SOME PROBLEMS IN
THE KINETIC THEORY OF PLASMAS

BY

M. C. TAPP.

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PREFACE

This thesis describes the work done by the author, in part at the Department of Mathematics at the University of Leicester, during the period 1969 to 1973.

Throughout this period the author has worked under the supervision of Dr. A. J. Willson whom he would like to thank for his help and encouragement. In particular the author would like to thank Dr. Willson for suggesting the topic undertaken in Chapter 2.

Finally the author is indebted to the Science Research Council for the award of a grant during the tenure of which part of this work was carried out.

This thesis is not substantially the same as one which has already been submitted at any other University.

M. C. Tapp
INTRODUCTION

This thesis covers essentially two problems in the kinetic theory of plasmas, covered respectively in Chapter 2 and Chapters 3 to 6. The first concerns the investigation of plasma oscillations in a constant electric field - a topic investigated by Akheizer and Sitenko as early as 1936 \cite{19}. More recently Stenflo \cite{22, 23, 26} has considered the problem in which he replaces the collision integral of the Boltzmann equation by a Fokker-Planck term and a B.G.K. \cite{25} term. In terms of the eigenfunctions of this operator he is able to perform an exact calculation of the dispersion relation. However, the relations he derived contained a number of parameters the relative importance of which Stenflo did not clearly define. Our analysis, we hope, goes some way in clarifying the physics of the problem and the work done by previous authors. The conclusions made by Stenflo and others regarding possible instabilities of the plasma needs modification, certainly in the context of a weakly ionised electron-ion gas.

The second topic covered here concerns the transport theory of relativistic gases. This has received increasing interest in recent years \cite{35, 36a-1}. Much attention has been devoted to calculating the first order relativistic effects on the transport coefficients. Up to now, only the 'Maxwellian' model, investigated by Israel \cite{35}, has been considered. The method of attack is via the Chapman-Enskog approach adapted to the needs of the relativistic Boltzmann equation. In this second topic we develop a more general approach to the problem by generalising the classical spherical harmonic solution of the Boltzmann equation to the relativistic case. The theory is applied to transport problems of fully ionised plasmas in the Coulomb field.

It is proposed now to give a brief summary of each of the chapters in turn.

Since the main theme of this thesis concerns the tensor expansion
solution of the Boltzmann equation, we devote Chapter 1 to a historical review and exposition of the classical theory. The steady state of a weakly ionised plasma in a uniform electric field is also discussed. Such a study is of interest in the discussion of many plasma problems including gas discharge physics, plasma heating and wave propagation.

Chapter 2 is concerned with the investigation of longitudinal plasma oscillations in a constant electric field. A discussion of this topic encompasses a wide variety of physical applications ranging, for example, from the propagation of radio waves through the ionosphere, to considering certain semiconductor effects associated with electron devices. It is our purpose here to consider the response of a weakly ionised gas to harmonic perturbations in the steady electric field. A dispersion relation is derived relating the angular frequency $\omega$ and the wave number $k$ ($2\pi$ divided by the wavelength) of longitudinal waves which vary harmonically in space and time. A general discussion of the relation is given together with some computed Nyquist diagrams for instabilities.

Chapter 3 is concerned with the development of the tensor (or equivalent spherical harmonic) expansion for the relativistic Boltzmann equation. The generalisation presents no immediate difficulty, following on directly from the classical theory given in Chapter 1. The first three moment equations for the relativistic case are derived.

Chapter 4 deals with the relativistic Boltzmann and Fokker-Planck equations. Further use of the tensor expansions developed in Chapter 3 can only be made if the collision term is specified. Here we discuss the interaction term given by Belyaev and Budker $^{[37]}$ for Coulomb plasmas and its derivation from impact theory by Akama $^{[38]}$. A discussion of the screening term, which represents the limit of Coulomb interactions, is given together with some invariance relations for collision terms.

In order to develop further the equations of Chapter 3 the
appropriate collision moments \( \sigma^{(l)} \) must be calculated. This we do in Chapter 5. This involves much analysis and labour in tensor manipulations - many of which we have relegated to appendices. On first reading, the reader can omit the actual derivations and proceed directly to special cases starting in \( \S 4 \).

Finally in Chapter 6 we give some applications of the theory to transport problems of the relativistic plasma. The basic aim is an investigation of the first order relativistic effects on the coefficients of conductivity, thermo-electricity and those associated with energy flow. Both classical and relativistic coefficients are tabulated for various degrees of ionisation of the plasma.

Throughout this work a rationalised M.K.S. system of units has been employed. Thus densities are measured per cubic metre, while electric fields \( E \) are measured in volts per metre. The units for other quantities where used, will be explicitly stated together with the notation usual in this system.
CHAPTER 1

CLASSICAL THEORY OF BOLTZMANN'S EQUATION
IN POLYADIC TENSOR FORM.

§1 INTRODUCTION.

The starting point of most work on gases and plasmas, in which the theory is handled statistically, is the Boltzmann transport equation. It is the one adopted here. We do not propose to give a general discussion of the equation as there are many excellent articles and books on the subject to which the reader may refer. In particular there is the article by Grad [1], and the books by Shkarofsky et al [2], Chapter 2, and Jeans [3], Chapter 8. Others include Chapman and Cowling [4], Chapter 3, and Clarke and McChesney [5]. Many other references given within these cited above can also be consulted. Our list is by no means exhaustive, but merely serves to indicate the wealth of literature available for reference in this field. Where needed, basic definitions and properties are given to maintain continuity within the text itself.

Much of our work in the following chapters involves the expanded form of the Boltzmann equation in spherical harmonics, or their equivalent polyadic tensor form, and results derived therefrom. Thus, because of its basic importance, we devote Chapter 1 to a review of the classical theory together with a derivation of the associated moment equations. In Chapter 3 we undertake a reformulation of the theory to the relativistic regime, while Chapters 4 and 5 follow on with the development of appropriate collision terms.

Historically the first step towards the development of the spherical harmonic expansion was taken by Lorentz in 1906 [6]. He was dealing with the conductivity of electrons in metals and employed a term equivalent to the first order spherical harmonic, \(\cos \theta\) - the ratio
of the electron velocity parallel to the field to the total velocity. A similar treatment was adopted by Pidduck in 1916 when he considered the problem of a slightly ionised gas permeated by a constant electric field. Pidduck's paper seems to have been the first to actually expand the Boltzmann equation to the first order in \[ \cos \theta \] - and was quite advanced, for its time, in the mathematical treatment. In 1935, Morse, Allis and Lamar applied this formulation to the plasma. Other workers in the field about this time were Davydov and Druyvesteyn. These treatments were all one-dimensional, with a steady electric field.

With the advent of microwave techniques more advanced experiments became possible to investigate the response of a plasma to time dependent fields. This pioneer work began during and in the Second World War when microwave research became important in the development of radar. The early leading investigators in this field were Allis and Brown working in America at the Massachusetts Institute of Technology. The group also included Margenau, Hartmann, Rosen, Van Zandt and Kelly. Allis eventually published a well organised exposition of the theory, using the compact vector form, in 1956.

In France, Jancel and Kahan and Bayet, Delcroix and Denisse obtained the same formulation apparently independently. Similar concepts had been developed earlier by Ikenberry, hinted at by Sommerfeld for gas transport theory, and given first by Wallace for neutron transport theory. Wallace seems to be the first author to have converted the spherical harmonic expansion into irreducible tensor form and vice versa. The general theory of harmonic and polyadic tensor expansions applied to the Boltzmann equation was expounded in a paper by Johnston in 1960. Since then, it has been much used in discussing transport properties of gases and plasmas (e.g. Shkarofsky et al., Chapter 8).

That, briefly, gives an historical development of the subject.
In this chapter we consider in some detail the spherical harmonic expansion and list some useful tensor definitions and relations in the appendices for reference. § 2 is concerned with some preliminary remarks on the velocity distribution function and the derived principal gas parameters as averages over velocity space. In § 3 we return to the problem of solving Boltzmann's equation and in this connection state how the expansion in velocity angle space can be converted to a fully symmetric tensor form. This is then substituted into the Boltzmann equation and the first three tensor moment equations derived. In § 4 the problem of a weakly ionised gas permeated by a uniform electric field is discussed. The electron-velocity distribution in the steady state is given and subsequently used in Chapter 2 for discussing plasma oscillations when the system is perturbed by small-amplitude plane waves.

Before beginning the exposition we should indicate as to why the spherical harmonics and their related tensors are used at all. Firstly, for convenience we want a set of orthogonal expansion functions. When particles are scattered by particles of comparable mass, with an approximately Maxwellian distribution in velocities, the orthogonality property is provided by Laguerre polynomials with spherical harmonics. An example of the use of these functions for electron-electron interactions is given in Chapter 6 when we discuss the problem of relativistic transport phenomena. For electron-molecule collisions the dominant feature is an angular deflection with very little change in the electron's speed. We thus look for such functions in velocity angle space, viz spherical harmonics.

The drift velocity, in most situations proves to be relatively small and the distribution function is mostly spherical in velocity space, i.e. nearly independent of direction. Thus the distribution function can be well represented by the first two terms in the expansion
\[ f = f_0(v) + f_1(v) \frac{v_z}{v} + \ldots, \]

where \( f_0 \) and \( f_1 \) are functions of \( v \), the magnitude of \( v \). The z-axis of the system has been taken in the direction of the ensemble-averaged directed velocity \( \overline{v} \).

Finally we give a condition as to the validity and convergence of this expansion. For small drift velocities we require the electric field to be not too strong. Then the directed velocity gain between collisions (the acceleration times \( \gamma^{-1} \), where \( \gamma \) is an average collision frequency) must be much less than the thermal velocity \( v_T \), i.e.

\[ \left( \frac{eE}{mv_T} \right)^2 \ll 1. \]

For small fields the condition is automatically satisfied, while for strong fields the condition for electron-molecule scattering essentially reduces to

\[ \frac{m}{M_a} \ll 1. \]

where \( m \) and \( M_a \) are the masses of electron and molecule respectivity. This latter condition is well satisfied for most cases of interest to us.

**DEFINITIONS AND REMARKS.**

Before proceeding with the exposition of the tensor equations we devote this section to a brief summary of the role of the distribution function in gas kinetic theory. This will serve as both a review and a reference for the definitions of the principal plasma parameters which are used in the subsequent chapters. For more detailed information, the reader may consult any of the texts cited in the introduction.
Fundamentally, the motion of each microsystem (or particle) can be represented by the motion of a 'representative point' in six-dimensional phase-space (sometimes called μ-space). μ-space is simply made up of the configuration co-ordinates (J) and the velocity co-ordinates (V), so that a particle is represented by a particular point (J, V) in this space. The distribution function f (a scalar function of position, velocity and time) in essence determines the density of points in this μ-space for a particular gas species.

Denoting an element of volume of this space by

\[ d^3J d^3V = dx dy dz dv_1 dv_2 dv_3, \]

in standard notation, then \( f(J, V, t) d^3J d^3V \) represents the number of microsystems of given species which have velocities in the range \( d^3V \) about \( V \) and position co-ordinates in the range \( d^3J \) about \( J \). Naturally, this concept is only well defined if the smallest volume considered contains sufficient particles to make the operation continuous. For our purposes, densities are such that this is always so.

The number density \( n \) at a given position is obtained by adding together all the particles there regardless of velocity, i.e. integrating over \( d^3V \); this gives

\[ n = \int f d^3V. \]

Similarly the velocity average \( \bar{\phi} \) of some molecular property \( \phi(J, V, t) \) of the gas species is defined by the statement

\[ \bar{\phi} = \int \phi f d^3V / n. \]

It is of course a function of \( J \) and \( t \) only.

In dealing with plasma problems, three principal parameters are involved, namely current, charge density and temperature. For the net charge \( \rho \) in a multicomponent plasma we have
\[
\rho = \sum_i n_i e_i = \sum_i e_i \int f_i \, d^3 v_i , \]
where \( n_i , e_i , f_i \) and \( v_i \) are respectively the number density, particle charge, distribution function, and velocity for the \( i \) th species. The current is

\[
\mathbf{J} = \sum_i n_i e_i \mathbf{v}_i = \sum_i e_i \int v_i f_i \, d^3 v_i .
\]
Finally the temperature \( T_i \) of the \( i \) th species, is defined by

\[
\frac{3}{2} K T_i = \frac{1}{2} m_i \mathbf{\bar{v}}_i^2 - \frac{1}{2} m_i (\mathbf{\bar{v}}_i^2) ,
\]
where \( K \) is Boltzmann's constant and \( m_i \) is the mass of the particles. \( T_i \) is defined as a measure of the difference between the total energy of the system \( \frac{1}{2} m_i \mathbf{\bar{v}}_i^2 \) and the directed energy \( \frac{1}{2} m_i (\mathbf{\bar{v}}_i^2) \).

In the subsequent work, the above definitions are used frequently in the development of transport properties and the study of plasma waves.

\[ \S 3 \quad \text{BOLTZMANN'S EQUATION IN POLYADIC TENSOR FORM.} \]

In this section we shall consider the general distribution function, \( \mathbf{f}_e \), for electrons, which is a scalar function of position \( \mathbf{r} \), velocity \( \mathbf{v} \) and time \( t \). Denoting an element of volume in phase-space by

\[
d^3 \mathbf{r} d^3 \mathbf{v} = dx dy dz dv_x dv_y dv_z ,
\]
where \( x, y, z \) are the cartesian co-ordinates and \( v_x, v_y, v_z \) the cartesian components of velocity; then

\[
\mathbf{f}_e (\mathbf{r}, \mathbf{v}, t) \, d^3 \mathbf{r} d^3 \mathbf{v}
\]
represents the number of electrons with velocities in the range \( \mathbf{v} \) to \( \mathbf{v} + d^3 \mathbf{v} \) and position co-ordinates in the range \( \mathbf{r} \) to \( \mathbf{r} + d^3 \mathbf{r} \).
Boltzmann's equation is (e.g. see Shkarofsky \(\text{[2, Chapter 2]}\))

\[
\frac{\partial f_e}{\partial t} + \mathbf{v} \cdot \nabla f_e + \frac{e}{m} \mathbf{E} \cdot \nabla f_e = C. \tag{1}
\]

In (1) \(C\) represents the collision terms, \(\mathbf{F}\) the Lorentz force on an electron, \(m\) is the electron mass, and \(\nabla_r\) and \(\nabla_v\) are respectively the spatial and velocity gradient operators. Writing \(\omega = \frac{eB}{m}\) and \(\omega_c = \frac{eB}{m}\) (so that \(|\omega_c|\) is the cyclotron frequency and \(e\) the electronic charge), the Boltzmann equation takes the form

\[
\frac{\partial f_e}{\partial t} + \mathbf{v} \cdot \nabla f_e + \left[\frac{\omega_c^2}{\epsilon} + (\mathbf{v} \times \omega_c)\right] \cdot \nabla f_e = C. \tag{2}
\]

Now the most general expansion for the function \(f_e\) in velocity space is

\[
f_e(\mathbf{r}, \mathbf{v}, t) = \sum_{l,m,s} f_{l,m,s} (\mathbf{r}, \mathbf{v}, t) Y_{l,m,s} (\theta, \phi), \tag{3}
\]

where \(Y_{l,m,s} (\theta, \phi)\) is the spherical harmonic of order \(l\) and the \(f_{l,m,s}\) coefficients are functions of position \(\mathbf{r}\), time \(t\) and the velocity magnitude \(\mathbf{v}\) only. The spherical harmonic is defined \((\text{[2,7}, \text{p.77 et seq})\) by the equation

\[
Y_{l,m} (\theta, \phi) = P^m_l (\cos \theta) (\delta_{0s} \cos m \phi + \delta_{1s} \sin m \phi). \tag{4}
\]

\((\mathbf{v}, \theta, \phi)\) are the spherical components of the velocity \(\mathbf{v}\) and \(P^m_l (\cos \theta)\) is the associated Legendre function. \(\delta_{ij}\) is the usual kronecker delta and the subscripts \(l, m, s\) are subject to the conditions:

\[
l = 0, 1, 2, \ldots, \quad 1 \leq m \leq l, \quad s = 0, 1 \text{ AND } P^s_l (\cos \theta). \tag{5}
\]

The orthogonality relation for these functions is

\[
\int_0^{2\pi} \int_0^\pi Y_{l,m,s} (\theta, \phi) Y_{l',m',s'} (\theta, \phi) \sin \theta \, d\theta d\phi = \frac{2\pi (l+m)!(1+\delta_{ss'})}{(2l+1)(2l'-1)!} \delta_{ll'} \delta_{mm'} \delta_{ss'}. \tag{5}
\]
It can be shown, Appendix 2 this chapter, that the expression (3) may be written in the form

$$f_{l} = \sum_{i} \int_{i}^{(l)} \frac{V_{L}}{V_{k}} \cdot \ldots$$

(6)

The \( l \) th term on the right hand side of (6) is the "\( l \) th order dot product" of the symmetric tensors \( \int^{l} \) and \( \sqrt{V_{k}} \). For explicit definitions and examples see Appendix 1. The reader is referred to the references \( [27] p. 109 \) et seq and especially \( [15] \) for further analysis of the theory.

A comparison of (6) and (3) shows

$$\sum_{m,s} f_{m} Y_{m} = \int_{i}^{(l)} \frac{V_{L}}{V_{k}} \cdot \ldots$$

(7)

Writing the general element of \( \int^{(l)} \) as \( \int_{i}^{(l)} \), with \( l \) suffices implied, the right hand side of (7) is explicitly

$$\int_{i}^{(l)} \frac{V_{i} V_{j} V_{k}}{V_{l} V_{m}} \cdot \ldots$$

\( V_{i} \) is the \( i \) th cartesian component of \( \sqrt{V} \) (\( i=1,2,3 \) only).

The \( \int^{(l)} \) tensors are composed of the \( f_{m} \) terms of order \( l \) only. Further, it may be shown that the operation of contraction (equating two subscripts and summing) gives zero. An explicit proof of symmetry and contraction properties is given in Appendix 2.

Essentially we can start with equation (7) and derive a relation for the general element \( \int_{i}^{(l)} \) of \( \int^{(l)} \) in terms of the \( f_{m} \) elements. In order to ensure uniqueness we impose the restriction that \( \int^{(l)} \) be a fully symmetric tensor - that is interchanging any two subscripts yields the same result. Furthermore, from the symmetry property and the fact that \( \sqrt{V} Y_{m} \) satisfies Laplaces equation, we may show that the operation of contraction on \( \int^{(l)} \) gives zero.
The case \( l = 0 \) is trivial. \( f_0^{(2l)} \) is then just a scalar \( f_{00} \).

