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Fatma Al-Sirehy
Department of Mathematics and Computer Science
University of Leicester
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DEDICATION

To my husband and my children
ABSTRACT

In 1950, Laurent Schwartz marked a convenient starting point for the theory of generalized functions as a subject in its own right. He developed and unified much of the earlier work by Hadamard, Bochner, Sobolev and others. Since then an enormous literature dealing with both theory and applications has grown up, and the subject has undergone extensive further development. The original Schwartz treatment defined a distribution as a linear continuous functional on a space of test functions.

This thesis can be considered a part of the development going in that direction. It is partly an extension of earlier contributions by Fisher, Kuribayashi, Itano and others.

After introducing the background and basic definitions in Chapter One, we developed some basic results concerning the cosine integral \( \mathrm{Ci}(\lambda x) \) and its associated functions \( \mathrm{Ci}_+(\lambda x) \) and \( \mathrm{Ci}_-(\lambda x) \) as well as the neutrix convolution products of the cosine integral.

Chapter Three is devoted to similar results concerning the sine integral \( \mathrm{Si}(\lambda x) \).

In Chapter Four, we generalize some earlier results by Fisher and Kuribayashi concerning the product of the two distributions \( x^\lambda_+ \) and \( x^{\lambda-r}_+ \). Moreover, other results are obtained concerning the neutrix product of \( |x|^{\lambda-r} \ln^p |x| \) and \( \text{sgn} x |x|^{\lambda-r} \ln^q |x| \). Other theorems are proved about the neutrix product of some other distributions such as \( x^\lambda_+ \ln x_+ \) and \( x^{\lambda-r}_- \).

Chapter Five contains new results about the composition of distributions. It involves the application of the neutrix limit to establish such relationships between different distributions.
Chapter 1

INTRODUCTION

The first to use generalized functions in the explicit and presently accepted form was S. L. Sobolev in 1936 in studying the uniqueness of solutions of the Cauchy problem for linear hyperbolic equations.

In 1950-1951, Laurent Schwartz's published a monograph entitled "Theories des Distributions". In this book Schwartz systematized the theory of generalized functions, basing it on the theory of linear topological spaces, related all the earlier approaches, and obtained many important and far reaching results. Unusually soon after the appearance of "Theories des Distributions", in fact literally within two or three years, generalized functions attained an extremely wide popularity. Since then an enormous literature dealing with both theory and applications has grown up, and the subject has undergone extensive further development. Some mathematicians believe that distribution theory was one of the greatest revolutions in mathematical analysis in the 20th century. They think of it as the completion of differential calculus, just as the other great revolution, measure theory (or Lebesgue integration theory), can be thought of as the completion of integral calculus. It is sufficient just to point out the great increase in the number of mathematical works containing the delta function.
The Dirac delta function $\delta(x)$ has a long history. Its first appearance seems to have been in Fourier's *Theorie Analytique de la Chaleur*, (1822). Kirchoff [29] later defined $\delta(x)$ by

$$\delta(x) = \lim_{\mu \to \infty} \pi^{-1/2}\mu \exp(-\mu^2 x^2).$$

Clearly $\delta(x) = 0$ for $x \neq 0$ and $\delta(0) = \infty$. He also defined

$$\int_{-\infty}^{x} \delta(t) \, dt = \lim_{\mu \to \infty} \pi^{-1/2}\mu \int_{-\infty}^{x} \exp(-\mu^2 t^2) \, dt,$$

which implies that

$$\int_{-\infty}^{x} \delta(t) \, dt = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

In this sense, it is apparent that the delta function is not a function in the normal sense.

Heaviside's function $H(x)$ is defined by a locally summable function where

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Considering this function, Heaviside concluded that the derivative of $H$ is equal in some sense to $\delta$.

Later Dirac treated the delta function as the function which is everywhere equal to zero except at the origin where it is infinite, in such a sense that it satisfies

$$\int_{-\infty}^{\infty} \delta(x) \, dx = 1.$$ 

Dirac thought of $\delta$ as a unit point charge at the origin. Moreover, he thought of $\delta'$ the derivative of $\delta$ as a dipole of unit electric moment at the origin since

$$\int_{-\infty}^{\infty} x \delta'(x) \, dx = \lim_{\mu \to \infty} \pi^{-1/2}\mu \int_{-\infty}^{\infty} x \exp(-\mu^2 x^2) \, dx = -1.$$ 

Similarly, higher derivatives of $\delta$ can be used to represent more complicated multiple-layers and have been used in the physical and engineering science for some time.
1.1 DISTRIBUTIONS AND THE NEUTRIX CALCULUS

In this section, we give an overview of the concepts of distributions and neutrix calculus as used in evaluating some forms of improper integrals.

1.1.1 DISTRIBUTIONS

DEFINITION 1.1 Let \( f \) be a real (or complex) valued function defined on the real line. Then \( \text{supp} \ f \), the support of \( f \), is the closure of the set on which \( f(x) \neq 0 \).

DEFINITION 1.2 Let \( \varphi \) be an infinitely differentiable real valued function with compact support. Then \( \varphi \) is said to be a test function. The set of all test functions with the usual definition of sum and product by a scalar is a vector space and is denoted by \( \mathcal{D} \).

Note that if \( \varphi \in \mathcal{D} \) and \( g \) is any infinitely differentiable function, then \( g\varphi \in \mathcal{D} \).

DEFINITION 1.3 Let \( \{\varphi_n\} \) be a sequence of test functions in \( \mathcal{D} \). Then the sequence \( \{\varphi_n\} \) is said to converge to zero if there exists a bounded interval \([a,b]\) with \( \text{supp} \ \varphi_n \subseteq [a,b] \) for all \( n \) and \( \lim_{n \to \infty} \varphi_n^{(r)}(x) = 0 \) for all \( x \) and \( r = 0,1,2,\ldots \). If \( f \) is a linear functional on \( \mathcal{D} \) into the real (or complex) numbers, we denote its value at \( \varphi \in \mathcal{D} \) by \( \langle f, \varphi \rangle \).

DEFINITION 1.4 Let \( f \) be a linear functional on \( \mathcal{D} \). Then \( f \) is said to be continuous if \( \lim_{n \to \infty} \langle f, \varphi_n \rangle = 0 \) whenever \( \{\varphi_n\} \) is a sequence in \( \mathcal{D} \) converging to zero. A continuous linear functional on \( \mathcal{D} \) is said to be a distribution (generalized function). The set of all distributions is a vector space and is denoted by \( \mathcal{D}' \).
The famous mathematician Laurent Schwartz, during the period 1945 - 1950, did an extensive work on distributions. Schwartz gave this name after his research on generalizing the idea of electric density so that it could be applicable to these generalized functions of electricity.

Every locally summable function defines a functional. In fact if $f$ is a locally summable function then we can define a linear functional, which we will also denote by $f$ by putting

$$
\langle f(x), \varphi(x) \rangle = \int_a^b f(x)\varphi(x) \, dx
$$

(1.1)

if $\text{supp } \varphi \subseteq [a, b]$. Further, the functional $f$ is continuous and so a distribution because if $\{\varphi_n\}$ converges to zero, then for each $\epsilon > 0$ there exists an $N$ such that $|\varphi_n| < \epsilon$ for every $n > N$ and so

$$
|\langle f(x), \varphi_n(x) \rangle| \leq \epsilon \int_a^b |f(x)| \, dx
$$

if $\text{supp } \varphi_n \subseteq [a, b]$ for all $n$.

Distributions that are defined by equation (1.1) from locally summable functions are called regular distributions. All distributions that are not regular are called singular.

An example of a distribution that is not regular is the Dirac-delta function $\delta$ defined on $\mathcal{D}$ by

$$
\langle \delta(x), \varphi(x) \rangle = \varphi(0).
$$

(1.2)

This distribution can not be defined by a locally summable function. By changing the origin in equation (1.2), we get

$$
\langle \delta(x - a), \varphi(x) \rangle = \varphi(a),
$$

where $a$ is any real number.
DEFINITION 1.5 The distribution $f$ is said to vanish in the open neighbourhood $U$ of a point $x_0$ if $\langle f(x), \varphi(x) \rangle = 0$ for every $\varphi$ which has its support in $U$. The support of $f$ is the smallest closed set of points outside of which $f$ vanishes. For example $\delta(x-x_0)$ has the point $x = x_0$ as its support.

In order to define the derivative of the distribution, we first of all consider a continuous function $f$ which is differentiable everywhere and whose derivative is continuous. Its derivative $f'$ will define a bounded linear functional and

$$\langle f'(x), \varphi(x) \rangle = \int_{-\infty}^{\infty} f'(x) \varphi(x) \, dx = [f(x) \varphi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \varphi'(x) \, dx = -\langle f(x), \varphi'(x) \rangle$$

for all $\varphi \in \mathcal{D}$. This suggests that we define the derivative $f'$ of a distribution $f$ by the equation

$$\langle f'(x), \varphi(x) \rangle = -\langle f(x), \varphi'(x) \rangle.$$ 

It is easily seen that $f'$ is a distribution, since if the sequence $\{\varphi_n\}$ converges to zero, the sequence $\{\varphi'_n\}$ also converges to zero and so

$$\lim_{n \to \infty} \langle f'(x), \varphi_n(x) \rangle = -\lim_{n \to \infty} \langle f(x), \varphi'_n(x) \rangle = 0.$$ 

In general

$$\langle f^{(n)}(x), \varphi(x) \rangle = (-1)^n \langle f(x), \varphi^{(n)}(x) \rangle$$

and $f^{(n)}$ is a distribution for $n = 1, 2, \ldots$.

EXAMPLE 1.1 Let $x_+$ be the locally summable function defined by

$$x_+ = \begin{cases} x, & x > 0, \\ 0, & x \leq 0. \end{cases}$$
Then its derivative is Heaviside’s function $H$, since

$$-\langle x_+, \varphi'(x) \rangle = -\int_0^\infty x \varphi'(x) \, dx = \int_0^\infty \varphi(x) \, dx = \langle H(x), \varphi(x) \rangle.$$ 

The derivative $H'$ of $H$ is given by

$$\langle H'(x), \varphi(x) \rangle = -\langle H(x), \varphi'(x) \rangle = -\int_0^\infty \varphi'(x) \, dx = \varphi(0)$$

and so $H' = \delta$. In general the $r$-th derivative $\delta^{(r)}$ of $\delta$ is given by

$$\langle \delta^{(r)}(x), \varphi(x) \rangle = (-1)^r \langle \delta(x), \varphi^{(r)}(x) \rangle = (-1)^r \varphi^{(r)}(0).$$

**EXAMPLE 1.2** Let $x_+^\lambda$ ($\lambda > -1$) be the locally summable function defined by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x < 0. \end{cases}$$

If $\lambda > 0$, its derivative is the locally summable function $\lambda x_+^{\lambda-1}$ but if $-1 < \lambda < 0$, $x_+^{\lambda-1}$ is not a locally summable function. If $-1 < \lambda < 0$ we will still denote the derivative of $x_+^\lambda$ by $\lambda x_+^{\lambda-1}$ but it must be defined by

$$\langle (x_+^\lambda)', \varphi(x) \rangle = -\langle x_+^\lambda, \varphi'(x) \rangle = -\int_0^\infty x^\lambda d[\varphi(x) - \varphi(0)]$$

$$= \lambda \int_0^\infty x^{\lambda-1}[\varphi(x) - \varphi(0)] \, dx.$$

Thus if $-2 < \lambda < -1$, we have defined $x_+^\lambda$ by

$$\langle x_+^\lambda, \varphi(x) \rangle = \int_0^\infty x^\lambda [\varphi(x) - \varphi(0)] \, dx.$$ 

In general, it can be proved by induction that if $-r - 1 < \lambda < -r$, then

$$\langle x_+^\lambda, \varphi(x) \rangle = \int_0^\infty x^\lambda \left[ \varphi(x) - \sum_{i=0}^{r-1} \frac{x^i}{i!} \varphi^{(i)}(0) \right] \, dx.$$
EXAMPLE 1.3 The locally summable function $\ln x_+$ is defined by

$$\ln x_+ = \begin{cases} \ln x, & x > 0, \\ 0, & x < 0. \end{cases}$$

We define the distribution $x_+^{-1}$ to be the derivative of $\ln x_+$. Thus

$$\langle x_+^{-1}, \varphi(x) \rangle = -\langle (\ln x_+)', \varphi(x) \rangle = \langle (\ln x_+), \varphi'(x) \rangle$$

$$= -\int_0^1 \ln x [\varphi(x) - \varphi(0)] - \int_1^\infty \ln x \varphi(x) dx$$

$$= \int_0^1 x^{-1} [\varphi(x) - \varphi(0)] dx + \int_1^\infty x^{-1} \varphi(x) dx$$

$$= \int_0^\infty x^{-1} [\varphi(x) - \varphi(0)H(1-x)] dx.$$

More generally, we define the distribution $x_+^{-r}$ inductively by

$$x_+^{-r} = (-r + 1)^{-1}(x_+^{-r+1})'$$

for $r = 2, 3, \ldots$, see [12] and not as defined in Gel'fand and Shilov [25]. Denoting Gelfand and Shilov's definition of $x_+^{-r}$ by $F(x_+,-r)$, it follows that

$$x_+^{-r} = F(x_+,-r) + \frac{(-1)^r}{(r-1)!} \phi(r-1)\delta^{(r-1)}(x)$$

for $r = 2, 3, \ldots$, where

$$\phi(r) = \begin{cases} \sum_{i=1}^r i^{-1}, & r = 1, 2, \ldots, \\ 0, & r = 0. \end{cases}$$

It can be proved by induction that

$$\langle x_+^{-r}, \varphi(x) \rangle = \int_0^\infty x^{-r} [\varphi(x) - \sum_{i=0}^{r-2} \frac{x^i}{i!} \varphi^{(i)}(0) - \frac{x^{r-1}}{(r-1)!} \varphi^{(r-1)}(0)H(1-x)] dx$$

$$- \frac{1}{(r-1)!} \phi(r-1)\varphi^{(r-1)}(0)$$

for $r = 2, 3, \ldots$, see Fisher [12].
Now let $f$ be a locally summable function. Then

$$
\langle f(-x), \varphi(x) \rangle = \int_{-\infty}^{\infty} f(-x) \varphi(x) \, dx = \int_{-\infty}^{\infty} f(x) \varphi(-x) \, dx = \langle f(x), \varphi(-x) \rangle.
$$

This suggests the following definition:

**DEFINITION 1.6** Let $f$ be an arbitrary distribution. Then $f(-x)$ is the distribution defined by

$$
\langle f(-x), \varphi(x) \rangle = \langle f(x), \varphi(-x) \rangle.
$$

**EXAMPLE 1.4** The distribution $x^\lambda$ is defined by $x^\lambda = (-x)^\lambda$ so that

$$
\langle x^\lambda, \varphi(x) \rangle = \langle x^\lambda, \varphi(-x) \rangle.
$$

Thus

$$
\langle x^\lambda, \varphi(x) \rangle = \int_0^\infty x^\lambda \varphi(-x) \, dx
$$

for $\lambda > -1$, and

$$
\langle x^\lambda, \varphi(x) \rangle = \int_0^\infty x^\lambda \left[ \varphi(-x) - \sum_{i=0}^{r-1} \frac{(-x)^i}{i!} \varphi^{(i)}(0) \right] \, dx
$$

for $-r-1 < \lambda < -r$,

$$
\langle x^{-1}, \varphi(x) \rangle = \int_0^\infty x^{-1} \left[ \varphi(-x) - \varphi(0)H(1-x) \right] \, dx,
$$

$$
\langle x^{-r}, \varphi(x) \rangle =
\int_0^\infty x^{-r} \left[ \varphi(-x) - \sum_{i=0}^{r-1} \frac{(-x)^i}{i!} \varphi^{(i)}(0) - \frac{(-x)^{r-1}}{(r-1)!} \varphi^{(r-1)}(0)H(1-x) \right] \, dx + \frac{(-1)^r}{(r-1)!} \phi(r-1)\varphi^{(r-1)}(0)
$$

for $r = 2, 3, \ldots$. 
The distribution $|x|^\lambda$ is defined by

$$|x|^\lambda = x_+^\lambda + x_-^\lambda,$$

the distribution $\text{sgn} x|x|^\lambda$ is defined by

$$\text{sgn} x|x|^\lambda = x_+^\lambda - x_-^\lambda$$

and the distribution $x^r$ is defined by

$$x^r = x_+^r + (-1)^r x_-^r$$

for $r = 0, \pm 1, \pm 2, \ldots$. In particular

$$\langle x^{-2r}, \varphi(x) \rangle = \int_0^\infty x^{-2r} \left[ \varphi(x) + \varphi(-x) - 2 \sum_{i=0}^{r-1} \frac{x^{2i}}{(2i)!} \varphi^{(2i)}(0) \right] dx,$$

$$\langle x^{-2r+1}, \varphi(x) \rangle = \int_0^\infty x^{-2r+1} \left[ \varphi(x) - \varphi(-x) - 2 \sum_{i=0}^{r-1} \frac{x^{2i-1}}{(2i-1)!} \varphi^{(2i-1)}(0) \right] dx$$

for $r = 1, 2, \ldots$.

It follows that if $\ln |x| = \ln x_+ + \ln x_-$, then

$$(\ln |x|)' = x^{-1}, \quad (x^{-r})' = -rx^{-r-1}$$

for $r = 1, 2, \ldots$, see [25].

A very important concept in the theory of distributions is that of convergence.

**DEFINITION 1.7** Let $\{f_n\}$ be a sequence of distributions in $\mathcal{D}'$. Then $\{f_n\}$ is said to converge to the limit $f$ in $\mathcal{D}'$ if and only if

$$\lim_{n \to \infty} \langle f_n(x), \varphi(x) \rangle = \langle f(x), \varphi(x) \rangle$$

for all $\varphi \in \mathcal{D}$. 

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The following theorem is easily proved.

**THEOREM 1.1** If the sequence \( \{f_n\} \) of distributions converges to \( f \), then the sequence \( \{f_n^{(k)}\} \) converges to \( f^{(k)} \) for \( k = 1, 2, \ldots \).

**DEFINITION 1.8** A sequence of test functions \( \{\varphi_n\} \) is said to be a null sequence if

(i) the support of each \( \varphi_n \) is contained in some fixed domain \( D \) independent of \( n \),

(ii) the sequence \( \{\varphi_n^{(m)}\} \) converges uniformly to zero in \( D \), as \( n \) tends to infinity, for all \( m \).

**DEFINITION 1.9** A sequence of functions \( \{f_n\} \) is said to be regular if

(i) \( f_n \) is infinitely differentiable, for all \( n \),

(ii) the sequence \( \{\langle f_n, \varphi \rangle\} \) converges, to a limit \( \langle f, \varphi \rangle \) say, for each test function \( \varphi \),

(iii) \( \langle f, \varphi \rangle \) is continuous in the sense that \( \lim_{n \to \infty} \langle f, \varphi_n \rangle = 0 \) for each null sequence of test functions \( \{\varphi_n\} \), see Temple [33].

Two regular sequences \( \{g_n\} \) and \( \{h_n\} \) are said to be equivalent if

\[
\lim_{n \to \infty} \langle g_n - h_n, \varphi \rangle = 0
\]

for each test function \( \varphi \).
1.1.2 NEUTRIX CALCULUS

In 1932, see Temple [34] Hadamard was faced with the divergent integral

$$\int_0^1 \frac{A(x)}{x^{p+1/2}} dx,$$  \hspace{1cm} (1.3)

where $p$ is a positive integer. He therefore separated the integral

$$\int_0^1 \frac{A(x)}{x^{p+1/2}} dx, \hspace{1cm} (\epsilon > 0)$$

into two parts, namely

$$F(\epsilon) = \int_\epsilon^1 \frac{A(x) - B(x)}{x^{p+1/2}} dx$$

and

$$I(\epsilon) = \int_\epsilon^1 \frac{B(x)}{x^{p+1/2}} dx,$$

where

$$B(x) = \sum_{i=0}^{p-1} \frac{A^{(i)}(0)}{i!} x^i.$$  \hspace{1cm} (1.4)

$F(\epsilon)$ tends to a finite limit $F(0)$ as $\epsilon$ tends to 0, whereas $I(\epsilon)$ diverges as $\epsilon$ tends to 0. He therefore defined $F(0)$ as the finite part of the integral (1.3) and wrote

$$\text{f.p. } \int_0^1 \frac{A(x)}{x^{p+1/2}} dx = F(0).$$

However,

$$I(\epsilon) = - \sum_{i=0}^{p-1} \frac{A^{(i)}(0)}{i!(p-i-\frac{1}{2})} + \sum_{i=0}^{p-1} \frac{A^{(i)}(0)\epsilon^{-p+i+\frac{1}{2}}}{i!(p-i-\frac{1}{2})}$$

$$= K + I_1(\epsilon)$$

and so the divergent integral

$$\int_0^1 \frac{B(x)}{x^{p+1/2}} dx$$
has a finite part $K$ and a divergent part $I_1(\epsilon)$. Thus, we can write
\[
\text{f.p. } \int_0^1 \frac{B(x)}{x^{p+1/2}} \, dx = K
\]
and so we should in fact have
\[
\text{f.p. } \int_0^1 \frac{A(x)}{x^{p+1/2}} \, dx = F(0) + K.
\]

This was Temple’s interpretation in [34]. Hadamard’s original interpretation however is entirely correct; see for example The Prehistory of the Theory of Distributions by Jesper Lutzen (Springer, 1982). We will come back to this shortly.

In his study of asymptotics, van der Corput [5] came across similar problems. He noticed that certain terms he had calculated just cancelled out and so were superfluous. He called such terms negligible functions. From this he developed the neutrix calculus. In the following chapters, we will use the neutrix calculus to evaluate neutrix products, neutrix convolution products and compositions of distributions.

**DEFINITION 1.10** Let $N'$ be a set and let $N$ be a commutative, additive group of functions mapping $N'$ into a commutative, additive group $N''$. If $N$ has the property that the only constant function in $N$ is the zero function, then $N$ is said to be a neutrix and the functions in $N$ are said to be negligible.

**EXAMPLE 1.5** Let $N'$ be the closed interval $[0, 1] = \{x : 0 \leq x \leq 1\}$ and let $N$ be the set of all functions defined on $N'$ of the form
\[
a \sin x + bx^2,
\]
where $a$ and $b$ are arbitrary real numbers. Then $N$ is a neutrix, since if
\[
a \sin x + bx^2 = c
\]
for all $x$ in $N'$, then $a = b = c = 0$. 

Now suppose that $N'$ is a subspace of a topological space $X$ having a limit point $y$ which is not contained in $N'$. Let $N''$ be the real (or complex) numbers and let $N$ be a commutative, additive group of functions mapping $N'$ into $N''$ with the property that if $N$ contains a function $f(x)$ which converges to a finite limit $c$ as $x$ tends to $y$, then $c = 0$. $N$ is a neutrix, because if $f$ is in $N$ and $f(x) = c$ for all $x$ in $N'$, then $f(x)$ converges to the finite limit $c$ as $x$ tends to $y$ and so $c = 0$.

**Definition 1.11** Let $f(x)$ be a real (or complex) valued function defined on $N'$ and suppose it is possible to find a constant $c$ such that $f(x) - c$ is negligible in $N$. Then $c$ is called the neutrix limit or $N$-limit of $f(x)$ as $x$ tends to $y$ and we write

$$N - \lim_{x \to y} f(x) = c.$$  

Note that if a neutrix limit $c$ exists, then it is unique, since if $f(x) - c$ and $f(x) - c'$ are in $N$, then the constant function $c - c'$ is also in $N$ and so $c = c'$.

Also note that if $N$ is a neutrix containing the set of all functions which converge to zero in the normal sense as $x$ tends to $y$, then

$$\lim_{x \to y} f(x) = c \Rightarrow N - \lim_{x \to y} f(x) = c.$$  

In the final two examples, the neutrix $N$ we are using will have domain $N'$ the positive reals and range $N''$ the real numbers with negligible functions finite linear sums of the functions

$$e^{\lambda \ln r - 1} \epsilon, \quad \ln^r \epsilon \quad (\lambda < 0, \quad r = 1, 2, \ldots)$$  \hspace{1cm} (1.4)

and all functions which converge to zero in the normal sense as $\epsilon$ tends to 0.
**EXAMPLE 1.6** The Gamma function $\Gamma(\lambda)$ defined for $\lambda > 0$ by

$$\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}e^{-t} dt$$

and for $-r < \lambda < -r + 1$, $r \in \mathbb{N}$ by

$$\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}[e^{-t} - \sum_{i=0}^{r-1} \frac{(-t)^i}{i!}] dt.$$

Note that

$$\int_\epsilon^\infty t^{\lambda-1}e^{-t} dt = \int_\epsilon^\infty t^{\lambda-1}[e^{-t} - \sum_{i=0}^{r-1} \frac{(-t)^i}{i!}] dt + \sum_{i=0}^{r-1} \frac{(-1)^i \epsilon^{\lambda+i}}{i!(\lambda+i)}$$

and it follows that

$$\Gamma(\lambda) = \text{N} \lim_{\epsilon \to 0} \int_\epsilon^\infty t^{\lambda-1}e^{-t} dt = \text{f.p.} \int_0^\infty t^{\lambda-1}e^{-t} dt,$$

where f.p. $\int_0^\infty t^{\lambda-1}e^{-t} dt$ denotes Hadamard’s finite part of $\int_0^\infty t^{\lambda-1}e^{-t} dt$.

More generally, it can be shown that

$$\Gamma^{(r)}(\lambda) = \text{N} \lim_{\epsilon \to 0} \int_\epsilon^\infty t^{\lambda-1}\ln^r te^{-t} dt = \text{f.p.} \int_0^\infty t^{\lambda-1}\ln^r te^{-t} dt$$

for $\lambda \neq 0, -1, -2, \ldots$ and $r = 0, 1, 2, \ldots$.

This neutrix limit was also used to define $\Gamma^{(r)}(\lambda)$ for $\lambda = 0, -1, -2, \ldots$ and $r = 0, 1, 2, \ldots$, see Fisher and Kuribayashi [20].

**EXAMPLE 1.7** The Beta function is usually defined by

$$B(\lambda, \mu) = \int_0^1 t^{\lambda-1}(1-t)^{\mu-1} dt$$

for $\lambda, \mu > 0$, but more generally, if

$$B_{r,s}(\lambda, \mu) = \frac{\partial^{r+s}}{\partial \lambda^r \partial \mu^s} B(\lambda, \mu),$$

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it can be proved that

\[ B_{r,s}(\lambda, \mu) = N \lim_{\epsilon \to 0} \int_{\epsilon}^{1-\epsilon} t^{\lambda-1} \ln^r t (1-t)^{\mu-1} \ln^s (1-t) \, dt \]

for \( \lambda, \mu \neq 0, -1, -2, \ldots \) and \( r, s = 0, 1, 2, \ldots \).

Again

\[ B_{r,s}(\lambda, \mu) = \text{f.p.} \int_{0}^{1} t^{\lambda-1} \ln^r t (1-t)^{\mu-1} \ln^s (1-t) \, dt. \]

This neutrix limit was also used to define \( B_{r,s}(\lambda, \mu) \) for \( \lambda, \mu = 0, -1, -2, \ldots \) and \( r, s = 0, 1, 2, \ldots \), see Fisher and Kuribayashi [19].

We finally note that if the set of negligible functions is changed, then the neutrix limits will change. For example, if we look at Hadamard’s integral (1.3) again we can write

\[ \int_{\epsilon}^{1} \frac{A(x)}{x^{p+1/2}} \, dx = \int_{\epsilon}^{1} \frac{A(x) - B(x)}{x^{p+1/2}} \, dx + \sum_{i=0}^{p-1} \frac{A^{(i)}(0)}{i!(p-i-\frac{1}{2})} (\epsilon^{p+i+1/2} - 1). \]

Using the set of negligible functions (1.4) it follows that

\[ N \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \frac{A(x)}{x^{p+1/2}} \, dx = F(0) + K. \]

However, if we use the set of negligible functions consisting of

\[ (\epsilon^{\lambda} - 1) \ln^{r-1} \epsilon, \ln^r \epsilon \quad (\lambda < 0, r = 1, 2, \ldots) \]

we get

\[ N \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \frac{A(x)}{x^{p+1/2}} \, dx = F(0). \]

The former seems to be the correct answer using the standard set of negligible functions (1.4), otherwise incorrect answers would be given in Examples 1.6 and 1.7 if we used the set of negligible functions (1.5).

In addition to this introduction, there are four chapters in this thesis.
Chapter Two involves the neutrix convolution products which are introduced for the cosine integral \( \text{Ci}(\lambda x) \). Moreover, some new results are established in this regard.

In Chapter Three similar results concerning the sine integral \( \text{Si}(\lambda x) \) are obtained.

In Chapter Four, some earlier results by Fisher and Kuribayashi concerning the neutrix products are generalized.

In Chapter Five, new results are established regarding the composition of distributions such as \((x^2)^{-1/2}\) and \((|x|^{-\lambda})^{2r/\lambda}\).
Chapter 2

THE COSINE INTEGRAL

The cosine integral $\text{Ci}(x)$ is defined for $x > 0$ by

$$\text{Ci}(x) = - \int_x^\infty u^{-1} \cos u \, du. \quad (2.1)$$

This integral is divergent for $x \leq 0$. Equation (2.1) however can be rewritten in the form

$$\text{Ci}(x) = - \int_x^\infty u^{-1} \left[ \cos u - H(1-u) \right] \, du + H(1-x) \ln |x|. \quad (2.2)$$

The integral in this equation is convergent for all $x$ and $\ln |x|$ is a locally summable function on the real line. We will therefore use equation (2.2) to define $\text{Ci}(x)$ on the real line as a locally summable function.

