,1,1)} + e_{(1,2,1,2,1)} + e_{(1,1,2,1,2)}, \\
v_{(1,5)} &= e_{i(2,3)} + e_{i(2,4)} + e_{i(3,4)} = e_{(2,1,2,1,1)} + e_{(1,2,1,2,1)} + e_{(1,1,2,1,2)}, \\
v_{(2,3)} &= e_{i(1,4)} + e_{i(1,5)} + e_{i(4,5)} = e_{(2,1,1,2,1)} + e_{(2,1,1,1,2)} + e_{(1,1,1,2,2)}, \\
v_{(2,4)} &= e_{i(1,3)} + e_{i(1,5)} + e_{i(3,5)} = e_{(2,1,2,1,1)} + e_{(2,1,1,1,2)} + e_{(1,1,2,1,2)}, \\
v_{(2,5)} &= e_{i(1,3)} + e_{i(1,4)} + e_{i(3,4)} = e_{(2,1,2,1,1)} + e_{(2,1,1,2,1)} + e_{(1,1,2,1,2)}, \\
v_{(3,4)} &= e_{i(1,2)} + e_{i(1,5)} + e_{i(2,5)} = e_{(2,2,1,1,1)} + e_{(2,1,1,1,2)} + e_{(1,2,1,1,2)}, \\
v_{(3,5)} &= e_{i(1,2)} + e_{i(1,4)} + e_{i(2,4)} = e_{(2,2,1,1,1)} + e_{(2,1,1,2,1)} + e_{(1,2,1,1,2)}, \\
v_{(4,5)} &= e_{i(1,2)} + e_{i(1,3)} + e_{i(2,3)} = e_{(2,2,1,1,1)} + e_{(2,1,2,1,1)} + e_{(1,2,1,1,1)}.
\end{align*}

So we have decomposed the module $M^{(5,2)}$ into a direct sum of Young modules, and given an explicit construction for the Young modules that appear in the decomposition.

Note that, as the characteristic of the field $K$ is two, we see that $w_{(5)} = w_{(1)} + w_{(2)} + w_{(3)} + w_{(4)}$, therefore $Y^{(4,1)} = \text{span}\{w_{(1)}, w_{(2)}, w_{(3)}, w_{(4)}, w_{(5)}\} = \text{span}\{w_{(1)}, w_{(2)}, w_{(3)}, w_{(4)}\}$, and in fact this set is now a basis for $Y^{(4,1)}$.

Also, as the characteristic of $K$ is two, we see that

\begin{align*}
v_{(1,5)} &= v_{(1,2)} + v_{(1,3)} + v_{(1,4)}, \\
v_{(2,5)} &= v_{(1,2)} + v_{(2,3)} + v_{(2,4)}, \\
v_{(3,5)} &= v_{(1,3)} + v_{(2,3)} + v_{(3,4)}, \\
v_{(4,5)} &= v_{(1,4)} + v_{(2,4)} + v_{(3,4)}.
\end{align*}
Therefore \(Y^{(3,2)} = \text{span}\{v_a : a \in J(5,2)\} = \text{span}\{v_{(1,2)}, v_{(1,3)}, v_{(1,4)}, v_{(2,3)}, v_{(2,4)}, v_{(3,4)}\}\), and this set can be shown to be a basis for \(Y^{(3,2)}\).

The problem of constructing a basis for Young modules is considered more in Chapter 5.

By Theorem 2.4.9 we know that \(Y^{(r-s,s)} = \text{Im} e_{m,\gamma}\), where \(m = r - 2k\) and \(\gamma = k - s\).

So to prove Theorem 4.2.1 we will show that \(\text{Im} e_{m,\gamma} \subseteq \text{span}\{w_a : a \in J(r,s)\}\) (see Proposition 4.2.5) and \(\text{Im} e_{m,\gamma} \supseteq \text{span}\{w_a : a \in J(r,s)\}\) (see Proposition 4.2.9). We will do this using the action of \(e_{m,\gamma}\) on \(E^{\otimes r}\) as calculated in Section 3.2, but first we give some other lemmas which will be needed.

**Lemma 4.2.4** For all \(z \in \{0,1,\ldots,k\}\),

\[
\sum_{x=0}^{s} \binom{k-z}{x} \binom{r-2k+z}{s-x} \binom{k-x}{s} \equiv \binom{r-k-s+z}{s} \quad \text{mod } 2.
\]

**Proof.** Let \(z \in \{0,1,\ldots,k\}\) and recall that \(k \geq s\). Let

\[
A := \sum_{x=0}^{s} \binom{k-z}{x} \binom{r-2k+z}{s-x} \binom{k-x}{s} = \sum_{x=0}^{k} \binom{k-z}{x} \binom{r-2k+z}{s-x} \binom{k-x}{s},
\]

as if \(x > s\) then \(s-x < 0\) and therefore \(\binom{r-2k+z}{s-x} = 0\). For \(x \in \mathbb{N}_0\), by Lemma 1.3.1 (4) we have that \(\binom{r-2k+z}{s-x} \equiv \sum_{y} \binom{z}{y} \binom{r-2k+z+y}{s} \mod 2\). Therefore,

\[
A \equiv \sum_{x} \sum_{y} \binom{k-z}{x} \binom{r-2k+z+y}{s} \binom{k-x}{s} \mod 2
\]

\[
\equiv \sum_{y} \binom{k-z}{y} \binom{r-2k+z+y}{s} \sum_{x=0}^{k} \binom{k-x}{s} \binom{k-z-y}{x-y} \mod 2
\]

\[
\equiv \sum_{y} \binom{k-z}{y} \binom{r-2k+z+y}{s} \binom{z}{k-s-y} \mod 2
\]

\[
\equiv \sum_{w} \binom{k-z}{s-w} \binom{r-k-s+z-w}{s} \binom{z}{w} \mod 2,
\]

where \(w = k - s - y\). Above we have also used Lemma 1.3.1 (2) which tells us that \(\binom{k-z}{y} = \binom{k-z-y}{x-y}\) then Lemma 1.3.1 (5) which gives \(\sum_{x=0}^{k} \binom{k-z}{s} \binom{k-z-y}{z-y} \equiv \binom{z}{k-s-y} \mod 2\). Now, using Lemma 1.3.1 (4) and (2), for \(w \in \mathbb{N}_0\) we have
\[
\binom{k-z}{(k-s)-w} \equiv \sum_y \binom{w}{y} \binom{k-z+y}{k-s} \text{ modulo two, and } \binom{z}{w} \binom{w-y}{w-y} = \binom{z-y}{w-y}. \text{ So, also using Lemma 1.3.1 (5) and (1), which gives } \sum_{w=0}^{r-k-s+z} \binom{r-k-s+z-w}{s} \binom{z-y}{w-y} \equiv \binom{r-k-s}{r-k-2s+z-y} = \binom{r-k-s}{(s+z+y)} \text{ modulo two, we have that,}
\]

\[
A \equiv \sum_w \sum_y \binom{w}{y} \binom{k-z+y}{k-s} \binom{r-k-s+z-w}{s} \binom{z}{w} \mod 2
\]

\[
\equiv \sum_y \binom{k-z+y}{k-s} \binom{z}{y} \sum_{w=0}^{r-k-s+z} \binom{r-k-s+z-w}{s} \binom{z-y}{w-y} \mod 2
\]

\[
\equiv \sum_y \binom{k-z+y}{k-s} \binom{z}{y} \binom{r-k-s}{s-z+y} \mod 2
\]

\[
\equiv \sum_z \binom{k-x}{k-s} \binom{z}{s-x} \binom{r-k-s}{s-x} \mod 2,
\]

where \( x = z - y \). Now for all \( x \in \{0, \ldots, s\} \) we will show that \( \binom{k-z}{k-s} \equiv \binom{r-k-s}{s-z} \mod 2 \). This is clear if \( \binom{k-z}{k-s} \equiv 1 \mod 2 \). If \( \binom{k-z}{k-s} \equiv 0 \mod 2 \), by Lemma 1.3.1 (2) we have

\[
0 \equiv \binom{r-2s}{k-s} = \binom{r-2s}{k-s} \binom{(r-2s)-(k-s)}{(k-x)-(k-s)} \equiv 1 \cdot \binom{r-k-s}{s-x} \mod 2,
\]

as \( \binom{r-2s}{k-s} \equiv 1 \mod 2 \). Now, using Lemma 1.3.1 (1) and (3),

\[
A \equiv \sum_z \binom{z}{s-x} \binom{r-k-s}{s-x} = \sum_z \binom{z}{x} \binom{r-k-s}{s-x} \equiv \binom{r-k-s+z}{s} \mod 2,
\]

as required. \( \square \)

Recall that \( e_{m,\gamma} : M^{(r-k,k)} \to M^{(r-k,k)} \) is a primitive idempotent of \( \text{End}_{K^r}(M^{(r-k,k)}) \), and that for \( a \in J(r,k) \) the vector \( w_a \) is as defined in Equation (4.3).

**Proposition 4.2.5** For all \( j \in J(r,k) \),

\[
e_{m,\gamma}(e_{ij}) = \sum_{a \in J(r,s) \atop a \subseteq j} w_a,
\]

in particular \( \text{Im } e_{m,\gamma} \subseteq \text{span} \{w_a : a \in J(r,s)\} \).
Proof. Let \( j \in J(r, k) \), then by the definition given in (4.3)

\[
\sum_{a \in J(r, s)} w_a = \sum_{a \in J(r, s)} \sum_{l \in J(r, k)} \left( k - \psi(a, l) \right) e_{i_l}^s
\]

\[
= \sum_{l \in J(r, k)} \left( \sum_{a \in J(r, s)} \left( k - \psi(a, l) \right) \right) e_{i_l}^s
\]

\[
= \sum_{l \in J(r, k)} B e_{i_l}^s
\]

(4.4)

where, for \( l \in J(r, k) \), \( B \) is the coefficient of the vector \( e_{i_l}^s \). Now, for \( j = (j_1, \ldots, j_k) \) and \( l = (l_1, \ldots, l_k) \in J(r, k) \) we have

\[
B = \sum_{x=0}^{s} \text{Card}\{a \in J(r, s) : a \subseteq j', \psi(a, l) = x\} \left( \begin{array}{c} k - x \\ s \end{array} \right).
\]

Let \( j' = (j'_1, \ldots, j'_{r-k}) \in J(r, r - k) \) and let \( z = \psi(j, l) \), so \( \psi(j', l) = k - z \) (by definition of \( \psi \) and \( j' \)). Now, let \( \{j'_1, \ldots, j'_{r-k}\} \cap \{l_1, \ldots, l_k\} = \{\alpha_1, \ldots, \alpha_{k-z}\} \). For \( x \in \{0, 1, \ldots, s\} \) we need to calculate \( \text{Card}\{a \in J(r, s) : a \subseteq j', \psi(a, l) = x\} \), which is the number of \( a = (a_1, \ldots, a_s) \in J(r, s) \) with \( x \) values in \( \{\alpha_1, \ldots, \alpha_{k-z}\} \) and the other \( s - x \) values in \( \{j'_1, \ldots, j'_{r-k}\}\backslash\{\alpha_1, \ldots, \alpha_{k-z}\} \). This cardinality is \( \binom{k-z}{x} \binom{r-2k+z}{s-x} \).

So,

\[
B = \sum_{x=0}^{s} \binom{k - \psi(j, l)}{x} \binom{r - 2k + \psi(j, l)}{s - x} \binom{k - x}{s}
\]

\[
\equiv \binom{r - k - s + \psi(j, l)}{s} \mod 2,
\]

by Lemma 4.2.4. So, by (4.4) we have,

\[
\sum_{a \in J(r, s)} w_a = \sum_{l \in J(r, k)} \left( r - k - s + \psi(j, l) \right) e_{i_l}^s = e_{m, \gamma}(e_{i_l}^s),
\]

by Proposition 3.2.9.

Now we need to prove that \( \text{Im} e_{m, \gamma} \supseteq \text{span}\{w_a : a \in J(r, s)\} \), which we will do by proving that for all \( a \in J(r, s) \), \( e_{m, \gamma}(w_a) = w_a \) (see Proposition 4.2.9). We first need some other results.
Lemma 4.2.6 For non-negative integers \( z, w \) and \( a \) with \( z \leq w \leq s \) and \( a \leq k - w \) we have

\[
\binom{r - k - s}{k - s - z - a} \sum_y \binom{k - w - a}{y} \binom{r - s + z - w - y}{s - y} \binom{r - s - y}{s} \\
\equiv \binom{k - w - a}{s} \binom{r - k - s}{k - s - z - a} \pmod{2}.
\]

Proof. The result clearly holds if \( \binom{r - k - s}{k - s - z - a} \equiv 0 \pmod{2} \), so now let \( \binom{r - k - s}{k - s - z - a} \equiv 1 \pmod{2} \). Note that, by Lemma 1.3.1 (1), \( \binom{r - k - s}{k - s - z - a} \equiv 1 \pmod{2} \). We define

\[
A := \sum_y \binom{k - w - a}{y} \binom{r - s + z - w - y}{s - y} \binom{r - s - y}{s},
\]

and will show that \( A \equiv \binom{k - w - a}{s} \pmod{2} \). Using Lemma 1.3.1 (1) and (4) we have that

\[
\binom{k - w - a}{y} = \binom{k - w - a}{k - w - a - y} \\
= \binom{k - w - a}{(r - s + z - w - y) - (r - s + z - k + a)} \\
\equiv \sum_x \binom{r - s + z - k + a}{x} \binom{k - w - a + x}{r - s + z - w - y} \pmod{2}.
\]

Therefore, modulo two we have,

\[
A \equiv \sum_y \sum_x \binom{r - s + z - k + a}{x} \binom{k - w - a + x}{r - s + z - w - y} \binom{r + z - s - w - y}{r - 2s + z - w} \binom{r - s - y}{s} \\
= \sum_x \binom{r - s + z - k + a}{x} \binom{k - w - a + x}{r - 2s + z - w} \sum_y \binom{k - a + x - r + 2s - z}{s - y} \binom{r + z - s - w - y}{r - 2s + z - w} \binom{r - s - y}{s}
\]

as, by Lemma 1.3.1 (2), \( \binom{r - k - s}{r + z - s - w - y} \binom{r + z - s - w - y}{r - 2s + z - w} = \binom{k - w - a + x}{r + z - s - w - y} \binom{k - a + x - r + 2s - z}{s - y} \binom{r + z - s - w - y}{r - 2s + z - w} \binom{r - s - y}{s} \). Now, by taking \( \alpha = s - y \) and using Lemma 1.3.1 (4) we have,

\[
\sum_y \binom{k - a + x - r + 2s - z}{s - y} \binom{r - s - y}{s} \\
= \sum_x \binom{k - a - r + 2s - z + x}{\alpha} \binom{r - 2s + \alpha}{s} \\
\equiv \binom{r - 2s}{s - (k - a - r + 2s - z + x)} \pmod{2}.
\]

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Hence, using Lemma 1.3.1 (1),

\[
A = \sum_x \left( \frac{r - k - s + z + a}{r - k - s + z + a - x} \right) \left( \frac{k - w - a + x}{r - 2s + z - w} \right) \left( \frac{r - 2s}{r - k - s + a + z - x} \right)
\]

where \(y = r - k - s + z + a - x\).

