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ON THE COMPOSITION OF THE DISTRIBUTIONS $x_+^{-s}\ln^m x_+$ AND x_+^{μ}

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Let F be a distribution and let f be a locally summable function. The distribution $F(f)$ is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The composition of the distributions $x_+^{-s}\ln^m x_+$ and x_+^{μ} is proved to exist and be equal to $\mu^m x_+^{-s\mu}\ln^m x_+$ for $\mu > 0$ and $s, m = 1, 2, \ldots$.

1. INTRODUCTION AND PRELIMINARIES

In the following we let $\rho(x)$ be an infinitely differentiable function having the following properties:

$$\text{(i)}\quad \rho(x) = 0 \text{ for } |x| \geq 1, \qquad \text{(ii)}\quad \rho(x) \geq 0,$$

$$\text{(iii)}\quad \rho(x) = \rho(-x), \qquad \text{(iv)}\quad \int_{-1}^{1} \rho(x)\,\mathrm{d}x = 1.$$

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

We let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, let $\mathcal{D}(a, b)$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$, let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$ and let $\mathcal{D}'(a, b)$ be the space of distributions defined on $\mathcal{D}(a, b)$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t)\rangle$$

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for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

We now define the distribution $x_+^{-1}\ln^m x_+$ by

$$x_+^{-1}\ln^m x_+ = \frac{(\ln^{m+1} x_+)'}{m+1}$$

for $m = 1, 2, \ldots$ and we define the distribution $x_+^{-r-1}\ln^m x_+$ inductively by

$$x_+^{-r-1}\ln^m x_+ = \frac{mx^{-r-1}\ln^{m-1} x_+ - (x_+^{-r}\ln^m x_+)'}{r}$$

for $r, m = 1, 2, \ldots$. This is not the same as Gel'fand and Shilov [**8**].

Next, we define the locally summable function $x_+^\lambda \ln^m x_+$ for $\lambda > -1$, and $m = 0, 1, 2, \ldots$ by

$$x_+^\lambda \ln^m x_+ = \begin{cases} x^\lambda \ln^m x, & x > 0, \\ 0, & x < 0 \end{cases}.$$

The distribution $x_+^\lambda \ln^m x_+$ is then defined inductively for $\lambda < -1$, $\lambda \neq -2, -3, \ldots$ and $m = 0, 1, 2, \ldots$, by the equation

$$(x_+^\lambda \ln^{m+1} x_+)' = \lambda\, x_+^{\lambda-1}\ln^{m+1} x_+ + (m+1)x_+^{\lambda-1}\ln^m x_+.$$

The distribution $x_-^\lambda \ln^m x_-$ is then defined for $\lambda \neq -1, -2, \ldots$ and $m = 0, 1, 2, \ldots$, by

$$x_-^\lambda \ln^m x_- = (-x)_+^\lambda \ln^m (-x)_+$$

and the distribution $|x|^\lambda \ln^m |x|$ is then defined for $\lambda \neq -1, -2, \ldots$ and $m = 0, 1, 2, \ldots$, by

$$|x|^\lambda \ln^m |x| = x_+^\lambda \ln^m x_+ + x_-^\lambda \ln^m |x|.$$

It follows that if r is a positive integer and $-r-1 < \lambda < -r$, then

$$(1) \quad \langle x_+^\lambda \ln^m x_+, \varphi(x)\rangle = \int_0^1 x^\lambda \ln^m x\Big[\varphi(x) - \sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\Big]\,\mathrm{d}x + \sum_{k=0}^{r-1}\frac{(-1)^m m!\varphi^{(k)}(0)}{k!(\lambda+k+1)^{m+1}},$$

for an arbitrary φ in $\mathcal{D}[-1, 1]$.

