An investigation of the propagation of electromagnetic waves in some circular cylindrical waveguides using a finite difference formulation

by

P. J. Lawrence, M.Sc.

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CONTENTS

INTRODUCTION

CHAPTER 1 Mathematical formulation
  1.1 The configuration
  1.2 The permeability tensor
  1.3 Maxwell's equations
  1.4 The boundary conditions
  1.5 The boundary value problem

CHAPTER 2 The finite difference method
  2.1 Outline
  2.2 To find the propagation constant of TM modes in a circular waveguide completely filled with dielectric material

CHAPTER 3 The implementation of the finite difference method using a computer
  3.1 The sliding array method
  3.2 Adaption of the sliding array method for computer use
  3.3 The computer programmes

CHAPTER 4 A one region problem: the circular cylindrical waveguide completely filled with dielectric material
  4.1 The mathematical theory
  4.2 The finite difference formulations using the separate and simultaneous methods
  4.3 The computer programmes for the separate and simultaneous methods
  4.4 Numerical results

CHAPTER 5 The two region problem
  5.1 Boundary conditions
  5.2 The finite difference formulation of the boundary value problem of section 1.5
  5.3 The special case $\mu_1 = \mu_2 = \mu_0$
CHAPTER 6 The circular cylindrical waveguide filled with two coaxial dielectrics

6.1 The derivation of the characteristic equation
6.2 Solution of the characteristic equation
6.3 The finite difference method
6.4 The special case $\mu_2 = \mu_1$ and $\varepsilon_2 = \varepsilon_1$

CHAPTER 7 Some numerical results for the two region problem

7.1 Scaling and the two types of zero
7.2 The computer programmes
7.3 The special case of section 6.4
7.4 The case of two coaxial dielectrics

CHAPTER 8 Transversely magnetised ferrite tube in circular waveguide

8.1 Numerical results using the finite difference method
8.2 Conclusion

APPENDIX 1 The computer programme for the separate method

APPENDIX 2 The computer programme for the simultaneous method

APPENDIX 3 The computer programme to evaluate the function $f(n, B)$, the left-hand side of equation (6.18)

APPENDIX 4 Programme 4. The computer programme to evaluate the determinant $|I_0|$ of equation (5.15)

APPENDIX 5 Programme 5. The computer programme to evaluate the determinant $|I_0|$ of equation (5.15) in the special case $\mu_1 = \mu_2 = \mu_0$

APPENDIX 6 The propagation of plane waves in an infinite transversely magnetised ferrite medium

APPENDIX 7 Notation

BIBLIOGRAPHY
INTRODUCTION

The study of the theory of the propagation of electromagnetic waves through waveguides started at the end of the last century. One of the earliest important papers was written by Lord Rayleigh (1897). Work continued on the subject but it was not until just before the second World War that the waveguide became a practical way of transmitting electromagnetic waves. G. Southworth (1936), J. R. Carson, S. P. Mead and S. A. Schelkunoff (1936) and W. L. Barrow (1936) all made significant contributions to this development. The demand for radar for military purposes during World War II gave a great boost to work throughout the field. The problem of a waveguide longitudinally filled with two coaxial dielectrics was studied by H. Bucholz (1943) and L. Pincherle (1944) amongst others. All the work relating to dielectric loaded guides has been extensively reviewed by S. K. Chatterjee and Mrs. R. Chatterjee (1965).

In the post-war period, the magnetic properties of magnetised ferrites were studied. D. Polder (1949) and C. L. Hogan (1952) derived the susceptibility tensor of ferrite. The problem of propagation through a waveguide containing ferrite has also been tackled by a number of researchers. A. A. Van Trier (1953) and H. Suhl and L. R. Walker (1954) have done important work in this field. The circular waveguide containing a concentric longitudinally magnetised ferrite rod has been comprehensively studied by M. L. Kales (1953) and R. A. Waldron (1958). A. J. Baden Fuller (1961) has studied other configurations of longitudinally magnetised ferrite in a circular guide.

More generally, P. J. B. Clarricoats (1961), R. A. Waldron (1961) and A. G. Gurevich (1963) have ranged over the whole subject of ferrites and the influence of the material on guided waves for different directions of magnetisation. R. A. Waldron (1969) contains a survey of most of the work on electromagnetic wave propagation through waveguides.
At the present time, the problem of finding the propagation constant of electromagnetic waves travelling through a circular guide partially filled with concentric longitudinally magnetised ferrite material has been reduced to the solution of a characteristic equation. The equation can easily be solved using a computer.

On the other hand, in the case of ferrite being transversely magnetised, it does not seem to be possible to derive such an equation. The purpose of this thesis is to illustrate a numerical method for obtaining the propagation constant in a transversely magnetised case. The problem is expressed in boundary value form in section 1.5 and in finite difference form in section 5.2. Numerical results are given and a hypothesis is offered to explain them in Chapter 8. The generality of the adopted approach is also discussed.

Throughout the thesis the S.I. system of units is used. In this and basic electromagnetic theory the author follows J. A. Stratton (1941). All the numerical work was carried out on the computer configuration at the University of Manchester Regional Computer Centre. This configuration is based upon a CDC 7600 coupled to an ICL 1906A. The computer programmes used are given in the appendices. The term "word" is used to mean four bytes (thirty-two bits) in the appendices.
CHAPTER 1

Mathematical formulation

1.1 The configuration

A circular cylindrical waveguide of radius $a$, bounded by a perfectly conducting wall, contains a circular ferrite tube of inner radius $b$ and outer radius $a$, as shown in Figure 1.1. In a system of cylindrical polar coordinates, $r, \theta, z$ the $z$-axis coincides with the axis of the guide, the region $0 \leq r < b$ is lossless, homogeneous, isotropic dielectric and the ferrite is magnetised by a transverse static magnetic field $H_0(r, \theta)$. The direction of the field is shown in Figure 1.2.

For electromagnetic waves travelling in the positive $z$-direction along the guide with temporal and spatial dependence $\exp(j(\omega t - \beta z))$, any component of the electric or magnetic field can be assumed to have the form $f(r, \theta)\exp(j(\omega t - \beta z))$. The problem then reduces to a two-dimensional one since all dependence on $z$ and $t$ will cancel out in Maxwell's equations.
Figure 1.1. The configuration

Figure 1.2. The static magnetic field

Figure 1.3. The two regions
1.2 The permeability tensor

In general, the permeability tensor of ferrite material saturated by a static magnetic field can be written in a rectangular cartesian coordinate system in the form

$$\begin{bmatrix} \mu_2 & -j\kappa & 0 \\ j\kappa & \mu_2 & 0 \\ 0 & 0 & \mu_0 \end{bmatrix}$$

(1.1)

when the direction of the static magnetic field and hence of the magnetisation is along the z-axis. The gyroaxis at any point is defined to be the axis in the direction of the static magnetic field at that point, and, following J. A. Stratton (1941)

$$B = [\mu]H = \mu_0 (H + M).$$

(1.2)

Throughout this work energy losses are assumed small enough to be neglected. This is a reasonable approximation at a frequency not close to the frequency of gyromagnetic resonance $\mu_0\gamma H_0$ and then $\mu_2$ and $\kappa$ are real and given by

$$\mu_2 = \mu_0 \left( 1 + \frac{\mu_0^2 \gamma_0^2 H_0 M_0}{\mu_0^2 \gamma_0^2 H_0^2 - \omega^2} \right)$$

(1.3)

and

$$\kappa = \frac{\mu_0^2 \gamma_0 M_0}{\mu_0^2 \gamma_0^2 H_0^2 - \omega^2}$$

(1.4)

where $\gamma$ denotes the gyromagnetic ratio and $H_0$ and $M_0$ are, respectively, the magnitudes of the static magnetic field and of the saturation magnetisation of the ferrite. The results outlined above were derived by Polder (1949).

By putting $\mu_2 = \mu_0$ and $\kappa = 0$ in (1.1) the permeability tensor for vacuum is obtained. If the static magnetic field $H_0$ is reversed in direction the permeability tensor will be that of (1.1) with $\kappa$ replaced by $-\kappa$, as can easily be seen by substituting $-H_0$ and $-M_0$ for $H_0$ and $M_0$ in (1.3) and (1.4).
We consider the unmagnetised ferrite material to be isotropic. When magnetised to saturation it displays rotational symmetry about the axis of static magnetisation, as shown by the form of the permeability tensor. Even if the ferrite were not saturated by an applied static magnetic field it would still display rotational symmetry about the axis of that field and hence the permeability tensor would have a similar form to that given by (1.1) without the parameters $\nu_2$ and $K$ having the specific values of (1.3) and (1.4).

Returning to the description in section 1.1, the direction of the transverse static magnetic field $H_0$ applied to the ferrite varies with $\theta$ as indicated in Figure 1.2. In order to achieve a mathematical representation of the permeability tensor at any point in the ferrite, the following assumptions are made concerning the magnetic field $H_0$;

a) $|H_0|$ is constant throughout the ferrite,

b) the direction of the field is constant along any line $\theta = \text{constant}$, $b \leq r \leq a$,

c) the direction of the field varies at a uniform rate with $\theta$, i.e. $\frac{d\alpha}{d\theta} = \text{constant}$ where $\alpha$ is the angle between the field direction and the $\theta$-axis (see Figure 1.2).

These assumptions seem reasonable especially if the tube thickness $a - b$ is small.

If $\hat{\chi}$ denotes the unit vector in the direction of magnetisation

$$\hat{\chi} = \sin 2\theta \hat{c} + \cos 2\theta \hat{c}.$$  \hspace{1cm} (1.5)

A right-hand cartesian coordinate system $S'$, is formed by taking $\hat{c}$ along the guide and $\hat{\chi}$ as the remaining axis.

i.e.

$$\hat{\chi} = \cos 2\theta \hat{c} - \sin 2\theta \hat{c}.$$  \hspace{1cm} (1.6)
Denoting the original cylindrical polar coordinate system by $S$, it follows that

$$\chi' = [P] \chi$$  \hspace{1cm} (1.7)

where $\chi'$ is a vector measured in $S'$, $\chi$ is the same vector measured in $S$ and

$$[P] = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (1.8)

using (1.5) and (1.6).

In $S'$ the permeability tensor $[\mu']$ can be written in the form

$$[\mu'] = \begin{bmatrix} \mu_2 & 0 & jk \\ 0 & \mu_0 & 0 \\ -jk & 0 & \mu_2 \end{bmatrix}$$  \hspace{1cm} (1.9)

in a similar way to (1.1) and the relationship

$$\mathcal{B}' = [\mu'] \mathcal{H}'$$  \hspace{1cm} (1.10)

holds.

In $S$,

$$\mathcal{B} = [\mu'] \mathcal{H}$$

i.e

$$[P]^{-1} \mathcal{B}' = [\mu'] \mathcal{H}$$

$$[P]^{-1} [\mu'] \mathcal{H}' = [\mu'] \mathcal{H}$$

$$[P]^{-1} [\mu'][P] \mathcal{H} = [\mu'] \mathcal{H}$$

It follows that

$$[\mu] = [P]^{-1} [\mu'][P]$$  \hspace{1cm} (1.11)
i.e. $[\mu] = \begin{bmatrix} 
\cos 2\theta & \sin 2\theta & 0 \\
-\sin 2\theta & \cos 2\theta & 0 \\
0 & 0 & 1 
\end{bmatrix} \begin{bmatrix} 
\mu_2 & 0 & jk \\
0 & \mu_0 & 0 \\
-jk & 0 & \mu_2 
\end{bmatrix} \begin{bmatrix} 
\cos 2 & -\sin 2 & 0 \\
\sin 2\theta & \cos 2\theta & 0 \\
0 & 0 & 1 
\end{bmatrix}$

$[\mu] = \begin{bmatrix} 
\mu_2 \cos 2\theta & \mu_0 \sin 2\theta & jk \cos 2\theta \\
-\mu_2 \sin 2\theta & \mu_0 \cos 2\theta & -jk \sin 2\theta \\
-jk & 0 & \mu_2 
\end{bmatrix} \begin{bmatrix} 
\cos 2\theta & -\sin 2\theta & 0 \\
\sin 2\theta & \cos 2\theta & 0 \\
0 & 0 & 1 
\end{bmatrix}$

$[\mu] = \begin{bmatrix} 
\mu_2 \cos^2 2\theta + \mu_0 \sin^2 2\theta & (\mu_0 - \mu_2) \sin 2\theta \cos 2\theta & jk \cos 2\theta \\
(\mu_0 - \mu_2) \sin 2\theta \cos 2\theta & \mu_2 \sin^2 2\theta + \mu_0 \cos^2 2\theta & -jk \sin 2\theta \\
-jk \cos 2\theta & jk \sin 2\theta & \mu_2 
\end{bmatrix}$
1.3 Maxwell's equations

The interior of the waveguide can be split into two regions. Region (1) is specified by $0 \leq r \leq b$ and here there is both a real scalar permittivity $\varepsilon_1$, and a real scalar permeability $\mu_1$. In region (2), $b \leq r \leq a$ the permittivity is denoted by $\varepsilon_2$, a constant real scalar and the permeability is $[\mu]$ given by (1.12) (see Figure 1.3).

An electromagnetic wave propagating along the guide will satisfy Maxwell's equations in each region. With an $\exp(j(\omega t - \beta z))$ dependence, it is apparent that $\beta$ must be the same in the ferrite and dielectric in order to satisfy any boundary conditions on the interface between them for all values of $z$. Also with such a dependence,

$$\frac{\partial}{\partial t} \equiv j\omega \quad \text{and} \quad \frac{\partial}{\partial z} \equiv -j\beta.$$ 

In region (2) Maxwell's equations give

$$\nabla \times \mathbf{H} = j\omega \mathbf{D} = j\omega \varepsilon_2 \mathbf{E}_r$$

and

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B} = -j\omega [\mu] \mathbf{H}_r$$

From (1.13)

$$\frac{1}{r} \frac{\partial \mathbf{H}_r}{\partial \theta} + j\beta \mathbf{H}_\phi = j\omega \varepsilon_2 \mathbf{E}_r$$

and

$$\frac{1}{r} \frac{\partial \mathbf{H}_\phi}{\partial r} - j\beta \mathbf{H}_r = j\omega \varepsilon_2 \mathbf{E}_\theta$$

From (1.14)

$$\frac{1}{r} \frac{\partial \mathbf{E}_r}{\partial \theta} + j\beta \mathbf{E}_\phi = -j\omega (\mu_{11}\mathbf{H}_r + \mu_{12}\mathbf{H}_\phi + \mu_{13}\mathbf{H}_z)$$

and

$$\frac{1}{r} \frac{\partial \mathbf{E}_\phi}{\partial r} - j\beta \mathbf{E}_r = -j\omega (\mu_{12}\mathbf{H}_r + \mu_{22}\mathbf{H}_\phi + \mu_{23}\mathbf{H}_z)$$

and

$$\frac{1}{r} \frac{\partial \mathbf{E}_\theta}{\partial r} + \frac{\partial \mathbf{E}_r}{\partial \theta} = -j\omega (\mu_{31}\mathbf{H}_r + \mu_{32}\mathbf{H}_\phi + \mu_{33}\mathbf{H}_z)$$
where

\[
\begin{bmatrix}
\mu & \nu_1 & \nu_2 \\
\nu_1 & \nu_2 & \nu_3 \\
\nu_2 & \nu_3 & \nu_4
\end{bmatrix}
\]

is given by (1.12).

$E_\tau$ and $E_\theta$ can be easily eliminated by substituting from (1.15) and (1.16) into (1.18), (1.19) and (1.20). This gives the following four equations for the four unknowns $E_z$, $H_\tau$, $H_\theta$, and $H_z$:

\[
\left(\omega^2 \varepsilon_2 \mu_{11} - \beta^2 H_\tau \right)_r + \omega^2 \varepsilon_2 \mu_{12} H_\theta = j \omega \varepsilon_2 \left( \frac{1}{r} \frac{\partial H_\theta}{\partial r} - j \beta \frac{\partial H_\tau}{\partial \theta} - \omega^2 \varepsilon_2 \mu_{13} H_z \right) \tag{1.22}
\]

\[
\omega^2 \varepsilon_2 \mu_{21} H_\tau + (\omega^2 \varepsilon_2 \mu_{22} - \beta^2) H_\theta = - j \omega \varepsilon_2 \left( \frac{1}{r} \frac{\partial E_z}{\partial r} - j \beta \frac{1}{r} \frac{\partial E_\theta}{\partial \theta} - \omega^2 \varepsilon_2 \mu_{23} H_z \right) \tag{1.23}
\]

and

\[
- j \beta \frac{\partial H_z}{\partial \tau} - j \beta \frac{1}{r} \frac{\partial H_\theta}{\partial \theta} - (j \beta \frac{1}{r} + \omega^2 \varepsilon_2 \mu_{31}) H_\tau - \omega^2 \varepsilon_2 \mu_{32} H_\theta = \frac{3}{r^2} \frac{\partial^2 H_z}{\partial \tau^2} + \frac{1}{r} \frac{\partial^2 H_z}{\partial \tau \partial \theta} + \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \theta^2} + \omega^2 \varepsilon_2 \mu_{23} H_z \tag{1.24}
\]

After some manipulation $H_\tau$ and $H_\theta$ are eliminated yielding

\[
\beta \omega \varepsilon_2 \left\{ - \mu_{12} \frac{\partial^2 E_z}{\partial r^2} + (\mu_{11} - \mu_{22}) \frac{1}{r} \frac{\partial^2 E_z}{\partial r \partial \theta} + \mu_{12} \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} \right\} + (\omega^2 \varepsilon_2 \mu_{00} - \beta^2 \mu) \frac{3}{r^2} \frac{\partial^2 H_z}{\partial r^2} - 2 \beta \mu_{12} \frac{1}{r} \frac{\partial^2 H_z}{\partial r \partial \theta} + (\omega^2 \varepsilon_2 \mu_{02} - \beta^2 \mu_{22}) \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \theta^2} + \omega \varepsilon_2 \left\{ k \sin \theta \cos^2 \theta - \mu_{12} \frac{1}{r} \frac{\partial E_z}{\partial r} + \left( \omega^2 \varepsilon_2 \mu_{02} - \beta^2 [ \mu_{11} + 2(\mu_{02} - \mu_2)(\cos^2 \theta - \sin^2 \theta)] \right) \right\} \frac{1}{r} \frac{\partial H_z}{\partial r} + \omega \varepsilon_2 \left\{ k \cos \theta \sin^2 \theta \frac{1}{r} + 2 \beta (\mu_0 - \mu_2) (\cos^2 \theta - \sin^2 \theta) \frac{1}{r^2} \frac{\partial E_z}{\partial \theta} + 4 \beta^2 \mu_{12} \frac{1}{r^2} \frac{\partial H_z}{\partial \theta} \right\} + k^2 \left( \omega^2 \varepsilon_2 (\mu_2^2 - k^2) - \beta k \cos \theta \sin^2 \theta \frac{1}{r} - \mu_2 \beta^2 \right) H_z = 0 \tag{1.25}
\]
\[-(\beta^2 - \omega^2 \varepsilon_2 \mu_{11}) \frac{\partial^2 E_z}{\partial r^2} + 2\omega^2 \varepsilon_2 \mu_{12} \frac{1}{r} \frac{\partial^2 E_z}{\partial r \partial \theta} - (\beta^2 - \omega^2 \varepsilon_2 \mu_{22}) \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} \]

\[+ \omega \left\{ -\mu_{12} \frac{\partial^2 H_z}{\partial r^2} + (\mu_{11} - \mu_{22}) \frac{1}{r} \frac{\partial^2 H_z}{\partial r \partial \theta} + \mu_{12} \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \theta^2} \right\} \]

\[+ \left\{ 2\omega^2 \varepsilon_2 (\mu_0 - \mu_2) (\cos^2 \theta - \sin^2 \theta) - (\beta^2 - \omega^2 \varepsilon_2 \mu_{11}) \right\} \frac{1}{r} \frac{\partial E_z}{\partial \theta} \]

\[+ \omega \left( -\kappa \sin 2\theta k^2 + 3\mu_{12} \frac{1}{r} \frac{\partial H_z}{\partial r} \right) \]

\[- 4\omega^2 \varepsilon_2 \mu_{12} \frac{1}{r^2} \frac{\partial E_z}{\partial \theta} + \omega \left\{ 2\beta (\mu_0 - \mu_2) (\cos^2 \theta - \sin^2 \theta) \frac{1}{r^2} - \kappa \cos 2\theta k^2 \frac{1}{r} \frac{\partial E_z}{\partial \theta} \right\} \]

\[+ k^2 k_2^2 E_z + \omega \kappa \sin 2\theta k^2 \frac{1}{r} H_z = 0 \quad (1.26) \]

with \( k^2 = \omega^2 \varepsilon_2 \mu_0 - \beta^2 \) and \( k_2^2 = \omega^2 \varepsilon_2 \mu_2 - \beta^2 \).

On putting \( \mu_2 = \mu_0 = \mu_1, \kappa = 0 \) and \( \varepsilon_2 = \varepsilon_1 \), these equations reduce to

\[ \frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r} \frac{\partial H_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \theta^2} + k_1^2 H_z = 0 \quad (1.27) \]

and

\[ \frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r} \frac{\partial E_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} + k_1^2 E_z = 0 \quad (1.28) \]

with \( k_1^2 = \omega^2 \varepsilon_1 \mu_1 - \beta^2 \).

The equations have uncoupled to give Bessel's equation for both \( H_z \) and \( E_z \). The parameters \( \mu_2, \mu_0, \kappa \) and \( \varepsilon_2 \) have been given their respective values for the dielectric region (1) and hence Maxwell's equations give (1.27) and (1.28) there. These arise naturally from Maxwell's equations for region (1) as is well-known.

Two more useful relationships obtainable from (1.15) to (1.20) give \( H_\theta \) and \( E_\theta \) in terms of \( H_z \) and \( E_z \) and their first derivatives with respect to \( r \) and \( \theta \), viz:

\[ jk^2 k_2^2 H_\theta = -\omega^2 \varepsilon_2 \kappa \sin 2\theta k^2 H_z + \omega^3 \varepsilon_2 \mu_{12} \frac{1}{r} \frac{\partial E_z}{\partial \theta} - (\beta^2 - \omega^2 \varepsilon_2 \mu_{11}) \omega \varepsilon_2 \frac{3E_z}{3r} \]

\[ - \omega^2 \varepsilon_2 \mu_{12} \frac{\partial H_z}{\partial r} - \beta (\beta^2 - \omega^2 \varepsilon_2 \mu_{11}) \frac{1}{r} \frac{\partial H_z}{\partial \theta} \quad (1.29) \]

and
\[ j k^2 k^2 E_{\theta} = -\omega \beta k \cos 2\theta k^2 H_z - \beta (\beta^2 - \omega^2 \varepsilon_2 \mu_{22}) \left( \frac{1}{r} \frac{\partial E_z}{\partial \theta} + \frac{\beta}{r} \frac{\omega}{\varepsilon_2 \mu_1 \mu_2} \frac{\partial E_z}{\partial r} \right) + \beta \omega \mu_{12} \frac{1}{r} \frac{\partial H_z}{\partial \theta} + \omega (\beta^2 \mu_{11} - \omega^2 \varepsilon_2 \mu_{22} \mu_0) \frac{\partial H_z}{\partial r}. \] (1.30)

On putting \( \mu_0 = \mu_2 \) and \( k = 0 \) and interchanging \( \varepsilon_1 \) and \( \mu_1 \) for \( \varepsilon_2 \) and \( \mu_2 \) these equations reduce to

\[ j k^1 H_{\theta} = \omega \varepsilon_1 \frac{\partial E_z}{\partial r} + \frac{\beta}{r} \frac{\partial H_z}{\partial \theta}, \] (1.31)

and

\[ j k^1 E_{\theta} = -\omega \mu_1 \frac{\partial H_z}{\partial r} + \frac{\beta}{r} \frac{\partial E_z}{\partial \theta}. \] (1.32)

The parameter substitution chosen has given the respective parameter values for region (1) and hence (1.31) and (1.32) hold in that region. Like (1.27) and (1.28) they could have been obtained directly from Maxwell's equations for region (1).

In this section (1.25), (1.26), (1.29) and (1.30) have been obtained in region (2), and (1.27), (1.28), (1.31) and (1.32) in region (1).
1.4 The boundary conditions

For two adjacent dielectric media of finite conductivities it is well known (e.g. see Stratton (1941)) that at the interface between them

\[ \mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \quad (1.33) \]

and

\[ \mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = 0 \quad (1.34) \]

where \( \mathbf{n} \) is the unit normal from medium (1) into medium (2) and \( \mathbf{E}_i \) and \( \mathbf{H}_i \) (i = 1, 2) are the usual field vectors suffixed to denote the medium in which they lie.

Also at the interface of a dielectric and conductor it is known (e.g. Stratton (1941)) that

\[ \mathbf{n} \times \mathbf{E} = 0 \quad (1.35) \]

with the usual notation.

Applying these results to the present problem it should be first noted that \( \mathbf{n} = \hat{\mathbf{r}} \) and hence on the interface between region (1) and region (2),

from (1.33)

\[ E_{\theta 1} = E_{\theta 2} \quad \text{at } r = b \quad (1.36) \]

\[ E_{z1} = E_{z2} \quad \text{at } r = b \quad (1.37) \]

and from (1.34)

\[ H_{\theta 1} = H_{\theta 2} \quad \text{at } r = b \quad (1.38) \]

\[ H_{z1} = H_{z2} \quad \text{at } r = b \quad (1.39) \]

where again and throughout this work the numbers in the suffices refer to the region in which the vector component lies.

Also the waveguide wall, which is assumed to be a perfect conductor, is adjacent to region (2) and hence from (1.35)

\[ E_{\theta 2} = 0 \quad \text{at } r = a \quad (1.40) \]

\[ E_{z2} = 0 \quad \text{at } r = a \quad (1.41) \]

A further condition on the problem is the fact that the fields must be finite at the origin.

i.e.

\[ E_{z1} \text{ is finite at } r = 0 \quad (1.42) \]

\[ H_{z1} \text{ is finite at } r = 0 \quad (1.43) \]
The relations (1.30), (1.32), (1.29) and (1.31) can be used to eliminate $E_\theta$ and $H_\theta$ from the boundary conditions. Equations (1.40), (1.36) and (1.38) can then be rewritten in the form

\[-\omega k \cos 2\theta k^2 H_z + \beta \omega^2 \varepsilon_2 \mu_{12} \frac{\partial E_z}{\partial r} + \omega (\beta^2 \mu_{11} - \omega^2 \varepsilon_2 \mu_{22}) \frac{\partial H_z}{\partial r} \]
\[-\beta (\beta^2 - \omega^2 \varepsilon_2 \mu_{22}) \frac{1}{r} \frac{\partial E_z}{\partial \theta} + \beta^2 \omega \mu_{12} \frac{1}{r} \frac{\partial H_z}{\partial \theta} = 0 \quad \text{at } r = a \ (1.44)\]

\[-\frac{1}{k^2 k_{22}} \left( -\omega \mu_1 \frac{\partial H_{z1}}{\partial r} + \beta \frac{\partial E_{z1}}{\partial \theta} \right) = \frac{1}{k^2 k_{22}} \left\{ -\omega k \cos 2\theta k^2 H_{z2} - \beta (\beta^2 - \omega^2 \varepsilon_2 \mu_{22}) \frac{1}{r} \frac{\partial E_{z2}}{\partial \theta} + \beta \omega^2 \varepsilon_2 \mu_{12} \frac{\partial E_{z2}}{\partial \theta} + \beta^2 \omega \mu_{12} \frac{1}{r} \frac{\partial H_{z2}}{\partial \theta} + \omega (\beta^2 \mu_{11} - \omega^2 \varepsilon_2 \mu_{22}) \frac{\partial H_{z2}}{\partial \theta} \right\} \quad \text{at } r = b \ (1.45)\]

and

\[-\frac{1}{k^2 k_{22}} \left( \omega \varepsilon_1 \frac{\partial E_{z1}}{\partial r} + \beta \frac{\partial H_{z1}}{\partial \theta} \right) = \frac{1}{k^2 k_{22}} \left\{ -\omega^2 \varepsilon_2 k^2 H_{z2} + \omega^3 \varepsilon_2 \mu_{12} \frac{1}{r} \frac{\partial E_{z2}}{\partial \theta} + \beta^2 \omega^2 \varepsilon_2 \mu_{12} \frac{\partial H_{z2}}{\partial \theta} - \beta (\beta^2 - \omega^2 \varepsilon_2 \mu_{11}) \frac{1}{r} \frac{\partial H_{z2}}{\partial \theta} \right\} \quad \text{at } r = b \ (1.46)\]
1.5 The boundary value problem

All the equations so far derived have now been expressed in terms of two dependent variables $E_z(r, \theta)$ and $H_z(r, \theta)$, the factor $\exp(j(\omega t - \beta z))$ being understood throughout. Collecting together and simplifying these equations by introducing $B$, which can be thought of as a scaled propagation constant, and $c$ where $\beta^2 = B^2 \omega^2 \epsilon_1 \mu_1$ and $c^2 = \frac{1}{\epsilon_1 \mu_1}$, gives the following:

from (1.25)

$$
\varepsilon_2 B c \left\{ - \mu_1 \frac{\partial^2 E_z}{\partial r^2} + (\mu_{11} - \mu_{22}) \frac{1}{r} \frac{\partial^2 E_z}{\partial r \partial \theta} + \mu_1 \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} \right\}
- \frac{\omega \varepsilon_2}{\omega} \sin 2\theta E_b + \frac{3Bc}{\omega} \mu_1 \frac{1}{r} \frac{\partial E_z}{\partial r} + \left\{ \mu_0 \varepsilon_r \mu_r - B^2 [\mu_{11} + 2(\mu_0 - \mu_2) \cos^2 \theta - \sin^2 \theta] \right\} \frac{1}{r} \frac{\partial H_z}{\partial r}
+ \varepsilon_2 \left\{ - \frac{\omega \kappa}{\omega} \cos 2\theta E_b \frac{1}{r} + 2Bc(\mu_0 - \mu_2) \cos^2 \theta \frac{1}{r^2} \frac{\partial E_z}{\partial \theta} + 4B^2 \mu_1 \frac{1}{r^2} \frac{\partial H_z}{\partial \theta} \right\}
- E_b \left\{ \omega^2 \varepsilon_2 (\mu_2 - \kappa^2) - \frac{B \omega \kappa}{c} \cos 2\theta \frac{1}{r} - \mu_2 B^2 \omega^2 \epsilon_1 \mu_1 \right\} H_z = 0
$$

$$
b \leq r \leq a, \quad 0 \leq \theta \leq 2\pi; \quad (1.47)
$$

from (1.26)

$$
- \frac{\varepsilon_r}{\mu_1} \frac{\partial^2 E_z}{\partial r^2} + 2 \varepsilon_2 \frac{\mu_1}{r^2} \frac{\partial^2 E_z}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} + B c \left\{ - \mu_1 \frac{1}{r^2} \frac{\partial^2 H_z}{\partial r \partial \theta} + (\mu_{11} - \mu_2) \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \theta^2} + \mu_1 \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \theta^2} \right\}
+ \frac{2}{\mu_1} \frac{\partial E_z}{\partial r} + \left\{ \frac{\omega \kappa}{\omega} \sin 2\theta E_b + 3Bc \mu_1 \frac{1}{r} \right\} \frac{\partial H_z}{\partial r}
- \frac{4}{\mu_1} \frac{\varepsilon_r}{r^2} \frac{\partial E_z}{\partial \theta} + \left\{ 2Bc(\mu_0 - \mu_2) \cos^2 \theta \sin^2 \theta \frac{1}{r^2} + \omega \kappa \cos 2\theta E_b \frac{1}{r} \right\} \frac{\partial H_z}{\partial \theta}
+ \omega^2 \epsilon_1 \mu_1 E_b E_z - \omega \kappa \sin 2\theta E_b \frac{1}{r} H_z = 0
$$

$$
b \leq r \leq a, \quad 0 \leq \theta \leq 2\pi \quad (1.48)
$$
from (1.27)

\[
\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r} \frac{\partial H_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \theta^2} + \omega^2 \mu_1 \mu_2 H_z = 0
\]

\(0 \leq r \leq b, \ 0 \leq \theta \leq 2\pi; \quad (1.49)\)

from (1.28)

\[
\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r} \frac{\partial E_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} + \omega^2 \mu_1 \mu_2 E_z = 0
\]

\(0 \leq r \leq b, \ 0 \leq \theta \leq 2\pi; \quad (1.50)\)

from (1.41)

\[E_z = 0 \quad r = a, \ 0 \leq \theta \leq 2\pi; \quad (1.51)\]

from (1.44)

\[
\frac{\omega B_k}{c} \cos 2\theta E_z H_z - B \epsilon \mu_1 \mu_2 \frac{\partial E_z}{\partial r} + E_f \frac{\partial H_z}{\partial r} - \frac{B}{c} E \mu_1 \mu_2 \frac{\partial E_z}{\partial \theta} + B^2 \mu_1 \mu_2 \frac{\partial^2 E_z}{\partial \theta^2} = 0
\]

\(r = a, \ 0 \leq \theta \leq 2\pi; \quad (1.52)\)

from (1.42)

\[E_{z1} \text{ is finite at } r = 0; \quad (1.53)\]

from (1.43)

\[H_{z1} \text{ is finite at } r = 0; \quad (1.54)\]

from (1.45)

\[
E \left\{ \frac{\omega B_k}{c} \cos 2\theta E_z H_z \right. \\
- \left. \frac{B}{c} E \mu_1 \mu_2 \frac{\partial E_z}{\partial r} + \frac{B}{c} \mu_1 \mu_2 \frac{\partial^2 E_z}{\partial \theta^2} + \frac{B^2 \mu_1 \mu_2}{r} \frac{\partial^2 E_z}{\partial \theta^2} \right\}
\]

\(r = b, \ 0 \leq \theta \leq 2\pi; \quad (1.55)\)

from (1.37)

\[E_{z1} = E_{z2} \quad r = b, \ 0 \leq \theta \leq 2\pi; \quad (1.56)\]
from (1.46)
\[
E_a \left\{ \omega \varepsilon_2 \sin 2 \theta E_b H_z^2 + \frac{\varepsilon_2 E_{12}}{\mu_1} \frac{\partial E_z^2}{\partial \theta} - \frac{\varepsilon_2 E_{12}}{c} \frac{\partial E_z^2}{\partial r} - \frac{\varepsilon_r B}{\mu_1} \frac{\partial H_z^2}{\partial r} + \frac{B}{c} E_d \left( \frac{1}{r} \frac{\partial H_z^2}{\partial \theta} \right) \right\}
\]
\[
= E_b E_c \left( \frac{\varepsilon_1}{r} \frac{\partial E_z^2}{\partial r} + \frac{B}{c} \frac{1}{r} \frac{\partial H_z^2}{\partial \theta} \right)
\]
\[
r = b, \quad 0 \leq \theta \leq 2\pi ; \quad (1.57)
\]
from (1.38)
\[
H_{z1} = H_{z2} \quad r = b, \quad 0 \leq \theta \leq 2\pi ; \quad (1.58)
\]
where
\[
\varepsilon_r = \frac{\varepsilon_2}{\varepsilon_1} ,
\]
\[
\mu_r = \frac{\mu_2}{\mu_1} ,
\]
\[
E_a = 1 - B^2 ,
\]
\[
E_b = B^2 - \varepsilon_r \frac{\mu_0}{\mu_1} ,
\]
\[
E_c = B^2 - \varepsilon_r \frac{\mu_1}{\mu_1} ,
\]
\[
E_d = B^2 - \varepsilon_r \frac{\mu_{11}}{\mu_1} ,
\]
\[
E_e = B^2 - \varepsilon_r \frac{\mu_{22}}{\mu_1} ,
\]
\[
E_f = B^2 \mu_{11} - \varepsilon_r \mu_r \mu_0 ,
\]
\[
and \quad E_g = B^2 \mu_{22} - \varepsilon_r \mu_r \mu_0 .
\]

It should be remembered that \( \mu_{11}, \mu_{12} \) and \( \mu_{22} \) and hence \( E_d, E_e, E_f \) and \( E_g \) are functions of \( \theta \) so that many of the coefficients of the partial derivatives of \( E_z \) and \( H_z \) in the equations are functions of \( r \) and \( \theta \).

The problem is given completely by equations (1.47) to (1.58) in a boundary value form. All the coefficients in the equations are real. This simplifies the later numerical work on the problem. It is necessary to solve the equations for the separate regions subject to the appropriate
boundary conditions in order to obtain a complete solution.

In region (1) the solution is straightforward. As can be seen from (1.49), (1.50), (1.53) and (1.54) the dependent variables $E_z$ and $H_z$ are completely uncoupled and so can be dealt with independently of each other. Equations (1.49) and (1.50) are simply Bessel's equation and so with the boundary conditions (1.53) and (1.54) and asserting that the field be periodic in $\theta$, it follows that

$$E_z = \sum_{n=0}^{\infty} A_n J_n(k_1 r) e^{-jn\theta}$$

and

$$H_z = \sum_{n=0}^{\infty} B_n J_n(k_1 r) e^{-jn\theta}$$

The $A_n$ and $B_n$ are arbitrary complex constants to be determined by the conditions at $r = b$.

In region (2) the basic equations (1.47) and (1.48) are two simultaneous second order partial differential equations with coefficients which are functions of the independent variables $r$ and $\theta$. It appears to be impossible to uncouple them and very unlikely that an analytic solution exists. A valid solution will contain arbitrary constants. Elimination of these together with $A_n$ and $B_n$ above, via (1.55) to (1.58), will yield a condition on $B$, and hence on $\beta$, for a non-zero solution. This condition, the characteristic equation for the problem, will determine the admissible values of $B$. The problem may be regarded as one to determine the eigenvalues $B$ of the system of equations (1.47) to (1.58). In particular, only positive real values of $B$ are required to give unattenuated propagation down the guide.
2.1 Outline

The calculus of finite differences (L. M. Milne-Thomson (1933)) is concerned with the changes in value of a function, the dependent variable, due to changes in the independent variable. It has wide application as the basis of numerical methods including the solution of both ordinary and partial differential equations. A differential equation for a function over a domain is converted into a difference equation at each of a number of chosen points in the domain. The difference equation is just an algebraic equation between the values of the function at the chosen points. In order to carry out this conversion the differential coefficients of the function at any chosen point are approximated by differences between its values at neighbouring points. The boundary conditions are also put into difference equation form. Then by solving the system of difference equations it is hoped that one obtains an approximation to the solution of the differential equation boundary value problem.