Now for \(0 \leq y \leq r - 2s\) we are going to show that \(\binom{r - 2s}{y} \equiv \binom{r - 2s}{y} \mod 2\). This is clear if \(\binom{r - 2s}{y} \equiv 1 \mod 2\), so now let \(\binom{r - 2s}{y} \equiv 0 \mod 2\). We have,

\[
\binom{r - 2s}{y} \binom{r - 2s - y}{k - s - z - a} \binom{r - k - s + z + a}{k - s - y} \binom{r - k - s + z + a - y}{k - s} \binom{r - 2s}{y} \binom{r - k - s + z + a}{k - s} = \binom{r - 2s}{y} \binom{r - 2s - y}{k - s - z - a} \binom{r - k - s + z + a}{k - s - y} \binom{r - k - s + z + a - y}{k - s},
\]

where (4.5) and (4.7) follow from Lemma 1.3.1 (2), and (4.6) follows from Lemma 1.3.1 (1). Now by Lemma 1.3.1 (1), \(\binom{r - 2s}{y} = \binom{r - 2s}{y} \mod 2\), therefore we have that \(\binom{r - 2s}{y} \binom{r - k - s}{y} \binom{r - k - s + z + a}{k - s} \equiv 0 \mod 2\). Now, using the assumption at the beginning of this proof, and the assumption throughout this section we have \(\binom{r - 2s}{y} \equiv 1 \mod 2\) and \(\binom{r - 2s}{y} \equiv 1 \mod 2\). Therefore we must have that \(\binom{r - 2s}{y} \equiv 0 \mod 2\).

So, for \(\binom{r - 2s}{y} \equiv 1 \mod 2\) we have \(\binom{r - 2s}{y} \binom{r - k - s + z + a}{y} \equiv \binom{r - k - s + z + a}{y} \mod 2\). Therefore, using Lemma 1.3.1 (5),

\[
A = \sum_{y=0}^{r - s - z - w} \left( \frac{r - k - s + z + a}{y} \right) \left( \frac{r - s - z - w - y}{r - 2s + z - w} \right) \equiv \left( \frac{k - w - a}{s} \right) \mod 2,
\]
as required. \(\square\)

**Lemma 4.2.7** For \(0 \leq z \leq w \leq s\) we have,

\[
\sum_y \left( \frac{k - w}{y} \right) \left( \frac{r - s + z - w - y}{s - y} \right) \left( \frac{r - s - y}{s} \right) \left( \frac{r - s - w - y}{k - s - z} \right) \equiv \left( \frac{k - w}{s} \right) \left( \frac{r - 2s - w}{k - s - z} \right) \mod 2.
\]
Proof. Let

\[ A := \sum_y \binom{k-w}{y} \left( \binom{r-s+z-w-y}{s-y} \binom{r-s-y}{s} \binom{r-s-w-y}{k-s-z} \right). \]

Firstly, by Lemma 1.3.1 (3) we have, \( \binom{k-w-y}{k-s-z-a} \) so that

\[ A = \sum_y \binom{k-w}{y} \left( \binom{r-s+z-w-y}{s-y} \sum_a \binom{k-w-y}{a} \binom{r-s-k}{k-s-z-a} \right) = \sum_a \binom{k-w-a}{a} \sum_y \binom{k-w-a-y}{y} \left( \binom{r-s+z-w-y}{s-y} \binom{r-s-y}{s} \right) \]

as, by Lemma 1.3.1 (1) and (2), we have \( \binom{k-w}{y} \binom{k-w-y}{a} = \binom{k-w}{y} \binom{k-w-y-a}{a} = \binom{k-w-y}{y} \). Hence, by Lemma 4.2.6,

\[ A \equiv \sum_a \binom{k-w}{a} \binom{k-w-a}{s} \binom{r-k-s}{k-s-z-a} \mod 2 \]

\[ \equiv \binom{k-w}{s} \sum_a \binom{k-w-s}{a} \binom{r-k-s}{k-s-z-a} \mod 2 \]

\[ \equiv \binom{k-w}{s} \left( \binom{r-2s-w}{k-s-z} \right) \mod 2 \]

where \( \binom{k-w}{s} \binom{k-w-a}{a} = \binom{k-w-a}{a} \), \( \binom{k-w-s}{a} \binom{r-k-s}{k-s-z-a} = \binom{k-w-s}{a} \) using Lemma 1.3.1 (1) and (2), and \( \sum_a \binom{k-w-s}{a} \binom{r-k-s}{k-s-z-a} = \binom{r-2s-w}{k-s-z} \) by Lemma 1.3.1 (3). So \( A \equiv \binom{k-w}{s} \binom{r-2s-w}{k-s-z} \mod 2 \), as required.

\[ \square \]

Lemma 4.2.8 For \( 0 \leq z \leq w \leq s \) we have,

\[ \sum_{x,y,z} \binom{w}{x} \binom{s-w}{y} \binom{k-w}{z} \binom{r-k-s+w}{y} \binom{k-x}{s} \binom{r-k-s+y}{s} = \binom{k-w}{s} \]

modulo two.

Proof. Let

\[ A := \sum_{x,y,z} \binom{w}{z} \binom{s-w}{x-z} \binom{k-w}{y-z} \binom{r-k-s+w}{k-y-x+z} \binom{k-x}{s} \binom{r-k-s+y}{s}. \]

For \( x \in \{0,1,\ldots,s\} \) and \( z \in \{0,1,\ldots,w\} \), we will now calculate

\[ B := \sum_y \binom{k-w}{y-z} \binom{r-k-s+w}{k-y-x+z} \binom{r-k-s+y}{s}. \]

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By Lemma 1.3.1 (4) we have
\[
\left( \frac{r - k - s + w}{k - x} - (y - z) \right) \equiv \sum_a \left( \frac{y - z}{a} \right) \left( \frac{r - k - s + w + a}{k - x} \right) \mod 2,
\]
so
\[
B \equiv \sum_y \sum_a \left( \frac{k - w}{y - z} \right) \left( \frac{r - k - s + w + a}{a} \right) \left( \frac{r - k - s + w + a}{k - x} \right) \left( \frac{r - k - s + y}{s} \right)
\equiv \sum_a \left( \frac{k - w}{a} \right) \left( \frac{r - k - s + w + a}{k - x} \right) \sum_y \left( \frac{y - z - a}{a} \right) \left( \frac{r - k - s + y}{s} \right)
\equiv \sum_a \left( \frac{k - w}{a} \right) \left( \frac{r - k - s + w + a}{k - x} \right) \sum_b \left( \frac{k - w - a}{b} \right) \left( \frac{r - k - s + a + z + b}{s} \right)
\equiv \sum_a \left( \frac{k - w}{a} \right) \left( \frac{r - k - s + w + a}{k - x} \right) \left( \frac{r - k - s + a + z}{s} \right) \left( \frac{s - (k - w - a)}{a} \right)
\equiv \sum_y \left( \frac{k - w}{y} \right) \left( \frac{r - s - y}{k - x} \right) \left( \frac{r - s + z - w - y}{s - y} \right) \mod 2.
\]
Above we have used Lemma 1.3.1(2) which gives that \(\binom{k-w}{y-z} = \binom{k-w}{a} \binom{k-w-a}{y-z-a}\), then taken \(b = y - z - a\) and used Lemma 1.3.1(4) to get \(\sum_b \binom{k-w-a}{r-k-s+a+s+b} \equiv \binom{r-k-s+a+z}{s-(k-w-a)} \mod 2\). We also used Lemma 1.3.1 (1) to give \(\binom{k-w}{a} \equiv \binom{k-w}{k-w-a} \mod 2\), and finally taken \(y = k - w - a\). So we have,
\[
A \equiv \sum_{x,y,z} \left( \frac{w}{z} \right) \left( \frac{s - w}{x - z} \right) \left( \frac{k - x}{s} \right) \left( \frac{k - w}{y} \right) \left( \frac{r - s - y}{k - x} \right) \left( \frac{r - s + z - w - y}{s - y} \right)
\equiv \sum_z \left( \frac{w}{z} \right) \sum_y \left( \frac{k - w}{y} \right) \left( \frac{r - s + z - w - y}{s - y} \right) \sum_x \left( \frac{s - w}{x - z} \right) \left( \frac{k - x}{s} \right)
\equiv \sum_z \left( \frac{w}{z} \right) \sum_y \left( \frac{k - w}{y} \right) \left( \frac{r - s + z - w - y}{s - y} \right) \left( \frac{r - s - y}{s} \right) \left( \frac{r - s - w - y}{k - s - z} \right) \mod 2,
\]

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as, by Lemma 1.3.1 (2) and (3) we have \((r^{s-y})^{(k-z)} = (r^{s-y})^{(r-2s-y)}\) and \(\sum_{z=0}^s (s-w) (r^{2s-y})^{(k-s-z)} = \sum_{\alpha} (s-w) (r^{2s-y})^{(k-s-z)} = (r^{s-w-y})^{(k-s-z)}\). Now, by Lemma 4.2.7,

\[
A \equiv \sum_{z=0}^w (w) (k-w) (s) (r-2s-w) (k-s-z) \\
= (k-w) \sum_{z=0}^w (w) (r-2s-w) (k-s-z) \\
= (k-w) (r-2s) (k-s) \\
= (k-w) \mod 2,
\]

using Lemma 1.3.1 (3) and the fact that \((r^{2s-y})^{(k-s)} \equiv 1 \mod 2\). \(\square\)

**Proposition 4.2.9** For all \(a \in J(r, s)\), \(e_{m,\gamma}(w_a) = w_a\). In particular

\[\Im e_{m,\gamma} \supseteq \text{span}\{w_a : a \in J(r, s)\}.\]

**Proof.** Let \(a \in J(r, s)\), then by definition of \(w_a\) and using Proposition 3.2.9 we have,

\[
e_{m,\gamma}(w_a) = e_{m,\gamma}(\sum_{j \in J(r,k)} (k-\psi(a,j)) e_{i,j}) \\
= \sum_{j \in J(r,k)} (k-\psi(a,j)) e_{m,\gamma}(e_{i,j}) \\
= \sum_{j \in J(r,k)} (k-\psi(a,j)) \sum_{l \in J(r,k)} (r-k-s+\psi(j,l)) e_{i,l} \\
= \sum_{l \in J(r,k)} \sum_{j \in J(r,k)} (k-\psi(a,j)) (r-k-s+\psi(j,l)) e_{i,l} \\
= \sum_{l \in J(r,k)} B e_{i,l}
\]

where, for \(l \in J(r,k)\), \(B\) is the coefficient of the vector \(e_{i,l}\). We now want to simplify the expression for \(B\), firstly we have

\[
B = \sum_{x=0}^s \sum_{y=0}^k \text{Card}\{j \in J(r,k) : \psi(a,j) = x, \psi(j,l) = y\} (k-x) (r-k-s+y).
\]

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Now, for $0 \leq x \leq s$ and $0 \leq y \leq k$ we need to calculate

$$\text{Card}\{j \in J(r, k) : \psi(a, j) = x, \psi(j, l) = y\}.$$ 

Let $a = (a_1, \ldots, a_s) \in J(r, s)$, $a' = (a'_1, \ldots, a'_{s-r}) \in J(r, r-s)$, $l = (l_1, \ldots, l_k) \in J(r, k)$ and $l' = (l'_1, \ldots, l'_{r-k}) \in J(r, r-k)$. Let $w = \psi(a, l)$, then $\psi(a, l') = s - w$, and $\psi(a', l) = k - w$. Let

\[
\begin{align*}
\{a_1, \ldots, a_s\} \cap \{l_1, \ldots, l_k\} &= \{b_1, \ldots, b_w\}, \\
\{a_1, \ldots, a_s\} \cap \{l'_1, \ldots, l'_{r-k}\} &= \{c_1, \ldots, c_{s-w}\}, \\
\{a'_1, \ldots, a'_{r-s}\} \cap \{l_1, \ldots, l_k\} &= \{d_1, \ldots, d_{k-w}\}.
\end{align*}
\]

Now to calculate $\text{Card}\{j \in J(r, k) : \psi(a, j) = x, \psi(j, l) = y\}$, let $j \in J(r, k)$ with $\psi(a, j) = x$ and $\psi(j, l) = y$. Therefore $j = (j_1, \ldots, j_k)$ has $x$ values from $\{b_1, \ldots, b_w, c_1, \ldots, c_{s-w}\}$ and $y$ values from $\{b_1, \ldots, b_w, d_1, \ldots, d_{k-w}\}$. Let $z \in \{0, 1, \ldots, w\}$ be such that $j$ has $z$ values from $\{b_1, \ldots, b_w\}$, then $j$ has $x - z$ values from $\{c_1, \ldots, c_{s-w}\}$, $y - z$ values from $\{d_1, \ldots, d_{k-w}\}$ and the other $k - x - y + z$ from $\{1, \ldots, r\}\{b_1, \ldots, b_w, c_1, \ldots, c_{s-w}, d_1, \ldots, d_{k-w}\}$. The number of $j$ of this form is

$$\binom{w}{z} \binom{s-w}{x-z} \binom{k-w}{y-z} \binom{r-s-k+w}{k-x-y+z}.$$ 

Therefore we have,

$$\text{Card}\{j \in J(r, k) : \psi(a, j) = x, \psi(j, l) = y\} = \sum_{z=0}^{w} \binom{w}{z} \binom{s-w}{x-z} \binom{k-w}{y-z} \binom{r-k+w-s}{k-y-x+z},$$

where $w = \psi(a, l)$. So

$$B = \sum_{x,y,z} \binom{\psi(a, l)}{z} \binom{s-\psi(a, l)}{x-z} \binom{k-\psi(a, l)}{y-z} \binom{r-k+\psi(a, l)-s}{k-y-x+z} \binom{k-x}{s} \binom{r-k-s+y}{s} \mod 2,$$

by Lemma 4.2.8. Therefore we have that

$$e_{m,\gamma}(w_a) = \sum_{l \in J(r, k)} \binom{k-\psi(a, l)}{s} e_{il} = w_a,$$

as required. □
Throughout this section we have therefore completed the proof of Theorem 4.2.1, which enables us to construct the Young module $Y^{(r-s,s)}$ whenever it occurs as a direct summand in the permutation module $M_{(r-k,k)}$. 
Chapter 5

On a Basis for Young Modules

In the previous chapter we constructed generators for Young modules over a field of characteristic two. In this chapter we would like to refine these results and investigate whether there is some natural basis, or a natural algorithm providing a basis for Young modules. We cannot answer this question in general at the moment, therefore we have collected a series of examples where the answer can be given. First we will look at some Young modules over a field of characteristic \( p \) where \( p \) is any prime, then specialize to the case where \( p \) is two.