More generally, see [8], if $A \neq 0$, we define $\text{Ci}(Ax)$ by

$$\text{Ci}(Ax) = - \int_{Ax}^\infty u^{-1} \left[ \cos u - H(1-u) \right] \, du + H(1-Ax) \ln |Ax|. \quad (2.3)$$

In particular, if $A > 0$, it follows that

$$\text{Ci}(\lambda x) = - \int_x^\infty u^{-1} \left[ \cos u - H(1-u) \right] \, du + H(1-\lambda x) \ln |\lambda x| \quad (2.4)$$

and if $A < 0$, it follows that

$$\text{Ci}(\lambda x) = - \int_{-\infty}^x u^{-1} \left[ \cos u - H(1-u) \right] \, du + H(1-\lambda x) \ln |\lambda x| \quad (2.5)$$

$$= - \int_x^{-\infty} u^{-1} \left[ \cos u - H(1+u) \right] \, du + H(1-\lambda x) \ln |\lambda x|. \quad (2.6)$$
If \( \lambda \neq 0 \), we define \( \text{Ci}_+(\lambda, x) \) to be the locally summable function given by

\[
\text{Ci}_+(\lambda, x) = H(x) \text{Ci}(\lambda x).
\]  

(2.7)

It follows that if \( \lambda > 0 \), then

\[
\text{Ci}_+(\lambda, x) = \begin{cases} 
-\int_x^{\infty} u^{-1}[\cos(\lambda u) - H(1 - \lambda u)] \, du + H(1 - \lambda x) \ln |\lambda x|, & x > 0, \\
0, & x < 0.
\end{cases}
\]

(2.8)

Alternatively, as a generalization of equation (2.1), equation (2.8) can be expressed in the simpler form

\[
\text{Ci}_+(\lambda, x) = \begin{cases} 
-\int_x^{\infty} u^{-1} \cos(\lambda u) \, du, & x > 0, \\
0, & x < 0.
\end{cases}
\]

(2.9)

If \( \lambda < 0 \), it follows that

\[
\text{Ci}_+(\lambda, x) = \begin{cases} 
-\int_{-\infty}^{-x} u^{-1}[\cos(\lambda u) - H(1 + \lambda u)] \, du + \ln |\lambda x|, & x > 0, \\
0, & x < 0.
\end{cases}
\]

\[
= -c + \int_0^x u^{-1}[\cos(\lambda u) - 1] \, du + \ln |\lambda x|, \quad x > 0,
\]

(2.10)

where

\[
c = \begin{cases} 
\int_0^{\infty} u^{-1}[\cos(\lambda u) - H(1 + \lambda u)] \, du, & \lambda < 0, \\
\int_0^{\infty} u^{-1}[\cos(\lambda u) - H(1 - \lambda u)] \, du, & \lambda > 0.
\end{cases}
\]

We next define the locally summable function \( \text{Ci}_-(\lambda, x) \) for \( \lambda \neq 0 \) by

\[
\text{Ci}_-(\lambda, x) = H(-x) \text{Ci}(\lambda x) = \text{Ci}(\lambda x) - \text{Ci}_+(\lambda, x)
\]

(2.11)

so that if \( \lambda > 0 \),

\[
\text{Ci}_-(\lambda, x) = \begin{cases} 
-c + \int_0^{-x} u^{-1}[\cos(\lambda u) - 1] \, du + \ln |\lambda x|, & x < 0, \\
0, & x > 0.
\end{cases}
\]

(2.12)
If $\lambda < 0$, it follows that

$$
\text{Ci}_-(\lambda, x) = \left\{ \begin{array}{ll}
\int_{-\infty}^{x} u^{-1}[\cos(\lambda u) - H(1 - \lambda u)] du + H(1 - \lambda x) \ln |\lambda x|, & x < 0, \\
0, & x > 0,
\end{array} \right. \qquad (2.13)
$$

$$
= \int_{-\infty}^{x} u^{-1} \cos(\lambda u) du, \quad x < 0.
$$

For future reference, we note that if we replace $x$ by $-x$ in equation (2.3), we see that $\text{Ci}(\lambda(-x)) = \text{Ci}((-\lambda)x) = \text{Ci}(-\lambda x)$ and so if we replace $x$ by $-x$ in equation (2.7) we get

$$
\text{Ci}_+(\lambda, (-x)) = H(-x) \text{Ci}(\lambda(-x)) = H(-x) \text{Ci}((-\lambda)x).
$$

It follows that

$$
\text{Ci}_+(\lambda, (-x)) = \text{Ci}_-((-\lambda), x) \quad (2.14)
$$
for all $\lambda$. Similarly

$$
\text{Ci}_-(\lambda, (-x)) = \text{Ci}_+((-\lambda), x) \quad (2.15)
$$
for all $\lambda$.

We will now find the derivative $[\text{Ci}(\lambda x)]'$ of $\text{Ci}(\lambda x)$ as a distribution for $\lambda \neq 0$. We let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Putting

$$
f(x) = \int_{\lambda x}^{\infty} u^{-1}[\cos u - H(1 - u)] du
$$
and letting $\varphi$ be an arbitrary test function in $\mathcal{D}$, we have

$$
\langle [f(x)]', \varphi(x) \rangle = -\langle f(x), \varphi'(x) \rangle
$$

$$
= -\int_{-\infty}^{\infty} \varphi'(x) \int_{\lambda x}^{\infty} u^{-1}[\cos u - H(1 - u)] du dx
$$

$$
= -\int_{-\infty}^{\infty} u^{-1}[\cos u - H(1 - u)] \int_{-\infty}^{u/\lambda} \varphi'(x) dx du
$$

$$
= -\int_{-\infty}^{\infty} u^{-1}[\cos u - H(1 - u)] \varphi(u/\lambda) du
$$

$$
= -\int_{0}^{\infty} u^{-1} \cos(\lambda u)[\varphi(u) - \varphi(-u)] du +
$$

$$
+ \int_{0}^{\infty} u^{-1}[H(1 - \lambda u)\varphi(u) - H(1 + \lambda u)\varphi(-u)] du
$$

$$
= -\langle \cos(\lambda x)x^{-1} - H(1 - \lambda x)x^{-1}, \varphi(x) \rangle, \quad (2.16)
$$
where \(\cos(\lambda x)x^{-1}\) is the product of the infinitely differentiable function \(\cos(\lambda x)\) and the distribution \(x^{-1}\), see Gel'fand and Shilov [25]. It now follows from equation (2.3) that

\[
[Ci(\lambda x)]' = \cos(\lambda x)x^{-1} = x^{-1} + \sum_{i=1}^{\infty} \frac{(-1)^i \lambda^{2i}}{(2i)!} x^{2i-1},
\]

(2.17)
since \(x^i x^{-1} = x^{i-1}\) for \(i = 1, 2, \ldots\). More generally, we have

\[
[Ci(\lambda x)]^{(2r-1)} = (2r - 2)! x^{-2r+1} + \sum_{i=1}^{\infty} \frac{(-1)^i \lambda^{2i}}{2i(2i - 2r + 1)!} x^{2i-2r+1},
\]

(2.18)

\[
[Ci(\lambda x)]^{(2r)} = -(2r-1)! x^{-2r} + \sum_{i=1}^{\infty} \frac{(-1)^i \lambda^{2i}}{2i(2i - 2r)!} x^{2i-2r},
\]

(2.19)
for \(r = 1, 2, \ldots\).

We next find the derivative of \(Ci_+(\lambda, x)\) for \(\lambda > 0\). Using equation (2.9) we have

\[
\langle [Ci_+(\lambda, x)]', \varphi(x) \rangle = -\langle Ci_+(\lambda, x), \varphi'(x) \rangle
\]

\[
= \int_0^\infty \varphi'(x) \int_x^\infty u^{-1} \cos(\lambda u) \, du \, dx
\]

\[
= \int_0^\infty u^{-1} \cos(\lambda u) \int_0^u \varphi'(x) \, dx \, du
\]

\[
= \int_0^\infty u^{-1} \cos(\lambda u) [\varphi(u) - \varphi(0)] \, du
\]

\[
= \int_0^\infty u^{-1} [\cos(\lambda u) \varphi(u) - H(1-u)\varphi(0)] \, du +
\]

\[
-\varphi(0) \int_0^\infty u^{-1} [\cos(\lambda u) - H(1-\lambda u)] \, du +
\]

\[
+\varphi(0) \int_0^\infty u^{-1} [H(1-u) - H(1-\lambda u)] \, du
\]

\[
= \langle \cos(\lambda x)x_+^{-1} - (c - \ln |\lambda|)\delta(x), \varphi(x) \rangle.
\]

It follows that

\[
[Ci_+(\lambda, x)]' = \cos(\lambda x)x_+^{-1} - (c - \ln |\lambda|)\delta(x)
\]

(2.20)

\[
= x_+^{-1} + \sum_{i=1}^{\infty} \frac{(-1)^i \lambda^{2i}}{(2i)!} x_+^{2i-1} - (c - \ln |\lambda|)\delta(x),
\]

(2.21)
since \(x^i x^{-1}_+ = x^i x^{-1}_+\) for \(i = 1, 2, \ldots\), see Gel’fand and Shilov [25]. More gener­ly, we have

\[
[C_\iota(\lambda, x)]'' = -x_i - \sum_{i=1}^{\infty} \frac{(-1)^i \lambda^{2i}}{2i(2i - 2)!} x_i^{2i - 2} - (c - \ln |\lambda|) \delta'(x) \tag{2.22}
\]

and

\[
[C_\iota(\lambda, x)]^{(2r-1)} = (2r - 2)! x_i^{-2r+1} + \sum_{i=1}^{\infty} \frac{(-1)^i \lambda^{2i}}{2i(2i - 2r + 1)!} x_i^{2i - 2r+1} + \sum_{i=1}^{r-1} \frac{(-1)^i \lambda^{2i}}{2i} \delta^{(2r-2i-2)}(x) - (c - \ln |\lambda|) \delta^{(2r-2)}(x), \tag{2.23}
\]

\[
[C_\iota(\lambda, x)]^{(2r)} = -(2r - 1)! x_i^{-2r} + \sum_{i=1}^{\infty} \frac{(-1)^i \lambda^{2i}}{2i(2i - 2r)!} x_i^{2i - 2r} + \sum_{i=1}^{r-1} \frac{(-1)^i \lambda^{2i}}{2i} \delta^{(2r-2i-1)}(x) - (c - \ln |\lambda|) \delta^{(2r-1)}(x) \tag{2.24}
\]

for \(r = 2, 3, \ldots\).

For the case \(\lambda < 0\), we put

\[
f(x) = \begin{cases} 
\int_0^x u^{-1} [\cos(\lambda u) - 1] du, & x > 0, \\
0, & x < 0.
\end{cases}
\]

Then for arbitrary test function \(\varphi\) in \(D\), we have

\[
\langle [f(x)]', \varphi(x) \rangle = -\langle f(x), \varphi'(x) \rangle \\
= -\int_0^\infty \varphi'(x) \int_0^x u^{-1} [\cos(\lambda u) - 1] du \ dx \\
= -\int_0^\infty u^{-1} [\cos(\lambda u) - 1] \int_u^\infty \varphi'(x) \ dx \ du \\
= \int_0^\infty u^{-1} [\cos(\lambda u) - 1] \varphi(u) \ du \\
= \int_0^\infty u^{-1} [\cos(\lambda u) \varphi(u) - H(1 - u) \varphi(0)] \ du + \int_0^\infty u^{-1} [\varphi(u) - H(1 - u) \varphi(0)] \ du \\
= \langle \cos(\lambda x) x_i^{-1} - x_i^{-1}, \varphi(x) \rangle \tag{2.25}
\]

and it follows that equations (2.20) to (2.24) also hold for \(\lambda < 0\).
It now follows from equations (2.11), (2.17) and (2.20) that for all values of $\lambda \neq 0$

$$[C_i(\lambda, x)]' = -\cos(\lambda x)x^{-1} + (c - \ln \lambda)\delta(x)$$
$$= -x^{-1} \sum_{i=1}^{\infty} \frac{(-1)^i \lambda^{2i}}{(2i)!} x^{2i-1} + (c - \ln \lambda)\delta(x),$$

$$[C_i(\lambda, x)]'' = -x^{-2} + \sum_{i=1}^{\infty} \frac{(-1)^i \lambda^{2i}}{2i(2i-2)!} x^{2i-2} + (c - \ln \lambda)\delta'(x)$$

and more generally

$$[C_i(\lambda, x)]^{(2r-1)} = -(2r - 2)!x^{-2r+1} - \sum_{i=r}^{\infty} \frac{(-1)^i \lambda^{2i}}{2i(2i-2r+1)!} x^{2i-2r+1} +$$
$$+ \sum_{i=1}^{r-1} \frac{(-1)^i \lambda^{2i}}{2i} \delta^{(2r-2i-2)}(x) + (c - \ln \lambda)\delta^{(2r-2)}(x),$$

$$[C_i(\lambda, x)]^{(2r)} = -(2r - 1)!x^{-2r} + \sum_{i=r}^{\infty} \frac{(-1)^i \lambda^{2i}}{2i(2i-2r)!} x^{2i-2r} +$$
$$+ \sum_{i=1}^{r-1} \frac{(-1)^i \lambda^{2i}}{2i} \delta^{(2r-2i-1)}(x) + (c - \ln \lambda)\delta^{(2r-1)}(x)$$

for $r = 2, 3, \ldots$

**Convolution of distributions**

The classical definition of the convolution product of two functions $f$ and $g$ is as follows:

**DEFINITION 2.1** Let $f$ and $g$ be functions. Then the convolution product $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x - t) \, dt$$

for all points $x$ for which the integral exists.

It follows easily from the definition that if $f * g$ exists then $g * f$ exists and

$$f * g = g * f$$

(2.29)
and if \((f * g)'\) and \(f * g'\) (or \(f' * g\)) exists, then

\[
(f * g)' = f * g' \quad \text{(or } f' * g). \quad (2.30)
\]

Definition 2.1 can be extended to define the convolution product \(f * g\) of two distributions \(f\) and \(g\) in \(D'\) with the following definition, see Gel'fand and Shilov [25].

**DEFINITION 2.2** Let \(f\) and \(g\) be distributions in \(D'\). Then the convolution product \(f * g\) is defined by the equation

\[
\langle (f * g)(x), \varphi \rangle = \langle f(y), \langle g(x), \varphi(x + y) \rangle \rangle
\]

for arbitrary \(\varphi\) in \(D\), provided \(f\) and \(g\) satisfy either of the conditions

(i) either \(f\) or \(g\) has bounded support,

(ii) the supports of \(f\) and \(g\) are bounded on the same side.

In the following, the locally summable functions \(\sin_\pm(\lambda x)\) and \(\cos_\pm(\lambda x)\) are defined by

\[
\sin_+(\lambda x) = H(x) \sin(\lambda x), \quad \sin_-(\lambda x) = H(-x) \sin(\lambda x),
\]

\[
\cos_+(\lambda x) = H(x) \cos(\lambda x), \quad \cos_-(\lambda x) = H(-x) \cos(\lambda x).
\]

It follows as above that

\[
\sin_+(\lambda(-x)) = \sin_-((-\lambda)x), \quad \sin_-(\lambda(-x)) = \sin_+((-\lambda)x),
\]

\[
\cos_+(\lambda(-x)) = \cos_-((-\lambda)x), \quad \cos_-(\lambda(-x)) = \cos_+((-\lambda)x).
\]

We need the following theorems and corollaries which were proved by Fisher and Al-Sirehy, see [8].
**Theorem 2.1** If $\lambda \neq 0$, then

$$
\text{Ci}_+(\lambda, x) \ast x^r_+ = -\frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} a_{r-i}(x, \lambda)x^i_+ + \frac{1}{r+1} x^{r+1} \text{Ci}_+(\lambda, x) 
$$

(2.31)

for $r = 0, 1, 2, \ldots$, where

$$
a_i(x, \lambda) = \int_0^x (-u)^i \cos(\lambda u) \, du
$$

for $i = 0, 1, 2, \ldots$. In particular,

$$
\text{Ci}_+(\lambda, x) \ast H(x) = -\lambda^{-1} \sin_+(\lambda x) + x \text{Ci}_+(\lambda, x),
$$

(2.32)

$$
\text{Ci}_+(\lambda, x) \ast x_+ = -\frac{1}{2}[\lambda^{-2}H(x) - \lambda^{-2} \cos_+(\lambda x) + \lambda^{-1} x \sin_+(\lambda x)] +
$$

$$
+ \frac{1}{2} x^2 \text{Ci}_+(\lambda, x).
$$

(2.33)

**Corollary 2.1** If $\lambda \neq 0$, then

$$
[\cos(\lambda x)x_+^{-1}] \ast x^r_+ = (c - \ln |\lambda|)x^r_+ - \sum_{i=0}^{r-1} \binom{r}{i} a_{r-i-1}(x, \lambda)x^i_+ + x^r \text{Ci}_+(\lambda x)
$$

(2.34)

for $r = 1, 2, \ldots$. In particular,

$$
[\cos(\lambda x)x_+^{-1}] \ast H(x) = (c - \ln |\lambda|)H(x) + \text{Ci}_+(\lambda, x),
$$

(2.35)

$$
[\cos(\lambda x)x_+^{-1}] \ast x_+ = (c - \ln |\lambda|)x_+ - \lambda^{-1} \sin_+(\lambda x) + x \text{Ci}_+(\lambda, x).
$$

(2.36)

**Theorem 2.2** If $\lambda \neq 0$, then

$$
\text{Ci}_-(\lambda, x) \ast x_+^r = \frac{(-1)^r}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^i a_{r-i}(x, \lambda)x^i_+ + \frac{(-x)^{r+1}}{r+1} \text{Ci}_-(\lambda, x)
$$

(2.37)

for $r = 0, 1, 2, \ldots$. In particular,

$$
\text{Ci}_-(\lambda, x) \ast H(-x) = \lambda^{-1} \sin_-(\lambda x) - x \text{Ci}_-(\lambda, x),
$$

(2.38)

$$
\text{Ci}_-(\lambda, x) \ast x_+ = -\frac{1}{2}[\lambda^{-2}H(-x) - \lambda^{-2} \cos_-(\lambda x) + \lambda^{-1} x \sin_-(\lambda x)] +
$$

$$
+ \frac{1}{2} x^2 \text{Ci}_-(\lambda, x).
$$

(2.39)
COROLLARY 2.2 If \( \lambda \neq 0 \), then the convolution product \([\cos(\lambda x)x^{-s}] * x_r\) exists for \( r = 0, 1, 2, \ldots \) and \( s = 1, 2, \ldots \). In particular,

\[
[\cos(\lambda x)x^{-1}] * H(-x) = (c - \ln |\lambda|)H(-x) + \text{Ci}_-(\lambda, x), \quad (2.40)
\]

\[
[\cos(\lambda x)x^{-1}] * x_+ = (c - \ln |\lambda|)x_+ + \lambda^{-1}\sin_-(\lambda x) - x\text{Ci}_-(\lambda, x) \quad (2.41)
\]

and in general

\[
[\cos(\lambda x)x^{-1}] * x_r = (c - \ln |\lambda|)x_r + \sum_{i=0}^{r-1} \binom{r}{i}(-1)^{r-i}a_{r-i-1}(x, \lambda)x_i + (-1)^r x^r \text{Ci}_-(\lambda, x) \quad (2.42)
\]

for \( r = 1, 2, \ldots \).

The above definition of the convolution product is rather restrictive and so a neutrix convolution product was introduced in [10] and was later modified in [17]. In order to define the neutrix convolution product we first of all let \( \tau \) be a function in \( \mathcal{D} \) satisfying the following properties:

(i) \( \tau(x) = \tau(-x) \),

(ii) \( 0 \leq \tau(x) \leq 1 \),

(iii) \( \tau(x) = 1 \) for \( |x| \leq \frac{1}{2} \),

(iv) \( \tau(x) = 0 \) for \( |x| \geq 1 \).

The function \( \tau_n \) is now defined by

\[
\tau_n(x) = \begin{cases} 
1, & |x| \leq n, \\
\tau(n^n x + n^{n+1}), & x < -n, \\
\tau(n^n x - n^{n+1}), & x > n,
\end{cases}
\]

for all real \( n > 0 \).

The following definition of the neutrix convolution product was given in [10].
DEFINITION 2.3 Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) and let \( f_n = f \tau_n \) for \( n > 0 \). Then the neutrix convolution product \( f \boxplus g \) is defined as the neutrix limit of the sequence \( \{f_n \ast g\} \), provided that the limit \( h \) exists in the sense that
\[
N \lim_{n \to \infty} (f_n \ast g, \varphi) = \langle h, \varphi \rangle
\]
for all \( \varphi \) in \( \mathcal{D} \), where \( N \) is the neutrix, see van der Corput [5], having domain \( N' \) the positive reals and range \( N'' \) the real numbers, with negligible functions finite linear sums of the functions
\[
n^\lambda \ln^{r-1} n, \ln^r n, \quad (\lambda > 0, r = 1, 2, \ldots)
\]
and all functions which converge to zero in the usual sense as \( n \) tends to infinity.

We now increase the set of negligible functions given here to include finite linear sums of the functions
\[
n^\mu \cos(\lambda n), \ n^\mu \sin(\lambda n), \ n^\mu \text{Ci}[\lambda(\alpha + n)] \quad (\mu \neq 0).
\]

Note that in this definition the convolution product \( f_n \ast g \) is defined in Gel’fand and Shilov’s sense, the distribution \( f_n \) having bounded support.

It is easily seen that any results proved with the original definition hold with the new definition. The following two theorems were proved in [10], the first showing that the neutrix convolution product is a generalization of the convolution product.

THEOREM 2.3 Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) satisfying either condition \( (a) \) or condition \( (b) \) of Gel’fand and Shilov’s definition. Then the neutrix convolution product \( f \boxplus g \) exists and
\[
f \boxplus g = f \ast g.
\]
THEOREM 2.4 Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution product $f \circledast g$ exists. Then the neutrix convolution product $f \circledast g'$ exists and

$$
(f \circledast g)' = f \circledast g'.
$$

(2.43)

Note however that equation (2.29) does not necessarily hold for the neutrix convolution product and that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$, but we do have the following lemma, see Fisher and Al-Sirehy [8].

LEMMA 2.1. Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution product $f \circledast g$ exists. If $N \lim_{n \to \infty} \langle (f \tau_n') * g, \varphi \rangle$ exists and equals $\langle h, \varphi \rangle$ for all $\varphi$ in $\mathcal{D}$, then $f' \circledast g$ exists and

$$
(f \circledast g)' = f' \circledast g + h.
$$

(2.44)

PROOF. Using equation (2.30) we have

$$
\langle (f_n * g)', \varphi \rangle = \langle (f' \tau_n) * g, \varphi \rangle + \langle (f \tau_n') * g, \varphi \rangle
$$

and it follows that

$$
N \lim_{n \to \infty} \langle (f' \tau_n) * g, \varphi \rangle = -N \lim_{n \to \infty} \langle f_n * g', \varphi \rangle - N \lim_{n \to \infty} \langle f \tau_n' * g, \varphi \rangle,
$$

proving the existence of $f' \circledast g$ and equation (2.44).

The next theorem was proved in [8].

THEOREM 2.5 If $\lambda \neq 0$, then

$$
\text{Ci}_+(\lambda, x) \circledast x^r = -\frac{1}{r + 1} \sum_{i=0}^{r} \binom{r + 1}{i} L_{r-i}x^i
$$

(2.45)

for $r = 0, 1, 2, \ldots$, where

$$
L_{2i} = 0, \quad L_{2i+1} = (2i + 1)!(-1)^i \lambda^{-2i-2}
$$
for \( i = 0, 1, 2, \ldots \). In particular,

\[
\begin{align*}
\text{Ci}_+(\lambda, x) \oplus 1 &= 0, \quad (2.46) \\
\text{Ci}_+(\lambda, x) \odot x &= -\frac{1}{2} \lambda^{-2}. \quad (2.47)
\end{align*}
\]

**Proof.** We put \([\text{Ci}_+(\lambda, x)]_n = \text{Ci}_+(\lambda, x) \tau_n(x)\). Then the convolution product \([\text{Ci}_+(\lambda, x)]_n * x^r\) exists by Definition 2.1 and

\[
\begin{align*}
[\text{Ci}_+(\lambda, x)]_n * x^r &= \int_0^n \text{Ci}_+(\lambda, t)(x - t)^r \, dt + \int_n^{n+n-n} \tau_n(t) \text{Ci}_+(\lambda, t)(x - t)^r \, dt. \\
&= \int_0^n \text{Ci}_+(\lambda, t)(x - t)^r \, dt + \int_n^{n+n-n} \tau_n(t) \text{Ci}_+(\lambda, t)(x - t)^r \, dt.
\end{align*}
\]  

(2.48)

If \( \lambda > 0 \), we have

\[
\begin{align*}
\int_0^n \text{Ci}_+(\lambda, t)(x - t)^r \, dt &= -\int_0^n (x - t)^r \int_t^\infty u^{-1} \cos(\lambda u) \, du \, dt \\
&= -\int_0^n u^{-1} \cos(\lambda u) \int_0^u (x - t)^r \, dt \, du + \\
&\quad -\int_n^\infty u^{-1} \cos(\lambda u) \int_n^n (x - t)^r \, dt \, du \\
&= -\frac{1}{r+1} \sum_{i=0}^r \binom{r+1}{i} x^i \left[ \int_0^n (-u)^{r-i} \cos(\lambda u) \, du + \\
&\quad + (-n)^{r-i+1} \text{Ci}(\lambda n) \right].
\end{align*}
\]

(2.49)

We now put

\[
I_i = \int_0^n (-u)^i \cos(\lambda u) \, du
\]

\[
= \lambda^{-1} (-n)^i \sin(\lambda n) + \lambda^{-1} i \int_0^n (-u)^{i-1} \sin(\lambda u) \, du
\]

\[
= \lambda^{-1} (-n)^i \sin(\lambda n) - i \lambda^{-2} (-n)^{i-1} \cos(\lambda n) - i(i-1) \lambda^{-2} I_{i-2}
\]

for \( i \geq 2 \). In particular

\[
I_0 = \lambda^{-1} \sin(\lambda n), \quad I_1 = -\lambda^{-1} n \sin(\lambda n) - \lambda^{-2} \cos(\lambda n) + \lambda^{-2}
\]

and so

\[
L_i = \lim_{n \to \infty} I_i = -i(i-1) \lambda^{-2} L_{i-2}
\]

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for \( i \geq 2 \) and
\[
L_0 = 0, \quad L_1 = \lambda^{-2}.
\]

Thus
\[
L_{2i} = 0, \quad L_{2i+1} = (2i+1)!(-1)^i\lambda^{-2i-2}
\]
for \( i = 0, 1, 2, \ldots \). It now follows from equation (2.49) that
\[
\lim_{n \to \infty} \int_0^n \text{Ci}_+(\lambda, t)(x-t)^r \, dt = -\frac{1}{r+1} \sum_{i=0}^r \binom{r+1}{i} L_{r-i}x^i.
\]  
(2.50)

Further, with \( n \geq \lambda^{-1} \) and \( K = \sup\{|\text{Ci}_+(x)| : x \geq 1\} \) we have
\[
\left| \int_0^n \tau_n(t) \text{Ci}_+(\lambda, t)(x-t)^r \, dt \right| \leq K(n+n^{-n}+|x|)^r n^{-n}
\]
for each fixed \( x \) and so
\[
\lim_{n \to \infty} \int_0^n \tau_n(t) \text{Ci}_+(\lambda, t)(x-t)^r \, dt = 0.
\]  
(2.51)

Equations (2.45), (2.46) and (2.47) now follow immediately from equations (2.48), (2.49), (2.50) and (2.51) for the case \( \lambda > 0 \).

If \( \lambda < 0 \), equation (2.48) still holds but this time we have
\[
\int_0^n \text{Ci}_+(\lambda, t)(x-t)^r \, dt = (\ln|\lambda| - c) \int_0^n (x-t)^r \, dt + \int_0^n (x-t)^r \int_0^t u^{-1}[\cos(\lambda u) - 1] \, du \, dt + \int_0^n (x-u)^r \ln u \, du
\]
\[
= J_1 + J_2 + J_3.
\]  
(2.52)

Now
\[
\int_0^n (x-t)^r \, dt = -\frac{(x-n)^{r+1} - x^{r+1}}{r+1}
\]
and it follows that
\[
\lim_{n \to \infty} J_1 = 0.
\]  
(2.53)
Next

\[ J_2 = \int_0^n u^{-1}[\cos(\lambda u) - 1]\int_0^n (x - t)^r dt \, du \]

\[ = -\frac{1}{r + 1} \int_0^n [(x - n)^{r+1} - (x - u)^{r+1}]u^{-1}[\cos(\lambda u) - 1] \, du \]

\[ = -\frac{1}{r + 1} \sum_{i=0}^r \binom{r + 1}{i} (-n)^{r-i+1}x^i \int_0^n u^{-1}[\cos(\lambda u) - 1] \, du + \]

\[ -\frac{1}{r + 1} \sum_{i=0}^r \binom{r + 1}{i} x^i \int_0^n (-u)^{r-i}[\cos(\lambda u) - 1] \, du \]

\[ = -\frac{1}{r + 1} \sum_{i=0}^r \binom{r + 1}{i} x^i (-n)^{r-i+1}[C_{i+}(\lambda, n) + c - \ln |\lambda n|] + \]

\[ -\frac{1}{r + 1} \sum_{i=0}^r \binom{r + 1}{i} x^i [I_{r-i} + (r - i + 1)^{-1}(-n)^{r-i+1}] \]

and it follows that

\[ \lim_{n \to \infty} J_2 = -\frac{1}{r + 1} \sum_{i=0}^r \binom{r + 1}{i} L_{r-i}x^i. \] (2.54)

Finally we have

\[ J_3 = -\frac{1}{r + 1} \int_0^n \ln u d[(x - u)^{r+1} - x^{r+1}] \]

\[ = -\frac{\ln n}{r + 1} [(x - n)^{r+1} - x^{r+1}] + \frac{1}{r + 1} \sum_{i=0}^r \binom{r + 1}{i} \frac{x^i(-n)^{r-i+1}}{r - i + 1} \]

and it follows that

\[ \lim_{n \to \infty} J_3 = 0. \] (2.55)

It is easily seen that equation (2.51) still holds and equations (2.45), (2.46) and (2.47) now follow from equations (2.48), (2.51), (2.52), (2.53), (2.54) and (2.55) for the case \( \lambda < 0. \)

**Corollary 2.3** If \( \lambda \neq 0, \) then

\[ C_{i+}(\lambda, x) \bigoplus x_r = \frac{(-1)^{r+1}}{r + 1} \sum_{i=0}^r \binom{r + 1}{i} [L_{r-i}x^i - a_{r-i}(x, \lambda)x^i] + \]

\[ + \frac{(-1)^{r+1}}{r + 1} x^{r+1}C_{i+}(\lambda, x) \] (2.56)
for \( r = 0, 1, 2, \ldots \). In particular,

\[
\text{Ci}_+ (\lambda, x) \odot H(-x) = \lambda^{-1} \sin_+ (\lambda x) - x \text{Ci}_+(\lambda, x), \quad (2.57)
\]

\[
\text{Ci}_+ (\lambda, x) \odot x_- = \frac{1}{2} [\lambda^{-2} H(-x) + \lambda^{-2} \cos_+ (\lambda x) - \lambda^{-1} x \sin_+ (\lambda x)] + \\
+ \frac{1}{2} x^2 \text{Ci}_+(\lambda, x). \quad (2.58)
\]

**PROOF.** Since the neutrix convolution product is distributive with respect to addition, we have

\[
\text{Ci}_+ (\lambda, x) \odot x^r = \text{Ci}_+ (\lambda, x) \ast x_+^r + (-1)^r \text{Ci}_+ (\lambda, x) \odot x_-^r
\]

and equation (2.56) follows from equations (2.31) and (2.45). Equation (2.57) follows from equations (2.32) and (2.46) and equation (2.58) follows from equations (2.33) and (2.47).