If now $f(x)$ is an infinitely differentiable function having a single simple root at the point $x = x_0$, with $f'(x) > 0$, then putting $t = f(x)$ and $\psi(x) = f'(x)\varphi(f(x))$, we have

$$\begin{aligned}\int_{-\infty}^{\infty}\delta_n(t)\varphi(t)\,\mathrm{d}t &= \int_{-\infty}^{\infty}\delta_n(f(x))f'(x)\varphi(f(x))\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\delta_n(f(x))\psi(x)\,\mathrm{d}x = \langle\delta_n(f(x)), \psi(x)\rangle.\end{aligned}$$

Defining the distribution $\delta(f(x))$ by

$$\langle \delta(f(x)), \psi(x) \rangle = \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(f(x))\psi(x)\,\mathrm{d}x$$

we would get

$$\langle \delta(f(x)), \psi(x) \rangle = \frac{1}{|f'(x_0)|} \langle \delta(x - x_0), \psi(x) \rangle$$

so that

$$\delta(f(x)) = \frac{1}{|f'(x_0)|} \langle \delta(x - x_0)$$

and more generally

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(x_0)|} \Big[\frac{1}{f'(x)} \frac{\mathrm{d}}{\mathrm{d}x} \Big]^r \delta(x - x_0)$$

for $r = 0, 1, 2, \ldots$. This is of course in agreement with Gel'fand and Shilov's definition of $\delta^{(r)}(f(x))$, see [**8**].

In order to generalize this definition of $\delta^{(r)}(f(x))$, the following definition was given in [**2**].

Definition 1. *Let f be an infinitely differentiable function. We say that the distribution $\delta^{(r)}(f(x))$ exists and is equal to h on the open interval (a, b) if*

$$\mathrm{N}\!-\!\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,\mathrm{d}x = \langle h(x), \varphi(x) \rangle$$

for all functions $\varphi(x)$ in $\mathcal{D}(a, b)$, where N is the neutrix, see [**1**], *having domain N' the positive integers and range the real numbers, with negligible functions which are finite linear sums of the functions*

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \qquad \lambda > 0, \; r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Note that taking a neutrix limit of a function is equivalent to picking out the Hadamard finite part from the function and taking the usual limit of Hadamard's finite part.

The following definition generalizing Definition 1 was given in [**3**] and was originally called the composition of distributions.

Definition 2. *Let F be distribution in $\mathcal{D}'$ and let f be a locally summable function. We say the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if*

$$\mathrm{N}\!-\!\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)\,\mathrm{d}x = \langle h(x), \varphi(x) \rangle$$

for all functions $\varphi(x)$ *in* $\mathcal{D}(a,b)$, *where* $F_n(x)=F(x)*\delta_n(x)$ *for* $n=1,2,\ldots$.

In particular, we say that the composition $F(f(x))$ *exists and is equal to* h *on the open interval* (a,b) *if*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}F_n(f(x))\varphi(x)\,\mathrm{d}x=\langle h(x),\varphi(x)\rangle$$

for all functions φ *in* $\mathcal{D}(a,b)$.

The following theorems were proved in [**9**], [**6**] and [**5**] respectively.

Theorem 1. *The neutrix composition* $(x^r)^{-s}$ *exists and*

$$(x^r)^{-s}=x^{-rs}$$

for $r,s=1,2,\ldots$.

Theorem 2. *If* $F_s(x)$ *denotes the distribution* $x^{-s}\ln|x|$, *then the neutrix composition* $F_s(x^r)$ *exists and*

$$F_s(x^r)=rx^{-rs}\ln|x|$$

for $r,s=1,2,\ldots$.

Theorem 3. *If* $F_m(x)$ *denotes the distribution* $x^{-1}\ln^m|x|$, *then the neutrix composition* $F_m(x^r)$ *exists and*

$$F_m(x^r)=r^m x^{-r}\ln^m|x|$$

for $m,r=1,2,\ldots$.

The next theorem was proved in [**7**].

Theorem 4. *If* $F_{\lambda,m}(x)$ *denotes the distribution* $x_+^\lambda\ln^m x_+$, *then the neutrix composition* $F_{\lambda,m}(x^\mu)$ *exists and*

$$F_{\lambda,m}(x_+^\mu)=\mu^m x_+^{\lambda\mu}\ln^m x_+$$

for $m=1,2,\ldots$, $-1<\lambda<0$, $\mu>0$ *and* λ, $\lambda\mu\neq-1,-2,\ldots$.