5.1 The Modules \( Y^{(r,0)} \) and \( Y^{(r-1,1)} \)

Let \( r \) be a natural number and let \( p \) be prime. We know that \( Y^{(r,0)} = M^{(r,0)} \) is one-dimensional with basis \( \{e_i\} \) where \( i = (1,1,\ldots,1) \in I(2,r) \). Also \( \Sigma_r \) acts on \( Y^{(r,0)} \) by \( e_i \cdot \sigma = e_i \) for all \( \sigma \in \Sigma_r \), and so \( Y^{(r,0)} \) is the trivial \( K\Sigma_r \)-module. Let \( k \in \mathbb{N} \) with \( 2k \leq r \), so that \((r-k,k)\) is a partition of \( r \). Recall from Section 3.1 that \( M^{(r-k,k)} \) has basis \( \{e_{ij} : j \in J(r,k)\} \) over \( K \), and \( \Sigma_r \)-action given by \( e_{ij} \cdot \sigma = e_{ij+\sigma} \) for all \( j \in J(r,k) \) and all \( \sigma \in \Sigma_r \). We now consider the module \( Y^{(r,0)} \) inside \( M^{(r-k,k)} \).

**Lemma 5.1.1** If \( Y^{(r,0)} \) appears as a direct summand in \( M^{(r-k,k)} \) then \( Y^{(r,0)} \) is a one-dimensional \( K\Sigma_r \)-module with \( K \)-basis

\[
v = \sum_{j \in J(r,k)} e_{ij}
\]
in $M^{(r-k,k)}$, and $\Sigma_r$-action $v \cdot \sigma = v$ for all $\sigma \in \Sigma_r$.

**Proof.** Suppose that $Y^{(r,0)}$ does appear as a direct summand of $M^{(r-k,k)}$. We want to find a basis for $Y^{(r,0)}$ in $M^{(r-k,k)}$.

As $Y^{(r,0)}$ is one dimensional, $Y^{(r,0)}$ in $M^{(r-k,k)}$ will have one basis element,

$$v := \sum_{j \in J(r,k)} \alpha_j e_{ij},$$

for some $\alpha_j \in K$, not all zero. Now, for $\sigma \in \Sigma_r$:

$$v \cdot \sigma = \left( \sum_{j \in J(r,k)} \alpha_j e_{ij} \right) \cdot \sigma = \sum_{j \in J(r,k)} \alpha_j e_{ij \cdot \sigma} = \sum_{l \in J(r,k)} \alpha_l e_{il}.$$

As $Y^{(r,0)}$ is the trivial module, $v \cdot \sigma = v$ for all $\sigma \in \Sigma_r$. Therefore $\alpha_j \cdot \sigma -1 = \alpha_j$ for all $\sigma \in \Sigma_r$, and hence all of the $\alpha_j$’s for $j \in J(r,k)$ must be equal. So as $v \neq 0$, we can take each $\alpha_j$ to be equal to one, and so

$$v = \sum_{j \in J(r,k)} e_{ij}$$

is the basis element for $Y^{(r,0)}$. \hfill \Box

Now we consider the module $M^{(r-1,1)}$, which has $K$-basis $\{e_{ij} : 1 \leq j \leq r\}$, and $\Sigma_r$-action $e_{ij} \cdot \sigma = e_{ij \cdot \sigma}$ for all $1 \leq j \leq r$ and $\sigma \in \Sigma_r$, where $(j) \cdot \sigma = (\sigma^{-1}(j))$. As $k = 1$ here we will just write these basis elements of $M^{(r-1,1)}$ as $e_{ij}$ where $1 \leq j \leq r$.

**Lemma 5.1.2** $M^{(r-1,1)}$ is indecomposable if and only if $r \equiv 0 \mod p$.

**Proof.** By Corollary 2.1.2, $M^{(r-1,1)}$ is indecomposable if and only if there are no non-trivial idempotents in the algebra $\text{End}_{K\Sigma_r}(M^{(r-1,1)})$. Let $\phi \in \text{End}_{K\Sigma_r}(M^{(r-1,1)})$, then we have that $\phi(e_{i1}) = \sum_{j=1}^{r} \lambda_j e_{ij}$ for some $\lambda_1, \lambda_2, \ldots, \lambda_r \in K$, and $\phi(e_{i\sigma^{-1}(1)}) = \phi(e_{i1}) = \phi(e_{i1} \cdot \sigma) = \phi(e_{i1}) \cdot \sigma$ for all $\sigma \in \Sigma_r$. In particular, let $\sigma \in \Sigma_r$ be such that $\sigma^{-1}(1) = 1$, then

$$\sum_{j=1}^{r} \lambda_j e_{ij} = \phi(e_{i1}) = \phi(e_{i1} \cdot \sigma) = \phi(e_{i1}) \cdot \sigma = \sum_{j=1}^{r} \lambda_j e_{ij} \cdot \sigma = \sum_{j=1}^{r} \lambda_j e_{ij} \cdot \sigma = \sum_{l=1}^{r} \lambda_{\sigma(l)} e_{il},$$

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and therefore $\lambda_l = \lambda_{\sigma(l)}$ for all $l$. In particular taking $\sigma = (2 \ 3 \ \ldots \ r)$ gives $\lambda_2 = \lambda_3 = \ldots = \lambda_r$. Therefore $\phi(e_{ii}) = \lambda_1 e_{ii} + \sum_{j=2}^{r} \lambda_2 e_{ij}$, and we get, for all $1 \leq l \leq r$, that

$$\phi(e_{ii}) = \phi(e_{ii} \cdot (1 \ l)) = \phi(e_{ii}) \cdot (1 \ l) = \lambda_1 e_{ii} + \sum_{j \in \mathbb{F} \atop j \neq l} \lambda_2 e_{ij}.$$ 

Now, suppose that $\phi$ an idempotent, then $\phi = \phi^2$. We have,

$$\phi^2(e_{ii}) = \phi(\lambda_1 e_{ii} + \sum_{j \in \mathbb{F} \atop j \neq 1} \lambda_2 e_{ij})$$

$$= \lambda_1 \phi(e_{ii}) + \sum_{j \in \mathbb{F} \atop j \neq 1} \lambda_2 \phi(e_{ij})$$

$$= \lambda_1 (\lambda_1 e_{ii} + \sum_{j \in \mathbb{F} \atop j \neq 1} \lambda_2 e_{ij}) + \sum_{j \in \mathbb{F} \atop j \neq l} \lambda_2 (\lambda_1 e_{ij} + \sum_{l \in \mathbb{F} \atop l \neq j} \lambda_2 e_{ij})$$

$$= \lambda_1^2 e_{ii} + 2\lambda_1 \lambda_2 \sum_{j \in \mathbb{F} \atop j \neq 1} e_{ij} + \lambda_2^2 \sum_{j \in \mathbb{F} \atop j \neq l} \sum_{l \in \mathbb{F} \atop l \neq j} e_{ij}$$

$$= (\lambda_1^2 + \lambda_2^2 (r - 1)) e_{ii} + (2\lambda_1 \lambda_2 + \lambda_2^2 (r - 2)) \sum_{j \in \mathbb{F} \atop j \neq 1} e_{ij}.$$ 

So $\phi(e_{ii}) = \phi^2(e_{ii})$ if and only if

$$\lambda_1^2 + \lambda_2^2 (r - 1) = \lambda_1 \quad \text{and} \quad 2\lambda_1 \lambda_2 + \lambda_2^2 (r - 2) = \lambda_2. \quad (5.1)$$

Now, for $1 \leq l \leq r$ let $\sigma = (1 \ l)$. If $\phi^2(e_{ii}) = \phi(e_{ii})$ then we have $\phi^2(e_{ii}) = \phi^2(e_{i \cdot \sigma}) = \phi^2(e_{ii} \cdot \sigma) = \phi(e_{ii}) \cdot \sigma = \phi(e_{i \cdot \sigma}) = \phi(e_{ii} \cdot \sigma) = \phi(e_{ii})$. So $\phi$ is an idempotent if and only if Equations (5.1) and (5.2) hold. We now consider the following cases separately.

Let $r \equiv 0 \mod p$. Then (5.2) gives $2\lambda_1 \lambda_2 - 2\lambda_2^2 = \lambda_2$. Suppose that $\lambda_2 \neq 0$, then we have $2\lambda_1 - 2\lambda_2 = 1$. If $p = 2$ then this gives $0 = 1$ which is a contradiction. If $p \neq 2$ then we have $2\lambda_1 - 2\lambda_2 = 1$, and from (5.1) we get $(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2) = \lambda_1^2 - \lambda_2^2 = \lambda_1$, which gives $2(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2) = 2\lambda_1$. We know that $2(\lambda_1 - \lambda_2) = 1$, so we have $\lambda_1 + \lambda_2 = 2\lambda_1$, hence $\lambda_2 = \lambda_1$. Then, from $2\lambda_1 - 2\lambda_2 = 1$ we get $0 = 1$, a contradiction.
So for all $p$ we must have $\lambda_2 = 0$. Then (5.1) gives that $\lambda_1^2 = \lambda_1$, hence $\lambda_1 = 1$ or 0. Therefore, $\phi$ is a trivial idempotent. So, if $r \equiv 0 \mod p$, then $M_{(r-1,1)}$ is indecomposable.

Now consider the case where $r \not\equiv 0 \mod p$. Then take $\lambda_1 = \lambda_2 = r^{-1} \in K$, then

$$
\begin{align*}
\lambda_1^2 + \lambda_2^2(r - 1) &= (r^{-1})^2 + (r^{-1})^2(r - 1) = (r^{-1})^2 + (r^{-1})^2(r - (r^{-1})^2
\end{align*}
$$

$$
\begin{align*}
2\lambda_1\lambda_2 + \lambda_2^2(r - 2) &= 2(r^{-1})^2 + (r^{-1})^2(r - 2) = (r^{-1})^2(r = r^{-1} = \lambda_2.
\end{align*}
$$

Therefore $\phi$ given by $\phi(e_{ij}) = r^{-1} \sum_{j \in \mathbb{Z}} e_{ij}$ for $1 \leq i \leq r$ is a non-trivial idempotent in $\text{End}_{K}\mathcal{E}_r(M_{(r-1,1)})$. So, if $r \not\equiv 0 \mod p$, then $M_{(r-1,1)}$ is not indecomposable. \ □

**Corollary 5.1.3**

$$
M_{(r-1,1)} = \begin{cases} 
Y_{(r-1,1)} & \text{if } r \equiv 0 \mod p \\
Y_{(r-1,1)} \oplus Y_{(r,0)} & \text{if } r \not\equiv 0 \mod p
\end{cases}
$$

**Proof.** By Theorem 1.7.2, we know that $Y_{(r-1,1)}$ occurs exactly once as a direct summand of $M_{(r-1,1)}$, and only other possible direct summand is $Y_{(r,0)}$, and this occurs at most once. Therefore either $M_{(r-1,1)} = Y_{(r-1,1)}$ or $M_{(r-1,1)} = Y_{(r-1,1)} \oplus Y_{(r,0)}$, so the result follows from Lemma 5.1.2. \ □

**Remark 5.1.4** The proof of Lemma 5.1.2 that we have given illustrates the methods we want to generalize. It should however be noted that Lemma 5.1.2, and hence Corollary 5.1.3, could be obtained in an easier way using different methods. By Theorem 1.7.4 we have that $Y_{(r,0)}$ is a direct summand of $M_{(r-1,1)}$ if and only if 1 is $p$-contained in $r$. Now, by Lemma 1.3.2, 1 is $p$-contained in $r$ if and only if $\left(\frac{r}{p}\right) \neq 0 \mod p$. So this gives that $Y_{(r,0)}$ is a direct summand of $M_{(r-1,1)}$ if and only if $r \not\equiv 0 \mod p$, which gives us Lemma 5.1.2 and Corollary 5.1.3.

So if $r \equiv 0 \mod p$ we have that $Y_{(r-1,1)} = M_{(r-1,1)}$, which has basis $\{e_{ij} : 1 \leq j \leq r\}$. For the case $r \not\equiv 0 \mod p$, we have the following

**Lemma 5.1.5** For $r \in \mathbb{N}$, with $r \not\equiv 0 \mod p$, let $0 < s \leq p - 1$ with $r \equiv s \mod p$. 69
Define $\alpha, \beta \in K$ by $\beta = s^{-1}, \alpha = \beta - 1$. For $1 \leq l \leq r - 1$ let

$$v_l := \alpha e_{il} + \sum_{j \notin r}^{j \neq l} \beta e_{ij}.$$ 

Then $Y^{(r-1,1)}$ has basis $\{v_1, v_2, \ldots, v_{r-1}\}$.

**Proof.**

We know that, as $r \neq 0 \mod p$, $M^{(r-1,1)} = Y^{(r-1,1)} \oplus Y^{(r,0)}$, and recall from Lemma 5.1.1 that $Y^{(r,0)}$ is one-dimensional with basis vector $v = \sum_{j \in r} e_{ij}$. We define a $K$-vector space $Y$ by $Y := \text{span}\{v_1, v_2, \ldots, v_{r-1}\}$, and we will show that $M^{(r-1,1)} = Y \oplus Y^{(r,0)}$ is a module decomposition. We will first show that this is a vector space decomposition, by showing that $v_1, v_2, \ldots, v_{r-1}, v$ are linearly independent. Suppose we have $a_1, a_2, \ldots, a_r \in K$ with

$$0 = a_1 v_1 + a_2 v_2 + \ldots + a_{r-1} v_{r-1} + a_r v$$

$$= a_1 (\alpha e_{il} + \sum_{j \notin r}^{j \neq l} \beta e_{ij}) + \ldots + a_{r-1} (\alpha e_{i(r-1)} + \sum_{j \notin r}^{j \neq l} \beta e_{ij}) + a_r \sum_{j \notin r}^{j \neq l} e_{ij}$$

$$= e_{il} (a_1 \alpha + a_r + \sum_{j \notin r}^{j \neq l} a_j \beta) + \ldots + e_{i(r-1)} (a_{r-1} \alpha + a_r + \sum_{j \notin r}^{j \neq l} a_j \beta) +$$

$$e_{ir} (a_r + \sum_{j \notin r}^{j \neq l} a_j \beta).$$

So we have

$$a_1 \alpha + a_r + \beta \sum_{j \notin r}^{j \neq l} a_j = 0 \quad \text{(for } 1 \leq l \leq r - 1), \quad (5.3)$$

$$a_r + \beta \sum_{j \notin r}^{j \neq l} a_j = 0. \quad (5.4)$$

Let $1 \leq l \leq r - 1$, then from (5.4) we have

$$0 = a_r + \beta a_l + \beta \sum_{j \notin r}^{j \neq l} a_j = \beta a_l - a_l \alpha = (\beta - \alpha) a_l = a_l$$

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(by substituting in (5.3), and using $\beta - \alpha = 1$). So we must have that $a_l = 0$ for all $1 \leq l \leq r - 1$, and so (5.4) gives that $a_r = 0$ also. Therefore, $v_1, v_2, \ldots, v_{r-1}, v$ are linearly independent.