**THEOREM 2.6** If \( \lambda \neq 0 \), then

\[
[\cos(\lambda x)x_+^{-1}] \odot x^r = (c - \ln |\lambda|) x^r - \sum_{i=0}^{r-1} \binom{r}{i} L_{r-i-1} x^i \quad (2.59)
\]

for \( r = 0, 1, 2, \ldots \). In particular

\[
[\cos(\lambda x)x_+^{-1}] \odot 1 = c - \ln |\lambda|, \quad (2.60)
\]

\[
[\cos(\lambda x)x_+^{-1}] \odot x = (c - \ln |\lambda|) x. \quad (2.61)
\]

**PROOF.** We have

\[
[\text{Ci}_+ (\lambda, x) \tau_n^r (x)] \ast x^r = \int_n^{n+n-n} \text{Ci}(\lambda t)(x - t)^r d\tau_n (t) \\
= - \text{Ci}(\lambda n)(x - n)^r - \int_n^{n+n-n} \cos(\lambda t)t^{-1}(x - t)^r \tau_n (t) dt \\
+ r \int_n^{n+n-n} \text{Ci}(\lambda t)(x - t)^{r-1} \tau_n (t) dt. \quad (2.62)
\]
Now
\[ \left| \int_n^{n+n^{-n}} \cos(\lambda t)t^{-1}(x-t)^r \tau_n(t) \, dt \right| \leq n^{-n-1}(|x| + 2n)^r \]
and so
\[ \lim_{n \to \infty} \int_n^{n+n^{-n}} \cos(\lambda t)t^{-1}(x-t)^r \tau_n(t) \, dt = 0. \tag{2.63} \]

Further
\[ \left| \int_n^{n+n^{-n}} \text{Ci}(\lambda t)(x-t)^{r-1} \tau_n(t) \, dt \right| \leq (|x| + 2n)^{r-1} \int_n^{n+n^{-n}} |\text{Ci}(\lambda t)| \, dt. \]

If \( \lambda > 0 \), we put \( K = \sup\{|\text{Ci}(x)| : x \geq 1\} \). Then with \( n \geq \lambda^{-1} \), we have
\[ \int_n^{n+n^{-n}} |\text{Ci}(\lambda t)| \, dt \leq Kn^{-n}. \]

If \( \lambda < 0 \), we put
\[ K_1 = \int_0^1 u^{-1} |\cos(\lambda u) - 1| \, du. \]

Then with \( n \geq \lambda^{-1} \), we have
\[ \int_n^{n+n^{-n}} |\text{Ci}(\lambda t)| \, dt \leq \|[c] + |\ln(2\lambda n)| + K_1 + 2\ln(2n)\]n^{-n}.\]

It follows that in either case
\[ \lim_{n \to \infty} \int_n^{n+n^{-n}} \text{Ci}(\lambda t)(x-t)^{r-1} \tau_n(t) \, dt = 0. \tag{2.64} \]

It now follows from equations (2.62), (2.63) and (2.64) that
\[ N - \lim_{n \to \infty} [\text{Ci}_+(\lambda, x)\tau_n'(x)] * x^r = 0. \tag{2.65} \]

Equation (2.59) now follows from Lemma 2.1, equation (2.65) and the equation
\[ [\cos(\lambda x)x_+^{-1} - (c - \ln |\lambda|)\delta(x)] \otimes x^r = r \text{Ci}_+(\lambda, x) \otimes x^{r-1} = -\sum_{i=0}^{r-1} \binom{r}{i} L_{r-i-1} x^i. \]

Equations (2.60) and (2.61) follow immediately.
COROLLARY 2.4 If $\lambda \neq 0$, then

$$\left[\cos(\lambda x)x_+^{-1}\right] \otimes x_-^r = (-1)^{r-1} \sum_{i=0}^{r-1} \binom{r}{i} [L_{r-i-1}x_i - a_{r-i-1}(x, \lambda)x_i] +$$

$$+ (c - \ln |\lambda|)x_-^r + (-1)^{r-1}x^r \text{Ci}_+(\lambda, x) \quad (2.66)$$

for $r = 0, 1, 2, \ldots$. In particular

$$\left[\cos(\lambda x)x_+^{-1}\right] \otimes H(-x) = (c - \ln |\lambda|)H(-x) - \text{Ci}_+(\lambda, x), \quad (2.67)$$

$$\left[\cos(\lambda x)x_+^{-1}\right] \otimes x_- = (c - \ln |\lambda|)x_- + \lambda^{-1}\sin_+(\lambda x) + x \text{Ci}_+(\lambda, x). \quad (2.68)$$

PROOF. We have

$$\left[\cos(\lambda x)x_+^{-1}\right] \otimes x_-^r = (-1)^r[\cos(\lambda x)x_+^{-1}] \otimes x^r - (-1)^r[\cos(\lambda x)x_+^{-1}] \ast x_+^r$$

and equation (2.66) follows from equations (2.34) and (2.59). Equations (2.67) and (2.68) follow immediately.

THEOREM 2.7 If $\lambda \neq 0$, then

$$\text{Ci}_-(\lambda, x) \otimes x^r = -\frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i}L_{r-i}x^i \quad (2.69)$$

for $r = 0, 1, 2, \ldots$. In particular,

$$\text{Ci}_-(\lambda, x) \otimes 1 = 0, \quad (2.70)$$

$$\text{Ci}_-(\lambda, x) \otimes x = \frac{1}{2} \lambda^{-2}. \quad (2.71)$$

PROOF. Replacing $\lambda$ by $-\lambda$ in equation (2.45) we get

$$\text{Ci}_+((-\lambda), x) \otimes x^r = -\frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} L_{r-i}x^i$$

and equation (2.69) follows by replacing $x$ by $-x$ in this equation. Equations (2.70) and (2.71) follow immediately.
**COROLLARY 2.5** If \( \lambda \neq 0 \), then
\[
\text{Ci}(\lambda x) \odot x^r = 0
\]  
(2.72)
for \( r = 0, 1, 2, \ldots \).

**PROOF.** We have
\[
\text{Ci}(\lambda x) \odot x^r = \text{Ci}_+ (\lambda, x) \odot x^r + \text{Ci}_- (\lambda, x) \odot x^r
\]
and equation (2.72) follows from equations (2.45) and (2.69).

**COROLLARY 2.6** If \( \lambda \neq 0 \), then
\[
\text{Ci}_- (\lambda, x) \odot x^r_+ = \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \left[ (-1)^{r-i} L_{r-i} x^i + (-1)^i a_{r-i} (x, \lambda) x^i \right] + \frac{1}{r+1} x^{r+1} \text{Ci}_- (\lambda, x)
\]  
(2.73)
for \( r = 0, 1, 2, \ldots \). In particular,
\[
\text{Ci}_- (\lambda, x) \odot H(x) = -\lambda^{-1} \sin_-(\lambda x) + x \text{Ci}_- (\lambda, x),
\]  
(2.74)
\[
\text{Ci}_- (\lambda, x) \odot x_+ = \frac{1}{2} [\lambda^{-2} H(x) + \lambda^{-2} \cos_-(\lambda x) - \lambda^{-1} x \sin_-(\lambda x)] + \frac{1}{2} x^2 \text{Ci}_- (\lambda, x).
\]  
(2.75)

**PROOF.** Equation (2.73) follows on replacing \( \lambda \) by \( -\lambda \) and then \( x \) by \( -x \) in equation (2.56). Equations (2.74) and (2.75) follow immediately.

**COROLLARY 2.7** If \( \lambda \neq 0 \), then
\[
\text{Ci}(\lambda x) \odot x^r_+ = -\frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \left[ (-1)^{r-i} L_{r-i} x^i + a_{r-i} (x, \lambda) x^i \right] + \frac{1}{r+1} x^{r+1} \text{Ci}(\lambda x)
\]  
(2.76)
for \( r = 0, 1, 2, \ldots \). In particular,
\[
\text{Ci}(\lambda x) \odot H(x) = -\lambda^{-1} \sin(\lambda x) + x \text{Ci}(\lambda x),
\]  
(2.77)
\[
\text{Ci}(\lambda x) \odot x_+ = \frac{1}{2} [\lambda^{-2} \cos(\lambda x) - \lambda^{-1} x \sin(\lambda x)] + \frac{1}{2} x^2 \text{Ci}(\lambda x).
\]  
(2.78)
**PROOF.** Equation (2.76) follows from equations (2.31) and (2.73), equation (2.77) follows from equations (2.32) and (2.33) and equation (2.78) follows from equations (2.33) and (2.35).

**COROLLARY 2.8** If \( \lambda \neq 0 \), then

\[
Ci(\lambda x) \otimes x_r^r = \frac{(-1)^{r+1}}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \left[ L_{r-i} \frac{x^i}{x} - a_{r-i}(x, \lambda) x^i \right] + \frac{(-1)^{r+1}}{r+1} x^{r+1} Ci(\lambda x)
\]  

(2.79)

for \( r = 0, 1, 2, \ldots \). In particular,

\[
Ci(\lambda x) \otimes H(-x) = \lambda^{-1} \sin(\lambda x) - x \, Ci(\lambda x),
\]  

(2.80)

\[
Ci(\lambda x) \otimes x_\infty = \frac{1}{2} [\lambda^{-2} \cos(\lambda x) - \lambda^{-1} x \sin(\lambda x)] + \frac{1}{2} x^2 Ci(\lambda x).
\]  

(2.81)

**PROOF.** Equation (2.79) follows from equation (2.76) on replacing \( \lambda \) by \(-\lambda\) and \( x \) by \(-x\). Equations (2.80) and (2.81) follow similarly from equations (2.77) and (2.78).

**COROLLARY 2.9** If \( \lambda \neq 0 \), then

\[
[\cos(\lambda x) x^{-1}_\infty] \otimes x^r = (c - \ln |\lambda|) x^r - \sum_{i=0}^{r-1} \binom{r}{i} (-1)^{r-i} L_{r-i-1} x^i
\]  

(2.82)

for \( r = 0, 1, 2, \ldots \). In particular

\[
[\cos(\lambda x) x^{-1}_\infty] \otimes 1 = (c - \ln |\lambda|),
\]  

(2.83)

\[
[\cos(\lambda x) x^{-1}_\infty] \otimes x = (c - \ln |\lambda|) x.
\]  

(2.84)

**PROOF.** Equation (2.82) follows on replacing \( \lambda \) by \(-\lambda\) and \( x \) by \(-x\) in equation (2.59). Equations (2.83) and (2.84) follow immediately.
COROLLARY 2.10 If \( \lambda \neq 0 \), then

\[
[\cos(\lambda x)x^{-1}] \odot x^r = 0
\]  \hspace{1cm} (2.85)

for \( r = 0, 1, 2, \ldots \).

PROOF. We have

\[
[\cos(\lambda x)x^{-1}] \odot x^r = [\cos(\lambda x)x_+^{-1}] \odot x^r - [\cos(\lambda x)x_-^{-1}] \odot x^r = 0
\]
on using equations (2.59) and (2.82).

COROLLARY 2.11 If \[ \sum_{i=0}^{r-1} \binom{r}{i} \] then

\[
[\cos(\lambda x)x_-^{-1}] \odot x_+^r = - \sum_{i=0}^{r-1} \binom{r}{i} \left[ (-1)^{r-i} L_{r-i-1} x^i - (-1)^i a_{r-i-1}(x, \lambda) x^i \right] + \\
+ (c - \ln |\lambda|) x_+^r - x_+^r \text{Ci}_- (\lambda, x)
\]  \hspace{1cm} (2.86)

for \( r = 0, 1, 2, \ldots \). In particular

\[
[\cos(\lambda x)x_-^{-1}] \odot H(x) = (c - \ln |\lambda|) H(x) - \text{Ci}_- (\lambda, x),
\]  \hspace{1cm} (2.87)

\[
[\cos(\lambda x)x_-^{-1}] \odot x_+ = (c - \ln |\lambda|) x_+ + \lambda^{-1} \sin_-(\lambda x) - x_+ \text{Ci}_- (\lambda, x).
\]  \hspace{1cm} (2.88)

PROOF. Equation (2.86) follows on replacing \( \lambda \) by \(-\lambda\) and then \( x \) by \(-x\) in equation (2.66). Equations (2.87) and (2.88) follow immediately.

COROLLARY 2.12 If \( \lambda \neq 0 \), then

\[
[\cos(\lambda x)x_-^{-1}] \odot x_+^r = \sum_{i=0}^{r-1} \binom{r}{i} \left[ (-1)^{r-i} L_{r-i-1} x^i - a_{r-i-1}(x, \lambda) x^i \right] + \\
+ x_+^r \text{Ci}(\lambda x)
\]  \hspace{1cm} (2.89)

for \( r = 0, 1, 2, \ldots \). In particular

\[
[\cos(\lambda x)x_-^{-1}] \odot H(x) = \text{Ci}(\lambda x),
\]  \hspace{1cm} (2.90)

\[
[\cos(\lambda x)x_-^{-1}] \odot x_+ = -\lambda^{-1} \sin(\lambda x) + x \text{Ci}(\lambda x).
\]  \hspace{1cm} (2.91)
PROOF. Equation (2.89) follows from equations (2.34) and (2.86). Equation (2.90) follows from equations (2.35) and (2.87). Equation (2.91) follows from equations (2.36) and (2.88).

COROLLARY 2.13 If \( \lambda \neq 0 \), then

\[
[\cos(\lambda x)x^{-1}] \ast x_r^- = (-1)^r \sum_{i=0}^{r-1} \binom{r}{i} [L_{r-i-1}x^i - a_{r-i-1}(x, \lambda) x_r^i] + \\
-(c - \ln |\lambda|) x_r^r + (-1)^r x_r^r \text{Ci}_-(\lambda, x)
\]  

for \( r = 0, 1, 2, \ldots \). In particular

\[
[\cos(\lambda x)x^{-1}] \ast H(-x) = -(c - \ln |\lambda|) H(-x) - \text{Ci}_-(\lambda, x),
\]  

(2.92)

\[
[\cos(\lambda x)x^{-1}] \ast x_- = -(c - \ln |\lambda|) x_- + \lambda^{-1} \text{sin}_+(\lambda x) + x \text{Ci}_+(\lambda, x).
\]  

(2.93)

(2.94)

PROOF. Equation (2.92) follows on replacing \( \lambda \) by \(-\lambda\) and then \( x \) by \(-x\) in equation (2.89). Equations (2.93) and (2.94) follow similarly.

THEOREM 2.8 If \( \lambda \neq 0 \), then

\[
x_r^r \ast \text{Ci}_+(\lambda, x) = -\frac{1}{r + 1} \sum_{i=0}^{r} \binom{r + 1}{i} L_{r-i} x^i
\]  

for \( r = 0, 1, 2, \ldots \). In particular

\[
1 \ast \text{Ci}_+(\lambda, x) = 0,
\]  

(2.95)

(2.96)

\[
x \ast \text{Ci}_+(\lambda, x) = -\frac{1}{2} \lambda^{-2}.
\]  

(2.97)

PROOF. Put \((x^r)_n = x^r \tau_n(x)\). Then the convolution product \((x^r)_n \ast \text{Ci}_+(\lambda, x)\) exists by Definition 2.1 and if \( n > |x| \), we have

\[
(x^r)_n \ast \text{Ci}_+(\lambda, x) = \int_{-n}^{x} \text{Ci}_+[\lambda, (x - t)] t^r dt + \\
+ \int_{-n}^{-n} \tau_n(t) \text{Ci}_+[\lambda, (x - t)] t^r dt.
\]  

(2.98)
If $\lambda > 0$, we have

$$
\int_{-n}^{x} \text{Ci}_{1}[\lambda, (x-t)]t^{r} \, dt = -\int_{-n}^{x} t^{r} \int_{x-t}^{\infty} u^{-1} \cos(\lambda u) \, du \, dt
$$

$$
= - \int_{0}^{x+n} u^{-1} \cos(\lambda u) \int_{x-u}^{x} t^{r} \, dt \, du + \int_{x+n}^{\infty} u^{-1} \cos(\lambda u) \int_{-n}^{x} t^{r} \, dt \, du
$$

$$
= - \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} x^{i} \int_{0}^{x+n} (-u)^{r-i} \cos(\lambda u) \, du + \frac{x^{r+1} - (-n)^{r+1}}{r+1} \text{Ci}[\lambda(x+n)].
$$

(2.99)

If we now put

$$
I_{i} = \int_{0}^{x+n} (-u)^{i} \cos(\lambda u) \, du,
$$

it follows as in the proof of Theorem 2.5 that

$$
I_{0} = \lambda^{-1} \sin[\lambda(x+n)],
$$

$$
I_{1} = -\lambda^{-1} n \sin[\lambda(x+n)] - \lambda^{-2} \cos[\lambda(x+n)] + \lambda^{-2},
$$

$$
I_{i} = \lambda^{-1} (-n)^{i} \sin[\lambda(x+n)] - i \lambda^{-2} (-n)^{i-1} \cos[\lambda(x+n)] - i(i-1) \lambda^{-2} I_{i-2}
$$

for $i \geq 2$ and so

$$
\lim_{n \to \infty} I_{i} = L_{i}
$$

(2.100)

for $i = 0, 1, 2, \ldots$. It follows from equations (2.99) and (2.100) that

$$
\lim_{n \to \infty} \int_{-n}^{x} \text{Ci}_{1}[\lambda, (x-t)]t^{r} \, dt = -\frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} L_{r-i} x^{i}.
$$

(2.101)

Further with $K = \sup \{|\text{Ci}(\lambda x)| : \lambda x \geq 1\}$ and $n \geq \lambda^{-1} - x$ we have

$$
|\int_{-n-n}^{-n} \tau_{n}(t) \text{Ci}_{1}[\lambda, (x-t)]t^{r} \, dt| \leq K(n + n^{-n})^{r} n^{-n}
$$

and it follows that

$$
\lim_{n \to \infty} \int_{-n-n}^{-n} \tau_{n}(t) \text{Ci}_{1}[\lambda, (x-t)]t^{r} \, dt = 0.
$$

(2.102)
Equation (2.95) now follows for $\lambda > 0$ from (2.98), (2.101) and (2.102).

Now suppose that $\lambda < 0$. Equation (2.98) still holds but this time we have

\[
\int_{-n}^{x} \text{Ci}_{r}[\lambda, (x - t)]t^r \, dt = (\ln |\lambda| - c) \int_{-n}^{x} t^r \, dt + \int_{-n}^{x} t^r \int_{0}^{x-t} u^{-1}[\cos(\lambda u) - 1] \, du \, dt + \int_{-n}^{x} t^r \ln(x - t) \, dt
\]

\[= J_1 + J_2 + J_3, \quad (2.103)\]

where

\[J_1 = \frac{\ln |\lambda| - c}{r+1} [x^{r+1} - (-n)^{r+1}], \quad (2.104)\]

\[J_2 = \int_{0}^{x+n} u^{-1}[\cos(\lambda u) - 1] \int_{-n}^{x-u} t^r \, dt \, du
\]

\[= \frac{1}{r+1} \int_{0}^{x+n} u^{-1}[(x - u)^{r+1} - (-n)^{r+1}][\cos(\lambda u) - 1] \, du
\]

\[= \frac{1}{r+1} \int_{0}^{x+n} u^{-1}[(x - u)^{r+1} - x^{r+1}][\cos(\lambda u) - 1] \, du
\]

\[+ \frac{1}{r+1} [x^{r+1} - (-n)^{r+1}] \int_{0}^{x+n} u^{-1}[\cos(\lambda u) - 1] \, du
\]

\[= -\frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} x^i \int_{0}^{x+n} (-u)^{r-i} \cos(\lambda u) \, du +
\]

\[-\frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} x^i(-x - n)^{r-i+1} +
\]

\[+ \frac{1}{r+1} [x^{r+1} - (-n)^{r+1}] \int_{0}^{x+n} u^{-1}[\cos(\lambda u) - 1] \, du. \quad (2.105)\]

\[J_3 = \frac{1}{r+1} \int_{-n}^{x} \ln(x - t) \, d(t^{r+1} - x^{r+1})
\]

\[= \frac{\ln(x + n)}{r+1} [x^{r+1} - (-n)^{r+1}] + \frac{1}{r+1} \int_{-n}^{x} (x - t)^{-1}(t^{r+1} - x^{r+1}) \, dt
\]

\[= \frac{\ln(x + n)}{r+1} [x^{r+1} - (-n)^{r+1}] + \frac{1}{r+1} \int_{0}^{x+n} u^{-1}[(x - u)^{r+1} - x^{r+1}] \, du
\]

\[= \frac{\ln(x + n)}{r+1} [x^{r+1} - (-n)^{r+1}] +
\]

\[+ \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \frac{x^i(-x - n)^{r-i+1}}{r-i+1}. \quad (2.106)\]
It follows from equations (2.103), (2.104), (2.105) and (2.106) that

\[
\int_{-n}^{x} C_{i+}[\lambda, (x-t)] t^{r} dt = \frac{x^{r+1} - (-n)^{r+1}}{r+1} C_{i+}[\lambda, (x+n)] + \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} x^i \int_{0}^{x+n} (-u)^{r-i} \cos(\lambda u) du. \tag{2.107}
\]

It can be proved as above that

\[
N - \lim_{n \to \infty} \int_{-n}^{x} (-u)^{r-i} \cos(\lambda u) du = L_i \tag{2.108}
\]

for \(i = 0, 1, 2, \ldots\) and it follows from equations (2.107) and (2.108) that

\[
N - \lim_{n \to \infty} \int_{-n}^{x} C_{i+}[\lambda, (x-t)] t^{r} dt = -\frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} L_{r-i} \tag{2.109}
\]

Also as above, we can prove that

\[
\lim_{n \to \infty} \int_{-n}^{x} \tau_{n}(t) C_{i+}[\lambda, (x-t)] t^{r} dt = 0. \tag{2.110}
\]

Equation (2.95) now follows for \(\lambda < 0\) from equations (2.98), (2.99) and (2.100). Equations (2.94) and (2.95) follow immediately.

The proofs of the following corollaries follow easily.

**COROLLARY 2.14** If \(\lambda \neq 0\), then

\[
x_{r} \odot C_{i+}(\lambda, x) = \frac{(-1)^{r+1}}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} [L_{r-i} x^{i} - a_{r-i}(x, \lambda) x^{i}] + \frac{(-1)^{r+1}}{r+1} x^{r+1} C_{i+}(\lambda, x)
\]

for \(r = 0, 1, 2, \ldots\) In particular,

\[
H(-x) \odot C_{i+}(\lambda, x) = \lambda^{-1} \sin_{+}(\lambda x) - x C_{i+}(\lambda, x),
\]

\[
x_{-} \odot C_{i+}(\lambda, x) = \frac{1}{2} [\lambda^{-2} H(-x) + \lambda^{-2} \cos_{+}(\lambda x) - \lambda^{-1} x \sin_{+}(\lambda x)] + \frac{1}{2} x^{2} C_{i+}(\lambda, x).
\]
**COROLLARY 2.15** If $\lambda \neq 0$, then

\[ x^r \oplus [\cos(\lambda x)x_+^{-1}] = (c - \ln |\lambda|)x^r - \sum_{i=0}^{r-1} \binom{r}{i} L_{r-i-1}x^i \]

for $r = 0, 1, 2, \ldots$. In particular

\[ 1 \oplus [\cos(\lambda x)x_+^{-1}] = (c - \ln |\lambda|), \]
\[ x \oplus [\cos(\lambda x)x_+^{-1}] = (c - \ln |\lambda|)x. \]

**COROLLARY 2.16** If $\lambda \neq 0$, then

\[ x_+^r \oplus [\cos(\lambda x)x_+^{-1}] = (-1)^{r-1} \sum_{i=0}^{r-1} \binom{r}{i} [L_{r-i-1}x^i - a_{r-i-1}(x, \lambda)x_+^i] + (c - \ln |\lambda|)x^r \]

\[ + (c - \ln |\lambda|)x_+^r (\lambda x) + (-1)^{r-1}x^r \mathrm{Ci}_+(\lambda, x) \]

for $r = 0, 1, 2, \ldots$. In particular

\[ H(-x) \oplus [\cos(\lambda x)x_+^{-1}] = (c - \ln |\lambda|)H(-x) - \mathrm{Ci}_+(\lambda, x), \]
\[ x_- \oplus [\cos(\lambda x)x_+^{-1}] = (c - \ln |\lambda|)x_- + \lambda^{-1}\sin_+(\lambda x) + x \mathrm{Ci}_+(\lambda, x). \]

Theorem 2.9 and its corollary follow as above.

**THEOREM 2.9** If $\lambda \neq 0$, then

\[ x^r \oplus \mathrm{Ci}_-(\lambda, x) = -\frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i} L_{r-i}x^i \]

for $r = 0, 1, 2, \ldots$. In particular

\[ 1 \oplus \mathrm{Ci}_-(\lambda, x) = 0, \]
\[ x \oplus \mathrm{Ci}_-(\lambda, x) = \frac{1}{2} \lambda^{-2}. \]

**COROLLARY 2.17** If $\lambda \neq 0$, then

\[ x^r \oplus \mathrm{Ci}(\lambda x) = 0 \]

for $r = 0, 1, 2, \ldots$. 

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Chapter 3

THE SINE INTEGRAL AND THE NEUTRIX CONVOLUTION PRODUCT

The *sine integral* $\text{Si}(\lambda x)$, see Sneddon [31], can be defined on the real line for $\lambda \neq 0$ by

$$\text{Si}(\lambda x) = \int_0^{\lambda x} u^{-1} \sin u \, du = \int_0^x u^{-1} \sin(\lambda u) \, du. \quad (3.1)$$

The function $\text{Si}_+(\lambda, x)$ is defined by

$$\text{Si}_+ (\lambda, x) = H(x) \text{Si}(\lambda x) = \begin{cases} \int_0^x u^{-1} \sin(\lambda u) \, du, & x > 0, \\ 0, & x < 0 \end{cases} \quad (3.2)$$

and $\text{Si}_-(\lambda, x)$ is defined by

$$\text{Si}_- (\lambda, x) = H(-x) \text{Si}(\lambda x) = \begin{cases} -\int_x^0 u^{-1} \sin(\lambda u) \, du, & x < 0, \\ 0, & x > 0. \end{cases} \quad (3.3)$$

For future reference, we note that if we replace $x$ by $-x$ in equation (3.1) we see that

$$\text{Si}(\lambda(-x)) = \text{Si}((-\lambda)x) = \text{Si}(-\lambda x)$$

and so if we replace $x$ by $-x$ in equation (3.2) we get

$$\text{Si}_+(\lambda, (-x)) = H(-x) \text{Si}(\lambda(-x)) = H(-x) \text{Si}((-\lambda)x).$$
It follows that

$$S_{i+}(\lambda, (-x)) = S_{i-}((-\lambda), x)$$  \hspace{1cm} (3.4)$$

for all $\lambda \neq 0$. Similarly,

$$S_{i-}(\lambda, (-x)) = S_{i+}((-\lambda), x)$$  \hspace{1cm} (3.5)$$

for all $\lambda \neq 0$. The derivative of $S_i(\lambda x)$ is given by

$$[S_i(\lambda x)]' = \sin(\lambda x) x^{-1}$$  \hspace{1cm} (3.6)$$

and further,

$$[S_{i+}(\lambda, x)]' = \sin(\lambda x) x_{+}^{-1},$$  \hspace{1cm} (3.7)$$

$$[S_{i-}(\lambda, x)]' = -\sin(\lambda x) x_{-}^{-1}. \hspace{1cm} (3.8)$$

In the following results, which were proved in [4],

$$b_i(x, \lambda) = \int_0^x (-u)^i \sin(\lambda u) \, du$$

for $i = 0, 1, 2, \ldots$

$$S_{i+}(\lambda, x) * x_{+}^r = -\frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} b_{r-i}(x, \lambda) x_{+}^i + \frac{1}{r+1} x^{r+1} S_{i+}(\lambda, x), \hspace{1cm} (3.9)$$

$$[\sin(\lambda x) x_{+}^{-1}] * x_{+}^r = -\sum_{i=0}^{r-1} \binom{r}{i} b_{r-i-1}(x, \lambda) x_{+}^i + x^r S_{i+}(\lambda, x), \hspace{1cm} (3.10)$$

$$S_{i-}(\lambda, x) * x_{-}^r = \frac{(-1)^r}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^i b_{r-i}(x, \lambda) x_{-}^i + \frac{(-x)^{r+1}}{r+1} S_{i-}(\lambda, x), \hspace{1cm} (3.11)$$

$$[\sin(\lambda x) x_{-}^{-1}] * x_{-}^r = (-1)^{r-1} \sum_{i=0}^{r-1} \binom{r}{i} (-1)^i b_{r-i-1}(x, \lambda) x_{-}^i + (-x)^r S_{i-}(\lambda, x). \hspace{1cm} (3.12)$$
To prove the next theorem we now increase the set of negligible functions given above to include finite linear sums of the functions

\[ n^\mu \cos(\lambda n), \quad n^\mu \sin(\lambda n), \quad n^\mu \text{Si}[\lambda(\alpha + n)] \quad (\mu \neq 0). \]

**THEOREM 3.1** If \( \lambda \neq 0 \), then the neutrix convolution product \( x^r \otimes \text{Si}_+(\lambda, x) \) exists and

\[
x^r \otimes \text{Si}_+(\lambda, x) = \frac{-1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} L_{r-i} x^i
\]

for \( r = 0, 1, 2, \ldots \), where

\[
L_{2i} = (-1)^i (2i)! \lambda^{-2i-1}, \quad L_{2i+1} = 0
\]

for \( i = 0, 1, 2, \ldots \). In particular

\[
1 \otimes \text{Si}_+(\lambda, x) = -\lambda^{-1},
\]

\[
x \otimes \text{Si}_+(\lambda, x) = -\lambda^{-1} x.
\]

**PROOF.** We put \( (x_r)_n = x^n_\tau_n(x) \). Then the convolution product \( (x_r)_n \ast \text{Si}(\lambda x) \) exists by Definition 2.2 and if \( |x| < n \),

\[
(x_r)_n \ast \text{Si}_+(\lambda, x) = \int_{-n}^{x} \text{Si}_+(\lambda, (x-t)) t^r \, dt + \int_{-n}^{-n-n} \tau_n(t) \text{Si}_+(\lambda, (x-t)) t^r \, dt.
\]

We have

\[
\int_{-n}^{x} \text{Si}_+(\lambda, (x-t)) t^r \, dt = \int_{-n}^{x} t^r \int_{0}^{x-t} u^{-1} \sin(\lambda u) \, du \, dt
\]

\[
= \int_{0}^{x+n} u^{-1} \sin(\lambda u) \int_{-n}^{x-u} t^r \, dt \, du
\]

\[
= \frac{1}{r+1} \int_{0}^{x+n} u^{-1} \sin(\lambda u) [(x-u)^{r+1} - (-n)^{r+1}] \, du
\]

\[
= \frac{1}{r+1} \int_{0}^{x+n} u^{-1} \sin(\lambda u) \sum_{i=0}^{r+1} \binom{r+1}{i} x^i (-u)^{r+1-i} \, du +
\]

\[
- \frac{(-n)^{r+1}}{r+1} \int_{0}^{x+n} u^{-1} \sin(\lambda u) \, du
\]

\[
= \frac{x^{r+1}}{r+1} \text{Si}_+(\lambda, (x+n)) - \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} x^i \int_{0}^{x+n} (-u)^{r+1-i} \sin(\lambda u) \, du +
\]

\[
- \frac{(-n)^{r+1}}{r+1} \text{Si}_+(\lambda, (x+n)).
\]

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Putting
\[ I_i = \int_0^{x+n} (-u)^i \sin(\lambda u) \, du, \]
it follows easily that
\[ I_0 = -\lambda^{-1} \cos(\lambda(x+n)) + \lambda^{-1}, \quad I_1 = \lambda^{-1}(x+n) \cos(\lambda(x+n)) - \lambda^{-2} \sin(\lambda(x+n)) \]
and
\[ I_i = \begin{cases} (1)^{i-1} \lambda^{-1}(x+n)^i \cos(\lambda(x+n)) + (1)^i \lambda^{-2} i (x+n)^{i-1} \sin(\lambda(x+n)) \\ -\lambda^{-2} i (i-1) I_{i-2} \end{cases} \]
for \( i \geq 2. \) Thus
\[ N - \lim_{n \to \infty} I_i = L_i \tag{3.18} \]
for \( i = 0, 1, 2, \ldots \) and it follows from equations (3.17) and (3.18) that
\[ N - \lim_{n \to \infty} \int_{-n}^{x} S_i(\lambda, (x - t)) t^r \, dt = \frac{-1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} L_{r-i} x^i. \tag{3.19} \]
Further, with \( K = \sup\{|S_i(\lambda, x)| : \lambda x \geq 0\}, \) we have
\[ \left| \int_{-n-n^{-n}}^{x-n} \tau_n(t) S_i(\lambda, (x - t)) t^r \, dt \right| \leq K(n + n^{-n})^r n^{-n} \]
and it follows that
\[ \lim_{n \to \infty} \int_{-n-n^{-n}}^{x-n} \tau_n(t) S_i(\lambda, (x - t)) t^r \, dt = 0. \tag{3.20} \]
Equation (3.13) now follows from equations (3.16), (3.19) and (3.20). Equa­
tions (3.14) and (3.15) follow immediately.