Theorem 4 was then generalized in [**4**] with the following theorem.

Theorem 5. *If* $F_{\lambda,m}(x)$ *denotes the distribution* $x_+^\lambda\ln^m x_+$, *then the neutrix composition* $F_{\lambda,m}(x^\mu)$ *exists and*

$$F_{\lambda,m}(x_+^\mu)=\mu^m x_+^{\lambda\mu}\ln^m x_+$$

for $m=1,2,\ldots$, $\lambda<0$, $\mu>0$ *and* λ, $\lambda\mu\neq-1,-2,\ldots$.

The following lemma can be proved easily by induction.

Lemma 1.

$$\int_{-1}^{1} v^i \rho^{(r)}(v)\,\mathrm{d}v = \begin{cases} 0, & 0 \le i < r, \\ (-1)^r r!, & i = r \end{cases}$$

for $r = 0, 1, 2, \ldots$.

The next two lemmas can be proved easily by induction.

Lemma 2. *If* φ *is an arbitrary function in* $\mathcal{D}$ *with support contained in the interval* $[-1, 1]$, *then*

$$\langle x^{-r} \ln^m |x|, \varphi(x)\rangle = \int_{-1}^{1} x^{-r} \ln^m |x| \Big[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!}\, x^k\Big]\,\mathrm{d}x - \sum_{k=0}^{r-2} \frac{[1 + (-1)^{r+k}]m!}{(r-k-1)^{m+1}k!}\, \varphi^{(k)}(0),$$

for $m, r = 1, 2, \ldots$, *where the second sum is empty when* $r = 1$.

Lemma 3.

$$\int_{1}^{n} u^{\alpha} \ln^r u\,\mathrm{d}u = \frac{(-1)^{r+1} r!(1 - n^{\alpha+1})}{(\alpha+1)^{r+1}} + E(\ln n),$$

for $\alpha \neq -1$ *and* $r = 1, 2, \ldots$, *where* $E(\ln n)$ *denotes a sum of negligible functions, each containing a positive power of* $\ln n$.

2. MAIN RESULTS

We now prove the following extension of Theorem 5.

Theorem 6. *If* $F_{s,m}(x)$ *denotes the distribution* $x_+^{-s} \ln^m x_+$, *then the neutrix composition* $F_{s,m}(x_+^{\mu})$ *exists and*

$$F_{s,m}(x_+^{\mu}) = \mu^m x_+^{-s\mu} \ln^m x_+ \tag{2}$$

for $m = 0, 1, 2, \ldots$, $s = 1, 2, \ldots$, $\mu > 0$ *and* $s\mu \neq 1, 2, \ldots$.

Proof. We first of all put $G_{s,m}(x) = (\ln^{m+1} x_+)^{(s)}$ and note that $G_{s,m}(x)$ is of the form

$$G_{s,m}(x) = \sum_{i=0}^{m} c_{s,m,i} x_+^{-s} \ln^i x_+, \tag{3}$$

where $c_{s,m,i} = 0$ if $i \le m - s$. Since $x_+^{-s} \ln^i x_+$ is an infinitely differentiable function on any closed interval not containing the origin, it follows that

$$F_{s,i}(x_+^{\mu}) = \mu^i x_+^{-s\mu} \ln^i x_+$$

and then

$$G_{s,m}(x_+^\mu) = \sum_{i=0}^{m} c_{s,m,i}\,\mu^i x_+^{-s\mu}\ln^i x_+ \tag{4}$$

on any closed interval not containing the origin.