In particular $v_1, v_2, \ldots, v_{r-1}$ are linearly independent, and so form a $K$-basis of the vector space $Y$. Now, we have $r$ linearly independent vectors in $M^{(r-1,1)}$, which has dimension $r$, so these vectors form a basis for $M^{(r-1,1)}$. Therefore $M^{(r-1,1)}$ has basis $v_1, v_2, \ldots, v_{r-1}, v$, where $v_1, v_2, \ldots, v_{r-1}$ is a basis of $Y$ and $v$ is a basis of $Y^{(r,0)}$. Therefore $M^{(r-1,1)} = Y \oplus Y^{(r,0)}$ is a vector space decomposition. To show that this is actually a module decomposition, we need to show that $Y$ is a $K\Sigma_r$-module. We will now show that $v_l \cdot \sigma \in Y$ for all $\sigma \in \Sigma_r$ and all $1 \leq l \leq r - 1$.

Now for all $\sigma \in \Sigma_r$, and for each $l \in \{1, 2, \ldots, r - 1\}$ we have that

$$v_l \cdot \sigma = (\alpha e_{i_l} + \sum_{j \neq l} \beta e_{i_j}) \cdot \sigma = \alpha e_{i_l \sigma} + \sum_{j \in \mathbb{Z}} \beta e_{i_j \sigma} = \alpha e_{i_l \sigma^{-1}(l)} + \sum_{m \in \mathbb{Z}} \beta e_{i_m},$$

using $i^l \sigma = i^{l \sigma}$, and so $i^l \sigma = i^{l \sigma^{-1}(j)}$. Let

$$v_r := \alpha e_{i_r} + \beta \sum_{j \neq r} e_{i_j},$$

then we have $v_l \cdot \sigma \in \{v_1, v_2, \ldots, v_{r-1}, v_r\}$. If we can show that $v_r \in Y$ then we know that $Y$ is a $K\Sigma_r$-module. We see that that:

$$v_1 + v_2 + \ldots + v_{r-1} = (\alpha e_{i_1} + \sum_{j \neq 1} \beta e_{i_j}) + \ldots + (\alpha e_{i_{r-1}} + \sum_{j \neq r-1} \beta e_{i_j})$$

$$= e_{i_1}(\alpha + (r-2)\beta) + e_{i_2}(\alpha + (r-2)\beta)$$

$$+ \ldots + e_{i_{r-1}}(\alpha + (r-2)\beta) + e_r((r-1)\beta).$$

Now in $K$ we have: $\alpha + (r-2)\beta = \alpha + r\beta - 2\beta = (\beta - 1) + 1 - 2\beta = -\beta$, using the fact that $\alpha = \beta - 1$ and $\beta \equiv r^{-1} \mod p$. Similarly $(r-1)\beta = r\beta - \beta = 1 - \beta = -\alpha$. So we have that

$$v_1 + v_2 + \ldots + v_{r-1} = -\beta e_{i_1} - \beta e_{i_2} - \ldots - \beta e_{i_{r-1}} - \alpha e_r = -v_r,$$

therefore $v_r = -v_1 - v_2 - \ldots - v_{r-1} \in Y$.

Hence $Y$ is a $K\Sigma_r$-module. Therefore $M^{(r-1,1)} = Y \oplus Y^{(r,0)}$ is a module decomposition, hence $Y = Y^{(r-1,1)}$. \qed
5.2 Finding The Dimension of Young Modules

Throughout this section let $r$ and $k$ be natural numbers such that $r \geq 2k$, therefore $(r - k, k)$ is a partition of $r$, and let $p = 2$ so that $K$ is a field of characteristic two. We can construct the Young module $Y^{(r-k,k)} = \text{span}\{v_a : a \in J(r,k)\}$ (see Theorem 4.1.4), but we would like to actually find a basis for this Young module of the form $\{v_a : a \in \overline{J}(r,k)\}$ for some set $\overline{J}(r,k) \subseteq J(r,k)$. So we remove some vectors from the set $\{v_a : a \in J(r,k)\}$, which spans $Y^{(r-k,k)}$, to obtain a basis. Before we do this, we need to know how many vectors need to be removed, and our aim throughout this section is to determine the dimensions of Young modules in some particular cases.

Note that for $k = 0$ the dimension of the Young module $Y^{(r,0)}$ is one, which is why we now only consider natural numbers $k$. For the fixed natural number $k$ we define a non-negative integer $x$ such that

$$2^x \leq k < 2^{x+1}.$$ 

For a real number $l$ define $[l]$ to be the largest integer less than or equal to $l$ (so $l - 1 < [l] \leq l$).

**Lemma 5.2.1** For $r - 2k \equiv 0 \mod 2^{x+1}$ we have that $M^{(r-k,k)} = Y^{(r-k,k)}$, so $\dim Y^{(r-k,k)} = \binom{r}{k}$.

**Proof.** As $x \in \mathbb{N}_0$ is such that $2^x \leq k < 2^{x+1}$, we can write

$$k = k_0 \cdot 2^0 + k_1 \cdot 2^1 + k_2 \cdot 2^2 + \ldots + k_{x-1} \cdot 2^{x-1} + 1 \cdot 2^x$$

for some $k_0, k_1, \ldots, k_{x-1} \in \{0, 1\}$. For $r - 2k \equiv 0 \mod 2^{x+1}$, there is an integer, $y$, such that $r - 2k = 2^{x+1}y$. Let $0 \leq s \leq k$ and let $l_0, l_1, \ldots, l_x \in \{0, 1\}$ such that

$$k - s = l_0 \cdot 2^0 + l_1 \cdot 2^1 + l_2 \cdot 2^2 + \ldots + l_{x-1} \cdot 2^{x-1} + l_x \cdot 2^x.$$ 

Then

$$r - 2s = (r - 2k) + 2(k - s)$$

$$= 0 \cdot 2^0 + l_0 \cdot 2^1 + l_1 \cdot 2^2 + \ldots + l_{x-1} \cdot 2^{x-1} + 2^x \cdot (l_x + y).$$
So $k - s$ is 2-contained in $r - 2s$ if and only if $l_0 \leq 0, l_1 \leq l_0, \ldots, l_x \leq l_{x-1}$, i.e. $k - s = 0$. So, by Theorem 1.7.4, the Young module $Y^{(r-s,s)}$ is a direct summand of $M^{(r-k,k)}$ if and only if $k = s$. Therefore $M^{(r-k,k)} = Y^{(r-k,k)}$, so $\dim Y^{(r-k,k)} = \dim M^{(r-k,k)} = \binom{k}{r-k}$.

Recall that in Section 2.4 we defined what it means for a module to be liftable. The Young module as a direct summand of a permutation module is liftable, and therefore we can consider the module as a module in characteristic zero. There it has an ordinary character (ordinary character means the character in characteristic zero) which we denote by $\chi_{Y^{(r-k,k)}}$ (see, for example, [29] Lemma 4.6.2 (i)). Let $\chi^{(r-k,k)}$ be the associated ordinary character of the Specht module $S^{(r-k,k)}$. The ordinary characters of Young modules corresponding to two-part partitions are determined in [17] Section 5.2 (also see [19] Section 4). This is done for a general prime $p$, but here we will just consider the case where $p = 2$. Following [19] Section 4, but specializing to the case $p = 2$, we define a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ by $f(x,y) = \prod_{i \in \mathbb{N}_0} \left( \frac{1-x}{1-y} \right)^{i}$ for all $x, y \in \mathbb{N}_0$ with 2-adic decompositions $x = \sum_{i \in \mathbb{N}_0} x_i 2^i$ and $y = \sum_{i \in \mathbb{N}_0} y_i 2^i$. We then have the following theorem:

**Theorem 5.2.2 (see [19], Proposition 4.1 and 4.2)** The Young module $Y^{(r-k,k)}$ has the associated ordinary character

$$\chi_{Y^{(r-k,k)}} = \sum_{t=0}^{[r/2]} k_{[(r-2t)/2],[(r-2k)/2]} \chi^{(r-t,t)},$$

where

$$k_{[(r-2t)/2],[r-2k)/2]} = \begin{cases} 1 & \text{if } f(r-2k, r-k-t) = 1 \\ 0 & \text{otherwise} \end{cases}$$

In particular this gives the following corollary:

**Corollary 5.2.3** The Young module $Y^{(r-k,k)}$ has dimension

$$\dim Y^{(r-k,k)} = \sum_{t=0}^{k} \rho_t \dim S^{(r-t,t)}$$

over $K$ where, for $t \in \{0, 1, \ldots, k\}$, $\rho_t \in \{0, 1\}$ with $\rho_t \equiv \frac{(r-k-t)}{r-2k} \mod 2$. 

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Proof. Let \( t \in \{0,1, \ldots, \lfloor r/2 \rfloor \} \), and define \( \rho_t \in \{0,1\} \) to be such that \( \rho_t \equiv \binom{r-k-t}{r-2k} \) mod 2. Let \( r-2k \) and \( r-k-t \) have 2-adic decompositions \( r-2k = \sum_{i \in \mathbb{N}_0} \alpha_i 2^i \) and \( r-k-t = \sum_{i \in \mathbb{N}_0} \beta_i 2^i \). Then, by definition, \( f(r-2k, r-k-t) = \prod_{i \in \mathbb{N}_0} \binom{\alpha_i}{1-\beta_i} \). Now, for each \( i \), \( \binom{1-\alpha_i}{1-\beta_i} \) is zero if \( \alpha_i = 1 \) and \( \beta_i = 0 \), and one otherwise. So

\[
f(r-2k, r-k-t) = \begin{cases} 0 & \text{if there exists an } i \in \mathbb{N}_0 \text{ with } \alpha_i = 1 \text{ and } \beta_i = 0 \\ 1 & \text{otherwise} \\
\end{cases}
= \begin{cases} 0 & \text{if } \binom{r-k-t}{r-2k} \equiv 0 \mod 2 \\
1 & \text{otherwise},
\end{cases}
\]
as by Lemma 1.3.2 we have \( \binom{r-k-t}{r-2k} \equiv \prod_{i \in \mathbb{N}_0} \binom{\beta_i}{\alpha_i} \mod 2 \). So, by the definition given in Theorem 5.2.2, we have

\[
k_i(\lfloor (r-2t)/2 \rfloor, \lfloor (r-2k)/2 \rfloor) = \begin{cases} 1 & \text{if } f(r-2k, r-k-t) = 1 \\ 0 & \text{otherwise} \\
\end{cases}
= \begin{cases} 1 & \text{if } \binom{r-k-t}{r-2k} \equiv 1 \mod 2 \\
0 & \text{otherwise} \\
= \rho_t.
\]

So, using Theorem 5.2.2 we have that

\[
\dim Y^{(r-k,k)} = \sum_{t=0}^{\lfloor r/2 \rfloor} \rho_t \dim S^{(r-t,k)}.
\]

In particular, if \( t > k \) then \( 0 \leq r-k-t < r-2k \), so \( \binom{r-k-t}{r-2k} = 0 \) by Lemma 1.3.1 (1), and therefore \( \rho_t = 0 \). So,

\[
\dim Y^{(r-k,k)} = \sum_{t=0}^{k} \rho_t \dim S^{(r-t,k)}.
\]

The following lemma then tells us us what the dimensions of the Specht modules are for two-part partitions, and can be proved using the Hook Length Formula given in [10].

**Lemma 5.2.4** (see for example [23], Example 14.4) For \( r \) a natural number and \( t \) a non-negative integer with \( t \leq r/2 \) we have that

\[
\dim S^{(r-t,t)} = \binom{r}{t} - \binom{r}{t-1}.
\]
Example 5.2.5 Let \( r - 2k \equiv 0 \mod 2^{x+1} \), so that \( r - 2k = 2^{x+1} \alpha \) for some \( \alpha \in \mathbb{N}_0 \). Let \( t \in \{0, 1, \ldots, k\} \) and let \( 0 \leq \beta < 2^{x+1} \) be such that \( r - k - t \equiv \beta \mod 2^{x+1} \), then \( r - k - t = \beta + 2^{x+1} \gamma \) for some \( \gamma \in \mathbb{N}_0 \). Now, \( 0 \leq t \leq k \) so we have \( 0 \leq (r - k - t) - (r - 2k) = (r - k) - (r - 2k) < 2^{x+1} \), by definition of \( x \). So that \( 0 \leq (\beta + 2^{x+1} \gamma) - (2^{x+1} \alpha) < 2^{x+1} \), hence \( 0 \leq \beta + 2^{x+1} (\gamma - \alpha) < 2^{x+1} \). So, as \( 0 \leq \beta < 2^{x+1} \) we have \( \gamma = \alpha \). Therefore, \( r - 2k = 2^{x+1} \alpha \) and \( r - k - t = \beta + 2^{x+1} \alpha \) with \( 0 \leq \beta < 2^{x+1} \).

Hence, for all \( t \in \{0, 1, \ldots, k\} \) we have \( \binom{r-k-t}{r-2k} \equiv 1 \mod 2 \), so by Corollary 5.2.3 and using Lemma 5.2.4 we have

\[
\dim Y^{(r-k,k)} = \sum_{t=0}^{k} \dim S^{(r-t,t)}
\]

\[
= \sum_{s=0}^{k} \left( \binom{r}{k-s} - \binom{r}{k-s-1} \right)
\]

\[
= \binom{r}{k} - \binom{r}{k-1} + \binom{r}{k-1} - \binom{r}{k-2} + \ldots + \binom{r}{1} - \binom{r}{0} + \binom{r}{0} - \binom{r}{-1}
\]

\[
= \binom{r}{k}.
\]

So in this case we have \( \dim Y^{(r-k,k)} = \binom{r}{k} \), as we had already shown in Lemma 5.2.1.

Now we will look at some more specific cases where we can apply Corollary 5.2.3.