**COROLLARY 3.1** If \( \lambda \neq 0, \) then
\[ x^n \odot S_i(\lambda, x) = \frac{(-1)^{r+1}}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} [L_{r-i} x^i - b_{r-i}(x, \lambda) x^i] + \frac{(-1)^{r+1}}{r+1} x^{r+1} S_i(\lambda, x) \tag{3.21} \]
for \( r = 0, 1, 2, \ldots \). In particular,

\[
H(-x) \oplus \text{Si}_+(\lambda, x) = -\lambda^{-1}H(-x) - \lambda^{-1}\cos_+ (\lambda x) - x \text{Si}_+(\lambda, x),
\]

(3.22)

\[
x_- \oplus \text{Si}_+(\lambda, x) = \frac{1}{2}[\lambda^{-1}x \cos_+ (\lambda x) + \lambda^{-2} \sin_+ (\lambda x)] - \lambda^{-1}x_- + \frac{1}{2} x^2 \text{Si}_+(\lambda, x).
\]

(3.23)

**Proof.** We have

\[
(-1)^r x_- \oplus \text{Si}_+(\lambda, x) = x^r \oplus \text{Si}_+(\lambda, x) - x^r \ast \text{Si}_+(\lambda, x)
\]

and equation (3.21) follows from equations (3.9) and (3.13). Equations (3.22) and (3.23) follow immediately.

**Corollary 3.2** If \( \lambda \neq 0 \), then

\[
x^r \oplus [\sin(\lambda x)x^i_+] = -\sum_{i=0}^{r-1} \binom{r}{i} L_{r-i-1} x^i
\]

(3.24)

for \( r = 0, 1, 2, \ldots \). In particular,

\[
1 \oplus [\sin(\lambda x)x^i_+] = 0,
\]

(3.25)

\[
x \oplus [\sin(\lambda x)x^i_+] = -\lambda^{-1}.
\]

(3.26)

**Proof.** Using equations (3.7), (2.43) and (3.13), we have

\[
[x^r \oplus \text{Si}_+(\lambda, x)]' = x^r \oplus [\sin(\lambda x)x^i_+]
\]

\[
= \frac{-1}{r+1} \sum_{i=0}^{r} \binom{r}{i} L_{r-i} x^i = -\sum_{i=0}^{r-1} \binom{r}{i} L_{r-i-1} x^i,
\]

giving equation (3.24). Equations (3.25) and (3.26) follow immediately.

**Corollary 3.3** If \( \lambda \neq 0 \), then

\[
x_- \oplus [\sin(\lambda x)x^i_+] = (-1)^{r-1} \sum_{i=0}^{r-1} \binom{r}{i} [L_{r-i-1} x^i - b_{r-i-1}(x, \lambda)x^i_+] + (-1)^{r-1} x^r \text{Si}_+(\lambda, x)
\]

(3.27)
for $r = 0, 1, 2, \ldots$. In particular,

$$H(-x) \boxdot [\sin(\lambda x)x_+^{r-1}] = -\text{Si}_+(\lambda, x),$$  \hspace{1cm} (3.28)

$$x_- \boxdot [\sin(\lambda x)x_+^{r-1}] = \lambda^{-1}H(-x) + \lambda^{-1}\cos_+(\lambda x) + x\text{Si}_+(\lambda, x).$$  \hspace{1cm} (3.29)

**Proof.** Using equations (2.43), (3.7) and (3.21) we have

$$[x^-_\oplus \text{Si}_+(\lambda, x)x_+^{r-1}]' = x^-_\oplus [\sin(\lambda x)x_+^{r-1}]$$

$$= (-1)^{r+1} \sum_{i=0}^{r} \left( \frac{r+1}{i} \right) [iL_{r-i}x_i - (-1)^{r-i}\sin(\lambda x)x_+^r +$$

$$-ib_{r-i}(x, \lambda)x_+^{r-1}] +$$

$$+(-1)^{r+1}x^r\text{Si}_+(\lambda, x) + \frac{(-1)^{r+1}}{r+1}\sin(\lambda x)x_+^r.$$

Noting that

$$\sum_{i=0}^{r} \left( \frac{r+1}{i} \right) (-1)^i + (-1)^{r+1} = (1 - 1)^{r+1} = 0,$$

and equation (3.27) follows. Equations (3.28) and (3.29) follow immediately.

**Theorem 3.2** If $\lambda \neq 0$, then

$$x^r \boxdot \text{Si}_-(\lambda, x) = \frac{1}{r+1} \sum_{i=0}^{r} \left( \frac{r+1}{i} \right) (-1)^{r-i}L_{r-i}x_i$$  \hspace{1cm} (3.30)

for $r = 0, 1, 2, \ldots$. In particular

$$1 \boxdot \text{Si}_-(\lambda, x) = \lambda^{-1},$$  \hspace{1cm} (3.31)

$$x \boxdot \text{Si}_-(\lambda, x) = \lambda^{-1}x.$$  \hspace{1cm} (3.32)

**Proof.** Replacing $\lambda$ by $-\lambda$ in equation (3.13) gives us

$$x^r \boxdot \text{Si}_+((-\lambda), x) = \frac{1}{r+1} \sum_{i=0}^{r} \left( \frac{r+1}{i} \right)L_{r-i}x_i$$

and equation (3.30) follows on replacing $x$ by $-x$ in this equation. Equations (3.31) and (3.32) follow immediately.
COROLLARY 3.4 If $\lambda \neq 0$, then

$$x^r \boxplus \text{Si}(\lambda x) = 0$$  \hspace{1cm} (3.33)

for $r = 0, 1, 2, \ldots$.

PROOF. We have

$$x^r \boxplus \text{Si}(\lambda x) = x^r \boxplus \text{Si}_+(\lambda, x) + x^r \boxplus \text{Si}_-(\lambda, x)$$

and equation (3.33) follows from equations (3.13) and (3.30).

COROLLARY 3.5 If $\lambda \neq 0$, then

$$x_+^r \boxplus \text{Si}_-(\lambda, x) = \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \left( (-1)^{r-i} L_{r-i} x^i - (-1)^i b_{r-i}(x, \lambda) x_i \right) +$$

$$+ \frac{x^{r+1}}{r+1} \text{Si}_-(\lambda, x)$$  \hspace{1cm} (3.34)

for $r = 0, 1, 2, \ldots$. In particular

$$H(x) \boxplus \text{Si}_-(\lambda, x) = \lambda^{-1} H(x) + \lambda^{-1} \cos_-(\lambda x) + x \text{Si}_-(\lambda, x),$$  \hspace{1cm} (3.35)

$$x_+ \boxplus \text{Si}_-(\lambda, x) = \frac{1}{2} [\lambda^{-1} x \cos_-(\lambda x) + \lambda^{-2} \sin_-(\lambda x)] + \lambda^{-1} x_+ +$$

$$+ \frac{1}{2} x^2 \text{Si}_-(\lambda, x).$$  \hspace{1cm} (3.36)

PROOF. We have

$$x_+^r \boxplus \text{Si}_-(\lambda, x) = x^r \boxplus \text{Si}_-(\lambda, x) - (-1)^r x^r \boxplus \text{Si}_-(\lambda, x)$$

and equation (3.34) follows from equations (3.11) and (3.30). Equations (3.35) and (3.36) follow immediately.

The following two corollaries follow similarly.
COROLLARY 3.6 If $\lambda \neq 0$ then

$$x^r_+ \odot \text{Si}(\lambda x) = \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \left[ (-1)^{r-i} L_{r-i} - b_{r-i}(x, \lambda) \right] x^i + \frac{x^{r+1}}{r+1} \text{Si}(\lambda x)$$

for $r = 0, 1, 2, \ldots$. In particular

$$H(x) \odot \text{Si}(\lambda x) = \lambda^{-1} \cos(\lambda x) + x \text{Si}(\lambda x), \quad (3.38)$$

$$x_+ \odot \text{Si}(\lambda x) = \frac{1}{2} \left[ \lambda^{-1} x \cos(\lambda x) + \lambda^{-2} \sin(\lambda x) + x^2 \text{Si}(\lambda x) \right]. \quad (3.39)$$

COROLLARY 3.7 If $\lambda \neq 0$, then

$$x^r_- \odot \text{Si}(\lambda x) = \frac{(-1)^{r+1}}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \left[ L_{r-i} - b_{r-i}(x, \lambda) \right] x^i + \frac{(-1)^{r+1}}{r+1} x^{r+1} \text{Si}(\lambda x)$$

for $r = 0, 1, 2, \ldots$. In particular

$$H(-x) \odot \text{Si}(\lambda x) = -\lambda^{-1} \cos(\lambda x) - x \text{Si}(\lambda x), \quad (3.41)$$

$$x_- \odot \text{Si}(\lambda x) = \frac{1}{2} \left[ \lambda^{-1} x \cos(\lambda x) + \lambda^{-2} \sin(\lambda x) + x^2 \text{Si}(\lambda x) \right]. \quad (3.42)$$

COROLLARY 3.8 If $\lambda \neq 0$, then

$$x^r \odot \left[ \sin(\lambda x)x_-^{-1} \right] = \sum_{i=0}^{r-1} \binom{r}{i} (-1)^{r-i} L_{r-i-1} x^i$$

for $r = 0, 1, 2, \ldots$. In particular

$$1 \odot \left[ \sin(\lambda x)x_-^{-1} \right] = 0, \quad (3.44)$$

$$x \odot \left[ \sin(\lambda x)x_-^{-1} \right] = -\lambda^{-1}. \quad (3.45)$$
PROOF. Using equations (3.8), (2.43) and (3.30), we have

\[
[x^r \odot \text{Si}_-(\lambda, x)]' = -x^r \odot [\sin(\lambda x)x^{-1}]
\]

\[
= \frac{1}{r + 1} \sum_{i=0}^{r} \binom{r + 1}{i} (-1)^{r-i} L_{r-i} i x^{i-1}
\]

\[
= \sum_{i=0}^{r-1} \binom{r}{i} (-1)^{r-i} L_{r-i-1} x^i
\]

and equation (3.43) follows. Equations (3.44) and (3.45) follow immediately.

COROLLARY 3.9 If \( \lambda \neq 0 \), then

\[
x^r \odot [\sin(\lambda x)x^{-1}] = 0
\]

for \( r = 0, 1, 2, \ldots \).

PROOF. Equation (3.46) follows immediately on using equations (3.43) and (3.24).

COROLLARY 3.10 If \( \lambda \neq 0 \), then

\[
x_+^r \odot [\sin(\lambda x)x^{-1}] = \sum_{i=0}^{r-1} \binom{r}{i} \left[\left((-1)^r L_{r-i-1} x^i\right) + (-1)^i b_{r-i-1}(x, \lambda)x^i\right] +
\]

\[-x^r \text{Si}_-(\lambda, x)
\]

for \( r = 0, 1, 2, \ldots \). In particular

\[
H(x) \odot [\sin(\lambda x)x^{-1}] = -\text{Si}_-(\lambda, x),
\]

\[
x_+ \odot [\sin(\lambda x)x^{-1}] = -\lambda^{-1}[H(x) + \cos(\lambda x)] - x \text{Si}_-(\lambda, x).
\]

PROOF. Equation (3.47) follows similarly on using equations (2.43) and (3.34). Equations (3.48) and (3.49) then follow immediately.
**COROLLARY 3.11** If \( \lambda \neq 0 \), then
\[
x_+^r \otimes [\sin(\lambda x)x^{-1}] = -\sum_{i=0}^{r-1} \binom{r}{i} [b_{r-i-1}(x, \lambda) + (-1)^{r-i}L_{r-i-1}]x^i + x^r \text{Si}(\lambda x)
\]  
(3.50)

for \( r = 0, 1, 2, \ldots \). In particular
\[
H(x) \otimes [\sin(\lambda x)x^{-1}] = \text{Si}(\lambda x), \quad (3.51)
\]
\[
x_+ \otimes [\sin(\lambda x)x^{-1}] = \lambda^{-1} \cos(\lambda x) + x \text{Si}(\lambda x). \quad (3.52)
\]

**PROOF.** Equation (3.50) follows similarly from equations (2.43) and (3.37).
Equations (3.51) and (3.52) follow immediately.

The proof of the final corollary follows similarly from equations (2.43) and (3.40).

**COROLLARY 3.12** If \( \lambda \neq 0 \), then
\[
x_-^r \otimes [\sin(\lambda x)x^{-1}] = (-1)^r \sum_{i=0}^{r-1} \binom{r}{i} [L_{r-i-1} - b_{r-i-1}(x, \lambda)]x^i + (-1)^r x^r \text{Si}(\lambda x)
\]  
(3.53)

for \( r = 0, 1, 2, \ldots \). In particular
\[
H(-x) \otimes [\sin(\lambda x)x^{-1}] = -\text{Si}(\lambda x),
\]
\[
x_- \otimes [\sin(\lambda x)x^{-1}] = \lambda^{-1} \cos(\lambda x) + x \text{Si}(\lambda x).
\]
Chapter 4

THE NON-COMMUTATIVE NEUTRIX PRODUCT OF DISTRIBUTIONS

The product of an arbitrary distribution by an ordinary infinitely differentiable function is defined as follows.

**DEFINITION 4.1** Let $f$ be a distribution and let $g$ be an infinitely differentiable function. Then the product $fg = gf$ is defined by

$$\langle fg, \varphi \rangle = \langle gf, \varphi \rangle = \langle f, g\varphi \rangle$$

for all $\varphi$ in $\mathcal{D}$.

It then follows easily by induction that

$$f^{(r)}g = \sum_{i=0}^{r} \binom{r}{i} (-1)^i [fg^{(i)}]^{(r-i)}$$

for $r = 1, 2, \ldots$.

This suggests the following extension of Definition 4.1, see for example [11].
DEFINITION 4.2 Let $f$ be the $r$-th derivative of a locally summable function $F$ in $L^p(a,b)$ and let $g^{(r)}$ be a locally summable function in $L^q(a,b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ on the interval $(a,b)$ is defined by

$$fg = gf = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left[ F^{(i)} \right]^{(r-i)}.$$ 

It should be noted that the product is not in general associative, see Schwartz [32].

For our next definition we let $\rho(x)$ be a fixed infinitely differentiable function in $\mathcal{D}$ having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x) \, dx = 1$.

The function $\delta_n$ is then defined by $\delta_n(x) = n \rho(nx)$ for $n = 1, 2, \ldots$. It follows that $\{\delta_n\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta$.

Now let $f$ be an arbitrary distribution in $\mathcal{D}'$ and define the function $f_n$ by

$$f_n(x) = f * \delta_n(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f$.

Fisher [11] generalized Definition 4.1 as follows:

DEFINITION 4.3 Let $f$ and $g$ be arbitrary distributions in $\mathcal{D}'$ and let

$$f_n(x) = (f * \delta_n)(x), \quad g_n = (g * \delta_n)(x).$$

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We say that the product $f \cdot g$ of $f$ and $g$ exist and is equal to the distribution $h$ on the interval $(a, b)$ if $\{f_n g_n\}$ is a regular sequence converging to $h$ on the open interval $(a, b)$.

The product which is defined above is clearly commutative if it exists. Later Fisher [13] gave the following non-commutative definition of the product:

**DEFINITION 4.4** Let $f$, $g$ be distributions in $\mathcal{D}'$ and $g_n(x) = (g * \delta_n)(x)$. We say that the product $f \cdot g$ of $f$ and $g$ exists and is equal to the distribution $h$ on the interval $(a, b)$ if

$$\lim_{n \to \infty} \langle f(x) g_n(x), \varphi(x) \rangle = \langle h(x), \varphi(x) \rangle$$

for all $\varphi$ in $\mathcal{D}$ with support contained in the interval $(a, b)$.

It was proved in [13] that if $fg$ exists by Definition 4.2 then it exists by Definition 4.4 and $fg = f \cdot g$.

Definition 4.4 was generalized by Fisher [14] with the following.

**DEFINITION 4.5** Let $f$ and $g$ be distributions in $\mathcal{D}'$ and $g_n(x) = (g * \delta_n)(x)$. We say that the neutrix product $f \circ g$ of $f$ and $g$ exists and is equal to the distribution $h$ on the interval $(a, b)$ if

$$\lim_{n \to \infty} N - \lim_{n \to \infty} \langle f(x) g_n(x), \varphi(x) \rangle = \langle h(x), \varphi(x) \rangle$$

for all functions $\varphi$ in $\mathcal{D}$ with support contained in the interval $(a, b)$, where $N$ is the neutrix, see van der Corput [5], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \quad \lambda > 0, \quad r = 1, 2, \ldots$$

and all functions which converge to zero in the normal sense as $n$ tends to infinity.
This definition of the neutrix product is in general non-commutative. It is obvious that if the product \( f \cdot g \) exists then the neutrix product \( f \circ g \) exists and \( f \cdot g = f \circ g \). The next theorem was proved in [13].

**THEOREM 4.1** Let \( f \) and \( g \) be distributions and suppose that the neutrix products \( f \circ g \) and \( f \circ g' \) exist on the interval \((a, b)\). Then the neutrix product \( f' \circ g \) exists and

\[
(f \circ g)' = f' \circ g + f \circ g'
\]
on the interval \((a, b)\).

The following theorem was proved in [23].

**THEOREM 4.2** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) and suppose that the neutrix product \( f \circ g^{(i)} \) (or \( f^{(i)} \circ g \)) exists on the interval \((a, b)\) for \( i = 0, 1, 2, \ldots, r \). Then the neutrix product \( f^{(r)} \circ g \) (or \( f \circ g^{(r)} \)) exists on the interval \((a, b)\) and

\[
f^{(r)} \circ g = \sum_{i=0}^{r} \binom{r}{i} (-1)^i [f \circ g^{(i)}]^{(r-i)},
\]
or

\[
f \circ g^{(r)} = \sum_{i=0}^{r} \binom{r}{i} (-1)^i [f^{(i)} \circ g]^{(r-i)}
\]
on the interval \((a, b)\).

The next theorem was proved in [18].

**THEOREM 4.3** The neutrix product \( x_+^{r-1/2} \circ x_+^{-r-1/2} \) exists and

\[
x_+^{r-1/2} \circ x_+^{-r-1/2} = x_+^{-1} + a_r \delta(x)
\] (4.1)

for \( r = 0, \pm 1, \pm 2, \ldots \), where

\[
a_0 = 2[\ln 2 - c(\rho)],
\]

\[
a_r = a_{-r} = 2\left[\ln 2 - c(\rho) - \sum_{i=1}^{r} \frac{1}{2i - 1}\right]
\]
for $r = 1, 2, \ldots$ and

$$c(\rho) = \int_0^1 \ln t \rho(t) \, dt.$$  

The next theorem is a generalization of Theorem 4.3 and was proved in [6].

**Theorem 4.4** The neutrix product $x_+^\lambda \circ x_+^{-\lambda - 1}$ exists and

$$x_+^\lambda \circ x_+^{-\lambda - 1} = x_+^{-1} + a_1(\lambda) \delta(x) \quad (4.2)$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$, where

$$a_1(\lambda) = \frac{1}{2} B(0, \lambda + 1) + \frac{1}{2} B(0, -\lambda) - 2c_1(\rho)$$

$$= -[\gamma + \frac{1}{2} \psi(-\lambda) + \frac{1}{2} \psi(\lambda + 1) + 2c(\rho)], \quad (4.3)$$

where

$$\psi(\lambda) = \frac{\Gamma'(\lambda)}{\Gamma(\lambda)}$$

and $\gamma$ denotes Euler’s constant.

We now generalize Theorem 4.4

**Theorem 4.5** The neutrix product $x_+^\lambda \circ x_+^{-\lambda - r}$ exists and

$$x_+^\lambda \circ x_+^{-\lambda - r} = x_+^{-r} + a_r(\lambda) \delta^{(r-1)}(x) \quad (4.4)$$

for $r = 1, 2, \ldots$ and $\lambda \neq 0, \pm 1, \pm 2, \ldots$, where

$$a_r(\lambda) = \frac{(-1)^r[\gamma + 2c(\rho) + \frac{1}{2} \psi(\lambda + 1) + \frac{1}{2} \psi(-\lambda - r + 1) - \phi(r - 1)]}{(r - 1)!} +$$

$$+ \frac{\Gamma(\lambda + 1)}{2\Gamma(\lambda + r)} \sum_{j=0}^{r-2} \binom{\lambda + r - 1}{j} \frac{(-1)^j}{r - j - 1}.$$
Proof. We first of all suppose that $-1 < A < 0$. We require to evaluate

$$\lim_{n \to \infty} \langle x_+^\lambda(x_+^{-\lambda-r})_n, \varphi(x) \rangle,$$

where

$$(x_+^{-\lambda-r})_n = x_+^{-\lambda-r} \ast \delta_n(x)$$

$$= \begin{cases} 
  (1)^{-1} \Gamma(\lambda + 1) \int_{-1/n}^{1/n} (x - t)^{-\lambda-1} \delta_n(r-1)(t) \, dt, & x > 1/n, \\
  (1)^{-1} \Gamma(\lambda + 1) \int_{-1/n}^x (x - t)^{-\lambda-1} \delta_n(r-1)(t) \, dt, & -1/n \leq x \leq 1/n, \\
  0, & x < -1/n.
\end{cases}$$

We note that since $x_+^\lambda \cdot x_+^{-\lambda-r} = x_+^{-r}$ on any closed interval not containing the origin, we need only consider $\varphi \in \mathcal{D}$ with supp $\varphi \subset [-1, 1]$. With such a $\varphi$, we have by Taylor’s Theorem

$$\varphi(x) = \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i + \frac{\varphi^{(r)}(\xi x)}{r!} x^r,$$

where $0 < \xi < 1$ and so

$$\langle x_+^\lambda(x_+^{-\lambda-r})_n, \varphi(x) \rangle = \int_0^1 x^\lambda(x_+^{-\lambda-r})_n \varphi(x) \, dx$$

$$= \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} \int_0^1 x^{\lambda+i}(x_+^{-\lambda-r})_n \, dx +$$

$$+ \frac{1}{r!} \int_0^1 x^\lambda[x^r(x_+^{-\lambda-r})_n] \varphi^{(r)}(\xi x) \, dx. \quad (4.5)$$

We have

$$\frac{(1)^{-1} \Gamma(\lambda + r)}{\Gamma(\lambda + 1)} \int_{-1}^1 x_+^\lambda(x_+^{-\lambda-r})_n x^i \, dx =$$

$$= \int_{1/n}^1 x^{\lambda+i} \int_{-1/n}^x (x - t)^{-\lambda-1} \delta_n(r-1)(t) \, dt \, dx +$$

$$+ \int_{1/n}^1 x^{\lambda+i} \int_{-1/n}^{1/n} (x - t)^{-\lambda-1} \delta_n(r-1)(t) \, dt \, dx$$

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\[
\begin{aligned}
&= \int_0^{1/n} \delta_n^{(r-1)}(t) \int_t^1 x^{\lambda+i}(x-t)^{-\lambda-1} \, dx \, dt + \\
&\quad + \int_{-1/n}^0 \delta_n^{(r-1)}(t) \int_0^1 x^{\lambda+i}(x-t)^{-\lambda-1} \, dx \, dt \\
&= n^{r-i-1} \int_0^1 \rho^{(r-1)}(u) \int_u^n x^{\lambda+i}(u-v)^{-\lambda-1} \, du \, dv + \\
&\quad -(-1)^r n^{r-i-1} \int_0^1 \rho^{(r-1)}(v) \int_0^n x^{\lambda+i}(u+v)^{-\lambda-1} \, du \, dv, \quad (4.6)
\end{aligned}
\]

where the substitutions \( nt = v \) and \( nx = u \) have been made in the first integral
and \( nt = -v \) and \( nx = u \) in the second integral.

We have

\[
\int_v^n x^{\lambda+i}(u-v)^{-\lambda-1} \, du = (-1)^r \int_0^n x^{\lambda+i}(u+v)^{-\lambda-1} \, du
\]

and it follows for the cases \( i = 0, 1, \ldots, r-2 \) that

\[
N - \lim_{n \to \infty} n^{r-i-1} \left[ \int_v^n x^{\lambda+i}(u-v)^{-\lambda-1} \, du - (-1)^r \int_0^n x^{\lambda+i}(u+v)^{-\lambda-1} \, du \right] = 
\]

\[
N - \lim_{n \to \infty} n^{r-i-1} \int_v^n x^{\lambda+i}[(u-v)^{-\lambda-1} - (-1)^r(u+v)^{-\lambda-1}] \, du
\]

\[
= \sum_{j=0}^{\infty} \binom{-\lambda-1}{j} [(-1)^j - (-1)^r] v^j \int_v^n x^{i-j-1} \, du
\]

\[
= \frac{2(-1)^r}{r-i-1} \binom{-\lambda-1}{r-1} v^{r-1}
\]

\[
= -\frac{2\Gamma(\lambda+r)}{(r-i-1)(r-1)!\Gamma(\lambda+1)} v^{r-1}.
\]

It follows that

\[
N - \lim_{n \to \infty} \int_{-1}^1 x_+^{\lambda}(x_+^{\lambda-r})_n x^i \, dx = -\frac{1}{r-i-1} \quad (4.7)
\]

for \( i = 0, 1, \ldots, r-2 \), since it is easily proved by induction that

\[
\int_0^1 v^r \rho^{(r)}(v) \, dv = \frac{1}{r}(-1)^r r!.
\]
When \( i = r - 1 \), we have on making the substitution \( u = v/y \)

\[
\int_v^n u^{\lambda+r-1}(u-v)^{-\lambda-1} du = v^{r-1} \int_{v/n}^1 y^{-r}(1-y)^{-\lambda-1} dy
\]

\[
= v^{r-1} \int_{v/n}^1 y^{-r} \left[ (1-y)^{-\lambda-1} - \sum_{j=0}^{r-1} \binom{-\lambda-1}{j} (-y)^j \right] dy + v^{r-1} \sum_{j=0}^{r-1} (-1)^j \binom{-\lambda-1}{j} \int_{v/n}^1 y^{j-r} dy
\]

\[
= v^{r-1} \int_{v/n}^1 y^{-r} \left[ (1-y)^{-\lambda-1} - \sum_{j=0}^{r-2} \binom{-\lambda-1}{j} \frac{1-(n/v)^{r-j-1}}{j-r+1} \right] + (-\frac{-\lambda-1}{r-1})(-v)^{r-1}(\ln v - \ln n).
\]

It follows that

\[
N \lim_{n \to \infty} \int_v^n u^{\lambda+r-1}(u-v)^{-\lambda-1} du = v^{r-1} B(-r+1, -\lambda) + \left(-\frac{-\lambda-1}{r-1}\right)(-v)^{r-1} \ln v
\]

see Fisher and Kuribayashi [19], and so

\[
N \lim_{n \to \infty} \int_0^1 \rho^{(r-1)}(v) \int_v^n u^{\lambda+r-1}(u-v)^{-\lambda-1} du dv =
\]

\[
= \frac{1}{2} (-1)^{r-1} (r-1)! B(-r+1, -\lambda) - \left(-\frac{-\lambda-1}{r-1}\right) (r-1)! \left[ \frac{1}{2} \phi(r-1) + c(\rho) \right]
\]

\[
= \frac{1}{2} (-1)^{r-1} (r-1)! B(-r+1, -\lambda) + \frac{(-1)^r \Gamma(\lambda+r)}{\Gamma(\lambda+1)} \left[ \frac{1}{2} \phi(r-1) + c(\rho) \right], \quad (4.8)
\]

since it is easily proved by induction that

\[
\int_0^1 v^r \ln v \rho^{(r)}(v) dv = \frac{1}{2} (-1)^r r! \phi(r) + (-1)^r r! c(\rho).
\]
Further, making the substitution \( u = v(y^{-1} - 1) \), we have

\[
\int_0^n u^{\lambda+r-1}(u+v)^{-\lambda-1} \, du = v^{r-1} \int_{v/(n+v)}^1 y^{-r}(1-y)^{\lambda+r-1} \, dy
\]

\[
= v^{r-1} \int_{v/(n+v)}^1 y^{-r} \left[ (1-y)^{\lambda+r-1} - \sum_{j=0}^{r-1} \binom{\lambda + r - 1}{j} (-y)^j \right] \, dy +
\]

\[
+ v^{r-1} \sum_{j=0}^{r-1} (-1)^j \binom{\lambda + r - 1}{j} \int_{v/(n+v)}^1 y^{j-r} \, dy
\]

\[
= v^{r-1} \int_{v/(n+v)}^1 y^{-r} \left[ (1-y)^{\lambda+r-1} - \sum_{j=0}^{r-1} \binom{\lambda + r - 1}{j} (-y)^j \right] \, dy +
\]

\[
+ v^{r-1} \sum_{j=0}^{r-2} (-1)^j \binom{\lambda + r - 1}{j} \frac{1 - (n/v + 1)^{r-j-1}}{j - r + 1} +
\]

\[- \left( \frac{\lambda + r - 1}{r - 1} \right) (-v)^{r-1} \left[ \ln v - \ln(n + v) \right].
\]