This approach has been widely used in the field of partial differential equations (G. E. Forsythe and W. R. Wasow (1959)). The domain of the partial differential equation is covered by a mesh or grid of points and then the first order partial differential coefficients of the dependent variable \( z(x,y) \) at any mesh point \((x_i,y_j)\) may be approximated by the forward difference formulae

\[
\frac{\partial z}{\partial x}(x_i,y_j) = \frac{z(x_{i+1},y_j) - z(x_i,y_j)}{hx} + O(hx) \quad (2.1)
\]

and

\[
\frac{\partial z}{\partial y}(x_i,y_{j+1}) = \frac{z(x_i,y_{j+1}) - z(x_i,y_j)}{hy} + O(hy) \quad (2.2)
\]

where \( hx = x_{i+1} - x_i \) and \( hy = y_{j+1} - y_j \). Alternatively central or backward difference formulae may be used. The second order
coefficients are given by

\[ \frac{\partial^2 z}{\partial x^2}(x_i, y_j) = \frac{z(x_{i+1}, y_j) - 2z(x_i, y_j) + z(x_{i-1}, y_j)}{hx^2} + 0(hx^2) \]  

(2.3)

\[ \frac{\partial^2 z}{\partial x \partial y}(x_i, y_j) = \frac{z(x_{i+1}, y_{j+1}) - z(x_i, y_{j+1}) - z(x_{i+1}, y_j) + z(x_i, y_j)}{hxhy} + 0(hx) + 0(hy) \]  

(2.4)

and

\[ \frac{\partial^2 z}{\partial y^2}(x_i, y_j) = \frac{z(x_{i+1}, y_{j+1}) - 2z(x_i, y_j) + z(x_{i+1}, y_{j-1})}{hy^2} + 0(hy^2) \]  

(2.5)

where \( hx = x_{i+1} - x_i = x_i - x_{i-1} \) and \( hy = y_{j+1} - y_j = y_j - y_{j-1} \).

Similar formulae can be derived for higher order coefficients. Using these approximations the partial differential equation is converted into a difference equation at each mesh point within the domain. The boundary conditions are transformed into difference equations. This transformation is usually straightforward in the case of a regular boundary. In other cases it can be extremely difficult.

After the conversion of the partial differential equation and boundary conditions into difference equations, the boundary value problem is reduced to one of a large number of algebraic equations, \( n \) say, between the values of the function at the same large number of mesh points. A large number is needed to ensure a reasonable approximation. If all the \( n \) equations are linear and homogeneous they can be written in the form

\[ \mathbf{M} \mathbf{Z} = \mathbf{0} \]  

(2.6)

where \( \mathbf{Z} \) is the \( n \) column vector whose components are the values of the function at the mesh points, \( \mathbf{0} \) is the corresponding zero \( n \) column vector and \( \mathbf{M} \) is an \((n \times n)\) matrix whose elements are the coefficients of the variables, the values at the mesh points, in the equations.

Then by the general theory of equations, for a non-trivial solution of system (2.6) the determinant of \( \mathbf{M} \) must be zero

i.e.

\[ |\mathbf{M}| = 0 \]  

(2.7)
(2.7) is the condition on the coefficients of the \( n \) algebraic equations. It is this condition which will give the characteristic equation for the propagation constant for the electromagnetic wave problems considered in this thesis. Since the difference representations of the differential coefficients are not exact, it would perhaps be more precise to consider the condition as an "approximate characteristic equation". The existence of the equation (2.7) depends only upon the fact that all the derived difference equations are linear and homogeneous.

The theory can be extended to cases including two or more dependent variables connected by simultaneous partial differential equations on the domain or domains, the latter case involving the matching of solutions across domains by means of boundary conditions. There are three conditions which any problem must satisfy before the theory is applicable. They are

1. The problem must be defined on a finite domain. This ensures that a mesh with a finite number of points can be used.
2. All the differential equations on the domain (in every part of it) must be capable of transformation into linear, homogeneous difference equations between the values of the dependent variable (or variables) at the mesh points.
3. All the boundary conditions (including those on the interface between any two adjacent regions if the domain is split up) must also be capable of transformation into linear, homogeneous difference equations between the values of the dependent variable (or variables) at the mesh points.

It will be shown that the problem of Chapter 1 satisfies these criteria.

More generally, it can be observed that Maxwell's equations with no sources of current and charge are linear and homogeneous in the field vectors and hence it will always be possible to convert them into difference equations of the required form in a finite domain. Also for many
electromagnetic wave problems in such a domain the boundary conditions can be arranged to satisfy the condition 3. Hence for any such problem it is possible to derive an equation similar to (2.7). This equation will be an approximate characteristic equation for the propagation constant of the waves. There are several examples in the remainder of this thesis. For illustration of the general method a simple problem is considered in the next section.
2.2 To find the propagation constant of TM modes in a circular waveguide completely filled with dielectric material

The dielectric material is assumed to be lossless, homogeneous and isotropic. The waveguide of radius, \( a \), is assumed to have a perfectly conducting wall and to be completely filled with the dielectric of real scalar permittivity, \( \varepsilon_0 \), and real scalar permeability, \( \mu_0 \) (see Figure 2.1). This problem and its solution are well known (e.g. R. A. Waldron (1969)), and so the theory will be dealt with only briefly here.

Assuming an \( \exp\{j(\omega t-\beta z)\} \) dependence for all field components, Maxwell's equations are

\[
\nabla \times \mathbf{E} = -j\omega\mathbf{D} = -j\omega\mu_0\mathbf{H}
\]

and

\[
\nabla \times \mathbf{H} = j\omega\mathbf{B} = j\omega\varepsilon_0\mathbf{E}.
\]

These equations give, on eliminating \( E_r \), \( E_\theta \), \( H_r \) and \( H_\theta \), the transverse field components,

\[
\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r} \frac{\partial E_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} + k_0^2 E_z = 0 \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi \tag{2.8}
\]

and

\[
\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r} \frac{\partial H_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \theta^2} + k_0^2 E_z = 0 \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi \tag{2.9}
\]

where \( k_0^2 = \omega^2\varepsilon_0\mu_0 - \beta^2 \).

For transverse magnetic (TM) modes \( H_z \) is zero everywhere and so (2.9) is satisfied and the field is given by (2.8) subject to the conditions that, one, \( E_z \) is finite at the origin, from (1.42), and, two, from (1.35)

\[
\nabla \times \mathbf{E} = \mathbf{Q} \quad \text{at } r = a
\]

since the waveguide wall is perfectly conducting.

Therefore \( E_z = 0 \) at \( r = a \). \tag{2.10}
The field must be periodic in $\theta$. Assuming an $\exp\{-jn\theta\}$ dependence, and solving (2.8) under the finite field assumption yields

$$E_z = \sum_{n=0}^{\infty} A_n J_n(k_0 r)e^{-jn\theta}$$

(2.11)

the $e^{j(\omega t - \beta z)}$ factor being understood and the $A_n$ being arbitrary complex constants.

The condition (2.10) on substitution into (2.11) implies

$$J_n(k_0 a) = 0$$

(2.12)

and this gives the values of $k_0$ and hence of $\beta$, since $\omega$, $\varepsilon_0$ and $\mu_0$ are fixed constants, for which propagation is possible.

(2.12) is the characteristic equation for the propagation constant of the TM modes.

This analytic approach to the problem works because equations (2.8) and (2.9) are of a relatively straightforward form and so can be solved without much difficulty. If, on the other hand, they had been of a complicated form, similar to (1.25) and (1.26) for example, then this approach would be much more difficult if not impossible. In contrast, a finite difference method can be used to approximate the partial differential equations (2.8) and (2.9) however involved they might be. The accuracy of any solution derived by such a method is uncertain and needs to be considered very carefully.

The finite difference approach of section 2.1 will now be adopted for the problem of this section. The work proceeds as before up to equation (2.10). So

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r} \frac{\partial E_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} + k_0^2 E_z = 0$$

$$0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi$$

(2.8)

$$E_z = 0$$

$$r = a, \quad 0 \leq \theta \leq 2\pi$$

(2.10)

and $E_z$ is finite at $r = 0$. 
A mesh of points must first be chosen on the domain $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$. The obvious one for such an area comprises radii and concentric circles, centre the origin, with the outside circle coinciding with $r = a$. For simplicity let there be $m_q$ circles regularly spaced from the origin to $r = a$ and $m_p$ radii uniformly distributed around the domain. Then two neighbouring radii are at an angle $\frac{2\pi}{m_p} = \alpha$ say, radians to each other, and two adjacent circles are a distance $\frac{a}{m_q}$, $= h$ say, apart. Numbering the circles outwards from 1 to $m_q$, if $r_i$ is the radius of the $i^{th}$ circle then

$$r_i = ih.$$ 

An example of a mesh with $m_q = 4$ and $m_p = 8$ is illustrated in Figure 2.2.

Let $P_{ij}$ be the point at the intersection of the $i^{th}$ circle and $j^{th}$ radius. There are $m_p m_q$ such points and with the origin a total of $(m_p m_q + 1)$ points. These are the chosen points for the problem and it is converted into a system of equations between the values of $E_z$ at these points. Let $E_{ij}$ denote the value of $E_z$ at $P_{ij}$ and $E_0$ the value of $E_z$ at the origin. Then for $1 \leq j \leq m_p$ the following approximations can be made.

$$\frac{\partial^2 E_z}{\partial r^2} \bigg|_{P_{ij}} = \frac{E_{i+lj} - 2E_{ij} + E_{i-lj}}{h^2} 1 \leq i \leq m_q$$  \hspace{1cm} (2.13) 

$$\frac{\partial^2 E_z}{\partial r^2} \bigg|_{P_{ij}} = \frac{E_{i+lj} - 2E_{ij} + E_{i-lj}}{h^2} 2 \leq i \leq m_q$$  \hspace{1cm} (2.14) 

$$\frac{\partial^2 E_z}{\partial r^2} \bigg|_{P_{ij}} = \frac{E_{2j} - 2E_{ij} + E_0}{h^2}$$  \hspace{1cm} (2.15) 

and

$$\frac{\partial^2 E_z}{\partial \theta^2} \bigg|_{P_{ij}} = \frac{E_{ij+} - 2E_{ij} + E_{ij-}}{a^2} 1 \leq i \leq m_q$$ \hspace{1cm} (2.16) 

where $j+ = 1$ if $j = m_p$ and $j+ = j + 1$ otherwise, and $j- = m_p$ if $j = 1$ and $j- = j - 1$ otherwise.
Equation (2.8) holds at \( P_{ij} \) for \( 1 \leq i \leq m_q - 1 \) and \( 1 \leq j \leq m_p \), so using the above approximations, an equation is obtained at each of these \((m_q-1) \cdot m_p\) mesh points. The equations are

\[
\frac{1}{h^2}(E_{ij} - 2E_{1j} + E_0) + \frac{1}{r_1 h}(E_{ij} - E_{1j}) + \frac{1}{r_1^2 a^2}(E_{ij} + 2E_{1j} + E_{1j}) + k_0^2 E_{1j} = 0 \\
1 \leq j \leq m_p \quad (2.17)
\]

and

\[
\frac{1}{h^2}(E_{ij} + 2E_{ij} + E_{1j}) + \frac{1}{r_1 h}(E_{ij} + E_{1j}) + \frac{1}{r_1^2 a^2}(E_{ij} - 2E_{ij} + E_{1j}) + k_0^2 E_{1j} = 0 \\
2 \leq i \leq m_q - 1, 1 \leq j \leq m_p, \quad (2.18)
\]

\((m_q-1) \cdot m_p\) equations between the \( m_q \cdot m_p+1 \) unknowns, i.e. the values of \( E_z \) at the mesh points.

The boundary condition (2.10) gives another \( m_p \) equation for the values on the outside circle.

\[
E_{mq,j} = 0 \quad 1 \leq j \leq m_p \quad (2.19)
\]

Another equation is obtained by the approximation

\[
E_0 = \frac{1}{m_p} \sum_{j=1}^{m_p} E_{1j} = 0 \quad (2.20)
\]

i.e. \( E_0 \) is the average value of the \( E_{ij}(s) \) on the first circle.

This obeys the condition of a finite field at the origin.

Let \( \mathbf{E} \) denote the \((m_q \cdot m_p+1)\) column vector consisting of all the values of \( E_z \) at the mesh points,

i.e. \( \mathbf{E}^T = (E_0, E_{11}, E_{12}, \ldots, E_{1m_p}, E_{21}, \ldots, E_{2m_p}, \ldots, E_{m_q m_p}) \);

the equations (2.17), (2.18), (2.19) and (2.20) may be written in the form

\[
\mathbf{M} \mathbf{E} = \mathbf{0} \quad (2.21)
\]

where \( \mathbf{M} \) is the \((m_q \cdot m_p+1) \times (m_q \cdot m_p+1)\) matrix of the coefficients of the equations and \( \mathbf{0} \) is the \((m_q \cdot m_p+1)\) zero column vector. It can be seen from the equations that \( \mathbf{M} \) contains only real elements.
By the elementary theory of equations, the system (2.21) will have a non-trivial solution if and only if

\[ |M| = 0 \]  \hspace{1cm} (2.22)

This condition is a polynomial equation for \( k_0^2 \), and hence for \( \beta \), in terms of \( h, \alpha \) and \( r_i \) (\( i=1...m \)) which are all known quantities given by the particular choice of mesh. Thus by solving (2.22) values for the propagation constant \( \beta \) may be found.

Equation (2.22) can be considered to be an "approximate characteristic equation" parallel to (2.12) for this problem and equivalent to (2.7) of the last section.

To illustrate the above method and the form of the resulting determinant consider the particular case \( m_p = 4, m_q = 3 \) with just 13 mesh points. Clearly the separations between so-called neighbouring mesh points are much too large for the approximations (2.13), (2.14), (2.15) and (2.16) to have any high degree of accuracy, but all the derived equations can be written out in full as below.

The mesh is as shown in Figure 2.3 with \( h = \frac{a}{3} \), \( \alpha = \frac{\pi}{2} \) and \( r_i = ih = \frac{ia}{3} \) (\( i = 1,2,3 \)). Working outwards from the centre,

from (2.20) \[ E_0 - \frac{1}{2}(E_{11} + E_{12} + E_{13} + E_{14}) = 0, \]

from (2.17)

for \( j=1 \), \[ \frac{1}{h^2}(E_{21}-2E_{11}+E_0) + \frac{1}{r_1h}(E_{21}-E_{11}) + \frac{1}{r_1^2\alpha^2}(E_{12}-2E_{11}+E_{14}) + k_0^2E_{11} = 0, \]

for \( j=2 \), \[ \frac{1}{h^2}(E_{22}-2E_{12}+E_0) + \frac{1}{r_1h}(E_{22}-E_{12}) + \frac{1}{r_1^2\alpha^2}(E_{13}-2E_{12}+E_{11}) + k_0^2E_{12} = 0, \]

for \( j=3 \), \[ \frac{1}{h^2}(E_{23}-2E_{13}+E_0) + \frac{1}{r_1h}(E_{23}-E_{13}) + \frac{1}{r_1^2\alpha^2}(E_{14}-2E_{13}+E_{12}) + k_0^2E_{13} = 0, \]

and

for \( j=4 \), \[ \frac{1}{h^2}(E_{24}-2E_{14}+E_0) + \frac{1}{r_1h}(E_{24}-E_{14}) + \frac{1}{r_1^2\alpha^2}(E_{11}-2E_{14}+E_{13}) + k_0^2E_{14} = 0, \]
from (2.18)

for j=1, \( \frac{1}{h^2}(E_{31}-2E_{21}+E_{11}) + \frac{1}{r_2h}(E_{31}-E_{21}) + \frac{1}{r_2^2\alpha^2}(E_{22}-2E_{21}+E_{24}) + k_0^2E_{21} = 0, \)

for j=2, \( \frac{1}{h^2}(E_{32}-2E_{22}+E_{12}) + \frac{1}{r_2h}(E_{32}-E_{22}) + \frac{1}{r_2^2\alpha^2}(E_{23}-2E_{22}+E_{21}) + k_0^2E_{22} = 0, \)

for j=3, \( \frac{1}{h^2}(E_{33}-2E_{23}+E_{13}) + \frac{1}{r_2h}(E_{33}-E_{23}) + \frac{1}{r_2^2\alpha^2}(E_{24}-2E_{23}+E_{22}) + k_0^2E_{23} = 0, \)

for j=4, \( \frac{1}{h^2}(E_{34}-2E_{24}+E_{14}) + \frac{1}{r_2h}(E_{34}-E_{24}) + \frac{1}{r_2^2\alpha^2}(E_{21}-2E_{24}+E_{23}) + k_0^2E_{24} = 0, \)

and from (2.19)

\[ E_{31} = 0 \]
\[ E_{32} = 0 \]
\[ E_{33} = 0 \]
\[ E_{34} = 0 . \]

The column vector \( \mathbf{E} \) is defined by

\[ \mathbf{E}^T = (E_0 \ E_{11} \ E_{12} \ E_{13} \ E_{14} \ E_{21} \ E_{22} \ E_{23} \ E_{24} \ E_{31} \ E_{32} \ E_{33} \ E_{34}) \]

and the equations can be rewritten in the form (2.21)

\[ \mathbf{M} \mathbf{E} = \mathbf{0} \]

where

\[ \mathbf{M} = \begin{bmatrix} 1 & -\frac{1}{h} & -\frac{1}{h} & -\frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \[1/2] \end{bmatrix} \]

\[ \begin{bmatrix} 1/h^2 & D_1 & R_1 & 0 & R_1 & C_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \[1/2] \end{bmatrix} \]

\[ \begin{bmatrix} 1/h^2 & R_1 & D_1 & R_1 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 & 0 \[1/2] \end{bmatrix} \]

\[ \begin{bmatrix} 1/h^2 & 0 & R_1 & D_1 & R_1 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 \[1/2] \end{bmatrix} \]

\[ \begin{bmatrix} 1/h^2 & 0 & 1/h^2 & D_1 & R_1 & 0 & R_2 & C_2 & 0 & 0 \[1/2] \end{bmatrix} \]

\[ \begin{bmatrix} 0 & 0 & 0 & 0 & 1/h^2 & R_2 & 0 & R_2 & D_2 & R_2 & 0 & 0 & C_2 & 0 \[1/2] \end{bmatrix} \]

\[ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_2 & 0 & 0 & 0 & 0 & 1 \[1/2] \end{bmatrix} \]

\[ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \[1/2] \end{bmatrix} \]

where \( D_i = \frac{2}{h^2} - \frac{1}{r_i h} - \frac{2}{r_i^2 \alpha^2} + k_0^2, \) \( C_i = \frac{1}{h^2} + \frac{1}{r_i h} \) and \( R_i = \frac{1}{r_i^2 \alpha^2} \) (i=1,2)
In this particular case the equation \( \det M = 0 \) is a polynomial of degree sixteen in \( k_0 \) and hence in \( \beta \), the propagation constant. Solving it will give the possible values of \( \beta \); also from equation (2.23) it can be seen that the (13 × 13) matrix \( M \) is real and has all its non-zero elements in a band of width nine elements, four to the left and four to the right of the main diagonal. In the general case, \( M \), a real \((m \times m + 1)\) square matrix, is banded with \( m \) elements on each side of the main diagonal. This is explained by the fact that in the difference equation at any point no values of \( E_x \) that are more than \( m \) positions in the \( E_x \) vector away from the value at that point occur.

Banding also occurs in more complicated problems and is a very important factor in making a solution on a computer tractable. All such problems are reduced to an equation of the form

\[
\begin{align*}
M_{\xi} Z &= \xi
\end{align*}
\]

In this equation both the ordering of the components of the vector \( Z \) and the ordering of the constituent difference equations are free to be chosen. In general, they can be arranged to ensure that \( M_{\xi} \) is banded. A difference equation at a mesh point links only values at neighbouring points; hence by ordering the components of \( Z \) so that values at neighbouring points are always near one another in that vector, each row of \( M_{\xi} \) will have the positions of its non-zero elements extending over a limited range. Then an ordering of the equations will ensure that \( M_{\xi} \) has a banded form. From the point of view of the solution on a computer, to be discussed in chapter 3, the optimal arrangement of \( M_{\xi} \) is one which minimises the product \((m_1 + 1)(m_1 + 1 + m_2)\) where \( m_1 \) and \( m_2 \) are the numbers of elements required to the left and right respectively of the main diagonal to form a band containing all the non-zero elements of the matrix.
Returning to the general case of the problem of this section, where \( m_q \) and \( m_p \) are unspecified, it should be remembered that the values of \( \beta \) found from equation (2.22) will be only approximations to the true values obtained from solving the exact characteristic equation (2.12). The method used to obtain them has approximated the differential coefficients of the function \( E_z \) in terms of the values of the function at neighbouring points. By the general theory of finite differences these approximations improve as the separation between neighbouring points decreases. In fact, the more mesh points, and therefore the more difference equations, the more accurate the method is likely to be.

This finite difference formulation of the original problem has reduced it to that of solving

\[
|M| = 0 ,
\]

(2.22)

where \( M \) is a real square matrix of high order, perhaps several hundreds, for an acceptable approximation. The elements of the matrix are just the coefficients of the equations (2.17), (2.18), (2.19) and (2.20) and are all real quantities which are easily evaluated. In theory equation (2.22) can be multiplied out to give

\[
F(k_0^2) = 0 \quad \text{where } F \text{ is a polynomial,}
\]

i.e.

\[
P(\beta^2) = 0
\]

(2.24)

where \( P \) is a polynomial of high power in \( \beta \).

It can be seen from the form of the coefficients in \( M \) that \( P(\beta^2) \) is a continuous function of \( \beta^2 \). This suggests that equation (2.24) may be solved by a trial and error method, substituting different values of \( \beta \) and evaluating \( P(\beta^2) \) and, from the results, choosing other values of \( \beta \) to work towards a zero of \( P(\beta^2) \). For example if, for two values \( \beta_1 \) and \( \beta_2 \), \( P(\beta_1^2) < 0 \) and \( P(\beta_2^2) > 0 \),
then it can be argued that since \( P(\beta^3) \) is continuous, there exists a value of \( \beta \) between \( \beta_1 \) and \( \beta_2 \) for which \( P(\beta^3) = 0 \). By selecting appropriate values of \( \beta \) the zero can be found to lie in as small an interval as is desirable.

In conclusion, this chapter has shown how an approximate characteristic equation may be derived for a certain type of differential equation boundary value problem by using a finite difference method. In particular, this approach can be applied to a large number of problems to determine the propagation constant of electromagnetic waves travelling through a completely or partially filled circular waveguide. One such problem has been discussed but the value of the method lies in tackling more complicated problems for which an analytical solution is very difficult if not impossible.

The form of the approximate characteristic equation (2.7) is not simple when the order of \( M \) is high. It can be solved by a trial and error method provided there is some economical way of evaluating the determinant, \( |M| \), for any substituted value of \( \beta \). A method of achieving this which takes advantage of the banded nature of such a determinant is discussed in the next chapter.
Figure 2.1  The configuration

Figure 2.2  A typical mesh

Figure 2.3  The particular mesh used in the illustration
3.1 The sliding array method

To evaluate the determinant of a large square matrix, one having perhaps a hundred or so rows and columns, it is most practical to use a computer. The value of this method is, of course, limited by the finite extent of the store and there may be limitations on the time available to the user. As a general rule, by minimising the amount of store employed, the time can be reduced and the computer thereby used most efficiently. The method presented here - the sliding array method - takes advantage of the banded nature of the considered matrix and greatly reduces the computer store required.

The determinant of any square matrix having all real elements can be transformed into one having all zeros below the main diagonal by elementary row operations as in Gaussian Elimination (A. S. Householder (1953)). For example, using this general method,

\[
\begin{bmatrix}
1 & 2 & 0 & 0 \\
2 & -1 & 1 & 0 \\
-2 & 1 & 2 & 2 \\
0 & -5 & 2 & 1
\end{bmatrix}
= \begin{bmatrix}
2 & -1 & 1 & 0 \\
0 & -5 & 2 & 1 \\
0 & 0 & 3 & 2 \\
0 & 0 & 0 & 1/6
\end{bmatrix}
= -5
\]

The procedure of reducing the elements below the main diagonal column by column is also adopted in the sliding array method for the evaluation of the determinant of a real banded matrix, but the number of elements to be considered at any one time is reduced by the banded nature. Suppose that \( A \) is an \( n \times n \) real matrix with a band consisting of, at most, \( m_1 \) elements to the left and \( m_2 \) elements to the right of the main diagonal, all other elements being zero as in Figure 3.1. Then at any one time, only \((m_1+1)(m_1+1+m_2)\) elements need be considered as
opposed to \( n^2 \) elements in the general method. For a 100 square matrix with a band of 20 elements on each side of the main diagonal this would mean 861 instead of 10,000 elements, a reduction of over 90\%. This is the great advantage of the sliding array method. It is outlined below.

To carry out the Gaussian Elimination procedure on matrix \( A \), first find the largest element in absolute value in the first column. The largest element must be in the first \((m_1+1)\) rows. Therefore, at this stage the array consisting of the first \((m_1+1)\) rows and their first \((m_1+1+m_2)\) columns is considered. This array includes all the non-zero elements in these particular rows. After interchanging rows, if necessary, to get the largest element in the leading position, the next step is to subtract multiples of the first row from the remaining ones to make their first elements zero. The banded nature requires that the rows after the first \((m_1+1)\) are unchanged. Only the elements in the considered array have been used at this stage.

Although it is not strictly necessary to choose the largest element in the column it is general computing practice to do so. This choice is made to minimise rounding errors during the subsequent computation. When multiples of a row are subtracted from the remaining rows the multipliers will not be greater than unity and so any rounding error in the row will not be increased.

The same procedure is now applied to the array composed of rows 2 to \((m_1+2)\) and columns 2 to \((m_1+m_2+2)\). In effect, the array has "slid" one place down the main diagonal. Continuing this process, the product of the chosen diagonal elements, including any changes of sign due to row interchange, is evaluated and the relevant array slides one place lower at each stage until the \((n-m_1-m_2)^{th}\) stage. See Figure 3.2. After this the array is decreased by one column at each successive stage so as to stay within \( A \). By the \((n-m_1)^{th}\) stage the array is square and comprises the bottom right-hand \((m_1+1)\) square of the elements of the matrix at this
point. The determinant of the square matrix is then evaluated by the general method and multiplied by the product of the diagonal elements in the first \((n-m_1-1)\) rows (the product formed after the \((n-m_1-1)\)th stage) to give the value of \(|A|\).

A numerical example follows.

\[
\Delta = \begin{vmatrix}
1 & 2 & 0 & 0 \\
2 & -1 & 1 & 0 \\
-2 & 1 & 2 & 2 \\
0 & -5 & 2 & 1 \\
\end{vmatrix}
\]

Here \(n = 4, \ m_1 = 2, \ m_2 = 1\).

Considering just the elements indicated,

\[
\Delta = \begin{vmatrix}
1 & 2 & 0 & 0 \\
2 & -1 & 1 & 0 \\
-2 & 1 & 2 & 2 \\
0 & -5 & 2 & 1 \\
\end{vmatrix} = \begin{vmatrix}
-2 & -1 & 1 & 0 \\
0 & 0 & 3 & 2 \\
0 & -5 & 2 & 1 \\
\end{vmatrix} = -2 \ (5.3.1/6) = -5
\]

12 elements are the most considered at any one time; in the general method all 16 elements would have to be considered.
Figure 3.1 The banded matrix $A$

Figure 3.2 The sliding array method
3.2 Adaption of the sliding array method for computer use

In the sliding array method of the previous section, using the same notation, a maximum of \((m_1+1)(m_1+1+m_2)\) elements of the banded matrix need to be considered at any one time, as opposed to the \(n^2\) elements in the general method. In evaluating the determinant, in computer terms, this means that the store required is reduced by a factor \((m_1+1)(m_1+1+m_2)/n^2\) for a programme based on this sliding array method instead of on the general method. The amount of store used for other purposes is usually negligible when compared with that required for large arrays.

This saving of computer store is the fundamental reason why the problems in this thesis can be handled in practice as well as in theory by a finite difference approach. As may be recalled from chapter 2, for reasonable accuracy, such an approach requires a mesh with a large number of points and hence a large number of equations, each of which forms one row of the matrix \(M\) of equation (2.7). Thus \(M\) must be large but as only a small part of it is stored at any one time in using the sliding array method the evaluation of its determinant can be accomplished economically. To illustrate this point, in an example discussed in chapter 7, \(M\) is a 578 square matrix but the array needed to evaluate its determinant has just 37 rows and 74 columns. This requires a store of 2738 locations instead of over a third of a million which would otherwise be necessary.

A computer programme so based (on the sliding array method) is more complicated than an equivalent one based on the general method. For the latter, all the elements of the matrix may first be calculated and placed in the store. (To prevent tedious repetition throughout this section, it has been assumed that the elements of the matrix have to be calculated. The same considerations apply if they are read in directly.) Then the evaluation of the determinant is carried out. The whole matrix is stored at one time. For the sliding array method
this is not possible. The calculation of the elements and the evaluation of the determinant must take place simultaneously. Each row of the matrix is calculated when it is required to go into the array. A brief outline of the process follows.

Initially, the coefficients of the first \((m_1+1)\) equations, and hence the first \((m_1+1)\) rows of the matrix are calculated. The array is then filled by the elements in the top left-hand corner of the matrix. The first stage of the procedure (see the previous section) is now carried out. The first element of the selected first row is stored separately, being the first term in the product which will eventually give the value of the determinant. The first row is replaced by row \((m_1+2)\) which is calculated at this stage. Its leading diagonal element is positioned in column \((m_1+1)\) and so all its non-zero elements are included in the first row of the array. The elements in the remaining rows of the array and not in the first column are all moved one place to the left. The gap created at the right-hand side is filled by zeros. The operation of replacing the first row and moving the other rows one place to the left is equivalent to sliding the array one position down the main diagonal of the matrix and interchanging the rows then included so that the last becomes the first and the remainder all move down one. A total of \(m_1\) row interchanges is required and so the sign of the determinant is reversed if \(m_1\) is odd. The procedure is repeated systematically working down the main diagonal and at each stage multiplying the product of the diagonal elements already produced by the first element of the currently chosen first row of the array. In the latter stages the row replacing the first will not completely fill the available locations in that row, as the array overlaps the edge of the matrix; the empty locations, which are on the right-hand side of the row, are filled by zeros. Eventually the final row of the matrix and \(m_2\) zeros are substituted for the first row in the array which then has the form shown in (d) below. It is, in fact, a square
matrix as is obtained in section 3.1. Its determinant is evaluated by the general method and multiplied by the product of the diagonal elements at this stage to give the determinant of the whole matrix.

The array stored in the computer during the evaluation of the determinant of the banded matrix \((a_{ij})\) takes the forms shown below at various stages during that evaluation.

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{1m_2+1} & 0 & \ldots & 0 \\
  a_{21} & a_{22} & a_{2m_2+1} & 0 & 0 & \ldots \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
  a_{m_1+1,1} & \cdots & \cdots & \cdots & \cdots & a_{m_1+1,m_1+1+m_2}
\end{bmatrix}
\]

a) Initially (before the first stage)

\[
\begin{bmatrix}
  a_{jj-m_1} & \ldots & \ldots & a_{jj} & \ldots & \ldots & a_{jj+m_2} \\
  t_{21} & \ldots & \ldots & t_{2m_1+m_2} & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  t_{m_1+1,1} & \ldots & \cdots & \cdots & \cdots & \cdots & t_{m_1+1,m_1+1+m_2} & 0
\end{bmatrix}
\]

b) Before the \((j-m_1)\)th stage \((1+m_1 < j \leq n-m_2)\)

\[
\begin{bmatrix}
  a_{jj-m_1} & a_{jj} & a_{jn} & 0 & \ldots & 0 \\
  v_{21} & \ldots & \ldots & v_{2n-j+m_1+1} & 0 & \ldots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  v_{m_1+1,1} & \cdots & \cdots & \cdots & \cdots & \cdots & v_{m_1+1,n-j+m_1+1}
\end{bmatrix}
\]

c) Before the \((j-m_1)\)th stage \((n-m_2 < j < n)\)
d) Before the $(j-n)^{th}$ stage; the final row of the matrix has just been put into the array.

In b), c) and d) $t_{k\xi}, v_{k\xi}, u_{k\xi}$ $(2 \leq k \leq m_1+1, 1 \leq \xi \leq m_1+m_2)$ are values which have been generated by the programme during its already completed stages.

To illustrate the method by a numerical example consider the evaluation of

$$
\Delta = \begin{bmatrix}
1 & 2 & 0 & 0 & 0 \\
2 & -1 & 1 & 0 & 0 \\
1 & -2 & 3 & 1 & 0 \\
0 & 1 & 0 & 1 & 2 \\
0 & 0 & 1 & 2 & 4
\end{bmatrix}
$$

Here $n = 5$, $m_1 = 2$ and $m_2 = 1$. The elements in the array are transformed as below during the process and $\Delta_d$ is the current value of the product of the diagonal elements already obtained.

$$
\begin{pmatrix}
1 & 2 & 0 & 0 & 0 \\
2 & -1 & 1 & 0 & 0 \\
1 & -2 & 3 & 1 & 0 \\
0 & 1 & 0 & 1 & 2 \\
0 & 0 & 1 & 2 & 4
\end{pmatrix} \rightarrow \begin{pmatrix}
2 & -1 & 1 & 0 \\
0 & 2\frac{1}{5} & -\frac{1}{5} & 0 \\
0 & -1\frac{1}{5} & 2\frac{1}{5} & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 0 & 1 & 2 \\
2\frac{1}{5} & -\frac{1}{5} & 0 & 0 \\
-1\frac{1}{5} & 2\frac{1}{5} & 1 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
2\frac{1}{5} & -\frac{1}{5} & 0 & 0 \\
0 & 1/5 & 1 & 2 \\
0 & 11/5 & 1 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 2 & 4 & 0 \\
1/5 & 1 & 2 & 0 \\
11/5 & 1 & 0 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 2 & 4 & 0 \\
1/5 & 1 & 2 & 0 \\
11/5 & 1 & 0 & 0
\end{pmatrix}
$$

and $\Delta_d = -2$

and $\Delta_d = 2$

and $\Delta_d = 5$
Since the last row is now in the array the determinant
\[
\begin{vmatrix}
  1 & 2 & 4 \\
  1/5 & 1 & 2 \\
  11/5 & 1 & 0
\end{vmatrix}
\]

is evaluated and multiplied by \( \Delta_d \).

In fact
\[
\begin{vmatrix}
  1 & 2 & 4 \\
  1/5 & 1 & 2 \\
  11/5 & 1 & 0
\end{vmatrix} = \begin{vmatrix}
  11/5 & 1 & 0 \\
  17/11 & 4 \\
  0 & 0 & -6/17
\end{vmatrix} = -\frac{6}{5}
\]

and \( \Delta = -\frac{6}{5} \cdot 5 = -6 \).

An array of twelve locations has been used to evaluate the determinant of a matrix containing twenty-five elements.
3.3 The computer programmes

The problems discussed in this thesis are tackled by a finite difference method as outlined in section 2.1 and hence are reduced to a condition of the form

\[ |M| = 0 \]  \hspace{1cm} (2.7)

a polynomial of high order in \( \beta \), the propagation constant.

It follows that the respective computer programmes will have several features in common. Their basic design is now briefly discussed.

First, a mesh is set up and all the required values that are independent of \( \beta \) (a scaled form of \( \beta \)) are calculated. The choice of mesh size is constrained only by practical considerations such as the amount of computer store which would be necessary to carry out the method with such a mesh. Successive different values of \( \beta \) are read in and \( |M| \) is evaluated for each, all the other parameters being held fixed.

The evaluation is then carried out using the method of sections 3.1 and 3.2 and therefore the elements of the matrix are calculated only when they are needed. The determinant is scaled during the evaluation into the form

\[ d \times 2^i \quad \text{where} \quad 0.0625 \leq d < 1.0, \quad i \in \text{integer} \]

and the values of \( d, i \) and \( \beta \) are printed out. In the computer programmes \( d \) is written as DI and \( i \) as ID. Then the next value of \( \beta \) is read in and the procedure continues until the list of values of \( \beta \) has been exhausted.

From the series of values of \( |M| \) produced in this way the location of a \( \beta \) which makes the determinant zero (\( M = 0 \)) is sought. The determinant is a continuous real function of \( \beta \), simply being a polynomial. Hence a change in its sign between two values of \( \beta \) indicates a zero in the interval between them. By repeatedly subdividing such an interval, solutions of the approximate characteristic equation (2.7) for any particular problem may be found, each in the form of lying within an
interval of arbitrarily small extent. It is always possible to miss a change of sign if two roots happen to be very close together. Therefore there is no way of ensuring that all the roots have been found. Also, in practice, certain other difficulties arise and these will be discussed in a later chapter.
CHAPTER 4

A one region problem: the circular cylindrical waveguide completely filled with dielectric material

4.1 The mathematical theory

Here the situation considered is identical to that of section 2.2 except that the propagation constants for all the modes are sought, not just for the TM modes. To avoid repetition results from section 2.2 will be quoted where appropriate and the same notation used. These results are well-known, e.g. R. A. Waldron (1969).

From equations (2.8) and (2.9)

\[
\frac{\delta^2 E_z}{\delta r^2} + \frac{1}{r} \frac{\delta E_z}{\delta r} + \frac{1}{r^2} \frac{\delta^2 E_z}{\delta \theta^2} + k_0^2 E_z = 0 \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi
\]

(4.1)

and

\[
\frac{\delta^2 H_z}{\delta r^2} + \frac{1}{r} \frac{\delta H_z}{\delta r} + \frac{1}{r^2} \frac{\delta H_z}{\delta \theta^2} + k_0^2 H_z = 0 \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi
\]

(4.2)

Also as in section 2.2, \(E_z\) and \(H_z\) are finite at the origin and boundary condition (1.35) holds.

i.e.