Putting

$$\begin{aligned} G_{s,m,n}(x) &= (\ln^{m+1} x_+)^{(s)} * \delta_n(x) \\ &= \int_{-1/n}^{1/n} \ln^{m+1}(x-t)_+\,\delta_n^{(s)}(t)\,\mathrm{d}t \\ &= \begin{cases} \int_{-1/n}^{1/n} \ln^{m+1}(x-t)\,\delta_n^{(s)}(t)\,\mathrm{d}t, & 1/n < x, \\ \int_{-1/n}^{x} \ln^{m+1}(x-t)\,\delta_n^{(s)}(t)\,\mathrm{d}t, & -1/n \le x \le 1/n, \\ 0, & x < -1/n, \end{cases} \end{aligned}$$

we have

$$G_{s,m,n}(x_+^\mu) = \begin{cases} \int_{-1/n}^{1/n} \ln^{m+1}(x^\mu - t)\,\delta_n^{(s)}(t)\,\mathrm{d}t, & 1/n < x^\mu, \\ \int_{-1/n}^{x^\mu} \ln^{m+1}(x^\mu - t)\,\delta_n^{(s)}(t)\,\mathrm{d}t, & 0 \le x^\mu \le 1/n, \\ \int_{-1/n}^{0} \ln^{m+1}(-t)\,\delta_n^{(s)}(t)\,\mathrm{d}t, & x < 0. \end{cases}$$

Our problem now is to evaluate

$$\begin{aligned} \int_{-1}^{1} G_{s,m,n}(x_+^\mu)x^k\,\mathrm{d}x &= \int_0^{n^{-1/\mu}} x^k \int_{-1/n}^{x^\mu} \ln^{m+1}(x^\mu - t)\,\delta_n^{(s)}(t)\,\mathrm{d}t\,\mathrm{d}x \\ &\quad + \int_{n^{-1/\mu}}^{1} x^k \int_{-1/n}^{1/n} \ln^{m+1}(x^\mu - t)\,\delta_n^{(s)}(t)\,\mathrm{d}t\,\mathrm{d}x \\ &\quad + \int_{-1}^{0} x^k \int_{-1/n}^{0} \ln^{m+1}(-t)\,\delta_n^{(s)}(t)\,\mathrm{d}t\,\mathrm{d}x \\ &= \frac{n^{s-(k+1)/\mu}}{\mu} \int_{-1}^{1} \rho^{(s)}(u) \int_0^1 u^{-(\mu-k-1)/\mu} \ln^{m+1}[(v-u)/n]\,\mathrm{d}u\,\mathrm{d}v \\ &\quad + \frac{n^{s-(k+1)/\mu}}{\mu} \int_{-1}^{1} \rho^{(s)}(u) \int_1^n v^{-(\mu-k-1)/\mu} \ln^{m+1}[(v-u)/n]\,\mathrm{d}v\,\mathrm{d}u \\ &\quad + n^{-s} \int_{-1}^{0} x^k \int_{-1/n}^{0} \ln^{m+1}(-u/n)\rho^{(s)}(u)\,\mathrm{d}u\,\mathrm{d}x \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{5}$$

on using the substitutions $u = nt$ and $v = nx^\mu$.

It is easily seen that

$$\operatorname*{N-lim}_{n\to\infty} I_1 = \operatorname*{N-lim}_{n\to\infty} I_3 = 0, \tag{6}$$

for $k = 0, 1, 2 \ldots .$

Now,

$$
\begin{aligned}
(7) \qquad I_2 &= \frac{n^{s-(k+1)/\mu}}{\mu} \int_{-1}^{1} \rho^{(s)}(u) \int_{1}^{n} v^{-(\mu-k-1)/\mu} [\ln(1-u/v) \\
&\qquad\qquad + \ln v - \ln n]^{m+1} \, dv \, du \\
&= \sum_{i=1}^{m+1} \binom{m+1}{i} \frac{n^{s-(k+1)/\mu}}{\mu} \int_{-1}^{1} \rho^{(s)}(u) \int_{1}^{n} v^{-(\mu-k-1)/\mu} \\
&\qquad\qquad \times \ln^{i}(1-u/v) \ln^{m-i+1} u \, dv \, du + E(\ln n) \\
&= \sum_{i=1}^{m+1} J_i + E(\ln n),
\end{aligned}
$$

where $E(\ln n)$ denotes the terms containing powers of $\ln n$ and so are negligible and the term containing $\ln^{m+1} u$ is zero, since $\int_{-1}^{1} \rho^{(s)}(v) \, dv = 0$ for $s = 1, 2, \ldots$, by Lemma 1.