It follows that

\[
\lim_{n \to \infty} \int_0^n u^{\lambda+r-1}(u+v)^{-\lambda-1} \, du = v^{r-1} B(-r + 1, \lambda + r) +
\]

\[
+ v^{r-1} \sum_{j=0}^{r-2} \left( \frac{\lambda + r - 1}{j} \right) \frac{(-1)^j}{r - j - 1} + \left( \frac{\lambda + r - 1}{r - 1} \right) (-1)^r v^{r-1} \ln v
\]

and so

\[
\lim_{n \to \infty} \int_0^1 \rho^{(r-1)}(v) \int_0^n u^{\lambda+r-1}(u+v)^{-\lambda-1} \, du \, dv =
\]

\[
= \frac{1}{2} (-1)^{r-1}(r - 1)! B(-r + 1, \lambda + r) +
\]

\[
+ \frac{1}{2} (-1)^{r-1}(r - 1)! \sum_{j=0}^{r-2} \binom{\lambda + r - 1}{j} \frac{(-1)^j}{r - j - 1} +
\]

\[- \left( \frac{\lambda + r - 1}{r - 1} \right) (r - 1)! \left[ \frac{1}{2} \phi(r - 1) + c(\rho) \right]
\]

\[
= \frac{1}{2} (-1)^{r-1}(r - 1)! B(-r + 1, \lambda + r) +
\]

\[
+ \frac{1}{2} (-1)^{r-1}(r - 1)! \sum_{j=0}^{r-2} \binom{\lambda + r - 1}{j} \frac{(-1)^j}{r - j - 1} +
\]

\[- \frac{\Gamma(\lambda + r)}{\Gamma(\lambda + 1)} \frac{1}{2} \phi(r - 1) + c(\rho)],
\]

(4.9)
since it was proved in [19] that

\[ B(0, \mu) = -\gamma - \psi(\mu) \]  

(4.10)

for \( \mu \neq 0, \pm 1, \pm 2, \ldots \). It follows that

\[
B(-r + 1, \lambda + r) = \frac{(-1)^{r-1}\Gamma(\lambda + r)}{(r-1)!\Gamma(\lambda + 1)}[\phi(r-1) - \gamma - \psi(\lambda + 1)],
\]

\[
B(-r + 1, -\lambda) = \frac{(-1)^{r-1}\Gamma(-\lambda)}{(r-1)!\Gamma(-\lambda - r + 1)}[\phi(r-1) - \gamma - \psi(-\lambda - r + 1)]
\]

and so

\[
B(-r + 1, -\lambda) - (-1)^r B(-r + 1, \lambda + r) =
\]

\[
= \frac{\Gamma(\lambda + r)}{(r-1)!\Gamma(\lambda + 1)}[2\phi(r-1) - 2\gamma - \psi(\lambda + 1) - \psi(-\lambda - r + 1)]. \tag{4.11}
\]

It now follows from equations (4.6), (4.8), (4.9) and (4.11) that

\[
N \lim_{n \to \infty} \frac{\Gamma(\lambda + r)}{\Gamma(\lambda + 1)} \int_{-1}^{1} x^n x^{\lambda-r} x^{r-1} dx =
\]

\[
= \frac{1}{2}(r-1)!\left[ B(-r + 1, -\lambda) - (-1)^r B(-r + 1, \lambda + r) \right] +
\]

\[
- \frac{1}{2}(r-1)! \sum_{j=0}^{r-2} \binom{\lambda + r - 1}{j} \frac{(-1)^{r-j}}{r-j-1} +
\]

\[
- \frac{\Gamma(\lambda + r)}{\Gamma(\lambda + 1)}[\phi(r-1) + 2c(\rho)]
\]

\[
= \frac{(-1)^{r-1}(r-1)!\Gamma(\lambda + r)}{\Gamma(\lambda + 1)} b_r(\lambda), \tag{4.12}
\]

where

\[
b_r(\lambda) = \frac{(-1)^r[\gamma + 2c(\rho) + \frac{1}{2} \psi(\lambda + 1) + \frac{1}{2} \psi(-\lambda - r + 1)]}{(r-1)!} +
\]

\[
+ \frac{\Gamma(\lambda + 1)}{2\Gamma(\lambda + r)} \sum_{j=0}^{r-2} \binom{\lambda + r - 1}{j} \frac{(-1)^j}{r-j-1}.
\]

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It was proved in [12] that

\[
\langle x_{+}^{-r}, \varphi(x) \rangle = \int_{0}^{\infty} x^{-r} \left[ \varphi(x) - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!} x^i - \frac{\varphi^{(r-1)}(0)}{(r-1)!} x^{r-1} H(1-x) \right] dx + \frac{\phi(r-1)}{(r-1)!} \varphi^{(r-1)}(0)
\]

for all \( \varphi \) in \( \mathcal{D} \). In particular, if the support of \( \varphi \) is contained in the interval \([-1,1]\), we have

\[
\langle x_{+}^{-r}, \varphi(x) \rangle = \int_{0}^{1} x^{-r} \left[ \varphi(x) - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!} x^i \right] dx - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!} \int_{1}^{\infty} x^{-r+i} dx + \frac{\phi(r-1)}{(r-1)!} \varphi^{(r-1)}(0)
\]

\[
= \int_{0}^{1} x^{-r} \left[ \varphi(x) - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!} x^i \right] dx - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} + \frac{\phi(r-1)}{(r-1)!} \varphi^{(r-1)}(0) . \tag{4.13}
\]

Since the sequence of continuous functions \( \{x^r(x_{+}^{-\lambda-r})_n\} \) converges distributionally to \( x^{-\lambda} \) on the closed interval \([0,1]\), it follows on using equations (4.7), (4.12) and (4.13) that

\[
N \lim_{n \to \infty} (x_{+}^{\lambda}(x_{+}^{-\lambda-r})_n, \varphi(x)) = N \lim_{n \to \infty} \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} \int_{1}^{1} x_{+}^{\lambda+i}(x_{+}^{-\lambda-r})_n dx + \\
+ \lim_{n \to \infty} \frac{1}{r!} \int_{1}^{1} x_{+}^{\lambda}[x^r(x_{+}^{-\lambda-r})_n]\varphi^{(r)}(\xi x) dx
\]

\[
= \frac{1}{r!} \int_{0}^{1} \varphi^{(r)}(\xi x) dx - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} + (-1)^{r-1} b_{r}(\lambda) \varphi^{(r-1)}(0)
\]

\[
= \int_{0}^{1} x^{-r} \left[ \varphi(x) - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!} x^i \right] dx - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} + \frac{\phi(r-1)}{(r-1)!} \varphi^{(r-1)}(0) + \\
+ (-1)^{r-1} b_{r}(\lambda) \varphi^{(r-1)}(0)
\]

\[
= \langle x_{+}^{-r}, \varphi(x) \rangle + (-1)^{r-1} a_{r}(\lambda) \varphi^{(r-1)}(0),
\]

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giving equation (4.4) on the interval \([-1, 1]\) and hence on the real line when \(-1 < \lambda < 0\).

Now suppose that (4.4) holds when \(-k < \lambda < -k + 1\) and \(r = 1, 2, \ldots\), for some positive integer \(k\). This is true for \(k = 1\). Differentiating equation (4.4) with \(-k < \lambda < -k + 1\), we get

\[
\lambda x_+^{\lambda-1} \circ x_+^{\lambda-r} - (\lambda + r)x_+^{\lambda-1} = -rx_+^{r-1} + a_r(\lambda)\delta^{(r)}(x).
\]

It follows from our assumptions that

\[
\lambda x_+^{\lambda-1} \circ x_+^{\lambda-r} = \lambda x_+^{\lambda-1} + [a_r(\lambda) + a_r(\lambda)]\delta^{(r)}(x).
\]

We have

\[
(\lambda + r)a_{r+1}(\lambda) + a_r(\lambda) = \frac{(-1)^{r-1}(\lambda + r)}{r!} + \frac{\Gamma(\lambda + 1)}{2\Gamma(\lambda + r)} \sum_{j=0}^{r-1} \binom{\lambda + r}{j} \frac{(-1)^j}{r - j} + \frac{(-1)^r(\lambda + r)\phi(r)}{r!} + \frac{\Gamma(\lambda + 1)}{2\Gamma(\lambda + r)} \sum_{j=0}^{r-2} \binom{\lambda + r - 1}{j} \frac{(-1)^j}{r - j - 1} - \frac{(-1)^r\phi(r - 1)}{(r - 1)!}.
\]

Noting that

\[
(\lambda + r)\psi(-\lambda - r) = 1 + (\lambda + r)\psi(-\lambda - r + 1),
\]

\[
\lambda\psi(\lambda + 1) = 1 + \lambda\psi(\lambda),
\]

\[
\sum_{j=0}^{r-1} \binom{\lambda + r - 1}{j} \frac{(-1)^j}{r - j} = \sum_{j=0}^{r-1} \binom{\lambda + r}{j} \frac{(-1)^j}{r - j} + \sum_{j=0}^{r-2} \binom{\lambda + r - 1}{j} \frac{(-1)^j}{r - j - 1},
\]

it follows that

\[
(\lambda + r)a_{r+1}(\lambda) + a_r(\lambda) = \lambda a_{r+1}(\lambda - 1)
\]

and we see that equation (4.4) holds when \(-k - 1 < \lambda < -k\).
Equation (4.4) therefore holds by induction for negative $\lambda \neq -1, -2, \ldots$ and $r = 1, 2, \ldots$. A similar argument shows that equation (4.4) holds for positive $\lambda \neq 1, 2, \ldots$. This completes the proof of the theorem.

**COROLLARY 4.1** For $\lambda \neq 0, \pm 1, \pm 2, \ldots$ we have

$$x_+^\lambda \circ x_-^\lambda = x_-^{-r} - (-1)^r a_r(\lambda) \delta^{(r-1)}(x).$$  

**(4.14)**

**PROOF.** Equation (4.14) follows immediately on replacing $x$ by $-x$ in equation (4.4).

In the next corollary, the distribution $(x + i0)^\lambda$ is defined by

$$(x + i0)^\lambda = x_+^\lambda + e^{i\lambda \pi} x_-^\lambda$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and

$$(x + i0)^{-r} = x^{-r} + \frac{(-1)^r i \pi}{(r - 1)!} \delta^{(r-1)}(x)$$

**COROLLARY 4.2** For $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$ we have

$$(x + i0)^\lambda \circ (x + i0)^{-\lambda - r} = (x + i0)^{-r}. $$

**(4.16)**

**PROOF.** The neutrix product is distributive with respect to addition and so

$$(x + i0)^\lambda \circ (x + i0)^{-\lambda - r} = x_+^\lambda \circ x_+^{-\lambda - r} + (-1)^r x_-^\lambda \circ x_-^{-\lambda - r} +$$

$$+ (-1)^r e^{-i\lambda \pi} x_+^\lambda \circ x_-^{-\lambda - r} + e^{i\lambda \pi} x_-^\lambda \circ x_+^{-\lambda - r}.$$

Further, it was proved in [12] that

$$x_+^\lambda \circ x_-^{-\lambda - r} = (-1)^{r-1} x_-^\lambda \circ x_+^{-\lambda - r} = -\frac{\pi \cos \pi \lambda}{2(r - 1)!} \delta^{(r-1)}(x)$$

**(4.18)**
for $\lambda \neq 0, \pm 1, \pm 2, \ldots$. It follows from equations (4.4), (4.14), (4.15), (4.17) and (4.18) that

$$(x + i0)^\lambda \circ (x + i0)^{-\lambda - r} = x^{-r} + \frac{(-1)^r i\pi}{(r - 1)!} \delta^{(r-1)}(x) = (x + i0)^{-r},$$

proving equation (4.16).

We finally note that the following results can be proved similarly.

$$|x|^\lambda \circ (\text{sgn } x |x|^{-\lambda} - 2r + 1) = x^{-2r + 1},$$

(4.19)

$$(\text{sgn } x |x|^\lambda) \circ |x|^{-\lambda} - 2r + 1 = x^{-2r + 1},$$

(4.20)

$$|x|^\lambda \circ |x|^{-\lambda - 2r} = x^{-2r},$$

(4.21)

$$(\text{sgn } x |x|^\lambda) \circ (\text{sgn } x |x|^{-\lambda - 2r}) = x^{-2r}$$

(4.22)

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$.

To prove the next theorem, we need to prove the following three equations.

$$B(0, \lambda) = B(0, \lambda + 1) + \lambda^{-1},$$

(4.23)

$$B_{0,0}(0, \lambda) = B_{0,0}(0, \lambda + 1) + \lambda^{-1} B(0, \lambda + 1),$$

(4.24)

$$B_{0,1}(0, \lambda) = B_{0,1}(0, \lambda + 1) - \lambda^{-2}$$

(4.25)

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

To prove (4.23), we note that

$$t^{-1}(1 - t)^{\lambda - 1} = t^{-1}(1 - t)^{\lambda} + (1 - t)^{\lambda - 1}$$

and it follows that

$$B(0, \lambda) = N \lim_{n \to \infty} \int_{1/n}^{1-1/n} t^{-1}(1 - t)^{\lambda - 1} dt$$

$$= N \lim_{n \to \infty} \int_{1/n}^{1-1/n} t^{-1}(1 - t)^{\lambda} dt + N \lim_{n \to \infty} \int_{1/n}^{1-1/n} (1 - t)^{\lambda - 1} dt$$

$$= B(0, \lambda + 1) + \lambda^{-1},$$

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proving equation (4.23).

Similarly, we have

\[ B_{1,0}(0, \lambda) = N \lim_{n \to \infty} \int_{1/n}^{1-1/n} t^{-1} \ln t (1-t)^{\lambda-1} dt \]
\[ = N \lim_{n \to \infty} \int_{1/n}^{1-1/n} t^{-1} \ln t (1-t)^{\lambda} dt + N \lim_{n \to \infty} \int_{1/n}^{1-1/n} \ln t (1-t)^{\lambda-1} dt \]
\[ = B_{1,0}(0, \lambda + 1) - N \lim_{n \to \infty} \lambda^{-1} \left( \ln t (1-t)^{\lambda} \right)^{1-1/n} + \]
\[ + N \lim_{n \to \infty} \lambda^{-1} \int_{1/n}^{1-1/n} t^{-1} (1-t)^{\lambda} dt \]
\[ = B_{1,0}(0, \lambda + 1) + \lambda^{-1} B(0, \lambda + 1), \]

proving equation (4.24). Finally, we have

\[ B_{0,1}(0, \lambda) = N \lim_{n \to \infty} \int_{1/n}^{1-1/n} t^{-1} (1-t)^{\lambda-1} \ln(1-t) dt \]
\[ = N \lim_{n \to \infty} \int_{1/n}^{1-1/n} t^{-1} (1-t)^{\lambda} \ln(1-t) dt + \]
\[ + N \lim_{n \to \infty} \int_{1/n}^{1-1/n} (1-t)^{\lambda-1} \ln(1-t) dt \]
\[ = B_{0,1}(0, \lambda + 1) - N \lim_{n \to \infty} \lambda^{-1} \left( (1-t)^{\lambda} \ln(1-t) \right)^{1-1/n} + \]
\[ - N \lim_{n \to \infty} \lambda^{-1} \int_{1/n}^{1-1/n} (1-t)^{\lambda-1} dt \]
\[ = B_{0,1}(0, \lambda + 1) - \lambda^{-2} \]

proving equation (4.25).

We now prove the following theorem.

**THEOREM 4.6** For \( \lambda \neq 0, \pm 1, \pm 2, \ldots \), we have

\[ (x_+^\lambda \ln x_+) \circ x_+^{-\lambda-1} = x_+^{-1} \ln x_+ + a(\lambda) \delta(x), \] (4.26)
\[ = x_+^{-\lambda-1} \circ (x_+^\lambda \ln x_+), \] (4.27)

where

\[ a(\lambda) = \frac{1}{2} [B_{0,1}(0, \lambda + 1) - B_{1,0}(0, \lambda + 1) - B_{1,0}(0, -\lambda)] + \]
\[ + c_1(\rho) [B(0, -\lambda) + B(0, \lambda + 1)] - c_2(\rho) \]

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and
\[ c_1(\rho) = \int_0^1 \ln u \rho(u) \, du, \quad c_2(\rho) = \int_0^1 \ln^2 u \rho(u) \, du. \]

**Proof.** The proof of this theorem follows the same general strategy as that of Theorem 4.5. We first of all suppose that \(-1 < \lambda < 0\) and put
\[
(x_+^{\lambda-1})_n = x_+^{\lambda-1} \ast \delta_n(x)
\]
\[
= \begin{cases} 
    \int_{-1/n}^{1/n} (x-t)^{-\lambda-1} \delta_n(t) \, dt, & x > 1/n, \\
    \int_{-1/n}^x (x-t)^{-\lambda-1} \delta_n(t) \, dt, & -1/n \leq x \leq 1/n, \\
    0, & x < -1/n.
\end{cases}
\]

Then
\[
\int_{-1}^1 x^\lambda \ln x_+(x_+^{\lambda-1})_n \, dx = \int_0^{1/n} x^\lambda \ln x \int_{-1/n}^x (x-t)^{-\lambda-1} \delta_n(t) \, dt \, dx + \\
+ \int_{1/n}^1 x^\lambda \ln x \int_{-1/n}^{1/n} (x-t)^{-\lambda-1} \delta_n(t) \, dt \, dx \\
= \int_0^{1/n} \delta_n(t) \int_t^1 x^\lambda \ln x (x-t)^{-\lambda-1} \, dx \, dt + \\
+ \int_{-1/n}^0 \delta_n(t) \int_0^t x^\lambda \ln x (x-t)^{-\lambda-1} \, dx \, dt \\
= \int_0^1 \rho(v) \int_v^n u^\lambda \ln u (u-v)^{-\lambda-1} \, du \, dv + \\
- \ln n \int_0^1 \rho(v) \int_v^n u^\lambda (u-v)^{-\lambda-1} \, du \, dv + \\
+ \int_0^1 \rho(v) \int_0^v u^\lambda \ln u (u+v)^{-\lambda-1} \, du \, dv + \\
- \ln n \int_0^1 \rho(v) \int_0^v u^\lambda (u+v)^{-\lambda-1} \, du \, dv, \quad (4.28)
\]

where the substitutions \(nt = v\) and \(nx = u\) have been made in the first integral
and \(nt = -v\) and \(nx = u\) in the second integral.

Making the substitution \(u = v/y\), we have
\[
\int_v^n u^\lambda \ln u (u-v)^{-\lambda-1} \, du = \ln v \int_{v/n}^1 y^{-1} (1-y)^{-\lambda-1} \, dy + \\
- \int_{v/n}^1 y^{-1} \ln y (1-y)^{-\lambda-1} \, dy
\]
\[
= \ln v \int_{v/n}^{1} y^{-1}[(1 - y)^{-\lambda-1} - 1] dy - \ln v(\ln v - \ln n) + \\
- \int_{v/n}^{1} y^{-1} \ln y[(1 - y)^{-\lambda-1} - 1] dy + \frac{1}{2}(\ln v - \ln n)^2
\]

and it follows that

\[
N - \lim_{n \to \infty} \int_{v}^{n} u^\lambda \ln u(u - v)^{-\lambda-1} du = \ln v \int_{0}^{1} y^{-1}[(1 - y)^{-\lambda-1} - 1] dy + \\
- \ln^2 v - \int_{0}^{1} y^{-1} \ln y[(1 - y)^{-\lambda-1} - 1] dy + \frac{1}{2} \ln^2 v
\]

\[
= B(0, -\lambda) \ln v - B_{1,0}(0, -\lambda) - \frac{1}{2} \ln^2 v. 
\tag{4.29}
\]

Next, we have

\[
\int_{v}^{n} u^\lambda(u - v)^{-\lambda-1} du = \sum_{i=0}^{\infty} \left(\frac{-\lambda - 1}{i}\right)(-v)^i \int_{v}^{n} u^{-i-1} du
\]

\[
= \sum_{i=1}^{\infty} \left(\frac{-\lambda - 1}{i}\right) \frac{(-v)^i}{i}(v^{-i} - n^{-i}) + \ln n - \ln v
\]

and it follows that

\[
N - \lim_{n \to \infty} \ln n \int_{v}^{n} u^\lambda(u - v)^{-\lambda-1} du = 0. 
\tag{4.30}
\]

Further, making the substitution \( u = v(y^{-1} - 1) \), we have

\[
\int_{0}^{1} u^\lambda \ln u(u + v)^{-\lambda-1} du \\
= \ln v \int_{v/(n+v)}^{1} y^{-1}(1 - y)^\lambda dy - \int_{v/(n+v)}^{1} y^{-1} \ln y(1 - y)^\lambda dy + \\
+ \int_{v/(n+v)}^{1} y^{-1}(1 - y)^\lambda \ln(1 - y) dy
\]

\[
= \ln v \int_{v/(n+v)}^{1} y^{-1}[(1 - y)^\lambda - 1] dy - \ln v[\ln v - \ln(n + v)] + \\
- \int_{v/(n+v)}^{1} y^{-1} \ln y[(1 - y)^\lambda - 1] dy + \frac{1}{2} [\ln v - \ln(n + v)]^2 + \\
+ \int_{v/(n+v)}^{1} y^{-1}(1 - y)^\lambda \ln(1 - y) dy
\]

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and it follows that
\[
N \lim_{n \to \infty} \int_0^n u^\lambda \ln u(u + v)^{-\lambda - 1} \, du = \ln v \int_0^1 y^{-1}[(1 - y)^\lambda - 1] \, dy - \ln^2 v + 
- \int_0^1 y^{-1} \ln y[(1 - y)^\lambda - 1] \, dy + \frac{1}{2} \ln^2 v + 
+ \int_0^1 y^{-1}(1 - y)^\lambda \ln(1 - y) \, dy
\]
\[
= B(0, \lambda + 1) \ln v - B_{1,0}(0, \lambda + 1) + 
+ B_{0,1}(0, \lambda + 1) - \frac{1}{2} \ln^2 v.
\] (4.31)

Finally, with \( n > v \), we have
\[
\int_0^n u^\lambda(u + v)^{-\lambda - 1} \, du = \sum_{i=0}^{\infty} \binom{-\lambda - 1}{i} u^\lambda i \int_0^u u^\lambda i \, du + 
+ \sum_{i=0}^{\infty} \binom{-\lambda - 1}{i} v^i \int_v^n u^\lambda i \, du
\]
\[
= \sum_{i=0}^{\infty} \binom{-\lambda - 1}{i} (\lambda + i + 1)^{-1} + 
+ \sum_{i=1}^{\infty} \binom{-\lambda - 1}{i} \frac{v^i}{i} (v^{-i} - n^{-i}) + \ln n - \ln v
\]
and it follows that
\[
N \lim_{n \to \infty} \left[ \ln n \int_0^n u^\lambda(u + v)^{-\lambda - 1} \, du \right] = 0. 
\] (4.32)

It now follows from equations (4.28), (4.29), (4.30), (4.31) and (4.32) that
\[
N \lim_{n \to \infty} \int_{-1}^1 x_+^\lambda \ln x_+(x_+^{-\lambda - 1})_n \, dx = 
= \frac{1}{2} [B_{0,1}(0, \lambda + 1) - B_{1,0}(0, \lambda + 1) - B_{1,0}(0, -\lambda)] + 
+ c_1(\rho)[B(0, -\lambda) + B(0, \lambda + 1)] - c_2(\rho)
\]
\[
= a(\lambda). 
\] (4.33)

Now let \( \varphi \) be an arbitrary function in \( \mathcal{D} \) with support contained in the interval \([-1, 1]\). By the Mean Value Theorem
\[
\varphi(x) = \varphi(0) + x \varphi'(\xi x),
\]
where \(0 < \xi < 1\) and so

\[
\langle x^\lambda \ln x_+ (x_+^{-\lambda-1})_n, \varphi(x) \rangle = \int_0^1 x^\lambda \ln x (x_+^{-\lambda-1})_n \varphi(x) \, dx
\]

\[
= \varphi(0) \int_0^1 x^\lambda \ln x (x_+^{-\lambda-1})_n \, dx + \int_0^1 x^\lambda \ln x [x(x_+^{-\lambda-1})_n] \varphi'(\xi x) \, dx.
\]

Since the sequence of continuous functions \(\{x(x_+^{-\lambda-1})_n\}\) converges distributionally to \(x^{-\lambda}\) on the closed interval \([0,1]\), it follows on using equation (4.33) that

\[
N \lim_{n \to \infty} (x^\lambda \ln x_+ (x_+^{-\lambda-1})_n, \varphi(x)) = N \lim \varphi(0) \int_0^1 x^\lambda \ln x (x_+^{-\lambda-1})_n \, dx + \lim_{n \to \infty} \int_0^1 x^\lambda \ln x [x(x_+^{-\lambda-1})_n] \varphi'(\xi x) \, dx
\]

\[
= a(\lambda) \varphi(0) + \int_0^1 \ln x \varphi'(\xi x) \, dx
\]

\[
= a(\lambda) \varphi(0) + \int_0^1 x^{-1} \ln x [\varphi(x) - \varphi(0)] \, dx
\]

\[
= a(\lambda) \varphi(0) + \langle x_+^{-1} \ln x_+, \varphi(x) \rangle,
\]

giving equation (4.26) on the interval \([-1,1]\) and hence on the real line when \(-1 < \lambda < 0\) and \(r = 1, 2, \ldots,\).

Now suppose that equation (4.26) holds when \(-k < \lambda < -k + 1\). If \(-k - 1 < \lambda < -k\), then the product \((x_+^{k+1} \ln x_+)x_+^{-\lambda-1}\) exists by Definition 4.2 and

\[
(x_+^{k+1} \ln x_+)x_+^{-\lambda-1} = \ln x_+.
\]

By Theorem 4.1 we can differentiate this equation to get

\[
[(\lambda+1)x_+^\lambda \ln x_+ + x_+^\lambda] \circ x_+^{-\lambda-1} - (\lambda+1)(x_+^{k+1} \ln x_+) \circ x_+^{-\lambda} = x_+^{-1}
\]

and it follows from our assumption and equation (4.2) that

\[
(\lambda+1)(x_+^\lambda \ln x_+) \circ x_+^{-\lambda-1} = (\lambda+1)x_+^{-1} \ln x_+ + [(\lambda+1)a(\lambda+1) - a_1(\lambda)] \delta(x). \quad (4.34)
\]
Now

\[(\lambda + 1)a(\lambda + 1) - a'_1(\lambda) = \]

\[= \frac{1}{2}(\lambda + 1)[B_{0,1}(0, \lambda + 2) - B_{1,0}(0, \lambda + 2) - B_{1,0}(0, -\lambda - 1)] +
\]

\[+ (\lambda + 1)c_1(\rho)[B(0, -\lambda - 1) + B(0, \lambda + 2)] +
\]

\[- (\lambda + 1)c_2(\rho) + 2c_1(\rho) - \frac{1}{2}[B(0, -\lambda) + B(0, \lambda + 1)]
\]

\[= \frac{1}{2}(\lambda + 1)[B_{0,1}(0, \lambda + 1) - B_{1,0}(0, \lambda + 1) - B_{1,0}(0, -\lambda)] +
\]

\[+ (\lambda + 1)c_1(\rho)[B(0, -\lambda) + B(0, \lambda + 1)] - (\lambda + 1)c_2(\rho)
\]

\[= (\lambda + 1)a(\lambda),\]

on using equations (4.3), (4.23), (4.24) and (4.25). Equation (4.26) follows by induction for negative \(\lambda \neq -1, -2, \ldots\).

A similar proof shows that equation (4.26) holds for positive \(\lambda \neq 1, 2, \ldots\).

To prove equation (4.27), we differentiate equation (4.2) partially with respect to \(\lambda\). This gives

\[x_+^\lambda \circ (x_+^{-\lambda-1} \ln x_+) = (x_+^\lambda \ln x_+) \circ x_+^{-\lambda-1} - a'_1(\lambda)\delta(x)
\]

\[= x_+^{-1} \ln x_+ + \frac{1}{2}[B_{0,1}(0, -\lambda) - B_{1,0}(0, \lambda + 1) - B_{1,0}(0, -\lambda)]\delta(x)
\]

\[+ c_1(\rho)[B(0, -\lambda) + B(0, \lambda + 1)]\delta(x) - c_2(\rho)\delta(x), \quad (4.35)
\]

on using equations (4.3) and (4.26). Replacing \(\lambda\) by \(-\lambda - 1\) in equation (4.35) gives equation (4.27). This completes the proof of the theorem.

**COROLLARY 4.3** For \(\lambda \neq 0, \pm 1, \pm 2, \ldots\) we have

\[(x_+^\lambda \ln x_+) \circ x_+^{-\lambda-1} = x_+^{-\lambda-1} \circ x_+^\lambda \ln x_+ = x_+^{-1} \ln x_+ + a(\lambda)\delta(x). \quad (4.36)\]
**PROOF.** Equation (4.36) follows immediately on replacing \( x \) by \(-x\) in equations (4.26) and (4.27).

To prove the next theorem we need the following lemmas which are easily proved.

**LEMMA 4.1.** For \( r = 0, 1, 2, \ldots \) we have
\[
\int_{-1}^{1} x^r \rho^{(r)}(x) \, dx = \int_{-1}^{0} x^r \rho^{(r)}(x) \, dx = \frac{1}{2} (-1)^r r!.
\]

**LEMMA 4.2** For \( k = 0, 1, \pm 2, \pm 3, \ldots \) and \( p = 1, 2, \ldots \) we have
\[
\int x^k \ln^p |x| \, dx = \sum_{i=0}^{p} \frac{(-1)^i p!}{(k+1)^{i+1}(p-i)!} x^{k+1} \ln^{p-i} |x|.
\]

**LEMMA 4.3** Let \( \varphi \) be in \( \mathcal{D} \) with support contained in \([-1, 1]\). Then
\[
\langle x^{-2r+1} \ln^p |x|, \varphi(x) \rangle = \int_{-1}^{1} x^{-2r+1} \ln^p |x| \left[ \varphi(x) - \sum_{i=0}^{2r-2} \frac{\varphi^{(i)}(0)}{i!} x^i \right] \, dx + \frac{(-1)^{p+1} p!}{(2i)! (2r-2i-1)^{p+1}},
\]
\[
\langle x^{-2r} \ln^p |x|, \varphi(x) \rangle = \int_{-1}^{1} x^{-2r} \ln^p |x| \left[ \varphi(x) - \sum_{i=0}^{2r-1} \frac{\varphi^{(i)}(0)}{i!} x^i \right] \, dx + \frac{(-1)^{p+1} p!}{(2i)! (2r-2i-1)^{p+1}}
\]
for \( r = 1, 2, \ldots \) and \( p = 0, 1, 2, \ldots \).