**Lemma 5.2.6** If \( r - 2k \equiv -2^y \mod 2^{x+1} \) for some non-negative integer \( y \leq x \) then \( \dim Y^{(r-k,k)} = \binom{r}{k} - \binom{r}{k-2y} \).

*Proof.* Let \( y \leq x \) and \( 0 \leq t \leq k \). As \( r - 2k \equiv -2^y \mod 2^{x+1} \) we can write

\[
r - 2k = 0 \cdot 2^0 + \ldots + 0 \cdot 2^{y-1} + 1 \cdot 2^y + 1 \cdot 2^{y+1} + \ldots + 1 \cdot 2^x + \alpha \cdot 2^{x+1},
\]

for some integer \( \alpha \). Now let \( r - k - t \equiv \beta \mod 2^{x+1} \), with \( 0 \leq \beta < 2^{x+1} \), so that

\[
r - k - t = \beta_0 \cdot 2^0 + \ldots + \beta_{y-1} \cdot 2^{y-1} + \beta_y \cdot 2^y + \beta_{y+1} \cdot 2^{y+1} + \ldots + \beta_x \cdot 2^x + \gamma \cdot 2^{x+1},
\]

for some integer \( \gamma \), where each \( \beta_i \in \{0,1\} \). Then \( k - t = (r - k - t) - (r - 2k) = (\beta + 2^{x+1} \gamma) - (2^{x+1} - 2^y + 2^{x+1} \alpha) = \beta + 2^y + 2^{x+1} (\gamma - \alpha - 1) \). So, as \( k - t \geq 0 \) and
\( \beta < 2^{x+1} \) we have that

\[
0 \leq \beta + 2^y + 2^{x+1}(\gamma - \alpha - 1) < 2^y + 2^{x+1}(\gamma - \alpha).
\]

Therefore, as \( y \leq x \), we must have that \( \gamma - \alpha \geq 0 \), hence \( \gamma \geq \alpha \).

In Corollary 5.2.3 we defined \( \rho_t \in \{0, 1\} \), and now we want to see when \( \rho_t = 0 \) and when \( \rho_t = 1 \). We have (using Lemma 1.3.2) that

\[
\binom{r - k - t}{r - 2k} \equiv \binom{\beta_0}{0} \cdots \binom{\beta_{y-1}}{0} \cdot \binom{\beta_y}{1} \cdots \binom{\beta_z}{1} \cdot \binom{\gamma}{\alpha} \mod 2.
\]

Therefore \( \rho_t = 1 \) if and only if \( \beta_y = \beta_{y+1} = \ldots = \beta_z = 1 \) and \( \binom{\gamma}{\alpha} \equiv 1 \mod 2 \). Now, \( \beta < 2^{x+1} \), so \( \rho_t = 1 \) if and only if \( \beta \geq 2^{x+1} - 2^y \) and \( \binom{\gamma}{\alpha} \equiv 1 \mod 2 \).

For \( 0 \leq t \leq k \) we consider two cases. Firstly, let \( t > k - 2^y \). From above we know that \( \gamma \geq \alpha \), and we will prove by contradiction that we must have \( \gamma = \alpha \). Assume that \( \gamma > \alpha \), then \( \gamma - \alpha > 0 \), therefore \( \gamma - \alpha - 1 \geq 0 \). So \( k - t = \beta + 2^y + 2^{x+1}(\gamma - \alpha - 1) \geq \beta + 2^y \geq 2^y \), which contradicts the fact that \( t > k - 2^y \). Therefore we must have that \( \gamma = \alpha \), so \( \binom{\gamma}{\alpha} = 1 \). Moreover, \( 0 \leq k - t = \beta + 2^y - 2^{x+1} \), so \( \beta \geq 2^{x+1} - 2^y \), and therefore in this case \( \rho_t = 1 \).

Now, consider \( t \leq k - 2^y \). Therefore \( 2^y \leq k - t \leq k < 2^{x+1} \), by definition of \( x \), and \( k - t = \beta + 2^y + 2^{x+1}(\gamma - \alpha - 1) \). So \( 0 \leq \beta + 2^{x+1}(\gamma - \alpha - 1) \), therefore as \( 0 \leq \beta < 2^{x+1} \) we must have that \( \gamma - \alpha - 1 \geq 0 \), hence \( 2^{x+1} > k - t \geq \beta + 2^y \) so that \( \beta < 2^{x+1} - 2^y \), hence \( \rho_t = 0 \) in this case.

So, for \( 0 \leq t \leq k \) we have \( \rho_t = 1 \) if \( t > k - 2^y \) and zero otherwise. Therefore, by Corollary 5.2.3 and Lemma 5.2.4 we have that

\[
\dim Y^{(r-k,k)} = \sum_{t=k-2^y+1}^{k} \dim S^{(r-t,t)}
\]

\[
= \dim S^{(r-k,k)} + \dim S^{(r-(k-1),k-1)} + \ldots + \dim S^{(r-(k-(2^y-1)),k-(2^y-1))}
\]

\[
= \binom{r}{k} - \binom{r}{k-1} + \binom{r}{k-1} - \binom{r}{k-2} + \ldots + \binom{r}{k-2^y+1} - \binom{r}{k-2^y}
\]

\[
= \binom{r}{k} - \binom{r}{k-2^y}.
\]
as required.

We now consider some applications of Lemma 5.2.6.

**Example 5.2.7** If \( r - 2k \equiv -1 \mod 2^{x+1} \), then by Lemma 5.2.6 (with \( y = 0 \)) we have \( \dim Y(r-k,k) = \binom{r}{k} - \binom{r}{k-1} \). So, by Lemma 5.2.4 \( \dim Y(r-k,k) = \dim S(r-k,k) \), therefore \( Y(r-k,k) = S(r-k,k) \) in this case.

**Remark 5.2.8** If \( r - 2k \equiv 2^z \equiv -2^x \mod 2^{x+1} \), then using Theorem 1.7.4 it can be shown that \( M(r-k,k) = Y(r-k,k) \oplus Y((r-(k-2^x)),(k-2^x)) \). Let \( l = k - 2^x \), then we want to determine \( \dim Y(r-l,l) \). Let \( z \) be the integer such that \( 2^z \leq l < 2^{z+1} \), then as \( 2^z \leq k < 2^{z+1} \) we have that \( l < 2^z \) so \( z < x \). Now, \( r - 2l = r - 2(k - 2^x) = r - 2k + 2^{z+1} \equiv 2^z \mod 2^{z+1} \), so \( r - 2l \equiv 0 \mod 2^{z+1} \), hence by Lemma 5.2.1 we have that \( \dim Y(r-l,l) = \binom{r}{l} \). Therefore, as \( \dim M(r-k,k) = \binom{r}{k} \) we have that \( \dim Y(r-k,k) = \binom{r}{k} - \binom{r}{k-1} \), which is consistent with Lemma 5.2.6 with \( y = x \).

**Lemma 5.2.9** For \( r - 2k \equiv 2 \mod 2^{x+1} \),

\[
\dim Y(r-k,k) = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{r}{k-2i}.
\]

**Proof.** Let \( r - 2k \equiv 2 \mod 2^{x+1} \) and let \( t \) be an integer with \( 0 \leq t \leq k \). We can write \( r - 2k = 0 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + \ldots + 0 \cdot 2^x + 2^{x+1} \cdot \alpha \), for some \( \alpha \in \mathbb{N}_0 \). Now, let \( r - k - t \equiv \beta \mod 2 \), where \( 0 \leq \beta < 2^{x+1} \) and \( \beta \) has \( 2 \)-adic decomposition \( \beta = \sum_{i=0}^{x} b_i \cdot 2^i \). Then we can write \( r - k - t = \beta_0 \cdot 2^0 + \ldots + \beta_x \cdot 2^x + 2^{x+1} \cdot \gamma \) for some \( \gamma \in \mathbb{N}_0 \). We have that \( k - t = (r - k - t) - (r - 2k) = (k - t) \cdot 2^{x+1}(\gamma - \alpha) \).

Note that, by Lemma 1.3.2, \( \binom{r-k-t}{r-2k} \equiv (\beta_0 \cdot 0) \binom{\beta_1}{0} (\beta_2 \cdot 0) \ldots (\beta_x \cdot 0) (\gamma_0) \equiv 1 \mod 2 \) if and only if \( \beta_1 = 1 \) and \( (\gamma) \equiv 1 \mod 2 \). Therefore \( \rho_t \), as defined in Corollary 5.2.3, is one if \( \beta_1 = 1 \) and \( (\gamma) \equiv 1 \mod 2 \), and zero otherwise. We will look at which values of \( t \) give \( \rho_t = 1 \).

If \( \beta = 0 \) or \( 1 \) then clearly \( \beta_1 = 0 \), so \( \rho_t = 0 \). Now consider \( \beta \geq 2 \) so that \( \beta - 2 \geq 0 \). Then \( 2^{x+1}(\gamma - \alpha) \leq \beta - 2 + 2^{x+1}(\gamma - \alpha) = k - t \leq k < 2^{x+1} \), therefore \( \gamma - \alpha < 1 \), so that \( \gamma \leq \alpha \). If \( \gamma < \alpha \) then clearly \( (\alpha) = 0 \) so \( \rho_t = 0 \), and if \( \gamma = \alpha \) then \( (\alpha) = 1 \) so \( \rho_t = 1 \) if and only if \( \beta_t = 1 \) where \( k - t = \beta - 2 \) so \( \beta = k - t + 2 \).
Therefore $\rho_t = 1$ if and only if $(k - t)_1 = 0$, so by Corollary 5.2.3 and Lemma 5.2.4 we have,

$$\dim Y^{(r-k,k)} = \sum_{0 \leq s \leq k, s_1 = 0} \dim S^{(r-(k-s),k-s)}$$

$$= \dim S^{(r-k,k)} + \dim S^{(r-(k-1),k-1)} + \dim S^{(r-(k-4),k-4)} + \dim S^{(r-(k-5),k-5)} + \ldots$$

$$= \binom{r}{k} - \binom{r}{k-1} + \binom{r}{k-1} - \binom{r}{k-2} + \binom{r}{k-4} - \binom{r}{k-5} + \binom{r}{k-5} - \binom{r}{k-6} + \ldots$$

$$= \binom{r}{k} - \binom{r}{k-2} + \binom{r}{k-4} - \binom{r}{k-6} + \ldots$$

$$= \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{r}{k-2i}.$$

In Corollary 5.2.12 we will give an alternative way of writing the dimension of the Young module $Y^{(r-k,k)}$ in the case where $r - 2k \equiv 2 \mod 2^s + 1$, which will be useful in the next section. Before we can do this, we need to consider the following Lemma and Proposition.

**Lemma 5.2.10** For all natural numbers $s$,

$$\sum_{i=1}^{s} 2^{i-1} \binom{2s - 2i}{s - i} = \sum_{i=1}^{\lfloor \frac{s+1}{2} \rfloor} (-1)^{i+1} \binom{2s}{s + 1 - 2i}.$$

**Proof.** We prove this by induction on $s$. First let $s = 1$, then $2^{1-1}(\binom{2-2}{1-1}) = 1 = (-1)^{1+1}(\binom{2}{1 + 1 - 2})$, so the result holds for $s = 1$.

Let $s \in \mathbb{N}$ and suppose the result holds for $s - 1$, so

$$\sum_{i=1}^{s-1} 2^{i-1} \binom{2(s - 1) - 2i}{s - 1 - i} = \sum_{i=1}^{\lfloor \frac{s}{2} \rfloor} (-1)^{i+1} \binom{2(s - 1)}{s - 1 - 2i}.$$

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Now,
\[
\sum_{i=1}^{s} 2^{i-1} \left( \frac{2s - 2i}{s - i} \right) = 2 \sum_{i=1}^{s} 2^{i-2} \left( \frac{2s - 2(i - 1) - 2}{s - (i - 1) - 1} \right)
\]
\[
= 2 \sum_{j=0}^{s-1} 2^{j-1} \left( \frac{2s - 2j - 2}{s - j - 1} \right)
\]
\[
= \left( \frac{2s - 2(0) - 2}{s - 0 - 1} \right) + 2 \sum_{j=1}^{s-1} 2^{j-1} \left( \frac{2(s - 1) - 2j}{(s - 1) - j} \right)
\]
\[
= \left( \frac{2s - 2}{s - 1} \right) + 2 \sum_{j=1}^{s-1} (-1)^{j+1} \left( \frac{2(s - 1)}{s - 2i} \right), \quad (5.5)
\]
by the induction hypothesis. Now, by Lemma 1.3.1 (3) with \( r = 2 \), for all integers \( i \) we have
\[
\left( \frac{2s}{s + 1 - 2i} \right) = \left( \frac{2s - 2}{s + 1 - 2i} \right) + 2 \left( \frac{2s - 2}{s - 2i} \right) + \left( \frac{2s - 2}{s - 1 - 2i} \right).
\]
Therefore,
\[
\sum_{i=1}^{\lfloor \frac{s+1}{2} \rfloor} (-1)^{i+1} \left( \frac{2s}{s + 1 - 2i} \right) = \sum_{i=1}^{\lfloor \frac{s+1}{2} \rfloor} (-1)^{i+1} \left( \frac{2s - 2}{s + 1 - 2i} \right) +
\]
\[
2 \sum_{i=1}^{\lfloor \frac{s+1}{2} \rfloor} (-1)^{i+1} \left( \frac{2s - 2}{s - 2i} \right) +
\]
\[
\sum_{i=1}^{\lfloor \frac{s+1}{2} \rfloor} (-1)^{i+1} \left( \frac{2s - 2}{s - 1 - 2i} \right). \quad (5.6)
\]
Now,
\[
\sum_{i=1}^{\lfloor \frac{s+1}{2} \rfloor} (-1)^{i+1} \left( \frac{2s - 2}{s + 1 - 2i} \right) + \sum_{i=1}^{\lfloor \frac{s+1}{2} \rfloor} (-1)^{i+1} \left( \frac{2s - 2}{s - 1 - 2i} \right)
\]
\[
= \sum_{i=1}^{\lfloor \frac{s+1}{2} \rfloor} (-1)^{i+1} \left( \frac{2s - 2}{s + 1 - 2i} \right) - \sum_{i=1}^{\lfloor \frac{s+1}{2} \rfloor} (-1)^{i} \left( \frac{2s - 2}{s + 1 - 2(i + 1)} \right)
\]
\[
= \left( \frac{2s - 2}{s - 1} \right) + \sum_{i=2}^{\lfloor \frac{s+1}{2} \rfloor} (-1)^{i+1} \left( \frac{2s - 2}{s + 1 - 2i} \right) - \sum_{j=2}^{\lfloor \frac{s+1}{2} \rfloor+1} (-1)^{j-1} \left( \frac{2s - 2}{s + 1 - 2j} \right)
\]
\[
= \left( \frac{2s - 2}{s - 1} \right) - (-1)^{\lfloor \frac{s+1}{2} \rfloor} \left( s + 1 - 2(\lfloor \frac{s+1}{2} \rfloor + 1) \right),
\]
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using the fact that for all integers $j$ we have $(-1)^{j+1} = (-1)^{j-1}$. Now, $\left\lfloor \frac{s+1}{2} \right\rfloor > \frac{s+1}{2} - 1$, so $2(\left\lfloor \frac{s+1}{2} \right\rfloor + 1) > s + 1$, therefore $s + 1 - 2(\left\lfloor \frac{s+1}{2} \right\rfloor + 1) < 0$, and so $\left(\frac{2s-2}{s+1-2(\left\lfloor \frac{s+1}{2} \right\rfloor + 1)}\right) = 0$. Therefore by (5.6) we have,

$$\sum_{i=1}^{\left\lfloor \frac{s+1}{2} \right\rfloor} (-1)^{i+1} \left( \frac{2s}{s + 1 - 2i} \right) = \left( \frac{2s - 2}{s - 1} \right) + 2 \sum_{i=1}^{\left\lfloor \frac{s+1}{2} \right\rfloor} (-1)^{i+1} \left( \frac{2s - 2}{s - 2i} \right).$$