We note that $\ln^{i}(1-u/v)$ can be expanded in the form

$$
\ln^{i}(1-u/v) = \sum_{p=0}^{\infty} \frac{a_{i,p} u^p}{v^p},
$$

where $a_{i,p} = 0$ for $p = 0, 1, \ldots, i-1$ and then

$$
\begin{aligned}
& n^{s-(k+1)/\mu} \int_{-1}^{1} \rho^{(s)}(u) \int_{1}^{n} v^{-(\mu-k-1)/\mu} \ln^{i}(1-u/v) \ln^{m-i+1} u \, dv \, du \\
&= \sum_{p=0}^{\infty} n^{s-(k+1)/\mu} a_{i,p} \int_{-1}^{1} u^p \rho^{(s)}(u) \int_{1}^{n} v^{-p-(\mu-k-1)/\mu} \ln^{m-i+1} v \, dv \, du \\
&= \sum_{p=0}^{\infty} \frac{a_{i,p}(-1)^{m-i+1}(m-i+1)! \, [n^{s-p} - n^{s-(k+1)/\mu}]}{[-p-(\mu-k-1)/\mu+1]^{m-i+2}} \int_{-1}^{1} u^p \rho^{(s)}(u) \, du + E(\ln n)
\end{aligned}
$$

on using Lemma 3.

Using Lemma 1, it follows that

$$
(8) \qquad \mathrm{N}\!-\!\lim_{n\to\infty} J_i = 0,
$$

if $i > s$ and

$$
(9) \qquad \mathrm{N}\!-\!\lim_{n\to\infty} J_i = \binom{m+1}{i} \frac{a_{i,s}(-1)^{s+1}(m-i+1)! \, s! \, \mu^{m-i+1}}{(s\mu-k-1)^{m-i+2}}
$$

if $i \le s$.

It then follows from equations (7) to (9) that

$$
(10) \qquad \mathrm{N}\!-\!\lim_{n\to\infty} I_2 = \sum_{i=1}^{m+1} \binom{m+1}{i} \frac{a_{i,s}(-1)^{s+1}(m-i+1)! \, s! \, \mu^{m-i+1}}{(s\mu-k-1)^{m-i+2}},
$$

for $k = 0, 1, 2, \dots$ and it then follows from equations (5), (6) and (10) that

$$\text{(11)}\qquad \operatorname*{N-lim}_{n\to\infty}\int_{-1}^{1} G_{s,m,n}(x_+^\mu)x^k\,\mathrm{d}x = \sum_{i=1}^{m+1}\binom{m+1}{i}\frac{a_{i,s}(-1)^{s+1}(m-i+1)!\,s!\,\mu^{m-i+1}}{(s\mu-k-1)^{m-i+2}} = \sum_{i=0}^{m}\binom{m+1}{i}\frac{a_{m-i+1,s}(-1)^{s+1}i!\,s!\,\mu^{i}}{(s\mu-k-1)^{i+1}}$$

for $k = 0, 1, 2, \dots$.

We now consider the case $k = r$, where r is chosen so that $s\mu - r - 1 < 0$, and let $\psi(x)$ be an arbitrary continuous function. Then

$$\int_0^{n^{-1/\mu}} x^r\psi(x)G_{s,m,n}(x_+^\mu)\,\mathrm{d}x = \frac{n^{s-(r+1)/\mu}}{\mu}\int_0^1 v^{(r+1)/\mu-1}\int_{-1}^{v}\psi[(v/n)^{1/\mu}] \times[\ln(v-u)-\ln n]^{m+1}\,\mathrm{d}u\,\mathrm{d}v$$

and it follows that

$$\text{(12)}\qquad \lim_{n\to\infty}\int_0^{n^{-1/\mu}} x^r\psi(x)\,\mathrm{d}x\,G_{s,m,n}(x_+^\mu) = 0.$$