The following neutrix products were proved in [9].
\[
|x|^\lambda \circ (\text{sgn} \, x |x|^{-\lambda-2r+1}) = x^{-2r+1}, \quad (4.37)
\]
\[
(\text{sgn} \, x |x|^{\lambda}) \circ |x|^{-\lambda-2r+1} = x^{-2r+1}, \quad (4.38)
\]
\[
|x|^\lambda \circ |x|^{-\lambda-2r} = x^{-2r}, \quad (4.39)
\]
\[
(\text{sgn} \, x |x|^{\lambda}) \circ (\text{sgn} \, x |x|^{-\lambda-2r}) = x^{-2r} \quad (4.40)
\]

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for \( \lambda \neq 0, \pm 1, \pm 2, \ldots \) and \( r = 1, 2, \ldots \).

We now give the following generalization of these equations.

**THEOREM 4.7** For \( \lambda \neq 0, \pm 1, \pm 2, \ldots \), \( r = 1, 2, \ldots \) and \( p, q = 0, 1, 2, \ldots \) we have

\[
(|x|^\lambda \ln^p |x|) \circ (\text{sgn} x |x|^{-\lambda-2r+1} \ln^q |x|) = x^{-2r+1} \ln^{p+q} |x|, \quad (4.41)
\]

\[
(\text{sgn} x |x|^\lambda \ln^p |x|) \circ (|x|^{-\lambda-2r+1} \ln^q |x|) = x^{-2r+1} \ln^{p+q} |x|, \quad (4.42)
\]

\[
(|x|^\lambda \ln^p |x|) \circ (|x|^{-\lambda-2r} \ln^q |x|) = x^{-2r} \ln^{p+q} |x|, \quad (4.43)
\]

\[
(\text{sgn} x |x|^\lambda \ln^p |x|) \circ (\text{sgn} x |x|^{-\lambda-2r} \ln^q |x|) = x^{-2r} \ln^{p+q} |x|. \quad (4.44)
\]

**PROOF.** We first of all prove equations (4.41) to (4.44) for the case \( q = 0 \) and suppose that \(-1 < \lambda - m < 0\) for some non-negative integer \( m \), and put

\[
(x_+^{-\lambda-r})_n = x_+^{-\lambda-r} \ast \delta_n(x)
\]

\[
= \begin{cases} 
\frac{(-1)^{r+m} \Gamma(\lambda - m)}{\Gamma(\lambda + r)} \int_{-1/n}^{x/n} (x - t)^{-\lambda+m} \delta_n^{(r+m)}(t) \, dt, & x > 1/n, \\
\frac{1}{\Gamma(\lambda + r)} \int_{-1/n}^{x} (x - t)^{-\lambda+m} \delta_n^{(r+m)}(t) \, dt, & |x| \leq 1/n, \\
0, & x < -1/n.
\end{cases} \quad (4.45)
\]

Then,

\[
\frac{(-1)^{r+m} \Gamma(\lambda + r)}{\Gamma(\lambda - m)} \int_{-1/n}^{1/n} |x|^\lambda \ln^p |x|(x_+^{-\lambda-r})_n x^i \, dx =
\]

\[
= \int_{0}^{1/n} x^{\lambda+i} \int_{-1/n}^{x} (x - t)^{-\lambda+m} \ln^p x \delta_n^{(r+m)}(t) \, dt \, dx + \int_{1/n}^{1} x^{\lambda+i} \int_{-1/n}^{x} (x - t)^{-\lambda+m} \ln^p x \delta_n^{(r+m)}(t) \, dt \, dx + (-1)^i \int_{1/n}^{0} |x|^{\lambda+i} \int_{-1/n}^{0} (x - t)^{-\lambda+m} \ln^p |x| \delta_n^{(r+m)}(t) \, dt \, dx
\]

\[
= \int_{0}^{1/n} \delta_n^{(r+m)}(t) \int_{t}^{1} x^{\lambda+i} (x - t)^{-\lambda+m} \ln^p x \, dx \, dt + \int_{1/n}^{0} \delta_n^{(r+m)}(t) \int_{-t}^{1} x^{\lambda+i} (x - t)^{-\lambda+m} \ln^p x \, dx \, dt +
\]

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+ \int_{-1/n}^{0} \delta_{n}^{r+m}(t) \int_{0}^{-t} x^{\lambda+i}(x-t)^{-\lambda+m} \ln^{p} x \, dx \, dt + \\
\quad + (-1)^i \int_{-1/n}^{0} \delta_{n}^{r+m}(t) \int_{t}^{0} |x|^{\lambda+i}(x-t)^{-\lambda+m} \ln^{p} |x| \, dx \, dt \\
= I_{1} + I_{2} + I_{3} + (-1)^i I_{4}. \quad (4.46)

Making the substitutions \(nt = v\) and \(nx = u\) in the first term of (4.46), we obtain

\[
I_{1} = n^{r-i-1} \int_{0}^{1} \rho^{r+m}(v) \int_{v}^{n} u^{\lambda+i}(\ln u - \ln n)^{p}(u-v)^{-\lambda+m} \, du \, dv \\
= \sum_{k=0}^{p} \sum_{j=0}^{\infty} \frac{p}{k} \left( -\lambda + m \right) \left( -1 \right)^{p-k-j} n^{r-i-1} \ln^{p-k} n \times \\
\quad \times \int_{0}^{1} v^{j} \rho^{r+m}(v) \int_{v}^{n} u^{m+i-j} \ln^{k} u \, du \, dv.
\]

It follows from Lemmas 4.1 and 4.2 that

\[
N - \lim_{n \to \infty} I_{1} = \frac{(-1)^{r+m+p} (r + m)}{(i - r + 1)^{p+1}} \left( -\lambda + m \right) \int_{0}^{1} v^{r+m} \rho^{r+m}(v) \, dv \\
= - \frac{(r + m)! p!}{2(r - i - 1)^{p+1}} \left( -\lambda + m \right) \quad (4.47)
\]

for \(i = 0, 1, 2, \ldots, r - 2\).

Similarly, replacing \(t\) by \(-t\) in \(I_{2}\), and making the above substitutions, we have

\[
I_{2} = (-1)^{r+m} n^{r-i-1} \int_{0}^{1} \rho^{r+m}(v) \int_{v}^{n} u^{\lambda+i}(\ln u - \ln n)^{p}(u+v)^{-\lambda+m} \, du \, dv
\]

and it follows that

\[
N - \lim_{n \to \infty} I_{2} = - \frac{(r + m)! p!}{2(r - i - 1)^{p+1}} \left( -\lambda + m \right) \quad (4.48)
\]

for \(i = 0, 1, 2, \ldots, r - 2\).

Further, with the above substitutions, we obtain

\[
I_{3} = (-1)^{r+m} \int_{0}^{1/n} \delta_{n}^{r+m}(t) \int_{0}^{t} x^{\lambda+i}(x+t)^{-\lambda+m} \ln^{p} x \, dx \, dt \\
= (-1)^{r+m} n^{r-i-1} \int_{0}^{1} \rho^{r+m}(v) \int_{0}^{v} u^{\lambda+i}(u+v)^{-\lambda+m} \ln^{p}(u/n) \, du \, dv
\]

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and it follows immediately that

$$\lim_{n \to \infty} I_3 = 0$$  \hspace{1cm} (4.49)

for \(i = 0, 1, 2, \ldots, r - 2\).

Similarly

$$\lim_{n \to \infty} I_4 = 0$$  \hspace{1cm} (4.50)

for \(i = 0, 1, 2, \ldots, r - 2\).

It now follows from equations (4.46) to (4.50) that

$$\lim_{n \to \infty} \int_{-1/n}^{1/n} |x|^\lambda \ln^p|x|(\text{sgn } x |x|^{-\lambda-2r+1})_n x^i \, dx = -\frac{p!}{(r-i-1)p+1},$$  \hspace{1cm} (4.51)

since

$$\binom{-\lambda+m}{r+m} = \frac{(-1)^r m \Gamma(\lambda+r)}{(r+m)! \Gamma(\lambda-m)}.$$

We now note that

$$\lim_{n \to \infty} \int_{-1}^{1} |x|^\lambda \ln^p|x|(\sgn x |x|^{-\lambda-2r+1})_n x^i \, dx =$$

$$= \lim_{n \to \infty} \int_{-1/n}^{1/n} |x|^\lambda \ln^p|x|[(x_{+}\lambda-r)_n - (x_{-}\lambda-r)_n] x^i \, dx$$

$$= \lim_{n \to \infty} 2 \int_{-1/n}^{1/n} |x|^\lambda \ln^p|x|(x_{+}\lambda-r)_n x^i \, dx$$

$$= -\frac{2p!}{(2r-i-2)p+1}$$  \hspace{1cm} (4.52)

on using equation (4.51), for \(i = 1, 3, 5, \ldots, 2r-3\), since the integrand is even and

$$\lim_{n \to \infty} \int_{-1}^{1} |x|^\lambda \ln^p|x|(\sgn x |x|^{-\lambda-2r+1})_n x^i \, dx = 0$$  \hspace{1cm} (4.53)

for \(i = 0, 2, 4, \ldots, 2r-2\), since the integrand is odd.

Now let \(\varphi\) be an arbitrary function in \(\mathcal{D}\) with support contained in the interval \([-1,1]\). By Taylor's Theorem, we have

$$\varphi(x) = \sum_{i=0}^{2r-2} \frac{\varphi^{(i)}(0)}{i!} x^i + \frac{\varphi^{(2r-1)}(\xi x)}{(2r-1)!} x^{2r-1},$$
where $0 < \xi < 1$. Then

$$
\langle |x|^\lambda \ln^p |x| (\text{sgn } x|x|^{-\lambda-2r+1})_n, \varphi(x) \rangle = 
$$

$$
= \int_{-1}^{1} |x|^\lambda \ln^p |x| (\text{sgn } x|x|^{-\lambda-2r+1})_n \varphi(x) \, dx 
$$

$$
= \sum_{i=0}^{2r-2} \frac{\varphi(i)(0)}{i!} \int_{-1}^{1} |x|^\lambda \ln^p |x| (\text{sgn } x|x|^{-\lambda-2r+1})_n x^i \, dx + 
$$

$$
\frac{1}{(2r - 1)!} \int_{-1}^{1} |x|^\lambda \ln^p |x| (\text{sgn } x|x|^{-\lambda-2r+1})_n x^{2r-1} \varphi^{(2r-1)}(\xi x) \, dx. 
$$

Since the sequence $\{|x|^\lambda x^{2r-1} (\text{sgn } x|x|^{-\lambda-2r+1})_n\}$ converges to 1 on the interval $[-1, 1]$, we have on using equations (4.52) and (4.53)

$$
N - \lim_{n \to \infty} (\langle |x|^\lambda \ln^p |x| (\text{sgn } x|x|^{-\lambda-2r+1})_n, \varphi(x) \rangle) = 
$$

$$
= N - \lim_{n \to \infty} \sum_{i=0}^{2r-2} \frac{\varphi(i)(0)}{i!} \int_{-1}^{1} |x|^\lambda \ln^p |x| (\text{sgn } x|x|^{-\lambda-2r+1})_n \, dx + 
$$

$$
+ \lim_{n \to \infty} \frac{1}{(2r - 1)!} \int_{-1}^{1} |x|^\lambda \ln^p |x| (\text{sgn } x|x|^{-\lambda-2r+1})_n x^{2r-1} \varphi^{(2r-1)}(\xi x) \, dx 
$$

$$
= \frac{1}{(2r - 1)!} \int_{-1}^{1} \ln^p |x| \varphi^{(2r-1)}(\xi x) \, dx - 2 \sum_{i=0}^{r-2} \frac{p! \varphi^{(i+1)}(0)}{(2r - 2i - 3)^{p+1}(2i + 1)!} 
$$

$$
= \int_{-1}^{1} x^{-2r+1} \ln^p |x| [\varphi(x) - \sum_{i=0}^{2r-2} \frac{\varphi(i)(0)x^i}{i!}] \, dx + 
$$

$$
-2 \sum_{i=0}^{r-2} \frac{p! \varphi^{(i+1)}(0)}{(2r - 2i - 3)^{p+1}(2i + 1)!} 
$$

$$
= \langle x^{-2r+1} \ln^p |x|, \varphi(x) \rangle. 
$$

It follows that

$$
\langle |x|^\lambda \ln^p |x| \rangle (\text{sgn } x|x|^{-\lambda-2r+1}) = x^{-2r+1} \ln^p |x| 
$$

(4.54)

for $\lambda > -1$, $\lambda \neq 0, 1, 2, \ldots$, $r = 1, 2, \ldots$ and $p = 0, 1, 2, \ldots$.

We now consider the product $\langle \text{sgn } x|x|^\lambda \ln^p |x| \rangle \circ |x|^{-\lambda-2r+1}$. It follows from equation (4.46) that

$$
\frac{(-1)^{r+m} \Gamma(\lambda + r)}{\Gamma(\lambda - m)} \int_{-1}^{1} \text{sgn } x|x|\ln^p |x|(x_{-}^{-\lambda-\tau})_n x^i \, dx = I_1 + I_2 + I_3 - (-1)^{i} I_4 
$$

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and so

\[ N \lim_{n \to \infty} \int_{-1/n}^{1} \text{sgn } x |x|^\lambda \ln^p |x| \left( x^{\lambda - r} \right) x^i \, dx = -\frac{p!}{(r - i - 1)^{p+1}}. \]

Then

\[ N \lim_{n \to \infty} \int_{-1}^{1} \text{sgn } x |x|^\lambda \ln^p |x| \left( |x|^{-\lambda - 2r + 1} \right) x^i \, dx = \frac{2p!}{(2r - i - 2)^{p+1}} \]

for \( i = 1, 3, 5, \ldots, 2r - 3 \) and

\[ N \lim_{n \to \infty} \int_{-1}^{1} \text{sgn } x |x|^\lambda \ln^p |x| \left( |x|^{-\lambda - 2r + 1} \right) x^i \, dx = 0 \]

for \( i = 0, 2, 4, \ldots, 2r - 2 \). Thus

\[ (\text{sgn } x |x|^\lambda \ln^p |x|) \circ |x|^{-\lambda - 2r + 1} = x^{-2r+1} \ln^p |x| \quad (4.55) \]

for \( \lambda > -1, \lambda \neq 0, 1, 2, \ldots, r = 1, 2, \ldots \) and \( p = 0, 1, 2, \ldots \).

Next, we consider the product \( (|x|^\lambda \ln^p |x|) \circ |x|^{-\lambda - 2r} \). It follows from above that

\[ \frac{(-1)^{r+m} \Gamma(\lambda + r)}{\Gamma(\lambda - m)} \int_{-1/n}^{1} |x|^\lambda \ln^p |x| \left( x^{\lambda - r} \right) x^i \, dx = I_1 + I_2 + I_3 + (-1)^i I_4 \]

and so

\[ N \lim_{n \to \infty} \int_{-1/n}^{1} |x|^\lambda \ln^p |x| \left( x^{\lambda - r} \right) x^i \, dx = -\frac{p!}{(r - i - 1)^{p+1}}. \]

Then

\[ N \lim_{n \to \infty} \int_{-1}^{1} |x|^\lambda \ln^p |x| \left( |x|^{-\lambda - 2r} \right) x^i \, dx = \frac{2p!}{(2r - i - 1)^{p+1}} \]

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for \( i = 0, 2, 4, \ldots, 2r - 2 \), since the integrand is even, and

\[
N - \lim_{n \to \infty} \int_{-1}^{1} |x|^\lambda \ln^p |x|(|x|^{-\lambda - 2r})_n x^i \, dx = 0
\]

for \( i = 1, 3, 5, \ldots, 2r - 1 \), since the integrand is odd. It now follows as above that

\[
N - \lim_{n \to \infty} \langle |x|^\lambda \ln^p |x|(|x|^{-\lambda - 2r})_n \rangle = \langle x^{-2r} \ln^p |x|, \varphi(x) \rangle
\]

for arbitrary \( \varphi \) in \( \mathcal{D} \) and so

\[
(|x|^\lambda \ln^p |x|) \circ |x|^{-\lambda - 2r} = x^{-2r} \ln^p |x|
\]  

(4.56)

for \( \lambda > -1, \lambda \neq 0, 1, 2, \ldots, r = 1, 2 \ldots \) and \( p = 0, 1, 2, \ldots \).

Finally, it follows similarly that

\[
(\text{sgn } x|x|^\lambda \ln^p |x|) \circ (\text{sgn } x|x|^{-\lambda - 2r}) = x^{-2r} \ln^p |x|
\]  

(4.57)

for \( \lambda > -1, \lambda \neq 0, 1, 2, \ldots, r = 1, 2, \ldots \) and \( p = 0, 1, 2, \ldots \).

Now suppose that (4.54) to (4.57) hold for \( -m < \lambda < -m + 1 \) and \( p = 0, 1, 2, \ldots \). This is true when \( m = 1 \). Also suppose that

\[
(|x|^\lambda \ln^k |x|) \circ (\text{sgn } x|x|^{-\lambda - 2r + 1}) = x^{-2r+1} \ln^k |x|,
\]  

(4.58)

\[
(\text{sgn } x|x|^\lambda \ln^k |x|) \circ |x|^{-\lambda - 2r + 1} = x^{-2r+1} \ln^k |x|,
\]  

(4.59)

\[
(|x|^\lambda \ln^k |x|) \circ |x|^{-\lambda - 2r} = x^{-2r} \ln^k |x|,
\]  

(4.60)

\[
(\text{sgn } x|x|^\lambda \ln^k |x|) \circ (\text{sgn } x|x|^{-\lambda - 2r}) = x^{-2r} \ln^k |x|
\]  

(4.61)

for \( -m - 1 < \lambda < -m \) and some \( k \). This is also true for all \( m \) when \( k = 0 \).

Then with \( -m - 1 < \lambda < -m \), we have

\[
(|x|^\lambda + 1 \ln^{k+1} |x|) \circ (\text{sgn } x|x|^{-\lambda - 2r}) = x^{-2r+1} \ln^{k+1} |x|
\]

for \( k = 0, 1, 2, \ldots \).
Differentiating this equation we get

\[
[(\lambda + 1) \text{sgn } x |x|^{\lambda} \ln^{k+1} |x| + (k + 1) \text{sgn } x |x|^{\lambda} \ln^{k} |x|] \circ (\text{sgn } x |x|^{-\lambda-2r}) + \\
-(\lambda + 2r)(|x|^{\lambda+1} \ln^{k+1} |x|) \circ |x|^{-\lambda-2r-1}
\]

\[
= (-2r + 1)x^{-2r} \ln^{k+1} |x| + (k + 1)x^{-2r} \ln^{k} |x|.
\]

Using our assumptions, Theorem 4.1 and equations (4.56) and (4.61), we see that the neutrix product \((\text{sgn } x |x|^{\lambda} \ln^{k+1} |x|) \circ (\text{sgn } x |x|^{-\lambda-2r})\) exists and

\[
(\text{sgn } x |x|^{\lambda} \ln^{k+1} |x|) \circ (\text{sgn } x |x|^{-\lambda-2r}) = x^{-2r} \ln^{k+1} |x|
\]

for \(k = 0, 1, 2, \ldots\). It follows by induction that equation (4.57) holds for \(-m-1 < \lambda < -m\) and \(p = 0, 1, 2, \ldots\). It then follows by induction that (4.57) holds for \(\lambda < -1, \lambda \neq -2, -3, \ldots\) and \(p = 0, 1, 2, \ldots\). Equations (4.54), (4.55) and (4.56) follow similarly for \(\lambda < -1, \lambda \neq -2, -3, \ldots\), and \(p = 0, 1, 2, \ldots\).

Finally, suppose that (4.41) holds for \(\lambda \neq 0, \pm 1, \pm 2, \ldots, p = 0, 1, 2, \ldots\) and some \(q\). This is true when \(q = 0\). Then differentiating equation (4.41) partially with respect to \(\lambda\), we get

\[
(|x|^\lambda \ln^{p+1} |x|) \circ (\text{sgn } x |x|^{-\lambda-2r+1} \ln^{p+1} |x|) + \\
-(|x|^\lambda \ln^{p} |x|) \circ (\text{sgn } x |x|^{-\lambda-2r+1} \ln^{p+1} |x|) = 0
\]

and so

\[
(|x|^\lambda \ln^{p} |x|) \circ (\text{sgn } x |x|^{-\lambda-2r+1} \ln^{q+1} |x|) = \\
= (|x|^\lambda \ln^{p+1} |x|) \circ (\text{sgn } x |x|^{-\lambda-2r+1} \ln^{q} |x|).
\]

Equation (4.41) now follows by induction for \(\lambda \neq 0, \pm 1, \pm 2, \ldots\) and \(p, q = 0, 1, 2, \ldots\).

Equations (4.42), (4.43) and (4.44) follow similarly for \(\lambda \neq 0, \pm 1, \pm 2, \ldots\) and \(p, q = 0, 1, 2, \ldots\). This completes the proof of the theorem.
**THEOREM 4.8** For \( \lambda \neq 0, \pm 1, \pm 2, \ldots \) and \( r = 1, 2, \ldots \) we have

\[
(x_+^\lambda \ln x_+) \circ x_-^{-\lambda-r} = -\frac{\pi \cosec(\pi \lambda)}{2(r-1)!}[2c(\rho) + \psi(\lambda + r) - \Gamma'(1)]\delta^{(r-1)}(x) \quad (4.62)
\]

where \( c(\rho) = \int_0^1 \ln t \rho(t) \, dt. \)

**PROOF.** We will first of all suppose that \( -1 < \lambda < 0 \). Then \( x_+^\lambda \ln x_+ \) and \( x_-^{-\lambda-1} \) are locally summable functions and

\[
x_-^{-\lambda-r} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + r)}(x_-^{-\lambda-1})^{(r-1)}.
\]

Thus

\[
(x_-^{-\lambda-r})_n = x_-^{-\lambda-r} \ast \delta_n(x) = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + r)} \int_x^{1/n} (t - x)^{-\lambda-1}\delta_n^{(r-1)}(t) \, dt
\]

for \( r = 1, 2, \ldots \) and so

\[
\frac{\Gamma(\lambda + r)}{\Gamma(\lambda + 1)} \int_{-\infty}^{\infty} x_+^\lambda \ln x_+ (x_-^{-\lambda-r})_n x^i \, dx = \]

\[
= \int_0^{1/n} x_+^{\lambda+i} \ln x \int_x^{1/n} (t - x)^{-\lambda-1}\delta_n^{(r-1)}(t) \, dt \, dx
\]

\[
= \int_0^{1/n} \delta_n^{(r-1)}(t) \int_0^1 x_+^{\lambda+i} \ln x(t - x)^{-\lambda-1} \, dx \, dt
\]

\[
= \int_0^{1/n} t^i \delta_n^{(r-1)}(t) \int_0^1 v^{\lambda+i} \ln(1-v)^{-\lambda-1} \, dv \, dt
\]

\[
= B(\lambda + i + 1, -\lambda) \int_0^{1/n} t^i \ln t \delta_n^{(r-1)}(t) \, dt +
\]

\[
+ B_{1,0}(\lambda + i + 1, -\lambda) \int_0^{1/n} t^i \delta_n^{(r-1)}(t) \, dt, \quad (4.63)
\]

where the substitution \( x = tv \) has been made, \( B \) denotes the Beta function and in general

\[
B_{p,q}(\lambda, \mu) = \frac{\partial^{p+q}}{\partial^p \lambda \partial^q \mu} B(\lambda, \mu).
\]
Making the substitution $nt = y$ we have

\[ \int_{0}^{1/n} t^i \delta_{n}^{(r-1)}(t) \, dt = n^{r-i-1} \int_{0}^{1} y^i \rho^{(r-1)}(y) \, dy, \]  
\[ \int_{0}^{1/n} t^i \ln t \delta_{n}^{(r-1)}(t) \, dt = -n^{r-i-1} \ln n \int_{0}^{1} y^i \rho^{(r-1)}(y) \, dy + 
+ n^{r-i-1} \int_{0}^{1} y^i \ln y \rho^{(r-1)}(y) \, dy \]  

(4.65)

for $i = 0, 1, 2, \ldots$.

In particular, when $i = r - 1$, it is easily proved by induction that

\[ \int_{0}^{1} y^{r-1} \rho^{(r-1)}(y) \, dy = \frac{1}{2} (-1)^{r-1} (r-1)!, \]  
\[ \int_{0}^{1} y^{r-1} \ln y \rho^{(r-1)}(y) \, dy = (-1)^{r-1} (r-1)! \left[ \frac{1}{2} \phi(r-1) + c(\rho) \right] \]  

(4.67)

for $r = 1, 2, \ldots$.

Further, putting

\[ K = -\frac{\Gamma(\lambda + 1)}{\lambda \Gamma(\lambda + r)} \sup_{x} \{ |\rho^{(r-1)}(x)| \} > 0, \]

we obtain

\[ |(x^{-\lambda-r})_n| = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + r)} \int_{nx}^{1} n^{\lambda+1} (u - nx)^{-\lambda-1} n^{r-1} \rho^{(r-1)}(u) \, du \]
\[ \leq -\lambda Kn^{\lambda+r} \int_{nx}^{1+nx} (u - nx)^{-\lambda-1} \, du \]
\[ = Kn^{\lambda+r} \]

and so when $i = r$, we obtain

\[ \left| \int_{-\infty}^{\infty} x^{\lambda} \ln x (x^{-\lambda-r})_n x^r \, dx \right| \leq \int_{0}^{1/n} |x^\lambda \ln x (x^{-\lambda-r})_n x^r| \, dx \]
\[ \leq Kn^{-1} \ln n. \]  

(4.68)

Now let $\varphi$ be an arbitrary function in $C$. Then by Taylor's Theorem, we have

\[ \varphi(x) = \sum_{i=0}^{r-1} \frac{x^i}{i!} \varphi^{(i)}(0) + \frac{x^r}{r!} \varphi^{(r)}(\xi x), \]  

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where $0 < \xi < 1$ and so

\[
\langle x_+^\lambda \ln x_+, (x_{-}^{-\lambda-r})_n \varphi(x) \rangle = \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} \int_{-\infty}^\infty x_+^\lambda \ln x_+ (x_{-}^{-\lambda-r})_n x^i \, dx + \\
+ \frac{1}{r!} \int_{-\infty}^\infty x_+^\lambda \ln x_+ (x_{-}^{-\lambda-r})_n x^r \varphi^{(r)}(\xi x) \, dx.
\]  

(4.69)

Since

\[
\left| \int_{-\infty}^\infty x_+^\lambda \ln x_+ (x_{-}^{-\lambda-r})_n x^r \varphi^{(r)}(\xi x) \, dx \right| \leq \sup_x \{|\varphi^{(r)}(x)|\} Kn^{-1} \ln n,
\]

it follows from equations (4.63) to (4.69) that

\[
N^{-1} \lim_{n \to \infty} \frac{\Gamma(\lambda + r)}{\Gamma(\lambda + 1)} \langle x_+^\lambda \ln x_+, (x_{-}^{-\lambda-r})_n \varphi(x) \rangle = \\
= (-1)^{r-1} B(\lambda + r, -\lambda) \left[ \frac{1}{2} \phi(\rho - 1) + c(\rho) \right] \varphi^{(r-1)}(0) + \\
+ \frac{1}{2} (-1)^{r-1} B_{1,0}(\lambda + r, -\lambda) \varphi^{(r-1)}(0).
\]  

(4.70)

Using the well-known result (see e.g. [1, Equation 6.3.1])

\[
\frac{\Gamma'(r)}{(r-1)!} = \phi(r - 1) + \Gamma'(1),
\]

(4.71)

it follows that

\[
\frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + r)} B_{1,0}(\lambda + r, -\lambda) = \frac{\Gamma(\lambda + 1) \Gamma(-\lambda)}{(r-1)!} \left[ \frac{\Gamma'(\lambda + r)}{\Gamma(\lambda + r)} - \frac{\Gamma'(r)}{(r-1)!} \right] \\
= -\frac{\pi \csc(\pi \lambda)}{(r-1)!} \left[ \psi(\lambda + r) - \phi(r - 1) - \Gamma'(1) \right],
\]

(4.72)

since (see e.g. [1, Equation 6.1.17])

\[
\Gamma(-\lambda) \Gamma(\lambda + 1) = -\pi \csc(\pi \lambda).
\]

Further,

\[
\frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + r)} B(\lambda + r, -\lambda) = \frac{\Gamma(\lambda + 1) \Gamma(-\lambda)}{(r-1)!} = -\frac{\pi \csc(\pi \lambda)}{(r-1)!},
\]

(4.73)

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and equation (4.62) now follows from equations (4.70), (4.72) and (4.73) for the case $-1 < \lambda < 0$.

Now let us suppose that equation (4.62) holds when $-k < \lambda < -k + 1$ and $r = 1, 2, \ldots$, where $k$ is a positive integer. This is true when $k = 1$. Thus if $-k - 1 < \lambda < -k$, it follows from our assumption that

$$x_+^{\lambda+1} \ln x_+ \circ x_-^{\lambda-1-r} = \frac{\pi \csc(\pi \lambda)}{2(r-1)!} [2c(\rho) + \psi(\lambda + 1 + r) - \Gamma'(1)] \delta^{(r-1)}(x),$$

for $r = 1, 2, \ldots$. It follows from Theorem 4.1 that

$$[(\lambda + 1)x_+^{\lambda} \ln x_+ + x_+^\lambda] \circ x_-^{\lambda-r-1} + (\lambda + r + 1)x_+^{\lambda+1} \ln x_+ \circ x_-^{\lambda-r} =$$

$$= \frac{\pi \csc(\pi \lambda)}{2(r-1)!} [2c(\rho) + \psi(\lambda + r + 1) - \Gamma'(1)] \delta^{(r)}(x) +$$

$$\frac{(\lambda + r + 1) \pi \csc(\pi \lambda)}{2r!} [2c(\rho) + \psi(\lambda + r + 2) - \Gamma'(1)] \delta^{(r)}(x).$$

Thus

$$(\lambda + 1)x_+^{\lambda} \ln x_+ \circ x_-^{\lambda-r-1} =$$

$$= \frac{(\lambda + 1) \pi \csc(\pi \lambda)}{2r!} [2c(\rho) + \psi(\lambda + r + 2) - \Gamma'(1)] \delta^{(r)}(x) +$$

$$\frac{\pi \csc(\pi \lambda)}{2(r-1)!} [r^{-1} + \psi(\lambda + r + 1) - \psi(\lambda + r + 2)] \delta^{(r)}(x)$$

$$= \frac{(\lambda + 1) \pi \csc(\pi \lambda)}{2r!} [2c(\rho) + \psi(\lambda + r + 1) - \Gamma'(1)] \delta^{(r)}(x),$$

since

$$\psi(\lambda + r + 2) - (\lambda + r + 1)^{-1} = \psi(\lambda + r + 1)$$

and so

$$r^{-1} + \psi(\lambda + r + 1) - \psi(\lambda + r + 2) = \frac{\lambda + 1}{r(\lambda + r + 1)}. $$

Equation (4.62) now follows by induction for $\lambda < 0$, $\lambda \neq -1, -2, \ldots$ and $r = 2, 3, \ldots$. 