Now, if $s$ is even then $\left\lfloor \frac{s+1}{2} \right\rfloor = \frac{s}{2}$. If $s$ is odd then $\left\lfloor \frac{s+1}{2} \right\rfloor = \left\lfloor \frac{s}{2} \right\rfloor + 1$ and $\left( \frac{2s-2}{s-2(\left\lfloor \frac{s}{2} \right\rfloor + 1)} \right) = (\frac{s-2}{s-1}) = 0$. So we have,

$$\sum_{i=1}^{\left\lfloor \frac{s+1}{2} \right\rfloor} (-1)^{i+1} \left( \frac{2s}{s + 1 - 2i} \right) = \left( \frac{2s - 2}{s - 1} \right) + 2 \sum_{i=1}^{\left\lfloor \frac{s}{2} \right\rfloor} (-1)^{i+1} \left( \frac{2s - 2}{s - 2i} \right),$$

so by (5.5) we have

$$\sum_{i=1}^{s} 2^{i-1} \left( \frac{2s - 2i}{s - i} \right) = \sum_{i=1}^{\left\lfloor \frac{s+1}{2} \right\rfloor} (-1)^{i+1} \left( \frac{2s}{s + 1 - 2i} \right),$$

and the result is proved by induction. □

**Proposition 5.2.11** For any natural numbers $r$ and $s$ with $1 \leq s \leq r/2$,

$$\sum_{i=1}^{s} 2^{i-1} \left( \frac{r - 2i}{s - i} \right) = \sum_{i=1}^{\left\lfloor \frac{s+1}{2} \right\rfloor} (-1)^{i+1} \left( \frac{r}{s + 1 - 2i} \right).$$

**Proof.** First we note that if $s = 1$ then

$$\sum_{i=1}^{s} 2^{i-1} \left( \frac{r - 2i}{s - i} \right) = 2^{1-1} \left( \frac{r - 2(1)}{1 - 1} \right) = \left( \frac{r - 2}{0} \right) = 1,$$

and

$$\sum_{i=1}^{\left\lfloor \frac{s+1}{2} \right\rfloor} (-1)^{i+1} \left( \frac{r}{s + 1 - 2i} \right) = (-1)^{1+1} \left( \frac{r}{1 + 1 - 2(1)} \right) = \left( \frac{r}{0} \right) = 1.$$

Similarly, if $s = 2$ then $\sum_{i=1}^{s} 2^{i-1} \left( \frac{r - 2i}{s - i} \right) = r = \sum_{i=1}^{\left\lfloor \frac{s+1}{2} \right\rfloor} (-1)^{i+1} \left( \frac{r}{s+1-2i} \right)$. So the result holds for $s = 1$ and $s = 2$.

We now give a proof by induction on $r$. If $r = 2, 3, 4$ or $5$ then as $1 \leq s \leq r/2$ we have $s = 1$ or $2$, so we know that the result holds in this case.

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Let \( r \in \mathbb{N} \) and assume that the result holds for \( r - 1 \), so for all \( s \in \mathbb{N} \) with \( 1 \leq s \leq (r - 1)/2 \) we have,

\[
\sum_{i=1}^{s} 2^{i-1} \left( \frac{r - 1 - 2i}{s - i} \right) = \sum_{i=1}^{\left\lfloor \frac{s+1}{2} \right\rfloor} (-1)^{i+1} \left( \frac{r - 1}{s + 1 - 2i} \right).
\]

Let \( s \in \mathbb{N} \) with \( 1 \leq s \leq (r - 1)/2 \). We know that \( \binom{r - 2i}{s-1} = \binom{r - 2i - 1}{s-1} + \binom{r - 2i - 1}{s-1-i} \), so

\[
\sum_{i=1}^{s} 2^{i-1} \left( \frac{r - 2i}{s - i} \right) = \sum_{i=1}^{s} 2^{i-1} \left( \binom{r - 2i - 1}{s-1} + \binom{r - 2i - 1}{s-1-i} \right)
= \sum_{i=1}^{s} 2^{i-1} \left( \frac{r - 1 - 2i}{s - i} \right) + \sum_{i=1}^{s-1} 2^{i-1} \left( \frac{r - 1 - 2i}{s - 1 - i} \right)
= \sum_{i=1}^{\left\lfloor \frac{s+1}{2} \right\rfloor} (-1)^{i+1} \left( \frac{r - 1}{s + 1 - 2i} \right) + \sum_{i=1}^{\left\lfloor \frac{s}{2} \right\rfloor} (-1)^{i+1} \left( \frac{r - 1}{s - 2i} \right),
\]

using the induction assumption. Now, if \( s \) is even then \( \lfloor s/2 \rfloor = \lfloor (s + 1)/2 \rfloor - 1 \), and \( \binom{r - 1}{s-2\lfloor (s+1)/2 \rfloor} = \binom{r - 1}{s-1} = 0 \). So we have,

\[
\sum_{i=1}^{s} 2^{i-1} \left( \frac{r - 2i}{s - i} \right) = \sum_{i=1}^{\left\lfloor \frac{s+1}{2} \right\rfloor} (-1)^{i+1} \left( \frac{r - 1}{s + 1 - 2i} \right) + \sum_{i=1}^{\left\lfloor \frac{s}{2} \right\rfloor} (-1)^{i+1} \left( \frac{r - 1}{s - 2i} \right)
= \sum_{i=1}^{\left\lfloor \frac{s+1}{2} \right\rfloor} (-1)^{i+1} \left( \frac{r - 1}{s + 1 - 2i} \right) + \sum_{i=1}^{\left\lfloor \frac{s}{2} \right\rfloor} (-1)^{i+1} \left( \frac{r - 1}{s - 2i} \right)
= \sum_{i=1}^{\left\lfloor \frac{s+1}{2} \right\rfloor} (-1)^{i+1} \left( \frac{r}{s + 1 - 2i} \right).
\]

Therefore, we have proved by induction that the proposition holds for all \( r \) and any \( s \leq (r - 1)/2 \). The other case is if \( s = r/2 \) where \( r \) is even. The result in this case follows from Lemma 5.2.10.

\[\Box\]

**Corollary 5.2.12** If \( r - 2k \equiv 2 \mod 2^{k+1} \) then

\[
\dim Y^{(r-k,k)} = \binom{r}{k} - \sum_{i=1}^{k-1} 2^{i-1} \binom{r - 2i}{k - 1 - i}.
\]

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Proof. Let \( r - 2k \equiv 2 \mod 2^{n+1} \). Then, by Lemma 5.2.9,

\[
\dim Y^{(r-k,k)} = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{r}{k-2i} \\
= \binom{r}{k} + \sum_{i=1}^{\lfloor k/2 \rfloor} (-1)^i \binom{r}{k-2i} \\
= \binom{r}{k} - \sum_{i=1}^{\lfloor k/2 \rfloor} (-1)^{i+1} \binom{r}{k-2i} \\
= \binom{r}{k} - \sum_{i=1}^{k-1} 2^{i-1} \binom{r-2i}{k-1-i},
\]

using Proposition 5.2.11 with \( s = k - 1 \). \(\square\)

Lemma 5.2.13 \ Let \( r \) be odd. Then \( \dim Y^{(r-k,k)} = \dim Y^{(r-1-k,k)} \).

Proof. Let \( r \) be odd, and let \( r - 2k \) have 2-adic decomposition given by \( r - 2k = \sum_{i \in \mathbb{N}_0} \alpha_i 2^i \), where each \( \alpha_i \in \{0,1\} \). Firstly, by Corollary 5.2.3, we have that

\[
\dim Y^{(r-k,k)} = \sum_{t=0}^{k} \rho_t \left( \binom{r}{t} - \binom{r}{t-1} \right) \\
\dim Y^{(r-k-1,k)} = \sum_{t=0}^{k} \tilde{\rho}_t \left( \binom{r-1}{t} - \binom{r-1}{t-1} \right),
\]

where, for \( t \in \{0,1, \ldots, k\} \),

\[
\rho_t = \begin{cases} 
1 & \text{if } \binom{r-k-t}{r-2k} \equiv 1 \mod 2, \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
\tilde{\rho}_t = \begin{cases} 
1 & \text{if } \binom{r-1-k-t}{r-1-2k} \equiv 1 \mod 2, \\
0 & \text{otherwise}.
\end{cases}
\]

As \( r - 2k = \sum_{i \in \mathbb{N}_0} \alpha_i 2^i \), let \( x \in \mathbb{N}_0 \) and \( v_1, \ldots, v_x \) be integers with \( 0 \leq v_1 < v_2 < \ldots < v_x \) such that \( \alpha_{v_1} = \ldots = \alpha_{v_x} = 1 \) and \( \alpha_i = 0 \) for all \( i \not\in \{v_1, \ldots, v_x\} \). As \( r - 2k \) is odd we must have that \( \alpha_0 = 1 \), therefore \( v_1 = 0 \).

For \( 0 \leq t \leq k \), let \( r - k - t = \sum_{i \in \mathbb{N}_0} \beta_i 2^i \), then by Lemma 1.3.2 we have that \( \rho_t = \prod_{i \in \mathbb{N}_0} \binom{\beta_i}{\alpha_i} \)). Therefore \( \rho_t = 1 \) if \( \binom{\beta_i}{\alpha_i} = 1 \) for all \( i \), and zero otherwise. Hence
\[ \rho_t = 1 \text{ if and only if } \beta_{v_1} = \ldots = \beta_{v_x} = 1. \] Now, by Lemma 1.3.1(1), for \(0 \leq t \leq k\) we have that \( \binom{r}{t} = \binom{r}{r-t} \) and \( \binom{r}{t-1} = \binom{r}{r-t+1} \), then by taking \( \beta = r - k - t \) we have that

\[
\dim Y^{(r-k,k)} = \sum_{0 \leq t \leq k} \rho_t \left( \binom{r}{r-t} - \binom{r}{r-t+1} \right)
= \sum_{\substack{r-2k \leq \beta \leq r-k \\ \beta_{v_0} = \beta_{v_1} = \ldots = \beta_{v_x} = 1}} \left( \binom{r}{k+\beta} - \binom{r}{k+\beta+1} \right)
= \sum_{\substack{r-2k \leq \beta \leq r-k \\ \beta_{v_0} = \beta_{v_1} = \ldots = \beta_{v_x} = 1}} \left( \binom{r-1}{k+\beta-1} - \binom{r-1}{k+\beta+1} \right),
\]

as \( \binom{r}{k+\beta} = \binom{r-1}{k+\beta} + \binom{r-1}{k+\beta+1} \) and \( \binom{r}{k+\beta+1} = \binom{r-1}{k+\beta+1} + \binom{r-1}{k+\beta} \).

Now, \( \binom{r}{k+\beta}, \ldots, \binom{r}{k+\beta+i} \) and \( \binom{r}{k+\beta-1}, \ldots, \binom{r}{k+\beta+i} \) have 2-adic decomposition \( r - k - t - 1 = \sum_{i \in \mathbb{N}_0} \gamma_i 2^i \), then \( \bar{\rho}_t = 1 \) if and only if \( \gamma_{v_2} = \ldots = \gamma_{v_x} = 1 \) and \( \alpha_i = 0 \) for all \( i \not\in \{v_2, \ldots, v_x\} \). For \( t \in \{0, 1, \ldots, k\} \) let \( r - k - t - 1 \) have 2-adic decomposition \( r - k - t - 1 = \sum_{i \in \mathbb{N}_0} \gamma_i 2^i \), then \( \bar{\rho}_t = 1 \) if and only if \( \gamma_{v_2} = \ldots = \gamma_{v_x} = 1 \). Now, by Lemma 1.3.1(1) we have \( \binom{r-1}{t} = \binom{r-1}{r-t} \) and \( \binom{r-1}{t} = \binom{r-1}{r-t} \), so by taking \( \gamma = r - k - t - 1 \),

\[
\dim Y^{(r-k-1,k)} = \sum_{0 \leq t \leq k} \bar{\rho}_t \left( \binom{r-1}{r-1-t} - \binom{r-1}{r-1-t} \right)
= \sum_{\substack{r-2k-1 \leq \gamma \leq r-k-1 \\ \gamma_{v_2} = \ldots = \gamma_{v_x} = 1}} \left( \binom{r-1}{\gamma + k} - \binom{r-1}{\gamma + k+1} \right)
= \sum_{\substack{r-2k-1 \leq \gamma \leq r-k-1 \\ \gamma_{v_2} = \ldots = \gamma_{v_x} = 1}} \left( \binom{r-1}{\gamma + k} - \binom{r-1}{\gamma + k+1} \right)
+ \sum_{\substack{r-2k-1 \leq \gamma \leq r-k-1 \\ \gamma_{v_2} = \ldots = \gamma_{v_x} = 1}} \left( \binom{r-1}{\gamma + k} - \binom{r-1}{\gamma + k+1} \right).
\]

Now, we start by considering,

\[
A := \sum_{\substack{r-2k-1 \leq \gamma \leq r-k-1 \\ \gamma_{v_2} = \ldots = \gamma_{v_x} = 1}} \left( \binom{r-1}{\gamma + k} - \binom{r-1}{\gamma + k+1} \right).
\]

Note that \( \gamma_{v_1} = \ldots = \gamma_{v_x} \) implies that \( \gamma \geq r - 2k = 2^{v_1} + \ldots + 2^{v_x} \). Also if \( \gamma = r - k \)
then \((r-1)_{\gamma+k} - (r-1)_{\gamma+k+1} = 0\), so we have that
\[
A = \sum_{r-2k \leq \gamma \leq r-k} \left( \left( \frac{r-1}{\gamma+k} \right) - \left( \frac{r-1}{\gamma+k+1} \right) \right) .
\]

Now let
\[
B := \sum_{r-2k-1 \leq \gamma \leq r-k-1} \left( \left( \frac{r-1}{\gamma+k} \right) - \left( \frac{r-1}{\gamma+k+1} \right) \right)
\]
by taking \(\delta = \gamma + 1\). Therefore we have that
\[
\dim Y^{(r-1-k,k)} = A + B = \sum_{r-2k \leq s \leq r-k} \left( \left( \frac{r-1}{s+k-1} \right) - \left( \frac{r-1}{s+k+1} \right) \right)
\]
\[= \dim Y^{(r-k,k)}, \]
as required. □

Therefore we only really need to consider the problem of \(r\) being even.