Consider now the case \( l = 1 \). \( f_0^{(1)} \) is then a cartesian vector, and we can write, using \( x, y \) and \( z \) suffices for convenience,

\[
\mathbf{f}_0^{(1)} = (f_x^{(1)}, f_y^{(1)}, f_z^{(1)}).
\]

From (7)

\[
\sum_{n,s} f_n^{(1)} Y_n^{(1)} = f_x^{(1)} V_x + f_y^{(1)} V_y + f_z^{(1)} V_z.
\]  

But,

\[
Y_{00} = \cos \Theta, \quad Y_{10} = 0, \\
Y_{11} = \sin \Theta \cos \phi, \quad Y_{22} = \sin \Theta \sin \phi,
\]

where \( \Theta \) and \( \phi \) are the usual polar angles in velocity space. Therefore, substituting into the left hand side of (8) gives

\[
V_x f_{00}^{(1)} + 0 f_{10}^{(1)} + V_y f_{11}^{(1)} + V_z f_{12}^{(1)}.
\]

Since this is identically equal to

\[
f_x^{(1)} V_x + f_y^{(1)} V_y + f_z^{(1)} V_z,
\]

we deduce at once that

\[
f_x^{(1)} = f_{x0}^{(1)}, \quad f_y^{(1)} = f_{y1}^{(1)}, \quad f_z^{(1)} = f_{z0}^{(1)}.
\]

Now consider the case \( l = 2 \). \( f_0^{(2)} \) is now a 3x3 matrix and we can write

\[
f_0^{(2)} = \begin{bmatrix} f_{i,j}^{(2)} \end{bmatrix}, \quad i, j = 1, 2, 3.
\]

Then from (7)

\[
\sum_{n,s} f_n^{(2)} Y_n^{(2)} = \sum_{n,s} f_n^{(2)} Y_n^{(2)} = \sum_{n,s} f_n^{(2)} Y_n^{(2)} = \sum_{n,s} f_n^{(2)} Y_n^{(2)}.
\]
If we define \( \cos \theta_x \), \( \cos \theta_y \) and \( \cos \theta_z \) by the relations:

\[
\begin{align*}
\cos \theta_x &= \cos \theta, \quad \cos \theta_y = \sin \theta \sin \phi, \quad \cos \theta_z = \sin \theta \cos \phi,
\end{align*}
\]

then it is easy to show that the five spherical harmonics of order 2 can be written as

\[
\begin{align*}
\gamma_{200} &= \cos^2 \theta_x - \frac{1}{2} \cos^2 \theta_x - \frac{1}{2} \cos^2 \theta_y, \\
\gamma_{220} &= 3 \left( \cos^2 \theta_x - \cos^2 \theta_y \right), \\
\gamma_{210} &= 3 \left( \cos \theta_z \cos \theta_x \right), \\
\gamma_{211} &= 3 \left( \cos \theta_z \cos \theta_y \right), \\
\gamma_{221} &= 6 \left( \cos \theta_y \cos \theta_x \right).
\end{align*}
\]

Inserting these values in the left hand side of equation (7) (for \( J=2 \)), and using the definitions \( \forall \cos \theta_x = \gamma_x \), \( \forall \cos \theta_y = \gamma_y \) and \( \forall \cos \theta_x = \gamma_x \) we find, by equating coefficients of \( \gamma_x \gamma_y \) etc,

\[
\left[ f^{(2)}_{i,j} \right] = \begin{bmatrix}
-\frac{1}{2} f_{200} + 3 f_{220} & 3 f_{210} & 3 f_{211} \\
3 f_{221} & -\frac{1}{2} f_{200} - 3 f_{210} & 3 f_{211} \\
\frac{3}{2} f_{210} & \frac{3}{2} f_{211} & f_{200}
\end{bmatrix}
\]

Note \( f^{(2)}_{i,i} = 0 \). In deriving these expressions for the elements of \( f^{(2)} \) we have used the fact that it is a symmetric matrix.

For understanding the bulk of the work in the remaining chapters, the reader needs a knowledge of the \( f^{(2)} \) and \( f^{(1)} \) terms only.

It is assumed that the collision term \( C \), a function of \( \gamma \), \( \gamma \) and \( t \), can similarly be expressed in the expanded form.
The procedure to obtain the moment equations is to substitute the expansions (6) and (9) into Boltzmann's equation (1), multiply through by the tensor $\frac{v^{(l)}}{V^l}$ and integrate over velocity angles letting $l$ take on successively the values 0, 1, 2, ... The disadvantage of the spherical harmonic expansion in this respect, is that when multiplying by $Y^l_{l^1}^{m^1}$ and integrating over angles, terms like $\frac{\partial}{\partial \theta} (Y^l_{lm}) Y^l_{l^1m^1}$ are involved, for which no simple recursion relation exists.

Using the results

$$C = \sum_{l} \frac{C^{(l)}_{i}}{l} \frac{v^{(l)}}{V^l}, \quad (9)$$

\[ \frac{\partial}{\partial t} \left\{ f^{(l)}_{i} \frac{v}{V^l} \right\} = \frac{\partial f^{(l)}_{i}}{\partial t} \frac{v}{V^l}, \]

\[ V^l_{i} \frac{\partial}{\partial x_j} \left\{ f^{(l)}_{i} \frac{v}{V^l} \right\} = V^l_{i} \nabla \cdot \int_{\frac{v}{V^l}} f^{(l)}_{j} \frac{v}{V^l} \frac{v^{(l)}}{V^{l+1}}, \]

\[ (0 \cdot \nabla) \left\{ f^{(l)}_{i} \frac{v}{V^l} \right\} = V^l_{i} \frac{\partial}{\partial v} \left( f^{(l)}_{i} \frac{v}{V^l} \right) \frac{v^{(l+1)}}{V^{l+1}}, \]

\[ + \frac{\partial a}{\partial v} f^{(l)}_{i} \frac{v^{(l-0)}}{V^{l-1}}, \]

AND

\[ V \wedge \omega \cdot \nabla \left\{ f^{(l)}_{i} \frac{v}{V^l} \right\} = l \left( \omega \wedge f^{(l)}_{i} \right) \frac{v^{(l)}}{V^l}, \]
Boltzmann's equation becomes

$$\sum_{l=0}^{\infty} \left\{ \frac{\partial f}{\partial t} + \nabla \cdot \left( \frac{\partial f}{\partial V} \right) + \frac{1}{\sqrt{V}} \nabla \cdot f^{(l)} \right\} \cdot \frac{V^{(l)}}{\sqrt{V}} = 0 \quad (10)$$

In (10) the second, third, and fifth terms are summed for $l \geq 1$. Furthermore the quantity $\frac{V^{(l)}}{V}$ is defined as being equal to unity.

Multiplying (10) through by $\sin \theta d \theta d \phi$ and integrating over angles, using the relation:

$$\int_0^{2\pi} \int_0^{\pi} \cos^2 \theta x \cos^2 \theta y \cos^2 \theta z \cdot \sin \theta d \theta d \phi$$

we obtain the scalar equation

$$\frac{\partial f^{(0)}}{\partial t} + \nabla \cdot f^{(0)} + \frac{1}{3} \nabla \cdot \left( \sqrt{V} \frac{\partial f^{(0)}}{\partial V} \right) = C^{(0)} \quad (11)$$

Contributions to (11) come from the $l=0$ and $l=2$ terms in (10) only. Terms that are odd in the components of $\nabla$ integrate to zero, and, for a fully symmetric tensor $A^{(2l)}$

$$\int_0^{2\pi} \int_0^{\pi} A^{(2l)} \cdot \frac{V^{(2l)}}{\sqrt{V}} \sin \theta d \theta d \phi = \phi,$$

if the operation of contraction on $A^{(2l)}$ gives zero.
Next we multiply (10) by \( \frac{\nu^{(2)}}{\sqrt{V}} \) and integrate over angles obtaining

\[
\frac{\partial f^{(0)}}{\partial t} + \nabla \cdot f^{(0)} + \frac{\partial f^{(6)}}{\partial V} + \nu \cdot \nabla \cdot f^{(3)} + \frac{2}{5} \nabla \cdot f^{(2)} = \frac{2}{5} \frac{\partial}{\partial V} \left( \nu^3 \frac{\partial f^{(0)}}{\partial V} \right) = Z^{(0)}.
\]

(12)

Terms in (12) come from the \( l=1 \) and \( l=3 \) terms of (10) only.

The results

\[
\int_0^{2\pi} \int_0^{\pi} V \nabla_{\phi} f^{(2)} \frac{V}{V^2} \sin \theta d\phi d\theta = \frac{2}{15} \cdot 4\pi V \nabla_{\phi} f^{(2)}
\]

and

\[
\int_0^{2\pi} \int_0^{\pi} V^2 \frac{\partial}{\partial V} \left( \frac{f^{(2)}}{V^2} \right) \frac{V}{V^2} \sin \theta d\phi d\theta = \frac{2}{15} \cdot 4\pi V^2 \frac{\partial}{\partial V} \left( \frac{\partial f^{(2)}}{V^2} \right)
\]

are found useful, together with the general result

\[
\int_0^{2\pi} \int_0^{\pi} A^{\frac{(2l-1)}{2L}} \frac{V^{(2L)}}{V^{2L}} \sin \theta d\phi d\theta = 0, \quad (l \geq 2),
\]

for a fully symmetric tensor upon which the operation of contraction is zero.

Again, for the 3rd moment equation, we multiply (10) by \( \nu^{(2)} \sqrt{V} \) and integrate over angles to yield the final result.
\[
I^{(2)} = \left[ \frac{5}{2} \frac{\partial f^{(6)}}{\partial \epsilon} - \frac{5}{2} C^{(6)} + \frac{1}{2} \nabla f^{(1)} + \frac{1}{2\sqrt{V}} \frac{\partial}{\partial V} \left( \frac{\partial f^{(1)}}{\partial V} \right) \right]_2^{(2)} \nonumber \\
+ \frac{\partial \mathbf{f}^{(2)}}{\partial \epsilon} + \sqrt{\frac{2}{V}} \frac{\partial}{\partial V} \left( \frac{\mathbf{f}^{(1)}}{V} \right) + \sqrt{\nabla f^{(1)}}^{(1)} + 2 \mathbf{\omega} \cdot \mathbf{f}^{(2)} \\
+ \frac{3}{7} \left\{ \nabla \mathbf{\omega} \cdot \mathbf{f}^{(3)} + \nabla^{4} \frac{\partial}{\partial V} \left( \nabla^{4} \mathbf{f}^{(3)} \right) \right\} - C^{(2)}_2 = 0.
\]

In equation (13), \( I^{(2)}_2 \) is the 3x3 unit matrix. \( \frac{1}{2} \left[ \mathbf{A}^{(2)} + \mathbf{A}^{(2)}^T \right] \) and \( T \) denote the usual matrix transpose.

Contributions to (13) come from the \( J = 0, 2 \) and \( 4 \) terms of (10) only. The results

\[
\int \int_{0}^{2\pi} \int_{0}^{\pi} A^{(2)}_{\omega} \cdot \nabla^{(1)}_{\omega} \sin \theta \cos \phi \ d\theta \ d\phi = \frac{4\pi}{15} \left\{ \mathbf{A}^{(1)} + \mathbf{A}^{(2)} + \left( \mathbf{A}^{(2)} - \mathbf{A}^{(1)} \right) \mathbf{I}^{(2)}_2 \right\},
\]

for any tensor \( \mathbf{A}^{(2)} \), and

\[
\int \int_{0}^{2\pi} \int_{0}^{\pi} A^{(1)}_{\omega} A^{(3)}_{\omega} \cdot \nabla^{(4)}_{\omega} \sin \theta \cos \phi \ d\theta \ d\phi = \frac{8\pi}{35} \mathbf{A}^{(1)}_{\omega} \mathbf{A}^{(3)}_{\omega}
\]

for any symmetric tensor \( \mathbf{A}^{(3)} \) which gives zero upon contraction, are found useful.

Although, in subsequent chapters, the 3rd moment equation will not be used, we feel it is worthwhile to include here in order to demonstrate the extreme complexity of the moment equations as they are unravelled beyond \( J = 1 \). In most of the subsequent work we shall be dealing with the \( \mathbf{f}^{(2)} \) and \( \mathbf{f}^{(1)} \) terms only, truncating the higher order tensors in the series. The \( \mathbf{f}^{(2)} \) equation will be rederived in relativistic form in Chapter 3.

\section{EQUILIBRIUM STATE IN A UNIFORM ELECTRIC FIELD.}

In this section we briefly describe the application of equations
(11) and (12) to the problem of a weakly ionised gas permeated by a constant electric field. There have been numerous investigations of this situation, the topic being first investigated by Pidduck in 1916. For a comprehensive account of the historical development of this problem the reader is referred to the article in Loeb's book, and also p. 345 et seq.

It is necessary to amplify our restriction to slightly ionised gases used above. In a gas mixture of neutral molecules, electrons and positive ions, the various intermolecular forces brought into play during collisions will depend on the particles involved. Between charged particles the classical law of interaction is the Coulomb inverse square law; whereas between electron and molecule, scattering is governed by some other law, which it is reasonable to suppose is a short range force. Now it is an essential feature of Boltzmann's equation that only short range forces can be dealt with in the collision terms, without additional assumptions being invoked. Thus in application to ionised gases, we can expect good results only where Coulomb interactions are an inessential feature of the general collision process. In this respect, then, we confine our attention to an ionised gas where the electron density \( n_e \) is much less than the neutral molecule density \( N_m \). An electrically neutral situation in which the electron and positive ion charge densities are equal, is assumed.

In reality Coulomb forces are 'screened' at large distances, and it is possible to incorporate such features as 'Debye-Screening' or a more simplified 'collision cut-off' into the theory. The second method is discussed in Chapter 4 in the application of the relativistic Boltzmann equation to plasmas.

The determination of the properties of a gas or plasma in uniform fields of any strength is of interest in the discussion of various problems, concerning for example, wave propagation, plasma heating, and gas discharge physics. The steady state situation discussed in
this chapter will be used to investigate plasma oscillations when the electric field is slightly perturbed.

Let us return to the physics of the problem in hand. Due to widely differing masses of electron and molecule, in an elastic collision (one in which particle identities are unaltered) the electron can transfer only a small fraction of its energy to the molecule. Thus in a strong electric field, a slowly moving electron will gain far more energy from the field during a free path (distance between collisions) than it can lose elastically at an encounter. The mean energy of an electron thus grows until, in equilibrium, an energy balance is restored. Thus in a steady-state, the mean energy of the electrons will be much larger than the thermal (molecular) energy. The electronic temperature \( T_e \) will greatly exceed the molecular temperature \( T \) and the electrons become 'heated'.

Since under the above assumptions, the effect of a collision is to randomise the electron velocity, the electron distribution function \( f_e \) can be assumed very nearly isotropic in velocity space. On account of their much greater mass, the velocity distribution function \( f_n \) for molecules can be taken as the Maxwellian form for a gas at rest, namely

\[
N_n \left( \frac{M_n}{2\pi kT} \right)^{3/2} \exp \left( -\frac{M_n v^2}{2kT} \right)
\]

Here \( M_n \) and \( v \) are the molecular mass and speed respectively. With the situation steady in time and uniform in space the first two moment equations to determine \( f_e \) to the first order reduce to:

\[
\begin{align*}
\ell &= 0, & \frac{eE}{3nv^2} \frac{\partial}{\partial v} \left( v^2 f_e \right) &= C^{(e)}, \\
\text{AND} & \quad \ell &= 1, & \frac{eE}{m} \frac{\partial f_e^{(e)}}{\partial v} &= C^{(i)}.
\end{align*}
\]

(14a-b)
with the neglect of \( \int_{\gamma}^{(2)} \) and higher terms in the expansion for \( f_e \).

This is well justified if \( f_e \) is very nearly isotropic in electron velocity space.

Because the electric field is uniform, there is only one preferred direction, and for convenience we choose the z-axis of a fixed cartesian frame parallel to \( \mathbf{E} \). Thus we write \( \mathbf{E} = (0,0,E) \), and by axial symmetry,

\[
\bar{f}_e(v_x, v_y, v_z) = \bar{f}_e(v, v_x / \sqrt{v}) = \bar{f}_e(v, \cos \theta),
\]

where \( v \) is the magnitude of the velocity \( \mathbf{v} \). The expansion for \( \bar{f}_e \) to the first order is written as

\[
\bar{f}_e = \bar{f}_e^{(2)} + \bar{f}_e^{(1)} / v.
\]

The collision terms \( C^{(0)} \) and \( C^{(1)} \) in equations (14a, b) have the following forms \( \gamma^2, \text{p. 81 et seq.}, \gamma^4, \text{p. 345 et seq.} \):

\[
C^{(0)} = \frac{1}{v^2} \frac{\partial}{\partial v} \left\{ \frac{v^2 \gamma(v) K \pi}{M_n} \left[ \frac{\partial \bar{f}_e^{(0)}}{\partial v} + \frac{nv}{K \pi} \bar{f}_e^{(0)} \right] \right\},
\]

\[
C^{(1)} = - \gamma(v) \bar{f}_e^{(1)},
\]

AND

\[
\gamma(v) = 2N_n v \pi \int_0^\pi (1 - \cos \chi) \tilde{I}(v, \chi) d\chi.
\]

Here \( \gamma \) is an effective collision frequency, \( \tilde{I}(v, \chi) \) the cross-section for electron-molecule scattering, and \( \chi \) the angle of scatter.

Finally we derive explicit forms of the distribution function for the general case \( \gamma = \gamma(v) \), and in particular for constant \( \gamma \).
Returning to equations (14a-b), we see that from (14b) (with a co-ordinate system as indicated above),

\[ \int f^{(1)} = -\frac{eE}{m_e} \frac{d}{dv} f^{(0)} \]  

using \( c^{(1)} \) as given by (16b). Hence on elimination of \( f^{(1)} \) between (14a) and (14b) we obtain, with \( c^{(0)} \) given by (16a),

\[ f^{(0)} = A \exp \left\{ -\int_0^v \frac{mzdz}{kT + \left( \frac{eE}{n} \right)^2} \right\} \]

and

\[ \int f^{(1)} = \frac{eE}{\sqrt{v}} \frac{f^{(0)}}{\sqrt{v} (kT + \left( \frac{eE}{n} \right)^2)} \int f^{(0)} \]

\( A \) is a constant of integration to be determined by the condition

\[ 4\pi \int_0^\infty f^{(0)} v^2 dv = n \]

the electron number density.

The special case \( \sqrt{v} = \) constant reduces the distribution to the Maxwellian form, but with an enhanced electron temperature. For this case, integration of (18) is trivial giving

\[ f^{(0)} = n \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{mv^2}{2kT}} \]

\[ \text{and} \]

\[ f^{(1)} = \frac{eE \sqrt{v}}{\sqrt{kT}} \int f^{(0)} \]
where 

$$T^* = T + \left( \frac{eE}{mV} \right)^2 \frac{M_n}{3K} \tag{19c}$$

is the effective electron temperature.

Other results can be derived using different forms for the collision frequency $\nu$ when this depends on the speed $V$. In Chapter 2 the problem of longitudinal oscillations in a weakly ionised gas permeated by a constant electric field is discussed. The steady state situation is described by the distributions $(i a-c)$, these being the easiest to handle in the subsequent derivation of the dispersion relation.

The condition for validity of the expansion, namely

$$\left( \frac{eE}{mV^2} \right)^2 \ll 1,$$

with $V_c \sim \sqrt{\frac{kT^*}{m}}$ and $T^*$ given by (19c), essentially reduces to

$$\frac{m}{M_n} \ll 1$$

in strong fields.

This is well satisfied for electron-neutral scattering which is the predominant collision process with which we are at present concerned.

For further reference on the topic of weakly ionised gases, the reader may consult Shkerofsky et al. [2, Chapter 4], and Chapman and Cowling [4], p. 345 et seq.

APPENDIX 1.

In §3 use is made of some tensor relations and definitions. Chapters 3, 4 and 5 also draw heavily on tensor theory and manipulations. Thus for reference we give some of the more basic formulae and ideas.

The cartesian tensor of order $\tau$ in an $n$-dimensional space is a set of $\bar{n}^\tau$ quantities each associated with one of the $\bar{n}^\tau$ distinct orderings of the $n$-co-ordinate axes taken $\tau$ at a time. Typically an
element of this tensor is written as

\[ A_{i_1 \cdots i_r}^{j_1 \cdots j_s} \]

where \( j_k (k=1, \ldots, r) = 1, 2, \ldots, n \) in any order (repeats being allowed), and \( \hat{z}_k \) is the unit vector along the \( k \)th co-ordinate axis of the system. The tensor is written as \( A^{(r)} \) and its rank (number of subscripts) is \( r \). For \( r = 0 \) it is a scalar, \( r = 1 \) an \( n \)-dimensional vector, and \( r = 2 \) it can be represented by an \( nxn \) matrix.