\( \mathbf{R} \times \mathbf{E} = 0 \)

at \( r = a, \quad 0 \leq \theta \leq 2\pi \)

In the present case \( \mathbf{R} = \hat{\mathbf{z}} \) and this condition yields

\( E_z = 0 \) and \( E_\theta = 0 \)

at \( r = a, \quad 0 \leq \theta \leq 2\pi \)

(4.3)

From equation (1.32)

\( jk_0^2 E_\theta = -\omega_0 \frac{\delta H_z}{\delta r} + \frac{1}{r} \frac{\delta E_z}{\delta \theta} \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi; \)

but at \( r = a, \ E_z = 0 \) for all \( \theta \),

hence

\( \frac{\delta E_z}{\delta \theta} = 0, \quad \text{at} \ r = a, \quad 0 \leq \theta \leq 2\pi. \)

It follows that

\( \frac{\delta H_z}{\delta r} = 0, \quad \text{at} \ r = a, \quad 0 \leq \theta \leq 2\pi \)

(4.4)
The problem is now completely specified by the system of equations (4.1) to (4.4) in terms of the two variables $E_z$ and $H_z$. As these variables occur independently in the equations, the problem can be considered in two parts viz,

(a) $H_z = 0$ TM modes and
(b) $E_z = 0$ TE modes

Then, any complete solution will be a linear combination of these modes. For TM modes the solution is

$$E_z = \sum_{n=0}^{\infty} A_n J_n(k_0 r)e^{-jn\theta} \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi \quad (4.5)$$

the $A_n$ being arbitrary complex constants,

where $J_n(k_0a) = 0$. \hspace{1cm} (4.6)

Likewise for TE modes

$$H_z = \sum_{n=0}^{\infty} B_n J'_n(k_0 r)e^{-jn\theta} \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi \quad (4.7)$$

the $B_n$ being arbitrary complex constants,

where $J'_n(k_0a) = \left. \frac{d}{dr} \{J_n(k_0 r)\} \right|_{r=a} = 0$. \hspace{1cm} (4.8)

The factor $\exp j(\omega t - \beta z)$ in the field components is understood.

Any value of $k_0$ and hence of $\beta$ which satisfies either equation (4.6) or (4.8) for some integral $n$ will determine a mode. For modes to be unattenuated and to propagate down the guide, $\beta$ must be real and positive; therefore the required propagation constants are given by real positive values of $\beta$ which satisfy either (4.6) or (4.8). The zeros of Bessel functions and their derivatives are tabulated (Mrs. C. L. Beattie (1958)) and hence such values of $\beta$ can be found by a simple calculation.

Equations (4.6) and (4.8) are the characteristic equations for $\beta$ for TM and TE modes respectively.
4.2 The finite difference formulations using the separate and simultaneous methods

If a finite difference approach is adopted to the problem of determining the propagation constants, there are two distinct ways of proceeding. One is to consider the two types of mode separately. It will be called the "separate method". As was shown in section 2.2, putting \( H_z = 0 \) reduces the system to equations (2.8) and (2.10) for \( E_z \) and TM modes only. Now following section 2.2 a mesh is set up on the waveguide cross-section and equations (2.17), (2.18), (2.19) and (2.20) derived. These can be written in the form (2.21) where \( E_\alpha \) is a certain vector composed of the values of \( E_z \) at the mesh points.

Then the equation (2.22) gives a polynomial for the values of \( \beta \) at which propagation is possible. This is the approximate characteristic equation for TM modes and from it approximate values for the propagation constants can be obtained.

Then the TE modes are considered. Putting \( E_z = 0 \) reduces the system to the two equations (4.2) and (4.4) for \( H_z \). A similar method to the one for TM modes is employed. Again a mesh is set up and the equations transformed into difference equations. Since (4.2) is identical to (4.1) with \( H_z \) replacing \( E_z \) the derived difference equations are the same. However (4.4) is not the same as (4.3) and it is crudely approximated by

\[
\frac{1}{h}(H_{m_0,j} - H_{m_0-1,j}) = 0 \quad 1 \leq j \leq m_p
\]

The notation is that used in section 2.2 with \( H_{i,j} \) denoting the value of \( H_z \) at mesh point \((i,j)\), and \( H_0 \) the value at the origin.

Boundary conditions and their transformation into difference equations are more fully discussed in section 5.1.

The difference equations are now

\[
H_0 - \frac{1}{m_p} \sum_{j=1}^{m_p} H_{i,j} = 0 \quad (4.9)
\]
\[
\frac{1}{\hbar^2} (H_{2j} - 2H_{1j} + H_0) + \frac{1}{r_1 \hbar} (H_{2j} - H_{1j}) + \frac{1}{r_1^2 \alpha^2} (H_{1j} + 2H_{1j} - H_{1j}) + k_0^2 H_{1j} = 0
\]

\[1 \leq j \leq m_p \tag{4.10}\]

\[
\frac{1}{\hbar^2} (H_{i+1j} - 2H_{ij} + H_{i-1j}) + \frac{1}{r_1 \hbar} (H_{i+1j} - H_{ij}) + \frac{1}{r_1^2 \alpha^2} (H_{ij} - 2H_{ij} + H_{ij}) + k_0^2 H_{ij} = 0
\]

\[2 \leq i \leq m-1, \quad 1 \leq j \leq m_p \tag{4.11}\]

and

\[
\frac{1}{\hbar} (H_{q.m} - H_{q.m-1}) = 0 \quad 1 \leq j \leq m_p \tag{4.12}\]

Again there are \((m \cdot m + 1)\) equations between the same number of unknowns. These equations are equivalent to (2.17), (2.18), (2.19) and (2.20) and can be written in a form similar to (2.21) by making \(\mathbf{H}_0\) the \(m \cdot m + 1\) column vector

\[\mathbf{H}_0^T = (H_0, H_{11}, H_{12}, H_{1m_p}, H_{21} \ldots H_{m_qm_p})\]

Then

\[\mathbf{N} \mathbf{H} = \mathbf{Q}\]

where \(\mathbf{N}\) is the \((m \cdot m + 1)\) real square matrix of the coefficients of the difference equations and \(\mathbf{Q}\) is the corresponding zero column vector.

Again

\[|\mathbf{N}| = 0 \tag{4.14}\]

is the condition for a non-trivial solution and gives a polynomial for the values of \(\beta\) for which propagation is possible. It is the approximate characteristic equation for TE modes.

In the above separate method two polynomial equations in \(\beta\), namely (2.22) and (4.14), are derived. Their solutions are approximate values of \(\beta\) for which propagation is possible for respective TM and TE modes. They are approximations to the exact characteristic equations (4.6) and (4.8). As noted previously, for non-attenuated propagation down the guide the solutions must be real and positive and so it is only these values that are sought.
The second finite difference method ignores the fact that the modes separate into two types and tackles the simultaneous problem. This suggests the name "simultaneous method". A similar mesh to the one of section 2.2 is set up, but now two values $E_{ij}$ and $H_{ij}$ are considered at every mesh point and there are two simultaneous partial differential equations (4.1) and (4.2) defined on the domain. These are converted into two linear difference equations in the usual way and now every interior mesh point contributes two equations to the system. Similarly both boundary conditions (4.3) and (4.4) are used to give two difference equations at each boundary mesh point. At the centre of the guide both $E_z$ and $H_z$ are averaged over the innermost circle to give the values $E_0$ and $H_0$. A mesh of $m_q$ circles and $m_p$ radii leads to $2 + 2m_p \cdot m_q$ equations between the values of $E_z$ and $H_z$ at the $1 + m_q \cdot m_p$ mesh points.

In the notation of section 2.2, these equations are

\begin{align*}
E_0 - \frac{1}{m_p} \sum_{j=1}^{m_p} E_{1j} &= 0 \quad (4.15) \\
H_0 - \frac{1}{m_p} \sum_{j=1}^{m_p} H_{1j} &= 0 \quad (4.16)
\end{align*}

\begin{align*}
\frac{1}{h^2}(E_{2j} - 2E_{1j} + E_0) + \frac{1}{r_1 h}(E_{2j} - E_{1j}) + \frac{1}{r_1^2 a^2} (E_{1j} - 2E_{1j} + E_{1j}) + k_0^2 E_{1j} &= 0 \\
& \quad 1 \leq j \leq m_p ,
\end{align*}

\begin{align*}
\frac{1}{h^2}(H_{2j} - 2H_{1j} + H_0) + \frac{1}{r_1 h}(H_{2j} - H_{1j}) + \frac{1}{r_1^2 a^2} (H_{1j} - 2H_{1j} + H_{1j}) + k_0^2 H_{1j} &= 0 \\
& \quad 1 \leq j \leq m_p \quad (4.18)
\end{align*}

\begin{align*}
\frac{1}{h^2}(E_{i+1j} - 2E_{ij} + E_{i-1j}) + \frac{1}{r_1 h}(E_{i+1j} - E_{ij}) + \frac{1}{r_1^2 a^2} (E_{ij} - 2E_{ij} + E_{ij}) + k_0^2 E_{ij} &= 0 \\
& \quad 2 \leq i \leq m_q - 1, \quad 1 \leq j \leq m_p \quad (4.19)
\end{align*}

\begin{align*}
\frac{1}{h^2}(H_{i+1j} - 2H_{ij} + H_{i-1j}) + \frac{1}{r_1 h}(H_{i+1j} - H_{ij}) + \frac{1}{r_1^2 a^2} (H_{ij} - 2H_{ij} + H_{ij}) + k_0^2 H_{ij} &= 0 \\
& \quad 2 \leq i \leq m_q - 1, \quad 1 \leq j \leq m_p \quad (4.20)
\end{align*}
These equations can be written in the form

$$\mathbf{M}_q \mathbf{F} = \mathbf{0}$$

(4.23)

where $\mathbf{M}_q$ is a certain $2(m_q . m + 1)$ square real matrix, $\mathbf{0}$ is the $2(m_q . m + 1)$ zero column vector and $\mathbf{F}$ is a $2(m_q . m + 1)$ column vector whose components are the values of $E_z$ and $H_z$ at the mesh points given by

$$\mathbf{F}^T = (E_0 H_0 E_{11} E_{12} \ldots H_{m_q m_p}) .$$

Taking the components in this order ensures that $\mathbf{M}_q$ has the narrowest possible band of non-zero elements, a band of $2m_p$ elements at most on each side of the main diagonal.

Again

$$|\mathbf{M}_q| = 0$$

(4.24)

is the condition for a non-trivial solution and is the approximate characteristic equation for $\beta$. It gives both the TM and the TE modes.

The two methods above are mathematically equivalent. The first, the separate method, is in two parts which are very similar and simpler than the single part of the second or simultaneous method. The derived matrices in the separate method are each only a quarter of the size of one for the same mesh in the simultaneous method. This is a considerable advantage from the point of view of solution on a computer. The separate method seems to be the better one for this problem and one would expect it to be more accurate.

However, from a theoretical point of view and in the context of other problems, the simultaneous method is the more powerful one.
It does not depend on the separation of the $E_z$ and $H_z$ components of the field as happens in equations (4.1) to (4.4). Hence it can be applied to two coupled simultaneous partial differential equations. In fact, any problem in one finite region can be tackled provided that it satisfies the conditions 1, 2 and 3 given in section 2.1.
4.3 The computer programmes for the separate and the simultaneous methods

For the first part of the separate method, a computer programme was constructed to find the propagation constants of the TM modes only. This programme is included in Appendix 1. It is based on the considerations of section 3.3. The parameters $m^q$, $m^p$, $\omega$, $\varepsilon_0$, $\mu_0$ and $a$ are read in and kept fixed throughout. From the values of $m^q$ and $m^p$ the mesh is set up and the size of the matrix calculated. This matrix is banded with $m_1 = m_p$ and $m_2 = m_p$ in the notation of section 3.1. Also, from that section, the array (denoted by BM in the programme) in which the elements of the matrix are placed must have at least $(m_1+1)$ rows and $(m_1+1+m_2)$ columns. Hence there must be, at least, $(m_1+1)(2m_p+1)$ locations reserved for it in the computer store. Each of the other arrays used in the programme contains $m^q$ elements and so needs that number of store locations. These requirements must be borne in mind when the dimension statements are written so that sufficient space is reserved for the arrays for the particular values of $m^q$ and $m^p$ to be used. The programme printed in Appendix 1 will handle any mesh with $m_p \leq 12$ and $m_q \leq 20$. If values of $m^q$ and $m^p$ are chosen so that either of these conditions is violated then the dimension statements must be adjusted to increase the sizes of the arrays to the requisite amount. Throughout this thesis, all the programmes which include arrays have some similar conditions on the size of the reserved computer store. Ideally, to minimise computer time and store, the arrays should be exactly the right size with no redundant locations.

For the second part of the separate method (the TE modes), an almost identical programme is used. As was shown in section 4.2, all the difference equations are the same as those for the case of TM modes except equations (2.19) which are replaced by (4.12). Thus if $m^q$ and $m^p$ are chosen to be the same in both cases the matrices $M$ of (2.21) and $N$ of (4.13) will be identical except for the $m^p$ rows given by equations (2.19)
for $M_\mathbb{C}$ and by equations (4.12) for $N_\mathbb{C}$. By taking the simplest form of (4.12), namely,

$$H_{m,j}^{q} - H_{m-1,j}^{q} = 0 \quad 1 \leq j \leq m$$

the only alteration required to the TM mode programme is the insertion of just one line immediately after that labelled 400. The inserted line is

$$BM(1,LL1) = -1.0$$

The considerations on the values of $m_q$ and $m_p$ in relation to the number of locations reserved for the arrays in the computer store are the same here as in the programme for TM modes.

The computer programme used for the simultaneous method is contained in Appendix 2. It follows the design of the ones used in the separate method. All the equations of both parts of that method are included in the single part of the simultaneous method, and the programme is developed to evaluate the determinant of the correspondingly enlarged matrix. As has been stated in section 4.2, this matrix has a non-zero band of $2m_p$ elements at most on each side of the main diagonal. So here $m_1 = m_2 = 2m_p$ and an array with $(2m_p + 1)$ rows and $(4m_p + 1)$ columns is used in the sliding array evaluation of the determinant of the matrix. This requires $(2m_p + 1)(4m_p + 1)$ locations in the computer store as opposed to the $(m_p + 1)(2m_p + 1)$ locations for the array used in each part of the separate method. The remaining arrays each require $m_q$ locations as in the separate method. The difference in the computer store requirements is the main reason why the separate method is preferable to the simultaneous method for the problem of this chapter. The programme in Appendix 2 will handle any mesh with $m_p \leq 12$ and $m_q \leq 20$. As before, a larger mesh is dealt with by changing the dimension statements appropriately.
4.4 Numerical results

In section 4.1 it was shown that the solution of the problem posed in this chapter is given by equations (4.6) and (4.8). It was also stated that these equations could easily be solved using tables of Bessel's functions. In fact, if for some integral \( n \), either the Bessel function \( J_n \), or its derivative with respect to \( r \), \( J'_n \), has a zero at \( x = k_0 a > 0 \) then, for the propagation constant \( \beta \),

\[
k_0^2 = \omega^2 \varepsilon_0 \mu_0 - \beta^2
\]

i.e.

\[
\beta^2 = \omega^2 \varepsilon_0 \mu_0 - \frac{x^2}{a^2}
\]  

(4.25)

For an unattenuated mode \( \beta \) is real and then

\[
\omega^2 \varepsilon_0 \mu_0 > \frac{x^2}{a^2}
\]

i.e.

\[
x < \omega \sqrt{\varepsilon_0 \mu_0}
\]

This inequality restricts the number of modes in any particular case. The possible values of \( x \) are well-known; the first few in order of increasing magnitude are given in Table 4.1.

It follows that any result derived by one of the finite difference methods can be compared with a corresponding calculated value and its accuracy checked. The calculated value can be made as accurate as is desired by taking a sufficient number of significant figures.

In this section the values of \( B \) (the scaled propagation constant \( = \frac{\beta}{\omega \sqrt{\varepsilon_0 \mu_0}} \)) are calculated correct to two decimal places. The values of \( B \) found from the finite difference methods each lie within an interval of length 0.005 or 0.01. Throughout the results in this thesis the \( \rightarrow \) (arrow) notation is used to mean "between", i.e. "0.85 \( \rightarrow \) 0.86" means "between the two values 0.85 and 0.86". Also for all the results in this section the permittivity and permeability have their values in vacuo, i.e.

\[
\varepsilon_0 = 8.854 \times 10^{-12}
\]

and \( \mu_0 = 1.257 \times 10^{-6} \)
Table 4.2 is a typical computer print-out giving the values $D_1$ and $ID$ for different values of $B$. The value of the determinant $|M|$ of equation (4.24) (or a corresponding one in the separate method) is $D_1 \times 2^{ID}$. The function which is the expansion of the determinant has no significance other than when it is zero and its behaviour between such values is of no importance. All that can be said of it is that it is a continuous function of $B$. There appears to be no reason why the function should have opposite signs on either side of a zero and, in fact, this is not always the case. Therefore finding zeros is much less straightforward than might have been hoped, especially since the function can change in value very rapidly. This can be seen, for example, from Table 4.2 by comparing the values at $B = 0.835$ and $B = 0.84$. A discussion of this difficulty will be given in section 7.1. Here it should simply be noted that not all the zeros are obtained at a change of sign of the determinant.

Table 4.3 illustrates the convergence of a zero towards its true value as the number of points in the mesh is increased. This result was predicted in chapter 2 for any finite difference method. All such methods used in this thesis display a similar convergence as the number of mesh points is increased. After reaching a certain size of mesh a zero appears to stay fixed in an interval for any further decrease in size. A best approximation has then been obtained.

From a practical point of view it should be remembered that more computer time and store are required as more mesh points are used. Hence any increase in accuracy must be weighed against the corresponding increase of computer resources necessary to accomplish it. A mesh size that strikes a balance between these two considerations is clearly desirable. This can only be achieved by trial and error in any particular case.

Tables 4.4 to 4.7 give a comparison between the zeros found by the finite difference methods and their calculated values for a number of
cases. They show that such methods work but with a varying degree of accuracy. In general, it can be seen from the results that the accuracy is good for values of $B$ near unity and deteriorates as the values of $B$ decrease.

There are two main causes of inaccuracy. Since a finite difference method employs approximations there are intrinsic errors in it. It is very difficult to quantify these errors, and bounds for them have been found to orders of magnitude in only the simplest cases. Even then the bounds are often too large to be useful. Round off errors occur in any computation involving decimal numbers because any value can be stored only to a certain number of figures. In evaluating the determinant these errors are limited by choosing the largest element in a column as explained in section 3.1.

The complexity of the difference equations and the size of the corresponding matrix suggests that any rigorous error analysis would be very difficult if not impossible for this problem. None is attempted here. Comparison with known results is the only indication of the accuracy of the finite difference methods used.

However a mechanism which leads to decreasing accuracy with decreasing $B$ will be discussed. It is included to show that the errors in the given results could arise in a straightforward manner. There is no suggestion that they do, in fact, come about in this way but perhaps this or a similar mechanism has some influence on the results.

In the computer programmes $B$ is read in and from it the quantity $1 - B^2 = E_a$ say, is calculated. The parameters of the physical problem are given to three or four significant figures at most, and so on a large computer working to many more figures calculating $E_a$ is equivalent to reading it in directly. $B$ occurs only in the factor $E_a$ in the coefficients of the difference equations which form the matrix. Therefore the programme is equivalent to one which reads in values of
E and evaluates the determinant of the matrix. The method can be considered to find values of \( E_a \) for which the determinant is zero. Since the method is only approximate there will be errors in these values.

Suppose, for simplicity, there is a constant percentage error in \( E_a \) for all values of \( B \). The influence on \( B \) of such an error in \( E_a \) will now be investigated. At some zero of the determinant let \( B \) and \( E_a \) be the computed values and \( \hat{B} \) and \( \hat{E}_a \) the corresponding true values with their respective small errors \( \delta \hat{B} \) and \( \delta \hat{E}_a \).

i.e. \[ B = \hat{B} + \delta \hat{B} \]
and \[ E_a = \hat{E}_a + \delta \hat{E}_a \).

\( \delta \hat{E}_a \) is by the assumption a fixed percentage \( p \) say, of \( \hat{E}_a \),

\[
\delta \hat{E}_a = \frac{p}{100} \hat{E}_a \tag{4.26}
\]

Now \[ E_a = 1 - B^2 \tag{4.27} \]
and \[ \hat{E}_a = 1 - \hat{B}^2 \tag{4.28} \]

Subtracting \[ \delta \hat{E}_a = -2\hat{B}\delta \hat{B} + 0(\delta \hat{B}^2) \tag{4.29} \]

For unattenuated propagation down the guide \( \hat{B} > 0 \) and so from equation (4.29)

if \( \delta \hat{E}_a > 0 \) then \( \delta \hat{B} < 0 \)

and if \( \delta \hat{E}_a < 0 \) then \( \delta \hat{B} > 0 \)

Combining (4.29), (4.26) and (4.28)

\[
\delta \hat{E}_a = \frac{p}{100} \left(1 - \hat{B}^2\right) = -2\hat{B}\delta \hat{B}
\]

neglecting the term of order \( \delta \hat{B}^2 \)

\[
\therefore \quad \frac{\delta \hat{B}}{\hat{B}} = -\frac{p}{200} \left(\frac{1}{\hat{B}^2} - 1\right) \tag{4.30}
\]
The equation (4.30) gives the fractional error in $\hat{B}$ in terms of the percentage error in $\hat{E}_a$ and the value $\hat{B}$. If there is no error in $\hat{E}_a$ ($p = 0$) then there is no error in $\hat{B}$. For values of $\hat{B}$ near unity and $p$ small, the error in $\hat{B}$ is small. For a fixed $p$ the error in $\hat{B}$ increases with decreasing $\hat{B}$ since $0 < \hat{B} < 1$.

Taking $p$ as a parameter, equation (4.30) can be plotted as a family of curves of $\frac{\delta\hat{B}}{\hat{B}}$ against $\hat{B}$. The equation is valid only when $\frac{\delta\hat{B}}{\hat{B}}$ is small. The three curves with $p = -2$, -5 and -10 are plotted in Figure 4.1 from the calculated values of $\frac{\delta\hat{B}}{\hat{B}}$ for values of $\hat{B}$ given in Table 4.8. The curves show that the fractional error $\frac{\delta\hat{B}}{\hat{B}}$ is quite small for high values of $\hat{B}$ and increases more and more rapidly for lower values. This is similar to the error pattern in the numerical results as can be seen from Figure 4.2. The error pattern in the results is depicted in the form of a curve termed the average fractional error curve. It is obtained as follows: the results from Tables 4.4 to 4.7 are placed into sets depending on their values of $\hat{B}$. Six sets are defined by

$$r \leq \hat{B} \leq r + 0.1$$

$r = 0.4, 0.5, \ldots, 0.9$

For each set a mean of the fractional errors of the members is calculated. This mean is plotted at $\hat{B} = r + 0.5$ where $r$ takes the appropriate value for the set. By joining up the six plotted values the average fractional error curve is formed.

Figure 4.2 suggests that the assumption of a constant percentage error in $E_a$ or a very similar assumption may play a part in accounting for the observed errors. It also suggests a correction procedure for the results. Eliminating $\hat{E}_a$ and $\delta\hat{E}_a$ from equations (4.26), (4.27) and (4.28)

$$1 - B^2 = \left[1 + \frac{p}{100}\right](1 - \hat{B}^2)$$

i.e.

$$\hat{B}^2 \left[1 + \frac{p}{100}\right] = B^2 + \frac{p}{100}$$
and since $\hat{B} > 0$

$$\hat{B} = \sqrt{\frac{B^2 + \frac{p}{100}}{1 + \frac{p}{100}}} \quad (4.31)$$

Equation (4.31) is a correction formula which can be applied to the computed value $B$ to obtain a better approximation to the true value $\hat{B}$. In fact, this $\hat{B}$ would be the true value as given by equation (4.6) or (4.8) if the error in the numerical results for $B$ were due solely to the fixed percentage error $p$, in $E_a$. Of course, this is not the case, there being intrinsic errors in any finite difference method, as has already been noted. However, the above correction formula does improve the results when it is applied to them using an appropriate value of $p$. (See Table 4.9). It may perhaps be considered to reduce the errors to those of a more fundamental kind.

In conclusion, the numerical results show that the finite difference methods give approximate values of $B$ for the problem of this chapter. It seems likely that the errors in the results can be accounted for by the method's intrinsic inaccuracy, which is reduced by increasing the number of mesh points, and by a mechanism similar to the one outlined above. It has been shown that this latter source of error can be tackled using a correction formula.
Figure 4.1  Error curves
Figure 4.2  The average fractional error curve
### Table 4.1
Zeros of the Bessel functions $J_n(x)$ and $\frac{d}{dx}J_n(x)$

<table>
<thead>
<tr>
<th>Value of $x$ (to 2 dec.pl.)</th>
<th>$m^{th}$ Zero</th>
<th>of</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.84</td>
<td>1</td>
<td>$J_1'(x)$</td>
<td>TE$_{11}$</td>
</tr>
<tr>
<td>2.40</td>
<td>1</td>
<td>$J_0(x)$</td>
<td>TM$_{01}$</td>
</tr>
<tr>
<td>3.05</td>
<td>1</td>
<td>$J_2'(x)$</td>
<td>TE$_{21}$</td>
</tr>
<tr>
<td>3.83</td>
<td>1</td>
<td>$J_1(x)$</td>
<td>TM$_{11}$</td>
</tr>
<tr>
<td>3.83</td>
<td>1</td>
<td>$J_0'(x)$</td>
<td>TE$_{01}$</td>
</tr>
<tr>
<td>4.20</td>
<td>1</td>
<td>$J_3'(x)$</td>
<td>TE$_{31}$</td>
</tr>
<tr>
<td>5.14</td>
<td>1</td>
<td>$J_2(x)$</td>
<td>TM$_{21}$</td>
</tr>
<tr>
<td>5.32</td>
<td>1</td>
<td>$J_4'(x)$</td>
<td>TE$_{41}$</td>
</tr>
<tr>
<td>5.33</td>
<td>2</td>
<td>$J_1'(x)$</td>
<td>TE$_{12}$</td>
</tr>
<tr>
<td>5.52</td>
<td>2</td>
<td>$J_0(x)$</td>
<td>TM$_{02}$</td>
</tr>
</tbody>
</table>

### Table 4.2
A typical computer print-out
The three columns contain values of $D1$, $ID$ and $B$ respectively.

<table>
<thead>
<tr>
<th>$W$</th>
<th>$Q0$</th>
<th>$EO$</th>
<th>$RA$</th>
<th>$MQ$</th>
<th>$MP$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.534E+11</td>
<td>.126E-05</td>
<td>.885E-11</td>
<td>.260E-01</td>
<td>9</td>
<td>8</td>
<td>73</td>
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### Table 4.3
Convergence of a zero with increasing number of mesh points.
The value of $B$ by analytical theory is 0.85 by the separate method for TM modes with $\omega = 5.341 \times 10^{10}$ and $a = 2.6 \times 10^{-2}$.
The TM$_{01}$ mode

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<td>0.84 $\rightarrow$ 0.845</td>
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Table 4.4 The separate method - TM modes with $\omega = 5.341 \times 10^{10}$
Computed values and the corresponding values calculated from analytical theory

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<th>$m_p$</th>
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<th>mode</th>
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Table 4.5 The separate method - TE modes with $\omega = 5.341 \times 10^{10}$.

Computed values and the corresponding values calculated from analytical theory

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<th>mode</th>
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Table 4.6 The separate method - TE modes

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Table 4.7 The simultaneous method - TM and TE modes

Computed values and the corresponding values calculated from analytical theory

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<td>0.28</td>
<td></td>
<td>TE$_{01}$</td>
</tr>
</tbody>
</table>
Table 4.8 $\frac{\delta \hat{B}}{\hat{B}}$, the fractional error in $\hat{B}$ for different values of $p$, the percentage error in $E_{\theta}$.

<table>
<thead>
<tr>
<th>$\hat{B}$</th>
<th>$p = -2$</th>
<th>$p = -5$</th>
<th>$p = -10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9</td>
<td>0.002</td>
<td>0.006</td>
<td>0.012</td>
</tr>
<tr>
<td>0.8</td>
<td>0.006</td>
<td>0.014</td>
<td>0.029</td>
</tr>
<tr>
<td>0.7</td>
<td>0.011</td>
<td>0.026</td>
<td>0.054</td>
</tr>
<tr>
<td>0.6</td>
<td>0.019</td>
<td>0.045</td>
<td>0.093</td>
</tr>
<tr>
<td>0.5</td>
<td>0.031</td>
<td>0.078</td>
<td>0.163</td>
</tr>
<tr>
<td>0.4</td>
<td>0.054</td>
<td>0.144</td>
<td>0.311</td>
</tr>
<tr>
<td>0.301</td>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>0.3</td>
<td>0.107</td>
<td>0.296</td>
<td></td>
</tr>
<tr>
<td>0.218</td>
<td></td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.279</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.14</td>
<td>1.0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.9 The correction formula applied to the results of Table 4.7 with $p = -8$

<table>
<thead>
<tr>
<th>$B$</th>
<th>corrected $B$</th>
<th>calculated value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.56</td>
<td>0.605</td>
<td>0.61</td>
</tr>
<tr>
<td>0.905</td>
<td>0.91</td>
<td>0.92</td>
</tr>
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<td>0.84</td>
<td>0.85</td>
<td>0.85</td>
</tr>
<tr>
<td>0.74</td>
<td>0.76</td>
<td>0.75</td>
</tr>
<tr>
<td>0.47</td>
<td>0.53</td>
<td>0.56</td>
</tr>
<tr>
<td>0.435</td>
<td>0.50</td>
<td>0.56</td>
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<tr>
<td>0.91</td>
<td>0.915</td>
<td>0.92</td>
</tr>
<tr>
<td>0.845</td>
<td>0.855</td>
<td>0.85</td>
</tr>
<tr>
<td>0.75</td>
<td>0.765</td>
<td>0.75</td>
</tr>
<tr>
<td>0.545</td>
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</tr>
<tr>
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<tr>
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<td>0.89</td>
<td>0.89</td>
</tr>
<tr>
<td>0.785</td>
<td>0.80</td>
<td>0.80</td>
</tr>
<tr>
<td>0.645</td>
<td>0.68</td>
<td>0.65</td>
</tr>
</tbody>
</table>
CHAPTER 5

The two region problem

5.1 Boundary conditions

In a problem defined on a domain consisting of two regions there are boundary conditions to be satisfied by the dependent variables, both on the interface between the regions and on the boundary of the domain. It is necessary that all these conditions can be written as linear homogeneous difference equations between the values of the dependent variables at the chosen mesh points in order to ensure that the finite difference method adopted here can be used.

To discuss the choice of mesh points near a boundary, consider first the simple one region problem of Chapter 4. The same considerations will need to hold for a two region problem both on the domain boundary and on the interface. Quoting from section 4.1, the boundary conditions at \( r = a \) are

\[
E_z = 0 \quad r = a, \quad 0 \leq \theta \leq 2\pi \tag{4.3}
\]

and

\[
\frac{\partial H_z}{\partial r} = 0 \quad r = a, \quad 0 \leq \theta \leq 2\pi \tag{4.4}
\]

and the mesh chosen was one with equal steps of length, \( h \), in the radial direction. Then equation (4.4) was approximated by

\[
\frac{1}{h^2}(H_{m,q}^{j+1} - H_{m,q}^{j-1}) = 0 \quad 1 \leq j \leq m_p \tag{4.12}
\]

This equation links the value of \( H_z \) at a distance \( h \) from the boundary to the value at the boundary. But equation (4.4) holds only on the boundary and certainly not at a distance \( h \) from it. On the other hand equation (4.2),

\[
\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r} \frac{\partial H_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \theta^2} + k_0^2 H_z = 0 \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi,
\]

holds throughout the domain, and when \( \frac{\partial H_z}{\partial r} \) at an interior mesh point \((i,j)\) is approximated by
\[ \frac{\partial H_z}{\partial r} = \frac{1}{h} (H_{i+1,j} - H_{ij}) \]

the equation (4.11) holds at both \((i,j)\) and at point \((i+1,j)\) a distance \(h\) away. It follows that a distinction should be made between equations over a domain and those just on a boundary and that this should show in the approximations adopted. One way to do this is to make the radial step adjacent to the boundary very small. A step of \(h/100\) is taken where \(h\) is the step length in the domain. Then the boundary condition links only the value of \(H_z\) at the boundary with a value very close to it. The boundary condition is thus being assumed to hold over a very thin ring of width \(h/100\) next to the boundary.

A better approximation to the problem of Chapter 4 is then obtained by taking \(m_q^q\) radial steps, \((m_q-1)\) of which are equal and the outside one a small fraction of one of the others. The mesh needs to be set up slightly differently in order to do this and equations (4.1) and (4.2) have a different difference approximation for the points \(i = m_q - 1, 1 \leq j \leq m_q^q\). On that circle the approximations used are

\[ \frac{\partial H_z}{\partial r} = \frac{1}{\delta h} \left( \frac{H_{mq,j} - H_{mq-1,j}}{\delta h} \right) \]

and

\[ \frac{\partial^2 H_z}{\partial r^2} = \frac{1}{\delta h} \left( \frac{H_{mq,j} - H_{mq-1,j}}{\delta h} - \frac{H_{mq-1,j} - H_{mq-2,j}}{h} \right) \]

\[ = \frac{1}{\delta h} \left( \frac{1}{\delta h} H_{mq,j} - \left[ \frac{1}{\delta h} + \frac{1}{h} \right] H_{mq-1,j} + \frac{1}{h} H_{mq-2,j} \right) \]

and two similar expressions for \(\frac{\partial E_z}{\partial r}\) and \(\frac{\partial^2 E_z}{\partial r^2}\);

\(\delta h\) and \(h\) are the respective lengths of the outermost radial step and one of the others. The concentric circles in such a mesh are illustrated in Figure 5.1 where \(m_q = 6\).

It could also be argued that the centre step should also be a very small one to improve the accuracy of the approximation there. The value at the centre is taken as an average of neighbouring values. However, such a centre step would increase the size of the other steps assuming
a fixed $m_q$, and thereby accuracy would be lost. Perhaps the only way to settle such questions of step length is by trial in each particular case.

For the two region problem the above considerations are more important because there are six boundary conditions instead of two. Three very small steps, one on each side of the interface and one inside the domain boundary are used when these conditions are approximated. The necessity of small steps on each side of the interface is demonstrated in section 6.4.

The boundary value problem of section 1.5 will now be tackled using the finite difference method. From it several special cases of interest will be derived.
Figure 5.1. Mesh circles for a one region problem

Figure 5.2. Mesh circles for a two region problem
5.2 The finite difference formulation of the boundary value problem of section 1.5

In this section the equations (1.47) to (1.58) on the domain $0 \leq r \leq a$, $0 < \theta < 2\pi$, are approximated by a set of finite difference equations between the values of $E_z$ and $H_z$ at points of a chosen mesh on the domain. The mesh is composed of $m_p$ radial arms equally spaced around the origin and $m_q$ circles each with centre the origin, where $m_p$ and $m_q$ are two chosen integers. The circles are numbered outwards $i = 1, 2, \ldots, m_q$ and the radial arms from some chosen one $j = 1, \ldots, m_p$ and usually the axes are chosen so that $\theta = 0$ where $j = m_p$. The outside boundary of the domain, $r = a$, is the circle $i = m_q$ and the interface, $r = b$, between the two regions is the circle $i = m_b$. The concentric circles of a possible mesh are illustrated in Figure 5.2 where $m_q = 9$. The radius of the $i^{th}$ circle is $r_i$ and the radial step lengths, $h_i$, are defined by

$$h_1 = r_1$$

and

$$h_i = r_i - r_{i-1} \quad i = 2, \ldots, m_q$$

No other restrictions are placed on the $r_i$ so there is considerable freedom in choosing the radial step lengths. However it should be noted that $h_i$ for $i = m_b, m_b + 1$ and $i = m_q$ must be relatively small to ensure that the boundary condition approximations operate very near the boundaries as discussed in section 5.1.

The mesh has been taken with a constant angle between two adjacent radial arms for simplicity. There seems to be little point in having a variable angle in this problem but the theory could be extended easily to cover such a case. The angle $\alpha = \frac{2\pi}{m_p}$ between two adjacent radial arms would be replaced by $\alpha_j$ where $\alpha_j$ is the angle between the $(j-1)^{th}$ and $j^{th}$ arm.

There are $(m_p, m_q + 1)$ mesh points of which $2m_p$ are on $r = a$ and $r = b$ and these will be called boundary mesh points. The remaining
m_{p}(m_{p}-2)+1 will be called domain mesh points. At one of these latter points where \(1 \leq i \leq m_{b}-1\) or \(m_{b}+1 \leq i \leq m_{q}-1\) and \(1 \leq j \leq m_{p}\) the following approximations are made:

\[
\frac{\partial E_{z}}{\partial r} = \frac{1}{h_{i}+1} \left( E_{i+1j} - E_{ij} \right)
\]

\[
\frac{\partial^{2} E_{z}}{\partial r^{2}} = \frac{1}{h_{i}+1} \left\{ \frac{1}{h_{2}} E_{2j} - \left( \frac{1}{h_{2}} + \frac{1}{h_{1}} \right) E_{1j} + \frac{1}{h_{1}} E_{0} \right\} \quad \text{for } i = 2
\]

\[
\frac{\partial^{2} E_{z}}{\partial r^{2}} = \frac{1}{h_{i}+1} \left\{ \frac{1}{h_{i+1}} E_{i+1j} - \left( \frac{1}{h_{i+1}} + \frac{1}{h_{i}} \right) E_{ij} + \frac{1}{h_{i}} E_{i-1j} \right\} \quad \text{for } i > 2
\]

\[
\frac{\partial E_{z}}{\partial \theta} = \frac{1}{\alpha} \left( E_{ij+} - E_{ij} \right)
\]

\[
\frac{\partial^{2} E_{z}}{\partial r \partial \theta} = \frac{1}{\alpha h_{i+1}} \left\{ E_{i+1j+} - E_{ij+} - E_{i+1j} + E_{ij} \right\}
\]

and \[
\frac{\partial^{2} E_{z}}{\partial \theta^{2}} = \frac{1}{\alpha^{2}} \left( E_{ij+} - 2E_{ij} + E_{ij-} \right)
\]

where \(E_{ij}\) is the value of \(E_{z}\) at \((i,j)\) and \(E_{0}\) is the value at the origin and

\[
\begin{align*}
  j+ & = j + 1 & 1 \leq j \leq m_{p}-1 \\
  j+ & = 1 & j = m_{p} \\
  j- & = j - 1 & 2 \leq j \leq m_{p} \\
  \text{and} & & j = 1
\end{align*}
\]

throughout this work.

The same equations hold for \(H_{z}'\), \(H\) replacing \(E\) throughout the above.

Using these approximations, the equations (1.47) to (1.50) can be put into difference form.