**Example 5.2.14** Let \(r = 11\) and \(k = 2\). Then \(r-2k = 7 \equiv -1 \mod 4\), so by Lemma 5.2.6 with \(y = 0\) we have that \(\dim Y^{(9,2)} = \binom{11}{2} - \binom{11}{1} = 55 - 11 = 44\). Now \(r-1 = 10\) and \((r-1) - 2k = 6 \equiv -2 \mod 4\), so by Lemma 5.2.6 with \(y = 1\) we have that \(\dim Y^{(8,2)} = \binom{10}{2} - \binom{10}{0} = 45 - 1 = 44\).

### 5.3 Finding a Basis of the Young Module \(Y^{(r-k,k)}\) in Some Specific Cases

As in the previous section, let \(K\) be a field of characteristic two, and let \(r\) and \(k\) be natural numbers with \(r \geq 2k\). As \(k\) is a natural number we can define \(x \in \mathbb{N}_0\) to be
such that $2^n \leq k < 2^{n+1}$. We know from Theorem 4.1.4 that $Y^{(r-k,k)} = \text{span}\{v_a : a \in J(r,k)\}$ where, for $a \in J(r,k)$, $v_a$ is as defined in Definition 4.1.2. We want to use this to find a basis for $M^{(r-k,k)}$, at least in some cases.

**Lemma 5.3.1** For $r - 2k \equiv 0 \mod 2^n+1$, $Y^{(r-k,k)}$ has $K$-basis $\{v_a : a \in J(r,k)\}$.

**Proof.** Let $r - 2k \equiv 0 \mod 2^n+1$. By Theorem 4.1.4, $Y^{(r-k,k)} = \text{span}\{v_a : a \in J(r,k)\}$ and by Lemma 5.2.1 $Y^{(r-k,k)}$ has dimension $\binom{r}{k}$. We know that $\text{Card}(J(r,k)) = \binom{r}{k}$, so $\{v_a : a \in J(r,k)\}$ has $\binom{r}{k}$ elements, and therefore must be a basis for $Y^{(r-k,k)}$. □

In the case where $r - 2k \equiv 0 \mod 2^n+1$ we have $M^{(r-k,k)} = Y^{(r-k,k)}$ and therefore $Y^{(r-k,k)}$ has $K$-basis $\{e_j : j \in J(r,k)\}$, but in Lemma 5.3.1 we construct a different basis. We now consider some other cases, firstly the case $r - 2k \equiv -2^y \mod 2^n+1$ for some integer $y$ with $0 < y < x$. In this case we need the following conjecture.

**Conjecture 5.3.2** Let $r - 2k \equiv -2^y \mod 2^n+1$ for some integer $y$ with $0 \leq y \leq x$. Let $l$ be an integer with $2^y \leq l \leq k$. Then, for $b \in J(r - 2l + 2^y, k - l)$ and $c \in J(r, r - l + 2^y)$ with $1, 2, \ldots, r - 2l + 2^y + 2 \subseteq c$ we have

$$\sum_{\substack{a \in J(r,k) \\ b \subseteq a \subseteq c}} v_a = 0.$$

Assuming that this conjecture is true, a basis for the Young module $Y^{(r-k,k)}$ in the case where $r - 2k \equiv -2^y \mod 2^n+1$ for some integer $y$ with $0 \leq y \leq x$ is given by the following proposition.

**Proposition 5.3.3** Assume that Conjecture 5.3.2 is true. If $r - 2k \equiv -2^y \mod 2^n+1$ for some integer $y$ with $0 \leq y \leq x$ then $Y^{(r-k,k)}$ has $K$-basis

$$\{v_a : a = (a_1, \ldots, a_k) \in J(r,k), a_i \neq r - 2k + 2^y - 1 + 2i \text{ for } 1 \leq i \leq k - (2^y - 1)\}.$$

Before we give an idea of how this could be proved, we consider an example.

**Example 5.3.4** Let $r = 14$ and $k = 4$ then $x = 2$ and $r - 2k = 6 \equiv -2 \mod 8$, so that $r - 2k \equiv -2^y \mod 2^n+1$ where $y = 1$. Now, $r - 2k + 2^y - 1 = 7$, and therefore by
Proposition 5.3.3 $Y^{(10,4)}$ has $K$-basis \( \{v_a : a = (a_1, a_2, a_3, a_4) \in J(14, 4), a_1 \neq 9, a_2 \neq 11, a_3 \neq 13 \} \).

We now sketch how Proposition 5.3.3 could be proved, assuming that Conjecture 5.3.2 holds. First we note that, by taking $l = k - i + 1$ in Proposition 5.3.3 we can rewrite the basis for $Y^{(r-k,k)}$ in this case as

\[ \{v_a : a = (a_1, \ldots, a_k) \in J(r, k), a_{k-l+1} \neq r + 2^y + 1 - 2l \text{ for } 2^y \leq l \leq k \}. \]

Let $r - 2k \equiv -2^y \pmod{2^{x+1}}$, and let $l$ be an integer with $2^y \leq l \leq k$. Fix $b = (b_1, \ldots, b_{k-l}) \in J(r - 2l + 2^y, k - l)$ and $c = (1, \ldots, r - 2l + 2^y + 1, r - 2l + 2^y + 2, c_{r-2l+2^y+3}, \ldots, c_{r-1+2^y}) \in J(r, r - l + 2^y)$. Using Conjecture 5.3.2 and the fact that $K$ has characteristic two, we have that

\[
\sum_{a \in J(r, k) \atop a_{k-l+1} = r - 2l + 2^y + 1} v_a = \sum_{a \in J(r, k) \atop b \subseteq a \subseteq c \atop a_{k-l+1} \neq r - 2l + 2^y + 1} v_a.
\]

Now, for $a \in J(r, k)$ with $a_{k-l+1} = r - 2l + 2^y + 1$ and $b \subseteq a \subseteq c$ we must have that $a = (b_1, \ldots, b_{k-l}, r - 2l + 2^y + 1, r - 2l + 2^y + 2, c_{r-2l+2^y+3}, \ldots, c_{r-1+2^y})$. Therefore we have that

\[
v(b_1, \ldots, b_{k-l}, r - 2l + 2^y + 1, r - 2l + 2^y + 2, c_{r-2l+2^y+3}, \ldots, c_{r-1+2^y}) = \sum_{a \in J(r, k) \atop b \subseteq a \subseteq c \atop a_{k-l+1} \neq r - 2l + 2^y + 1} v_a.
\]

This will allows us to remove the required vectors from that set $\{v_a : a \in J(r, k)\}$, which spans $Y^{(r-k,k)}$, to obtain a basis. To see how this works, we will consider the particular case of $y = 1$.

As $2^y = 2 \leq l \leq k$, first let $l = 2$. For $b = (b_1, \ldots, b_{k-2}) \in J(r - 2, k - 2)$ and $c = (1, 2, \ldots, r) \in J(r, r)$ we have that

\[
v(b_1, \ldots, b_{k-2}, r-1, r) = \sum_{a \in J(r, k) \atop b \subseteq a \subseteq c \atop a_{k-1} \neq r-1} v_a.
\]

Therefore any vector $v_a$ with $a \in J(r, k)$ and $a_{k-1} = r - 1$ can be written as a sum of vectors $v_a$ with $a_{k-1} \neq r - 1$. Hence $\{v_a : a \in J(r, k), a_{k-1} \neq r - 1\}$ spans
\( Y(r-k,k) \). Next consider \( l = 3 \), then for \( b = (b_1, \ldots, b_{k-3}) \in J(r-4,k-3) \) and \( c = (1,2,\ldots,r-2,c_{r-1}) \in J(r,r-1) \) we have
\[
v(b_1,\ldots,b_{k-3},r-3,r-2,c_{r-1}) = \sum_{a \in J(r,k) \atop b \subseteq a \subseteq c \atop a_{k-2} \neq r-3} v_a.
\]

Therefore we can write any element of \( \{v_a : a \in J(r,k), a_{k-1} \neq r-1, a_{k-2} = r-3\} \) as a sum of elements of the set \( \{v_a : a \in J(r,k), a_{k-1} \neq r-1, a_{k-2} \neq r-3\} \). Hence we have \( Y^{(r-k,k)} = \text{span}\{v_a : a \in J(r,k), a_{k-2} \neq r-3, a_{k-1} \neq r-1\} \). Continuing in the same way we obtain \( Y^{(r-k,k)} = \text{span}\{v_a : a \in J(r,k), a_{k-l+1} \neq r + 3 - 2l \text{ for } 2 \leq l \leq k\} \), and this set will actually give a basis for \( Y^{(r-k,k)} \) in this case. In general it will work in a similar way.

**Conjecture 5.3.5** Let \( r \) be even and suppose that \( Y^{(r-k,k)} \) has \( K \)-basis \( \{v_a : a \in J(r,k)\} \) for some set \( J(r,k) \subseteq J(r,k) \). Then \( J(r,k) \subseteq J(r+1,k) \) and \( Y^{(r+1-k,k)} \) has \( K \)-basis \( \{v_a : a \in J(r,k)\} \). (Note that the vectors \( v_a \) in each case are not actually the same as they depend on \( r \) - see the next example).

**Example 5.3.6** Let \( r = 4 \) and \( k = 2 \), then the Young module \( Y^{(2,2)} \) has \( K \)-basis \( \{v_a : a \in J(4,2)\} = \{v(1,2), v(1,3), v(1,4), v(2,3), v(2,4), v(3,4)\} \) where for example \( v(1,2) = e_i(3,4) \). By Conjecture 5.3.5 \( Y^{(3,2)} \) has \( K \)-basis \( \{v_a : a \in J(4,2)\} = \{v(1,2), v(1,3), v(1,4), v(2,3), v(2,4), v(3,4)\} \), where here for example \( v(1,2) = e_i(3,4) + e_i(3,5) + e_i(4,5) \).

Note that, from Example 4.2.3, we know that \( \{v(1,2), v(1,3), v(1,4), v(2,3), v(2,4), v(3,4)\} \) is a basis for \( Y^{(3,2)} \).

We will now consider the case where \( r - 2k \equiv 2 \mod 2^{l+1} \). In this case the basis for \( Y^{(r-k,k)} \) is given by the following lemma.

**Lemma 5.3.7** For \( l \) an integer with \( 1 \leq l \leq k - 1 \), define
\[
X_l := \{a = (a_1, a_2, \ldots, a_k) \in J(r,k) : a_{k-l} = r - (2l - 1), \ a_{k-(l-1)} = r - (2l - 2), \ a_{k-(l-m)} = r - (2l - 2m + 1) \text{ or } r - (2l - 2m) \text{ for } m = 2,3,\ldots,l\}.
\]
Now define

\[ X := \bigcup_{i=1}^{k-1} X_i, \]

where the union is disjoint. Then for \( r - 2k \equiv 2 \mod 2^{x+1} \), \( Y^{(r,k)} \) has basis

\[ \{v_a : a \in J(r,k) \setminus X \}. \]

Before we show how this was obtained, we give an example.

**Example 5.3.8** Let \( r = 10 \) and \( k = 4 \), then \( x = 2 \) and \( r - 2k \equiv 2 \mod 2^3 \). For \( 1 \leq l \leq 3 \) we define \( X_i \) as in Lemma 5.3.7. Then we have

\[
\begin{align*}
X_1 &= \{a \in J(r, k) : a_3 = r - 1, a_4 = r \}, \\
X_2 &= \{a \in J(r, k) : a_2 = r - 3, a_3 = r - 2, a_4 = r - 1 \text{ or } r \}, \\
X_3 &= \{a \in J(r, k) : a_1 = r - 5, a_2 = r - 4, a_3 = r - 3 \text{ or } r - 2, a_4 = r - 1 \text{ or } r \}.
\end{align*}
\]

So

\[
\begin{align*}
X_1 &= \{(a_1, a_2, 9, 10) : 1 \leq a_1 < a_2 \leq 8\}, \\
X_2 &= \{(a_1, 7, 8, 9), (a_1, 7, 8, 10) : 1 \leq a_1 \leq 6\}, \\
X_3 &= \{(5, 6, 7, 9), (5, 6, 7, 10), (5, 6, 8, 9), (5, 6, 8, 10)\}.
\end{align*}
\]

Now by Lemma 5.3.7 for \( r - 2k \equiv 2 \mod 8 \), \( Y^{(6,4)} \) has basis \( \{v_a : a \in J(10, 4) \setminus X \} \), where \( X = X_1 \cup X_2 \cup X_3 \).

Note that, by Lemma 5.2.9, \( \dim Y^{(6,4)} = \binom{10}{4} - \binom{10}{2} + \binom{10}{0} = 166 \). We have \( \text{Card}(X) = 28 + 12 + 4 = 44 \), so \( \text{Card}(J(10, 4) \setminus X) = \binom{10}{4} - 44 = 210 - 44 = 166 \), so we have the correct amount of elements for a basis of \( Y^{(6,4)} \).

**Proposition 5.3.9** Let \( r - 2k \equiv 2 \mod 4 \). Then for \( b \in J(r, k - 2) \) we have that

\[
\sum_{a \in J(r,k) \atop b \leq a} v_a = 0.
\]
Proof. First let \( b = (b_1, b_2, \ldots, b_{k-2}) \in J(r, k - 2) \). Then
\[
\sum_{a \in J(r,k)} v_a = \sum_{a \in J(r,k)} \sum_{j \in J(r,k)} e_{ij} = \sum_{a \in J(r,k)} \sum_{j \in J(r,k)} e_{ij}, \tag{5.7}
\]
using Lemma 4.1.1. Now, for \( j = (j_1, j_2, \ldots, j_k) \in J(r, k) \), \( e_{ij} \) appears in the right hand side of (5.7) for all \( a \in J(r, k) \) with \( b = (b_1, b_2, \ldots, b_{k-2}) \subseteq a \) and \( a \subseteq j' = (j_1', \ldots, j_{r-k}') \). So for \( e_{ij} \) to occur, we need \( b \subseteq j' \) which means that \( j \subseteq b' \) (by Lemma 4.1.1).