As a generalisation of the familiar scalar product for vectors, we define the multiple scalar product of two fully symmetric tensors as

\[ \tilde{A}^{(\tau)} \cdot \tilde{B}^{(s)} = A_{i_1 \cdots i_r} B_{j_1 \cdots j_s} \]

Here \( \tau \) indicates \( \tau \) suffices have been paired symmetrically around the junction; repeated suffices on components have been summed (from 1 to \( n \) for an \( n \)-dimensional system), and the corresponding unit vectors have been dropped. The ordering system employed here is only well defined for completely symmetric tensors. Since we are concerned only with these anyway our remarks will be limited to this case. The rank of (2) is \( \tau + s - 2 \rho > 0 \). We give some examples to illustrate this operation.

1. For two vectors \( \tilde{A}^{(1)} \) and \( \tilde{B}^{(1)} \) we have the usual scalar product

\[ \tilde{A}^{(1)} \cdot \tilde{B}^{(1)} = \sum_{i} A_i B_i \]

2. \[ \tilde{A}^{(2)} \cdot \tilde{B}^{(2)} = \sum_{i, j} A_{ij} B_{ij} \]

and

3. \[ \tilde{A}^{(3)} \cdot \tilde{B}^{(3)} = \sum_{i, j, k} A_{ijk} B_{ijk} \]
For the examples (2), (3) and (4) the tensors used have been assumed fully symmetric.

A useful tensor is the $2r$-order identity tensor

$$\mathbf{I} = \delta_{i,j} \delta_{k,l} \cdots i_i j_j \cdots i_i j_j \cdot$$

The elements of (3) are non-zero only for $i = j$, $j = m$, $k = n$, and so on. From this definition

$$A^{(r)} \cdot I^{(2r)} = A^{(r)}.$$

A product of two tensors is simply

$$A^{(r)} \otimes B^{(s)} = A_{i,j} \cdots B_{i,k} \cdots i_i j_j \cdots i_i j_j,$$

and has rank $r+s$.

Specifically for a 3-dimensional system the gradient of a tensor is defined as

$$\nabla A^{(r)} = \frac{\partial}{\partial x_i} \left( A_{i,j,k} \cdots \right) i_i j_j k_k \cdots \cdot$$

The divergence is the $\nabla \cdot A^{(r)}$ scalar product of $\nabla$ with the tensor in question, that is

$$\nabla \cdot A^{(r)} = \frac{\partial}{\partial x_j} \left( A_{i,j,k} \cdots \right) i_i j_j k_k \cdots \cdot$$

where summation of the repeated index $j$ is implied. The divergence of a tensor reduces the rank by one, while the gradient does the opposite.
One last operation must be considered in connection with magnetic fields - the cross product. For 3-dimensional cartesian vectors this is

\[ \vec{A} \times \vec{B} = e_{ijk} A_j A_k \hat{z}_i, \]

where \( e_{ijk} \) is the alternating tensor (\( \neq 0 \) if \( i,j,k \) are all different, and \( i,j,k \) is an \( \text{EVEN}_{\text{ODD}} \) permutation of \( x, y \) and \( z \) and is otherwise zero). We define the product of a vector with a symmetric tensor by the relation

\[ A \otimes B^{(r)} = e_{ijk} A_j B_k \hat{z}_i \hat{z}_j \hat{z}_k \ldots \hat{z}_m. \]  

(6)

Notice the rank is the same as that of \( B^{(r)} \).

The definitions given here are essentially those given in \( \text{APPENDIX} \). See also the references therein for a more general discussion in this topic.

\text{APPENDIX} 2.

Starting with the expansion

\[ \vec{f}_e = \sum \vec{f}_{\text{EMS}} Y_{\text{EMS}}, \]  

(1)

we wish to convert this to a tensor form

\[ \vec{f}_e = \sum \vec{f}_{l}^{(l)} Y_{l}^{(l)}. \]  

(2)

Now it is a well known theorem that \( V^l Y_{\text{EMS}} \) can be uniquely expressed as a homogeneous polynomial of degree \( l \) in the components \( V_x, V_y \) and \( V_z \) of \( V \). Then we can write

\[ Y_{\text{EMS}} = \sum \alpha^{l}_{\text{EMS}} \cos^l \theta_x \cos^l \theta_y \cos^l \theta_z, \]  

(3)
where \( \rho + \varphi + \tau = \lambda \) and \( \cos \theta_i = \sqrt{\lambda} \) \( (i=x, y, z) \). From (1), (2) and (3) we have

\[
\sum_{i,j,k} f_{i,j,k} \frac{v_i v_j v_k ...}{u v v ...} = \sum_{pq,r} \sum_{mn} \alpha_{pq,r} f_{mn} \cos \theta_p \cos \theta_q \cos \theta_r.
\]

Equation (4), by itself, does not define the \( f^{(l)} \) elements uniquely. If we impose the restriction that \( f \) be a fully symmetric tensor, then from (4),

\[
f_{i,j,k}^{(l)} = \rho ! \varphi ! \tau ! \sum_{m,n} \alpha_{pq,r} f_{m,n}.
\]

Defined in this way, the \( f^{(l)} \) tensors are unique.

Further, since \( \nabla^l \mathcal{Y}_{\lambda \mu s} \) satisfies Laplace's equation

\[
\nabla^2 \left( \nabla^l \mathcal{Y}_{\lambda \mu s} \right) = 0,
\]

upon substitution of

\[
\nabla^l \mathcal{Y}_{\lambda \mu s} = \sum_{pq,r} \alpha_{pq,r} \nabla^{\rho} \nabla^{\varphi} \nabla^{r} \nabla^{\mu} \nabla^{l} \mathcal{Y}_{\lambda \mu s}.
\]

into (6), the following recursion relation between the \( \alpha \)'s is found:

\[
P(p-1)\alpha_{pq,r} + (q+1)(q+2)\alpha_{p-1,q+2,r} + (r+1)(r+2)\alpha_{p-2,q,r+2} = 0.
\]

It is now a relatively easy matter to prove the contraction property of the \( f^{(l)} \) tensors, namely

\[
f^{(l)}_{\lambda \mu s \tau \rho q r} = f^{(l)}_{\lambda \mu \lambda} + f^{(l)}_{\lambda \mu \lambda} + f^{(l)}_{\lambda \mu \lambda} = 0.
\]
In (8) we have equated the last two subscripts and summed from 1 to 3. This is not a restriction since the complete symmetry of the tensor permits us to permute the suffices at will. Before the contraction operation was performed the appropriate element was \( \mathbf{f}_{ij}^{(\ell)} \). Let us assume that originally \( i=1 \) and \( j=2 \) (\( x \) and \( y \) axes respectively). From (4) and (5) we observe that \( \rho \) represents the number of 1 subscripts appearing in \( \mathbf{f}_{ij}^{(\ell)} \), \( q \) the number of 2 subscripts, and \( r \) the number of 3 subscripts. Hence the elements in (8) become

\[
\mathbf{f}_{i_1 j_2 k_3}^{(\ell)} = \left( \frac{\rho+1}{\rho} \right) \left( \frac{\rho+2}{\rho+1} \right) \frac{\rho!}{\ell!} \sum_{m,n,s} \alpha_{\rho+1,\rho+1,\rho+1}^{\ell} \mathbf{f}_{i_1 j_2 k_3}^{(\ell,n,s)},
\]

and

\[
\sum_{m,n,s} \alpha_{\rho+1,\rho+1,\rho+1}^{\ell} \mathbf{f}_{i_1 j_2 k_3}^{(\ell,n,s)},
\]

Addition of these terms and use of the recursion relation (7) between the \( \alpha' \)'s gives zero. The result is easily proved for any initial combination of \( i \) and \( j \). We refer the reader to Johnston for further details on this expansion. However, he does not appear to give an explicit proof of the contraction property which we include in the above for completeness.

It is an easy matter to show that with the symmetry and contraction properties each \( \mathbf{f}_{ij}^{(\ell)} \) tensor has exactly \( 2\ell+1 \) independent elements. This provides an additional check on the expansion, since there are only \( 2\ell+1 \) independent harmonics of order \( \ell \).
CHAPTER 2

PLASMA OSCILLATIONS IN A CONSTANT ELECTRIC FIELD.

§1  INTRODUCTION.

In the previous chapter we briefly outlined the steady state situation of a weakly ionised gas permeated by a uniform electric field. As was stated, such a study is of interest in the discussion of many plasma problems including gas discharge physics, plasma heating and wave propagation. It is with this last topic that this chapter is concerned. A study of coherent oscillations of an ionised gas in the presence of external fields, both electric and magnetic, encompasses a wide variety of physical applications ranging, for example, from the propagation of radio waves through the ionosphere, to considering certain semi-conductor effects associated with electron devices.

Because a plasma contains charged particles which are nearly free, the presence of electric, magnetic or electromagnetic fields can have a pronounced effect. For our steady state situation above, the distribution function of electron-velocities will be slightly anisotropic in velocity space (equation (13), Chapter 1). It may also be unstable in that perturbations in the electric field may increase with time. It is our purpose here to consider the response of the system to harmonic perturbations in the steady state electric field.

Many theories of plasma oscillations use a perturbation method in which the governing equations are linearised. We adopt this approach here. Our discussion is limited to small longitudinal oscillations of a weakly ionised gas permeated by a constant electric field. Free oscillations in an unbounded medium are considered, the main result being a dispersion equation relating the angular frequency
Investigation into this topic appears to have begun with the work of Akheizer and Sitenko [19] in 1956. They considered the rigid sphere model for electron-molecule interaction to obtain the steady state distribution and used the simple collision term $-\nu f_e$ ($\nu$ constant) in order to get a dispersion relation. Their results were only valid for long wavelengths. Criticism may be levelled at their work on two accounts: firstly the collision term used does not conserve the number density of particles in a collision, and secondly their treatment of the uniform electric field ($E_0$) — where they expand only to the first order in field strength but quote results for both strong and weak fields. More recently Sumi [20] has considered this problem in connection with semi-conductors in electric fields. He assumed the collision term in the linearised Boltzmann equation can be treated as small compared with the other terms. Ross [21] and Stenflo [22] have essentially followed the same approach as Sumi in considering weakly ionised gases, but their solutions were only valid for the long wavelength modes. However, for a weakly ionised plasma, the neglect of collisions, while being mathematically simple, does not appear physically justifiable in view of the dominant role played by electron-molecule interactions.

Stenflo [23] has attempted a more complex analysis of the problem, in which he replaces the collision integral in Boltzmann's equation by a Fokker-Planck term and a B.G.K. term. The former is the simple collision operator derived by Chandrasekhar [24] and the B.G.K. term is so named after its originators Bhatnagar, Gross and Krook [25]. Stenflo finds a drifted Maxwellian distribution (that is one where $\gamma$ is replaced by $\gamma - \mathcal{U}, \mathcal{U}$ being the mean velocity) for the steady state situation described by his equations. He is then able to derive the dispersion relation by means of exact analytical
calculations in which he uses the eigenfunctions of the Fokker-Planck term. However his collision model for a weakly ionised gas (in its final form see [23] p. 557) contains a number of parameters, the various roles of which he does not clearly define. The dispersion relation derived by Stenflo, in consequence, has many unknown constants, making their relative importance on the dispersive properties of the medium difficult to discern. Indeed, Stenflo comments that with this model there are so many unknown parameters it precluded a discussion of possible cases. In a further model [26], Stenflo uses a modified Fokker-Planck term to describe electron-neutral interaction and obtains a heavily parameterised dispersion relation by techniques similar to those he used in [23]. However such a term is more appropriate for small angle collisions (eg those between electrons) than the more abrupt angular deflections of electron-molecule scattering.

Throughout, Stenflo's analysis appears to try and fit the physics of the problem to the mathematics of his particular collision operator and their associated eigenfunctions, rather than vice-versa. Stenflo finds that instability of longitudinal oscillations is governed by some critical value of the charge-carrier drift velocity (usually when this exceeds the thermal velocity). However it is well known that in weakly ionised plasmas, the drift velocity is much less than the thermal velocity - even for quite large fields. Thus a re-examination of the problem is required for possible unstable modes (i.e., perturbations increasing in time) in this situation and also the controlling parameters, if any, involved.

We have undertaken here a stability study of longitudinal oscillations of a weakly ionised gas permeated by a uniform electric field. We take as our steady state solution for the electrons the distributions derived in Chapter 1 (64) for which the electron-molecule collision
frequency \( V \) is independent of electron velocity. This enables the dispersion relation to be formulated in terms of error functions suitable for computation. While \( V = \) constant is not the most realistic law, we feel it should give an adequate description for our study of space-charge oscillations.

Our subsequent analysis of the above problem includes the ion-motion contribution to the dispersion relation as well as that of the electrons. Since there are a number of parameters of interest both for electrons and ions, we give in §2 a brief description of the notation employed throughout the rest of this chapter. In §3 we derive the linearised forms of the Boltzmann equation for both charged particle species and give the collision terms employed. §4 is concerned with the derivation of the dispersion relation and its formulation in terms of error-type functions. In §5 we consider some general aspects of the relation including long wavelength results and symmetry properties. Some computational studies are undertaken in §6 for various plasma parameters of interest to examine the possibility of unstable modes. The results are exhibited graphically in the form of Nyquist plots (Figures 5-12).

§2 NOTATION

It is necessary to distinguish quantities between electrons and the ions, and also between those in the perturbed and unperturbed states of the plasma. To minimise awkward notational problems we simply indicate with small characters those belonging to electrons, and use 'capitals' to denote corresponding ion terms. Other particular definitions, as needed, will be given in the text. The suffix \( 'o' \) is used to denote steady state values, and the suffix \( 'i' \) to denote perturbed values. For example we denote the electron distribution function by \( f \), the steady state value by \( f_o \) and the perturbed value by \( f_i \). \( n_o \) denotes the unperturbed electron number.
density and so on. Ion terms are indicated by the corresponding capital letters.

§3 COLLISION TERMS AND THE LINEARISED BOLTZMANN EQUATION.

We devote this section to the development of the linearised equations which determine the perturbation velocity distribution quantities of both electrons and ions.

In the steady state situation, an infinite, weakly ionised plasma is postulated, permeated by a uniform electric field \( \mathbf{E}_0 \). By 'weakly ionised' we of course imply that the electron number density \( N_e \) is much less than the neutral molecule density \( N_n \).

The system is assumed to be a three component one, consisting of electrons, neutral molecules, and singly charged positive ions (i.e. neutrals with one electron removed). The extension of the theory to cater for multiple ionisation is relatively straightforward, although we have not considered it in this current work. Finally, the plasma is considered to be everywhere electrically neutral, which of course implies equality of the electron and ion number densities.

Since we are considering a singly ionised gas, the ion mass \( M \) and the neutral molecule mass \( M_n \) are for all practical purposes identical, although their distinction in the text will be made clear where appropriate.

This chapter is concerned with the response of the above steady state system to a small applied electric field, with harmonic space and time variation of the form

\[
\mathbf{E} = e^{i(\omega t - kz)}
\]

where \( \omega \) is the angular frequency and \( k \) the wavenumber. \( \mathbf{E}_0 \) is a constant vector in the direction of \( \mathbf{E}_0 \), which, for convenience we take as defining the z-axis of a right-handed cartesian system. With the above form of the perturbation field (1), it follows at once from Maxwell's equations that there can be no magnetic field.
We consider first the equation for electrons. The Boltzmann equation for $f$ (the electronic distribution function) reads (see Chapter 1, equation (1)):

$$\frac{df}{dt} + v \cdot \nabla f + \frac{e}{m} \cdot \nabla V f = C^e(f).$$

(2)

Here $C^e$ denotes the collision term describing electron-molecule interaction. Since the plasma is assumed weakly ionised the electron-electron and electron-ion contributions may be omitted in this description. Our steady state solution $(f^0)$ for the electrons has been given in the previous chapter in the form

$$f^0 = f^{(0)} + f^{(i)} \sqrt{\frac{e}{m}} \sqrt{\gamma},$$

(3)

where the functions $f^{(0)}$ and $f^{(i)}$ are explicitly exhibited in Chapter 1 equations (19 a-c). With these definitions, equation (3) may be written as

$$f = n_o f^0 (1 + \frac{m_0 \gamma}{\gamma \sqrt{kT^*}} \sqrt{\gamma}),$$

(4)

where

$$f^0 = \left( \frac{m}{2\pi kT^*} \right)^{\frac{3}{2}} e^{-\frac{m_0 v^2}{2kT^*}}.$$

and $n_o$ is the unperturbed electron number density. $E_o(= |E_o|)$ is related to $\gamma_o$ by the relation

$$\gamma_o = \frac{E_o}{n_o}.$$

In the perturbed state, we note from equation (1), that the plasma is driven by the forcing term

$$0, 0, E_o + E_o e^{i(\omega t - \gamma t^2)},$$

(5)

when expressed in the standard component form. The Boltzmann transport
equation for the determination of $f$ reads, in consequence,

$$\frac{\partial f}{\partial \epsilon} + \chi \cdot \nabla f + \left( \frac{e E_x}{m} + \frac{e E_z}{m} \right) i^{(\omega \tau - \kappa z)} \frac{\partial f}{\partial x^2} = C_f(f).$$  \hspace{1cm} (6)$$

A solution to (6) is tried in the form

$$f = f_o + f_i e^{i(\omega - \kappa z)}$$  \hspace{1cm} (7)

where $f_i$ is a function of electron velocity only. Evidently $f_i$ vanishes with $E_1$ and is also to be treated as a small quantity. Upon substitution of (7) into (6) and the retention of only first order terms in small quantities (i.e. neglecting products of such terms) we arrive at an equation for $f_i$ of the form

$$i(\omega - \kappa z) f_i + \gamma_0 \frac{\partial f_i}{\partial x^2} = -\chi \frac{\partial f_o}{\partial x^2} + C^{\text{el}}(f_i).$$  \hspace{1cm} (8)

$\gamma_0$ is the force per unit mass produced by the perturbing field and is given by $\gamma_0 = eE_i/m$. The collision term $C^{\text{el}}$ has been assumed linear in $f$ and $f_o$ is a solution of the steady state situation given by

$$\gamma_0 \frac{\partial f_o}{\partial x^2} = C^{\text{el}}(f_o).$$  \hspace{1cm} (9)

The form of $f_o$ has been given in equation (4) above.

It remains to discuss the form of the collision term $C^{\text{el}}$. We propose to use in equation (8). An expansion in the tensor form described in Chapter 1 is not possible since, in general, $f_i$ is not so expandable. Thus we are faced with the problem of representing $C^{\text{el}}$ in a form suitable for describing electron-neutral collisions, and at the same time being simple enough to include in a dispersion analysis. We propose to represent $C^{\text{el}}$ in a phenomenological manner by a linearised B.G.K. term $\sqrt{2} f$ of the form
where \( \mathcal{J} \) is a suitably chosen Maxwellian and \( \mathcal{N} = \int \mathcal{J}_0 d\nu \) is the perturbation number density. \( \gamma_{eN} \) is the electron-molecule collision frequency, taken as independent of \( \nu \). The form of the collision term \( (10) \) conserves number density and after collision the electrons are assumed to be emitted with a Maxwellian distribution. The B.G.K. term thus simulates abrupt changes in electron velocity as would be expected by collision with much heavier neutrals. The form of \( \mathcal{J}_{\text{Max}} \) has, as yet, been left unspecified. We take the form

\[
\mathcal{J}_{\text{Max}} = \left( \frac{m}{2\pi k T} \right)^{\frac{3}{2}} e^{-\frac{m\nu^2}{2kT}},
\]

where \( T \) is the equilibrium electron temperature as defined in Chapter 1, equation (19c). That is, \( \mathcal{J}_{\text{Max}} \) is taken as equal to the \( \mathcal{J} \) defined in (4). Strictly speaking, an exact linearisation of the B.G.K. term should include a perturbation quantity in \( T \). However, since we are using (10) only for a description of collisions in the perturbed Boltzmann equation, we neglect such contributions in what follows. With this proviso, the form of \( \mathcal{C}^{eN} \) exhibited in (10) conserves energy but destroys momentum at a rate equal to \( \gamma_{eN} \). Thus we see this models fairly well electron-neutral interactions, which are predominantly an angular deflection with little change in the electron's speed.