From equation (1.50)

\[
\left\{ \omega^{2} \varepsilon_{1} \mu_{1} E_{a} - \frac{1}{h_{2}} \left( \frac{1}{h_{2}} + \frac{1}{h_{1}} \right) \right\} E_{1j} + \frac{1}{h_{2}} \left( \frac{1}{r_{1}h_{2}} + \frac{1}{r_{1}} \right) E_{2j} + \frac{1}{h_{1}h_{2}} E_{0} + \frac{1}{r_{1}^{2}a^{2}} E_{1j+} + \frac{1}{r_{1}^{2}a^{2}} E_{1j-} = 0 \\
1 \leq j \leq m_{p}
\]

(5.1)
\[
\left\{ \omega^2 \varepsilon_1 \mu_1 E_a - \frac{1}{h_{i+1}} \left( \frac{1}{h_{i+1}^2} + \frac{1}{h_i} \right) - \frac{1}{r_i h_{i+1}} - \frac{2}{r_i^2 r_i} \right\} E_{ij} + \frac{1}{h_{i+1}} \left( \frac{1}{h_{i+1}^2} + \frac{1}{r_i} \right) E_{i+1j} + \frac{1}{h_{i+1}^2} E_{i-1j} + \frac{1}{r_i^2 r_i^2} E_{ij+} + \frac{1}{r_i^2 r_i^2} E_{ij-} = 0
\]

From equation (1.49)

\[
\left\{ \omega^2 \varepsilon_1 \mu_1 E_a - \frac{1}{h_2 h_2} \left( \frac{1}{h_2^2} + \frac{1}{h_1} \right) - \frac{1}{r_1 h_2} - \frac{2}{r_1^2 r_1^2} \right\} H_{1j} + \frac{1}{h_2} \left( \frac{1}{h_2^2} + \frac{1}{r_1} \right) H_{2j} + \frac{1}{h_1 h_2} H_{0} + \frac{1}{r_1^2 r_1^2} E_{ij+} + \frac{1}{r_1^2 r_1^2} H_{ij-} = 0
\]

and

\[
\left\{ \omega^2 \varepsilon_1 \mu_1 E_a - \frac{1}{h_{i+1}} \left( \frac{1}{h_{i+1}^2} + \frac{1}{h_i} \right) - \frac{1}{r_i h_{i+1}} - \frac{2}{r_i^2 r_i^2} \right\} H_{1j} + \frac{1}{h_{i+1}} \left( \frac{1}{h_{i+1}^2} + \frac{1}{r_i} \right) H_{i+1j} + \frac{1}{h_i h_{i+1}} H_{i-1j} + \frac{1}{r_i^2 r_i^2} H_{ij+} + \frac{1}{r_i^2 r_i^2} H_{ij-} = 0
\]

From equation (1.48)

\[
-E_{d,j} \frac{1}{h_1 h_{i+1}} E_{i-1j} \mu_{12,j} B_c \frac{1}{h_1 h_{i+1}} H_{i-1j} - E_{e,j} \frac{1}{a^2 r_i^2} E_{ij-}
\]

\[
+ \left\{ E_{d,j} \frac{1}{h_{i+1}} \left( \frac{1}{h_{i+1}^2} + \frac{1}{h_i} \right) - \frac{1}{r_i h_{i+1}} + \frac{\varepsilon_r}{\mu_1} \left( \mu_{22,j} - \mu_{11,j} \right) \right\} B^2 + 2E_{e,j} \frac{1}{a^2 r_i^2} E_{ij+}
\]

\[
+ 2\varepsilon_r \frac{\mu_{12,j}}{\mu_1} \frac{1}{r_i} \left( \frac{1}{h_{i+1}^2} + \frac{2}{r_i} \right) + \omega^2 \varepsilon_1 \mu_1 E_b E_c
\]

\[
- 2\varepsilon_r \frac{\mu_{12,j}}{\mu_1} \left( \frac{1}{h_{i+1}^2} + \frac{2}{r_i} \right) + E_{e,j} \frac{1}{a r_i} \left( \frac{1}{r_i} - \mu_{12,j} \right) B_{-c} \frac{1}{a^2 r_i^2} H_{ij-}
\]

\[
+ \left\{ B_c \frac{1}{\alpha r_i} \left( \mu_{11,j} - \mu_{22,j} \right) \left( \frac{1}{h_{i+1}^2} + \frac{2}{r_i} \right) + \mu_{12,j} \left( \frac{1}{h_{i+1}^2} + \frac{1}{h_i} + \frac{3}{r_i} \right) - \frac{2}{a^2 r_i^2} \right\}
\]

\[
- \omega E_b \left[ \sin \theta \left( \frac{1}{h_{i+1}^2} + \frac{1}{r_i} \right) + \cos \theta \frac{1}{a r_i} \right]
\]
From equation (1.47)

\[-e_2Bc\mu_{12j}\frac{1}{r_i} + \frac{1}{h_i+1} H_{i-1j} - E_{f,j} \frac{1}{h_i+1} H_{i-1j} + e_2Bc\mu_{12j} \frac{1}{\alpha r_i^2} E_{ij} -
\]

\[+ \left\{ e_2Bc \frac{1}{\alpha r_i} \left( \mu_{11j} - \mu_{22j} \right) \left( \frac{1}{h_i+1} + \frac{2}{r_i} \right) + \mu_{12j} \left( \frac{1}{h_i+1} \left( \frac{1}{h_i} - \frac{3}{r_i} \right) - \frac{2}{\alpha r_i^2} \right) \right\} E_{ij} \]

\[+ \left\{ e_2Bc \frac{1}{\alpha r_i} \left( \mu_{11j} - \mu_{22j} \right) \left( \frac{1}{h_i+1} + \frac{2}{r_i} \right) - e_2\omega k E_b \cos 2\theta \right\} \frac{1}{\alpha r_i} E_{ij} +
\]

\[- E_{g,j} \frac{1}{\alpha^2 r_i^2} H_{ij} - \left\{ 2B^2\mu_{12j} \left( \frac{1}{h_i+1} + \frac{2}{r_i} \right) - \frac{1}{\alpha r_i} E_{g,j} \right\} \frac{1}{\alpha r_i} H_{ij} +
\]

\[+ \left\{ E_{f,j} \frac{1}{h_i+1} \left( \frac{1}{h_i+1} + \frac{1}{h_i} \right) - \frac{1}{r_i h_i+1} \left[ \mu_0 \varepsilon_\mu \varepsilon_\tau + B^2 \left( \mu_{11j} - 2\mu_{22j} \right) \right] \right. \]

\[+ \left. 2E_{g,j} \frac{1}{r_i r_i^2 \alpha^2} \right\} h_{ij} +
\]

\[+ \left\{ e_2Bc \left[ \mu_{12j} \left( \frac{3}{r_i} - \frac{1}{h_i+1} \right) + \left( \mu_{11j} - \mu_{22j} \right) \right] - \omega k E_2 \sin 2\theta \right\} H_{i+1j} \]

\[+ \left\{ \mu_0 \varepsilon_\mu \varepsilon_\tau \frac{1}{r_i} - E_{f,j} \frac{1}{h_i+1} + B^2 \left( \mu_{12j} \frac{1}{\alpha} + \mu_{11j} - 2\mu_{22j} \right) \frac{1}{r_i} \right\} \frac{1}{h_i+1} H_{i+1j} +
\]

\[+ e_2Bc \left[ \mu_{11j} - \mu_{22j} \right] \frac{1}{\alpha r_i} \frac{1}{h_i+1} E_{i+1j} - 2B^2\mu_{12j} \frac{1}{\alpha r_i} \frac{1}{h_i+1} H_{i+1j} = 0
\]

\[m_{b+1} \leq i \leq m_{q-1}, \quad 1 \leq j \leq m_p
\]
where $E_d,j, E_e,j, E_f,j, E_g,j, v_{11,j}, v_{12,j}$ and $u_{22,j}$ denote the values of the functions $E_d, E_e, E_f, E_g, v_{11}, v_{12}$ and $u_{22}$ at the particular value of $j$, that is at $\theta = ja$ provided $\theta = 0$ at $j = m_p$.

At the origin, from conditions (1.53) and (1.54) $E_z$ and $H_z$ are finite. They can be approximated by averaging the surrounding values as was done in the problem of Chapter 4.

Therefore

$$E_0 - \frac{1}{m_p} \sum_{j=1}^{m_p} E_{1j} = 0 \quad (5.7)$$

and

$$H_0 - \frac{1}{m_p} \sum_{j=1}^{m_p} H_{1j} = 0 \quad (5.8)$$

Two equations have now been derived at each of the $(m-2) m + 1$ domain mesh points.

The boundary conditions are given by equations (1.51), (1.52) and (1.55) to (1.58). On the outside boundary $r = a$, the approximation for $\frac{\partial E_z}{\partial r}$ based on the backward difference,

$$\frac{\partial E_z}{\partial r} = \frac{E_{mj} - E_{m-1j}}{h_m} \quad m = m_q, \quad 1 \leq j \leq m_p,$$

must be used as $E_{m+1,j}$ is undefined for $r \leq a$ and so the forward difference cannot be taken. Similarly for $\frac{\partial H_z}{\partial r}$. The other approximations used are the same as those at a domain mesh point.

Thus from equation (1.51)

$$E_{m,j} = 0 \quad m = m_q, \quad 1 \leq j \leq m_p \quad (5.9)$$

and from (1.52)

$$-B \varepsilon_2 \mu_{12,j} \frac{1}{h_m} E_{m-1j} - E_{fj} \frac{1}{h_m} H_{m-1j}$$
\[ + B \left( \varepsilon_2 \varepsilon_{12, j} \frac{1}{h_m} + \frac{1}{c E_j} e_{j, a} \frac{1}{a} \right) E_{mj} - \frac{B}{c E_j} e_{j, a} \frac{1}{a} E_{mj}^+ \]

\[ + \left( E_f, j \frac{1}{h_m} - B^2 \varepsilon_{12, j} \frac{1}{a} + \omega k \cos \theta \frac{1}{c E_b} \right) H_{mj} + B^2 \varepsilon_{12, j} \frac{1}{a} H_{mj}^+ = 0 \]

\[ m = m_q', \ 1 \leq j \leq m_p. \quad (5.10) \]

On the interface \( r = b \), the equations (1.55) to (1.58) hold. It follows that

\[ E_b E_c \left\{ \varepsilon_1 \frac{\partial E_z}{\partial r} \frac{1}{|1|} + \frac{B}{c} \frac{\partial E_z}{\partial \theta} \frac{1}{|1|} \right\} = E_a \left\{ \frac{\omega k}{c} \sin \theta \frac{\partial E_z}{\partial r} \frac{1}{|2|} + \varepsilon_2 \frac{\partial E_z}{\partial r} \frac{1}{|2|} \right\} - \varepsilon_2 \frac{\partial E_z}{\partial r} \frac{1}{|2|} \]

\[ \varepsilon r, \frac{\partial E_z}{\partial r} \frac{1}{|2|} - \frac{B}{c} E_d \frac{\partial E_z}{\partial r} \frac{1}{|2|} \right\} \]

\[ r = b, \ 0 \leq \theta \leq 2\pi \quad (5.11) \]

and

\[ E_b E_c \left\{ -\frac{\mu_1}{c} \frac{\partial E_z}{\partial r} \frac{1}{|1|} + \frac{B}{c} \frac{\partial E_z}{\partial \theta} \frac{1}{|1|} \right\} = E_a \left\{ \frac{\omega k}{c} \cos \theta \frac{\partial E_z}{\partial r} \frac{1}{|2|} - \frac{B}{c} E_e \frac{\partial E_z}{\partial r} \frac{1}{|2|} \right\} \]

\[ + \frac{B}{c} \varepsilon_r \frac{\mu_1}{c} \frac{\partial E_z}{\partial r} \frac{1}{|2|} + B^2 \varepsilon_{12} \frac{1}{r} \frac{\partial E_z}{\partial r} \frac{1}{|2|} + E_f \frac{\partial E_z}{\partial r} \frac{1}{|2|} \right\} \]

\[ r = b, \ 0 \leq \theta \leq 2\pi \quad (5.12) \]

where the suffixes 1,2 on the derivatives indicate the region to which they refer. These derivatives must now be approximated by differences.

Since \( E_{21} = E_{22} \), \( r = b, \ 0 \leq \theta \leq 2\pi \quad (1.56) \)

it follows that

\[ \frac{\partial E_z}{\partial \theta} \frac{1}{|1|} = \frac{\partial E_z}{\partial \theta} \frac{1}{|2|}, \quad r = b, \ 0 \leq \theta \leq 2\pi. \]

This derivative can be approximated by

\[ \frac{1}{a} (E_{mj}^+ - E_{mj}) \text{ at the interface mesh point } (m_j) \]

where \( m = m_b', \ 1 \leq j \leq m_p \).
Similarly using equation (1.58) one can write

\[
\frac{\partial H_z}{\partial \theta} \bigg|_1 = \frac{\partial H_z}{\partial \theta} \bigg|_2 = \frac{1}{\alpha} (H_{mj}^+ - H_{mj}) \quad m = m_b, \quad 1 \leq j \leq m_p.
\]

In region 2 the usual forward difference approximations for the derivatives with respect to \( r \) give

\[
\frac{\partial E_z}{\partial r} \bigg|_2 = \frac{1}{h_{m+1}} (E_{m+1j} - E_{mj}) \quad m = m_b, \quad 1 \leq j \leq m_p
\]

and

\[
\frac{\partial H_z}{\partial r} \bigg|_2 = \frac{1}{h_{m+1}} (H_{m+1j} - H_{mj}) \quad m = m_b, \quad 1 \leq j \leq m_p
\]

The derivatives \( \frac{\partial E_z}{\partial r} \bigg|_1 \) and \( \frac{\partial H_z}{\partial r} \bigg|_1 \) must be expressed in terms of values of \( E_z \) and \( H_z \) in region 1. A backward difference is used as on the outside boundary \( r = a \) to give

\[
\frac{\partial E_z}{\partial r} \bigg|_1 = \frac{1}{h_{m}} (E_{mj} - E_{m-1j}) \quad m = m_b, \quad 1 \leq j \leq m_p
\]

and

\[
\frac{\partial H_z}{\partial r} \bigg|_1 = \frac{1}{h_{m}} (H_{mj} - H_{m-1j}) \quad m = m_b, \quad 1 \leq j \leq m_p
\]

Using the above difference approximations, the equations (5.11) and (5.12) become at the interface mesh points

\[
\begin{align*}
-\varepsilon_1 E_b E_c \frac{1}{h_{m}} F_{m-1j} & + \left( \varepsilon_1 E_b E_c \frac{1}{h_{m}} - \varepsilon_2 E_{a,d,j} \frac{1}{h_{m+1}} + E_a \varepsilon_2 \varepsilon_r \frac{\mu_{12,j}}{\mu_1} \frac{1}{ab} \right) E_{mj} \\
- E_a \varepsilon_2 \varepsilon_r \frac{\mu_{12,j}}{\mu_1} \frac{1}{ab} E_{mj+} & + \varepsilon_2 E_{a,d,j} \frac{1}{h_{m+1}} E_{m+lj} \\
- \left( B \frac{1}{ab} (E_b E_c + E_{a,d,j}) + E_a \varepsilon_r \frac{B \mu_{12,j}}{\mu_1} \frac{1}{h_{m+1}} + E_a \mu_{2} \sin 2\theta \right)_b H_{mj} \\
+ (E_b E_c + E_{a,d,j}) \frac{B}{c} \frac{1}{ab} H_{mj+} & + E_a \varepsilon_r \frac{B \mu_{12,j}}{\mu_1} \frac{1}{h_{m+1}} H_{m+lj} &= 0
\end{align*}
\]

\[ m = m_b, \quad 1 \leq j \leq m_p \quad (5.13) \]
and
\[ \mu_1 \frac{d}{dx} E_c \frac{1}{h_m} H_m + B^2 \frac{d}{dx} \frac{1}{ab} H_m + \frac{1}{h_{m+1}} H_{m+1} \]

Thus another \(4m_p\) difference equations (5.9), (5.10), (5.13) and (5.14) have been obtained, two at each boundary mesh point.

In total, there are \(2(m_q.m_p+1)\) equations between the same number of unknowns, the values of \(E_z\) and \(H_z\) at the mesh points. The integers \(m_p\) and \(m_q\) and the radial step lengths \(h_i\) are free to be chosen and it is this choice which has a great bearing on the accuracy of the method.

The reader is referred to Chapter 2 and section 5.1.

As in the simultaneous method of Chapter 4 the \(2(m_q.m_p+1)\) column vector \(F\) is considered where
\[ F^T = (E_0 0 E_{11} \ldots E_{1m_p} H_{11} \ldots H_{1m_p} E_{21} \ldots H_{m_qm_p}) \]

Then the equations (5.1) to (5.10), (5.13 and (5.14) can be expressed in the form
\[ M F = \xi \]

where \(M\) is a real \(2(m_q.m_p+1)\) square matrix containing the coefficients of the equations and \(\xi\) is the corresponding zero column vector.

By listing the equations systematically, the matrix takes a banded form with all the non-zero elements in a band of width \(3m_p\) to the left and \(3m_p+1\) to the right of the main diagonal. The equations are written down working outwards from the origin, the order being
(5.7) j = 1\ldots p_m \\
(5.8) j = 1\ldots p_m \\
(5.1) j = 1\ldots p_m \\
(5.3) j = 1\ldots p_m \\
(5.2) j = 1\ldots p_m \quad \text{and} \quad (5.4) j = 1\ldots p_m \quad \text{alternately for} \quad i = 2, \ldots, m_b - 1 \\
(5.13) j = 1\ldots p_m \\
(5.14) j = 1\ldots p_m \\
(5.5) j = 1\ldots p_m \quad \text{and} \quad (5.6) j = 1\ldots p_m \quad \text{alternately for} \quad i = m_b + 1, \ldots, m_q - 1 \\
(5.9) j = 1\ldots p_m \\
and finally (5.10) j = 1\ldots p_m \\

The condition for a non-trivial solution is

$$|M| = 0 \quad (5.16)$$

and this gives the approximate characteristic equation for the problem.

For any reasonable approximation it is a polynomial of very high order in $B$.

The finite difference method has reduced the problem of finding the propagation constant for the system given in section 1.5 to one of solving equation (5.16).
5.3 The special case $\mu_1 = \mu_2 = \mu_0$

It is quite common for dielectrics to have a scalar permeability of very nearly $\mu_0$, the vacuum value. If region 1 contains such a material, air for example, then $\mu_1$ is equal to $\mu_0$, at least to within a good degree of accuracy. Also for a small static magnetic field below the saturation value the ratio of $\mu_2$ to $\mu_0$ is very near unity for many ferrites. In fact, for a small field $\mu_0 \gamma H_0 \ll \omega$ and so it can be seen from equation (1.3) that $\mu_2$ is very near to $\mu_0$. Therefore the condition $\mu_1 = \mu_2 = \mu_0$ will cover many different cases, at least to a very good approximation. Its effect on the difference equations derived in the previous section will now be investigated.

On putting $\mu_1 = \mu_2 = \mu_0$ it follows from the definitions in Chapter 1 that

\[
\mu_{11} = \mu_{22} = \mu_0 \\
\mu_{12} = 0 \\
\mu_\tau = 1
\]

\[
E_c = E_d = E_e = E_b = B^2 - \varepsilon_\tau
\]

and

\[
E_f = E_g = \mu_0 E_b
\]

Then the difference equations (5.1) to (5.10), (5.13) and (5.14) simplify to the following, working outwards from the origin.

Equations (5.7) and (5.8 are unchanged

\[
E_0 - \frac{1}{m_p} \sum_{j=1}^{m_p} E_{1j} = 0
\]

and

\[
H_0 - \frac{1}{m_p} \sum_{j=1}^{m_p} H_{1j} = 0.
\]

Equation (5.1) becomes

\[
\left\{ \omega^2 \varepsilon_1 \mu_0 E_a - \frac{1}{h_2} \left( \frac{1}{h_2} + \frac{1}{h_1} \right) + \frac{1}{r_1 h_2} - \frac{2}{r_1^2 a^2} \right\} E_{1j} \\
+ \frac{1}{h_2} \left( \frac{1}{h_2} + \frac{1}{r_1} \right) E_{2j} + \frac{1}{h_1 h_2} E_0 + \frac{1}{r_1^2 a^2} E_{1j} + + \frac{1}{r_1^2 a^2} E_{1j} - = 0, \ 1 \leq j \leq m_p
\]
Equation (5.3) becomes
\[
\left\{ \omega^2 \varepsilon_1 \mu_0 E_a - \frac{1}{h_2} \left( \frac{1}{h_2} + \frac{1}{h_1} \right) + \frac{1}{r_1 h_2} - \frac{2}{r_1^2 a^2} \right\} H_{1j} \\
+ \frac{1}{h_2(h_2 + r_1)} H_{2j} + \frac{1}{h_1 h_2} H_0 + \frac{1}{r_1^2 a^2} H_{1j} + \frac{1}{r_1^2 a^2} H_{1j-} = 0, \quad 1 \leq j \leq m_p \tag{5.20}
\]

Equation (5.2) becomes
\[
\left\{ \omega^2 \varepsilon_1 \mu_0 E_a - \frac{1}{h_{1+1}} \left( \frac{1}{h_{1+1}} + \frac{1}{h_i} \right) - \frac{1}{r_i h_{1+1}} - \frac{2}{r_i^2 a^2} \right\} E_{ij} \\
+ \frac{1}{h_{1+1}(h_{1+1} + r_i)} E_{i+1j} + \frac{1}{h_i h_{1+1}} E_{i-1j} + \frac{1}{r_i^2 a^2} E_{ij+} + \frac{1}{r_i^2 a^2} E_{ij-} = 0, \\
2 \leq i \leq m_{-1}, \quad 1 \leq j \leq m_p \tag{5.21}
\]

Equation (5.4) becomes
\[
\left\{ \omega^2 \varepsilon_1 \mu_0 E_a - \frac{1}{h_{1+1}} \left( \frac{1}{h_{1+1}} + \frac{1}{h_i} \right) - \frac{1}{r_i h_{1+1}} - \frac{2}{r_i^2 a^2} \right\} H_{ij} \\
+ \frac{1}{h_{1+1}(h_{1+1} + r_i)} H_{i+1j} + \frac{1}{h_i h_{1+1}} H_{i-1j} + \frac{1}{r_i^2 a^2} H_{ij+} + \frac{1}{r_i^2 a^2} H_{ij-} = 0, \\
2 \leq i \leq m_{-1}, \quad 1 \leq j \leq m_p . \tag{5.22}
\]

Equation (5.13) becomes
\[
- \varepsilon_1 E_b \frac{1}{h_m} E_{m-1j} + \left( \varepsilon_1 E_b \frac{1}{h_m} - \varepsilon_2 E_a \frac{1}{h_{m+1}} \right) E_{mj} + \varepsilon_2 E_a \frac{1}{h_{m+1}} E_{m+1j} \\
- \left( B \frac{1}{c ab} (E_b + E_a) + E_a \omega \varepsilon_2 \cos \theta \right) H_{mj} + (E_b + E_a) B \frac{1}{c ab} H_{mj+} = 0 , \\
m = m_b, \quad 1 \leq j \leq m_p . \tag{5.23}
\]

Equation (5.14) becomes
\[
\mu_0 E_b \frac{1}{h_m} H_{m-1j} + \left( \mu_0 E_a \frac{1}{h_m} - \mu_0 E_b \frac{1}{h_{m+1}} - E_a \omega \varepsilon_2 \cos \theta \right) H_{mj} - \mu_0 E_a \frac{1}{h_{m+1}} H_{m+1j} \\
- B \frac{1}{c ab} (E_b + E_a) E_{mj} + B \frac{1}{c ab} (E_b + E_a) E_{mj+} = 0 , \quad m = m_b, \quad 1 \leq j \leq m_p . \tag{5.24}
\]
Equation (5.5) becomes
\begin{align*}
&\frac{1}{h_i h_{i+1}} E_{i-1j} + \frac{1}{a^2 r_i} E_{ij} - \left( \frac{1}{h_{i+1}} \left( \frac{1}{h_i} + \frac{1}{h_i} \right) + \frac{1}{r_i h_{i+1}} + \frac{2}{r_i^2 a^2} + \omega^2 \varepsilon_1 \mu_0 \right) E_{ij} \\
&+ \frac{1}{a^2 r_i} E_{ij} + \omega \kappa \left( \sin \theta \left( \frac{1}{h_{i+1}} + \frac{1}{h_i} \right) + \cos \theta \frac{1}{a r_i} \right) H_{ij} - \omega \kappa \sin \theta \frac{1}{a r_i} H_{ij} \\
&+ \left( \frac{1}{h_{i+1}} + \frac{1}{r_i} \right) \frac{1}{h_i} E_{i+1j} - \omega \kappa \sin \theta \frac{1}{h_{i+1}} H_{i+1j} = 0,
\end{align*}
\begin{align*}
&\text{for } m_b+1 \leq i \leq m_q-1, \ 1 \leq j \leq m_p \quad (5.25)
\end{align*}

Equation (5.6) becomes
\begin{align*}
&\frac{\mu_0}{h_i h_{i+1}} H_{i-1j} - \omega \varepsilon_2 \left( \sin \theta \frac{1}{h_{i+1}} + \cos \theta \frac{1}{a r_i} \right) E_{ij} + \omega \varepsilon_2 \kappa \cos \theta \frac{1}{a r_i} E_{ij} + \mu_0 \frac{1}{a^2 r_i^2} H_{ij} \\
&- \left( \mu_0 \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) + \frac{1}{r_i h_{i+1}} + \frac{2}{a^2 r_i^2} \right) - \omega^2 \varepsilon_2 (\mu_0^2 - \kappa^2) + \frac{B}{c} \omega \kappa \cos \theta \frac{1}{r_i} + B^2 \varepsilon_1 \mu_0 \right) H_{ij} \\
&+ \mu_0 \frac{1}{a^2 r_i^2} H_{ij} + \omega \varepsilon_2 \sin \theta \frac{1}{h_i} E_{i+1j} + \mu_0 \frac{1}{h_{i+1}} \left( \frac{1}{h_i} + \frac{1}{r_i} \right) H_{i+1j} = 0,
\end{align*}
\begin{align*}
&\text{for } m_b+1 \leq i \leq m_q-1, \ 1 \leq j \leq m_p \quad (5.26)
\end{align*}

Equation (5.9) becomes
\begin{align*}
E_{mj} = 0, \quad m = m_q, \ 1 \leq j \leq m_p \quad (5.27)
\end{align*}

Finally equation (5.10) becomes
\begin{align*}
-\mu_0 \frac{1}{h_m} H_{m-1j} + \frac{B}{c} \frac{1}{aa} E_{mj} - \frac{B}{c} \frac{1}{aa} E_{mj} + \\
+ \left( \mu_0 \frac{1}{h_m} + \omega \kappa \frac{B}{c} \cos \theta \right) H_{mj} = 0, \quad m = m_q, \ 1 \leq j \leq m_p \quad (5.28)
\end{align*}

Equations (5.17) to (5.28) are \(2(m_q \cdot m_p + 1)\) equations between the same number of unknowns and the theory carries through as in the previous section to give the approximate characteristic equation (5.16). However here the matrix \(M\) has a narrower band of non-zero elements. They are all included in a band of width \(2m_p\) elements to the left and \(3m_p\) elements to the
right of the main diagonal. In the computer evaluation of $|M|$ the amount of store required is directly related to the size of the band containing all the non-zero elements (see section 3.2). It follows that this special case uses less computer store and so is to be preferred to the general case wherever possible. The relevant computer programmes are discussed in section 7.2

Another special case of the general problem is considered in the next chapter.
CHAPTER 6

The circular cylindrical waveguide filled with two coaxial dielectrics

6.1 The derivation of the characteristic equation

The waveguide is assumed to have perfectly conducting walls. Its cross-section is divided into two regions by the circle $r = b$ which is the interface between the two dielectrics. Let $0 \leq r \leq b$ be region (1) and $b \leq r \leq a$ be region (2). The dielectrics are assumed to be homogeneous, isotropic and lossless, having real scalar permeabilities and permittivities; $\mu_1$, $\mu_2$, $\varepsilon_1$ and $\varepsilon_2$ being the appropriate values as shown in Figure 6.1.

An electromagnetic wave propagating along the guide will satisfy Maxwell's equations in each region. Assuming an $\exp\{j(\omega t - \beta z)\}$ dependence on time $t$ and $z$, the distance along the guide, it is apparent that $\beta$ must be the same in each region in order to satisfy the boundary conditions on the interface for all $z$. The two regions differ only in the values of their permittivities and permeabilities. Region (1) is identical to the corresponding region considered in section 1.3 and so quoting from equations (1.27), (1.28), (1.31) and (1.32) Maxwell's equations give

\[
\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r} \frac{\partial H_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \theta^2} + j k_i \eta_{i} H_z = 0 \quad (6.1)
\]

\[
\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r} \frac{\partial E_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} + j k_i \eta_{i} E_z = 0 \quad (6.2)
\]

\[
j k_i \eta_{i}^2 = \frac{\omega \varepsilon_i}{\mu_i} \frac{\partial E_z}{\partial r} + \frac{\beta}{r} \frac{\partial H_z}{\partial \theta} \quad (6.3)
\]

and

\[
j k_i \eta_{i}^2 = -\omega \mu_i \frac{\partial H_z}{\partial r} + \frac{\beta}{r} \frac{\partial E_z}{\partial \theta}, \quad (6.4)
\]

where $k_i^2 = \omega^2 \varepsilon_i \mu_i - \beta^2$

for $0 \leq \theta \leq 2\pi$ and $i = 1, 2$

with $0 \leq r \leq b$ for $i = 1$

and $b \leq r \leq a$, for $i = 2$. 

Since the waveguide is assumed to have perfectly conducting walls, the boundary condition (1.35) applied here gives

\[ \mathbf{E} \times \mathbf{E} = 0 \quad r = a, \ 0 \leq \theta \leq 2\pi \]

i.e.

\[ E_z = 0 \quad r = a, \ 0 \leq \theta \leq 2\pi \]  \hspace{1cm} (6.5)

and \[ E_\theta = 0 \quad r = a, \ 0 \leq \theta \leq 2\pi \]  \hspace{1cm} (6.6)

Similarly the boundary conditions (1.33) and (1.34) give here

\[ E_{z1} = E_{z2} \quad r = b, \ 0 \leq \theta \leq 2\pi \]  \hspace{1cm} (6.7)

\[ E_{\theta 1} = E_{\theta 2} \quad r = b, \ 0 \leq \theta \leq 2\pi \]  \hspace{1cm} (6.8)

\[ H_{z1} = H_{z2} \quad r = b, \ 0 \leq \theta \leq 2\pi \]  \hspace{1cm} (6.9)

and \[ H_{\theta 1} = H_{\theta 2} \quad r = b, \ 0 \leq \theta \leq 2\pi \]  \hspace{1cm} (6.10)

where the suffices 1 and 2 denote the region.

The problem is now completely specified by the equations (6.1) to (6.10). It can be considered as a problem in two dependent variables \( E_z \) and \( H_z \) since the \( E_\theta \) and \( H_\theta \) components are easily eliminated.

The fields must be periodic in \( \theta \) and so a dependence of the form \( \exp\{-jn\theta\} \) (n integral) is assumed. Then equations (6.1) and (6.2) are just Bessel's equation and can be solved in the form of infinite series.

In region (1) remembering that the fields are finite at the origin,

\[ E_z = \sum_{n=0}^{\infty} A_n J_n(k_1r)e^{-jn\theta} \]

and

\[ H_z = \sum_{n=0}^{\infty} B_n J_n(k_1r)e^{-jn\theta} \]

\[ 0 \leq r \leq b, \ 0 \leq \theta \leq 2\pi \]

where the \( A_n \) and \( B_n \) are arbitrary complex constants.

In region (2)

\[ E_z = \sum_{n=0}^{\infty} \left\{ C_n J_n(k_2r) + D_n Y_n(k_2r) \right\} e^{-jn\theta} \]

and

\[ H_z = \sum_{n=0}^{\infty} \left\{ F_n J_n(k_2r) + G_n Y_n(k_2r) \right\} e^{-jn\theta} \]

\[ b \leq r \leq a, \ 0 \leq \theta \leq 2\pi \]
where \( C_n, D_n, F_n \) and \( G_n \) are arbitrary complex constants. The two further conditions (6.5) and (6.6) must be satisfied by this solution.

The four boundary conditions (6.7) to (6.10) at \( r = b \) must be satisfied. From the relations (6.3) and (6.4) it is clear that they must be satisfied for each integer \( n \) individually to hold for all \( \theta \).

Then the equations (6.5) to (6.10) imply that, for all \( n \),

\[
C_n J_n(k_2a) + D_n Y_n(k_2a) = 0 \quad (6.11)
\]

\[
-\omega_2 k_2 \{ F_n J_n(k_2a) + G_n Y_n(k_2a) \} - jn\theta \frac{1}{\alpha} \{ C_n J_n(k_2a) + D_n Y_n(k_2a) \} = 0
\]

(6.12)

\[
A_n J_n(k_1b) - C_n J_n(k_2b) - D_n Y_n(k_2b) = 0 \quad (6.13)
\]

\[
-\frac{\omega_1}{k_1} B_n J_n(k_1b) - \frac{jn\theta}{bk_1^2} A_n J_n(k_1b) + \frac{\omega_2}{k_2} \{ F_n J_n(k_2b) + G_n Y_n(k_2b) \} + \frac{jn\theta}{k_2^2} \{ C_n J_n(k_2b) + D_n Y_n(k_2b) \} = 0
\]

(6.14)

\[
B_n J_n(k_1b) - F_n J_n(k_2b) - G_n Y_n(k_2b) = 0 \quad (6.15)
\]

\[
\frac{\omega_1}{k_1} A_n J_n'(k_1b) - \frac{jn\theta}{bk_1^2} B_n J_n(k_1b) - \frac{\omega_2}{k_2} \{ C_n J_n'(k_2b) + D_n Y_n'(k_2b) \} + \frac{jn\theta}{k_2^2} \{ F_n J_n'(k_2b) + G_n Y_n'(k_2b) \} = 0
\]

(6.16)

where the dash means differentiation with respect to \( r \).

The condition for this set of equations to have a non-trivial solution is, after some simplification,

\[
\begin{vmatrix}
J_n(k_1b) & 0 & -J_n(k_2b) & -Y_n(k_2b) & 0 & 0 \\
0 & J_n(k_1b) & 0 & 0 & -J_n(k_2b) & -Y_n(k_2b) \\
0 & 0 & J_n(k_2a) & Y_n(k_2a) & 0 & 0 \\
0 & 0 & 0 & 0 & J_n'(k_2a) & Y_n'(k_2a) \\
\frac{k_2\omega_1}{k_1\epsilon_2} J_n(k_1b) & \frac{jn\psi}{\epsilon_2} J_n(k_1b) & -J_n'(k_2b) & -Y_n'(k_2b) & 0 & 0 \\
\frac{jn\psi}{\mu_2} J_n(k_1b) & -\frac{\mu_1 k_2}{\mu_2 k_1^2} J_n'(k_1b) & 0 & 0 & J_n'(k_2b) & Y_n'(k_2b)
\end{vmatrix} = 0
\]

(6.17)

\( n = 0, 1, 2, \ldots \).
where \( \psi = \frac{B \sqrt{\varepsilon_1 \mu_1 (k_1^2 - k_2^2)}}{bk_1^2 k_2} \) and \( \beta^2 = B^2 \omega^2 \varepsilon_1 \mu_1 \).

\( B \) is the scaled propagation constant of the previous chapter.

On multiplying out the determinant of equation (6.17) the following equation is obtained:

\[
\left\{ \frac{k_1 \varepsilon_2}{\mu_1} S J_n(k_1b) - k_2 R J'_n(k_1b) \right\} \left\{ \frac{k_1 \varepsilon_2}{\varepsilon_1} Q J_n(k_1b) - k_2 P J'_n(k_1b) \right\} - \frac{n^2 B^2 (k_1^2 - k_2^2)^2}{b^2 k_1^2 k_2^2} R P J_n^2(k_1b) = 0 \quad (6.18)
\]

where \( P = J_n(k_2a) Y_n(k_2b) - J_n(k_2b) Y_n(k_2a) \)

\( Q = J_n(k_2a) Y'_n(k_2b) - J'_n(k_2b) Y_n(k_2a) \)

\( R = J'_n(k_2a) Y_n(k_2b) - J_n(k_2b) Y'_n(k_2a) \)

and \( S = J'_n(k_2a) Y'_n(k_2b) - J'_n(k_2b) Y'_n(k_2a) \).

Equation (6.18) is the characteristic equation for the propagation constants. It has been derived by Bucholz (1943) in an equivalent form and by others later. Solving it for each \( n \) in turn will give all the possible propagation constants for a particular dielectric filled waveguide configuration. For lossless propagation down the guide only real positive values of \( B \) are required. Each such \( B \) gives one mode of propagation.

If the two regions are identical, \( \varepsilon_2 = \varepsilon_1 \) and \( \mu_2 = \mu_1 \), the problem reduces to the one of one region discussed in section 4.1. Equation (6.18) becomes

\[
J_n(k_1a) J'_n(k_1a) \{J_n(k_1b) Y'_n(k_1b) - J'_n(k_1b) Y_n(k_1b) \}^2 = 0
\]

and \( b \) can take any value \( 0 \leq b \leq a \) so the squared term is not always zero.
Hence

\[ J_n(k_1a) = 0 \]

or \[ J'_n(k_1a) = 0, \]

and these are the characteristic equations (4.6) and (4.8) of section 4.1.

A similar result can be obtained by putting \( b = a \) in equation (6.18).

Therefore that equation is consistent with the one region problem.
Figure 6.1. The configuration

Figure 6.2. The corresponding meshes
6.2 Solution of the characteristic equation

The characteristic equation (6.18) is very complicated and is best solved by a numerical method using a computer. The left-hand side of the equation can be considered to be a function, $f(n, B)$, of $n$ and $B$ for any given configuration of dielectric and waveguide, i.e. for fixed values of the parameters $a, b, \varepsilon_1, \mu_1, \varepsilon_2$ and $\mu_2$. It is a fairly straightforward matter to write a computer programme to evaluate $f(n, B)$ for any pair of values of $n$ and $B$ that may be read in. $f(n, B)$ is a continuous function of $B$ for a fixed $n$ since all its constituent functions are continuous. It follows that if, for any $n$, two values of $B$ give values of $f(n, B)$ of opposite sign then at least one root of equation (6.18) must lie between those values of $B$. Hence solutions of equation (6.18) can be found by successively trying appropriate values of $B$. In fact, for any root, an arbitrarily small interval in which it lies may be found.

Of course, with such a trial and error method there is no guarantee that all the roots will be found even for one $n$. Two solutions could conceivably be close enough together so that no chosen value of $B$ happened to fall between them and hence show a change of sign in $f(n, B)$. Also not every theoretical value of $n$ can be tried so again a solution could be missed.