When $x \le 0$, we have

$$\int_{-1}^{0} x^r\psi(x)G_{s,m,n}(x_+^\mu)\,\mathrm{d}x = \int_{-1}^{0} x^r\psi(x)\int_{-1/n}^{0}\ln^{m+1}(-t)\delta_n^{(s)}(t)\,\mathrm{d}t = n^{-s}\int_{-1}^{0} x^r\psi(x)\int_{-1}^{0}[\ln(-u)-\ln n]^{m+1}\rho^{(s)}(u)\,\mathrm{d}u\,\mathrm{d}x$$

and it follows that

$$\text{(13)}\qquad \operatorname*{N-lim}_{n\to\infty}\int_{-1}^{0} x^r\psi(x)G_{s,m,n}(x_+^\mu)\,\mathrm{d}x = 0.$$

Next, when $x^\mu \ge 1/n$, we have

$$\text{(14)}\qquad G_{s,m,n}(x_+^\mu) = \int_{-1/n}^{1/n}\ln^{m+1}(x^\mu-t)\delta_n^{(s)}(t)\,\mathrm{d}t = n^s\int_{-1}^{1}\Big[\ln x^\mu + \ln\Big(1-\frac{u}{nx^\mu}\Big)\Big]^{m+1}\rho^{(s)}(u)\,\mathrm{d}u = n^s\int_{-1}^{1}\left[\ln^{m+1}x^\mu + \sum_{i=1}^{m+1}\binom{m+1}{i}\ln^{m-i+1}x^\mu\ln^i\left(1-\frac{u}{nx^\mu}\right)\right]\rho^{(s)}(u)\,\mathrm{d}u$$

$$
\begin{aligned}
&= n^s \ln^{m+1} x^\mu \int_{-1}^{1} \rho^{(s)}(u)\,\mathrm{d}u \\
&\qquad + n^s \sum_{i=1}^{m+1} \binom{m+1}{i} \ln^{m-i+1} x^\mu \sum_{p=i}^{\infty} \int_{-1}^{1} \frac{a_{i,p} u^p}{n^p x^{\mu p}}\, \rho^{(s)}(u)\,\mathrm{d}u \\
&= \sum_{i=1}^{m+1} \binom{m+1}{i} \ln^{m-i+1} x^\mu \sum_{p=i}^{\infty} \int_{-1}^{1} \frac{a_{i,p} u^p}{n^{p-s} x^{\mu p}}\, \rho^{(s)}(u)\,\mathrm{d}u \\
&= (-1)^s s! \sum_{i=1}^{m+1} \binom{m+1}{i} \mu^{m-i+1}\, a_{i,s}\, x^{-s\mu} \ln^{m-i+1} x + O(n^{-1}) \\
&= (-1)^s s! \sum_{i=0}^{m} \binom{m+1}{i} \mu^i a_{m-i+1,s} x^{-s\mu} \ln^i x + O(n^{-1}).
\end{aligned}
$$

Letting n tend to infinity, it follows that

$$
\lim_{n\to\infty} G_{s,m,n}(x_+^\mu) = G_{s,m}(x_+^\mu) = (-1)^s s! \sum_{i=0}^{m} \binom{m+1}{i} \mu^i a_{m-i+1,s} x^{-s\mu} \ln^i x \tag{15}
$$

for $x > 0$.

Comparing equations (4) and (15) we see that

$$
c_{s,m,i} = (-1)^s s! \binom{m+1}{i} a_{m-i+1,s}, \tag{16}
$$

for $i = 0, 1, 2, \dots, m$.

We also see from equation (14) that

$$
\left| \int_{-1/n}^{1/n} \ln^{m+1}(x^\mu - t) \delta_n^{(s)}(t)\,\mathrm{d}t \right| \le s! \sum_{i=0}^{m} \binom{m+1}{i} \mu^i \left| a_{m-i+1,s} x^{-s\mu} \ln^i x \right| + O(n^{-1}),
$$

for $x^\mu \ge 1/n$.