Substitution of (10) into (8) yields the linearised equation for \( \mathcal{J}_i \) (the electronic perturbation) which, in its final form, reads:

\[
i (\omega' - i\nu_i) \mathcal{J}_i + \mathcal{N}_0 \frac{\partial \mathcal{J}_i}{\partial \nu_i} = -\mathcal{N}_1 \frac{\partial \mathcal{J}_0}{\partial \nu_i} + \mathcal{N}_1 \gamma_{eN} \mathcal{J}_{\text{Max}}.
\]
ω' is the modified angular frequency defined by

$$\omega' = \omega - i \nu_e \omega.$$

We now turn our attention to the ionic distribution function $F$. Subject to the same forcing term, the equivalent of equation (5) reads

$$\frac{\partial F}{\partial t} + \nabla \cdot \nabla F = \left( \frac{eF}{M} + \frac{eF}{M} \right) \frac{\partial F}{\partial \nabla_v} = C_{in}(F),$$

(13)

The ion charge is given by $-e(<0)$. $C_{in}$ denotes the ion-molecule collision term, and $\nabla_v$ the ion velocity. As in the case of the electrons previously considered, ion-ion and ion-electron contributions may be omitted since the plasma is weakly ionised.

The steady state situation for the ions needs to be considered next. On account of their much greater mass, the velocity distribution function for ions ($F_o$, in the steady state) can be taken as the Maxwellian form for a gas at rest, with a temperature $T$ equal to that of the gas molecules. Essentially this is stating the fact that the ions do not become 'heated' much above the average gas temperature. Energy given to the ions by electric fields is rapidly transferred to neutrals and lost to the boundaries. The electrons in contrast find it difficult to transfer energy by elastic collisions with much heavier molecules, and are readily accelerated (Shkarofsky, p.28). The ionic distribution $F_o$ may thus be expressed as

$$F_o = N_o \left( \frac{M}{2\pi kT} \right)^{3/2} e^{-\frac{M v^2}{2kT}}.$$

(14)

$N_o$ is the ion number density in the unperturbed state and is equal to its electronic counterpart for macroscopic charge neutrality. $M$ is, of course, the ion mass, which for most practical purposes is indistinguishable from the molecular mass ($M_o$) for single ionisation.
A solution to (13) is tried in the form

$$F = F_0 + F_1 e^{i(\omega t - k \cdot x)}$$

where $F_1$ is a function of ion velocity only. Again, $F_1$ vanishes with $E_1$ and is to be regarded as a small quantity. We substitute (15) into (13) and linearise terms that are small in perturbation quantities. If the collision term $C^{i\kappa}_i$ is assumed linear in $F$, the equation for $F_1$ takes the form

$$i(\omega - k \cdot v_x)F_1 - \delta \chi_0 \frac{\partial F_1}{\partial v_x} = \delta \chi_0 \frac{\partial F_0}{\partial v_x} + C^{i\eta}(F_0).$$

(16)

$\delta$ is the ratio of electron and ion masses. Of course, strictly speaking, (16) implies that $F_0$ is a solution to the equation

$$-\delta \chi_0 \frac{\partial F_0}{\partial v_x} = C^{i\eta}(F_0).$$

(17)

The left hand side of (17) is the forcing term responsible for any 'heating' effects of the ions. From what has been previously said this is small anyway. Returning to (16), $C^{i\kappa}$ is represented in a similar manner to that of $C^{e\kappa}$ (equation (10)). Hence

$$C^{i\kappa}(F) = \gamma^{i\kappa} \left( N_i \cdot F_{\text{max}} - F_1 \right),$$

(18)

where $\gamma^{i\kappa}$ is the ion-molecule collision frequency, assumed, as in the case of electrons, to be independent of velocity. The collision frequencies, both electron-molecule and ion-molecule, are taken as being independent of position; that is the scatterers (molecules) are assumed uniformly distributed in space. $N_i$ is the perturbation number density given by $N_i = \int F_i d^3 \mathbf{v}$ and $F_{\text{max}}$ is the Maxwellian
In conclusion we observe that $C_{\text{in}}$ has the same properties as $C_{\text{ea}}$ given previously.

Enough information is now in hand to specify completely the linearised equation for $F_i$. (16) combined with (18) yields the equation

$$i(\omega^+ - k V_x)F_i - \delta Y_e \frac{\partial F_i}{\partial V_e} = \delta Y_i \frac{\partial F_e}{\partial V_e} + N_i Y_{\text{in}} F_{\text{max}},$$

where $\omega^+$ is the modified frequency $\omega - i Y_{\text{in}}$.

Equations (12) and (19) are the final results of this section, being the linearised equations which describe the electron and ion perturbations of their respective distribution functions. In the next section these equations are formulated in integral form and with the aid of Poisson's equation, a dispersion relation is obtained relating $\omega$ and $k$ for possible modes of oscillation of the form $\sim e^{i(\omega t - k z)}$.

§ 4 DERIVATION OF THE DISPERSION EQUATION.

In the previous section the linearised equations governing the variation of the perturbation functions $f_i$ and $F_i$ were established. This section is devoted to their solution and the derivation of the dispersion relation. Our aim is not to solve for $f_i$ and $F_i$ explicitly, but rather their integrated quantities, as functions of $\omega$ and $k$; that is we require
\[ n_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_i \, d^3y, \]
\[ N_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_i \, d^3y. \]

(20 a-b)

Then, with the help of Poisson's equation in the form

\[ \nabla \cdot \varepsilon_0 \mathbf{E}_i = \rho \] (21)

(\( \rho \) is the charge density distribution) and the equations (20a, b), the dispersion relation is derived from

\[ -\frac{i k e n_i \mathbf{E}_i}{m} = \omega_e^2 (n_i - N_i). \] (22)

\( \omega_e \) is the electron plasma frequency defined by

\[ \omega_e^2 = \frac{n_e e^2}{m \varepsilon_0}, \]

and \( \varepsilon_0 \) is, of course, the permittivity of free space. An ion frequency can be similarly defined by

\[ \omega_i^2 = \frac{N_i e^2}{M \varepsilon_0} = \frac{8}{3} \omega_e^2, \]

since \( n_0 = N_0 \) for our situation. In passing we note that, numerically, for the electrons

\[ \omega_e \approx 565 \sqrt{n_0} \text{ rad/sec}. \]

The problem is to calculate the expressions (20a, b). A slight complication arises because of the presence of these quantities in the equations for \( f_i \) and \( F_i \) themselves. We will consider first the equation for \( f_i \) and calculate the necessary integral forms. The ion contribution can be easily inferred from the results obtained for the electron gas and hence will be treated at a later stage. In review, the differential equation for \( f_i \) is
where the various terms have been defined in §3. Our interest here centres on a stability study of longitudinal waves; that is, possible situations in which the perturbations grow in time. In this respect $k$ is regarded as a real quantity and $\omega$ as having possibly a complex part - positive for damped solutions and negative for growing (unstable) modes. Under these restrictions, the solution to (23) which vanishes at $V_2 = \pm \infty$ is

$$ f(z) = \left( 1 + \frac{m V_2}{V_c c n k T e} \right), $$

and

$$ f_M(s) \equiv f_{\text{MAX}}(s) = \left( \frac{m}{2 \pi k T} \right) e^{\frac{-m (V_{1 s}^2 + V_{c s}^2 + s^2)}{2 k T}}. $$

(24) is easily verified by differentiating with respect to $V_2$ and substituting into the left hand side of (23).

The condition that the function $f_1$ should tend to zero as $V_2 \rightarrow \pm \infty$ is easily seen to be true for (24). Firstly, as $V_2 \rightarrow +\infty$ the integration range tends to zero and hence so does $f_1$. For $V_2 \rightarrow -\infty$, we note that $\omega^2 = -\frac{i}{c c n} V_2 k T$, and hence, providing the imaginary part of $\omega$ is less than $V_c c n$, the term $e^{-\frac{i \omega V_2}{k T}} \rightarrow 0$.

The case of $\omega > V_c c n$ can be accommodated by suitable rearrangement of the integration limits in (24). However it can easily be shown that the value of $n_1$, obtained is identical to the
case for \( \mathcal{J}(\omega) < \gamma_{\text{th}} \), which we are considering here. The results obtained for the latter case can hence be analytically continued across to encompass the whole complex \( \omega \)-plane. For a detailed discussion on the analytic continuation of dispersion relations we refer the reader to Stix\( ^{29}, \text{p. 143} \).

The next step is to calculate \( \pi \), from (24). This is made up of two contributions, one of which depends on the steady state distribution \( f_0 \), the other arising from collisional effects in the equation for \( \mathcal{J} \). The former is considered first and contributes a term \( I(\omega', k) \) to \( \pi \), given by

\[
I(\omega', k) = \frac{\gamma}{8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left\{ -\frac{i\omega}{k} (V_x - S) + \frac{i\nu}{k} (V_y - S) \right\} \frac{f_0(s)}{\partial s} ds \, dV_x. \tag{25}
\]

(25) consists of a fourfold integration, firstly over the dummy variable \( S \), then over the components \( V_x, V_y \) and \( V_z \). Since \( f_0(s) \) is essentially Maxwellian, multiplied by a function of \( S \), the integrations over the components of \( V \) automatically uncouple themselves. Reversing the order of the \( S \) and \( V_z \) integrals yields the expression

\[
I(\omega', k) = \frac{\gamma}{8} \int_{-\infty}^{\infty} \int_{S}^{\infty} \int_{-\infty}^{\infty} \exp \left\{ -\frac{i\omega}{k} (V_x - S) + \frac{i\nu}{k} (V_y - S) \right\} \frac{f_0(s)}{\partial s} ds \, dV_x dV_y dV_z. \tag{26}
\]

The object now is to remove the variable \( V_z \) limit. This is achieved by defining a new variable \( \Theta \) given by

\[
V_z = S + \frac{\nu}{k} \Theta,
\]

remembering that \( \nu < 0 \). While the \( V_z \) integration is performed, \( S \) is treated as a constant, so that \( V_z \) is simply linearly related
to $\theta$. Thus writing $\text{SGN}(k) = \zeta ( = +1 \text{ if } k > 0 , = -1 \text{ if } k < 0 )$, (26) takes the form

$$I(\omega', k) = \frac{\chi}{k} \int_{\sigma}^{i+\infty} \left[ \int_{-\infty}^{\infty} e^{i \phi} \frac{\partial f(s)}{\partial s} ds \right] d\phi \left[ e^{-\frac{\sigma}{2} - i \frac{\phi}{2}} - e^{\frac{\sigma}{2} + i \frac{\phi}{2}} \right] d\phi.$$

(27)

The term in brackets is just the complex Fourier transform of $\frac{\partial f(s)}{\partial s}$ modified by the integration over $V_x$ and $V_y$ which just contributes a multiplying factor.

After two integrations by parts, and using the result

$$\int_{-\infty}^{\infty} e^{i \phi} e^{-s \sqrt{V_x^2 + V_y^2}} ds = \frac{e^{-\sigma \sqrt{V_x^2 + V_y^2}}}{\sqrt{V_x^2 + V_y^2}} ,$$

we see that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i \phi} \frac{\partial f(s)}{\partial s} ds = \alpha (-i \phi + U \phi^2) e^{-\theta^2 V_o^2 / 4} .$$

(28)

$V_o$ is defined as the electron thermal velocity ($= \sqrt{\frac{2kT}{m}}$) and $U$ is the corresponding directed velocity ($= \frac{V_o}{V_{\perp}}$). We note in passing that some other acceptable definition of $V_o$ (e.g. $V_o = \sqrt{\frac{kT}{m}}$ as used in [27], p. 4) would not subsequently alter the results. Thus (27) with (28) may be written in the form

$$I(\omega', k) = \frac{\chi}{k} \int_{\sigma}^{i+\infty} \left\{ i \int_{0}^{\infty} \phi e^{-a \phi - i \frac{\phi}{2}} d\phi - U \int_{0}^{\infty} \phi e^{-a \phi - i \frac{\phi}{2}} d\phi \right\} ,$$

(29)

where the integration limits have been reversed and a new variable $\phi$ defined equal to $\text{SGN}(k) \phi$, ($\zeta \phi$). $\zeta$ denotes the quantity $\frac{\omega}{k}$ and 'a' is a complex number depending on the thermal speed $V_o$, the
wave number $k$, and $\gamma_0$. Explicitly:

$$a = \frac{V_0^2}{4} (1 - \frac{2i\gamma_0}{kV_0^2}).$$

We can further introduce the Debye length ($\lambda_\text{D}$) for electrons by the relation

$$\lambda_\text{D} = \frac{\gamma_0}{\omega_\text{e}}.$$

This parameter represents the upper limit for microscopic interactions, and a minimum length for macroscopic charge separation effects in a plasma (see Eq. 3, p. 2). In the literature the definition $\lambda_\text{D} = \frac{\varepsilon_0 kT}{n e^2}$ is sometimes found. However this and our definition differ by only a numerical factor. With $\lambda_\text{D}$, 'a' can be written as

$$a = \frac{V_0^2}{4} \left(1 - \frac{2i\gamma_0}{kV_0^2}\right). \tag{30}$$

The final evaluation of $I(\omega', k)$ in terms of error functions can now proceed straightforwardly. In terms of the function $\gamma_0$

$$W(\zeta) = e^{-\zeta^2} \left\{1 + \frac{2i}{\sqrt{\pi}} \int_0^\infty e^{-t^2} \, dt\right\},$$

$I(\omega', k)$ is given by

$$I(\omega', k) = \frac{-\gamma_0 a^2}{k} \left\{ -i \frac{\omega'}{2a} \left( \frac{1}{\sqrt{\pi}} + i \int_0^\infty W(\xi) \, d\xi \right) + \frac{\omega'}{4a^2} \left[ W(\xi-2\gamma) + \frac{2i\gamma}{\sqrt{\pi}} \right] \right\}. \tag{31}$$

The complex argument $\gamma$, for electrons, is given by

$$\gamma = \frac{-(\omega - i\gamma_0)}{\omega_\text{c} \sqrt{k \lambda_\text{D}} (1 - \frac{2i\gamma_0}{kV_0^2})}. \tag{32}$$
The reduction of (29) into the final form (31) requires the manipulation of certain standard integrals. Since only the final form is required here, these details have been relegated to the appendix to which the reader may refer for further details.

To complete the calculation of $n_i$, we require to evaluate the contribution

$$n_i G(\omega', k) = -\frac{n_i \gamma a}{\epsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{E} \mathcal{F} \left\{ \frac{-i \omega'}{\epsilon_0} (V_z S) + \frac{i k}{2 \epsilon_0} (V_z^* - S^*) \right\} \mathcal{F}(s) ds \, d^3 V' ,$$

with $f_{\text{MAX}}$ as defined in (24). The evaluation of (33) proceeds in a very similar manner to our calculation for $I(\omega', k)$. First the order of the $S$ and $V_z$ integrations is reversed and then the substitution $V_z = S + \gamma \theta_k / k$ is made to remove the variable limits. This reduces the $S$ integration to simply the Fourier transform of $f_{\text{MAX}}(s)$. The integration by parts is not now required once the $S$ integral is already in a standard form. Finally the substitution $\theta = \xi \phi$ yields

$$G(\omega', k) = \frac{\gamma a \xi}{k} \int_{-\infty}^{\infty} e^{-a \phi^2 - i \xi \phi} d\phi .$$

The terms $\xi$, $\gamma$ and $a$ have the same definitions as used in $I(\omega', k)$. From the appendix, and with $\xi$ defined in (32), $G(\omega', k)$ is related to the error function $W(\xi)$ by

$$G(\omega', k) = \frac{\gamma a \xi}{2 \epsilon_0 k} \sqrt{\frac{\alpha}{\xi}} W(\xi) .$$

(34)

The electron perturbation density $n_i$, given by (20a), is composed of two terms (one containing $n_i$ itself) in the form

$$n_i = I(\omega', k) + n_i G(\omega', k) .$$

(35)
Solving for \( \mathcal{N}_1 \), we have

\[
\mathcal{N}_1 = \frac{i \mathcal{I}(\omega', k)}{(1 - G(\omega', k))}. \tag{36}
\]

Before proceeding with the evaluation of \( \mathcal{N}_1 \), the ion perturbation density, we define one further function, \( D(\xi, \beta) \), by the relation

\[
\mathcal{I}(\omega', k) = \frac{i \mathcal{I}_0 \mathcal{G}_0}{2\alpha k} D(\xi, \beta). \tag{37}
\]

where by comparison with (31),

\[
D(\xi, \beta) = i \mathcal{W}(\xi)[\gamma_0 + \beta(1 - 2\xi)] - \frac{1}{\sqrt{\alpha}}(-\gamma + \beta \xi). \]

\( \beta \) is the complex number \( \frac{\gamma_{\alpha}}{\sqrt{2} \gamma_{\alpha}} \), which written in terms of electron quantities is

\[
\beta = \frac{\gamma_{\alpha}}{\gamma_{\alpha} \sqrt{1 - \frac{2i \gamma_{\alpha}}{\hbar_{\alpha} \omega_{\alpha} V_0}}}. \]

The square root is to be interpreted as the positive branch, so that

\[
\sqrt{\alpha} \rightarrow \sqrt{1} \quad \text{as} \quad \gamma_{\alpha} \rightarrow 0.
\]

The functions \( I \) and \( G \), which determine \( \mathcal{N}_1 \) through (36), have now been completely defined in terms of electron plasma quantities by the equations (34) and (37). We now complete this section with the evaluation of the ion perturbation term \( \mathcal{N}_1 \). Most of the analysis has, formally, already been done in calculating \( \mathcal{N}_1 \). All that is necessary is to interpret corresponding ion quantities from electron ones, paying due regard to signs. Firstly, for reference, the equation determining \( F_i \) reads (\( \mathcal{S}_3 \), equation (19)):

\[
i(\omega^t - kV_z) F_i - \delta V_z \frac{\partial F_i}{\partial V_z} = \delta V_z \frac{\partial F_0}{\partial V_z} + \mathcal{N}_1 \gamma_{\alpha} F_{\text{MAX}}, \tag{38}
\]
where \( \omega^+ = \omega - i \nu_{\infty} \), and the distributions \( F_o \) and \( F_{\text{MAX}} \) are related by

\[
F_o = N_o \ F_{\text{MAX}}
\]

with

\[
F_{\text{MAX}} = \left( \frac{M}{2\xi K T} \right)^{3/4} \ e^{-\frac{M \nu^2}{2 K T}}
\]

The solution for \( F_i \), in integral form, can be expressed (under similar conditions) in a form similar to (24) for the electrons. With the appropriate change in signs, necessitated by the fact that the ion charge is \(-\xi\), \( F_i \) can be written as

\[
F_i = \frac{-\chi_i}{\chi_s} \int_{-\infty}^{s = \nu_s} e^{i \omega^+ (s - s') - \frac{i k}{2 \xi} (s - s')^2} \left[ \frac{\partial F_s(s)}{\partial s} + \frac{\nu_{\infty} N_i}{\delta \chi_i} F_{\text{MAX}}(s) \right] ds
\]

(39)

\( F_{\text{MAX}}(s) \) denotes the Maxwellian form

\[
\left( \frac{M}{2\xi K T} \right)^{\nu_s/2} e^{-\left[ -\left( -\nu_s^2 + \nu_s^2 + s'^2 \right) M / 2 K T \right]}
\]

Note that the integration limits of (39) are different from those in the electron case. This is a direct consequence of the change in sign necessary to accommodate the ion charge \((-\xi\)). (39) converges at the limit \( \nu_s \to \infty \) provided the imaginary part of \( \omega \) is less than \( \nu_{\infty A} \) (\( \Im(\omega) < \nu_{\infty A} \)). As briefly mentioned in the case for electrons the results may be analytically continued to cover the whole \( \omega \)-plane.