In practice, modes are plotted from easily found starting positions. For example, when $b = a$ the values of $B$ are calculated (the problem having reduced to the one region type) and then the mode is followed with decreasing $b$. The root moves continuously with a continuous variation in a parameter. Other modes which are cut off when $b = a$ may be introduced as $b$ decreases. The cut-off point, $B = 0$, is relatively easy to find since equation (6.18) is then much simplified and again the mode can be followed from there. For a detailed discussion the reader is referred to Waldron (1969).
Numerical results for the case of region (1) being a dielectric rod and region (2) air are given by Waldron (1958). The computer programme in the Appendix 3 evaluates and prints out $f(n,B)$ for any $n$ and $B$ provided that $k_2^2 > 0$. It uses the Bessel functions $J_n$ and $Y_n$ and also, when $k_1^2 < 0$, the modified Bessel function $I_n$. The results obtained are accurate when tested against calculated values for the one region case and values from Waldron (1958). See section 7.4.
6.3 The finite difference method

The problem discussed in this chapter is merely a special case of the boundary value problem of section 1.5. It is obtained from that one by making the substitutions $\kappa = 0$ and $\mu_0 = \mu_2$. Then it is not difficult to show that the equations in section 1.5 become equivalent to the equations (6.1) to (6.10). The difference equations (5.1) to (5.10), (5.13) and (5.14) in section 5.2 were derived from the equations of section 1.5. In this special case, these difference equations will be simplified in the following way.

When $\kappa = 0$ and $\mu_0 = \mu_2$,

$$
E_{c,d,j} = E_{e,j} = E_{b} = B^2 - \varepsilon_r \mu_r
$$

and

$$
E_{f,j} = E_{g,j} = \mu_2 E_b.
$$

The equations (5.1) to (5.4), (5.7) and (5.8) are unchanged. This is only to be expected since region(1) is the same in both cases and these are the equations in that region.

Equation (5.5) becomes

$$
- \frac{1}{h_{i+1}} E_{i-1,j} - \frac{1}{\alpha_r^2 r_{i}^2} E_{i,j -} + \left( \frac{1}{h_{i+1}} \left( \frac{1}{h_{i+1}} + \frac{1}{h_{i}} \right) + \frac{1}{r_i h_{i+1}} + \frac{2}{r_i^2 \alpha_r^2} + \omega^2 \varepsilon_1 \mu_1 E_b \right) E_{i,j} = 0
$$

Equation (5.6) becomes

$$
- \frac{1}{h_{i+1}} H_{i-1,j} - \frac{1}{\alpha_r^2 r_{i}^2} H_{i,j -} + \left( \frac{1}{h_{i+1}} \left( \frac{1}{h_{i+1}} + \frac{1}{h_{i}} \right) + \frac{1}{r_i h_{i+1}} + \frac{2}{r_i^2 \alpha_r^2} + \omega^2 \varepsilon_1 \mu_1 E_b \right) H_{i,j} = 0
$$
Equation (5.9) is unaltered

\[ E_{mj} = 0 \quad m = m_q, \quad 1 \leq j \leq m_p \]  

(6.21)

Equation (5.10) becomes

\[ \frac{B}{c} \frac{1}{\alpha \alpha} (E_{mj} - E_{mj+}) + \frac{\mu_2}{h_m} (H_{mj} - H_{m-1j}) = 0 \]

\[ m = m_q, \quad 1 \leq j \leq m_p \]  

(6.22)

Equation (5.13) becomes

\[ \varepsilon_1 E_{bh_m} \left( E_{mj} - E_{m-1j} \right) + \varepsilon_2 E_a \frac{1}{h_{m+1}} (E_{m+1j} - E_{mj}) \]

\[ + \left( E_b + E_a \right) \frac{B}{c} \frac{1}{\alpha \beta} (H_{mj+} - H_{mj}) = 0 \]

\[ m = m_b, \quad 1 \leq j \leq m_p \]  

(6.23)

Equation (5.14) becomes

\[ \mu_1 E_b \frac{1}{h_m} (H_{mj} - H_{m-1j}) + \mu_2 E_a \frac{1}{h_{m+1}} (H_{m+1j} - H_{mj}) \]

\[ - \left( E_b + E_a \right) \frac{B}{c} \frac{1}{\alpha \beta} (E_{mj+} - E_{mj}) = 0 \]

\[ m = m_b, \quad 1 \leq j \leq m_p \]  

(6.24)

The above set of difference equations (5.1) to (5.4), (5.7), (5.8) and (6.19) to (6.24) between values of \( E_z \) and \( H_z \) at points of a chosen mesh gives a finite difference approximation to the boundary value problem specified by the set of differential equations (6.1) to (6.10). The same assumptions, namely \( \kappa = 0 \) and \( \mu_0 = \mu_2 \), have been made in the derivation of both sets of equations. These two sets of equations are equivalent to each other just as the difference equations (5.1) to (5.10), (5.13) and (5.14) are equivalent to the differential equations of section 1.5.

In more detail, the equations (6.1) and (6.2) with \( i = 1 \) and the condition of a finite field at the origin are expressed as equations (5.1) to (5.4), (5.7) and (5.8) exactly as in section (5.2). Similarly,
equations (6.1) and (6.2) with \( i = 2 \) and (6.5) are equivalent to equations (6.19), (6.20) and (6.21) respectively. Combining equations (6.4) and (6.6) to eliminate \( E_\theta \) gives

\[
- \omega \mu_2 \frac{\partial H_z}{\partial r} + \frac{\beta}{r} \frac{\partial E_z}{\partial \theta} = 0 \quad r = a, \ 0 \leq \theta \leq 2\pi \quad (6.25)
\]

which is equivalent to equation (6.22). Combining equation (6.10) with (6.3) with \( i = 1 \) and 2 to eliminate \( H_{\theta 1} \) and \( H_{\theta 2} \) and equation (6.8) with (6.4) with \( i = 1 \) and 2 to eliminate \( E_{\theta 1} \) and \( E_{\theta 2} \) gives the two equations

\[
\begin{align*}
\frac{\omega \varepsilon_1}{k_1^2} \frac{\partial E_z}{\partial r} \bigg|_1 + \frac{\beta}{k_1^2 r} \frac{\partial H_z}{\partial \theta} \bigg|_1 - \frac{\omega \varepsilon_2}{k_2^2} \frac{\partial E_z}{\partial r} \bigg|_2 - \frac{\beta}{k_2^2 r} \frac{\partial H_z}{\partial \theta} \bigg|_2 &= 0 \\
r = b, \ 0 \leq \theta \leq 2\pi
\end{align*}
(6.26)
\]

and

\[
\begin{align*}
\frac{\omega \mu_1}{k_1^2} \frac{\partial H_z}{\partial r} \bigg|_1 - \frac{\beta}{k_1^2 r} \frac{\partial E_z}{\partial \theta} \bigg|_1 - \frac{\omega \mu_2}{k_2^2} \frac{\partial H_z}{\partial r} \bigg|_2 + \frac{\beta}{k_2^2 r} \frac{\partial E_z}{\partial \theta} \bigg|_2 &= 0 \\
r = b, \ 0 \leq \theta \leq 2\pi
\end{align*}
(6.27)
\]

As before the suffices on the derivatives indicate to which region they refer. Using the substitutions of section 5.2 for these derivatives, it is easily shown that equations (6.26) and (6.27) are equivalent to equations (6.23) and (6.24).

The verification of the equivalence of the two sets of equations in this special case provides a check on the correctness of the difference equations (5.1) to (5.10), (5.13) and (5.14) used for the more general case. Also, by putting \( \kappa = 0 \) and \( \mu_0 = \mu_2 \) in the computer programme for the general case, values given by the finite difference method can be tested against corresponding values from the exact characteristic equation (6.18) for this special case. The exact values are obtained by the computer method outlined in section 6.2. A comparison of the numerical results is given in section 7.5. A further special case will now be considered.
6.4 The special case $\mu_2 = \mu_1$ and $\varepsilon_2 = \varepsilon_1$

If now $\mu_2$ and $\varepsilon_2$ are put equal to $\mu_1$ and $\varepsilon_1$ the problem reduces to the one region one discussed in Chapter 4; $\mu_1$, $\varepsilon_1$ and $k_1$ replacing $\mu_0$, $\varepsilon_0$ and $k_0$ in the notation used. The equations (6.1) to (6.10) reduce to the equations (4.1) to (4.4) under this substitution.

For the finite difference method, in Chapter 4, a mesh with a constant radial step length was chosen. In this chapter, on the other hand, there is no such restriction but it will be assumed that the steps $h_i$ for $i = m_b, m_b + 1$ and $m_q$ are small compared with the remaining ones. This assumption is the one suggested in section 5.1. The effect of the substitutions $\mu_2 = \mu_1$ and $\varepsilon_2 = \varepsilon_1$ into the difference equations (5.1) to (5.4), (5.7), (5.8) and (6.19) to (6.24) will now be investigated.

Again the equations (5.1) to (5.4), (5.7) and (5.8) are unchanged since again region (1) is unaltered by the substitutions. Now however,

$$\mu_r = \varepsilon_r = 1$$

and $E_b = B^2 - 1 = -E_a$ and so

equations (6.19) and (6.20) become respectively

$$-\frac{1}{h_i h_{i+1}} E_{i-1j} - \frac{1}{a^2 r_i^2} E_{ij} - \left\{ \frac{1}{h_{i+1}} \left( \frac{1}{h_{i+1}} + \frac{1}{h_i} \right) + \frac{1}{r_i h_{i+1}} + \frac{2}{r_i^2 a^2 - \omega^2 \varepsilon_1 \mu_1 E_a} \right\} E_{ij}$$

$$- \frac{1}{a^2 r_i^2} E_{ij+} - \frac{1}{h_{i+1}} \left( \frac{1}{h_{i+1}} + \frac{1}{r_i} \right) E_{i+1j} = 0$$

$$m_b + 1 \leq i \leq m_q - 1, \ 1 \leq j \leq m_p, \quad (6.28)$$

and

$$-\frac{1}{h_i h_{i+1}} H_{i-1j} - \frac{1}{a^2 r_i^2} H_{ij} - \left\{ \frac{1}{h_{i+1}} \left( \frac{1}{h_{i+1}} + \frac{1}{h_i} \right) + \frac{1}{r_i h_{i+1}} + \frac{2}{r_i^2 a^2 - \omega^2 \varepsilon_1 \mu_1 E_a} \right\} H_{ij}$$

$$- \frac{1}{a^2 r_i^2} H_{ij+} - \frac{1}{h_{i+1}} \left( \frac{1}{h_{i+1}} + \frac{1}{r_i} \right) H_{i+1j} = 0,$$

$$m_b + 1 \leq i \leq m_q - 1, \ 1 \leq j \leq m_p. \quad (6.29)$$
Equation (6.21) is unaltered

$$E_{mj} = 0 \quad m = m_q, \quad 1 \leq j \leq m_p \quad (6.30)$$

Equation (6.22) becomes

$$\frac{B}{c \alpha a} (E_{mj} - E_{mj+1}) + \frac{\mu_1}{h_m} (H_{mj} - H_{m-1j}) = 0$$

$$m = m_q, \quad 1 \leq j \leq m_p$$

which on substituting from (6.30) simplifies to

$$H_{mj} - H_{m-1j} = 0 \quad m = m_q, \quad 1 \leq j \leq m_p \quad (6.31)$$

Equations (6.23) and (6.24) become respectively

$$\frac{1}{h_{m+1}} E_{m+1j} - \left( \frac{1}{h_{m+1}} + \frac{1}{h_m} \right) E_{mj} + \frac{1}{h_m} E_{m-1j} = 0$$

$$m = m_b, \quad 1 \leq j \leq m_p, \quad (6.32)$$

and

$$\frac{1}{h_{m+1}} H_{m+1j} - \left( \frac{1}{h_{m+1}} + \frac{1}{h_m} \right) H_{mj} + \frac{1}{h_m} H_{m-1j} = 0$$

$$m = m_b, \quad 1 \leq j \leq m_p \quad (6.33)$$

The set of difference equations (5.1) to (5.4), (5.7), (5.8) and (6.28) to (6.33) gives a finite difference approximation to the problem of Chapter 4. It includes values of both $E_z$ and $H_z$ at the mesh points and so leads to a simultaneous method of tackling the problem. Such a method has already been discussed in Chapter 4 and the set of difference equations (4.15) to (4.22) derived for it. The only difference between the two procedures is in the choosing of the mesh. Leaving this aside, the two sets of difference equations, equations (4.15) to (4.22) and equations (5.1) to (5.4), (5.7), (5.8) and (6.28) to (6.33) should correspond completely. It can be seen immediately that this is the case except, apparently, on the circle $i = m_b$ ($r = b$), the interface between the two regions which now have the same physical properties. This difficulty on the "fictitious" interface will be resolved by a more detailed consideration.

The equations (5.2) for $i = m_b - 1$, (6.28) for $i = m_b + 1$ and (6.32) are combined to eliminate the values of $E_z$ on the circles $i = m_b - 1$ and $i = m_b + 1$. 
The resulting equation links values of $E_x$ on the circles $i = m_b^2$, $i = m_b$ and $i = m_b^2$. This equation is equivalent to equation (4.19) when the circles $i = m_b^2$ and $i = m_b^2$ tend to the circle $i = m_b$.

Let A, B and C be three neighbouring mesh points with respective values of $i$ and $j$ given by $(m-1,j)$, $(m,j)$ and $(m+1,j)$ with $m = m_b$.

Equation (5.2) holds at A and equation (6.28) holds at C and they give

$$
\frac{1}{h_{m-1}} E_{m-2j} + \frac{1}{\alpha^2 r_{m-1}^2} E_{m-1j} - \left( \frac{1}{h_m} \left( \frac{1}{h_m} + \frac{1}{h_{m-1}} \right) + \frac{1}{r_{m-1} h_m} + \frac{2}{\alpha^2 r_{m-1}^2} - \omega^2 \epsilon_1 \mu_1 E_a \right) E_{m-1j}
$$

$$
+ \frac{1}{\alpha^2 r_{m-1}^2} E_{m-1j} + \frac{1}{h_m} \left( \frac{1}{h_m} + \frac{1}{h_{m-1}} \right) E_{mj} = 0
$$

(6.34)

and

$$
\frac{1}{h_{m+2}} E_{mj} + \frac{1}{\alpha^2 r_{m+1}^2} E_{mj} - \left( \frac{1}{h_{m+2}} \left( \frac{1}{h_{m+2}} + \frac{1}{h_{m+1}} \right) + \frac{1}{r_{m+1} h_{m+2}} + \frac{2}{\alpha^2 r_{m+1}^2} - \omega^2 \epsilon_1 \mu_1 E_a \right) E_{m+1j}
$$

$$
+ \frac{1}{\alpha^2 r_{m+1}^2} E_{m+1j} + \frac{1}{h_{m+2}} \left( \frac{1}{h_{m+2}} + \frac{1}{r_{m+1}} \right) E_{m+2j} = 0
$$

(6.35)

Multiplying equation (6.34) by $\frac{h_m}{h_{m+2}}$, adding equation (6.35) and simplifying using equation (6.32) which holds at B gives

$$
\frac{1}{h_{m+2}} E_{m-2j} + \frac{1}{h_{m+2}} E_{m-1j} + \frac{h_m}{h_{m+2}} \left( \frac{1}{h_{m+2}} + \frac{1}{r_{m+1}} \right) E_{m+2j}
$$

$$
+ \frac{1}{\alpha^2 h_{m+2}} \left( \frac{h_m}{r_{m+1}^2 - 1} E_{m-1j} + \frac{1}{r_{m+1}^2 - 1} E_{m+1j} - \frac{1}{h_{m+2}} \right) E_{m-1j} + \frac{1}{\alpha^2 r_{m+1}^2} \left( \frac{h_m}{r_{m+1}^2} E_{m+1j} + \frac{1}{r_{m+2}^2 - 1} E_{m+1j} \right)
$$

$$
- \frac{2}{\alpha^2} \left( \frac{h_m}{r_{m+1}^2} E_{m+1j} + \frac{1}{r_{m+2}^2} E_{m+1j} \right) \omega^2 \epsilon_1 \mu_1 E_a \left( \frac{h_m}{h_{m+2}} E_{m-1j} + E_{m+1j} \right) = 0
$$

(6.36)

By definition $r_{m-1} = r_m - \frac{h_m}{h_{m+2}}$ and $r_{m+1} = r_m + \frac{h_m}{h_{m+2}}$.

From the assumption made at the beginning of this section that the steps $h_m$ and $h_{m+2}$ are relatively small, it follows that
\[
\frac{1}{r_{m+1}} = \frac{1}{r_m} - \frac{h_{m+1}}{r_m^2} + O(h_m^2)
\]
\[
\frac{1}{r_{m-1}} = \frac{1}{r_m} + \frac{h_m}{r_m^2} + O(h_m^2)
\]
\[
\frac{1}{r_{m+1}^2} = \frac{1}{r_m^2} - \frac{2h_{m+1}}{r_m^3} + O(h_m^2)
\]
and
\[
\frac{1}{r_{m-1}^2} = \frac{1}{r_m^2} + \frac{2h_m}{r_m^3} + O(h_m^2)
\]

Also
\[
E_{m+1j} = E_z(r_{m+1}, \theta_j) = E_z(r_m \theta_j) + h_m \frac{\partial}{\partial r} E_z(r_m \theta_j) + O(h_m^2)
\]
i.e.
\[
E_{m+1j} = E_{mj} + h_m E'_{mj} + O(h_m^2)
\]
where the ' denotes differentiation with respect to \( r \).

Similarly
\[
E_{m-1j} = E_{mj} - h_m E'_{mj} + O(h_m^2)
\]
and also for \( j = j^+ \) and \( j = j^- \) the above two formulae obviously hold.

To \( O(h_m^2) \) and \( O(h_m^2) \) the equation (6.36) is
\[
\frac{1}{h_{m+2}} E_{m-2j} + \frac{1}{h_{m+2}} E_{mj} + \frac{1}{h_{m+2}} \frac{h_m}{r_m} E_{mj} + \frac{1}{h_{m+2}} \left( \frac{1}{h_m} + \frac{1}{r_m} \right) E_{m+2j}
\]
\[
+ \frac{1}{\alpha^2 r_m^2} E_{mj} - \frac{1}{\alpha^2 r_m^2} \left( \frac{h_m}{h_{m+2}} E_{mj} - \frac{h_m}{r_m} E'_{mj} - \frac{2h_{m+1}}{r_m} E_{mj} \right)
\]
\[
+ \frac{1}{\alpha^2 r_m^2} E_{mj} + \frac{1}{\alpha^2 r_m^2} \left( \frac{h_m}{h_{m+2}} E_{mj} + \frac{h_m}{r_m} E'_{mj} - \frac{2h_{m+1}}{r_m} E_{mj} \right)
\]
\[
- \frac{1}{h_m^2} \left( \frac{1}{h_{m-1}} + \frac{1}{r_m} \right) E_{mj} - \frac{h_m}{h_{m+2}} \left( \frac{1}{r_m^2} E_{mj} - \frac{1}{r_m} E'_{mj} - \frac{1}{h_{m-1}} E'_{mj} \right)
\]
\[
- \frac{1}{h_m^2} \left( \frac{1}{h_{m+2}} + \frac{1}{r_m} \right) E_{mj} - \frac{h_{m+1}}{h_{m+2}} \left( \frac{1}{r_m^2} E_{mj} - \frac{1}{r_m} E'_{mj} + \frac{1}{r_m} E'_{mj} \right)
\]
\[
- \frac{2}{\alpha^2 r_m^2} E_{mj} - \frac{2}{\alpha^2 r_m^2} \left( \frac{h_m}{h_{m+2}} E_{mj} + \frac{h_m}{r_m} E'_{mj} - \frac{2h_{m+1}}{r_m} E_{mj} \right)
\]
\[
+ \omega^2 \epsilon_1 \mu_1 E_{a mj} + \omega^2 \epsilon_1 \mu_1 E_{a} \left( \frac{h_{m+1}}{r_m} E'_{mj} + \frac{h_m}{h_{m+2}} E_{mj} \right) = 0
\]
(6.37)
Now letting $A$ and $C$ approach $B$ so that $h_m \to 0$ and $h_{m+1} \to 0$ the equation (6.37) becomes

\[
\frac{1}{h_m^2 + 2} E_{m+2j} - \frac{1}{h_{m+2}^2} \left( \frac{1}{h_m^2} + \frac{1}{h_{m-1}^2} \right) E_{mj} + \frac{1}{h_{m+2}^2 h_{m-1}^2} E_{m-2j} + \frac{1}{h_{m+2}^2} \left( E_{m+2j} - E_{mj} \right) + \frac{1}{\alpha^2 r_m^2} \left( E_{mj+} - 2E_{mj} + E_{mj-} \right) + \omega^2 \epsilon_1 u E_{amj} = 0
\]

(6.38)

This equation holds for all $j$, $1 \leq j \leq m$.  

The statement $h_m \to 0$ and $h_{m+1} \to 0$ is a mathematical expression for the circles $i = m-1$ and $i = m+1$ tending to the circle $i = m$.  

In the limiting case the mesh will have two fewer circles and if $h_{m+2} = h_{m-1} = h$ the equation (6.38) corresponds to the equation (4.19) with $i = m$.  A similar analysis holds for the other dependent variable $H_z$.  When $h_m \to 0$ and $h_{m+1} \to 0$ equations (5.4), (6.29) and (6.33) can be combined to give the same equation as (6.38) with "$H" replacing "E" which corresponds to equation (4.20) with $i = m$.  Then the two sets of difference equations, equations (4.15) to (4.22) and equations (5.1) to (5.4), (5.7), (5.8) and (6.28) to (6.33) correspond completely provided that $m_p$ is the same and the mesh circles are chosen appropriately for the two methods.  Such a choice is illustrated in Figure 6.2 which includes the circles $i = m-1$ and $i = m+1$ in the mesh for the two region method.  In the figure $m_p = 8$ and $m_q = 5$ for the one region method and $m_p = 8$, $m_q = 7$ and $m = 5$ for the two region method.  It has already been argued in section 5.1 that the steps $h_m$ and $h_{m+1}$ should be very small.  They may be taken as small as one pleases.  It follows that the finite difference formulation for two regions may be made to approach as nearly as one desires to that of one region by taking small enough values of $h_m$ and $h_{m+1}$. 
In this section, it has been shown that the difference equations in the general case (equations (5.1) to (5.10), (5.13) and (5.14)) reduce to the correct equations for a special case under the given assumptions. Another check on the numerical results of the finite difference method has been provided. Results are given in the next chapter along with the corresponding calculated values from equations (4.6) and (4.8). In theory, the position of the "fictitious" interface can be varied at will without affecting the results. This fact provides a further test of the finite difference method and again relevant numerical results are given in the next chapter.
CHAPTER 7

Some numerical results for the two region problem

7.1 Scaling and the two types of zero.

All the problems of this thesis are reduced by the finite difference method to finding the solution of an equation of the form (2.15)
i.e. \(|M_0| = 0\). \hspace{1cm} (7.1)

Equation (7.1) is a polynomial of very high order of \(B\), the scaled propagation constant. For lossless propagation along the waveguide, \(B\) must be real and positive. Suppose that one row of the matrix \(M_0\) is multiplied through by a continuous function \(f(B)\) of \(B\) and let the matrix so formed be \(M_1\). Then by the theory of determinants

\[ |M_1| = f(B) |M_0| \hspace{1cm} (7.2) \]

and \(|M_1|\) is a continuous function since it is the product of two continuous functions. Consider the equation

\[ |M_1| = 0, \hspace{1cm} (7.3) \]
i.e. \(f(B)|M_0| = 0\).

Then either \(f(B) = 0\) or \(|M_0| = 0\).

The solutions of equation (7.3) will be given by the solutions of \(f(B) = 0\) and the solutions of equation (7.1). For a simple choice of \(f(B)\) the solutions of \(f(B) = 0\) are easily found and so the solutions of equation (7.1) can be obtained from those of equation (7.3).

This argument naturally extends to the general case where a matrix \(M_0\) is formed from \(M_0\) by the multiplication of a number of its rows by continuous functions. The procedure of converting a matrix \(M_0\) into a matrix \(M_1\) in this way will be termed "scaling" and the continuous functions termed the "scaling functions". Two simple scaling functions
that are employed in obtaining the results in this thesis are \( E_a = 1 - B^2 \), and \( E_b = B^2 - \varepsilon \) as defined in section 1.5 with \( \mu_1 = \mu_0 \).

For positive \( B \) they introduce the additional solutions \( B = 1 \) and \( B = \sqrt{\varepsilon} \) respectively. For any matrix \( M \), there is an infinity of scaled matrices \( M' \) formed by different choices of scaling functions and the rows on which they operate. All the roots of equation (7.1) will also be roots of the equation

\[
|M'_{\text{rs}}| = 0
\]  

(7.4)

for all \( M'_{\text{rs}} \). A comparison of the numerical solutions found from equation (7.1) and those from any equation (7.4) will provide a way of testing the consistency of the results for a particular problem. An example is given in section 7.3.

The main purpose of scaling is to change the shape of the function \( |M|_0 \) into a more convenient one. Again it should be emphasised that the only points of interest are those at which \( |M|_0 = 0 \) and so nothing will be lost by considering a function \( |M'_{\text{rs}}| \) instead of the function \( |M|_0 \). Scaling can be used to "flatten" a function, i.e. to form a function with an apparently small slope over a range of values of \( B \). If from computed values \( |M|_0 \) appears to be monotonic increasing with \( B \) over an interval, then by multiplying it by a function of \( B \) which is decreasing over the same interval, a function \( |M'_{\text{rs}}| \) which appears to increase less rapidly is obtained. One use of this technique is to form a function which is easier to plot than the original one. Such a function is illustrated in section 7.3.

Another, and perhaps more important, use of scaling a function to "flatten" it is to help to overcome a certain practical difficulty in finding some of the zeros of the function \( |M|_0 \). Before this is considered in more detail a discussion of the zeros will be given.
The computer programmes evaluate the function $|m_\nu|$ at chosen values of $B$. Since the function is a very complicated one, it is highly unlikely that a value of $B$ will be chosen that satisfies equation (7.1) exactly. Therefore the zeros cannot be found exactly. As has been mentioned in section 4.4, there is no reason why the function $|m_\nu|$ should change sign at all of its zeros for that particular one-region problem. This is equally true of two region problems. The zeros can be divided into two types, those at which the function does change sign and those at which it does not. See Figure 7.1.

A zero of the first type is easily found. Whenever the function $|m_\nu|$ has a change of sign from one chosen value of $B$ to the next there must be at least one zero in the interval between those values because the function is continuous. Such a zero can be found to lie within as small an interval as is desired by successive subdivision of intervals. There is no doubt of its existence even though the value of $B$ will in general not be found exactly.

For a zero of the second type, where the function $|m_\nu|$ does not change sign the problem is not so simple. The numerical results will show that the function $|m_\nu|$ has a minimum or a maximum close to the axis since it is highly unlikely that the exact value of $B$ for a zero will ever be chosen. For definiteness suppose $|m_\nu|$ has a minimum. A similar argument holds for a maximum. There is no numerical way of distinguishing between a zero and a minimum infinitesimally close to the B axis. Scaling, as has been shown, will not move the zero but is likely to have very little effect on a minimum very close to the axis. All that can be done numerically is to show that the function $|m_\nu|$ tends nearer and nearer to zero as an interval containing the minimum is successively subdivided. By taking $B$ to a sufficient number of decimal places it should be possible to make $|m_\nu|$ smaller than any chosen value. Zeros of this type do appear to exist and one is investigated in section 7.3.
They can be difficult to find since there is no obvious indicator like the change of the sign of the function $|\mathbf{M}|$ for the zeros of the first type. Scaling can help to overcome this difficulty.

A typical situation where scaling may be of use is illustrated in Figure 7.2. If the function $|\mathbf{M}|$ were evaluated at the chosen values $B_i$ ($i = 1, \ldots, 7$) it would appear to be monotonic increasing and there would be no indication of the presence of a zero. Scaling to "flatten" such a function will make the finding of the zero easier.

The following simple example illustrates this.

Suppose that $|\mathbf{M}| = 4B^4 + 4B^3 - 3B^2 - 2B + 1$ and it is evaluated for $B = 1/4, 3/4, 5/4, \ldots$.

At $B = 1/4$, $|\mathbf{M}| = 25/64$,
$B = 3/4$, $|\mathbf{M}| = 49/64$,
$B = 5/4$, $|\mathbf{M}| = 729/64$, and so on.

$|\mathbf{M}|$ appears to be a monotonic increasing function with no zeros in the region $B > 0$.

Now scale $|\mathbf{M}|$ by multiplying one of its rows by the continuous function $f(B) = \frac{1}{B(B+1)}$.

$f(B)$ has no zeros in $B > 0$ and is monotonic decreasing there and so will "flatten" the function $|\mathbf{M}|$. Also the zeros of $|\mathbf{M}|$ will coincide with those of $|\mathbf{M}| = f(B).|\mathbf{M}|$ in $B > 0$.

Evaluating $|\mathbf{M}_1|$ at the chosen values of $B$ gives

$B = 1/4$, $|\mathbf{M}_1| = 5/4$,
$B = 3/4$, $|\mathbf{M}_1| = 7/12$,
$B = 5/4$, $|\mathbf{M}_1| = 81/20$,

and for $B > 5/4$, it appears to be monotonic increasing. From the numerical values it can be seen that the function $|\mathbf{M}_1|$ has a minimum
in $1/4 < B < 5/4$. This minimum can be restricted to a smaller interval by subdividing.

At $B = 1/2$, $|M_1| = 0$,

$B = 1$, $|M_1| = 2$,

and so $B = 1/2$ is a root of $|M_1| = 0$ and so of $|M| = 0$.

It follows that in some circumstances it may be easier to find the zeros of a function if it is scaled into a form in which it changes less rapidly. There is no guarantee that all the zeros of a scaled function will be found since values of $B$ have to be taken at finite intervals. Hence a situation similar to that of Figure 7.2 could still occur. However during the work for this thesis scaling has proved to be a useful aid in locating zeros and some have been found which would otherwise have been missed. It was carried out on what was largely an ad hoc basis for each particular problem.

The results given in this thesis were obtained using the computer programmes in the appendices and, where appropriate, scalings of them. The exact scaling used for each set of results is not recorded. Usually the rows of the matrix representing the equations

$$E_{mj} = 0 \quad m = m_q', \quad 1 \leq j \leq m_p$$

were multiplied through by a scaling function, $E_a$, $E_b$ or a combination of them. The scaling was done in this way for simplicity. The results include both types of zero and no distinction is drawn between them.
Figure 7.1 The two types of zero

Figure 7.2 A possible zero
The computer programme termed programme 4 given in Appendix 4 is based on the difference equations derived in section 5.2 and follows the general pattern discussed in section 3.3. A series of values of \( B \) are read in and the determinant \( |M| \) of equation (5.16) is evaluated for each in turn, the remaining parameters being kept fixed. Initially the parameters \( m_p, \omega, \mu_1, \epsilon_1, a, \frac{k}{\mu_0}, \epsilon_r, \frac{b}{a}, \mu_0 \) and \( \mu_2 \) are read in as well as an additional one, termed ANG in the programme, which allows the position of the line \( \theta = 0 \) to be chosen arbitrarily in relation to the radial arms of the mesh. When \( \text{ANG} = 0 \) the line \( \theta = 0 \) coincides with the radial arm \( j = m_p \). The mesh set up by the programme is an example of the type considered in section 5.2. The radii of its concentric circles are calculated in the subroutine RADII. This allows the radial step lengths to be altered to give another mesh of the same type without changing the main programme. The subroutine initially reads the values \( m_x, \) an integer, and \( d_x, \) a real number. It sets up a step length \( h \), given by

\[
h = \frac{a-b}{m_x + 2d_x},
\]

and a boundary step length \( \delta h = d_x h \). \( d_x \) is a small fraction usually taken to be 0.01 for the numerical results given. The subroutine then sets up the radii of the mesh circles. Working inwards from the outside boundary these are

\[
a, a-\delta h, a-\delta h-h, \ldots a-\delta h-m_x h, b, b-\delta h, b-\delta h-h, \ldots b-\delta h-m_d h
\]

where \( m_d \) is the integer which satisfies

\[
d_x h < b-\delta h-m_d h < (1+d_x) h;
\]

\( m_q \) is the total number of circles and is given by

\[
m_q = m_d + m_x + 4.
\]
Figure 7.3 illustrates such a mesh with \( m_q = 10 \) and \( m_p = 16 \).

In the programme \( h, \delta h \) and \( b-\delta h-m_dh \) are known as the regional step, the boundary step and the first step lengths respectively.

In the evaluation of the determinant \( |W| \) for a particular \( B \), the array \( BM \) stores the elements of the sliding array which has \( (m_1+1) \) rows and \( (m_1+1+m_2) \) columns in the usual notation. For the present problem \( m_1 = 3m_p \) and \( m_2 = 3m_p+1 \). It follows that \( BM \) must have at least \( 3m_p+1 \) rows and \( 6m_p+2 \) columns. Other conditions hold on the one-dimensional arrays used in the programme. As in the case of the one region problem (see section 4.3), there is a direct link between the sizes of the arrays used in the programme and the size of the mesh.

Programme 4 will handle any mesh with

\[
m_p \leq 16 \quad \text{and} \quad m_q \leq 30.
\]

The dimension statements would have to be adjusted to accommodate a larger mesh.

The computer programme termed programme 5 given in Appendix 5 is based on the difference equations derived in section 5.3. Clearly programme 4 with \( \mu_1 = \mu_2 = \mu_0 \) could be used in this special case. However programme 5 gives a more efficient method of tackling the problem from the point of view of computer resources. The parameters read in are: \( m_f, m_p, d_f, \omega, \mu_0, \varepsilon_1, a, k, \varepsilon_2, b \) and ANG and a series of values of \( B \). The mesh is the one used in programme 4 but in this case it is set up in the main part of the programme and not in a subroutine. The determinant \( |W| \) of equation (5.16) is evaluated for each value of \( B \) in turn using the usual sliding array method. Programme 5 will handle any mesh with

\[
m_p \leq 16 \quad \text{and} \quad m_q \leq 30.
\]

For a larger mesh the dimension statements in the programme would have to be altered.
Programmes 4 and 5 are limited by the same inequalities

\( m \leq 16 \) and \( m \leq 30 \) on the size of the mesh. However, programme 4 requires an appreciably smaller number of computer store locations.

See Appendices 4 and 5. It follows that in the interests of economy programme 5 should be used wherever possible in preference to programme 4.
Figure 7.3  A possible mesh
7.3 The special case of section 6.4

The problem here is that of Chapter 4. Its exact solutions are given by equations (4.6) and (4.8). The calculation of these solutions has been discussed in section 4.4. The Tables 7.1 to 7.4 give results obtained by using the two region finite difference method on this problem.

The example listed as 1 in Table 7.4 was considered in some detail. Table 7.1 shows an investigation of two of the supposed zeros of its determinant. One zero of each type has been included. Table 7.2 shows the values of the determinant $|M|_{\text{RS}}$ of a scaled matrix for the same example and the graphs in Figures 7.4 and 7.5 have been plotted from these results. The two different types of zero are illustrated in them. Table 7.3 shows the result of further subdivisions of the intervals around the same two supposed zeros as before under the scaling. It can be seen that the two determinants have zeros in the same intervals of $B$ even to 8 decimal places. Also, the determinant approaches zero very closely when the factor $2^{128}$ is discarded. The function $|M|_{\text{RS}}$ can be multiplied by any number without making the least difference to the problem as was pointed out in section 7.1. It is the relative values of the function which are important, not the absolute values.

The results support the suggestion of two types of zero.

Table 7.4 gives zeros for a variety of different examples and these are compared with the corresponding calculated values. Examples 1 and 3 show that the position of the "fictitious" interface ($r = b$) makes very little difference to the results. Examples 1 and 2 and examples 5, 6 and 7 illustrate the convergence of the zeros towards their correct values as the number of points in the mesh is increased. The results are similar to those for the simultaneous method of Chapter 4 as would be expected since the method used here reduces to that one except for a slight difference in the mesh (see section 6.4). It follows that the
discussion of section 4.4 concerning the results and their accuracy will hold equally well here.