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To cover the case \( r = 1 \), we note the product \((x_+^{\lambda+1} \ln x_+).x_-^{\lambda-1}\) exists by Definition 4.2 and
\[
(x_+^{\lambda+1} \ln x_+).x_-^{\lambda-1} = 0 \tag{4.74}
\]
for all \( \lambda \).

Let us suppose that equation (4.62) holds when \(-k < \lambda < -k+1\) and \(r = 1\), where \(k\) is a positive integer. This is true when \(k = 1\). Thus if \(-k-1 < \lambda < -k\), it follows from our assumption that
\[
(x_+^{\lambda+1} \ln x_+).x_-^{\lambda-2} = \frac{1}{2} \pi \csc(\pi \lambda)[2c(\rho) + \psi(\lambda + 2) - \Gamma'(1)]\delta(x).
\]

It follows from equation (4.74) and Theorem 4.1 that
\[
[(\lambda + 1)x_+^{\lambda} \ln x_+ + x_+^{\lambda}] \circ x_-^{\lambda-1} + (\lambda + 1)(x_+^{\lambda+1} \ln x_+) \circ x_-^{\lambda-2} = 0
\]
\[
= (\lambda + 1)(x_+^{\lambda} \ln x_+) \circ x_-^{\lambda-1} - \frac{1}{2} \pi \csc(\pi \lambda)\delta(x) +
+ \frac{1}{2}(\lambda + 1)\pi \csc(\pi \lambda)[2c(\rho) + \psi(\lambda + 2) - \Gamma'(1)]\delta(x)
\]
\[
= (\lambda + 1)(x_+^{\lambda} \ln x_+) \circ x_-^{\lambda-1} +
+ \frac{1}{2}(\lambda + 1)\pi \csc(\pi \lambda)[2c(\rho) + \psi(\lambda + 1) - \Gamma'(1)]\delta(x).
\]
Equation (4.62) now follows by induction for \(\lambda < 0\), \(\lambda \neq -1, -2, \ldots\) and \(r = 1\).

Now let us suppose that equation (4.62) holds when \(k-1 < \lambda < k\) and \(r = 1, 2, \ldots\), where \(k\) is a positive integer. This is true when \(k = 0\). Then for an arbitrary function \(\varphi\) in \(\mathcal{D}\) we have
\[
\langle(x_+^{\lambda+1} \ln x_+), (x_-^{\lambda-r-1})_n\varphi(x)\rangle = \langle(x_+^{\lambda} \ln x_+), (x_-^{\lambda-r-1})_n\chi(x)\rangle,
\]
where \(\chi(x) = x\varphi(x)\) is also in \(\mathcal{D}\). It follows from our assumption with \(k-1 < \lambda < k\) that
\[
\lim_{n \to \infty}(x_+^{\lambda} \ln x_+), (x_-^{\lambda-r-1})_n\chi(x)) =
\]
\[
= -\frac{(-1)^r \pi \csc(\pi \lambda)}{2r!}[2c(\rho) + \psi(\lambda + r + 1) - \Gamma'(1)]\chi^{(r)}(0)
\]
\[
= -\frac{(-1)^r \pi \csc(\pi \lambda)}{2(r-1)!}[2c(\rho) + \psi(\lambda + r + 1) - \Gamma'(1)]\varphi^{(r-1)}(0)
\]
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and so
\[
N \lim_{n \to \infty} (x_{+}^{\lambda+1} \ln x_{+}, (x_{-}^{-\lambda-r-1})_{n} \varphi(x)) = \\
= -\frac{(1)^{r} \pi \csc(\pi \lambda)}{2(r-1)!} [2c(\rho) + \psi(\lambda + r + 1) - \Gamma'(1)]\varphi^{(r-1)}(0).
\]
Equation (4.62) now follows by induction for \( \lambda > 0, \lambda \neq 1, 2, \ldots \) and \( r = 1, 2, \ldots \), completing the proof of the theorem.

**COROLLARY 4.4** For \( \lambda \neq 0, \pm 1, \pm 2, \ldots \) and \( r = 1, 2, \ldots \) we have
\[
(x_{+}^{\lambda} \ln x_{+}) \circ x_{+}^{-\lambda-r} = \frac{(1)^{r} \pi \csc(\pi \lambda)}{2(r-1)!} [2c(\rho) + \psi(\lambda + r) - \Gamma'(1)]\delta^{(r-1)}(x). \tag{4.75}
\]
**PROOF.** Equation (4.75) follows on replacing \( x \) by \(-x\) in equation (4.62).

**THEOREM 4.9** For \( \lambda \neq 0, \pm 1, \pm 2, \ldots \) and \( r = 1, 2, \ldots \) we have
\[
(x_{+}^{\lambda} \ln x_{+}) \circ x_{+}^{-\lambda-r} = -\frac{\pi \csc(\pi \lambda)}{2(r-1)!} [2c(\rho) + \psi(-\lambda - r + 1) - \Gamma'(1)]\delta^{(r-1)}(x). \tag{4.76}
\]
**PROOF.** Differentiating the equation
\[
x_{+}^{\lambda} \circ x_{+}^{-\lambda-r} = -\frac{\pi \csc(\pi \lambda)}{2(r-1)!} \delta^{(r-1)}(x),
\]
which was proved in [14] partially with respect to \( \lambda \) we get
\[
(x_{+}^{\lambda} \ln x_{+}) \circ x_{+}^{-\lambda-r} = (x_{-}^{\lambda} \ln x_{-}) = \frac{\pi^{2} \cot(\pi \lambda) \csc(\pi \lambda)}{2(r-1)!} \delta^{(r-1)}(x)
\]
and on using equation (4.62) it follows that
\[
x_{+}^{\lambda} \circ (x_{-}^{-\lambda-r} \ln x_{-}) = -\frac{\pi \csc(\pi \lambda)}{2(r-1)!} [\pi \cot(\pi \lambda) + 2c(\rho) + \psi(\lambda + r) - \Gamma'(1)]\delta^{(r-1)}(x). \tag{4.77}
\]
Taking logs and differentiating the identity
\[
\Gamma(-\lambda)\Gamma(\lambda + 1) = (-1)^{r-1} \Gamma(-\lambda - r + 1)\Gamma(\lambda + r) = -\pi \csc(\pi \lambda)
\]
gives
\[
-\psi(-\lambda - r + 1) + \psi(\lambda + r) = -\pi \cot(\pi \lambda) \tag{4.78}
\]
and equation (4.76) follows from equations (4.77) and (4.78).
COROLLARY 4.5 For $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$ we have

$$x_+^\lambda \circ (x_-^{\lambda-r} \ln x_+) =$$

$$= \frac{(-1)^r \pi \cosec(\pi \lambda)}{2(r - 1)!} [2c(\rho) + \psi(-\lambda - r + 1) - \Gamma'(1)] \delta^{(r-1)}(x). \quad (4.79)$$

PROOF. Equation (4.79) follows on replacing $x$ by $-x$ in equation (4.76).

We finally note that if we replace $\lambda$ by $-\lambda - r$ in equation (4.79), we get

$$x_-^{\lambda-r} \circ (x_+^\lambda \ln x_+) = -\frac{\pi \cosec(\pi \lambda)}{2(r - 1)!} [2c(\rho) + \psi(\lambda + 1) - \Gamma'(1)] \delta^{(r-1)}(x)$$

and we see that the product of the distributions $x_+^\lambda \ln x_+$ and $x_-^{\lambda-r}$ is commutative only when $r = 1.$
Chapter 5

THE COMPOSITION OF DISTRIBUTIONS

In the following the function $\delta_n(x)$ is defined as in Chapter 4 and $F$ is a distribution in $\mathcal{D}'$, $F_n(x) = (F * \delta_n)(x)$. We now define the composition of a distribution $F$ and a locally summable function $f$.

**DEFINITION 5.1** Let $F$ be in $\mathcal{D}'$ and let $f$ be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to the distribution $h(x)$ in $\mathcal{D}'$ on the interval $(a, b)$ if

$$\lim_{n \to \infty} \int_a^b F_n(f(x)) \varphi(x) \, dx = \langle h(x), \varphi(x) \rangle$$

for all $\varphi$ in $\mathcal{D}$ with support contained in the interval $(a, b)$, where $N$ is the neutrix defined in Chapter 4.

The following two theorems were proved in [15] and [16] respectively:

**THEOREM 5.1** The distributions $(x_+^\mu)_\lambda$ and $(x_+^\mu)_\lambda$ exists and

$$(x_+^\mu)_\lambda = (x_+^\mu)_\lambda = 0$$

for $\mu > 0$ and $\lambda \mu \neq -1, -2, \ldots$ and

$$(x_+^\mu)_\lambda = (-1)^{\lambda \mu} (x_+^\mu)_\lambda = \frac{\pi \cosec(\pi \lambda)}{2\mu(-\lambda \mu - 1)!} \delta(-\lambda \mu - 1)(x)$$

for $\mu > 0$, $\lambda \neq -1, -2, \ldots$ and $\lambda \mu = -1, -2, \ldots$.
THEOREM 5.2 The distribution $(x^2)^{s-1/2}$ exists and

$$(x^2)^{s-1/2} = x^{-2s-1} + \frac{2}{(2s)!} \ln 2 - c(\rho) \delta^{(2s)}(x)$$

for $s = 0, 1, 2, \ldots$.

Before proving the next theorem we note the following lemma which can be proved easily.

**LEMMA 5.1** Let $\varphi$ be a function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. Then

$$
\langle x^{-r}, \varphi(x) \rangle = \int_{0}^{1} x^{-r} \left[ \varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i \right] dx - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} + \frac{\varphi(r-1)}{(r-1)!} \varphi^{(r-1)}(0) \tag{5.1}
$$

for $r = 1, 2, \ldots$.

We now prove the following theorem.

**THEOREM 5.3** Let $F(x)$ denote the function $x^{-1/2}$. Then the distribution $F(x^{2r})$ exists and

$$
F(x^{2r}) = x^{-r} + \frac{(-1)^{r-1}}{r!} \left[ \ln 2 - c(\rho) + r\phi(r-1) \right] \delta^{(r-1)}(x) \tag{5.2}
$$

for $r = 1, 2, \ldots$.

**PROOF.** We have

$$
F_n(x) = \begin{cases} 
\int_{-1/n}^{1/n} (x-t)^{-1/2} \delta_n(t) dt, & x > 1/n, \\
\int_{-1/n}^{1/n} (x-t)^{-1/2} \delta_n(t) dt, & -1/n < x < 1/n, \\
0, & x < -1/n.
\end{cases}
$$
It follows that

\[ \int_{-1}^{1} F_n(x^2r)x^i \, dx = \int_{0}^{1} F_n(x^2r)x^i \, dx + \int_{-1}^{0} F_n(0)x^i \, dx \]

\[ = \int_{0}^{n^{-1/2r}} \int_{-1/n}^{1/n} (x^2r - t)^{-1/2} x^i \delta_n(t) \, dt \, dx + \]

\[ + \int_{n^{-1/2r}}^{1} \int_{-1/n}^{1/n} (x^2r - t)^{-1/2} x^i \delta_n(t) \, dt \, dx + \]

\[ + \int_{-1}^{0} \int_{-1/n}^{1/n} (-t)^{-1/2} x^i \delta_n(t) \, dt \, dx \]

\[ = \int_{0}^{1/n} \int_{t^{1/2r}}^{t} (x^2r - t)^{-1/2} x^i \, dt \, dx + \]

\[ + \int_{-1/n}^{0} \int_{t^{1/2r}}^{t} (x^2r - t)^{-1/2} x^i \, dt \, dx + \]

\[ + \frac{(-1)^i}{i + 1} \int_{-1/n}^{0} (-t)^{-1/2} \delta_n(t) \, dt \]

(5.3)

\[ = \int_{0}^{1/n} \delta_n(t) \int_{t^{1/2r}}^{t} [(x^2r - t)^{-1/2} + (x^2r + t)^{-1/2}] x^i \, dt \, dx + \]

\[ + \int_{-1/n}^{0} \delta_n(t) \int_{t^{1/2r}}^{t} (x^2r + t)^{-1/2} x^i \, dt \, dx + \]

\[ + \frac{(-1)^i}{i + 1} \int_{-1/n}^{1/n} t^{-1/2} \delta_n(t) \, dt \]

\[ = I_1 + I_2 + I_3. \]

Putting \( nt = u \) and \( n^{1/2r}x = v \), we have

\[ I_1 = n^{(r-i-1)/2r} \int_{0}^{1} \rho(u) \int_{u^{1/2r}}^{u^{1/2r}} (v^2r - u)^{-1/2} + (v^2r + u)^{-1/2} v^i \, dv \, du \]

\[ = 2n^{(r-i-1)/2r} \sum_{k=0}^{\infty} \left( \frac{-1/2}{2k} \right) \int_{0}^{1} u^{2k} \rho(u) \int_{u^{1/2r}}^{u^{1/2r}} v^{-4rk-r+i} \, dv \, du \]

\[ = 2n^{(r-i-1)/2r} \sum_{k=0}^{\infty} \left( \frac{-1/2}{2k} \right) \int_{0}^{1} \frac{n^{-4rk-r+i+1/2r} - u^{-(4rk-r+i+1)/2r}}{-4rk-r+i+1} u^{2k} \rho(u) \, du. \]

Since the non-zero powers \( n \) are negligible, it follows that

\[ \lim_{n \to \infty} I_1 = -\frac{2}{r-i-1} \int_{0}^{1} \rho(u) \, du = -\frac{1}{r-i-1} \]

(5.4)

for \( i = 0, 1, \ldots, r-2. \)
Next we have
\[
I_2 = n^{(r-1)/2r} \int_0^1 \rho(u) \int_0^{u^{1/2}} (v^{2r} + u)^{-1/2v^i} dv du
\]
and it follows that
\[
N - \lim_{n \to \infty} I_2 = 0 \quad (5.5)
\]
for \(i = 0, 1, \ldots, r - 2\).

Finally we have
\[
I_3 = \frac{(-1)^i n^{1/2}}{i+1} \int_0^1 u^{-1/2} \rho(u) du
\]
and it follows that
\[
N - \lim_{n \to \infty} I_3 = 0 \quad (5.6)
\]
for \(i = 0, 1, \ldots, r - 2\).

It now follows from equations (5.4), (5.5) and (5.6) that
\[
\int_{-1}^1 F_n(x^2) x^i dx = -\frac{1}{r-i-1} \quad (5.7)
\]
for \(i = 0, 1, \ldots, r - 2\).

We now consider the case \(i = r - 1\). We have
\[
\int_{t^{1/2r}}^1 (x^{2r} - t)^{-1/2x^{r-1}} dx = \frac{\ln[1 + (1-t)^{1/2}] - \ln t^{1/2}}{r}, \\
\int_0^1 (x^{2r} - t)^{-1/2x^{r-1}} dx = \frac{\ln[1 + (1-t)^{1/2}] - \ln |t|^{1/2}}{r}, \\
\int_{-1/n}^0 (-t)^{-1/2} \delta_n(t) dt = \int_{-1}^0 (-u)^{-1/2} \rho(u) du.
\]

It follows from equation (5.3) that
\[
\int_{-1}^1 F_n(x^2) x^{r-1} dx = \frac{1}{r} \int_{-1/n}^{1/n} \ln[1 + (1-t)^{1/2}] \delta_n(t) dt + \\
-\frac{1}{2r} \int_{-1/n}^{1/n} \ln |t| \delta_n(t) dt - \frac{(-1)^r n^{1/2}}{r} \int_{-1}^0 (-u)^{-1/2} du
\]
and so
\[ N \lim_{n \to \infty} \int_{-1}^{1} F_n(x^2r) x^{r-1} \, dx = \frac{\ln 2 - c(\rho)}{r}, \tag{5.8} \]

since
\[ \int_{-1/n}^{1/n} \ln |t| \delta_n(t) \, dt = \int_{-1}^{1} \ln |u| \rho(u) \, du - n \int_{-1}^{1} \rho(u) \, du. \]

When \( i = r \), we have
\[ \int_{0}^{n^{-1/2r}} |x^r F_n(x^2r)| \, dx = \int_{0}^{n^{-1/2r}} \int_{-1/n}^{1/n} x^r (x^2r - t)^{-1/2} \delta_n(t) \, dt \, dx \\
= n^{-1/2r} \int_{0}^{1} \int_{-1}^{n^2r} v^r (v^2r - u)^{-1/2} \rho(u) \, du \, dv \\
= O(n^{-1/2r}) \]

and it follows that if \( \psi \) is an arbitrary continuous function then
\[ \lim_{n \to \infty} \int_{0}^{n^{-1/2r}} x^r F_n(x^2r) \psi(x) \, dx = 0. \tag{5.9} \]

Further,
\[ \int_{-1}^{0} x^r F_n(x^2r) \psi(x) \, dx = \int_{-1}^{0} x^r \psi(x) \int_{-1/n}^{1/n} (-t)^{-1/2} \delta_n(t) \, dt \, dx \\
= n^{1/2} \int_{-1}^{0} x^r \psi(x) \int_{-1}^{0} (-u)^{-1/2} \rho(u) \, du \, dx \]

and it follows that
\[ N \lim_{n \to \infty} \int_{-1}^{0} x^r F_n(x^2r) \psi(x) \, dx = 0. \tag{5.10} \]

If now \( \eta \) is chosen so that \( n^{-1/2r} < \eta < 1 \), then we obtain
\[ \int_{n^{-1/2r}}^{0} |x^r F_n(x^2r)| \, dx = \int_{n^{-1/2r}}^{1/n} \int_{-1/n}^{1/n} x^r (x^2r - t)^{-1/2} \delta_n(t) \, dt \, dx \\
= \sum_{i=0}^{\infty} \left( -\frac{1}{2} \right)^i \int_{-1}^{1} \rho(u) \int_{n^{-1/2r}}^{1/n} (-u)^i \frac{(-u)^i}{n^i x^{2ri}} \, dx \, du \\
= \sum_{i=0}^{\infty} \left( -\frac{1}{2} \right)^i \int_{-1}^{1} \rho(u) \frac{(-u)^i (n^{-i} \eta^{1-2ri} - n^{-1/2r})}{1 - 2ri} \, du. \]
It follows that
\[
\lim_{n \to \infty} \int_{n^{-1/2r}}^\eta |x^r F_n(x_+^{2r})| \, dx = \eta
\]
and so if \( \psi \) is a continuous function then
\[
\lim_{n \to \infty} \int_{n^{-1/2r}}^\eta |x^r F_n(x_+^{2r})\psi(x)| \, dx = O(\eta). \tag{5.11}
\]

Now let \( \varphi \) be an arbitrary function in \( \mathcal{D} \) with support contained in the interval \([-1, 1]\). By Taylor’s Theorem we have
\[
\varphi(x) = \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i + \frac{\varphi^{(r)}(\xi x)}{r!} x^r,
\]
where \( 0 < \xi < 1 \). Then
\[
\langle F_n(x_+^{2r}), \varphi(x) \rangle = \int_{-1}^{1} F_n(x_+^{2r}) \varphi(x) \, dx = \\
\sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} \int_{-1}^{1} x^i F_n(x_+^{2r}) \, dx + \frac{1}{r!} \int_{-1}^{0} x^r F_n(x_+^{2r}) \varphi^{(r)}(\xi x) \, dx + \\
\frac{1}{r!} \int_{0}^{1} x^r F_n(x_+^{2r}) \varphi^{(r)}(\xi x) \, dx + \\
\frac{1}{r!} \int_{-1}^{n^{-1/2r}} x^r F_n(x_+^{2r}) \varphi^{(r)}(\xi x) \, dx + \frac{1}{r!} \int_{n^{-1/2r}}^{\eta} x^r F_n(x_+^{2r}) \varphi^{(r)}(\xi x) \, dx.
\]
Taking the neutrix limit as \( n \) tends to infinity, using equations (5.7), (5.8), (5.9), (5.10) and (5.11) and noting that the sequence \( \{x^r F_n(x_+^{2r})\} \) converges uniformly to 1 on the interval \([\eta, 1]\), it follows that
\[
N \lim_{n \to \infty} \langle F_n(x_+^{2r}), \varphi(x) \rangle = -\sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} + \frac{\ln 2 - c(\rho)}{r!} \varphi^{(r-1)}(0) + \\
\frac{1}{r!} \int_{\eta}^{1} \varphi^{(r)}(\xi x) \, dx + O(\eta)
\]

\[
= -\sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} + \frac{\ln 2 - c(\rho)}{r!} \varphi^{(r-1)}(0) + \\
\frac{1}{r!} \int_{0}^{1} \varphi^{(r)}(\xi x) \, dx,
\]
since \( \eta \) can be made arbitrarily small. Thus, on using Lemma 5.1, we have

\[
N_{n \to \infty} \langle F_n(x_+^r), \varphi(x) \rangle = \int_0^1 x^{-r} \left[ \varphi(x) - \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i \right] dx + \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} + \frac{\ln 2 - c(\rho)}{r!} \varphi^{(r-1)}(0)
\]

\[
= \langle x_+^{-r}, \varphi(x) \rangle + \frac{(-1)^{r-1} \ln 2 - c(\rho) + r \phi(r-1)}{r!} \langle \delta^{(r-1)}(x), \varphi(x) \rangle,
\]

proving equation (5.2) on the interval \([-1,1]\). However, since \( F(x_+^r) = x_+^{-r} \) on any closed interval not containing the origin, we have proved equation (5.2) on the real line.

**COROLLARY 5.1** The distribution \( F(x_+^r) \) exists and

\[
F(x_+^r) = x_+^{-r} + \ln 2 - c(\rho) + r \phi(r-1) \delta^{(r-1)}(x)
\]  

(5.12)

for \( r = 1, 2, \ldots \).

**PROOF.** Replacing \( x \) by \(-x\) in equation (5.2) we have

\[
F((-x)_+^r) = (-x)_+^{-r} + \frac{(-1)^{r-1} \ln 2 - c(\rho) + r \phi(r-1)}{r!} \delta^{(r-1)}(-x)
\]

and so

\[
F(x_+^r) = x_+^{-r} + \frac{\ln 2 - c(\rho) + r \phi(r-1)}{r!} \delta^{(r-1)}(x),
\]

proving equation (5.12).

**COROLLARY 5.2** Let \( G \) denote the function \( x_{-1/2}^{-1/2} \). Then the distribution \( G(-x_+^r) \) exists and

\[
G(-x_+^r) = x_+^{-r} + \frac{(-1)^{r-1} \ln 2 - c(\rho) + r \phi(r-1)}{r!} \delta^{(r-1)}(x)
\]

(5.13)

for \( r = 1, 2, \ldots \).
PROOF. Since $G(-x) = F(x)$, we have

\[ G(-x^{2r}) = F(x^{2r}) = x^{-r} + \frac{(-1)^{r-1} \ln 2 - c(\rho) + r\phi(r-1)}{r!} \delta^{(r-1)}(x), \]

from equation (5.2), proving equation (5.13).

COROLLARY 5.3 The distribution $G(-x^{2r})$ exists and

\[ G(-x^{2r}) = x^{-r} + \frac{\ln 2 - c(\rho) + r\phi(r-1)}{r!} \delta^{(r-1)}(x) \quad \text{(5.14)} \]

for $r = 1, 2, \ldots$.

PROOF. Replacing $x$ by $-x$ in equation (5.13) we obtain

\[ G(-(-x)^{2r}) = (-x)^{-r} + \frac{(-1)^{r-1} \ln 2 - c(\rho) + r\phi(r-1)}{r!} \delta^{(r-1)}(-x) \]

and so

\[ G(-x^{2r}) = x^{-r} + \frac{\ln 2 - c(\rho) + r\phi(r-1)}{r!} \delta^{(r-1)}(x), \]

proving equation (5.14).

THEOREM 5.4 Let $F_r(x)$ denote the function $|x|^{2r/\lambda}$ where $0 < \lambda < 1$.

Then the distribution $F_r(|x|^{-\lambda})$ exists and

\[ F_r(|x|^{-\lambda}) = x^{-2r} \quad \text{(5.15)} \]

for $r = 1, 2, \ldots$.

PROOF. We have

\[ F_{r,n}(x) = \begin{cases} 
\int_{-1/n}^{1/n} (x-t)^{2r/\lambda} \delta_n(t) \, dt, & x > 1/n, \\
\int_{-1/n}^{x} (x-t)^{2r/\lambda} \delta_n(t) \, dt + \int_{1/n}^{1} (t-x)^{2r/\lambda} \delta_n(t) \, dt, & |x| \leq 1/n, \\
\int_{-1/n}^{1/n} (t-x)^{2r/\lambda} \delta_n(t) \, dt, & x < -1/n. 
\end{cases} \]

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Since $F_{r,n}(|x|^{-\lambda})$ is an even function, it follows that

$$\int_{-1}^{1} F_{r,n}(|x|^{-\lambda})x^{i} dx = \begin{cases} 0, & i \text{ odd}, \\ 2 \int_{0}^{1} F_{r,n}(|x|^{-\lambda})x^{i} dx, & i \text{ even}. \end{cases}$$

Also

$$\int_{0}^{1} F_{r,n}(|x|^{-\lambda})x^{i} dx = \int_{0}^{n^{1/\lambda}} \int_{-1/n}^{x^{-\lambda}} x^{i}(x^{-\lambda} - t)^{2r/\lambda}\delta_{n}(t) dt dx +$$

$$+ \int_{0}^{n^{1/\lambda}} \int_{x^{-\lambda}}^{1/n} x^{i}(t - x^{-\lambda})^{2r/\lambda}\delta_{n}(t) dt dx +$$

$$+ \int_{n^{1/\lambda}}^{1} \int_{-1/n}^{1/n} x^{i}(x^{-\lambda} - t)^{2r/\lambda}\delta_{n}(t) dt dx$$

$$= I_{1} + I_{2} + I_{3}. \quad (5.16)$$

Putting $nt = u$ and $n^{-1/\lambda}x = v$, we have

$$I_{1} = n^{(i+1-2r)/\lambda} \int_{0}^{1} \int_{-1}^{v^{-\lambda}} v^{i}(v^{-\lambda} - u)^{2r/\lambda}\rho(u) du dv$$

and it follows that

$$N \lim_{n \to \infty} I_{1} = 0 \quad (5.17)$$

for $i = 0, 2, \ldots, 2r - 2$.

Next we have

$$I_{2} = n^{(i+1-2r)/\lambda} \int_{0}^{1} \int_{v^{-\lambda}}^{1} v^{i}(u - v^{-\lambda})^{2r/\lambda}\rho(u) du dv$$

and it follows that

$$N \lim_{n \to \infty} I_{2} = 0 \quad (5.18)$$

for $i = 0, 2, \ldots, 2r - 2$.

Finally we have

$$I_{3} = n^{(i+1-2r)/\lambda} \int_{1}^{n^{-1/\lambda}} \int_{-1}^{1} v^{i}(v^{-\lambda} - u)^{2r/\lambda}\rho(u) du dv$$

$$= n^{(i+1-2r)/\lambda} \int_{1}^{1} \rho(u) \int_{1}^{n^{-1/\lambda}} \sum_{k=0}^{\infty} \binom{2r/\lambda}{k} (-u)^{k} v^{-2r+\lambda k+i} dv du$$

$$= \sum_{k=0}^{\infty} \binom{2r/\lambda}{k} \frac{1}{-2r+\lambda k+i+1} \left[ n^{-k} - n^{(i+1-2r)/\lambda} \right] \int_{-1}^{1} (-u)^{k} \rho(u) du$$

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and it follows that
\[ N - \lim_{n \to \infty} I_3 = \frac{-1}{2r - i - 1} \quad (5.19) \]
for \( i = 0, 2, \ldots, 2r - 2 \). It now follows from equations (5.16), (5.17), (5.18) and (5.19) that
\[ \int_{-1}^{1} F_{r,n,}(|x|^{-\lambda}) x^{2i} \, dx = \frac{-2}{2r - 2i - 1} \quad (5.20) \]
for \( i = 0, 1, \ldots, r - 1 \).