Convergence of (39) at the other limit, namely \( \nu_s = -\infty \), is assured by the nature of the integration limits.

The evaluation of \( N_i \) form (39) can now proceed in a similar manner to the electronic case. By writing
the integrals over \( V_z \) and \( S \) can be reversed, and the variability of the \( V_z \) limits removed by a suitable change of variable. The final integral forms can be shown to be of the form (29) (with \( U=0 \), and (33) corresponding to terms \( \frac{\partial F_0(s)}{\partial S} \) and \( \int_{\text{max}}^{\text{max}} \) respectively in the integrand for \( F_i \). No equivalent \( U \) term occurs for the ions, since the steady state function is isotropic in velocity space. Since we have mentioned that the final integral forms for \( N_i \) and \( N \), are equivalent (with \( U=0 \)), the analysis of (39) need not be dealt with. It is only necessary to interpret particle identities correctly by replacing electron quantities with the appropriate ion term (paying due regard to sign). Thus the contribution to \( N_i \), arising from the term \( \frac{\partial F_0}{\partial S} \) in (39) can be written as

\[
-\frac{i \gamma_i S N_0 \nu_i}{2 \Lambda k} \left( \zeta_i^2 \right)
\]

This can be seen by observing that for the ions there is no explicit drift velocity, so that \( \beta_{i,o}=0 \). Table 2-1 serves to illustrate the various electron plasma terms arising in the calculation of \( N_i \) and the corresponding ion quantities.
Similar arguments, using Table 2-1 show that the contribution to \( N_1 \) from the \( N, F_{\text{max}}(\xi) \) term may be written as

\[
N_1 \frac{\gamma_{\text{ca}}^2}{2k} \frac{1}{\sqrt{\alpha}} W(\xi_i).
\]

Hence the equation for \( N_1 \) becomes

\[
N_1 = \frac{-i \gamma_{\text{ca}}^2 h_1,0}{2\pi k} D(\xi_i,0) + \frac{\gamma_{\text{ca}}^2}{2k} \frac{1}{\sqrt{\alpha}} W(\xi_i). \tag{40}
\]

Solving for \( N_1 \) yields the final expression

\[
N_1 = \frac{-i \gamma_{\text{ca}}^2 h_1,0}{2\pi k} D(\xi_i,0)}{1 - \frac{\gamma_{\text{ca}}^2}{2\pi k} \sqrt{\alpha} W(\xi_i)}. \tag{41}
\]
The lengthy calculations to determine $\Omega$, and $N$, have now been completed. Knowing these values, as functions of $\omega$ and $k$, they can be substituted into Poisson's equation (22) in the form

$$-i k n_o \chi_i = \omega e^i (n_i - N_i)$$

(42)

to yield the required dispersion relation. For reference the relevant equations are (36), (41) and (42). Written in full, we have for the $\omega$-$k$ relation (remembering $n_o = N_o$ for our case)

$$k^2 = -\frac{2}{Z_a} \frac{\omega e}{2a} D(c^o, \beta) - \frac{2}{A} \frac{\omega e}{2A} \left( \frac{2}{A} \right) W(c_i, 0)$$

(43)

where

$$D(c^o, \beta) = i W(c^o) \left[ \frac{c^o}{\beta^2} + \beta (1 - 2 c^o) \right] - \frac{1}{\sqrt{\pi}} (-2 + 2 \beta c^o)$$

and

$$W(c) = e^{-c^2} \left( 1 + \frac{2i}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{t^2} dt \right)$$

Equation (43) is the final result of this section. It relates the angular frequency $\omega$ and the wave number $k$ of possible longitudinal oscillations for a weakly ionised plasma in a uniform electric field. Collisions have been included phenomenologically through collision frequencies $\gamma_a$ and $\gamma_{in}$ describing electron-neutral and ion-neutral interactions respectively. In the next section some general remarks concerning equation (43) are made with some computational work on possible unstable modes being undertaken in §6.

§5 GENERAL DISCUSSION OF THE DISPERSION RELATION.

So far we have established the dispersion relation (43) valid for a collisional weakly ionised plasma in a uniform electric field.
In this section we discuss some particular cases derivable from (43), investigate results of an asymptotic nature for the long wavelength modes, and also derive some symmetry properties relating $\omega$ and $k$.

One of the simplest cases is when the external field ($E_0$) is zero. Neglecting, for the moment, the ion term in the dispersion relation, and also collisions, we have

$$\left( \frac{\omega}{\omega_0} \right)^2 = -2 \sqrt{\frac{\kappa}{k_e}} \frac{2}{\omega_0} W(\xi) - 2 \xi^2,$$

where the argument $\xi$ reduces to $\frac{\omega}{\omega_0 k_e}$. Equation (44) represents the usual Landau form for space charge oscillations in an electron plasma without collisions. We do not propose to discuss all the significant aspects here, but instead refer the reader to Tanenbaum 28, p. 181 et seq. for discussion of such topics as Landau damping related to (44). Of particular interest here (for comparison with results derived later) is the Bohm-Gross relation 28, p. 180 for long wavelengths. This is obtainable by an asymptotic expansion in powers of $|\xi|^{-1}$ when $|\xi|$ is large. Since the usual assumption of long wavelengths is the localisation requirement that the wave phase velocity is very much greater than the particle thermal velocity, we have $k V/\omega \ll 1$, giving consequently $|\xi| \gg 1$.

This localising condition is essentially the 'cold plasma' approximation in which the random thermal motion is sufficiently small (or the wavelength sufficiently long) to make the dominating characteristic length that of the electro-magnetic wave itself. Using the expansion 27:

$$W(\xi) \sim \frac{i}{\sqrt{\kappa \xi}} \left( \frac{1}{2 \omega_0^2} + \frac{3}{4 \kappa^2} + O\left( \frac{1}{\xi^2} \right) \right),$$

we can write (44) as
\[ \left( \frac{\omega}{\omega_{c}} \right)^{2} = 1 + \frac{3}{2} \frac{e}{\kappa} - \cdots + \mathcal{O} \left( \frac{e^{2}}{\kappa^{2}} \right). \]

In the absence of an external electric field, the argument \( \lambda_{0} \) is (from Table 2.1) given by

\[ \lambda_{0} = -\frac{\omega}{i\kappa \nu_{0}} = -\frac{\omega}{i\kappa \sqrt{2kT_{*}}} \cdot \]

The asymptotic expansion of (44) thus takes the well known form

\[ \omega^{2} = \omega_{c}^{2} + \frac{3k^{2}kT_{*}}{m}, \quad \text{(46)} \]

where \( \omega \) has been put equal to \( \omega_{c} \) in the small correction term and \( T_{*} = T \) for \( E_{0} = 0 \). \( K \) is of course the Boltzmann constant.

The ion contribution in (43) in the absence of collisions and the electric field \( (E_{0}) \) may be similarly inferred, giving a term analogous to that in (44). Thus with ion motion included, the dispersion relation becomes

\[ \omega^{2} = -2i\kappa \nu_{0} \chi^{c} \chi^{e} W(\chi) - 2\omega_{c}^{4} - 2i\kappa \nu_{0} \chi^{c} \chi^{e} W(\chi) - 2\omega_{c}^{4} \cdot \text{(47)} \]

For long wavelengths \( | \gg |k\lambda_{e}| \gg |k\lambda_{i}| \) since the electron and ion temperatures are the same when no forcing term is present. By applying the asymptotic expansion as above, the leading terms of (47) essentially give

\[ \omega^{2} \sim \omega_{c}^{2} + \omega_{i}^{2}. \]

Since \( \omega_{c}^{2}/\omega_{i}^{2} = \frac{m}{M} \ll 1 \) this correction for ion motion proves to be relatively unimportant. Thus the oscillations for long wavelengths are at the electron plasma frequency, whether or not ion motion is included. As we shall see later, when the electron temperature is much larger than that of the ions there are modes for which \( |k\lambda_{e}| \gg 1 \gg |k\lambda_{i}| \) and in this case the ion term is the more important.
Before proceeding to discuss the long wavelength modes when collisions and the forcing term $F_o$ are present, we here give an important symmetry property relating $\omega$ and $k$. Remembering that the dispersion relation has been derived for real values of $k$ and complex $\omega$, the modes represented by

$$\omega, k \rightarrow -\bar{\omega}, -k$$

(over-bar denoting complex conjugate) are identical. The direction of travel is unaltered since this depends on the phase velocity $\text{Real} (\omega)/k$, the sign of which is invariant in this type of transformation. The damping or growth of the wave as represented by the imaginary part of $\omega$ is obviously unchanged by definition. It remains to show that under this type of transformation the general relation (43) is invariant. This is not obvious from mere inspection of the dispersion relation since $\omega$ and $k$ are related in a complex functional way.

The simplest, if not the most elegant or general proof, starts with the observation that

$$\mathcal{W}(-z) = \overline{\mathcal{W}(z)}.$$  \hspace{1cm} (49)

Further, using the definitions given in Table 2.1, the error function argument obeys the transformation law

$$\xi \rightarrow -\overline{\xi}.$$  Parallel results, of course, hold for the ion terms. So that in proving the invariance of the electron term, the corresponding ion result follows immediately.

We next see how the expression $\mathcal{D}(\xi, \beta)$ behaves. Remembering that $\zeta = \text{SGN}(k)$, this must be replaced by $-\zeta$, while $\beta = \frac{\zeta \omega}{2 \alpha \sqrt{\mu}}$ transforms to its complex conjugate $\overline{\beta}$. It is now an easy matter to show that

$$\mathcal{D}(-\xi, \overline{\beta}) = -\overline{\mathcal{D}(\xi, \beta)}.$$  \hspace{1cm} (50)
Note that (50) holds when $\beta = 0$ which is the case for the ions. With (49) and (48) it is now any easy matter to show that the electron term on the right hand side of (43) transforms to its complex conjugate. A similar result is true for the ion term. Since $k^1$ is real, we have shown that under the transformation (48), the dispersion relation (43) is merely replaced by its conjugate and hence remains unaltered in essence. This is a satisfactory result from our point of view, implying as it does a degree of consistency in the way $\omega$ and $k$ are related through (43).

The next task is to examine the effects caused by the presence of the forcing term $E_0$. To simplify the results for the moment, only the electron motion will be considered and again collisions will be neglected. The latter assumption is not strictly valid for our assumption of a weakly ionised gas, since collisions play an important rôle. However their neglect will aid the analysis and at the same time serve to highlight some important terms. The dispersion equation (43), with the neglect of such terms, may be written as

$$\omega^2 = -2\sqrt{\kappa} \gamma \omega_e \sum_{\alpha} \mathcal{D}(\zeta, \rho).$$

(51)

With the definition of $D$ as given in (43) and the expansion (45) for long wavelengths, we have

$$\mathcal{D}(\zeta, \rho) \sim \frac{-e}{2\sqrt{\kappa} \zeta} + \frac{\beta}{\sqrt{\kappa} \zeta^2} - \frac{3\zeta}{4\kappa \zeta^4} + O\left(\frac{1}{\zeta^6}\right).$$

Since the forcing electric field has been retained the argument $\zeta$ is modified to the expression

$$\frac{\omega}{|k| V_0 \sqrt{1 - \frac{\sigma}{R_k e}}}$$

where $\sigma = \frac{2\beta}{\omega_0 V_0}$. Because the electron thermal velocity $V_e$ depends on $V_0$ through $Y$ (equation 19c, Chapter 1), we have, for
strong fields,

\[ \sigma = \frac{2m}{\sqrt{3M}} \frac{\gamma_e}{\gamma_0 \omega_e}. \]

Thus with \( \gamma_e < \omega_e \) for most situations in which we are interested and \( \frac{m}{k} \ll 1 \), we have \( \sigma \ll 1 \). In weak fields, by definition, \( \sigma \ll 1 \).

Thus substituting the above expansion for \( D(\omega, \gamma) \) into (51) yields the long wavelength result

\[ \omega^2 \sim \omega_e^2 \left\{ 1 + \frac{2u k}{\omega_e} - \frac{3}{2} i \sigma k \hbar \right\}, \]

where \( \omega \) is the directed velocity of the electrons. Terms like \( \sigma k \hbar v_e \) and \( \frac{u k}{\omega_e} \) have been ignored in (52) because of the smallness of \( \sigma \) and \( \frac{u}{\omega_e} \) for electrons.

We can now compare (52) with the Bohm Gross relation (46).

Evidently the presence of an external field appears in the extra terms \( \frac{2u k}{\omega_e} \) and \( -\frac{3}{2} i \sigma k \hbar \). The quantity \( \frac{2u k}{\omega_e} \) represents the effect of the drift or directed velocity of the electrons, while the imaginary term is essentially the action of the electric field \( E_0 \) on the wave. To lowest order in real and imaginary terms separately

\[ R(\omega) \sim \omega_e \]

and

\[ J(\omega) \sim -\frac{3}{2} i \omega_e \sigma k \hbar \].

The complex term gives damping or growth of the wave depending on the sign of \( k \), the rate being equal to \( \frac{3}{2} \frac{k |v_e|}{\omega_e} \), with \( \frac{u v_e}{\omega_e} \ll 1 \) for long wavelengths. Since \( \sigma \ll 1 \) for all field strengths the effect is small in any case. For \( k \ll 0 \) the wave travels in the sense of the directed electron velocity and is amplified (unstable mode), while for \( k \gg 0 \) the wave and drift velocities are in opposite phase giving temporal decay.
Some simple concepts of power flow coupled with the macroscopic momentum and continuity equations yield essentially the same result as (53) for the growth/decay terms and it is instructive to show this here. Starting with Maxwell's equation, we have, for longitudinal waves

\[ 0 = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \]

where \( \mathbf{J} \) is the current flow due to perturbation forces and \( \varepsilon_0 \) is the permittivity of free space. The scalar product of this equation with the wave field \( \mathbf{E} \) gives

\[ \frac{1}{c} \varepsilon_0 \frac{\partial}{\partial t} (|\mathbf{E}|^2) = -\mathbf{E} \cdot \mathbf{J}, \]

which is essentially an energy equation for the electric field \( \mathbf{E} \). It represents Poynting's equation (see Stratton [30]), in the absence of any magnetic field. The left hand side represents the rate of change of electrical energy per unit volume, and the right hand side the power absorption associated with the wave. For this latter term we may write

\[ \mathbf{E} \cdot \mathbf{J} = n_e \varepsilon \mathbf{E} \cdot <\mathbf{\omega}> , \]

where \( <\mathbf{\omega}> \) is the particle averaged perturbation electron velocity in the direction of \( \mathbf{E} \). The \( z \)-equation of motion to determine \( <\mathbf{\omega}> \) reads

\[ n_e \frac{dv_z}{dt} = -\nabla_z P + n_e \mathbf{F}_z , \]

in which \( n \) represents the electron number density and the forcing terms are due to the pressure \( P = n k T^* \) and the electromagnetic contribution (\( \mathbf{F}_z \)). Collisions have been ignored in this analysis to highlight the direction of energy flow due to the uniform electric field. Linearising the momentum equation yields the following relation between \( <\mathbf{\omega}> , n \) and \( \mathbf{\nu} \), namely
\( \eta_1 \langle \omega_1 \rangle (\omega - k_U) = \eta_1 \left( \frac{i k k_U}{\eta_1 \omega} + \chi \right) + \chi \eta_0 \), \hspace{1cm} (55)

assuming a harmonic variation (1) for all the variables. \( \omega - k_U \) represents the Doppler corrected frequency due to an electron drift velocity \( U \). A similar linearisation of the continuity equation yields a relationship between \( \eta_1 \) and \( \langle \omega_1 \rangle \) which can be written as

\[ \eta_1 (\omega - k_U) = k \eta_0 \langle \omega_1 \rangle . \] \hspace{1cm} (56)

Returning to the Poynting equation, we can time average this over a wave period to obtain, with the help of (54), the averaged energy balance equation

\[ \left\langle \frac{1}{i} \frac{\varepsilon_0 \partial}{\partial t} |E_1|^2 \right\rangle_{\text{aver}} = -\frac{\eta_0 k}{4} \left( \chi \langle \omega_1 \rangle + \langle \omega \rangle \chi \right), \] \hspace{1cm} (57)

where the overbar denotes the complex conjugate. Eliminating \( \eta_1 \) and \( \langle \omega_1 \rangle \) between (55), (56) and (57) evidently gives

\[ \left\langle \frac{1}{i} \frac{\varepsilon_0 \partial}{\partial t} |E_1|^2 \right\rangle_{\text{aver}} = \frac{\varepsilon_0 k \chi_0}{2 \omega_0} |E_1|^2, \] \hspace{1cm} (58)

where we have taken \( \omega - k \chi \sim \omega_0 \) \( \text{(i.e. small growth or damping)} \) in calculating terms that appear on the right-hand side of (58). The difference in numerical factor between (58) and the growth/decay term of (53) is attributable to the kinetic approach in deriving the latter and the macroscopic approach for the former, and also to the fact that we have time averaged (58). (58) shows that the waves absorbs energy from the electric field for \( k \chi < 0 \) \( \text{(for electrons)} \) and dissipates it for propagation with \( k \chi > 0 \).

Finally, we briefly look at some long wavelengths results with ion motion and collision terms included. Since collisions are the dominant mechanism for the redistribution of energy within a weakly ionised gas,
we would expect this to be reflected in damping terms for the various modes governed by the dispersion relation. The possibility of unstable oscillations thus becomes balanced between the relative dominance of either collisions or growth terms.

To include collisions and the ion motion we must return to the full equation (43). Rather than attempt to discuss all possible cases, we limit our remarks here to the cases when the modes are predominantly electron or ion-waves. In the next section some computational results are examined for various plasma parameters of interest in which both charged particle species contribute to the space charge oscillations. If the wavelengths are sufficiently long so that \( k h_e \ll 1 \), the modes are mainly electronic with \( \omega \) given by

\[
(\omega - i \gamma_{e\text{a}})^2 = \frac{\omega e^2}{(1 + \frac{i \gamma_{e\text{a}}}{\omega - i \gamma_{e\text{a}}})},
\]

to the lowest order in \( |\gamma'|^{-1} \). Equation (53) for the frequency \( \omega \) is subsequently modified to

\[
\omega = \pm \left( \omega_e^2 - \frac{\gamma_{e\text{a}}}{4} \right)^{1/2} + i \frac{\gamma_{e\text{a}}}{2} \tag{59}
\]

and the modes are stable.

In strong electric fields for which \( T^* \gg T \), we have \( h_e^2 = \frac{T^*}{T} \) so that possible modes exist for which \( k h_e \gg 1 \Rightarrow k h_c \). For high temperatures, when the electron thermal velocity is much larger than the wave phase velocity, the wavelength is small compared to the electron Debye length \( h_e = \frac{\sqrt{m_e}}{\omega_e \lambda_e} \). Now this latter distance marks the minimum length for macroscopic charge separation effects in a plasma (see \( ^2 \), p.2). Charged particles interact with other individual charged particles if they are closer than one Debye length, but interact with what amounts to a continuous charge cloud at greater distances. Hence at these high temperatures the electrons, by
themselves, cannot contribute to coherent macroscopic wave oscillations. In this case the electron contribution to the dispersion relation is small in comparison to the ion term, and the oscillations are mainly ion modes. In the absence of collisions, we have to lowest order in real and imaginary terms

\[ \omega = \omega_i \left( 1 + \frac{3}{4} i \sigma_i k h_i \right) \]  

where for the ions \( \sigma_i = \frac{2 \delta \nu_s}{\omega_i \nu_0} \). Collisions introduce damping in a similar way to the electrons above. However the restriction \( \sigma_i < 1 \) no longer applies since \( \nu_s \) is independent of \( \nu_0 \). Hence in sufficiently strong fields the growth term ( \( k > 0 \) for ions) could overbalance the damping term and give rise to unstable ion modes. However quantitative deductions from (60) would not be appropriate since the equation assumes the neglect of quantities of order \( \sigma_i (k h_i) \), which for large fields could be important. The next section deals with the investigation of instabilities via Nyquist plots of the dispersion relation (43).