The parameters listed below were given the values stated below for all the results in this section. This was done for convenience. The computer programme 5 was used.

\[
\begin{align*}
\mu_0 &= 1.257 \times 10^{-6} \\
\varepsilon_1 &= \varepsilon_2 = 8.854 \times 10^{-12} \\
K &= 0 \\
d_f &= 0.01 \\
\text{and} \quad \text{ANG} &= 0.0
\end{align*}
\]
Figure 7.4: The determinant $|M_{33}|$
Figure 7.5 The determinant $|M_3|$
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Table 7.2 The determinant $|M_{\text{US}}|$ Parameters of example 1 of Table 7.4

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<td>0.81</td>
<td>-0.904</td>
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Table 7.3 The location of two zeros of $|M_{CS}|$, the determinant of Table 7.2

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<th>Subdivision</th>
<th>D1</th>
<th>ID-4128</th>
<th>B</th>
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<td>-0.340</td>
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<td>-0.077</td>
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Table 7.4  Zeros found by the finite difference method and their corresponding values calculated from analytical theory

<table>
<thead>
<tr>
<th>Example</th>
<th>$\omega \times 10^{-10}$</th>
<th>$a \times 10^2$</th>
<th>$b \times 10^2$</th>
<th>$m_q$</th>
<th>$m_p$</th>
<th>B</th>
<th>Calculated value of B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.341</td>
<td>2.6</td>
<td>2.03</td>
<td>13</td>
<td>12</td>
<td>0.895 + 0.905</td>
<td>0.92</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.84 + 0.845</td>
<td>0.85</td>
</tr>
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<td></td>
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<td></td>
<td></td>
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<td>0.73 + 0.74</td>
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<td>0.56</td>
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<td></td>
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<td>0.45 + 0.46</td>
<td>0.56</td>
</tr>
<tr>
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<td></td>
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<td></td>
<td></td>
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<td>0.39 + 0.395</td>
<td>0.42</td>
</tr>
<tr>
<td>2</td>
<td>5.341</td>
<td>2.6</td>
<td>2.03</td>
<td>22</td>
<td>12</td>
<td>0.905 + 0.915</td>
<td>0.92</td>
</tr>
<tr>
<td></td>
<td></td>
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<td>0.845 + 0.85</td>
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</tr>
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<tr>
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<td>0.505 + 0.515</td>
<td>0.56</td>
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<td>0.485 + 0.49</td>
<td>0.42</td>
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<td>5.341</td>
<td>2.6</td>
<td>1.18</td>
<td>14</td>
<td>12</td>
<td>0.9 + 0.91</td>
<td>0.92</td>
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<td></td>
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<td></td>
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<td></td>
<td>0.84 + 0.845</td>
<td>0.85</td>
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<tr>
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<td></td>
<td></td>
<td></td>
<td>0.735 + 0.745</td>
<td>0.75</td>
</tr>
<tr>
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<td></td>
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<td></td>
<td></td>
<td>0.525 + 0.535</td>
<td>0.56</td>
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<tr>
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<td>0.47 + 0.48</td>
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<tr>
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<td></td>
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<td>0.42 + 0.425</td>
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</tr>
<tr>
<td>4</td>
<td>5.341</td>
<td>1.3</td>
<td>1.0</td>
<td>16</td>
<td>12</td>
<td>0.56 + 0.57</td>
<td>0.61</td>
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<tr>
<td>5</td>
<td>5.341</td>
<td>1.27</td>
<td>1.143</td>
<td>14</td>
<td>12</td>
<td>0.485 + 0.495</td>
<td>0.58</td>
</tr>
<tr>
<td>6</td>
<td>5.341</td>
<td>1.27</td>
<td>1.143</td>
<td>24</td>
<td>12</td>
<td>0.54 + 0.55</td>
<td>0.58</td>
</tr>
<tr>
<td>7</td>
<td>5.341</td>
<td>1.27</td>
<td>1.143</td>
<td>24</td>
<td>16</td>
<td>0.555 + 0.565</td>
<td>0.58</td>
</tr>
<tr>
<td>8</td>
<td>3.0</td>
<td>4.0</td>
<td>1.44</td>
<td>16</td>
<td>12</td>
<td>0.87 + 0.88</td>
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<td></td>
<td></td>
<td>0.785 + 0.79</td>
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<td></td>
<td>0.625 + 0.635</td>
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<tr>
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<td></td>
<td></td>
<td>0.205 + 0.215</td>
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<td>3.0</td>
<td>2.03</td>
<td>16</td>
<td>12</td>
<td>0.755 + 0.765</td>
<td>0.79</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td>0.565 + 0.57</td>
<td>0.60</td>
</tr>
</tbody>
</table>
7.4 The case of two coaxial dielectrics

For this particular case, the exact characteristic equation was derived earlier (section 6.1). In a brief discussion of the solution it was indicated that the computer programme in Appendix 3 may be used. This method of solution will be termed the formula method. It gives the results in Tables 7.5 to 7.9.

Tables 7.5 and 7.6 verify the accuracy of the formula method for several examples by comparing its results with known ones obtained independently. In the simpler case (Table 7.5) where the parameters are chosen so that the problem reduces to that of one region the solutions can be calculated (see section 4.4). Waldron (1958) gave numerical solutions for some examples of the problem having two distinct regions. He used the parameters \( \mu, \varepsilon, \frac{a}{\lambda_0} \) and \( \bar{\beta} \). In the notation of Chapter 6 they are given by

\[
\begin{align*}
\mu &= \frac{\mu_1}{\mu_2}, \\
\varepsilon &= \frac{\varepsilon_1}{\varepsilon_2}, \\
\frac{a}{\lambda_0} &= \frac{a \omega \sqrt{\varepsilon_0 \mu_0}}{2\pi}, \\
\text{and} \quad \bar{\beta} &= \frac{\beta}{\omega \sqrt{\varepsilon_0 \mu_0}}
\end{align*}
\]

where \( \varepsilon_0 \) and \( \mu_0 \) are respectively the permittivity and permeability of free space.

Since

\[ \beta^2 = B^2 \omega^2 \varepsilon_1 \mu_1 \]

it follows that

\[ \bar{\beta} = \frac{\varepsilon_1 \mu_1}{\sqrt{\varepsilon_0 \mu_0}} B. \]

In Table 7.6 results from the formula method are compared with the corresponding values given by Waldron (1958). In both Tables 7.5 and 7.6 the following values have been taken throughout.
\[ \varepsilon_2 = \varepsilon_0 = 8.854 \times 10^{-12} \]
\[ \mu_2 = \mu_0 = 1.257 \times 10^{-6} \]

The quoted results give one every confidence in the accuracy of the computer programme used and the validity of the formula method. For the remainder of this thesis values of \( B \), the scaled propagation constant, given by the method in any particular case will be taken as correct.

Tables 7.7, 7.8 and 7.9 each give the possible values of \( B \) as the value of the inner radius is varied and the remaining parameters \( a, \omega, \varepsilon_1, \varepsilon_2, \mu_1 \) and \( \mu_2 \) are held fixed. For convenience the parameters \( \varepsilon_1 \) and \( \mu_1 \) have been given the values of \( \varepsilon_0 \) and \( \mu_0 \) respectively so that region 1 is free space. The values, to two decimal places, of \( \frac{a}{\lambda_0} \) are 0.36, 0.32 and 0.40 respectively for Tables 7.7 to 7.9. Figures 7.6, 7.7 and 7.8 are outline sketches of the resulting mode configurations. The corresponding results obtained by the finite difference method are discussed in the next section.
Figure 7.6
The scaled propagation constant $B$; $\omega = 5.341 \times 10^{10}$ Hz, $a = 1.27 \times 10^{-2}$ m, $\varepsilon_r = 13$ and $\mu = 1$. 
Figure 7.7
The scaled propagation constant \( B \); \( \omega = 6.027 \) Hz, \( \sigma = 1.0 \times 10^{-2} \) m, 
\( \varepsilon_r = 12 \) and \( \mu_r = 0.98 \).
Figure 7.8
The scaled propagation constant \( B \), \( \omega = 5.932 \times 10^{10} \) Hz,
\( a = 1.27 \times 10^{-2} \) m, \( d_r = 12 \) and \( \mu_r = 0.98 \).
Table 7.5  Zeros by the formula method and the corresponding values calculated from analytical theory

<table>
<thead>
<tr>
<th>example</th>
<th>$\omega \times 10^{-10}$</th>
<th>$a \times 10^2$</th>
<th>$b \times 10^2$</th>
<th>$\varepsilon_r$</th>
<th>$\mu_r$</th>
<th>$n$</th>
<th>$B$</th>
<th>calculated value</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>5.341</td>
<td>2.6</td>
<td>2.03</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.915 → 0.92</td>
<td>0.92</td>
</tr>
<tr>
<td></td>
<td>0</td>
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<td></td>
<td>0.85</td>
<td></td>
<td></td>
<td></td>
<td>0.855</td>
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<tr>
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<td>2</td>
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<td></td>
<td>0.75</td>
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<td>0.755</td>
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<td>0.56</td>
<td></td>
<td></td>
<td></td>
<td>0.565</td>
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<td></td>
<td></td>
<td></td>
<td>0.565</td>
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<td>0.42</td>
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<td></td>
<td>0.425</td>
</tr>
<tr>
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<td>1.27</td>
<td>1.143</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.68 → 0.685</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>0.40 → 0.405</td>
<td>0.40</td>
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</table>

Table 7.6  Zeros by the formula method and the corresponding values given by Waldron (1958)

<table>
<thead>
<tr>
<th>example</th>
<th>$\omega \times 10^{-8}$</th>
<th>$a$</th>
<th>$\frac{b}{a}$</th>
<th>$\frac{a}{\lambda_0}$</th>
<th>$\varepsilon$</th>
<th>$\mu$</th>
<th>$n$</th>
<th>$B$</th>
<th>$\bar{\beta}$</th>
<th>Waldron</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>4.449</td>
<td>1.27</td>
<td>0.2</td>
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<td>1</td>
<td>0.14 → 0.145</td>
<td>0.443 → 0.459</td>
<td>0.445</td>
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<tr>
<td>2</td>
<td>4.449</td>
<td>1.27</td>
<td>0.5</td>
<td>0.3$\times$10$^{-1}$</td>
<td>3</td>
<td>1</td>
<td>0.45 → 0.455</td>
<td>0.779 → 0.788</td>
<td>0.786</td>
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</tr>
<tr>
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<td>5.932$\times$10$^2$</td>
<td>1.27$\times$10$^{-2}$</td>
<td>0.1</td>
<td>1$\times$10$^{-1}$</td>
<td>1</td>
<td>1</td>
<td>0.285 → 0.29</td>
<td>0.698 → 0.710</td>
<td>0.700</td>
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</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td>0</td>
<td></td>
<td>0.19 → 0.195</td>
<td>0.466 → 0.478</td>
<td>0.469</td>
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Table 7.7 Zeros by the formula method
Parameters $\omega = 5.341 \times 10^0$, $a = 1.27 \times 10^{-2}$, $c_r = 13$ and $\mu_r = 1$

<table>
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<th>$\frac{b}{a}$</th>
<th>n</th>
<th>B</th>
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</thead>
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<td>1.0</td>
<td>1</td>
<td>0.58 → 0.585</td>
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<td>1</td>
<td>0.62 → 0.625</td>
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<tr>
<td>0.9</td>
<td>1</td>
<td>0.7 → 0.705</td>
</tr>
<tr>
<td>0.85</td>
<td>1</td>
<td>0.905 → 0.91</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.485 → 0.49</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.34 → 0.345</td>
</tr>
<tr>
<td>0.8</td>
<td>1</td>
<td>1.53 → 1.54</td>
</tr>
<tr>
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<td>0</td>
<td>1.47 → 1.48</td>
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<td>3</td>
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<td>0</td>
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<td>3</td>
<td>1.99 → 2.00</td>
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<td>1.20 → 1.21</td>
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<tr>
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<td>2</td>
<td>0.54 → 0.55</td>
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Table 7.8  Zeros found by the formula method  
Parameters  \( \omega = 6.027 \times 10^{10} \)  
\( a = 1.00 \times 10^{-2}, \epsilon_r = 12 \) and \( \mu_r = 0.98 \)

<table>
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<th>( \frac{b}{a} )</th>
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<tr>
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<td>( \rightarrow 0.455 )</td>
</tr>
<tr>
<td>0.9</td>
<td>1</td>
<td>0.535</td>
</tr>
<tr>
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<td></td>
<td>( \rightarrow 0.54 )</td>
</tr>
<tr>
<td>0.85</td>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \rightarrow 0.705 )</td>
</tr>
<tr>
<td>0.8</td>
<td>1</td>
<td>1.03</td>
</tr>
<tr>
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<td></td>
<td>( \rightarrow 1.035 )</td>
</tr>
<tr>
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<td>2</td>
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<td>( \rightarrow 0.555 )</td>
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<td>( \rightarrow 0.64 )</td>
</tr>
<tr>
<td>0.75</td>
<td>1</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>( \rightarrow 1.635 )</td>
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<tr>
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</tr>
<tr>
<td></td>
<td></td>
<td>( \rightarrow 1.575 )</td>
</tr>
<tr>
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<td>2</td>
<td>1.57</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \rightarrow 1.53 )</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.13</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \rightarrow 1.14 )</td>
</tr>
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</table>

Table 7.9  Zeros found by the formula method  
Parameters  \( \omega = 5.932 \times 10^{10} \)  
\( a = 1.27 \times 10^{-2}, \epsilon_r = 12 \) and \( \mu_r = 0.98 \)

<table>
<thead>
<tr>
<th>( \frac{b}{a} )</th>
<th>n</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1</td>
<td>0.68</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \rightarrow 0.685 )</td>
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<tr>
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<tr>
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<td>( \rightarrow 0.295 )</td>
</tr>
<tr>
<td>0.95</td>
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<td></td>
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<td>0.9</td>
<td>1</td>
<td>0.785</td>
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<td></td>
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<td>( \rightarrow 0.79 )</td>
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<tr>
<td></td>
<td>0</td>
<td>0.41</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \rightarrow 0.42 )</td>
</tr>
<tr>
<td>0.85</td>
<td>1</td>
<td>0.99</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \rightarrow 1.00 )</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.78</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \rightarrow 0.79 )</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.7</td>
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<tr>
<td></td>
<td></td>
<td>( \rightarrow 0.71 )</td>
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<tr>
<td>0.8</td>
<td>1</td>
<td>1.67</td>
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<td>( \rightarrow 1.68 )</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1.67</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \rightarrow 1.68 )</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.60</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \rightarrow 1.61 )</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.41</td>
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<tr>
<td></td>
<td></td>
<td>( \rightarrow 1.42 )</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.61</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \rightarrow 0.62 )</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \rightarrow 0.63 )</td>
</tr>
</tbody>
</table>
CHAPTER 8
Tranversely magnetised ferrite tube in circular waveguide

8.1 Numerical results using the finite difference method.

In section 5.2 a finite difference formulation reduced the boundary value problem of section 1.5 to that of solving the equation (5.16), viz $|\frac{M}{\kappa}| = 0$. This may be solved by using programme 4, and for the special case of section 5.3, by using programme 5. In obtaining the results given in this section programme 5 was used wherever possible.

Before applying the finite difference method to the above problem it was used on the special case of two concentric dielectrics ($\kappa = 0$). Results can be compared with the corresponding values obtained by the formula method discussed previously. The Tables 8.1 (represented graphically in Figure 8.1), 8.2 and 8.3 illustrate the comparison. Quite good approximations to the true values have been obtained. These approximations improve in accuracy as the chosen number of mesh points is increased. Such convergence is to be expected from the general theory of finite difference methods discussed in Chapter 2.

The results in the three tables indicate that, for this special case, the finite difference method gives stable solutions of the equation (5.16) when the number of mesh points is sufficiently large. The solutions obtained are reasonable approximations to the true values especially for the dominant mode. From a practical point of view it should be remembered that increasing the number of mesh points increases both the computer time and store required to evaluate the determinant $|\frac{M}{\kappa}|$ for a value of $B$. Hence it is desirable to choose a mesh which gives a compromise between the amount of computer resources employed and the accuracy of the results.
In the formula method it is necessary to assign a value to \( n \) in order to compute a mode of propagation. The values of \( n \) distinguish the modes one from another. On the other hand using the finite difference method there is no way to distinguish between the modes. From the Tables 7.7, 7.8 and 7.9 it is evident that the values of \( B \) for different modes often become very close. In these circumstances the finite difference method is virtually useless. Even if a value of \( B \) could be found there would appear to be no way of assigning a particular mode to it. However the dominant mode is usually the one of most interest and multimode propagation is undesirable. Therefore it seems not unreasonable to choose the parameters of the waveguide configuration so that only two modes propagate at most. In such a situation the values of \( B \) will be well separated, assuming that the behaviour shown in Tables 7.7, 7.8 and 7.9 is typical.

Consider now the problem in which the outer region, \( b \leq r \leq a \), is occupied by transversely magnetised ferrite (\( \kappa \neq 0 \)). For a given waveguide configuration both the strength of magnetisation and the thickness of the ferrite will influence the value of the propagation constant of any mode. There are effectively two independent variables \( \kappa \) and \( \frac{b}{a} \). The case of two concentric dielectrics is merely the special case where one of these variables, \( \kappa \) is held fixed with value zero.

Values of \( B \) are given as the thickness of the ferrite and the strength of its magnetisation vary for three sets of fixed values of the parameters \( \varepsilon_1, \varepsilon_2, \mu_1, \mu_2, a \) and \( \omega \). The outer radius of the guide and the frequency of the waves were selected to limit the number of modes of propagation to one or two when the ferrite tube is absent (\( \frac{b}{a} = 1.0 \)). Also, by keeping the ratio of the thickness of the ferrite to the radius of the guide relatively small, few extra modes were introduced in the presence of unmagnetised ferrite. The ratio \( \frac{b}{a} \) was taken in the range \( 0.8 \leq \frac{b}{a} \leq 1.0 \), avoiding the complications of multimode propagation.
The three examples given by the three sets of fixed data will now be discussed in turn.

Example 1

The parameters $\varepsilon_1$, $\mu_1$, $a$ and $\omega$ are those of Table 7.7; the dielectric of region 2 is replaced by a ferrite having an equal permittivity and a tensor permeability which reduces to $\mu_2 I$ in the absence of any static magnetising field, $I_0$ being the identity tensor. Therefore when the ferrite is unmagnetised this example reduces to that of Table 7.7. There is then one mode for $\frac{b}{a} = 1.0$ but three modes for $\frac{b}{a} = 0.85$ and they increase in number as $\frac{b}{a}$ decreases. Also some of the modes lie very close together and so it would be virtually impossible to distinguish them by the finite difference method.

For these reasons it is advisable to concentrate on the range $0.85 < \frac{b}{a} < 1.0$ for different strengths of ferrite magnetisation. The results of such a policy are given in Table 8.4. The mesh used had $m_q = 24$ and $m_p = 16$, a size which appears to give a reasonable accuracy for the known results without using an excessive amount of computer resources.

The results show that $B$ changes continuously with the parameters $\frac{b}{a}$ and $\frac{K}{\mu_0}$ as would be expected. One wave appears to split into two waves with different propagation constants in the presence of transversely magnetised ferrite. One of the resulting waves appears to have the same propagation constant for all $\frac{K}{\mu_0}$ while the other has a propagation constant varying with $\frac{K}{\mu_0}$. The difference in the values of $B$ for the two waves increases both with increasing strength of magnetisation and with thickness of the ferrite. The value of $B$ which varies with $\frac{K}{\mu_0}$ is smaller than the one which apparently remains fixed. The behaviour of the mode is reminiscent of the case of waves propagating through an infinite transversely magnetised ferrite medium which is discussed briefly in Appendix 6. In parallel with the
terminology used there an ordinary and an extraordinary wave will be postulated here. Also in that appendix it is shown that the values of the propagation constant for the two waves are very close together for small values of $\frac{K}{\mu_0}$. This may explain why there appears to be only one wave when $\frac{K}{\mu_0} = 0.1$ here.

For values of $\frac{b}{a}$ smaller than those tabulated it can be seen from Table 7.7 that the modes in the case of unmagnetised ferrite are very close together. Thus it would be very difficult to identify, with any certainty, the modes obtained by using the finite difference method. In fact, values of $0.76 \rightarrow 0.77$ and $0.66 \rightarrow 0.67$ were given when $\frac{K}{\mu_0} = 0.4$ and $0.6$ respectively for $\frac{b}{a} = 0.8$, but corresponding values for the other values of $\frac{K}{\mu_0}$ were not found.

Figures 8.2 and 8.3 illustrate the results.

Example 2

The fixed parameters were chosen so that this example reduces to that of Table 7.8 when the ferrite is unmagnetised. To avoid a large number of modes $\frac{b}{a}$ was restricted to the range $0.8 \leq \frac{b}{a} \leq 1.0$.

The computed results for $\frac{b}{a}$ in this range and varying $\frac{K}{\mu_0}$ are given in Table 8.5. It displays a marked similarity to Table 8.4 although the values computed for known results are less accurate. Again it seems reasonable to postulate an ordinary and an extraordinary wave and the other observations made on Example 1 hold equally well here. The mesh used had $m_q = 24$ and $m_p = 15$.

Figures 8.4 and 8.5 illustrate the results and their similarity to Figures 8.2 and 8.3 is apparent.

Example 3

The fixed parameters were chosen to give an extension of the example of Table 7.9. There are two modes of propagation along the waveguide
when \( \frac{b}{a} = 1.0 \). However results for the dominant mode only are given in Table 8.6. Its behaviour follows the pattern set by the two previous examples. The mesh used had \( m_q = 24 \) and \( m_p = 12 \).

Let us return to some more general observations. Reversing the sign of \( \kappa \) had no effect on the values of the propagation constant given by the finite difference method. \( |M| \) for any \( B \) had the same value for both \( |\kappa| \) and \( -|\kappa| \) in the cases computed. Replacing \( \kappa \) by \(-\kappa\) reverses the direction of magnetisation throughout the system. It can be seen from Figure 1.2 that this is equivalent to a rotation of the static magnetic field through one right angle about the axis of the waveguide. Since the waveguide configuration displays symmetry about this axis one would not expect such a rotation to affect the value of the propagation constant. It seems that the finite difference method accurately simulates this stability in value.

In addition different values of \( \text{ANG} \) in the range \( 0.0 \leq \text{ANG} \leq 0.5 \) were taken in several examples. Although the value of \( |M| \) altered with \( \text{ANG} \) for a constant \( B \), the solutions of equation (5.16) were quite stable and appeared to vary less than the finite difference solution (with \( \text{ANG} = 0 \)) from the true value when the latter was known. Since varying \( \text{ANG} \) merely rotates the coordinate axis \( \theta = 0 \) it should have no effect on the value of the propagation constant. The finite difference method reflects this to within the accuracy already achieved by it.

The true value of the propagation constant for the case of two concentric dielectrics \( (\kappa = 0) \) is known and so the accuracy of the finite difference method can be assessed in this instance. Each of the three examples displays an almost constant percentage error as \( \frac{b}{a} \) is varied. The average values of these errors are approximately 4.6%, 18%, and 4% in Examples 1, 2 and 3 respectively. When \( \frac{b}{a} = 1.0 \) this case reduces to that of one region discussed in Chapter 4. From
the results there one would expect an error following the pattern
sketched in Figure 4.1. The errors in the three examples are
consistent with such a pattern. Varying \( \frac{b}{a} \) makes little difference
to the magnitudes of the percentage errors. They appear to be related
to the value of \( B \) when \( \frac{b}{a} = 1.0 \). If the formula (4.30) were true
in this case, a value of \( p = 7 \) would give an error larger than the
average values recorded for each example. Under such a hypothesis
a fixed 7% error in the quantity \( (1-B^2) \) would more than account for
the recorded errors in \( B \).

In the more general case when \( \kappa \) is varied the true value of the
propagation constant is not known. One can assume only that the error
in the finite difference method results will be similar to that in the
case \( \kappa = 0 \). The computed results have an overall consistency which
suggests that such an assumption is, at least, not unlikely. Under
such circumstances configurations which have a value of \( B \) near unity
when \( \frac{b}{a} = 1.0 \) seem to be quite accurately simulated by the finite
difference method. On the other hand, the accuracy is poor for values of
\( B \) less than 0.5 and rapidly deteriorates as \( B \) decreases further.
However, trends in the results are still clearly discernible when \( B \) is
small, as can be seen from Example 2.

To summarise, the numerical results suggest that a transversely
magnetised ferrite tube splits one wave into two; one with the same
propagation constant as that of a wave travelling down the guide in
the absence of a static magnetising field and the other with a smaller
propagation constant. The difference between the two values increases
both with the thickness of the tube and the strength of the magnetising
field. The accuracy of the finite difference method seems quite good
for waveguide configurations which give values of \( B \) near unity for
\( \frac{b}{a} = 1.0 \), but deteriorates for configurations for which \( B \) decreases
whilst \( \frac{b}{a} = 1.0 \).

The general trend is similar to that shown in Figure 4.1.
Figure 8.1
The scaled propagation constant $\beta$, $\omega = 5.341 \times 10^{16} \text{Hz}$, $a = 1.27 \times 10^{-2} \text{m}$, $\varepsilon_r = 13$ and $\mu_r = 1$
The modes $\frac{\lambda}{\lambda_0} = 0.7$ and 0.4

The extraordinary waves (parameter $\frac{\lambda}{\lambda_0}$)

Figure 8.2  Example 1
Fixed values of \( \frac{b}{a} \)

**Figure 8.3. Example 1**
The extraordinary waves (parameter $\frac{\omega}{c}$)

Figure 8.4 Example 2
Figure 8.5. Example 2
Table 8.1 A comparison of zeros found by the finite difference and formula methods
Parameters as in Table 7.7

<table>
<thead>
<tr>
<th>( \frac{b}{a} )</th>
<th>method 1</th>
<th>method 2</th>
<th>method 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.54 ± 0.55</td>
<td>0.555 ± 0.565</td>
<td>0.58 ± 0.585</td>
</tr>
<tr>
<td>0.95</td>
<td>0.575 ± 0.585</td>
<td>0.595 ± 0.605</td>
<td>0.62 ± 0.625</td>
</tr>
<tr>
<td>0.9</td>
<td>0.635 ± 0.645</td>
<td>0.66 ± 0.67</td>
<td>0.7 ± 0.705</td>
</tr>
<tr>
<td>0.85</td>
<td>0.84 ± 0.85</td>
<td>0.84 ± 0.85</td>
<td>0.905 ± 0.91</td>
</tr>
</tbody>
</table>

where method 1) finite difference with \( m_q = 24 \) and \( m_p = 12 \)
2) finite difference with \( m_q = 24 \) and \( m_p = 16 \)
3) formula

Table 8.2 Convergence of the finite difference method
Parameters \( \omega = 5.341 \times 10^{10} \) and \( \mu_r = 1 \)

<table>
<thead>
<tr>
<th>example</th>
<th>( a \times 10^{-2} )</th>
<th>( b \times 10^{-2} )</th>
<th>( \epsilon_r )</th>
<th>( m_q )</th>
<th>( m_p )</th>
<th>( B ) corresponding ( B ) by formula method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.27</td>
<td>1.268</td>
<td>13</td>
<td>16</td>
<td>12</td>
<td>0.505 ± 0.515</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>19</td>
<td>0.525 ± 0.535</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>22</td>
<td>0.535 ± 0.545</td>
</tr>
<tr>
<td>2</td>
<td>1.27</td>
<td>1.49^2</td>
<td>13</td>
<td>18</td>
<td>12</td>
<td>0.595 ± 0.605</td>
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<td></td>
<td></td>
<td></td>
<td>22</td>
<td>0.635 ± 0.645</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>24</td>
<td>0.635 ± 0.645</td>
</tr>
<tr>
<td>3</td>
<td>1.27</td>
<td>1.079</td>
<td>13</td>
<td>22</td>
<td>12</td>
<td>0.83 ± 0.84</td>
</tr>
<tr>
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<td></td>
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<td></td>
<td>24</td>
<td>0.84 ± 0.85</td>
</tr>
<tr>
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<td></td>
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<td></td>
<td></td>
<td>24</td>
<td>0.84 ± 0.85</td>
</tr>
<tr>
<td>4</td>
<td>2.6</td>
<td>2.03</td>
<td>3</td>
<td>22</td>
<td>12</td>
<td>0.835 ± 0.845</td>
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<td>26</td>
<td>0.835 ± 0.845</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td>31</td>
<td>0.825 ± 0.835</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>22</td>
<td>0.83 ± 0.84</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>26</td>
<td>0.825 ± 0.835</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>31</td>
<td>0.825 ± 0.835</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>35</td>
<td>0.825 ± 0.835</td>
</tr>
</tbody>
</table>

where method 1) finite difference with \( m_q = 32 \) and \( m_p = 12 \)
2) finite difference with \( m_q = 32 \) and \( m_p = 16 \)
3) formula

where method 1) finite difference with \( m_q = 48 \) and \( m_p = 12 \)
2) finite difference with \( m_q = 48 \) and \( m_p = 16 \)
3) formula
Table 8.3 A comparison of zeros found by the finite difference and formula methods
Parameters $\omega = 5.341 \times 10^{10}$, $\mu_r = 1$, $a = 2.6 \times 10^{-2}$ and $c_r = 3$

<table>
<thead>
<tr>
<th>$m_q$</th>
<th>$m_P$</th>
<th>$B$</th>
<th>corresponding $B$ by formula method</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>16</td>
<td>0.825 $\rightarrow$ 0.835</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.72 $\rightarrow$ 0.73</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.555 $\rightarrow$ 0.565</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.32 $\rightarrow$ 0.33</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>16</td>
<td>0.825 $\rightarrow$ 0.835</td>
<td>0.825 $\rightarrow$ 0.83</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.73 $\rightarrow$ 0.74</td>
<td>0.79 $\rightarrow$ 0.795</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.55 $\rightarrow$ 0.56</td>
<td>0.49 $\rightarrow$ 0.495</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.33 $\rightarrow$ 0.34</td>
<td>0.305 $\rightarrow$ 0.31</td>
</tr>
</tbody>
</table>

Table 8.4 Zeros by the finite difference method
Parameters $m_q = 24$, $m_P = 16$; other parameters as in Table 7.7

<table>
<thead>
<tr>
<th>value of $\frac{k}{\mu_0}$</th>
<th>$\frac{b}{a} = 1.0$</th>
<th>$\frac{b}{a} = 0.95$</th>
<th>$\frac{b}{a} = 0.9$</th>
<th>$\frac{b}{a} = 0.85$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(formula method)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.58 $\rightarrow$ 0.585</td>
<td>0.62 $\rightarrow$ 0.625</td>
<td>0.7 $\rightarrow$ 0.705</td>
<td>0.905 $\rightarrow$ 0.91</td>
</tr>
<tr>
<td>0.1</td>
<td>0.555 $\rightarrow$ 0.565</td>
<td>0.595 $\rightarrow$ 0.605</td>
<td>0.66 $\rightarrow$ 0.67</td>
<td>0.84 $\rightarrow$ 0.85</td>
</tr>
<tr>
<td>0.2</td>
<td>0.555 $\rightarrow$ 0.565</td>
<td>0.6 $\rightarrow$ 0.61</td>
<td>0.66 $\rightarrow$ 0.67</td>
<td>0.84 $\rightarrow$ 0.85</td>
</tr>
<tr>
<td></td>
<td>0.59 $\rightarrow$ 0.6</td>
<td>0.62 $\rightarrow$ 0.63</td>
<td>0.79 $\rightarrow$ 0.8</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.555 $\rightarrow$ 0.565</td>
<td>0.58 $\rightarrow$ 0.59</td>
<td>0.67 $\rightarrow$ 0.68</td>
<td>0.86 $\rightarrow$ 0.87</td>
</tr>
<tr>
<td></td>
<td>0.55 $\rightarrow$ 0.56</td>
<td>0.58 $\rightarrow$ 0.59</td>
<td>0.66 $\rightarrow$ 0.67</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.555 $\rightarrow$ 0.565</td>
<td>0.58 $\rightarrow$ 0.59</td>
<td>0.68 $\rightarrow$ 0.69</td>
<td>0.89 $\rightarrow$ 0.9</td>
</tr>
<tr>
<td></td>
<td>0.53 $\rightarrow$ 0.54</td>
<td>0.54 $\rightarrow$ 0.55</td>
<td>0.58 $\rightarrow$ 0.59</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.555 $\rightarrow$ 0.565</td>
<td>0.57 $\rightarrow$ 0.58</td>
<td>0.69 $\rightarrow$ 0.7</td>
<td>0.9 $\rightarrow$ 0.91</td>
</tr>
<tr>
<td></td>
<td>0.51 $\rightarrow$ 0.52</td>
<td>0.49 $\rightarrow$ 0.5</td>
<td>0.55 $\rightarrow$ 0.56</td>
<td></td>
</tr>
</tbody>
</table>
### Table 8.5  Zeros by the finite difference method
Parameters $m^q = 24$, $m^p = 15$; other parameters as in Table 7.8

<table>
<thead>
<tr>
<th>value of $\kappa / \nu_0$</th>
<th>$\frac{b}{a} = 1.0$</th>
<th>$\frac{b}{a} = 0.95$</th>
<th>$\frac{b}{a} = 0.9$</th>
<th>$\frac{b}{a} = 0.85$</th>
<th>$\frac{b}{a} = 0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.4 $\rightarrow$ 0.405</td>
<td>0.45 $\rightarrow$ 0.455</td>
<td>0.535 $\rightarrow$ 0.54</td>
<td>0.7 $\rightarrow$ 0.705</td>
<td>1.03 $\rightarrow$ 1.035</td>
</tr>
<tr>
<td>(formula method)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.315 $\rightarrow$ 0.325</td>
<td>0.37 $\rightarrow$ 0.38</td>
<td>0.45 $\rightarrow$ 0.46</td>
<td>0.6 $\rightarrow$ 0.61</td>
<td>0.78 $\rightarrow$ 0.79</td>
</tr>
<tr>
<td>0.2</td>
<td>0.315 $\rightarrow$ 0.315</td>
<td>0.38 $\rightarrow$ 0.39</td>
<td>0.46 $\rightarrow$ 0.47</td>
<td>0.63 $\rightarrow$ 0.64</td>
<td>0.84 $\rightarrow$ 0.85</td>
</tr>
<tr>
<td></td>
<td>0.36 $\rightarrow$ 0.37</td>
<td>0.42 $\rightarrow$ 0.43</td>
<td>0.54 $\rightarrow$ 0.55</td>
<td>0.79 $\rightarrow$ 0.8</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.315 $\rightarrow$ 0.325</td>
<td>0.37 $\rightarrow$ 0.38</td>
<td>0.47 $\rightarrow$ 0.48</td>
<td>0.65 $\rightarrow$ 0.66</td>
<td>0.83 $\rightarrow$ 0.84</td>
</tr>
<tr>
<td></td>
<td>0.35 $\rightarrow$ 0.36</td>
<td>0.40 $\rightarrow$ 0.41</td>
<td>0.49 $\rightarrow$ 0.5</td>
<td>0.7 $\rightarrow$ 0.71</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.315 $\rightarrow$ 0.325</td>
<td>0.37 $\rightarrow$ 0.38</td>
<td>0.47 $\rightarrow$ 0.48</td>
<td>0.67 $\rightarrow$ 0.68</td>
<td>0.86 $\rightarrow$ 0.87</td>
</tr>
<tr>
<td></td>
<td>0.34 $\rightarrow$ 0.35</td>
<td>0.38 $\rightarrow$ 0.39</td>
<td>0.45 $\rightarrow$ 0.46</td>
<td>0.6 $\rightarrow$ 0.61</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.315 $\rightarrow$ 0.325</td>
<td>0.37 $\rightarrow$ 0.38</td>
<td>0.46 $\rightarrow$ 0.47</td>
<td>0.60 $\rightarrow$ 0.7</td>
<td>0.87 $\rightarrow$ 0.88</td>
</tr>
<tr>
<td></td>
<td>0.33 $\rightarrow$ 0.34</td>
<td>0.35 $\rightarrow$ 0.36</td>
<td>0.41 $\rightarrow$ 0.42</td>
<td>0.54 $\rightarrow$ 0.55</td>
<td></td>
</tr>
</tbody>
</table>

### Table 8.6  Zeros by the finite difference method
Parameters $m^q = 24$, $m^p = 12$; other parameters as in Table 7.9

<table>
<thead>
<tr>
<th>value of $\kappa / \nu_0$</th>
<th>$\frac{b}{a} = 1.0$</th>
<th>$\frac{b}{a} = 0.95$</th>
<th>$\frac{b}{a} = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0 (formula method)</td>
<td>0.68 $\rightarrow$ 0.685</td>
<td>0.715 $\rightarrow$ 0.72</td>
<td>0.785 $\rightarrow$ 0.79</td>
</tr>
<tr>
<td>0.0</td>
<td>0.66 $\rightarrow$ 0.67</td>
<td>0.685 $\rightarrow$ 0.695</td>
<td>0.74 $\rightarrow$ 0.75</td>
</tr>
<tr>
<td>0.25</td>
<td>0.66 $\rightarrow$ 0.67</td>
<td>0.68 $\rightarrow$ 0.69</td>
<td>0.69 $\rightarrow$ 0.7</td>
</tr>
<tr>
<td></td>
<td>0.67 $\rightarrow$ 0.68</td>
<td>0.65 $\rightarrow$ 0.66</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.66 $\rightarrow$ 0.67</td>
<td>0.69 $\rightarrow$ 0.7</td>
<td>0.79 $\rightarrow$ 0.8</td>
</tr>
<tr>
<td></td>
<td>0.65 $\rightarrow$ 0.66</td>
<td>0.67 $\rightarrow$ 0.68</td>
<td></td>
</tr>
</tbody>
</table>
8.2 Conclusion

The finite difference method has given numerical results in a number of different cases of the propagation of electromagnetic waves through circular cylindrical waveguide. There seems to be a continuous variation in result as a parameter is varied continuously. In the cases where the results can also be calculated from analytical theory a comparison of results has shown that the finite difference method is valid within certain limitations. A major one is the decline in accuracy for waveguide configurations for which the scaled propagation constant $B$ is appreciably less than unity when $b = a$ (region 1 completely fills the guide). The error appears to follow the general pattern illustrated in Figure 4.2. Another limitation is the difficulty in simulating multimode propagation. Using the finite difference method it was very difficult to separate one mode from another when their propagation constants were close in value and changing rapidly with the parameter which was being varied. These conditions appear to be not infrequent where there are several possible modes, and satisfactory results were obtained by the method when there were at most two modes when $b = a$. This second limitation is, of course, no disadvantage in many cases where the waveguide configuration is designed to support one mode only.

Within the above limitations the finite difference method will give results which seem impossible to obtain from analytical theory. An example of this has been considered in this thesis. It is the circular waveguide containing a transversely magnetised ferrite tube adjacent to its wall. The numerical results suggest that the effect of transversely magnetising the ferrite is to allow the configuration to support two modes of propagation where it would previously support only a single mode. One of these two modes has the same propagation constant as the single one and the other has a propagation constant which differs from that of the single one by an amount which increases both with the strength of
magnetisation and the thickness of the tube.