If now $n^{-1/\mu} < \eta < 1$, then

$$
\begin{aligned}
&\int_{n^{-1/\mu}}^{\eta} x^{s\mu} \left| \int_{-1/n}^{1/n} \ln^{m+1}(x^\mu - t) \delta_n^{(s)}(t)\,\mathrm{d}t \right| \mathrm{d}x \\
&\qquad \le s! \sum_{i=0}^{m} \binom{m+1}{i} \mu^i \left| a_{m-i+1,s} \right| \int_{n^{-1/\mu}}^{\eta} |\ln^i x|\,\mathrm{d}x + O(n^{-1}) \\
&\qquad = s! \sum_{i=0}^{m} \binom{m+1}{i} \mu^i |a_{m-i+1,s}| (\eta |\ln^i \eta| + n^{-1/\mu} |\ln^i n^{-1/\mu}| + \dots) + O(n^{-1}) \\
&\qquad = O(\eta\, |\ln^m \eta|) + O(n^{-1/\mu}).
\end{aligned}
$$

It follows that if ψ is an arbitrary continuous function, then

$$
\lim_{n\to\infty} \int_{n^{-1/\mu}}^{\eta} x^r \psi(x) \int_{-1/n}^{1/n} \ln^{m+1}(x^\mu - t)\, \delta_n^{(s)}(t)\,\mathrm{d}t\,\mathrm{d}x = O(\eta |\ln^m \eta|) \tag{17}
$$

for $r = 1, 2, \ldots$.

Now let φ be an arbitrary function in $\mathcal{D}[-1,1]$. Then by Taylor's Theorem we have

$$\varphi(x) = \sum_{k=0}^{r-1} \frac{x^k}{k!}\,\varphi^{(k)}(0) + \frac{x^r}{r!}\,\varphi^{(r)}(\xi x),$$

where $0 < \xi < 1$. Then

$$\begin{aligned}
\langle G_{s,m,n}(x_+^\mu), \varphi(x)\rangle &= \int_{-1}^1 \varphi(x) G_{s,m,n}(x_+^\mu)\,\mathrm{d}x \\
&= \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^1 x^k G_{s,m,n}(x_+^\mu)\,\mathrm{d}x + \int_0^{n^{-1/\mu}} \frac{x^r\varphi^{(r)}(\xi x)}{r!}\,G_{s,m,n}(x_+^\mu)\,\mathrm{d}x \\
&\quad + \int_{n^{-1/\mu}}^{\eta} \frac{x^r\varphi^{(r)}(\xi x)}{r!}\,G_{s,m,n}(x_+^\mu)\,\mathrm{d}x + \int_\eta^1 \frac{x^r G_{s,m,n}(x_+^\mu)\varphi^{(s)}(\xi x)}{r!}\,\mathrm{d}x.
\end{aligned}$$

Using equations (1), (11) to (13) and (15) to (17), it follows that

$$\begin{aligned}
&\mathrm{N}\!-\!\lim_{n\to\infty}\langle G_{s,m,n}(x_+^\mu), \varphi(x)\rangle \\
&\quad = \sum_{k=0}^{r-1}\sum_{i=0}^{m} \binom{m+1}{i} \frac{(-1)^{s+1} i!\, s!\, \mu^i a_{m-i+1,s}}{(\mu s - k - 1)^{i+1} k!}\,\varphi^{(k)}(0) + O(\eta|\ln^m \eta|) \\
&\qquad\qquad + (-1)^s s! \sum_{i=0}^{m} \binom{m+1}{i} \int_\eta^1 \frac{\mu^i a_{m-i+1,s} x^{-s\mu} \ln^i x \varphi^{(s)}(\xi x)}{r!}\,\mathrm{d}x \\
&\quad = -\sum_{k=0}^{r-1}\sum_{i=0}^{m} \frac{c_{s,m,i}\mu^i i!}{(\mu s - k - 1)^{i+1} k!}\,\varphi^{(k)}(0) + \sum_{i=0}^{m} \int_0^1 \frac{c_{s,m,i}\mu^i \ln^i x \varphi^{(s)}(\xi x)}{r!}\,\mathrm{d}x,
\end{aligned}$$