\section*{6 Computational Results: Nyquist Plots for Instabilities}

In this section a numerical investigation into instabilities for some plasma parameters of interest is presented, with the results displayed graphically in the form of Nyquist diagrams.

We have seen in \( \section*{5} \), equation (43), that with collisions, both electron and ion motions and the uniform electric field included, the dispersion relation governing possible modes becomes rather complicated. By considering long wavelength results we have shown electron waves are always damped owing to the smallness of the ratio of the electron drift velocity to thermal velocity. The corresponding growth parameter for ion modes, equation (60), is not so restricted since the ion thermal velocity is independent of field strength. Hence the quantity \( \frac{\nu_s}{\omega_i \nu_0} \) in strong fields, could contribute significantly to the growth term.
to outweigh collisional damping and give rise to unstable modes. Quantitative deductions from (60) are inappropriate since it assumes the neglect of terms which become significant in large fields.

Essentially, in this section, we aim to answer the question: given a set of plasma parameters describing our system, does this describe an unstable mode? The answer is provided by the Nyquist method of instabilities; see Stix (p. 152 et seq). This method applied to our problem is described in the following.

For unstable modes the imaginary part of the frequency \( \omega \) must be negative. Figure 1 simply indicates the complex \( \omega \) -plane and the relevant unstable and stable regions. Now the dispersion relation can, essentially, be written in the form

\[
\varphi = \phi \left( \frac{\omega}{k} \right), \quad (61)
\]

where \( \phi \) represents some function, possibly complex, which is determined by the problem in hand. Given a real value of \( k \), (61) determines the corresponding \( \omega \), which is generally complex and may be unstable. By mapping the unstable part of the \( \omega \) -plane into the complex \( k^2 \) -plane, according to (61), we can determine whether the mapped area encloses any portion of the positive real \( k^2 \) axis, and if so, the corresponding unstable modes. Essentially the Nyquist method enables us to plot the right-hand side of (61) for a given real \( k \) and all possible unstable \( \omega \)'s provided \( \phi \) tends to zero as \( |\omega| \to \infty \) in \( \mathcal{F}(\omega) < 0 \). As will be indicated, it is only necessary to specify the parameter \( \frac{\omega}{\omega_e k^2 \omega_e} \) rather than \( k \) itself.

The contour enclosing \( \mathcal{F}(\omega) < 0 \) is as shown in Figure 1 and consists of the real axis and the 'semi-circle at infinity'. Points at infinity are mapped into the origin and the real axis becomes, in the \( k^2 \) -plane, a closed contour. The interior of the locus corresponds to the unstable \( \omega \) half-plane, the exterior to the stable portion. If the locus encloses a portion of the real axis, then to these values
of \( k \) correspond unstable growing waves. Of course values of other plasma parameters (e.g. \( \omega_e, h_e \) etc) also serve to limit the range of allowed \( k \) values.

Unfortunately \( \omega \) is not given in any simple form, but via the arguments \( \zeta \) (for electrons) and \( \zeta_i \) (for ions). We recall that \( \zeta \) and \( \zeta_i \), in terms of the respective plasma parameters, are given by the expressions

\[
\zeta = \frac{-(\omega - i \nu_e)}{\omega_e |k| e}\sqrt{1 - i \sigma/k},
\]

and

\[
\zeta_i = \frac{-(\omega - i \nu_i)}{\omega_i |k| e}\sqrt{1 + i \sigma/k},
\]

where \( \sigma = \frac{\lambda_{ei}}{\omega_e} \) and \( \sigma_i = \frac{\lambda_{ii}}{\omega_i} \). Hence there is a direct correspondence between the \( \omega \)-plane and the \( \zeta / \zeta_i \)-planes.

Figures 2 and 3 show the portions of the \( \zeta \) and \( \zeta_i \) planes corresponding to unstable and stable frequencies. The real \( \omega \)-axis corresponds to a line in the \( \zeta \) and \( \zeta_i \) planes whose slope depends on the sign of \( k \) as indicated and also the strength of the electric field. The 'semi-circle at infinity' enclosing \( I(\omega) < 0 \) maps similarly as indicated for the case \( k < 0 \) in Figure 2.

If we write \( \frac{\sigma}{|k| e} = \sinh \theta \) and \( \frac{\sigma_i}{|k| e} = \sinh \theta_i \) and define \( \Lambda = \lambda + i \gamma \) and \( \Lambda_i = \lambda_i + i \gamma_i \), then the dividing line \( \chi(\omega) = 0 \) in the \( \omega \)-plane is equivalent to the line

\[
\gamma + X \text{sgn}(k) \tanh \frac{\theta}{2} = \frac{\gamma_e}{\omega_e |k| e \cosh \frac{\theta}{2}} = C_e,
\]

in the \( \Lambda \)-plane, and the line

\[
\gamma_i - X_i \text{sgn}(k) \tanh \frac{\theta_i}{2} = \frac{\gamma_i}{\omega_i |k| e \cosh \frac{\theta_i}{2}} = C_i,
\]
in the \( \zeta_i \)-plane. We note that since the electronic charge is \( e(>0) \), both \( \sigma \) and \( \sigma_i \) are negative numbers. Since \( \tan \alpha \leq 1 \) the slopes of the lines (62a-b) lie within the range \( \pm 1 \). In order to apply the method of Nyquist diagrams, it is necessary for \( \phi \) to tend to zero as \( |\omega| \to \infty \) in \( \mathcal{G}(\omega) < 0 \), and so it remains to verify the behaviour of \( \phi \) (for our problem) in the region
\[
\frac{\pi}{4} \geq \arg \zeta \geq -\frac{\pi}{4}
\]
and similarly for \( \zeta_i \).

Essentially the behaviour of the right-hand side of the dispersion equation (43) depends on the error function \( W(\zeta) \). Referring to Figure 4, we have divided the plane into 4 regions defined by the lines \( y=\pm x \). The asymptotic expansion for \( W(\zeta) \) (i.e. for large \( |\omega| \) ) exhibits what is known as Stokke’s phenomena in that different expansions are needed for different parts of the complex plane (see Stix 297, p. 178-180). In the following, what is said about the \( \zeta \)-plane corresponds equally to the \( \zeta_i \)-plane. We recall that \( W(\zeta) \) is defined by
\[
W(\zeta) = e^{-\zeta^2} \left( 1 + \frac{2i}{\sqrt{\pi}} \int_0^\zeta e^t \, dt \right).
\]

Now Stix (297, p. 178 et seq) has shown that for the function
\[
S(\zeta) = e^{-\zeta^2} \int_0^\zeta e^t \, dt,
\]
the asymptotic expansion \( |\zeta| \to \infty \) reads
\[
S(\zeta) = T(\zeta) + U(\zeta),
\]
where
\[
T(\zeta) \approx \frac{1}{2\zeta^2} \left( 1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \ldots \right)
\]
and
\[
U(\zeta) = 0, \quad \text{if} \quad |\Re(\zeta)| > |\Im(\zeta)|.
\]
or
\[ \frac{\sqrt{k}}{2} e^{-\frac{\xi^2}{2}} \text{ sgn}(k), \text{ for } |R(\xi)| > |\xi(\xi)|. \]

The relevant series expansions for \( W(\xi) \) read: in quadrants A and B \( R(\xi) > \xi(\omega) \)
\[ W(\xi) = e^{-\frac{\xi^2}{2}} + \frac{i}{\sqrt{2} \xi} \left( 1 + \frac{1}{2 \xi^2} + \ldots \right); \]
in quadrant C
\[ W(\xi) = \frac{i}{\sqrt{2} \xi} \left( 1 + \frac{1}{2 \xi^2} + \ldots \right); \]
and in quadrant D
\[ W(\xi) = 2 e^{-\frac{\xi^2}{2}} + \frac{i}{\sqrt{2} \xi} \left( 1 + \frac{1}{2 \xi^2} + \ldots \right). \]

Thus we see that for large \( |\xi| \), in quadrants A and B, the exponential term can be neglected, while it is rigorously absent in quadrant C. In quadrant D \( W(\xi) \) diverges and the usual 'cold' plasma approximations are no longer valid. Clearly, within the unstable regions of the \( \xi \)-plane (for either sign of \( k \)), \( W(\xi) \) is well behaved and hence so is \( \phi(\omega, k) \) for our problem.

In plotting \( \phi \) along the line \( \xi(\omega) = 0 \) it is necessary to have some knowledge of the plasma parameters. Analysis showed that only the ratio of the collision frequencies \( \gamma_{ea} \) and \( \gamma_{el} \) need be given in terms of other quantities. We have taken the value
\[ \frac{\gamma_{el}}{\gamma_{ea}} \sim 2 \phi \left( \frac{m_T}{M_T} \right)^{\frac{1}{2}}, \]
as given by Tanenbaum [28] (p. 252). The electron-ion mass ratio \( \left( \frac{m}{M} \right) \) was taken as \( 10^{-4} \) throughout, while the temperature ratio was considered as a variable parameter reflecting the strength of the external field. The electron drift velocity \( (U) \) was needed only through the ratio \( \frac{U}{V_0} \) and this is completely determined via the
temperature and mass ratios. Explicitly, we have

\[ U^2 = \frac{3v_0^3 m}{M} \left( 1 - \frac{I}{I_0} \right). \]

The remaining value that needs specification is \( k \) itself and in the
showing we found only a value for \( \frac{\nu_e}{\omega_e k^2} \) need be given.

With the mass ratio fixed, the only values needed to specify \( \phi \)
numerically reduce to the temperature ratio and the term involving \( k \).

All ion parameters are derivable from these and the ratio \( \frac{\nu_e}{\nu_i} \).

By evaluating \( \phi \) along the line \( \phi^*(\omega) = 0 \), the unstable
half-plane can be mapped into the complex \( k^2 \)-plane as described
above. Nyquist plots for various parameters of interest have been
computed (Elliot 4130, University of Leicester) and plotted in
Figures 3-12. Plots in which only the electron motion and collisions
were included, have been displayed in Figures 5 and 6. For convenience
the axes were measured in units of \( kh_e \) rather than \( k \) itself.

Both curves correspond to the case when the electron temperature is
three times that of the ions, which obtains in low electric fields
of the order of tens of volts per metre. The parameter \( \frac{\nu_e}{\omega_e k^2} \)
is essentially a measure of the collisional damping - large values
corresponding to heavy damping effects and vice-versa. It is to be
noted that increasing the ratio of the collision to the plasma
frequency or increasing the wavelength produces the same effect in
effectively increasing collisions. Evidently longer wavelengths
result in an increased number of collisions per wave cycle enhancing
the overall damping of the wave. With only the electron motion
included, the modes are stable and the loci do not enclose the
positive \( k^2 \) axis.

Figure 5 corresponds to \( \frac{\nu_e}{\omega_e k^2} = 0.1 \) and light damping.

In Figure 6 we have set \( \frac{\nu_e}{\omega_e k^2} = 10 \) for strong collisional
effects and we note that now the locus lies almost entirely in
\( R(k^2) < 0 \).
The remaining plots (Figures 7-12) include the effects of both ion and electron motions. For convenience, in these cases, the axes in the \( k^1 \)-plane were plotted in units of \( k^1 h^1 \) (\( h^1 \) being the ion Debye length). The ratio of the electron to ion temperatures was chosen to range from 20 (moderate fields of the order of 100's V.m\(^{-1}\)) to 1000 (very strong fields of the order of 1000's V.m\(^{-1}\)). The sign of \( k \) was taken to be positive in anticipation of any unstable ion modes. Collisional damping was reflected through the term \( \gamma_{ee} (\omega_{ee} k h_e) \) as for the pure electron modes above. Values of the order of unity and above evidently give rise to strong damping with the Nyquist plots lying in the region \( R(k^2) < 0 \) (Figures 9 and 11). Smaller values of the order 0.1 lead to lighter damping with typical loci as indicated in the remaining figures. Only Figure 12 indicates the possibility of unstable modes with the locus crossing the real \( k^1 \)-axis at \( k^1 h^1 = 0.7 \). The double spiralling nature of the curve indicates that for certain values of \( k \) located within the inner loop there corresponds two frequencies in the \( \omega \)-plane. From the curve plotted, it would appear that very large values of \( T^* / T \), i.e. of electric field are needed to excite unstable modes in a weakly ionised plasma. Typically with the ion temperature of the order of a few hundred \( ^\circ \text{K} \), electron temperatures of tens of thousands of degrees are required for possible unstable oscillations. Now temperatures of this size correspond to energies of the order of several electron volts and is roughly the minimum energy necessary for excitation and ionisation to occur. Since such processes are not considered here, a necessary restriction on the upper limit of \( T^* \) would certainly be something like \( 10^4 ^\circ \text{K} \).

Apparently, within the definition of a weakly ionised gas, unstable modes in such a system subject to a uniform electric field, would appear to be very unlikely indeed. This is in contrast to Stenflo's \( \Sigma 23 \) findings (and that of \( \Sigma 31 \) and Reiter \( \Sigma 32 \)) in which he
finds unstable modes can be supported when the drift velocity of the species exceeds the thermal velocity. In support of his findings, Stenflo quotes Reiter whose conclusions were similar but applied to a fully ionised plasma. As we have indicated, for electrons the ratio of drift to thermal velocity is always small in a weakly ionised plasma, typically being of the order of $10^{-2}$. For the ions we have neglected their directed velocity in this analysis. In fact if $U_\perp$ denotes the ratio of the ion drift to the ion thermal velocity and $U$ similarly for the electrons, then with $\gamma_{\perp} \sim (\frac{m_i T}{m_e T_e}) \gamma_e$, we have approximately

$$U_\perp \sim U \frac{T_e}{T}.$$  

With $U$ typically of the order $10^{-2}$ and $T \sim 100 \text{ K}$ electron temperatures in excess of $10^6 \text{ K}$ are necessary to produce ion drift velocities larger than the corresponding thermal speeds. At such temperatures, ionisation and excitation of the molecules becomes important, and the plasma can no longer be considered as simply weakly ionised. Evidently the very high temperatures necessary for unstable modes result from the smallness of the ratio of the directed thermal velocity for the electrons. At such temperatures $U_\perp \sim 1$ and our assumption of an isotropic velocity distribution for the ions could be criticised.

Stenflo's conclusion that the system is unstable if the drift velocity of each species is larger than the corresponding thermal velocity requires modification. Certainly, as we have seen above, this can never happen for the electrons because of the dependence of the thermal speed on the external field. Indeed, in the first part of his paper Stenflo completely ignores such a dependence and proceeds to the above statement tacitly assuming a sufficient drift velocity was attainable in his system. Later he includes terms which in fact correspond to a variation in thermal speed with the
electric field. However no concluding remarks were made owing to the extreme complexity of the dispersion relation. For the ions we find that at sufficiently high temperatures the drift to thermal velocity ratio exceeds unity and the system may be unstable. However, in the showing, it appears that the temperatures required need to be of the order $10^{6} \text{ K}$ or above - that is at temperatures associated with ionisation and excitation.

In this chapter we have tried to present a coherent analysis of wave oscillations in a weakly ionised plasma permeated by a uniform electric field. Our main aim has been to derive a dispersion relation relating the angular frequency $\omega$ and the wave number $k$ of longitudinal harmonic waves. As regards conclusions with respect to possible instabilities of the system we have attempted to present some definite if, albeit, qualitative remarks. It is hoped that this chapter has gone some way in clarifying the physics of the problem and the work covered by previous authors.

**APPENDIX**

It is required to evaluate integrals of the form \( \int_{\mathcal{J}} \),

\[
\int_{0}^{\infty} \phi^{-\alpha} e^{-\frac{\phi}{\alpha}} d\phi, \quad \gamma = 1, 2, \tag{1}
\]

in calculating the dispersion relation. We start with the standard result

\[
\int_{0}^{\infty} e^{-at - bt} dt = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}} C \text{ERFC}\left(\frac{b}{2\sqrt{\alpha}}\right), \text{ REAL } \alpha > 0,
\]

\[
= \frac{1}{2} \sqrt{\frac{\pi}{\alpha}} W\left(\frac{i}{2\sqrt{\alpha}}\right). \tag{2}
\]

\(W(\mathcal{Z})\) is defined in \(\mathcal{J}\) and \(\text{ERFC}(\mathcal{Z})\) is the complementary error function, which are connected by the relation
Putting \( \zeta = \frac{1}{2j\alpha} \) and differentiating with respect to \( \alpha \) both sides of equation (2) gives

\[
\int_{0}^{\infty} t e^{-at-uc} \, dt = \frac{\sqrt{\pi}}{4a^2} \left\{ W'(\zeta) + \frac{2i}{\sqrt{\pi}} \right\}.
\]

(3)

\( W'(\zeta) \) may be eliminated from (3) using the result \( \int_{27} \)

\[
W'(\zeta) = -2\zeta W(\zeta) + \frac{2i}{\sqrt{\pi}}.
\]

(4)

This gives the final form

\[
\int_{0}^{\infty} t e^{-at-uc} \, dt = \frac{\sqrt{\pi}}{4a^2} \left\{ W'(\zeta)(1-2\zeta') + \frac{2i}{\sqrt{\pi}} \right\}.
\]

(5)

Similarly by differentiating (2) with respect to \( \zeta \) we have

\[
\int_{0}^{\infty} t e^{-at-uc} \, dt = -\frac{\sqrt{\pi}i}{4a} W'(\zeta)
\]

which by (4) is equal to

\[
\int_{0}^{\infty} t e^{-at-uc} \, dt = \frac{\sqrt{\pi}}{2a} \left( \frac{1}{\sqrt{\pi}} + i \zeta' W(\zeta) \right).
\]

(6)

Comparison of (1) and (2) yields \( \zeta = i \xi, \xi = i \zeta, \zeta' W(\zeta) \) and

\[ \zeta = \frac{i \sqrt{2 \pi}}{2j\alpha}. \]

The forms (3) and (6) can be directly substituted into equation (29), \( \zeta \), to yield the final form (30), together with the definition of \( \zeta \) in terms of \( \omega, k \) and the plasma parameters.
CHAPTER 3

TENSOR EXPANSION FOR THE RELATIVISTIC
BOLTZMANN EQUATION.

§1 INTRODUCTION.

The classical theory of the transport phenomena of gases, and
associated problems, has been a rapidly developing subject from its
earliest beginnings (due to Maxwell) in the 1850's to the present day.
The first general solutions to the transport coefficients were obtained
independently by Chapman \[33\] in 1916, and by Enskog \[34\] in 1917.
Since then, the ideas have been considerably developed and generalised.
Applications have been made to such problems as the determination of
interatomic forces, propagation of sound and, generalisations to high
density gases. The 'classic' work to be found on this subject is the
book by Chapman and Cowling \[4\] on the mathematical theory of non-
uniform gases.

In contrast, the transport phenomena of high temperature
(relativistic) gases is still, comparatively, an uncharted field. Only
recently has any detailed work been undertaken in the literature, and
it is to this topic that we devote these remaining chapters. The
first published work appears to have been by Israel \[35\] who derives
transport coefficients for his so-called 'Relativistic Maxwellian
Molecule'. These he defines by certain simplifying assumptions
concerning the collision cross-section relative to the force law
between the particles. Israel's development of the governing equation;
essentially follows the Chapman-Enskog approach adapted to the needs
of the relativistic Boltzmann equation. In a series of recent papers
\[36a-f\] from 1968-1971 de Groot and co-workers have essentially
extended Israel's work to a binary gas mixture. However, up to the
present time, as far as we know only Israel's model has been studied.
The assumption made by Israel regarding the collision cross section
enabled him to solve the eigenvalue problem of the linearised collision operator. The expressions obtained for the transport coefficients reduce, in the classical limit, to the well known result for the Maxwellian molecule (see [4], p. 173 et seq). Such classical molecules are known to exist with certain types of polarizable particles, but there appears to be no evidence to support the relativistic case.