The finite difference method used in this thesis can be applied to many other problems. The conditions which must be satisfied to make it applicable have been discussed in Chapter 2.
APPENDIX 1

The computer programme for the separate method

Notation

\[ W = \omega, \]
\[ Q_0 = \mu_0, \]
\[ E_0 = \epsilon_0, \]
\[ R_A = a, \]
\[ M_Q = m_q, \]
\[ M_P = m_p, \]
\[ N = n, \]
where the matrix \( M \) of equation (2.21) is \( n \times n \)

\[ B = B^2, \] the scaled propagation constant, \( \beta^2 = B^2 \omega^2 \epsilon_0 \mu_0 \)

and the value of the determinant \( |M| \) is given by

\[ |M| = D_1 \times 2^{10} \]

\[ 0.625 \leq D_1 < 1.0 \]

Size of programme

The main programme TMMODE occupies 403 words of store plus the space required for the arrays RL, RR and SR. The subroutine EVALDT occupies 120 words. The common block of data occupies 6 words plus the space required for the array BM. The sizes of the arrays RL, RR, SR and BM are related to the values of \( m_q \) and \( m_p \), the number of circles and radial arms respectively in the chosen mesh. The minimum sizes which must be allocated to the arrays are

\[ BM(m_p+1, 2m_q+1) \]
\[ RL(m_q) \]
\[ RR(m_q) \]
\[ \text{and} \]
\[ SR(m_q). \]

Then the whole programme will require

\[ 529 + 3m_q + m_p(2m_q+3) \text{ words of store.} \]
Notes

The programme evaluates and prints out the values D1, ID and B for each value of B read in. For the case of TE modes the main programme TMMODE will occupy a few more words because it contains one extra line of coding. Also for this case the matrix $M_\beta$ of equation (2.21) is replaced by the matrix $N_\beta$ of equation (4.13) in the above.
PROGRAM TMODE
DIMENSION BM(15,25), RL(20), RR(20), SR(20)
COMMON BM, M1, M2, EPS, MQ, MP, NC
READ(7,171) MQ, MP
171 FORMAT(13,13)
M1=MP
M2=MP
MA=M1+1
NC=M1+M2+1
N=MP*M0+1
EPS=7.75957614133D-12
LD=M1+1
LO=LD+NP
L1=LD+MP
READ(7,172) W, Q0, E0, RA
172 FORMAT(4(E9,3, 2X))
WRITE(2,175) W, Ou, Fo, RA, MQ, MP, N
175 FORMAT(1X, 2HW, E9,3,2X,3H00=,E9,3,2X,3H00=,E9,3,2X,3H00=,E9,3,02X,3H00=,12,2X,3HMP=,12,2X,2HNC,13)
173 FORMAT(F7,4)
P=MP
Q=MQ
P1=3.141592653589
RP=2.0*P1/P
SK0=W*W*E0*Q0
H=RA/Q
MQL=MQ-1
D1 140 I=1,MQL
Y=1
RL(I)=Y*H
RR(I)=1.0/(RL(I)*RL(I)*RP*RP)
SR(I)=1.0/(RR(I)*H)
180 CONTINUE
H2=1.0/(H*H)
167 READ(7,173) B
1 IF ( B.LT.0.0) GO TO 560
FA=1.0-B*B
CB=SK0*FA
CLEARING STORE
DO 191 I=1,M1
DO 193 J=1,NC
BM(I,J)=0.0
190 CONTINUE
191 CONTINUE
SETTING UP INITIAL VALUES
BM(1,1)=P
DO 200 J=1,MP
L=1+1
BM(1,L)=-1.0
200 CONTINUE
DO 210 J=2,M1
BM(I,1)=CB-2.0*H2*SR(I)-2.0*RR(I)
I=I+MP
I=$=I+1
IF (< I, E0, 2) IT=IT+MP
IV=IV+1
BM(I,IP)= H2+SR(1)
BM(I,IS)=H2
BM(I,IT)= RR(1)
BM(I,IV)= RR(1)

210 CONTINUE
XMAX=0.0
DO 274 I=1,MA
X=0.0
DO 272 J=1,NC
X=X+ABS(BM(I,J))
272 CONTINUE
IF (XMAX.LT.X) XMAX=X
274 CONTINUE
D1=1.0
ID=0
C START OF LOOP FOR FIRST PART OF EVALUATION
MM11=N-M1-1
DO 556 M=1,MM11
J=1
DO 295 J=2,MA
IF (ABS(BM(J,1)) .LE. ABS(X)) GO TO 295
X=B(A(J,1))
J=J
295 CONTINUE
D1=X*01
IF (D1 .NE. 0.0) GO TO 304
ID=0
BM(1,1)=XMAX*EPS
C SCALING REAL PART
304 IF (ABS(D1) .LT. 1.0) GO TO 309
ID=ID+4
D1=0.0625*D1
GO TO 304
309 IF (ABS(D1) .GE. 0.0625) GO TO 314
ID=ID-4
D1=D1*16.0
GO TO 309
314 IF (T1 .EQ. 1) GO TO 322
D1=-D1
C SWAPPING WITH FIRST ROW
DO 521 J=1,NC
X=BM(1,J)
BM(1,J)=BM(1,J)
X=BM(1,J)
521 CONTINUE
C SUBTRACTING MULTIPLES OF FIRST ROW FROM REMAINING ONES
C TO MAKE FIRST ELEMENT ZERO IN REMAINING ROWS
522 DO 355 I=2,MA
X=BM(I,1)/BM(1,1)
523 DO 353 J=2,NC
BM(I,J-1)= BM(I,J)-X*BM(I,J)
353 CONTINUE
BM(I,NC)=0.0
355 CONTINUE
I=I+M+1
C CLEARING FIRST ROW

DO 335 J=1,NC
BM(1,J)=0,0
335 CONTINUE
C FOR REPLACEMENT OF FIRST ROW AFTER EACH STEP
IF ( I.GT. MUL*MP+1 ) GO TO 400
K = (I-2)/MP+1
RM(1,LD)= CB-2,0*H2-SR(K)-2,0*RR(K)
RM(1,LD1)=H2-SR(K)
RM(1,LL1)=H2-L=(K-1)*MP
IF (L.EQ.1) GO TO 350
BM(1,LD-1)=RR(K)
GO TO 352
350 RM(1,LD1-1)=RR(K)
352 IF (L.EQ. MP) GO TO 354
BM(1,LD+1)=RR(K)
GO TO 354
354 RM(1,LL1+1)=RR(K)
GO TO 554
400 BM(1,LD)=1,0
554 MA=M/H2
556 CONTINUE
C END OF LOOP
C NOW HAVE FINAL DETERMINANT
CALL EVALT
WRITE(2,558) D1, ID, B
558 FORMAT(1X, F11,8, 5X, 15, 5X, F7,4)
C TAKING NEXT VALUE OF B
GO TO 187
560 STOP
END
SUBROUTINE EVALDT
C SUBROUTINE TO EVALUATE FINAL DETERMINANT
DIMENSION BM(13,25)
COMMON BM, M1, M2, EPS, D1, ID, NC
NR=M1+1
XMAX=0.0
DO 18 I=1, NR
X=0.0
DO 16 J=1, NC
X=X+ABS(BM(I,J))
16 CONTINUE
IF (XMAX.LT.X) XMAX=X
18 CONTINUE
L=M1
DO 73 K=1, NR
K1=K+1
X=BM(K,1)
1=K
IF (1.GE. NR) GO TO 27
L=L+1
27 IF (K+1.GT.L) GO TO 34
C FINDING LARGEST ELEMENT IN COLUMN
DO 33 J=K1, L
IF (ABS(BM(J,1)).LE. ABS(X)) GO TO 33
Y=BM(J,1)
1=J
33 CONTINUE
34 NT=X*D1
IF (X .NE. 0.0) GO TO 42
ID=0
BM(1,1)=XMAX*EPS
C SCALING REAL PART INTO REQUIRED RANGE
42 IF (ABS(D1).LT.1.0) GO TO 47
ID=ID+4
D1=0.0625*D1
GO TO 42
47 IF (ABS(D1).GF.0.0625) GO TO 52
ID=ID-4
D1=16.0*D1
GO TO 47
52 IF (1.EQ.0, X) GO TO 61
D1=1.1
C SHidding 1 AND K ROWS TO BRING MAX ELEMENT TO LEADING POSITION
DO 55 J=1, NC
X=BM(K,J)
BM(K,J)=BM(1,J)
BM(1,J)=X
55 CONTINUE
61 IF (K+1.GT.L) GO TO 73
C SUBTRACTING MULTIPLES OF KTH ROW FROM REMAINING ONES
C TO MAKE FIRST ELEMENT ZERO IN REMAINING ROWS
DO 72 T=K1, L
X=BM(1,1)/BM(K,1)
DO 69 J=2, NC
BM(1,J-1)=BM(1,J)-X*BM(K,J)
69 CONTINUE
BM(1,NC)=0.0
72 CONTINUE
73 CONTINUE
RETURN
END
APPENDIX 2

The computer programme for the simultaneous method

Notation

As in Appendix 1 except that \( N = n \) where here the matrix \( |M| \)
of equation (4.23) is \( n \times n \) and \( D_1 \) and \( ID \) give the value of its
determinant \( |M| \).

Size of programme

The main programme DBLVAC occupies 438 words of store plus the
space required for the arrays RL, RR and SR. The subroutine
EVALDT occupies 120 words and the common block of data occupies 6 words
plus the space required for the array BM. Here for an \( m_q \times m_p \) mesh
the minimum sizes which must be allocated to the arrays are

\[
BM(2m_p + 1, 4m_p + 1),
\]
\[
RL(m_q),
\]
\[
RR(m_q),
\]
and \( SR(m_q) \).

Then the whole programme will require

\[ 564 + 3m_q + 2m_p(4m_p + 3) \] words of store.

Notes

The programme evaluates and prints out \( D_1, ID \) and \( B \) for each
value of \( B \) read in.
PROGRAM DOLVAC(INPUT, OUTPUT, TAPE1=INPUT, TAPE2=OUTPUT, TAPE25)
DIMENSION BN(25,49), RL(20), RR(20), SR(20)
COMMON BM, M1, M2, EPS, EI, ID, NC
READ(7,171) MQ, MP
171 FORMAT(13,13)
MP2=2*MP
M1=MP2
M2=MP2
MA=M1+1
LB=M1+M2+1
N=MP2*M0+2
EPS=7.275957614183D-12
ID=M1+1
LD1=ID+MP
LD2=LD+MP2
LL1=ID+MP
LL2=LD+MP2
READ(7,172) W, QO, E0, RA
172 FORMAT(4(E9.3, 2X))
WRITE(2,175) W, 00, F0, RA, MQ, MP, N
173 FORMAT(F7.4)
P=PI
Q=M0
PI=3.141592653590
PD=2.0*PI/P
SK0=W*W*E0*00
H=MA/Q
NO=NO-1
DO 180 I=1, MO
180 W=1
RL(I)=Y*H
RR(I)=1.0/(RL(I)*PL(I)*RP*RP)
SR(I)=1.0/(RL(I)*H)
186 CONTINUE
H=1.0/(H*H)
187 READ(7,173) R
TF(R, LT.0, 0) GO TO 560
FA=1.0-3*B
CF=SK0*FA
C CLEARING STORE
DO 191 I=1, MA
DO 190 J=1, NC
BM(I, J)=0, 0
190 CONTINUE
191 CONTINUE
C SETTING UP INITIAL VALUES
BM(1, 1)=P
BM(2, 2)=P
DO 200 J=1, MP
L=L+2
PP(1, L)=-1, 0
L=L+2
PP(2, L)=-1, 0
200 CONTINUE
DO 200 J=1, MA
```
BM(1,1) = CB-2.0*H2-SF(1)-2.0*PR(1)
IP=1+MP
IS=1
IF (I.GT.MP) IS=2
IT=I-1
IF (I.EQ.3) IT=IT+MP
IF (I.EQ.3+MP) IT=I+1+MP
TV=I+1
IF (I.EQ.2+MP) TV=TV+MP
BM(I,IP)=H2+SP(1)
BM(I,IS)=H2
BM(I,IT)=RR(1)
BM(I,TV)=RR(1)
210 CONTINUE
XMAX=0.0
DO 274 I=1,MA
X=0.0
DO 272 J=1,NC
X=X+ABS(BM(I,J))
272 CONTINUE
IF (XMAX.LT.X) XMAX=X
274 CONTINUE
01=1.0
ID=0
C START OF LOOP FOR FIRST PART OF EVALUATION
NM1=M-MP-1
DO 556 M=1,NM1
X=BM(1,1)
1=1
DO 295 J=2,MA
IF (ABS(BM(J,1)).LE.ABS(X)) GO TO 295
X=BM(J,1)
1=J
295 CONTINUE
01=X*01
IF (01.NF.0.0) GO TO 304
ID=0
BM(1,1)=XMAX*EPS
C SCALING REAL PART
304 IF (ABS(01).LT.1.0) GO TO 309
ID=ID+1
01=0.0625*01
GO TO 304
309 IF (ABS(01).GT.0.0625) GO TO 314
ID=ID-1
01=01*16.0
GO TO 309
314 IF (I.EQ.1) GO TO 322
01=0
C SWAPPING WITH FIRST ROW
DO 371 J=1,NC
X=BM(1,J)
BM(1,J)=BM(I,J)
BM(I,J)=X
371 CONTINUE
C SUBTRACTING MULTIPLES OF FIRST ROW FROM REMAINING ONES'
C TO MAKE FIRST ELEMENT ZERO IN REMAINING ROWS
```
322 DO 333 J=2,MA
   XM=BM(I,1)/BM(1,1)
   DO 331 J=2,NC
   BM(I,J-1)=BM(I,J)-XM*BM(I,J)
331 CONTINUE
   BM(I,NC)=0.0
333 CONTINUE
   I=I+M1+1
C CLEARING FIRST ROW
   DO 335 J=1,NC
   BM(1,J)=0.0
335 CONTINUE
C FOR REPLACEMENT OF FIRST ROW AFTER EACH STEP
   IF (I .GT. MP2*MGL+2) GO TO 400
   K=(I-3)/MP2+1
   BM(1,LD)=CB-2.0*H2-SR(K)+2.0*RR(K)
   BM(1,LD2)=H2+SR(K)
   BM(1,LL2)=H2
   KK=(I-3)/MP
   I=(I-2)-1.*KK*MP
   IF (I .LE. 1) GO TO 350
   BM(I,LD-1)=RR(K)
   GO TO 352
350 BM(I,LD1-1)=RR(K)
352 IF (I .GT. MP) GO TO 354
   BM(I,LD+1)=RR(K)
   GO TO 354
354 BM(I,LL1+1)=RR(K)
   GO TO 354
400 IF (I .GT. MP2*MGL+2-MP) GO TO 450
   BM(1,LD)=1.0
   GO TO 354
450 BM(1,LD)=1.0
   BM(1,LL2)=1.0
554 MAH=FA/2
   IF (MAH .EQ. 2*MAH) D1=D1
556 CONTINUE
C END OF LOOP
C NOW HAVE FINAL DETERMINANT
   CALL EVALDT
   WRITE(2,558) D1, ID, B
558 FORMAT(1X, F11.8, 5X, 15, 5X, F7.4)
C TAKING NEXT VALUE OF B
   GO TO 167
500 STOP
END
SUBROUTINE EVAUDT
C SUBROUTINE TO EVALUATE FINAL DETERMINANT
DIMENSION BK(25,69)
COMMON BK, M1, M2, EPS, D1, ID, NC
N=1+1
XMAX=0.0
DO 18 I=1,NR
X=0.0
DO 17 J=1,NC
X=X+ABS(BM(I,J))
16 CONTINUE
IF (XMAX,LT,X) XMAX=X
18 CONTINUE
I=M1
J=1
K1=K+1
X=BM(K,1)
1=1
IF (L .GE. NR) GO TO 27
I=I+1
27 IF (K+1,GT,L) GO TO 34
C FINDING LARGEST ELEMENT IN COLUMN
50 15 S=K1,L
IF (ABS(BM(J,1)) .LE. ABS(X)) GO TO 33
X=BM(J,1)
1=J
33 CONTINUE
54 D1=X*0.1
IF (Y .NE. 0.0) GO TO 42
1=6
BM(1,1)=XMAX*EPS
C SCALING REAL PART INTO REQUIRED RANGE
42 IF (ABS(D1) .LT. 1.0) GO TO 47
1=10*6
D1=0.0625*0.1
GO TO 42
47 IF (ABS(D1) .GE. 0.0625) GO TO 52
1=10*6
D1=10.0*0.1
GO TO 47
52 IF (1 .GE. K) GO TO 61
1=1
C SWAPPING I AND K ROWS TO BRING MAX ELEMENT TO LEADING POSITION
70 5 S=K1,NC
Y=BM(K,J)
BM(K,J)=BM(I,J)
BM(I,J)=Y
59 CONTINUE
61 IF (K+1,GT,L) GO TO 73
C SUBTRACTING MULTIPLES OF KTH ROW FROM REMAINING ONES
C TO MAKE FIRST ELEMENT ZERO IN REMAINING ROWS
72 1=K1,L
Y=BM(1,1)/BM(K,1)
60 69 J=2,NC
BM(1,J-1)=BM(1,J)-X*BM(K,J)
59 CONTINUE
BM(1,NC)=0.0
72 CONTINUE
73 CONTINUE
RETURN
END
APPENDIX 3

The computer programme to evaluate the function, \( f(N,B) \) on the left-hand side of equation (6.18)

Notation

\[ W = \omega, \]
\[ QO = \mu_1, \]
\[ EO = \varepsilon_1, \]
\[ QU = \mu_2, \]
\[ E = \varepsilon_2, \]
\[ RA = a, \]
\[ RB = b, \]

\( B = B \) the scaled propagation constant, \( \beta^2 = B^2 \omega^2 \varepsilon_1 \mu_1, \)

\( N = n \) the order of the Bessel functions considered,

\( VALUE = \text{value of } f(n,B) \text{ for } B^2 \leq 1 \)

and \( CVAL = \text{value of } f(n,B) \text{ for } B^2 > 1. \)

Size of programme

Since there are no arrays used the programme occupies comparatively little store and its size will not generally be a problem.

Notes

The programme evaluates and prints out the value of the function \( f(n,B) \) for each pair of values of \( n \) and \( B \) read in. \( B \) must be chosen in the range given by \( 0 \leq B^2 \leq \frac{\varepsilon_2 \mu_2}{\varepsilon_1 \mu_1} \). The main programme calls the library subroutines BESJ and BESY to compute the \( J \) and \( Y \) Bessel functions respectively. These subroutines use an iterative process and stop when the difference between two successive approximations is less than \( D \) (a user supplied value) multiplied by the second approximation.

In the case \( B^2 > 1, \) the main programme also calls the functions S18AUF(X) and S18ABF(X) to evaluate the Bessel functions \( I_0(X) \) and
I_1(X). From these the value of I_n(X) is calculated using a recurrence relation. The functions S18AAF(X) and S18ABF(X) are available from the Nottingham Algorithms Group Library. See the N.A.G. Library Manual.
PROGRAM CHAREQ (INPUT, OUTPUT, TAPE7=INPUT, TAPE2=OUTPUT, TAPE25=
COMPLEX C1, CIN, B1N, B1NJ, CDJ, CX1, CX2, CX3, CVAL, CZ0
READ(7,2) W, QO, E, EO, RA, RB
READ(7,2) QU
C
W ANGULAR FREQUENCY
C
QO, QU PERMEABILITY
C
EO, E PERMITTIVITY
C
RA, RB RADIUS
C
ALL FOR INNER AND OUTER REGION RESPECTIVELY
2 FORMAT(6(E9.3,1X))
WP=W*W*EO*QO
READ(7,3) D
3 FORMAT(F9,6)
WRITE(2,32) W, QO, EO
32 FORMAT(1X, 2HW=, E10.4, 3X, 3HQU=, E10.4, 3X, 3HEO=, E10.4)
WRITE(2,33) E, RA, RB, D
33 FORMAT(1X, 2HE=, E10.4, 3X, 3HRA=, E9.3, 3X, 3HRB=, E9.3, 3X,
62HF=, F9.6)
WRITE(2,34) QU
34 FORMAT(1X, 3HQU=, E10.4)
C
B IS PROPAGATION CONSTANT SCALED ON THE INNER REGION
4 FORMAT(7,6) N, B
6 FORMAT(I5, 2X, F7.4)
IF (B .LT. 0.0) GO TO 10
Z02= WP*(1.0-B*B)
Z12= WP*(E*QU/(EO*QO)-B*B)
C
THIS PROGRAMME IS FOR Z12 .GE. 0.0
C
Z1= SQRT(Z12)
RB1= RB*Z1
AZ1= RA*Z1
N1= N+1
CALL BESJ(AZ1, N, BJ1, D, IER)
IF (IER .EQ. 0) GO TO 12
WRITE(2,11) IER
11 FORMAT(1X, 3HB1, 2X, I2)
GO TO 10
12 CALL BESJ(BZ1, N, BJ1, D, IER)
IF (IER .EQ. 0) GO TO 14
WRITE(2,13) IER
13 FORMAT(1X, 3HB1, 2X, I2)
GO TO 10
14 CALL BESJ(AZ1, N1, BJ1, D, IER)
IF (IER .EQ. 0) GO TO 16
WRITE(2,15) IER
15 FORMAT(1X, 4HB1, 2X, I2)
GO TO 10
16 CALL BESJ(BZ1, N1, B1J1, D, IER)
IF (IER .EQ. 0) GO TO 18
WRITE(2,17) IER
17 FORMAT(1X, 4HB1, 2X, I2)
GO TO 10
18 CALL BESY(AZ1, N, YA, IER)
IF (IER .EQ. 0) GO TO 24
WRITE(2,23) IER
23 FORMAT(1X, 2HYA, 2X, I2)
GO TO 10  
24 CALL BFSY(AZ1, N1, YA1, IER)  
  IF (IER .EQ. 0) GO TO 26  
  WRITE(2,25) IER  
25 FORMAT(1X, 3HYA1, 2X, I2)  
GO TO 10  
26 CALL BFSY(BZ1, N, YB, IER)  
  IF (IER .EQ. 0) GO TO 28  
  WRITE(2,27) IER  
27 FORMAT(1X, 2HYB, 2X, I2)  
GO TO 10  
28 CALL BESY(BZ1, N1, YB1, IER)  
  IF (IER .EQ. 0) GO TO 30  
  WRITE(2,29) IER  
29 FORMAT(1X, 3HYB1, 2X, I2)  
GO TO 10  
30 P = BJA*YB-BJR*YA  
  T = BJA*YB1-BJB1*YA  
  Q = N*P/BZ1-T  
  R = N*P/AZ1-U  
  S = N*N*P/(AZ1*BZ1)-N*T/AZ1-N/U/BZ1+V  
  IF (Z02 .GE. 0.0) GO TO 50  
  CI = (0,0,1,0)  
  RCP = SQRT(-Z02)  
  CZ0 = CI*RCP  
  CP = RB*RCP  
  BIO = S18A6F(CP)  
  BI1 = S18ABF(CP)  
  J = 0  
  BIN = BIO  
  BIN1 = BI1  
  IF (N .EQ. 0) GO TO 48  
45 J = J + 1  
  X = BIN  
  YJJ = J  
  BIN1 = X-2.0*YJJ*BIN/CP  
  IF (J .LT. N) GO TO 45  
48 CIN = CI**N  
  AJN = CIN/BIN  
  BJN1 = CI*CIN*BIN  
  CDJ = N*BJN/(CP*CI)-BJN1  
  CX1 = QU*CZ0*S*BJN/GO-Z1*R*CDJ  
  CX2 = E*CZ0*Q*BJN/EO-Z1*P*CDJ  
  CX3 = N*N*BJN/(Z02-Z12)*(Z02-Z12)*R*P*BJN/(RB*RB*Z02*Z12)  
  CVAL = CX1*CX2-CX3  
  WRITE(2,49) CVAL, N, B  
49 FORMAT(1X, E15.8, 3X, E15.8, 3X, I2, 3X, E7.4)  
GO TO 55  
50 Z0 = SQRT(Z02)  
  RZ0 = RB*Z0  
  CALL BESJ(BZ0, N, AJ, D, IER)  
  IF (IER .EQ. 0) GO TO 20  
  WRITE(2,19) IER  
19 FORMAT(1X, 2HYBJ, 2X, I2)
GO TO 10
20 CALL BEJSJ(BZ0, N1, BJ1, D, IER)
   IF (IER .EQ. 0) GO TO 22
   WRITE(2,21) IER
21 FORMAT(1X, 3HB1, 2X, 12)
   GO TO 10
22 DJ = N*BJ/BZ0-BJ1
   EX1 = QU*Z0*S*BJ/QU-Z1*R*DJ
   EX2 = E*Z0*Q*BJ/E0-Z1*P*DJ
   EX3 = N*N*BJ*(Z02-Z12)*(Z02-Z12)*R*P*BJ/(RB*RR*Z02*Z12)
   VALUE = EX1*EX2*EX3
   GO TO 4
   "PROPAGATION OCCURS WHEN VALUE =0"
   WRITE(2,8) VALUE, N, B
8 FORMAT(1X, E15.8, 3X, 12, 3X, F7.4)
55 GO TO 4
10 STOP
END
SUBROUTINE BESJ

PURPOSE
COMPUTE THE J BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER

USAGE
CALL BESJ(X,N,BJ,D,IER)

DESCRIPTION OF PARAMETERS
X - THE ARGUMENT OF THE J BESSEL FUNCTION DESIRED
N - THE ORDER OF THE J BESSEL FUNCTION DESIRED
BJ - THE RESULTANT J BESSEL FUNCTION
D - REQUIRED ACCURACY
IER - RESULTANT ERROR CODE WHERE
IER=0 NO ERROR
IER=1 N IS NEGATIVE
IER=2 X IS NEGATIVE OR ZERO
IER=3 REQUIRED ACCURACY NOT OBTAINED
IER=4 RANGE OF N COMPARED TO X NOT CORRECT (SEE REMARKS)

REMARKS
N MUST BE GREATER THAN OR EQUAL TO ZERO, BUT IT MUST BE
LESS THAN
20+10*X-X**2/3 FOR X LESS THAN OR EQUAL TO 15
90+X/2 FOR X GREATER THAN 15

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
NONE

METHOD
RECURRENC TECHNIQUE DESCRIBED BY H. GOLDSTEIN AND
R. N. THALER, RECURRENC TECHNIQUES FOR THE CALCULATION OF
BESSEL FUNCTIONS, M.T.A.C., V. 13, PP. 102-108 AND I. A. STEGUN
AND M. ABRAMOWITZ, GENERATION OF BESSEL FUNCTIONS ON HIGH
SPEED COMPUTERS, M.T.A.C., V. 11, 1957, PP. 255-257

SUBROUTINE BESJ(X,N,BJ,D,IER)

BJ=0
IF(N)10,20,20
10 IEK=1
RETURN
20 IF(X)30,50,31
30 IEK=2
RETURN
31 IF(X-15)32,34,32
32 NTEST=20+10.X-X**2/3
GO TO 36
34 NTEST=90+X/2
36 IF(N-NTEST)40,38,38
38 IER=4
RETURN
40 IER=0
BPREF = 0

C COMPUTE STARTING VALUE OF M

C IF(X=5,50,60,60
50 MA=X+6.
   GO TO 70
60 MA=1.4*X+60/X
70 MB=N+IFIX(X)/4+2
MZERO=MAX0(MA,MB)

C SET UPPER LIMIT OF M

C MMAX=NTEST
100 DO 190 M=MZERO,MMAX,3
C
C SET F(M),F(M-1)
C
FM1=1.0E-28
FK=0
ALPHA=0
IF(M-(M/2)*2)120,110,120
110 JJ=-1
   GO TO 130
120 JT=1
130 M2=M-2
   DO 160 K=1,M2
      MK=M-K
      BMK=2.*FLAT(MK)*FM1/XFM
      FM=FM1
      FM1=BMK
      IF(MK-N-1)150,140,150
140 BJ=BMK
150 JT=-JT
      S=1+JT
160 ALPHA=ALPHA+BMK*S
      BMK=2.*FM1/XFM
      IF(N)180,170,180
170 BJ=BMK
180 ALPHA=ALPHA+BMK
      BJ=BJ/ALPHA
      IF(ABS(BJ-BPREF)-ABS(DBJ))200,200,190
190 BPREF=BJ
      IER=3
200 RETURN
END

SUBROUTINE BESY

PURPOSE
   COMPUTE THE Y BESSSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER

USAGE
   CALL BESY(X,N,BY,IER).

DESCRIPTION OF PARAMETERS

N - THE ORDER OF THE Y BESSEL FUNCTION DESIRED
BY - THE RESULTANT Y BESSEL FUNCTION
TER - RESULTANT ERROR CODE WHERE
  IER=0 NO ERROR
  IER=1 N IS NEGATIVE
  IER=2 X IS NEGATIVE OR ZERO
  IER=3 BY HAS EXCEEDED MAGNITUDE OF 10**70

REMARKS
VERY SMALL VALUES OF X MAY CAUSE THE RANGE OF THE LIBRARY
FUNCTION ALOG TO BE EXCEEDED
X MUST BE GREATER THAN ZERO
N MUST BE GREATER THAN OR EQUAL TO ZERO

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
NONE

METHOD
RECURSION RELATION AND POLYNOMIAL APPROXIMATION TECHNIQUE
AS DESCRIBED BY A.J.M.HITCHCOCK, POLYNOMIAL APPROXIMATIONS
TO BESSEL FUNCTIONS OF ORDER ZERO AND ONE AND TO RELATED
FUNCTIONS, M.T.A.C., V.11, 1957, PP.86-88, AND G.N. WATSON,
A TREATISE ON THE THEORY OF BESSEL FUNCTIONS, CAMBRIDGE
UNIVERSITY PRESS, 1958, P. 62
SUBROUTINE BESY(X,N,BY,IER)

CHECK FOR ERRORS IN N AND X

IF(N).NE.10,10,10
10 IER=0
IF(X).GE.190,190,20

BRANCH IF X LESS THAN OR EQUAL 4

IF(X-.4).LE.40,40,30

COMPUTE Y0 AND Y1 FOR X GREATER THAN 4

T1=4.0/X
T2=T1*T1
PU=(...)

C COMPUTE Y0 AND Y1 FOR X LESS THAN OR EQUAL TO 4

XX=X/2.
X2=XX*XX
T=ALOG(XX)+.5772157
SUM=0.
TERM=T
Y0=T
DO 70 L=1,15
IF(L-1).LT.50,60,50
50 SUM=SUM+1./FLOAT(L-1)
60 FL=L
TS=T-SUM
TERM=(TERM*(-X2)/FL**2)*(1.,1./(FL*TS))
70 Y0=Y0+TERM
TERM=XX*(T-.5)
SUM=0.
Y1=TERM
DO 80 I=2,16
SUM=SUM+1./FLOAT(L-1)
FL=I
FL1=FL-1.
TS=T-SUM
TERM=(TERM*(-X2)/(FL1*FL1))*(TS*.5/FL)/(TS+.5/FL1))
80 Y1=Y1+TERM
Y0 = PI2 * Y0
Y1 = -PI2 / X + PI2 * Y1

C CHECK IF ONLY Y0 OR Y1 IS DESIRED
C 90 IF (N-1) 100, 100, 130
C RETURN EITHER Y0 OR Y1 AS REQUIRED
C 100 IF (N) 110, 120, 110
110 BY = Y1
    GO TO 170
120 BY = Y0
    GO TO 170
C
C PERFORM RECURRENT OPERATIONS TO FIND YN(X)
C 130 YA = Y0
    YB = Y1
    K = 1
140 T = FLOAT (2 * K) / X
    YC = T * YB - YA
    IF (ABS (YC) - 1.0E70) 145, 145, 141
141 IER = 3
    RETURN
145 K = K + 1
    IF (K - N) 150, 160, 150
150 YA = YB
    YB = YC
    GO TO 140
160 BY = YC
170 RETURN
180 IER = 1
    RETURN
190 IER = 2
    RETURN
END
APPENDIX 4

Programme 4. The computer programme to evaluate the determinant $|\mathbf{M}|$ of eqn. (5.15).

Notation

\begin{align*}
W &= \omega, \\
Q_0 &= \mu_1, \\
E_0 &= \epsilon_1, \\
Q_u &= \mu_2, \\
E &= \epsilon_2, \\
Q_V &= \mu_0, \\
X_K &= \kappa, \\
R_A &= a, \\
R_B &= b, \\
M_P &= m_p', \\
M_Q &= m_q \\
R_K &= \frac{K}{\mu_0} \\
BDA &= \frac{b}{a} \\
\end{align*}

MF and DF are parameters which are used to set up the radial step lengths,

\[ B = B \text{ the scaled propagation constant, } \beta^2 = B^2 \omega^2 \epsilon_1 \mu_1, \]

\[ N = n \text{ where the matrix } \mathbf{M} \text{ of equation (5.15) is } n \times n \]

and $|\mathbf{M}| = D_1 \times 2^{1D}$

Size of programme

The main programme occupies 1122 words of store plus the space required for the arrays RL, RR, SR, CS, SN, Q11, Q12, Q22, QE, QF, TR, HA, HA2, HAD and HAM. The subroutine RADII occupies 95 words plus the space required for the array RAD. The subroutine EVALDT occupies 120 words. The common block of data occupies 6 words plus the space required for the array BM. For an $m_q \times m_p$ mesh the
minimum sizes which must be allocated to the arrays are

\[ m_q \] for each of RL, RR, SR, TR, HA, HA2, HAD, HAM and RAD

\[ m_p \] for each of CS, SN, Q11, Q12, Q22, QE and QF and

BM \((3m_p + 1, 6m_p + 2)\).

Then the whole programme will require

\[ 1345 + 9m_q + m_p(18m_p + 19) \] words of store.

Notes

The programme evaluates and prints out the values D1, ID and B for each value of B read in. The subroutine RADII sets up the radii of the mesh circles. Here the radial step lengths have all been made equal except for the ones adjacent to the circles \( r = a \) and \( r = b \) and the one at the centre of the guide. In the programme the equal steps are known as regional steps, the steps adjacent to \( r = a \) and \( r = b \) as boundary steps and the step at the centre as the first step. MF is the number of regional steps in region 2 and DF is the ratio of the length of a boundary step to the length of a regional step. There is a considerable freedom in choosing radial step lengths and the subroutine RADII can be easily replaced by another one giving a different choice.
PROGRAM FINDIF (INPUT, OUTPUT, TAPE? = INPUT, TAPE? = OUTPUT, TAPE? = TAPE)

DIMENSION RM(49, 98), RL(30), RR(30), SR(30), CS(16), SN(16),
6HA(30), HAD(30), HAM(30)

COMMON HM, M1, M2, EPS, D1, ID, NC

READ(7, 171) MP

171 FORMAT(13)
M1 = 3*MP
M2 = 3*MP+1
MA = M1 + 1
NC = M1 + M2 + 1
EPS = 7.275957614183E-12
MP2 = 2*MP
LD = M1 + 1
LD1 = LD + MP
LD2 = LD + MP2
LD3 = LD + 3*MP
LL1 = LD - MP
LL2 = LD - MP2
LL3 = LD - 3*MP

READ(7, 172) W, Q0, EO, RA

172 FORMAT(4(E9.3, 2X))
READ(7, 172) PK, ER, BDA
READ(7, 172) OV, QU
E = ER*EO
XK = RK*QV
RB = BDA*RA
QR = QU/QO

READ(7, 173) ANG

173 FORMAT(F7.4)
P = MP
PI = 3.1415926535890
RP = 2.0*PI/P
SK0 = W*W*EO*Q0
C = 1.0/SQR(EO*Q0)

CALL RADI(RR, RA, RL, DH, MB, MQ)
HA(1) = RL(1)
DO 160 I = 1, MQ
HA(I) = RL(I) - RL(I-1)

160 CONTINUE
MQ = MQ - 1
DO 185 I = 1, MQ
TR(I) = 1.0/(HA(I+1)*RL(I))
HA2(I) = 1.0/(HA(I+1)*HA(I))
HAD(I) = 1.0/(HA(I+1)*HA(I))
HAM(I) = HA2(I) + HAD(I)
RR(I) = 1.0/(RL(I)*RL(I)*RP*RP)

185 CONTINUE
DO 160 I = 1, MP
Y = I
VAR = (Y+ANG)*2.0*RP
SN = SIN(VAR)
CSG = COS(VAR)
SN(I) = SN
CS(I) = CSG
SSC = SSC*SN
CSG = SSC*CSG
Q11(I) = QV*SSQ + QU*CSQ
Q22(I) = QU*SSQ + QV*CSQ
Q12(I) = (QV-QU)*SNG*CSG
QE(I) = (QV-QU)*(CSQ-SSQ)
QF(I) = 2.0*QE(I) + Q11(I)

186 CONTINUE
N=2+M0*MP2 
WRITE (2, 174) MP, M0, N, RA, RB, ANG 
174 FORMAT (1X, 3HMP=, 12, 2X, 3HMQ=, 12, 2X, 2HQN=, 13, 2X, 3HRAN=, 6F9.3, 2X, 3HRB=, E9.3, 2X, 4HAN=, F5.2) 
WRITE (2, 175) W, Q0, E0, E, XX, QQ 
175 FORMAT (1X, 2HQ=, E10.4, 2X, 3HGU=, E10.4, 2X, 3HE0=, E10.4, 2X, 62HC=, E10.4, 2X, 3HXK=, E10.4, 2X, 3HQU=, F10.4) 
WRITE (2, 176) QV 
176 FORMAT (1X, 3HQV=, E10.4) 
WRITE (2, 178) RK, QR, ER, BDA 
178 FORMAT (1X, 6HEK/QV=, 11F6.3, 2X, 4HEP/QV=, 11F6.3, 2X, 66HRB/RA*, F6.3) 
WRITE (2, 177) 
177 FORMAT (1X, 5X, 2HD1, 10X, 2HID, 10X, H2) 
191 CONTINUE 
190 CONTINUE 

C CLEARING STORE
DO 191 I=1,MA 
DO 190 J=1,NC 
BM(I, J) = 0.0
190 CONTINUE 
191 CONTINUE

C SETTING UP INITIAL VALUES
BM(1,1) = P 
BM(2,2) = P 
DO 200 J=1,MP 
L=J+2 
BM(1,L) = -1.0 
L=J+2+MP 
BM(2,L) = 1.0 
200 CONTINUE

DO 220 I=3,MA 
K=(I-3)/MP2+1 
KK=(I-3)/MP 
L=1-2*KK*MP 
RM(I,1) = CB-HAM(K)-2.0*RR(K)-TR(K) 
IT=I+MP 
IV=I-MP 
IP=I+MP2 
IS=1 
IF (I .GT. 2+MP) IS=2 
IF (I .GT. 2+MP2) IS=1-MP2 
RM(I,IP) = HA2(K)+TR(K) 
BM(I,IS) = HAD(K) 
IF (L .EQ. 1) GO TO 212 
BM(I,1-1) = RR(K)
GO TO 215
212 BM(I,IT-1)=RR(K)
215 IF (L .EQ. MP) GO TO 218
   BM(I,1+1)=RR(K)
   GO TO 220
218 BM(I,IV+1)=RR(K)
220 CONTINUE
XMAX=0.0
DO 274 I=1,MA
   X=0.0
   DO 272 J=1,NC
      X=X+ABS(BM(I,J))
   272 CONTINUE
IF (XMAX.LT.X) XMAX=X
274 CONTINUE
D1=1.0
C START OF LOOP FOR FIRST PART OF EVALUATION
NM1=N-M1-1
DO 556 M=1,NM1
   X=BM(1,1)
   I=1
   DO 295 J=2,MA
      IF (ABS(BM(J,1)).LE. ABS(X)) GO TO 295
      X=BM(J,1)
      I=J
   295 CONTINUE
   D1=X*D1
   IF (D1 .NE. 0.0) GO TO 304
   ID=0
   BM(1,1)=XMAX*EPS
C SCALING REAL PART
304 IF (ABS(D1).LT. 1.0) GO TO 309
   ID=ID+4
   D1=0.0625*D1
309 IF (ABS(D1).GE. 0.0625) GO TO 314
   ID=ID-4
   D1=D1*16.0
314 IF (I .EQ. 1) GO TO 322
   D1=-D1
C SWAPPING WITH FIRST ROW
   DO 321 J=1,NC
      X=BM(1,J)
      BM(1,J)=BM(1,J)/DM(1,1)
      BM(1,J)=X
321 CONTINUE
C SUBTRACTING MULTIPLES OF FIRST ROW FROM REMAINING ONES
C TO MAKE FIRST ELEMENT ZERO IN REMAINING ROWS
322 DO 333 I=2,MA
      X=BM(I,1)/BM(1,1)
      DO 331 J=2,NC
         BM(I,J-1)=BM(I,J)-X*BM(I,J)
      331 CONTINUE
   BM(I,NC)=0.0
333 CONTINUE
I=M+M+1