since η can be made arbitrarily small. It now follows that

$$\begin{aligned}
&\mathrm{N}\!-\!\lim_{n\to\infty}\langle G_{s,m,n}(x_+^r), \varphi(x)\Big\rangle \\
&\quad = -\sum_{k=0}^{r-1}\sum_{i=0}^{m} \frac{c_{s,m,i}\mu^i\, i!}{(\mu s - k - 1)^{i+1} k!}\,\varphi^{(k)}(0) \\
&\qquad\qquad + \sum_{i=0}^{m} c_{s,m,i}\mu^i \int_{-1}^1 x^{-s\mu} \ln^i x \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{x^k}{k!}\,\varphi^{(k)}(0)\right] \mathrm{d}x \\
&\quad = \sum_{i=0}^{m} c_{s,m,i}\,\mu^i \left\{ \int_{-1}^1 x^{-s\mu} \ln^i x \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{x^k}{k!}\,\varphi^{(k)}(0)\right] \mathrm{d}x \right. \\
&\qquad\qquad \left. + \sum_{k=0}^{r-1} \frac{(-1)^i\, i!}{(-s\mu + k + 1)^{i+1} k!}\,\varphi^{(k)}(0) \right\} \\
&\quad = \sum_{i=0}^{m} c_{s,m,i}\,\mu^i \langle x_+^{-s\mu} \ln^i x_+, \varphi(x)\rangle
\end{aligned}$$

on using Lemmas 2 and 3. This proves that $G_{s,m}(x_+^\mu)$ and the sum

$$\sum_{i=0}^{m} c_{s,m,i}\,\mu^i\,x_+^{-s\mu}\ln^i x_+$$

exist and

$$G_{s,m}(x_+^\mu) = \sum_{i=0}^{m} c_{s,m,i}\,\mu^i\,x_+^{-s\mu}\ln^i x_+ \tag{18}$$

on the interval $[-1,1]$ for $m,r,s = 1,2,\ldots$. However, equation (1) clearly holds on any closed interval not containing the origin.

Now suppose that

$$F_{s,i}(x_+^\mu) = \mu^i\,x_+^{-s\mu}\,\ln^i x_+ \tag{19}$$

for $i = 0,1,\ldots,m-1$ for some m and $s = 1,2,\ldots$. This is true when $m = 1$ since we then have $F_{s,0}(x_+^\mu) = G_{s,0}(x_+^\mu)$.

Note that equation (18) can be rewritten in the form

$$G_{s,m}(x_+^\mu) = \sum_{i=0}^{m} c_{s,m,i}\,\mu^i\,F_{s,i}(x_+^\mu). \tag{20}$$

Since $G_{s,m}(x_+^\mu)$ exists and $F_{s,i}(x_+^\mu)$ exists by our assumption for $i = 0,1,\ldots,m-1$, it follows that $F_{s,m}(x_+^\mu)$ exists and

$$\begin{aligned} G_{s,m}(x_+^\mu) &= c_{s,m,m}\,\mu^i F_{s,m}(x_+^\mu) + \sum_{i=0}^{m-1} c_{s,m,i}\,F_{s,i}(x_+^\mu) \\ &= c_{s,m,m}\,\mu^i F_{s,m}(x_+^\mu) + \sum_{i=0}^{m-1} c_{s,m,i}\,\mu^i x_+^{-s\mu}\ln^i x_+ \\ &= \sum_{i=0}^{m} c_{s,m,i}\,\mu^i x_+^{-s\mu}\ln^i x_+, \end{aligned}$$

on using equations (19) and (20). It follows that

$$F_{s,m}(x_+^\mu) = \mu^m\,x_+^{-s\mu}\ln^m x_+$$

and so equation (19) holds for m. Equation (1) now follows by induction, completing the proof of the theorem.

Replacing x by $-x$ in Theorem 4, we get

Theorem 7. *If $F_{s,m}(x)$ denotes the distribution $x_-^{-s}\ln^m x_-$, then the neutrix composition $F_{s,m}(x_-^\mu)$ exists and*

$$F_{s,m}(x_-^\mu) = \mu^m\,x_-^{-s\mu}\ln^m x_- \tag{21}$$

for $m = 0,1,2,\ldots$, $s = 1,2,\ldots$, $\mu > 0$ and $s\mu \neq 1,2,\ldots$.

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