This present work presents a study of the transport phenomena of high temperature plasmas in which the particle interactions are governed by the Coulomb inverse square law. At temperatures high enough to produce ionisation in any significant degree (\( > 10^6 \text{K} \)) it would seem that this is the most important force law to be considered. True enough, in attempting to present a unified theory for a physically more realisable interaction, we in fact take a step backwards in that the governing equations become that much more complex. The eigenvalue theory of Israel is lost and we are presented with a challenging, though not insurmountable, problem.

Up to now, work on the relativistic Maxwellian molecule has been undertaken using the Chapman-Enskog approach. It became apparent that a more useful approach for us would be in terms of a tensor expansion analogous to the classical case discussed in Chapter 1. This has the advantage that a wider class of problems can be tackled, although our application is limited to the determination of transport coefficients. Because of its generality and wider appeal, we have devoted this chapter to the development of the tensor expansion applied to the relativistic Boltzmann equation. In the showing it was found that this presented a fairly straightforward problem provided the collision term could be expanded in a similar tensorial manner. However, as with most transport equations, it is the collision terms that prove to be the most difficult to construct. As far as this chapter is concerned the appropriate properties will be assumed. Chapters 5 and 6 are devoted to the derivation of interaction terms for the Coulomb case.
In §2 we begin with some tensor definitions and nomenclature for the relativistic case. §3 is concerned with the representation of Boltzmann's equation in polyadic tensor form. The expansion of the distribution function is given in terms of 'reduced-velocity' tensors and this is substituted into the transport equation to give the first three moment equations. The procedure is very similar to that given in Chapter 1 for the classical case. The collision term is assumed expandable in tensor form but development of appropriate 'collision-moments' is deferred until Chapter 5.

§2 TENSOR DEFINITIONS AND NOMENCLATURE.

Throughout this and the remaining chapters a suffix notation and the summation convention are used. It is necessary to distinguish between the spatial three component vectors and the four vectors of special relativity. To denote the former we use a Greek suffix running from 1 to 3 and the latter is denoted by a Latin suffix running from 1 to 4. Repeated suffices imply, in the usual way, summation over all the values unless otherwise indicated. The associated four vectors \(\mathbf{x}_i, \mathbf{U}_i\) and \(\mathbf{F}_i\) are defined by

\[\mathbf{x}_4 = i c t, \quad t \text{ being time;}\]

\[\mathbf{C} \mathbf{U}_i = \mathbf{V}_i \sqrt{1 + \mathbf{U}^2}, \quad \mathbf{U}_4 = i \sqrt{1 + \mathbf{U}^2},\]

and

\[\mathbf{F}_i = \mathbf{F}_i \sqrt{1 + \mathbf{U}^2}, \quad \mathbf{F}_4 = i \mathbf{U}_4 \mathbf{F}_i.\]

\(c\) denotes the velocity of light. The vector \(\mathbf{U}_i\) is termed the reduced velocity and \(\mathbf{U}\) denotes its magnitude. \(\mathbf{V}_i\) is the normal velocity vector and \(\mathbf{F}_i\) denotes external forces.

Corresponding to the velocity tensors \(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}\) defined in Chapter 1 we define the relativistic reduced velocity tensor of order \(J\) by the statement

\[
\frac{\mathbf{U}^{(J)}}{\mathbf{U}^2} = \frac{\mathbf{U}_1 \mathbf{U}_2 \mathbf{U}_3 \cdots}{\mathbf{U} \mathbf{U} \mathbf{U} \cdots}, \quad (1)
\]
with $l$ terms implied in both the numerator and denominator. In
the next section the distribution function is expanded in terms of
the tensors (1).

The distribution function is denoted by $f$ and is regarded as a
function of $t$ and the components of $\mathbf{X}_\lambda$ and $\mathbf{U}_\lambda$. It is defined by
the statement that

$$ f(t, \mathbf{X}_\lambda, \mathbf{U}_\lambda) \, dx_1 \, dx_2 \, dx_3 \, du_1 \, du_2 \, du_3, $$

represents, at time $t$, the number of particles with spatial co-ordinates
in the range $X_\lambda$ to $X_\lambda + dx_\lambda$ and reduced velocities in the range
$U_\lambda$ to $U_\lambda + dU_\lambda$.

**RELATIVISTIC BOLTZMANN EQUATION IN POLYADIC TENSOR FORM.**

From the definition of $f$ (the distribution function) given by
(2) we see that its represents the number density of points in the
phase-space composed of the 6 orthonormal co-ordinate axes of $\mathbf{X}_\lambda$ and
$\mathbf{U}_\lambda$. The function $f$ defined on this space has a continuity equation
exactly analogous to the ordinary 3-dimensional counterpart.

In phase ($\mu$) -space, we define a 6-dimensional 'position'
vector by the statement

$$ \mathbf{w} = (X_\lambda, U_\lambda), $$

where $X_\lambda$ and $U_\lambda$ refer to the position and reduced velocity vectors
of a particle. From (3), the phase-space generalized velocity is

$$ \frac{d\mathbf{w}}{dt} = \dot{\mathbf{w}} = \left( \frac{c \, u_\lambda}{\sqrt{1 + u^2}}, \frac{F_\lambda}{mc} \right), $$

where $m$ is the rest mass and $F_\lambda$ is the external force acting on the
particle given by the equation of motion

$$ \frac{du_\lambda}{dt} = \frac{F_\lambda}{mc} . $$

Also the gradient operator in $\mu$-space is
\[ \nabla_\nu = (\nabla_\tau, \nabla_\nu), \]  

where \( \nabla_\tau \) and \( \nabla_\nu \) are the usual 3-dimensional gradients for position and reduced velocity respectively.

The continuity equation for \( f \) becomes

\[ \frac{\partial f}{\partial t} + \nabla_\nu (f \nu') = \text{SOURCE TERMS} = C(f). \]  

Here \( C(f) \) represents essentially the rate of gain of particles to unit volume of phase-space per unit time.

With the definitions (4) and (5), (6) becomes

\[ \frac{\partial f}{\partial t} + \frac{cu_\lambda}{\sqrt{1 + u^2}} \frac{\partial f}{\partial x_\lambda} + \frac{f}{mc} \frac{\partial F_\lambda}{\partial u_\lambda} + \frac{F_\lambda}{mc} \frac{\partial f}{\partial u_\lambda} = C(f), \]  

where we have used the fact that the co-ordinate axes of phase-space are orthonormal to each other.

The only forces considered in the remaining chapters are those of electromagnetic origin. The Lorentz force on a charged particle (of mass \( m \) and charge \( e \)) is

\[ F = \frac{e}{m} \left( E + \frac{cu}{\sqrt{1 + u^2}} \wedge B \right), \]

where \( E \) is the electric field and \( B \) the magnetic field. Clearly 

\[ \nabla_\nu \cdot F = 0, \]

so that (7) reduces to

\[ \frac{\partial f}{\partial t} + \frac{cu_\lambda}{\sqrt{1 + u^2}} \frac{\partial f}{\partial x_\lambda} + \frac{F_\lambda}{mc} \frac{\partial f}{\partial u_\lambda} = C(f), \]  

which is the final form of the relativistic Boltzmann equation that we shall subsequently be considering. The left hand side of this
equation was given by Clemmow and Willson [42, 7].

In non-relativistic mechanics, we recall (Chapter 1) that we can expand $f$ in the tensor form

$$f = \sum_{l} f^{(l)} \frac{V^{(l)}}{V^{l}},$$

where the $l$th term on the right represents the $l$th order dot product of the tensors $f^{(l)}$ and $\frac{V^{(l)}}{V^{l}}$.

Within the inertial frame we are considering, we can define the relativistic $3$-tensor of order $l$ in the reduced velocity by equation (1). Since the relativistic distribution $f$ is a function of $U^\lambda$, by analogy with (9), we can write the tensor expansion for $f$ as

$$f = \sum_{l} f^{(l)} \frac{V^{(l)}}{U^{l}},$$

where the $f^{(l)}$ are fully symmetric, irreducible tensors depending on $x^\lambda, u$ and $t$ only. The properties of the $f^{(l)}$ tensors may be deduced as for the classical case (Appendix 2, Chapter 1) with $U$ interpreted for $V$.

Our aim is to substitute the expansion (10) into the Boltzmann equation (8) and derive the associated moment equations. The actual analysis closely follows the classical case described in Chapter 1 so that the reader can refer there for clarification on any mathematical details.

The tensor expansion (10) is equivalent to the spherical harmonic expansion

$$f(x, x^\lambda, u) = \sum_{j, m} f_{j, m}(x, x^\lambda, u) Y_{j, m}(\theta, \phi).$$
where \( \gamma_{\ell m n}(\theta, \phi) \) is the spherical harmonic of order \( \ell \) and the \( f_{\ell m n} \) coefficients are functions of position \( \mathbf{x} \), time \( t \), and reduced velocity magnitude \( \tilde{U} \) only. The \( f^{(k)} \) tensors, as in the classical case, are composed of the \( f_{\ell m n} \) terms of order \( \ell \) only.

Before proceeding with the analysis, the collision term \( C(f) \) must be considered. In this chapter it is assumed that \( C(f) \) can be similarly expressed in tensorial form as

\[
C = \sum_{\ell} C^{(\ell)}(\mathbf{U}) \frac{\tilde{U}(\mathbf{U})}{U \ell}.
\]

Here the \( C^{(\ell)}(\mathbf{U}) \) are assumed fully symmetric, irreducible tensors.

Chapter 5 discusses the construction of the \( C^{(\ell)}(\mathbf{U}) \) and \( C^{(\ell)}(\mathbf{U}) \) terms for Coulomb interactions.

The procedure to obtain the moment equations is to substitute the expansion (10) into the transport equation (8), multiply through by the tensor \( \tilde{U}(\mathbf{U}) \), and integrate over angles, letting \( \ell \) take successively the values 0, 1, 2 \ldots.

Using the results

\[
\frac{\partial}{\partial \ell} \left\{ f^{(\ell)} (\mathbf{U}) \tilde{U}(\mathbf{U}) \right\} = \frac{\partial f^{(1)} (\mathbf{U}) \tilde{U}(\mathbf{U})}{\partial \ell} \frac{\tilde{U}(\mathbf{U})}{U \ell},
\]

\[
\sqrt{1 + \tilde{U}^2} \frac{\partial}{\partial \mathbf{x}} \left\{ f^{(\ell)} (\mathbf{U}) \tilde{U}(\mathbf{U}) \right\} = \sqrt{1 + \tilde{U}^2} \nabla f^{(\ell)} \tilde{U}(\mathbf{U}) \frac{\tilde{U}(\mathbf{U})}{U \ell + 1},
\]

\[
\frac{\partial}{\partial \mathbf{U}} \left\{ f^{(\ell)} (\mathbf{U}) \tilde{U}(\mathbf{U}) \right\} = \frac{\mathbf{U}(\mathbf{U})}{\mathbf{C}} \frac{\partial}{\partial \mathbf{U}} \left\{ f^{(\ell)} (\mathbf{U}) \tilde{U}(\mathbf{U}) \right\} \frac{\tilde{U}(\mathbf{U})}{U \ell + 1} + \ell \frac{\partial}{\partial \mathbf{U}} \left\{ f^{(\ell)} \tilde{U}(\mathbf{U}) \right\} \frac{\tilde{U}(\mathbf{U})}{U \ell + 1},
\]

and
Boltzmann's equation becomes

$$\frac{U \cdot \dot{\mathbf{w}}}{\sqrt{1 + U^2}} \cdot \nabla \left\{ \int \frac{f^{(l)} - f^{(l-1)}}{U} \right\} = \frac{1}{\sqrt{1 + U^2}} \left( \omega_c \wedge f^{(l)} \right) \cdot \frac{\mathbf{U}^{(l)}}{U}$$

\[ \sum_{l} \left\{ \frac{\partial f^{(l)}}{\partial t} + U^{l-1} \cdot \nabla \left( \frac{f^{(l-1)}}{U} \right) + \frac{l \omega_c \wedge f^{(l)}}{\sqrt{1 + U^2}} - \frac{C^{(l)}}{\sqrt{1 + U^2}} \right\}
+ (l+1) \left( \mathbf{a} \cdot \frac{f^{(l+1)}_{(l)}}{l+1} \right) \frac{\mathbf{U}^{(l)}}{U} = 0. \]

(12)

\( \mathbf{Q} \) is the electron force per unit mass (\( \mathbf{c} \mathbf{E} / m \)) and \( \omega_c \) is the quantity \( \mathbf{c} \mathbf{E} / m \) where \( |\omega_c| \) is the cyclotron frequency. In the limit as \( C \to \infty \) we can recover the classical equation by letting \( U \to 0 \) in such a way that \( C U \to V \) (the non-relativistic speed).

Having established the expanded form of the Boltzmann equation (12) we now proceed to derive the form of the moment equations. The procedure is very similar to the classical case discussed in Chapter 1. The scalar (\( l = 0 \)) equation is obtained from (12) by multiplying through by \( \sin \theta \cos \phi \) and integrating over angles using the relation

$$\int_0^{2\pi} \int_0^\pi \cos \theta_x \cos \theta_y \cos \theta_z \sin \theta \, \text{d} \theta \, \text{d} \phi = \frac{\pi}{2} \sum_{l=1}^\infty (l+1)^{(l+1)} \frac{1 \cdot 3 \cdot 5 \cdots (l-1) \cdots (3-l) \cdots (1-l)}{l \cdot 3 \cdot 5 \cdots (l+q+f+1)}$$

\( \cos \theta_i \) (i=1,2 or 3) represents the direction cosines of the reduced velocity \( \mathbf{U} \) and are defined by \( \mathbf{c} \mathbf{E} / m \). The scalar equation after division by such terms as \( \frac{4 \pi}{3} \), may be written as
The terms in (13) come from the $l=0$ and $l=2$ terms in (12) only.

Tensors that are odd in the components of $U$ integrate to zero; and
for a fully symmetric tensor $A^{(l_1,2)}$

\[
\int_0^{2\pi} \int_0^{\pi} A^{(l_1,2)} \frac{U^{(l_1)}}{U^{(l_1)}} \sin \theta d\theta d\phi = 0,
\]

if the operation of contraction on $A^{(l_1,2)}$ gives zero.

Next we multiply (12) by $RU/U$ and again integrate over angles

obtaining the vector equation

\[
\frac{\partial f^{(1)}}{\partial t} + \frac{C U}{\sqrt{1 + U^2}} V_r f^{(3)} + \frac{\omega U}{C} \frac{\partial f^{(15)}}{\partial U} + \omega U V^3 \omega f^{(2)} - C^{(1)}
\]

\[
+ \frac{2}{5} \frac{C U}{\sqrt{1 + U^2}} V_r f^{(5)} + \frac{2}{5C^2 U^2} \frac{\partial}{\partial U} \left(U^3 \omega f^{(5)}\right) = 0.
\]

The terms in (14) come from the $l=1$ and $l=3$ terms of (12) only.

Some integral results, useful in the derivation of (13), are as for
the classical case with $V$ replaced by $U$ given in Chapter 1, and need
not be repeated here.

Again for the 3rd moment equation, we multiply (12) by $UU/U^3$

and integrate over angles to yield the tensor equation for $f^{(3)}$:
\[
\left\{ \frac{5}{2} \frac{\partial f^{(0)}}{\partial \varepsilon} - \frac{5}{2} C^{(0)} + \frac{1}{2} \frac{c u}{\sqrt{1 + u^2}} V_r \cdot f^{(0)} + \frac{1}{2 c u^2} \frac{\partial}{\partial u} \left( u^4 a \cdot f^{(1)} \right) \right\} \nabla \cdot f^{(2)} \\
+ \frac{\partial f^{(a)}}{\partial \varepsilon} + \frac{c u}{\sqrt{1 + u^2}} V_r f^{(0)} + \frac{u a}{c} \frac{\partial}{\partial u} \left( \frac{f^{(0)}}{\varepsilon} \right) + \frac{2 w c \wedge f^{(2)}}{\sqrt{1 + u^2}} \\
\n- C^{(2)} + \frac{3}{7} \left\{ \frac{c u}{\sqrt{1 + u^2}} V_r \cdot f^{(0)} + \frac{1}{c u} \frac{\partial}{\partial u} \left( u^4 a \cdot f^{(1)} \right) \right\} \nabla \cdot f^{(2)} = 0.
\]

(15)

Here \( I^{(2)} \) denotes the 3x3 unit matrix and \( \Lambda^{(2)} \) means \( \frac{1}{2} (\Lambda^{(2)} + \Lambda^{(2) T}) \) with \( T \) denoting the usual matrix transpose. Terms in (15) come from the \( \dot{L} = 0, 2 \) and \( 4 \) terms of (12) only.

In the remaining chapters we shall be dealing with the \( f^{(0)} \) and \( f^{(1)} \) terms only, truncating the 2nd and higher order tensors in the expansion for \( f \). Henceforth we shall be working with the scalar and vector equations (13) and (14). Of course their form has not entirely been specified since the collision terms need to be constructed. This is undertaken in Chapter 5 and in Chapter 6 the moment equations are employed in some transport problems.

We note in passing, that the form of the moment equations holds in any frame of reference since the Boltzmann equation (8) is relativistically invariant. However, the usefulness of the expansion is limited to those cases where the \( f^{(1)} \) and higher tensor terms are small so that truncation of the series at some suitable point is meaningful. In particular, we would expect that if the electromagnetic fields and spatial gradient terms are sufficiently small, then a useful description of the distribution function is afforded by the \( f^{(0)} \) and \( f^{(1)} \) terms alone.
CHAPTER 4

RELATIVISTIC BOLTZMANN AND FOKKER-PLANCK EQUATIONS.

§1 INTRODUCTION.

In the previous chapter we formulated the moment expansion of the relativistic Boltzmann equation in terms of reduced velocity tensors. The collision term was assumed expandable in this form but was otherwise left unspecified. Thus the equations presented are quite general in application provided the collision moments \( \gamma^{(l)} \) can be specified for the particular problem in hand. Of course, in any specific application, the construction of the \( \gamma^{(l)} \) terms when carried beyond the first order \( (\ell = 1) \) expansion) can prove quite difficult. For this reason we treat only the \( \ell = 0 \) and \( \ell = 1 \) terms in the subsequent chapters.

Of interest to us in this present work is the application of the relativistic moment equations of plasmas with Coulomb interactions. In order to be able to construct appropriate collision moments the Boltzmann collision term \( C(f) \) must be specified. We choose here a form for \( C(f) \) derivable from the covariant Landau equation developed by Belyaev and Budker [37, 39]. With \( C(f) \) completely defined for electron-ion collisions, Chapter 5 undertakes the construction of the \( \gamma^{(l)} \) terms up to \( l = 1 \). Electron-electron interaction terms are derived as a special case.

This chapter is concerned with the development of \( C(f) \) into a suitable form from which the collision moments can be derived. It became apparent that an appropriate starting point would be with the work of Akama [38] in which he develops a relativistic formulation of Boltzmann's equation with a collision term constructed on the basis of binary elastic collisions. Akama is able to show that the Fokker-Planck approximation to the Boltzmann equation yields a covariant Landau equation which agrees with the result of Belyaev and Budker,
q.v.. For systems of charged particles another approach is possible based on such macroscopic parameters of the medium as the dielectric constant. This method (see for example Silin [39]) leads to a kinetic equation which yields the relativistic Landau equation in the limiting case. The Debye screening of charged particle interactions is shown to come naturally in this latter approach. In contrast, the impact method of Akama requires the cutting off the collision integral at an appropriate point to remove the Coulomb divergence as in the non-relativistic case. Both approaches yield identical results. We have chosen to use the somewhat simpler approach of Akama to develop our form of $C(f)$ for electron-ion collisions.