C CLEARING FIRST ROW
DO 335 J=1, NC
    BM(1,J)=0.0
335  CONTINUE

C FOR REPLACEMENT OF FIRST ROW AFTER EACH STEP
IF (I.GE. 3+(MB-1)*MP2) GO TO 400
K=(I-3)/MP2+1
KK=(I-3)/MP
L=(I-2)-KK*MP
FH= HA(K+1)/HA(K)
FR=RR(K)*FH
BM(1,LD)= (CB-HAM(K)-2.0*RR(K)-TR(K))*FH
BM(1,LD2)= (HA2(K)+TR(K))*FH
BM(1,LL2)= HAD(K)*FH
IF (L.EQ. 1) GO TO 370
BM(1,LD -1)=FR
GO TO 372
370 BM(1,LD1-1)=FR
372 IF (L.EQ. MP) GO TO 374
    BM(1,LD +1)=FR
    GO TO 554
374 BM(1,LL1+1)=FR
    GO TO 554
400 IF (I.GE. 3+MB*MP2-MP) GO TO 430
    L=I-(2+(MB-1)*MP2)
C BOUNDARY CONDITION AT INTERFACE
FD= R*R-FR*Q11(L)/Q0
B1= Q12(L)*E*ER/(Q0*EB*EC*RB*RP)
B2= WS/EA +ER*WS/(EB*EC)
BM(1,LD )= ER*E/(EB*EC*DH) +B1
BM(1,LD2)= ER*E/(EB*EC*DH)
BM(1,LL2)= ER*E/(EB*EC*DH)
BM(1,LD1)= -ER*B*Q12(L)/(C*Q0*EB*EC*DH) -W*E*XX*SN(L)/EC -B2
BM(1,LD3)= ER*B*Q12(L)/(C*Q0*EB*EC*DH)
IF (L.EQ. MP) GO TO 415
BM(1,LD1+1)= B2
BM(1,LD +1)= -B1
    GO TO 420
415 BM(1,LD1+1)= B2
    BM(1,LL1+1)= -B1
420 GO TO 554
430 IF (I.GE. 3+MB*MP2) GO TO 500
    L=I-(2+MB*MP2-MP)
EE= B*B-FR*Q22(L)/Q0
EF= B*B*Q11(L)-ER*QV*QR
F1= WS/EA +EE*WS/(EB*EC)
F2= Q12(L)*E*B/(EB*EC*RB*RP)
BM(1,LD)= Q0/(EA*DH) +B*W*XX*CS(L)/(EC*C)-EF/(EB*EC*DH)-F2
BM(1,LL2)= -Q0/(EA*DH)
BM(1,LD2)= EF/(EB*EC*DH)
BM(1,LD1)= Q12(L)*ER*EO*B*/(EB*EC*DH)
BM(1,LL1)= -BM(1,LD1) +F1
IF (L.EQ. MP) GO TO 445
BM(1,LL1+1)= -F1
BM(1,LD +1)= F2
    GO TO 450
IF (I .GE. 3*(MQ=1)*MP) GO TO 540
K = (I-3)/MP2 + 1
K = I - 2 - MP2*(KK-1)
IF (K .GT. MP) GO TO 520
L = K
ED = B*B*ER*Q11(L)/Q0
EE = B*B*ER*Q22(L)/Q0
E3 = EE*RR(KK)
E4 = 4.0*ER*Q12(L)/(RL(KK)*RL(KK)*RP*Q0)
E5 = Q12(L)*B*C*RR(KK)
E6 = 2.0**C*QS(L)/(RL(KK)*RL(KK)*RP) + W**K*CS(L)*EB/(RL(KK)*RP)
E1 = 2.0*Q12(L)*ER*TR(KK)/(Q0*RP)
E2 = Q01(L)*Q22(L)*B*C*TR(KK)/RP
E7 = W**K*SN(L)*EB/HA(KK+1) + 3.0*Q12(L)*B*C*TR(KK)
E8 = (2.0*ER*QE(L)/Q0=ED)*TR(KK)
BM(1,LD1) = -ED*HA2(KK) + F8 - E1
BM(1,LD3) = -Q12(L)*HA2(KK)*B*C + E7 - E2
BM(1,LD1+1) = EB*SQ0*EC + ED*HAM(KK) - 2.0*E3 - E8 + E1 + E4
BM(1,LL2) = -ED*HAM(KK)
BM(1,LL1) = Q12(L)*B*C*HAM(KK) - 2.0*E5 - W**K*SN(L)*EB/RL(KK)
E6 = E7 + E2
BM(1,LL1) = Q12(L)*B*C*HAM(KK)
IF (L .GE. MP) GO TO 505
BM(1,LD+1) = E3 - E4 - E1
BM(1,LD1+1) = E5 + E6 - E2
BM(1,LD2+1) = E1
BM(1,LD3+1) = E2
GO TO 510
505 BM(1,LL1+1) = E3 - E4 - E1
BM(1,LD+1) = E5 + E6 - E2
BM(1,LD1+1) = E1
BM(1,LD2+1) = E2
510 IF (L .EQ. 1) GO TO 515
BM(1,LD-1) = E3
BM(1,LD1-1) = E5
GO TO 514
515 BM(1,LD1-1) = E3
BM(1,LD2-1) = E5
518 GO TO 554
520 L = K + MP
EF = B*B*Q11(L) - ER*QV*QR
EG = B*B*Q22(L) - ER*QV*QR
E1 = 2.0*Q12(L)*B*C*TR(KK)/RP
E2 = (Q11(L)-Q22(L))*B*C*TR(KK)/RP
E3 = ER*RR(KK)
E4 = 4.0*B*B*Q12(L)/(RL(KK)*RL(KK)*RP)
E5 = Q12(L)*B*C*TR(KK)
E6 = W**X*K*CS(L)*EB/(RL(KK)*RP) - 2.0*QE(L)*B*C/(RL(KK)*RP)
E7 = 3.0*B*C*Q12(L)*TR(KK) - W**X*K*SN(L)*EB/HA(KK+1)
E8 = (EF2, 0.0*B*Q(E(L)))*TR(KK)
BM(1,LD) = EB*QU*B*B*SQ0-W**W*E*QU-QU-X*K*KK)+B*W**X*K*CS(L)/(C*
6R(L)*KK))/EF*HAM(KK) + E8 - E1 - 2.0*E3 - E4
BM(1,LL2) = B*C*E*Q12(L)*HAM(KK) + E2 - E7 + E6 - 2.0*E5
BM(1,LD2) = -EF*HA2(KK) + E1 - E8
GRAM FINDIF

IF (L, EQ, MP) GO TO 525
BM(1, LD +1) = E3+E4+E1
BM(1, LL1+1) = E5-E6-E2
BM(1, LD2+1) = E1
BM(1, LD1+1) = E2
GO TO 530

525 BM(1, LL1+1) = E3+E4+E1
BM(1, LL2+1) = E5-E6-E2
BM(1, LD1+1) = E1
BM(1, LD +1) = E2

530 IF (L, EQ, 1) GO TO 535
BM(1, LD -1) = E3
BM(1, LL1-1) = E5
GO TO 538

535 BM(1, LD1-1) = E3
BM(1, LD -1) = E5

538 GO TO 554

540 IF (I, GE, 3+MQ*MP2-MP) GO TO 545
BM(1, LD) = EA*EA
GO TO 554

545 L = I-(2+MQ*MP2-MP)
EF = B*B*Q11(L)-ER*QV*QR
EE = B*B*ER*Q22(L)/Q0
S1 = B*B*Q12(L)/(HA*RP)
S2 = B*EF/(C*RA*RP)
BM(1, LD) = W*B*XX*CS(L)*EB/C +EF/HA(MQ) -S1
BM(1, LL1) = B*E*C*Q12(L)/HA(MQ)+S2
BM(1, LL2) = -EF/HA(MQ)
BM(1, LL3) = B*E*C*Q12(L)/HA(MQ)
IF (L, EQ, MP) GO TO 550
BM(1, LD +1) = S1
BM(1, LL1+1) = -S2
GO TO 554

550 BM(1, LL1+1) = S1
BM(1, LL2+1) = -S2

554 MAH=MA/2
IF (MA .EQ. 7*MAH) D1=-D1

556 CONTINUE
C END OF LOOP
C NOW HAVF FINAL DETERMINANT
CALL EVALDT
WRITE (7,558) D1, 1D, B

558 FORMAT(1X, F11.8, 5X, 15, 5X, F7.4)
C TAKING NEXT VALUE OF B
GO TO 18/

560 STOP
END
SUBROUTINE RADII(RI, RO, RAD, SH, NB, NQ)
DIMENSION RAD(3U)
READ(7,2) MF, DF
2 FORMAT(I3, F6.3)
F = MF
H = (RO - RI) / (F + 2.0 * DF)
SH = DF * H
K = (RI - SH) / H
HIN = RI - SH - K * H
WRITE(2, 12). HIN, H, SH
12 FORMAT(1X, 11HFIRST STEP=, E10.4, 2X, 14HRE gional STEP=, E10.4,
   62X, 14HB enmary STEP=, E10.4)
   K1 = K + 1
   IF (HIN / H .LT. 0.01) GO TO 4
   GO TO 6
4 HIN = H + HIN
   NB = K1
   GO TO 6
6 NQ = NB + MF + 2
MBP = NB + 1
MBL = NB - 1
DO 8 I = 1, MBL
   Y = I - 1
   RAD(I) = HIN + Y * H
8 CONTINUE
RAD(NB) = RI
NQL = NQ - 1
DO 10 I = MBP, NQL
   Y = I - 1 - NB
   RAD(I) = RI + SH + Y * H
10 CONTINUE
RAD(NQ) = RO
RETURN
END
SUBROUTINE EVALDT
C SUBROUTINE TO EVALUATE FINAL DETERMINANT
DIMENSION BM(49,96)
COMMON BM, M1, M2, EPS, D1, ID, NC
NR=M1+1
XMAX=0.0
DO 16 I=1,NR
X=0.0
DO 16 J=1,NC
X=X+ABS(BM(I,J))
16 CONTINUE
IF (XMAX.LT.X) XMAX=X
18 CONTINUE
L=M1
DO 73 K=1,NR
K1=K+1
X=BM(K,1)
I=K
IF (L.GE.NR) GO TO 27
L=L+1
27 IF (K+1.GT.L) GO TO 34
C FINDING LARGEST ELEMENT IN COLUMN
DO 33 J=K1,L
IF (ABS(BM(J,1)).LE.ABS(X)) GO TO 33
X=BM(J,1)
I=J
33 CONTINUE
34 D1=X*D1
IF (X.NE.0.0) GO TO 42
ID=0
BM(1,1)=XMAX*EPS
C SCALING REAL PART INTO REQUIRED RANGE
42 IF (ABS(D1).LT.1.0) GO TO 47
ID=ID+4
D1=0.0625*D1
GO TO 42
47 IF (ABS(D1).GE.0.0625) GO TO 52
ID=ID-4
D1=16.0*D1
GO TO 47
52 IF (I.EQ.K) GO TO 61
D1=-D1
C SWAPPING I AND K ROWS TO BRING MAX ELEMENT TO LEADING POSITION
DO 59 J=1,NC
X=BM(K,J)
BM(K,J)=BM(I,J)
BM(I,J)=X
59 CONTINUE
61 IF (K+1.GT.L) GO TO 73
C SUBTRACTING MULTIPLES OF KTH ROW FROM REMAINING ONES
C TO MAKE FIRST ELEMENT ZERO IN REMAINING ROWS
DO 69 J=2,NC
BM(I,J-1)=BM(I,J)-X*BM(K,J)
69 CONTINUE
BM(I,NC)=0.0
72 CONTINUE
73 CONTINUE
RETURN
END
APPENDIX 5

Programme 5. The computer programme to evaluate the determinant $|M|_n$ of equation (5.15) in the special case $\nu_1 = \nu_2 = \nu_0$.

Notation

As in Appendix 4 except that

$$Q_0 = \nu_1 = \nu_2 = \nu_0$$

Size of programme

The main programme occupies 896 words of store plus the space required for the arrays RL, SR, TR, HA, RR, HA2, HAM, HAD, CS and SN. The subroutine EVALDT occupies 120 words. The common block of data occupies 6 words plus the space required for the array BM. For an $m \times m$ mesh the minimum sizes which must be allocated to the arrays are

- $m_q$ for each of RL, SR, RR, TR, HA, HA2, HAM and HAD,
- $CS(m_p')$, $SN(m_p')$ and $BM(2m_p+1, 5m_p+1)$.

Then the whole programme will require

$$1003 + 8m_p + m_p (10m_q +9)$$

words of store.

Notes

The programme evaluates and prints out the values $D_1$, $D_2$ and $B$ for each value of $B$ read in. The radial step lengths are set up within the main programme using the parameters $MF$ and $DF$ as in the programme in Appendix 4.
PROGRAM FINDIF(INPUT,OUTPUT, TAPE1=INPUT,TAPE2=OUTPUT,TAPE25)
DIMENSION BM(3,3,E1), RL(30), SR(30), TF(30), HA(30), RR(30), 
6HA2(30), HAM(30), CS(16), SN(16), HAD(30)
COMMON BM, M1, M2, EPS, D1, ID, NC
READ(7,171) MF, MF, DF
171 FORMAT(13, 13, Fe.3)
M1= 2*MP
M2= 3*MP
MA=M1+1
NC=M1+M2+1
EPS=7.275457614183E-12
MP2=2*MP
LD=M1+1
LD1=LD+MP
LD2= LD+MP2
LD3=LD+3*MP
LL1=LD+MP
LL2= LD+MP2
READ(7,172) UI, QO, E0, RA
172 FORMAT(4(F9.3, 2X))
READ(7,172) XE, F, RR
READ(7,173) ANG
173 FORMAT(F7.4)
P=MP
P1=3.141592653590
PP=2.0*PI/P
SQO=UI*E0*QO
C=1.0/SQRT(QO*FO)
F=MF
N=(RA-RB)/(F+7.0*DF)
DH=DF*H
K=(RB-DH)/H
HIN=RB-DH-K*H
KK=K+1
IF(HIN/H .LT. 0.01) GO TO 180
MB=1+KK
GO TO 182
180 HIN=HIN+H
MB=KK
182 MQ=MB+MF+2
MPB=MB+1
MB=MB-1
DO 183 I=1,MB
Y=I-1
RL(I)=DIN+Y*H
183 CONTINUE
RL(MP)=RB
MOL=MQ-1
DO 184 I=MPB,MOL
Y=I-1
MB
RL(I)= RB+DH+Y*H
184 CONTINUE
RL(MQ)=RA
HA(I)= HIN
DO 160 I=2,MQ
HA(I)= RL(I)-RL(I-1)
160 CONTINUE
DO 195 I=1,MOL
TR(I)= 1.0/(HA(I+1)*RL(I))
HA2(I)= 1.0/(HA(I+1)*HA(I+1))
HAD(I)= 1.0/(HA(I+1)*HA(I))
HAM(I)= HA2(I)+HAD(I)
RR(I)= 1.0/(RL(I)*RL(I)*RP*RP)
SR(I)= 1.0/(2.0*RL(I)*H)
185 CONTINUE
DO 186 I=1,MP
Y=1
VAR=(Y+ANG)*2.0*RP
SN(I)=SIN(VAR)
CS(I)=COS(VAR)
186 CONTINUE
N=2*M0*MP2
H2=1.0/(H+H)
HI=(H+HIN)/(H+HIN)
HJ=1.0/(H+HIN)
HK=1.0/(H+HIN)
HD=2.0*H2
WRITE(2,174) MF, MP, MQ, N, RA, RB, DF
174 FORMAT(1X,3HMF=,12,2X,3HMP=,12,2X,3HMQ=,12,2X,2HN=,
615,2X,3HRA=,E9.3,2X,3HRB=,E9.3,2X,3HDF=,F6.3)
WRITE(2,175) M, MQ, KO, E, XX
175 FORMAT(1X,2HWA=,E10.4,2X,3HMO=,E10.4,2X,3HEO=,E10.4,2X,
624,2X,3HKA=,E10.4,2X,3HEA=,E10.4)
WRITE(2,176) ANG, H, DH
176 FORMAT(1X,4HANG=,F5.2,2X,2HWH=,E10.4,2X,3HH=,E10.4)
WRITE(2,177)
177 FORMAT(1X,5X,2HD1,11X,2HID,10X,1HB)
147 READ(7,173) B
IF (B .LT. 0.0) GO TO 560
FA=1.0-B*B
FB=B*E/F0
CB=SQRT(E)
US=R/(C*RL(MR)*RP)
C CLEARING STORE
DO 191 I=1,MA
DO 190 J=1,NC
RM(I,J)=0.0
190 CONTINUE
191 CONTINUE
C SETTING UP INITIAL VALUES
RM(1,1)=P
RM(2,2)=P
DO 200 J=1,MP
L=J+2
RM(1,L)=1.0
L=J+2+MP
RM(2,L)=1.0
200 CONTINUE
DO 210 I=5,MA
RM(1,1)=CB*HI-2.0*KR(1)-2.0*SR(1)
1T=1+MP
1P=1+MP2
1V=1-MP
IS=1
IF (I .GT. 2+MP) IS=
RM(I,1P)= H2+2.0*SP(1)
BM( I , IS ) =M J
KK=(I-S)/4P
K = I-2+MP*KK
IF (x .EQ. 1) GO TO 202
BM(I, I-1)= RR(1)
GO TO 204
202 BM(I,IT-1)= RR(1)
204 IF (K .EQ. 1, MP) GO TO 206
BM(I, I+1)= RR(1)
GO TO 210
206 BM(I, IV+1)= RR(1)
210 CONTINUE
XMAX=0.0
DO 274 I=1,MA
X=0.0
DO 272 J=1,NC
X=X+ABS(BM(I, J))
272 CONTINUE
IF (XMAX.LT.X) XMAX=X
274 CONTINUE
D1=1.0
ID=0
C START OF LOOP FOR FIRST PART OF EVALUATION
MM1=N-M1-1
DO 556 M=1,MM1
X=RM(1,1)
I=1
DO 295 J=1,MA
IF (ABS(RM(J,1)) .LE. ABS(X)) GO TO 295
X=RM(J,1)
I=J
295 CONTINUE
D1=X*D1
IF (D1 .NE. 0.0) GO TO 304
ID=0
RM(1,1)=XMAX*EPS
C SCALING REAL PART
304 IF (ABS(D1) .LT. 1.0) GO TO 309
ID=ID+4
D1=0.0625*D1
GO TO 304
309 IF (ABS(D1) .GT. 0.0625) GO TO 314
ID=ID-4
D1=D1*10.0
GO TO 309
314 IF (I .EQ. 1) GO TO 322
D1=-D1
C SWAPPING WITH FIRST ROW
DO 321 J=1,NC
X=RM(1, J)
RM(1, J)=RM(I, J)
RM(I, J)=X
321 CONTINUE
C SUBTRACTING MULTIPLES OF FIRST ROW FROM REMAINING ONES
C TO MAKE FIRST ELEMENT ZERO IN REMAINING ROWS
C  CLEARING FIRST ROW
DO 335 J=1,NC
BM(1,J)=0,0
335  CONTINUE
C  FOR REPLACEMENT OF FIRST ROW AFTER EACH STEP
IF ( I .GE. 3+*(MB-1)*MP2 ) GO TO 400
K=(I-3)/MP2+1
KK=(I-2)/MP
L=I-KK*MP
IF ( I .GE. (MB-2)*MP2+3 ) GO TO 360
RM(1,LD)= CB-HD-2.0*RR(K)-2.0*SR(K)
RM(1,LD2)= H2+2.0*SR(K)
RM(1,LD2)= H2
IF ( L .EQ. 1 ) GO TO 350
RM(1,LD1-1)=RR(K)
GO TO 352
350 RM(1,LD1-1)=RR(K)
352 IF ( I .EQ. MP ) GO TO 354
BM(1,LD1)= RR(K)
GO TO 354
354 BM(1,LD1+1)=RR(K)
GO TO 354
360 BM(1,LD)= CB*DH/H -(H+DH)/(H*H+DH)-2.0*RR(K)*DH/H-1.0/(RL(K)*H)
RM(1,LD2)= H2+1.0/(DH*H)+1.0/(RL(K)*H)
RM(1,L1L2)= H2
IF ( L .EQ. 1 ) GO TO 370
RM(1,LD1)= RR(K)*DH/H
GO TO 372
370 RM(1,LD1-1)= RR(K)*DH/H
372 IF ( I .EQ. MP ) GO TO 374
BM(1,LD1)= RR(K)*DH/H
GO TO 354
374 BM(1,LD1+1)= RR(K)*DH/H
GO TO 354
400 IF ( I .GE. 3+ME*MP2-MP ) GO TO 430
L=I-(2+(MB-1)*MP2)
C  BOUNDARY CONDITION AT INTERFACE
RM(1,LD)=GO/(FA*DH)-F/(ER*DH)
RM(1,L1L2)=F/(EB*DH)
RM(1,L1L2)=F/(EA*DH)
RM(1,LD1)= WS*FX*K*SN(L)/FB-WS/EA-WS/EB
IF ( L .EQ. MP ) GO TO 415
BM(1,LD1+1)= WS/EA+WS/EB
GO TO 420
415 BM(1,LD1+1)= WS/EA+WS/EB
420 GO TO 554
430 IF ( I .GE. 3+MP*MP2 ) GO TO 500
L=I-(2+MB*MP2-MP)
RM(1,LD)= QO/(FA*DH)-QO/(EB*DH)+B*W*FX*K*CS(L)/(C*EB)
BM(1,LD2) = GO/(ER*DH)
BM(1,LL2) = Q0/(EA*DH)
BM(1,LL1) = WS/EB+WS/EA
IF (L,EQ.,MP) GO TO 445
BM(1,LL1+1) = -WS/EB-WS/EA
GO TO 450
445 BM(1,LL2+1) = -WS/EB-WS/EA
450 GO TO 554
500 IF (I,GE,3*(M0-1)*MP2) GO TO 540
KK = (I-3)/MP2+1
K = 2+MP2*(KK-1)
IF (K,GT,MP) GO TO 520
KK = (I-3)/MP2+1
BM(1,LD2) = HAD(KK)
BM(1,LL2) = HAD(KK)+TR(KK)
BM(1,LD1) = W*XK*SN(L)/RL(KK)+W*XK*SN(L)/HA(KK+1)+W*XK*CS(L)/
6(RL(KK)*RP)
BM(1,LD3) = -W*XK*SN(L)/HA(KK+1)
IF (L,EQ.,MP) GO TO 505
BM(1,LD1+1) = -W*XK*CS(L)/(RL(KK)*RP)
BM(1,LD 1) = RR(KK)
GO TO 510
505 BM(1,LD 1) = -W*XK*CS(L)/(RL(KK)*RP)
BM(1,LD 1+1) = RR(KK)
GO TO 515
510 IF (L,EQ.,1) GO TO 515
BM(1,LD 1) = RR(KK)
GO TO 518
515 BM(1,LD 1-1) = RR(KK)
518 GO TO 554
520 L = K-PP
BM(1,LD2) = W*XK*SN(L)/RL(KK)+W*XK*CS(L)/(C*RL(KK))
6-Q0*HAM(KK)-2.0*RP(KK)*Q0-Q0*TR(KK)
RM(1,LD2) = Q0*HAM(KK)
RM(1,LD2) = Q0*(HA2(KK)+TR(KK))
RM(1,LD1) = W*XK*SN(L)/HA(KK+1)
RM(1,LD1) = W*XK*CS(L)/(RL(KK)*RP)
IF (L,EQ.,MP) GO TO 525
RM(1,LD 1+1) = W*XK*CS(L)/(RL(KK)*RP)
RM(1,LD 1+1) = Q0*RR(KK)
GO TO 530
525 RM(1,LD 2+1) = W*XK*CS(L)/(RL(KK)*RP)
BM(1,LD 1) = Q0*RR(KK)
530 IF (L,EQ.,1) GO TO 535
BM(1,LD 1-1) = Q0*RR(KK)
GO TO 538
535 RM(1,LD 1-1) = Q0*RR(KK)
538 GO TO 554
540 IF (L,GE,3+M0*MP2-MP) GO TO 545
BM(1,LD) = FB*FA*EA
GO TO 554
545 1 = I-(2+M0*MP2-MP)
RM(1,LD) = Q0/HA(M0)+XK*B*WS(L)/C
RM(1,LD2) = -Q0/HA(M0)
S2 = B/(C*RA*RP)
RM(1,LL1) = S2
IF (L,EQ.,MP) GO TO 550
A M  FINDIF  76/76  OPT=1  TRACE

```
RM(1,IL1+1)=-S2
GO TO 551
550 RM(1,IL2+1)=-S2
551 DO 552 JJ=1,MC
   RM(1,JJ)=-EB*BM(1, JJ)
552 CONTINUE
554 MAH=HA/2
   TF (MA, EQ. 2*MAH) D1=D1
556 CONTINUE
C END OF LOOP
C NOW HAVE FINAL DETERMINANT
   CALL EVALDT
   URITF(2,558) D1, ID, B
558 FORMAT(IX, F11.8, 5X, 15, 5X, F7.4)
C TAKING NEXT VALUE OF B
   GO TO 187
560 STOP
END
```
SUBROUTINE EVALDT
C SUBROUTINE TO EVALUATE FINAL DETERMINANT
DIMENSION BM(33,81)
COMMON BM, M1, M2, EPS, D1, ID, NC
NR=M1+1
XMAX=0.0
DO 18 I=1,NR
X=0.0
DO 16 J=1,NC
X=X+ABS(BM(I,J))
16 CONTINUE
IF (XMAX.LT.1.0) XMAX=X
18 CONTINUE
I=M1
DO 73 K=1,NR
K1=K+1
X=BM(K,1)
1=I
IF (I.GE.NR) GO TO 27
L=I+1
27 IF (K+1.GT.L) GO TO 34
C FINDING LARGEST ELEMENT IN COLUMN
DO 33 J=K1,L
IF (ABS(BM(J,1)).LE.ABS(X)) GO TO 33
X=BM(J,1)
J=I
33 CONTINUE
34 D1=X*0.01
IF (X.NF.0.0) GO TO 42
ID=0
BM(1,1)=XMAX*EPS
C SCALING REAL PART INTO REQUIRED RANGE
42 IF (ABS(D1).LT.1.0) GO TO 47
ID=ID+4
D1=0.0625*D1
GO TO 42
47 IF (ABS(D1).GE.0.0625) GO TO 52
ID=ID-4
D1=16.0*D1
GO TO 47
52 IF (I.EQ.K) GO TO 61
D1=D1
C SWAPPING I AND K ROWS TO BRING MAX ELEMENT TO LEADING POSITION
DO 59 J=1,NC
X=BM(K,J)
BM(K,J)=BM(I,J)
BM(I,J)=X
59 CONTINUE
61 IF (K+1.GT.L) GO TO 73
C SUBTRACTING MULTIPLES OF ITH ROW FROM REMAINING ONES
C TO MAKE FIRST ELEMENT ZERO IN REMAINING ROWS
DO 72 I=1,K
X=BM(I,1)/BM(K,1)
DO 69 J=2,NC
BM(I,J-1)=BM(I,J)-X*BM(K,J)
69 CONTINUE
BM(1,NC)=0.0
72 CONTINUE
73 CONTINUE
RETURN
END
APPENDIX 6

The propagation of plane waves in an infinite transversely magnetised ferrite medium.

Taking the z-axis in the direction of propagation the waves will have a dependence on space and time of the form \( \exp\{j(\omega t - \beta z)\} \).

Suppose that the static magnetic field is uniform in strength and direction and perpendicular to the direction of propagation. Let the y-axis be in the direction of the field. The permeability tensor anywhere in the medium will then have the form

\[
\begin{bmatrix}
\mu & 0 & j\kappa \\
0 & \mu_0 & 0 \\
-j\kappa & 0 & \mu
\end{bmatrix}
\]

Let the scalar permittivity of the medium be \( \epsilon \). Maxwell's equations yield

\[
\begin{align*}
 j\beta E_y &= -j\omega(\mu H_x + j\kappa H_z) \\
-j\beta E_x &= -j\omega\mu_0 H_y \\
0 &= -j\kappa H_x + \mu H_z \\
\beta H_y &= \omega\epsilon E_x \\
-\beta H_x &= \omega\epsilon E_y \\
0 &= E_z
\end{align*}
\]

i.e.

\[
\begin{pmatrix}
0 & j\beta & j\omega\mu & 0 & -j\kappa \\
-\beta & 0 & 0 & \omega\mu_0 & 0 \\
0 & 0 & j\kappa & 0 & -\mu \\
\omega\epsilon & 0 & 0 & -\beta & 0 \\
0 & \omega\epsilon & \beta & 0 & 0
\end{pmatrix}
\begin{bmatrix}
E_x \\
E_y \\
H_x \\
H_y \\
H_z
\end{bmatrix}
= 0
\]
For a non-trivial solution to this set of equations the determinant of the above matrix must equal zero

i.e. \[ [\beta^2 - \omega^2 \varepsilon \mu_0][\beta^2 \mu - \omega^2 \varepsilon (\mu^2 \kappa^2)] = 0 \]

Either \( \beta^2 = \omega^2 \varepsilon \mu_0 \) or \( \beta^2 = \omega^2 \varepsilon (\frac{\mu^2 - \kappa^2}{\mu}) \)

It follows that there are two independent solutions, one with \( \beta^2 = \omega^2 \varepsilon \mu_0 \) which is called the ordinary wave and is unaffected by the ferrite properties; the other with \( \beta^2 = \omega^2 \varepsilon (\frac{\mu^2 - \kappa^2}{\mu}) \) is called the extraordinary wave.

The fact that the waves have different propagation constants is termed birefringence.

The above result may be generalised. Again let the plane waves have a spatial and temporal dependence \( \exp\{j(\omega t - \beta z)\} \). Suppose that the static magnetic field \( H_0 \) is of constant strength and its direction at any point is in the plane perpendicular to the direction of wave propagation

i.e. \( H_0 = \{H_x(x,y), H_y(x,y), 0\} \) \( (H_x \text{ and } H_y \text{ assumed continuous}) \)

\( H_x^2 + H_y^2 = \text{constant} \).

Let the direction of the field at point \((x,y)\) be at an angle \( \alpha \) to the \( y \)-axis then

\[ \tan \alpha = \frac{H_x(x,y)}{H_y(x,y)} \]

A cartesian coordinate system \((x',y',z')\) may be defined as follows.

Let the direction of the \( y' \)-axis be the direction of \( H_0 \). Let the \( z' \)-axis be in the direction of the \( z \)-axis and let the \( x' \)-axis make up the right-hand system of coordinates.

Then

\[
\begin{bmatrix}
    x' \\
    y' \\
    z'
\end{bmatrix} = \begin{bmatrix}
    \cos \alpha & -\sin \alpha & 0 \\
    \sin \alpha & \cos \alpha & 0 \\
    0 & 0 & 1
\end{bmatrix}\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
\]
Since the static magnetic field is of constant strength and in the $y'$ direction, the permeability tensor $[\mu']$ in the $(x'y'z')$ system may be written in the form

$$[\mu'] = \begin{pmatrix}
\mu & 0 & j\kappa \\
0 & \mu_0 & 0 \\
-j\kappa & 0 & \mu
\end{pmatrix}$$

It follows that in the $(xyz)$ system the permeability tensor $[\mu]$ is given by

$$[\mu] = \begin{pmatrix}
\mu_{11} & \mu_{12} & \mu_{13} \\
\mu_{21} & \mu_{22} & \mu_{23} \\
\mu_{31} & \mu_{32} & \mu_{33}
\end{pmatrix} = \begin{pmatrix}
cosa & sina & 0 \\
-sina & cosa & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\mu & j\kappa & 0 \\
0 & \mu_0 & 0 \\
-j\kappa & 0 & \mu
\end{pmatrix} \begin{pmatrix}
cosa & -sina & 0 \\
sina & cosa & 0 \\
0 & 0 & 1
\end{pmatrix}$$

i.e. $[\mu] = \begin{pmatrix}
ucos^2\alpha + \mu_0\sin^2\alpha & (\mu_0-\mu)\sin\alpha\cos\alpha & j\kappa\cos\alpha \\
(\mu_0-\mu)\sin\alpha\cos\alpha & \mu\sin^2\alpha + \mu_0\cos^2\alpha - j\kappa\sin\alpha & 0 \\
-j\kappa\cos\alpha & 0 & \mu
\end{pmatrix}$

Proceeding as before Maxwell's equations yield

$$\beta E_y = -\omega(\mu_{11}H_x + \mu_{12}H_y + \mu_{13}H_z)$$
$$-\beta E_x = -\omega(\mu_{21}H_x + \mu_{22}H_y + \mu_{23}H_z)$$
$$0 = \mu_{31}H_x + \mu_{32}H_y + \mu_{33}H_z$$
$$\beta H_y = \omega E_x$$
$$-\beta H_x = \omega E_y$$
$$0 = E_z$$

For a non-trivial solution to this set of equations

$$\begin{pmatrix}
0 & \beta & \omega \mu_{11} & \omega \mu_{12} & \omega \mu_{13} \\
-\beta & 0 & \omega \mu_{21} & \omega \mu_{22} & \omega \mu_{23} \\
0 & 0 & \mu_{31} & \mu_{32} & \mu_{33}
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}$$
\[ \beta^4 \mu_{33} - \beta^2 \omega^2 \varepsilon [\mu_{33}(\mu_{11} + \mu_{22}) - (\mu_{13} \mu_{31} + \mu_{23} \mu_{32})] + \omega^4 \varepsilon^2 |\mu| = 0 \]

Substituting for the elements of \([\mu]\) gives

\[ \beta^4 \mu - \beta^2 \omega^2 \varepsilon [\mu(\mu_0 + \mu) - \kappa^2] + \omega^4 \varepsilon^2 \mu_0 (\mu^2 - \kappa^2) = 0 \]

i.e.

\[ (\beta^2 - \omega^2 \varepsilon \mu_0)(\beta^2 \mu - \omega^2 \varepsilon [\mu^2 - \kappa^2]) = 0 \]

There are two independent solutions; the ordinary wave with \( \beta^2 = \omega^2 \varepsilon \mu_0 \)

and the extraordinary wave with \( \beta^2 = \omega^2 \varepsilon \frac{(\mu^2 - \kappa^2)}{\mu} \).

Putting the angle \( \alpha \) to zero for all \( x \) and \( y \) gives the special case with \( \mathbb{H}_0 \) uniform in the \( y \)-direction.
APPENDIX 7

Notation

a  inner radius of waveguide
B  scaled propagation constant  $\beta^2 = B^2 \omega^2 \varepsilon_1 \mu_1$
b  radius of region 1
D1 $D1 \times 2^ID = |\mathbf{M}|$
Ea 1 - B^2
Eb $B^2 - \frac{\varepsilon \mu_0}{\varepsilon_1 \mu_1}$
H0 static magnetic field
H0 static magnetic field vector
ID see D1
Jn Bessel function of the first kind, order n
k0 $k_0^2 = \omega^2 \varepsilon_0 \mu_0 - \beta^2$
M matrix, the determinant of which when equated to zero gives the approximate characteristic equation
M0 static magnetisation
m number of radial arms in the mesh
m number of concentric circles in the mesh
Yn Bessel function of the second kind, order n
$\beta$ propagation constant
$\gamma$ gyromagnetic ratio
$\varepsilon_0$ permittivity of free space (permittivity of dielectric in chapters 2, 3 and 4)
$\varepsilon_1$ permittivity of region 1
$\varepsilon_2$ permittivity of region 2
$\varepsilon_T$ $\varepsilon_T = \frac{\varepsilon_2}{\varepsilon_1}$
$\kappa$ cross-diagonal component of the tensor permeability of region 2
$\mu_0$ permeability of free space (permittivity of dielectric in chapters 2, 3 and 4)
\( \mu_{ij} \) components of the permeability tensor \([\mu]\) given by equation (1.12)

\( \mu_1 \) permeability of region 1

\( \mu_2 \) diagonal component of the tensor permeability of region 2

\( \mu_r \) \( \frac{\mu_2}{\mu_1} \)

\( \omega \) angular frequency

\( \Rightarrow x + y \) means between the values \( x \) and \( y \)

(arrow)
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This thesis is concerned with the propagation of electromagnetic waves through circular cylindrical waveguide having perfectly conducting walls. A finite difference approximation method is used to evaluate the propagation constant of the waves. The method is one of great generality. It may be used for any coaxial configuration of media inside the waveguide.

In particular, the effects on propagating electromagnetic waves of a transversely magnetised ferrite tube adjacent to the waveguide wall are studied. Ferrite material is taken to have a permeability tensor of the form

\[
\begin{bmatrix}
\mu & -j\kappa & 0 \\
 j\kappa & \mu & 0 \\
0 & 0 & \mu_0
\end{bmatrix}
\]

when it is subjected to a static magnetic field along its third coordinate axis. The ferrite tube is subject to a static magnetic field formed by four magnetic poles at the corners of a square centred on the axis of the guide, like poles being at opposite corners. In the ferrite, this field leads to a permeability tensor which is dependent upon the angle in cylindrical polar coordinates when the z-axis is taken along the guide and Maxwell's equations reduce to two simultaneous second order partial differential equations with non-constant coefficients in the \( E_z \) and \( H_z \) components of the propagating electromagnetic wave. The finite difference approximation method reduces the problem to one of solving the condition for consistency of a large number of difference equations. Values of the propagation constant which satisfy this condition are found by a trial method which involves evaluating a determinant of very high order. This evaluation is carried out by computer and use is made of
the banded nature of the determinant to prevent the amount of computer store required becoming prohibitive.

The validity of the method is tested by applying it to several special cases with known results and its limitations and accuracy are discussed. A hypothesis is suggested to explain the numerical results.