When \( i = 2r \) we have
\[
\int_{0}^{n^{1/\lambda}} x^{2r} F_{r,n,}(|x|^{-\lambda}) \, dx = \int_{0}^{n^{1/\lambda}} \int_{-1/n}^{x^{-\lambda}} x^{2r} (x^{-\lambda} - t)^{2r/\lambda} \delta_n(t) \, dt \, dx + \\
\quad + \int_{0}^{n^{1/\lambda}} \int_{x^{-\lambda}}^{1/n} x^{2r} (t - x^{-\lambda})^{2r/\lambda} \delta_n(t) \, dt \, dx = n^{1/\lambda} \int_{0}^{1} \int_{-1}^{v^{-\lambda}} v^{2r} (v^{-\lambda} - u)^{2r/\lambda} \rho(u) \, du \, dv + \\
\quad + n^{1/\lambda} \int_{0}^{1} \int_{v^{-\lambda}}^{1} v^{2r} (u - v^{-\lambda})^{2r/\lambda} \rho(u) \, du \, dv = O(n^{1/\lambda})
\]
and it follows that if \( \psi \) is an arbitrary continuous function then
\[ N - \lim_{n \to \infty} \int_{0}^{n^{1/\lambda}} x^{2r} F_{r,n,}(|x|^{-\lambda}) \psi(x) \, dx = 0. \quad (5.21) \]

Similarly
\[ N - \lim_{n \to \infty} \int_{-n^{1/\lambda}}^{0} x^{2r} F_{r,n,}(|x|^{-\lambda}) \psi(x) \, dx = 0. \quad (5.22) \]

If now \( \eta \) is chosen so that \( n^{1/\lambda} < \eta < 1 \), we have
\[
\int_{n^{1/\lambda}}^{\eta} |x|^{2r} F_{r,n,}(|x|^{-\lambda}) \, dx = \int_{n^{1/\lambda}}^{\eta} \int_{-1/n}^{1/n} x^{2r} (x^{-\lambda} - t)^{2r/\lambda} \delta_n(t) \, dt \, dx = \\
\quad = \sum_{i=0}^{\infty} \left( \begin{array}{c} 2r/\lambda \\ i \end{array} \right) n^{-i} \int_{-1}^{1} (-u)^i \rho(u) \int_{n^{1/\lambda}}^{\eta} x^{i\lambda} \, dx \, du = \\
\quad = \sum_{i=0}^{\infty} \left( \begin{array}{c} 2r/\lambda \\ i \end{array} \right) \frac{n^{-i}}{i\lambda + 1} \int_{-1}^{1} (-u)^i \rho(u) \eta^{i\lambda + 1} + \\
\quad - \sum_{i=0}^{\infty} \left( \begin{array}{c} 2r/\lambda \\ i \end{array} \right) \frac{n^{1/\lambda}}{i\lambda + 1} \int_{-1}^{1} (-u)^i \rho(u) \, du.
\]
It follows that

\[ N - \lim_{n \to \infty} \int_{n^{1/\lambda}}^{\eta} x^{2r} F_{r,n}(|x|^{-\lambda}) \, dx = \eta \]

and so if \( \psi \) is a continuous function then

\[ N - \lim_{n \to \infty} \int_{n^{1/\lambda}}^{\eta} x^{2r} F_{r,n}(|x|^{-\lambda}) \psi(x) \, dx = O(\eta). \quad (5.23) \]

Similarly

\[ N - \lim_{n \to \infty} \int_{-\eta}^{-n^{1/\lambda}} x^{2r} F_{r,n}(|x|^{-\lambda}) \psi(x) \, dx = O(\eta). \quad (5.24) \]

Now let \( \varphi \) be an arbitrary function in \( D \) with support contained in the interval \([-1, 1]\). By Taylor’s Theorem, we have

\[ \varphi(x) = \sum_{i=0}^{2r-1} \frac{\varphi^{(i)}(0)}{i!} x^i + \frac{\varphi^{(2r)}(\xi x)}{(2r)!} x^{2r}, \]

where \( 0 < \xi < 1 \). Then

\[
\langle F_{r,n}(|x|^{-\lambda}), \varphi(x) \rangle = \int_{-1}^{1} F_{r,n}(|x|^{-\lambda}) \varphi(x) \, dx \\
= \sum_{i=0}^{2r-1} \frac{\varphi^{(i)}(0)}{i!} \int_{-1}^{1} F_{r,n}(|x|^{-\lambda}) x^i \, dx + \\
\frac{1}{(2r)!} \int_{0}^{n^{1/\lambda}} x^{2r} F_{r,n}(|x|^{-\lambda}) \varphi^{(2r)}(\xi) \, dx + \\
\frac{1}{(2r)!} \int_{n^{1/\lambda}}^{\eta} x^{2r} F_{r,n}(|x|^{-\lambda}) \varphi^{(2r)}(\xi) \, dx + \\
\frac{1}{(2r)!} \int_{-1}^{1} x^{2r} F_{r,n}(|x|^{-\lambda}) \varphi^{(2r)}(\xi) \, dx + \\
\frac{1}{(2r)!} \int_{-\eta}^{-n^{1/\lambda}} x^{2r} F_{r,n}(|x|^{-\lambda}) \varphi^{(2r)}(\xi) \, dx + \\
\frac{1}{(2r)!} \int_{-\eta}^{0} x^{2r} F_{r,n}(|x|^{-\lambda}) \varphi^{(2r)}(\xi) \, dx + \\
\frac{1}{(2r)!} \int_{0}^{-n^{1/\lambda}} x^{2r} F_{r,n}(|x|^{-\lambda}) \varphi^{(2r)}(\xi) \, dx + \\
\frac{1}{(2r)!} \int_{-\eta}^{-n^{1/\lambda}} x^{2r} F_{r,n}(|x|^{-\lambda}) \varphi^{(2r)}(\xi) \, dx. \]

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Taking the neutrix limit as $n$ tends to infinity, using equations (5.20), (5.21), (5.22), (5.23) and (5.24) and noting that the sequence $\{x^{2r}F_{r,n}(|x|^{-\lambda})\}$ converges uniformly to 1 on the intervals $[\eta, 1]$ and $[-1, -\eta]$, it follows that

$$N\lim_{n \to \infty}(x^{-\lambda}, \varphi(x)) = -\sum_{i=0}^{\infty} \frac{2\varphi^{2i}(0)}{(2i)!}\frac{1}{(2r - 2i - 1)} + \int_{\eta}^{1} \varphi^{2r}(\xi x) dx + O(\eta) + \frac{1}{(2r)!}\int_{-1}^{-\eta} \varphi^{2r}(\xi x) dx.$$ 

Since $\eta$ can be made arbitrary small, it follows on using Lemma 4.3 that

$$N\lim_{n \to \infty}(x^{-\lambda}, \varphi(x)) = -\sum_{i=0}^{\infty} \frac{2\varphi^{2i}(0)}{(2i)!}\frac{1}{(2r - 2i - 1)} + \int_{-1}^{1} x^{-2r}[\varphi(x) - \sum_{i=0}^{2r-1} \frac{\varphi^{(i)}(0)}{i!} x^i] dx$$

$$= (x^{-2r}, \varphi(x)).$$

**Theorem 5.5** Let $F_{\lambda}(x)$ denote the function $|x|^{-\lambda}$ where $s - 1 < \lambda < s$. Then the distribution $F_{\lambda}(|x|^{2r/\lambda})$ exists and

$$F_{\lambda}(|x|^{2r/\lambda}) = x^{-2r}$$

for $r, s = 1, 2, \ldots$.

**Proof.** Putting

$$F_{\lambda,n}(x) = F_{\lambda}(x) \ast \delta_n(x),$$

we have

$$\frac{\Gamma(-\lambda + s)}{\Gamma(-\lambda + 1)} F_{\lambda,n}(x) =$$

$$= \begin{cases} 
\int_{-1/n}^{1/n} (x - t)^{-\lambda + s - 1}\delta_n^{(s-1)}(t) dt, & x > 1/n, \\
\int_{x/n}^{x} (x - t)^{-\lambda + s - 1}\delta_n^{(s-1)}(t) dt + \int_{-1/n}^{1/n} (t - x)^{-\lambda + s - 1}\delta_n^{(s-1)}(t) dt, & |x| \leq 1/n, \\
(-1)^{s-1} \int_{-1/n}^{1/n} (t - x)^{-\lambda + s - 1}\delta_n^{(s-1)}(t) dt, & x < -1/n. 
\end{cases}$$
Since $F_{\lambda,n}(x^{2r/\lambda})$ is an even function, it follows that

$$\int_{-1}^{1} F_{\lambda,n}(x^{2r/\lambda})x^i\,dx = \begin{cases} 0, & i \text{ odd}, \\ 2\int_{0}^{1} F_{\lambda,n}(x^{2r/\lambda})x^i\,dx, & i \text{ even}. \end{cases} \tag{5.26}$$

We have

$$\frac{\Gamma(-\lambda+s)}{\Gamma(-\lambda+1)} \int_{0}^{1} F_{\lambda,n}(x^{2r/\lambda})x^i\,dx =$$

$$= \int_{0}^{n^{-\lambda/2r}} \int_{-1/n}^{x^{2r/\lambda}} (x^{2r/\lambda} - t)^{-\lambda+s-1}x^i\delta_n^{(s-1)}(t)\,dt\,dx +$$

$$-(-1)^s \int_{0}^{n^{-\lambda/2r}} \int_{x^{2r/\lambda}}^{1/n} (t - x^{2r/\lambda})^{-\lambda+s-1}x^i\delta_n^{(s-1)}(t)\,dt\,dx +$$

$$+ \int_{n^{-\lambda/2r}}^{1} \int_{-1/n}^{1/n} (x^{2r/\lambda} - t)^{-\lambda+s-1}x^i\delta_n^{(s-1)}(t)\,dt\,dx$$

$$= I_1 - (-1)^s I_2 + I_3. \tag{5.27}$$

Putting $nt = u$ and $n^{\lambda/2r}x = v$, we have

$$I_1 = n^{\lambda(2r-i-1)/2r} \int_{0}^{1} \int_{-1}^{u^{2r/\lambda}} (u^{2r/\lambda} - u)^{-\lambda+s-1}v^i\rho^{(s-1)}(u)\,du\,dv$$

and it follows that

$$N \lim_{n \to \infty} I_1 = 0 \tag{5.28}$$

for $i = 0, 2, \ldots, 2r - 2$.

Next we have

$$I_2 = n^{\lambda(2r-i-1)/2r} \int_{0}^{1} \int_{u^{2r/\lambda}}^{1} (u - v^{2r/\lambda})^{-\lambda+s-1}v^i\rho^{(s-1)}(u)\,du\,dv$$

and it follows that

$$N \lim_{n \to \infty} I_2 = 0 \tag{5.29}$$

for $i = 0, 2, \ldots, 2r - 2$.  

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Finally we have

\[ I_3 = n^{(2r-i-1)/2r} \int_1^{n^{\lambda/2r}} \int_{-1}^{1} (v^{2r/\lambda} - u)^{-\lambda+s-1} \rho^{(s-1)}(u) v^i \, du \, dv \]

\[ = n^{(2r-i-1)/2r} \int_{-1}^{1} \rho^{(s-1)}(u) \int_1^{n^{\lambda/2r}} (v^{2r/\lambda} - u)^{-\lambda+s-1} v^i \, du \, dv \]

\[ = \lambda \sum_{k=0}^{\infty} \binom{-\lambda + s - 1}{k} \frac{n^{s-1-k}}{-2r \lambda + 2rs - 2r - 2rk + i\lambda + \lambda} \int_{-1}^{1} (-u)^k \rho^{(s-1)}(u) \, du \]

\[ - \lambda \sum_{k=0}^{\infty} \binom{-\lambda + s - 1}{k} \frac{n^{\lambda(2r-i-1)/2r}}{-2r \lambda + 2rs - 2r - 2rk + i\lambda + \lambda} \int_{-1}^{1} (-u)^k \rho^{(s-1)}(u) \, du \]

and so

\[ \lim_{n \to \infty} I_3 = -\frac{\Gamma(-\lambda + s)}{(2r - i - 1)\Gamma(-\lambda + 1)} \] (5.30)

for \( i = 0, 2, \ldots, 2r - 2 \), since

\[ \int_{-1}^{1} (-u)^{s-1} \rho^{(s-1)}(u) \, du = (s - 1)! . \]

It now follows from equations (5.26) to (5.30) that

\[ \int_{-1}^{1} F_{\lambda,n}(\lfloor x^{2r/\lambda} \rfloor x^{2i}) \, dx = -\frac{2}{2r - 2i - 1} \] (5.31)

for \( i = 0, 1, 2, \ldots, r - 1 \).

When \( i = 2r \) we have

\[ \frac{\Gamma(-\lambda + s)}{\Gamma(-\lambda + 1)} \int_0^{n^{-\lambda/2r}} |x^{2r}F_{\lambda,n}(\lfloor x^{2r/\lambda} \rfloor)\, dx = \]

\[ = \int_0^{n^{-\lambda/2r}} x^{2r} \left[ \frac{\Gamma(-\lambda+s-1)}{\lambda+s-1} \rho_{\lambda+s-1}(t) \right] \, dt \, dx + \int_0^{n^{-\lambda/2r}} x^{2r} \left[ (t - x^{2r/\lambda})^{-\lambda+s-1} \rho_{\lambda+s-1}(t) \right] \, dt \, dx \]

\[ = n^{-\lambda/2r} \int_0^{n^{-\lambda/2r}} x^{2r} (v^{2r/\lambda} - u)^{-\lambda+s-1} \rho^{(s-1)}(u) \, du \, dv + \]

\[ + n^{-\lambda/2r} \int_0^{n^{-\lambda/2r}} (u - v^{2r/\lambda})^{-\lambda+s-1} v^{2r} \rho^{(s-1)}(u) \, du \, dv \]

\[ = O(n^{-\lambda/2r}) \]
and it follows that if \( \psi \) is an arbitrary continuous function, then
\[
\lim_{n \to \infty} \int_{0}^{n^{-\lambda/2r}} x^{2r} F_{\lambda,n}(|x|^{2r/\lambda}) \psi(x) \, dx = 0. \quad (5.32)
\]

Similarly
\[
\lim_{n \to \infty} \int_{-n^{-\lambda/2r}}^{0} x^{2r} F_{\lambda,n}(|x|^{2r/\lambda}) \psi(x) \, dx = 0. \quad (5.33)
\]

If now \( \eta \) is chosen so that \( n^{-\lambda/2r} < \eta < 1 \), then we obtain
\[
\frac{\Gamma(-\lambda+s)}{\Gamma(-\lambda+1)} \int_{n^{-\lambda/2r}}^{n} |x^{2r} F_{\lambda,n}(|x|^{2r/\lambda})| \, dx = \int_{n^{-\lambda/2r}}^{1/n} x^{2r} (x^{2r/\lambda} - t)^{-\lambda+s-1} \rho_{\lambda}^{(s-1)}(t) \, dt dx
\]
\[
= \int_{-1/n}^{1/n} \int_{n^{-\lambda/2r}}^{\eta} \sum_{i=0}^{\infty} \left( -\lambda + s - 1 \right) x^{2(r-s-rt)/\lambda - i - 1} |u^{i} \rho^{(s-1)}(u)| \, dx \, du
\]
\[
= \sum_{i=0}^{\infty} \left( -\lambda + s - 1 \right) \frac{\lambda(\eta^{1+2r(s-1-i)/\lambda} - n^{-\lambda/2r})}{2r \lambda^2 - 2ri + \lambda} \int_{-1}^{1} |u^{i} \rho^{(s-1)}(u)| \, du.
\]

It follows that
\[
\lim_{n \to \infty} \int_{n^{-\lambda/2r}}^{\eta} |x^{2r} F_{\lambda,n}(|x|^{2r/\lambda})| \, dx = O(\eta)
\]
and so if \( \psi \) is a continuous function then
\[
\lim_{n \to \infty} \int_{n^{-\lambda/2r}}^{\eta} x^{2r} F_{\lambda,n}(|x|^{2r/\lambda}) \psi(x) \, dx = O(\eta). \quad (5.34)
\]

Similarly
\[
\lim_{n \to \infty} \int_{-\eta}^{-n^{-\lambda/2r}} x^{2r} F_{\lambda,n}(|x|^{2r/\lambda}) \psi(x) \, dx = O(\eta). \quad (5.35)
\]

Now let \( \varphi \) be an arbitrary function in \( \mathcal{D} \) with support contained in the interval \([-1, 1]\). By Taylor’s Theorem, we have
\[
\varphi(x) = \sum_{i=0}^{2r-1} \frac{\varphi^{(i)}(0)}{i!} x^{i} + \frac{\varphi^{(2r)}(\xi)}{(2r)!} x^{2r},
\]

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where $0 < \xi < 1$. Then

$$
\langle F_{\lambda,n}(|x|^{2r/\lambda}), \varphi(x) \rangle = \int_{-1}^{1} F_{\lambda,n}(|x|^{2r/\lambda}) \varphi(x) \, dx
$$

$$
= \sum_{i=0}^{2r-1} \frac{\varphi^{(i)}(0)}{i!} \int_{-1}^{1} F_{\lambda,n}(|x|^{2r/\lambda}) x^i \, dx +
+ \frac{1}{(2r)!} \int_{0}^{n-x/2r} x^{2r} F_{\lambda,n}(|x|^{2r/\lambda}) \varphi^{(2r)}(\xi x) \, dx +
+ \frac{1}{(2r)!} \int_{n-x/2r}^{\eta} x^{2r} F_{\lambda,n}(|x|^{2r/\lambda}) \varphi^{(2r)}(\xi x) \, dx +
+ \frac{1}{(2r)!} \int_{\eta}^{1} x^{2r} F_{\lambda,n}(|x|^{2r/\lambda}) \varphi^{(2r)}(\xi x) \, dx +
+ \frac{1}{(2r)!} \int_{-\eta}^{-n-x/2r} x^{2r} F_{\lambda,n}(|x|^{2r/\lambda}) \varphi^{(2r)}(\xi x) \, dx +
+ \frac{1}{(2r)!} \int_{-1}^{-n-x/2r} x^{2r} F_{\lambda,n}(|x|^{2r/\lambda}) \varphi^{(2r)}(\xi x) \, dx +
+ \frac{1}{(2r)!} \int_{-1}^{-\eta} x^{2r} F_{\lambda,n}(|x|^{2r/\lambda}) \varphi^{(2r)}(\xi x) \, dx.
$$

Taking the neutrix limit as $n$ tends to infinity, using equations (5.26) and (5.31) to (5.35), and noting that the sequence $\{x^{2r} F_{\lambda,n}(|x|^{2r/\lambda})\}$ converges uniformly to 1 on the intervals $[\eta, 1]$ and $[-1, -\eta]$, it follows that

$$
N \lim_{n \to \infty} \langle F_{\lambda,n}(|x|^{2r/\lambda}), \varphi(x) \rangle = -\sum_{i=0}^{2r-1} \frac{2\varphi^{(2i)}(0)}{(2i)!(2r-2i-1)} + \frac{1}{(2r)!} \int_{\eta}^{1} \varphi^{(2r)}(\xi x) \, dx
$$

$$
+ O(\eta) + O(\eta) + \frac{1}{(2r)!} \int_{-\eta}^{-1} \varphi^{(2r)}(\xi x) \, dx.
$$

Since $\eta$ can be made arbitrarily small, it follows on using Lemma 5.2 that

$$
N \lim_{n \to \infty} \langle F_{\lambda,n}(|x|^{2r/\lambda}), \varphi(x) \rangle = -\sum_{i=0}^{2r-1} \frac{2\varphi^{(2i)}(0)}{(2i)!(2r-2i-1)} +
$$

$$
+ \int_{-1}^{1} x^{-2r} \left[ \varphi(x) - \sum_{i=0}^{2r-1} \frac{\varphi^{(i)}(0)}{i!} x^i \right] \, dx
$$

$$
= \langle x^{-2r}, \varphi(x) \rangle,
$$

proving equation (5.25).
THEOREM 5.6 Let $F_{\lambda,s}(x)$ denote the function $x^{-\lambda} \ln^s x$, where $0 < \lambda < 1$ and $s = 0, 1, 2, \ldots$. Then the distributions $F_{\lambda,s}(x^{r/\lambda})$, $F_{\lambda,s}(x^{-r/\lambda})$ and $F_{\lambda,s}(\lvert x \rvert^{r/\lambda})$ exist and

$$F_{\lambda,s}(x^{r/\lambda}) = \frac{(-1)^{r-1}\lambda}{r!} \sum_{j=0}^{s} [B_{0,s-j}(\lambda, -\lambda + 1)c_j(\rho)]\delta^{(r-1)}(x), \quad (5.36)$$

$$F_{\lambda,s}(x^{-r/\lambda}) = \frac{\lambda}{r!} \sum_{j=0}^{s} [B_{0,s-j}(\lambda, -\lambda + 1)c_j(\rho)]\delta^{(r-1)}(x), \quad (5.37)$$

$$F_{\lambda,s}(\lvert x \rvert^{r/\lambda}) = \frac{[1 + (-1)^{r-1}]\lambda}{r!} \sum_{j=0}^{s} [B_{0,s-j}(\lambda, -\lambda + 1)c_j(\rho)]\delta^{(r-1)}(x) \quad (5.38)$$

for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$, where $B(\alpha, \beta)$ denotes the Beta function,

$$B_{p,q}(\alpha, \beta) = \frac{\partial^{p+q}}{\partial p \partial q} B(\alpha, \beta)$$

and

$$c_j(\rho) = \int_{0}^{1} \ln^j t \rho(t) \, dt.$$

In particular

$$F_{\lambda,0}(x^{r/\lambda}) = \frac{(-1)^{r-1}\pi \lambda \csc(\pi \lambda)}{2r!} \delta^{(r-1)}(x),$$

$$F_{\lambda,0}(x^{-r/\lambda}) = \frac{\pi \lambda \csc(\pi \lambda)}{2r!} \delta^{(r-1)}(x),$$

$$F_{\lambda,0}(\lvert x \rvert^{r/\lambda}) = \frac{[1 + (-1)^{r-1}]\pi \lambda \csc(\pi \lambda)}{2r!} \delta^{(r-1)}(x) \quad (5.39)$$

for $r = 1, 2, \ldots$.

PROOF. We have

$$[F_{\lambda,s}(x)]_n = \left\{ \begin{array}{cl}
\int_{x/n}^{1/n}(t-x)^{-\lambda} \ln^s(t-x)\delta_n(t) \, dt, & \lvert x \rvert \leq 1/n, \\
\int_{-1/n}^{x/n}(t-x)^{-\lambda} \ln^s(t-x)\delta_n(t) \, dt, & x < -1/n, \\
0, & x > 1/n \end{array} \right. \quad (5.40)$$

and it follows that if $n^{-\lambda/r} < a$, then

$$\int_{-a}^{a} x^i[F_{\lambda,s}(x^{r/\lambda})]_n \, dx = \int_{0}^{n^{-\lambda/r}} x^i[F_{\lambda,s}(x^{r/\lambda})]_n \, dx + \int_{-a}^{0} x^i[F_{\lambda,s}(0)]_n \, dx$$

$$= I_1 + I_2. \quad (5.41)$$

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On using (5.40) we have

\[ I_1 = \int_0^{1/n} \int_0^{1/n} x^n(t - x^{r/\lambda})^{-\lambda} \ln^s(t - x^{r/\lambda}) \delta_n(t) \, dt \, dx \]

\[ = \int_0^{1/n} \delta_n(t) \int_0^{t^{1/r}} x^n(t - x^{r/\lambda})^{-\lambda} \ln^s(t - x^{r/\lambda}) \, dx \, dt \]

\[ = \frac{\lambda}{r} \int_0^{1/n} \delta_n(t) \int_0^{t^{(i-r+1)/r}} u^{\lambda(i+1)/r-1}(1 - u)^{-\lambda} \ln t + \ln(1 - u) \, du \, dt \]

\[ = \frac{\lambda}{r} \sum_{j=0}^{s} \binom{s}{j} \int_0^{1/n} \int_0^{t^{(i+1-r)/r}} \ln^j t \delta_n(t) \int_0^{1/n} u^{\lambda(i+1)/r-1}(1 - u)^{-\lambda} \ln^{s-j}(1 - u) \, du \, dt \]

\[ = \frac{\lambda}{r} \sum_{j=0}^{s} \binom{s}{j} \rho_{0,s-j}(\lambda(i + 1)/r, -\lambda + 1) \int_0^{1/n} t^{(i-r+1)/r} \ln^j t \delta_n(t) \, dt, \]

on putting \( x = (tu)^{\lambda/r} \).

Next, putting \( nt = v \), we have

\[ \int_0^{1/n} t^{(i-r+1)/r} \ln^j t \delta_n(t) \, dt = n^{(r-i-1)/r} \int_0^{1/n} v^{(i+1-r)/r} (\ln v - \ln n)^j \rho(v) \, dv \]

and it follows that

\[ N - \lim_{n \to \infty} I_1 = 0 \]  \hspace{1cm} (5.42)

for \( i = 0, 1, 2, \ldots, r - 2 \) and

\[ N - \lim_{n \to \infty} I_1 = \frac{\lambda}{r} \sum_{j=0}^{s} \binom{s}{j} \rho_{0,s-j}(\lambda, -\lambda + 1) c_j(\rho) \]  \hspace{1cm} (5.43)

for \( i = r - 1 \).

Next, on putting \( nt = v \), we have

\[ I_2 = \int_{-\alpha}^{0} \int_0^{1/n} x^n t^{-\lambda} \ln^s t \delta_n(t) \, dt \, dx \]

\[ = \frac{(-1)^i \alpha^{i+1}}{i + 1} n^\lambda \int_0^{1} v^{-\lambda} \int_0^{1} v^{-\lambda}(\ln v - \ln n)^s \rho(v) \, dv \]

and it follows that

\[ N - \lim_{n \to \infty} I_2 = 0 \]  \hspace{1cm} (5.44)

for \( i = 0, 1, \ldots, r - 1 \).
When \( i = r \) we have
\[
\int_0^{n^{-\lambda/r}} |x^r[F_{\lambda,s}(x^{r/\lambda})]_n| \, dx = O(n^{-\lambda/r} \ln^s n)
\]
and consequently, for any continuous function \( \psi \), we have
\[
\lim_{n \to \infty} \int_0^{n^{-\lambda/r}} x^r F_{\lambda,s}(x^{r/\lambda}) \psi(x) \, dx = 0. \tag{5.45}
\]

Finally we have
\[
\int_{-a}^{0} [F_{\lambda,s}(x^{r/\lambda})]_n \psi(x) \, dx = \int_{-a}^{0} \psi(x) \int_0^1 t^{-\lambda} \ln^{s} t \delta_{n}(t) \, dt \, dx
\]
\[
= n^{\lambda} \int_{-a}^{0} \psi(x) \int_0^1 v^{-\lambda} (\ln v - \ln n)^{s} \rho(v) \, dv \, dx
\]
and so
\[
N - \lim_{n \to \infty} \int_{-a}^{0} [F_{\lambda,s}(x^{r/\lambda})]_n \psi(x) \, dx = 0. \tag{5.46}
\]

Now let \( \varphi \) be an arbitrary function in \( \mathcal{D} \) with support contained in \([-a, a]\). By Taylor’s Theorem we have
\[
\varphi(x) = \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} x^i + \frac{\varphi^{(r)}(\xi x)}{r!} x^r,
\]
where \( 0 < \xi < 1 \). Then with \( n^{-\lambda/r} < a \), we have
\[
\langle [F_{\lambda,s}(x^{r/\lambda})]_n, \varphi(x) \rangle = \int_{-a}^{a} [F_{\lambda,s}(x^{r/\lambda})]_n \varphi(x) \, dx
\]
\[
= \sum_{i=0}^{r-1} \frac{\varphi^{(i)}(0)}{i!} \int_0^{n^{-\lambda/r}} x^i [F_{\lambda,s}(x^{r/\lambda})]_n \, dx +
\]
\[
+ \frac{1}{r!} \int_0^{n^{-\lambda/r}} [F_{\lambda,s}(x^{r/\lambda})]_n x^r \varphi^{(r)}(\xi x) \, dx +
\]
\[
+ \int_{-a}^{0} [F_{\lambda,s}(x^{r/\lambda})]_n \varphi(x) \, dx.
\]
Taking the neutrix limit as \( n \) tends to infinity and using equations (5.40) to (5.46), it follows that
\[
N - \lim_{n \to \infty} \langle [F_{\lambda,s}(x^{r/\lambda})]_n, \varphi(x) \rangle = \frac{\lambda \varphi^{(r-1)}(0)}{r!} \sum_{j=0}^{s} \binom{s}{j} B_{0,s-j}(\lambda, -\lambda + 1) c_{j}(\rho)
\]
\[
= \frac{(-1)^{r-1} \lambda}{r!} \sum_{j=0}^{s} B_{0,s-j}(\lambda, -\lambda + 1) c_{j}(\rho) \langle \delta^{(r-1)}(x), \varphi(x) \rangle
\]

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giving equation (5.36) on the interval $[-a, a]$. Since $[-a, a]$ can be any closed interval containing the origin, equation (5.36) follows on the real line.

To prove equation (5.37), we note from equation (5.40) that

$$[F_{\lambda,s}(x^{-\lambda})]_n = \begin{cases} \int_{|x|/\lambda}^{1/n} (t - |x|/\lambda)^{-\lambda} \ln^s(t - |x|/\lambda) \delta(t) \, dt, & -n^{-\lambda/r} \leq x \leq 0, \\ 0, & x > 0, \\ 0, & x < -n^{-\lambda/r} \end{cases}$$

and it follows that

$$\int_{-a}^{a} x^i[F_{\lambda,s}(x^{-\lambda})]_n \, dx = \int_{-n^{-\lambda/r}}^{0} x^i[F_{\lambda,s}(|x|^{-\lambda})]_n \, dx + \int_{0}^{a} x^i[F_{\lambda,s}(0)]_n \, dx$$

$$= (-1)^i \int_{0}^{n^{-\lambda/r}} x^i[F_{\lambda,s}(x^{-\lambda})]_n \, dx + (-1)^i \int_{-a}^{0} x^i[F_{\lambda,s}(0)]_n \, dx$$

$$= (-1)^i(I_1 + I_2).$$

Equation (5.37) now follows as above.

Finally, to prove equation (5.38), we note from equation (5.40) that

$$[F_{\lambda,s}(|x|^{-\lambda})]_n = \begin{cases} \int_{|x|/\lambda}^{1/n} (t - |x|/\lambda)^{-\lambda} \ln^s(t - |x|/\lambda) \delta(t) \, dt, & |x| \leq n^{-\lambda/r}, \\ 0, & |x| > n^{-\lambda/r} \end{cases}$$

and it follows that

$$\int_{-a}^{a} x^i[F_{\lambda,s}(|x|^{-\lambda})]_n \, dx = \int_{-n^{-\lambda/r}}^{0} x^i[F_{\lambda,s}(|x|^{-\lambda})]_n \, dx + \int_{0}^{n^{-\lambda/r}} x^i[F_{\lambda,s}(x^{-\lambda})]_n \, dx$$

$$= [1 + (-1)^i] I_1.$$ \(\text{Equation (5.38) now follows as above.}\)

**COROLLARY 5.4** Let $G_{\lambda,s}(x)$ denote $x^\lambda \ln^s x$, where $0 < \lambda < 1$. Then the distributions $G_{\lambda,s}(-x_+^{r/\lambda})$, $G_{\lambda,s}(-x_-^{r/\lambda})$ and $G_{\lambda,s}(-|x|^{r/\lambda})$ exist and

$$G_{\lambda,s}(-x_+^{r/\lambda}) = \frac{(-1)^{r-1} \lambda}{r!} \sum_{j=0}^{s} B_{0,s-j}(\lambda, -\lambda + 1)c_j(\rho)\delta^{(r-1)}(x),$$

$$G_{\lambda,s}(-x_-^{r/\lambda}) = \frac{\lambda}{r!} \sum_{j=0}^{s} B_{0,s-j}(\lambda, -\lambda + 1)c_j(\rho)\delta^{(r-1)}(x),$$

$$G_{\lambda,s}(-|x|^{r/\lambda}) = \frac{[1 + (-1)^{r-1}] \lambda}{r!} \sum_{j=0}^{s} B_{0,s-j}(\lambda, -\lambda + 1)c_j(\rho)\delta^{(r-1)}(x).$$
for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$. In particular

\[
G_{\lambda,0}(x_r^{r/\lambda}) = \frac{(-1)^{r-1} \pi \lambda \csc(\pi \lambda)}{2r!} \delta^{(r-1)}(x),
\]
\[
G_{\lambda,0}(x_r^{-r/\lambda}) = \frac{\pi \lambda \csc(\pi \lambda)}{2r!} \delta^{(r-1)}(x),
\]
\[
G_{\lambda,0}(-|x|^{r/\lambda}) = \frac{[1 + (-1)^{r-1}] \pi \lambda \csc(\pi \lambda)}{2r!} \delta^{(r-1)}(x)
\]

for $r = 1, 2, \ldots$.

**PROOF.** We have

\[G_{\lambda,s}(-x) = F_{\lambda,s}(x)\]

and so

\[
[G_{\lambda,s}(x_r^{r/\lambda})]_n = [F_{\lambda,s}(x_r^{r/\lambda})]_n,
\]
\[
[G_{\lambda,s}(x_r^{-r/\lambda})]_n = [F_{\lambda,s}(x_r^{-r/\lambda})]_n,
\]
\[
[G_{\lambda,s}(-|x|^{r/\lambda})]_n = [F_{\lambda,s}(|x|^{r/\lambda})]_n.
\]

Thus equations (5.47), (5.48) and (5.49) follow from equations (5.36), (5.37) and (5.38) respectively.
Bibliography