If \( j \subseteq b' \), then \( j' \subseteq b \), therefore \( j' \in J(r, r - k) \) contains \( b_1, b_2, \ldots, b_{k-2} \) and \( r - 2k + 2 \) other values between 1 and \( r \). In this case \( e_{ij} \) appears in Equation (5.7) for all \( a \in J(r, k) \) with \( b \subseteq a \subseteq j' \), which is all \( a \) containing \( b_1, b_2, \ldots, b_{k-2} \) and 2 other values from \( j' \). Therefore \( e_{ij} \) occurs \( \binom{r-2k+2}{2} \) times in Equation (5.7). Now, as the characteristic of the field \( K \) is two, the coefficient of \( e_{ij} \) in (5.7) is 1 if and only if \( \binom{r-2k+2}{2} \) is odd.

As \( r - 2k \equiv 2 \mod 4 \), there is an integer \( y \) such that \( r - 2k = 2 + 4y \), then we have
\[
\binom{r - 2k + 2}{2} = \binom{4y + 4}{2} = \frac{(4y + 4)(4y + 3)}{2} = (2y + 2)(4y + 3),
\]
which is even. Therefore the coefficient of \( e_{ij} \) in (5.7) is zero. If \( j \not\subseteq b' \) then \( e_{ij} \) does not occur in Equation (5.7), so for all \( j \) the coefficient of \( e_{ij} \) in (5.7) is zero. Therefore we have,
\[
\sum_{a \in J(r,k)} v_a = 0.
\]
\( \square \)

We'll now explain, using this proposition, how we obtained the basis for \( Y^{(r-k,k)} \) in the case where \( r - 2k \equiv 2 \mod 2^{x+1} \). We start with \( Y^{(r-k,k)} = \text{span}\{v_a : a \in J(r, k)\} \), by Theorem 4.1.4, and we want to delete some vectors from this set to give a basis for \( Y^{(r-k,k)} \). By Proposition 5.3.9 we have that
\[
\sum_{a \in J(r,k)} v_a = 0 \tag{5.8}
\]
for all $b = (b_1, b_2, \ldots, b_{k-2}) \in J(r, k - 2)$. Splitting this sum up, and using the fact that the characteristic of the field $K$ is two, we have that

$$
\sum_{a=(a_1,\ldots,a_k) \in J(r,k) \atop b \subseteq a, a_{k-1} = r-1} v_a = \sum_{a=(a_1,\ldots,a_k) \in J(r,k) \atop b \subseteq a, a_{k-1} \neq r-1} v_a.
$$

Hence for $b_{k-2} \leq r - 2$ we have

$$
v_{b_1,b_2,\ldots,b_{k-2},r-1,r} = \sum_{a=(a_1,\ldots,a_k) \in J(r,k) \atop b \subseteq a, a_{k-1} \neq r-1} v_a. \tag{5.9}
$$

So we can delete all $v_a$ with $a_{k-1} = r - 1$ and $a_k = r$ from the original set (i.e. $\{v_a : a \in J(r,k)\}$) and still get a spanning set for $Y^{(r-k,k)}$. So if $X_1 = \{a = (a_1,\ldots,a_k) \in J(r,k) : a_{k-1} = r - 1$ and $a_k = r\}$, then we are left with $Y^{(r-k,k)} = \text{span}\{v_a : a \in J(r,k) \setminus X_1\}$.

Now for $b_{k-2} > r - 2$ (so $b_{k-2} = r - 1$ or $r$) and $b_{k-3} < r - 1$, we have that

$$
\sum_{a \in J(r,k) \atop b \subseteq a, a_{k-1} = r-1} v_a = \sum_{c_1=1,\ldots,r-2} \left( \sum_{a \in J(r,k) \atop c_1 \neq b_1,b_2,\ldots,b_{k-3}} v_a \right) = \sum_{c_1=1,\ldots,r-2} \left( \sum_{a \in J(r,k) \atop c_1 \neq b_1,b_2,\ldots,b_{k-3},c_1} v_a \right),
$$

by (5.9). Now from (5.8) we have

$$
0 = \sum_{a \in J(r,k) \atop (b_1,\ldots,b_{k-2}) \subseteq a, a_{k-1} \neq r-1} v_a + \sum_{c_1=1,\ldots,r-2} \left( \sum_{a \in J(r,k) \atop c_1 \neq b_1,b_2,\ldots,b_{k-3},c_1} v_a \right)
\quad = \sum_{c_1=1,\ldots,r-2,b_{k-2}} \left( \sum_{a \in J(r,k) \atop c_1 \neq b_1,b_2,\ldots,b_{k-3},c_1} v_a \right). \tag{5.10}
$$

Now for $b_{k-3} \leq r - 4$ we have that
\[\sum_{c_1=1,2,\ldots,r-2,b_k-2 \atop c_1 \neq b_1,b_2,\ldots,b_k-3} \left( \sum_{a \in J(r,k) \atop b_1,\ldots,b_{k-3},c_1 \subseteq a} v_a \right) = \sum_{c_1=1,2,\ldots,r-2,b_k-2 \atop c_1 \neq b_1,b_2,\ldots,b_k-3} \left( \sum_{a \in J(r,k) \atop a_{k-1} \neq r-1 \atop a_{k-2} = r-3} v_a \right)\]

\[= \sum_{c_1=r-3, r-2,b_k-2 \atop c_1 \neq b_1,b_2,\ldots,b_k-3} \sum_{a \in J(r,k) \atop b_1,\ldots,b_{k-3},c_1 \subseteq a} v_a = \sum_{a \in J(r,k) \atop b_1,\ldots,b_{k-3},c_1 \subseteq a} v_a. \]

So we have that

\[v_{b_1,b_2,\ldots,b_k-3,r-3,r-2,b_k-2} = \sum_{c_1=1,2,\ldots,r-2,b_k-2 \atop c_1 \neq b_1,b_2,\ldots,b_k-3} \left( \sum_{a \in J(r,k) \atop b_1,\ldots,b_{k-3},c_1 \subseteq a} v_a \right). \quad (5.11)\]

So now, let \(X_2 = \{(a_1,\ldots,a_{k-3},r-3,r-2,a_k) \in J(r,k)\}\). Then \(Y^{(r-k)} = \text{span}\{v_a : a \in J(r,k) \setminus (X_1 \cup X_2)\}\).

Now let \(b_{k-3} = r-3 \text{ or } r-2 \) and \(b_{k-4} < r-3\) (recall that we still have \(b_{k-2} = r-1 \text{ or } r\)). Then it can be shown that

\[\sum_{c_1=1,2,\ldots,r-2,b_k-2 \atop c_1 \neq b_1,b_2,\ldots,b_k-3} \left( \sum_{a \in J(r,k) \atop b_1,\ldots,b_{k-3},c_1 \subseteq a} v_a \right) = \sum_{a \in J(r,k) \atop b_1,\ldots,b_{k-3},c_1 \subseteq a} v_a \]

\[= \sum_{c_2=1,2,\ldots,r-4 \atop c_2 \neq b_1,b_2,\ldots,b_{k-4}} \sum_{a \in J(r,k) \atop b_1,\ldots,b_{k-4},c_2 \subseteq a} v_a = \sum_{a \in J(r,k) \atop a_{k-1} \neq r-1 \atop a_{k-2} = r-3} v_a, \]

\[= \sum_{c_2=1,2,\ldots,r-4 \atop c_2 \neq b_1,b_2,\ldots,b_{k-4}} \sum_{a \in J(r,k) \atop b_1,\ldots,b_{k-4},c_2 \subseteq a} v_a, \]

\[= \sum_{c_1=1,2,\ldots,r-2,b_k-2 \atop c_1 \neq b_1,b_2,\ldots,b_k-4} \left( \sum_{a \in J(r,k) \atop b_1,\ldots,b_{k-4},c_1 \subseteq a} v_a \right).\]
by (5.11). Hence we have (from (5.10)) that

\[
0 = \sum_{c_1=1,2,\ldots,r-2,b_{k-2}} \left( \sum_{a \in J(r,k)} (\sum_{a \in J(r,k)} v_a) \right)
\]

\[
+ \sum_{c_2=1,2,\ldots,r-4} \left( \sum_{c_1=1,2,\ldots,r-2,b_{k-2}} (\sum_{a \in J(r,k)} v_a) \right)
\]

\[
= \sum_{c_2=1,2,\ldots,r-4} \left( \sum_{c_1=1,2,\ldots,r-2,b_{k-2}} \left( \sum_{a \in J(r,k)} v_a \right) \right)
\]

Now for \( b_{k-4} \leq r - 6 \) we have

\[
\sum_{c_2=r-5,r-4,b_{k-3}} \left( \sum_{c_1=r-5,r-4,r-3,b_{k-2}} \left( \sum_{a \in J(r,k)} v_a \right) \right)
\]

\[
= \sum_{c_2=r-5,r-4,b_{k-3}} \left( \sum_{c_1=r-5,r-4,r-3,b_{k-2}} \left( \sum_{a \in J(r,k)} v_a \right) \right)
\]

\[
= \sum_{c_2=r-5,r-4,b_{k-3}} \left( \sum_{c_1=r-5,r-4,r-3,b_{k-2}} \left( \sum_{a \in J(r,k)} v_a \right) \right)
\]

\[
\sum_{b_{k-4}, b_{k-3}, b_{k-2}} v_{b_1,b_2,\ldots,b_{k-4},r-5,r-4,r-3,r-2} + v_{b_1,b_2,\ldots,b_{k-4},r-5,r-4,b_{k-3},b_{k-2}}
\]

Therefore,

\[
\sum_{b_{k-4}, b_{k-3}, b_{k-2}} v_{b_1,b_2,\ldots,b_{k-4},r-5,r-4,b_{k-3},b_{k-2}}
\]

\[
= \sum_{c_2=r-5,r-4,b_{k-3}} \left( \sum_{c_1=r-5,r-4,r-3,b_{k-2}} \left( \sum_{a \in J(r,k)} v_a \right) \right)
\]

\[
+ v_{b_1,b_2,\ldots,b_{k-4},r-5,r-4,r-3,r-2}
\]
Now let \( X_3 = \{(a_1, \ldots, a_{k-4}, r-5, r-4, a_{k-1}, a_k) \in J(r, k) : a_{k-1} = b_{k-3} = r - 3 \text{ or } r - 2, \ a_k = b_{k-2} = r - 1 \text{ or } r\} \). Then we are left with \( Y^{(r-k,k)} = \text{span}\{v_a : a \in J(r, k) \setminus (X_1 \cup X_2 \cup X_3)\} \).

In the same way as before, we can show that the following sum is zero
\[
\sum_{c_3=1,2,\ldots,r-6,b_{k-4}}^{c_3 \neq b_1, b_2, \ldots, b_{k-5}} \left( \sum_{c_2=1,2,\ldots,r-4,b_{k-3}}^{c_2 \neq b_1, b_2, \ldots, b_{k-4}, c_3} \left( \sum_{c_1=1,2,\ldots,r-2,b_{k-2}}^{c_1 \neq b_1, b_2, \ldots, b_{k-4}, c_3, c_2, c_3} \left( \sum_{(b_1, \ldots, b_{k-4}, c_3, c_2, c_1) \subseteq a}^{(b_1, \ldots, b_{k-4}, c_3, c_2, c_1) \neq \emptyset} \sum_{a \in J(r, k)}^{a_{k-1} \neq r-1} \sum_{a_{k-2} \neq r-3} \sum_{v_a} \right) \right) \right) \right) (5.14)
\]

We can then continue in the same way, so we are able to delete all vectors in \( X \). Here each \( X_l \), for \( 1 \leq l \leq k - 1 \) has \( 2^{l-1} \binom{r-2l}{k-l-1} \) elements, so the number of vectors we have deleted is the number of elements in \( X = \bigcup_{l=1}^{k-1} X_l \), which is \( \sum_{l=1}^{k-1} 2^{l-1} \binom{r-2l}{k-l-1} \).

By Corollary 5.2.12 we have,
\[
\dim Y^{(r-k,k)} = \binom{r}{k} - \sum_{i=1}^{k-1} 2^{i-1} \binom{r-2i}{k-i-1},
\]
which is the number of vectors we are left with after deleting all those in \( X \). Therefore, we are left with a basis for \( Y^{(r-k,k)} \).

We will now summarize our results for finding a basis of the Young module \( Y^{(r-k,k)} \) in the case where \( K \) has characteristic two. First recall that, for \( r \geq 2k \), by Theorem 4.1.4
\[
Y^{(r-k,k)} = \text{span}\{v_a : a \in J(r, k)\}
\]
where the vector \( v_a \) is defined in Definition 4.1.2. If \( k = 0 \) then \( Y^{(r,0)} \) is the trivial module, which is one dimensional with basis vector \( v() = e() = e_{(1,1,\ldots,1)} \).

If \( k = 1 \) then \( Y^{(r-1,1)} \) has \( K \)-basis
\[
\{v_a : a \in J(r, 1)\} \text{ if } r \text{ is even},
\{v_a : a \in J(r - 1, 1)\} = \{v_a : a = (a_1) \in J(r, 1), a_1 \neq r\} \text{ if } r \text{ is odd}.
\]

In the following table, we consider the module \( Y^{(r-k,k)} \) for \( k = 2 \) and \( k = 3 \). The module \( Y^{(r-k,k)} \) has basis consisting of all vectors \( v_a \) with \( a = (a_1, \ldots, a_k) \in J(r, k) \) subject to to conditions given in the table:
Note that the entry "−" in the table means no extra conditions, so for example the module $Y'(r-2,2)$ with $r - 2k \equiv 0 \mod 4$ has basis given by $\{v_a : a \in J(r,2)\}$ (see Lemma 5.3.1). The other rows of the table are filled using Conjecture 5.3.5 for the case $r - 2k \equiv 1 \mod 4$ and Proposition 5.3.3 with $y = 1$ and $y = 0$ for the cases $r - 2k \equiv 2 \mod 4$ and $r - 2k \equiv 3 \mod 4$ respectively.

In the following table we consider $k = 4, 5, 6, 7$. Note that we only need to consider the case where $r$ is even, then we can use Conjecture 5.3.5 to get the results for $r$ odd. Here, in the case where $r - 2k \equiv 2 \mod 8$, see Lemma 5.3.7 which defines a set $X$, and in all other cases the basis is given by all $v_a$ with $a = (a_1, \ldots, a_k) \in J(r, k)$ subject to the conditions in the table below. The first row follows from Lemma 5.3.1, the third from Proposition 5.3.3 with $y = 2$ and the fourth from Proposition 5.3.3 with $y = 1$.
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