In §2 we begin with an account of the notation employed in this and subsequent chapters. The following section is concerned with the definition of relativistic distribution functions in both 8-dimensional (relativistic) and 6-dimensional (spacelike) phase-space. In §4 we present Akama's work on the relativistic Boltzmann equation. §5 is concerned with the derivation of the Fokker-Planck equation for the 6-dimensional distribution function. It is this form of the transport equation that we use in the next chapter to derive the collision moments for the Coulomb case. Electron-electron effects come out as a special case. In §6 we show how to transform the Fokker-Planck terms into Landau form and that in the limiting case this reduces to the well known classical result for Coulomb plasmas. A brief mention is made in §7 of the screening term which results when the cut-off approximation is applied to charged particle interaction. Finally in §8 we give some invariance relations for collision terms.

In the collision processes between charged particles it is well known ([2], p. 203) that radiation effects are involved. In particular a photon can either be emitted or absorbed by a particle
in a two-body collision. The former process is termed Bremsstrahlung and corresponds, in quantum mechanics, to a radiative transition of the particle between two energy states. The inverse process is that in which a photon is absorbed upon collision. In a first approximation interaction between like particles produce no radiation effects. For particles of widely different masses the lighter one produces a dominant radiational contribution (e.g. collisions between electrons and ions). In the collision terms that we are considering all radiational contributions have been neglected. Indeed, their inclusion presents a formidable task which some writers (e.g. \[ \gamma^4 \]) have undertaken. The resulting equations for plasma and radiation appear in too cumbersome and complicated a form for much detailed work and are hence not considered here.

The question of radiation effects in collision processes between charged particles has been discussed by Krizan [44\[ and in connection with the Belyaev and Budker term [37\[. The lowest order approximation retains only the terms in \( \epsilon^4 \) (\( \epsilon \) is the electronic charge) and corresponds to a \( (\sqrt{\gamma})^4 \) approximation in a pure particle theory. Krizan indicates that the question of radiation effects will come in if one proceeds beyond the lowest order. Belyaev and Budker's collision term, in which radiation is neglected, effectively goes beyond this approximation in \( (\sqrt{\gamma})^4 \) (it is derived for all values) although only retains the lowest order in terms \( \epsilon^4 \). Thus the physical content beyond \( (\sqrt{\gamma})^4 \) is open to some criticism but use to this order seems justified. In this chapter the full collision term is considered. In Chapters 5 and 6 the form of the interaction term to order \( (\sqrt{\gamma})^4 \) is considered, and in the light of what has been said, the neglect of radiational effects to this order seems in order.

\[ \S 2 \]

**NOTATION.**

A suffix notation and summation convention are used. Spatial three-vectors are denoted by a Greek suffix running from 1 to 3 while
a Latin suffix running from 1 to 4 is used to denote the four vectors of special relativity. A four vector is denoted by \( A^\alpha \), with \( (A^1, A^2, A^3) \) the spacelike components and \( A^4 = iA_0 \) the timelike component.

In this and subsequent chapters it is necessary to distinguish between quantities appropriate to electrons and ions. We use a small letter to denote the former and a corresponding capital letter to denote the latter. Thus for the electron reduced velocity we write \( u \) and for the ions \( U \). Similarly the distribution functions are \( f(\chi_\lambda, U_\lambda, \epsilon) \) for electrons and \( F(\chi_\lambda; U_\lambda, \epsilon) \) for ions. Other usage, where appropriate, will be made apparent.

\section{Relativistic Distribution Functions}

In this section we present some of Akama's work on the relativistic distribution functions. Our derivations follow slightly different lines. We introduce an 8-dimensional phase-space with co-ordinates \( x_i, u_i \) (for electrons) and denote the volume element \( dV \) by

\[ dV = d^4x_i d\epsilon_i \]

where

\[ d^4x_i = d^3x_i (c dt) \]

and

\[ d^4u_i = d^4u_i (d\epsilon_0) \].

\( d^3x \) and \( d^3y \) are the usual cartesian position and reduced velocity volume elements respectively, and by definition \( u_4 = i\epsilon_0 \). We denote a distribution function on this space by \( f(\chi_i, U_i) \) whose properties are to be specified later.

In 8-dimensional phase-space, we defined a representative point by the statement

\[ \hat{\omega} = (\chi_i, U_i) \]

\[ = (\chi_\lambda, ic\epsilon, U_\lambda, i\epsilon_0). \]  

(1)
In terms of the energy \( \frac{\xi}{m c^2} \) of a particle of rest mass \( m \) we have
\[
U_0 = \frac{\xi}{m c^2}.
\]
The generalised phase-space velocity is given as
\[
\frac{d\mathbf{v}}{d\tau} = \left( \frac{dx_i}{d\tau}, \frac{du_i}{d\tau} \right),
\]
where \( \tau \) is the proper time given by \( d\tau = \frac{mc dt}{\xi} \). Hence from (1) and (2) we have
\[
\frac{d\mathbf{v}}{d\tau} = \left( c u_\lambda, \frac{\xi}{mc^2}, u_0 \frac{du}{dT}, i u_0 \frac{du}{dT} \right).
\]
The gradient operator in 8-dimensional space is given by
\[
\nabla_\mathcal{J} = \left( \nabla_\tau, \frac{\mathbf{v}}{c} \frac{\partial}{\partial T}, \nabla_\mathcal{J}, -i \frac{\partial}{\partial \mathcal{J}} \right).
\]
The continuity equation for \( \mathcal{J} \) can thus be written as
\[
\nabla_\mathcal{J} \cdot \left( \frac{d\mathbf{v}}{d\tau} \right) = \text{Source terms} = S(3).
\]
Substituting the defining equations (3) and (4) into (5) yields an equation for \( \mathcal{J} \) which can be written in the form:
\[
\frac{\partial}{\partial \mathcal{J}} \left( \frac{d\mathbf{v}}{d\tau} \right) + \nabla_\mathcal{J} \cdot \frac{du}{dT} + \nabla_\mathcal{J} \cdot \left( u_0 \frac{du}{dT} \right) + \frac{\partial}{\partial u_0} \left( u_0 \frac{du_0}{dT} \right) = S(3).
\]
In suffix notation (6) may be written as
\[
\frac{\partial}{\partial x_i} \left( \frac{dx_i}{d\tau} \right) + \frac{2}{\partial u_i} \left( \frac{du_i}{d\tau} \right) = S(3).
\]
We next multiply (6) by \( \frac{d^2 \mathcal{J}}{d\tau^2} \) and integrate out to obtain
\[
\frac{\partial}{\partial \mathcal{J}} \int \frac{\xi}{mc^2} \mathcal{J}(x_i, u_i) dt du_i = 0.
\]
If the source terms are only due to elastic binary collisions then
\[ \int S(3) \, d^4u_i = 0 \]
because of number conservation of particles.

(8) is the result which Akama uses to define the distribution \( \mathcal{F} \). We note that the value of the integral in (8) does not change with time. Akama thus sets it equal to the number of particles of the system under consideration, i.e.

\[ \int E \mathcal{F} \, d^3x \, d^4u_i = N, \tag{9} \]

where the integration is over the whole of real space and momentum space. Alternatively, the statement

\[ \mathcal{F}(x_i, u_i) \frac{E}{mc^2} \, d^3x \, d^4u_i \quad \tag{10} \]

represents the number of particles of the system with position co-ordinates in the range \( x_\lambda \) to \( x_\lambda + d\mathcal{X}_\lambda \) and reduced velocity in the range \( u_i \) to \( u_i + d^4u_i \). Now the volume elements \( \frac{E}{mc^2} \, d^3x \) and \( d^4u_i \) are relativistically invariant, so that the expression (10) becomes invariant if \( \mathcal{F}(x_i, u_i) \) is.

Up to now we have said little about the rest mass \( m \) of the particles. The above formulation of \( \mathcal{F} \) allows for variable \( m \) (as a function of time), but as far as we are concerned only constant \( m \) will be considered subsequently. With \( m \) constant the relation \( u_o = \sqrt{1 + u^2} \) holds and \( \mathcal{F} \) involves the delta function \( \delta(u_o - \sqrt{1 + u^2}) \), negative energies being excluded. Thus we write

\[ \mathcal{F}(x_i, u_i) = f(x_\lambda, u_\lambda, \epsilon) \frac{mc^2}{E} \delta(\epsilon - \sqrt{1 + u^2}) \]

where \( f(x_\lambda, u_\lambda, \epsilon) \) is the ordinary 6-dimensional distribution function as defined in Chapter 3 (§2). Clearly from (11) and (9)

\[ \int f(x_\lambda, u_\lambda, \epsilon) \, d^3x \, d^3y = N. \]
For constant rest mass, the external forces \( F_{\lambda} \) are related to the reduced velocities through the equations of motion

\[
m c \frac{dU_{\lambda}}{dt} = F_{\lambda}.
\]

Substitution of (11) and (12) into (6) gives, after integrating over \( U_{\lambda} \), the familiar equation for \( f \):

\[
\frac{\partial f}{\partial t} + \frac{cU_{\lambda}}{\sqrt{1 + U_{\lambda}^2}} \frac{\partial f}{\partial x_{\lambda}} + \frac{F_{\lambda}}{m c} \frac{\partial f}{\partial U_{\lambda}} = \int S(\mathcal{F}) dU_{\lambda} = C_f(f).
\]

In deriving (13) it is assumed that \( \nabla \cdot F = 0 \) as is the case for electromagnetic forces. Finally, from (11) we note that the distribution function \( f(x_{\lambda}, U_{\lambda}, t) \) is Lorentz invariant (the multiplying factor is essentially \( S(U_{\lambda}^2) \), \( U_{\lambda}^2 \) being the scalar product of four-vectors, is invariant).

§ 4 THE BOLTZMANN EQUATION.

In this section we present the work of Akama \(^{38}\) on the relativistic Boltzmann transport equation. This is essentially a conservation equation for the distribution function \( f \) in which the source term is constructed on the basis of binary elastic collisions. The derivation follows similar lines to the classical theory except that now consideration has to be given to the 8-dimensional character of the phase-space upon which \( f \) is defined. At this stage we will not specify the nature of the interacting particles apart from labelling them species '1' and species '2'.

We consider elastic collisions between both species with typical reduced velocities \( U_1 \) (species 1) and \( U_2 \) (species 2). The latter are chosen as target for bombardment by group 1 particles. Let us focus our attention on the group of species 1 particles to be found in the range \( d^3 \omega \ d^4 U_1 \) about the point \( x_{\lambda}, U_1 \). After a time \( dt \) if no collisions had taken place, all the particles would have entered the range \( d^3 (\omega + \frac{d\omega}{dt} dt) d^4 (U_1 + \frac{dU_1}{dt} dt) \) about \( r + \vec{v}dt \),
\( \dot{U}_i + \dot{U}_i \, dt \). However frequent collisions do occur and thus some of the particles do not continue on the original course laid down by the specification of \( U_i \) and \( F \) (external force) at time \( t \).

The total number of particles lost from this group is denoted by 
\[
S_{1,2}^{(c)} \, d^3 \chi \, d^4 U_i \, dt
\]
while that gained by collisions is 
\[
S_{1,2}^{(t)} \, d^3 \chi \, d^4 U_i \, dt
\]
\( S_{1,2}^{(c)} \) is essentially the rate of loss of species 1 particles per unit volume of 8-dimensional phase-space; similarly for \( S_{1,2}^{(t)} \).

We suppose a pair of particles (species 1 and 2) collide elastically to produce a pair with reduced velocities \( \dot{U_i} \) and \( \dot{U_i}' \). Momentum conservation holds and the differential cross-section for the interaction, denoted by \( \sigma(\dot{U_i}', \dot{U_i}'', U_i, U_i') \), involves the delta function \( \delta(\dot{U_i} + \dot{U_i}' - \dot{U_i}' - \dot{U_i}'') \).

The mutual flux of incident particles per unit space-time is given by (Akama 38)

\[
I = \int \, \int \beta \beta' \gamma \left( \frac{(p', \xi)}{\beta_i \beta_i'} \right) d^4 U_i \, d^4 \dot{U}_i', \tag{14}
\]
with reduced velocities in the range \( d^4 U_i \), \( d^4 \dot{U}_i \) about \( U_i \) and \( \dot{U}_i \). In (14) \( \gamma \) is the four product \( U_i U_i' \) and is invariant.

Further \( \beta = \frac{\xi}{\gamma} \quad (\alpha = 1, 2) \) where \( \xi \) represents the energy of a particle. It can be demonstrated that \( I \) is invariant with respect to Lorentz transformations in which the beam and target particle motions are collinear. When the scattering is observed in a frame in which the colliding particles are not initially parallel we do not have any a priori way of defining the cross-section. In view of the invariance that keeps the initial momenta collinear, it is customary to define it to be invariant with respect to all Lorentz transformations. With this in mind there up comes the expression (14) for the mutual flux.

If we bombard a stationary target of \( \rho_2 \) particles per unit volume
with a beam of density \( \rho \) and velocity \( \mathbf{v}_1 \), then the number \( dN \) of scattered particles counted in a given solid angle element \( d\Omega \) (\( = \int d\epsilon d\phi \)) about the direction \( \mathbf{v}_1 \), per unit time per unit volume (of target) is

\[
dN = I \, d\sigma.
\]  

\( d\sigma \) is the differential cross-section for the scattering \( (= \sigma(\Omega) \, d\Omega) \) and \( I \) is the mutual flux of incident particles \( \mathbf{v}_1 \). Expressed in a general frame in which the colliding particles are not initially moving collinearly, \( I \) takes the form as given by (14) and \( \sigma(\Omega) \, d\Omega \) is replaced by

\[
\sigma(u_i', u_i, u_i, v_i) \, du_i' \, dv_i \, d\epsilon_i \, d\epsilon_i'.
\]  

From (14), (15) and (16), the total number of collisions per unit time per unit volume of real space between species 1 and species 2 particles with initial reduced velocities in the range \( u_i \) to \( u_i + d\epsilon_i \) and \( v_i \) to \( v_i + d\epsilon_i \) is given by

\[
I \sigma(u_i', u_i, u_i, v_i) \, du_i' \, dv_i \, d\epsilon_i \, d\epsilon_i'.
\]  

with scattering into the range \( u_i' \) to \( u_i' + d\epsilon_i' \) and \( v_i' \) to \( v_i' + d\epsilon_i' \) respectively. For convenience we write

\[
W(u_i', u_i, u_i, v_i) = (p^2 - 1) \, \sigma(u_i', u_i, u_i, v_i) \, d\epsilon_i \, d\epsilon_i'.
\]  

Expression (17) represents the number of species 1 particles lost from the range \( u_i \) to \( u_i + d\epsilon_i \) per unit space-time. In order to find \( S_{12}^{(-)} \) we must integrate (17) over all scattering angles and all values of \( \epsilon_i \) (the scatterer's initial reduced velocity) obtaining

\[
S_{12}^{(-)} \, du_i \, = \int W \, d\epsilon_i \, d\epsilon_i' \, du_i' \, dv_i' \, d\epsilon_i \, d\epsilon_i',
\]  

(19)
where \( f_i = f_i(x_i, u_i) \), \( f_x = f_x(x_i, u_i) \) and \( W = W(\nu_i, \nu_i', u_i, u_i') \).

The derivation of \( S_{12}^{(t)} \) follows very similar lines. Initial 'velocities' before collision are assumed to be \( \nu_i' \) and \( \nu_i'' \) and scattering into the solid angle \( d\Omega \) with reduced velocities \( \nu_i \) and \( \nu_i'' \). Writing \( f_i' \) for \( f_i(x_i, \nu_i') \) and \( f_x' \) for \( f_x(x_i, \nu_i') \) we have

\[
S_{12}^{(t)} d^4u_i = \int W f_i f_x d^4\nu_i d^4\nu_i' d^4u_i' d^4u_i',
\]

where now

\[
W = W(\nu_i, \nu_i', \nu_i'', \nu_i').
\]

Clearly we must admit the possibility of 'inverse' encounters in which the reverse of the above collisions could happen. It follows then as a consequence of 'detailed balancing' in individual microscopic collisions that

\[
W(\nu_i, \nu_i', \nu_i'', \nu_i') = W(\nu_i', \nu_i'', \nu_i, \nu_i)
\]

and

\[
d^4\nu_i d^4\nu_i' = d^4\nu_i' d^4\nu_i'.
\]

From (19), (20) and (21a-b) the collision term \( S(f_i) \) in the relativistic Boltzmann equation may be written as:

\[
S_{12}(f_i) = S_{12}^{(t)} - S_{12}^{(-)}
\]

\[
= \int W(\nu_i, \nu_i', \nu_i, \nu_i') (f_i' f_x' - f_i f_x) d^4\nu_i' d^4\nu_i d^4u_i'.
\]

With the general form of the collision term \( S(f_i) \) now specified, the relativistic formulation of the Boltzmann transport equation is
completed. (22) represents the rate of gain of species 1 particles per unit volume of 8-dimensional phase-space due to collisions with species 2 particles. The nature of the particle interactions enter via the cross-section $\sigma$ in the $\sqrt{v}$ term.

§5 FOKKER PLANCK EQUATION.

This section is concerned with the development of the Fokker-Planck equation for Coulomb interactions from the collision integral (22) in the Boltzmann equation. From the point of view of collision statistics, the former describes continuous changes while the latter describes discontinuous abrupt changes.

Coulomb forces decrease slowly with distance so that distant collisions with small deflections are more frequent than close violent encounters. Furthermore only the cumulative effect of a number of encounters produce any significant change in the magnitudes and directions of the motions. A measure of the effective range of Coulomb interactions is afforded by the Debye length ($\lambda_D$) which is equal to the average distance travelled by a charged particle in about one plasma period (reciprocal of the plasma frequency). For plasmas of interest to us, in which the fluid or continuum approach applies, we require that the number of particles ($n_p$) in a sphere of radius equal to the Debye length ($\lambda_D$) be large. That is we require

$$n \lambda_D^3 \gg 1,$$

where $n$ is the number density per unit volume. Now the average inter-particle distance ($d$) is of order $n^{-\frac{1}{3}}$, so that for ionised plasmas in which the above condition holds, essentially the Debye length is very much greater than the distance $d$. Violent binary collisions occur in distances less than $d$ and for Coulomb plasmas are usually small in number. The bulk effect of interactions comes from simultaneous small deflections in the range $d < l < \lambda_D$, where $l$ is the impact parameter. The impact approach yields results identical to those of wave theory (eg Silin [39]) even though it ignores the
breakdown of binary collisions for distances beyond the interparticle
distance since the more important small deflections are additive
anyway.

Akama \cite{38} has shown that expanding the Boltzmann collision
term (22) to second order in powers of \( \Delta U_i' \), yields a Fokker-
Planck term. This form is suitable for describing the small angle
deflections of Coulomb interactions. The procedure is very reminiscent
of that in the classical theory which yields the Landau equation in
its final form (see \cite{47}). Akama's results may be written in a
slightly more convenient form for us in terms of the reduced velocities.
The final form of \( S_n(\mathcal{F}_i) \) yields

\[
S_n(\mathcal{F}_i) = -\frac{2}{\partial \mathcal{F}_i} (A_i \mathcal{F}_i) + \frac{1}{2} \frac{\partial^2}{\partial \mathcal{F}_i \partial \mathcal{F}_j} (D_{ij} \mathcal{F}_i),
\]  

where the \( A_i \) and \( D_{ij} \) are respectively the friction and diffusion
coefficients given by the expressions

\[
A_i = \int (\Delta U_i) W(u_{i^*}, u_{i^*}, u_{i_1}, u_{i_2}) d^* u_{i_1} d^* u_{i_1} d^* u_{i_2},
\]

and

\[
D_{ij} = \int (\Delta U_i) (\Delta U_j) W(u_{i^*}, u_{i^*}, u_{i_1}, u_{i_2}) d^* u_{i_1} d^* u_{i_1} d^* u_{i_2}.
\]

(\( \Delta U_i \)) denotes the change in the reduced velocity of a species 1
particle due to a collision with a species 2 particle. It is
convenient to write the integrations over scattering angles in the
form

\[
\langle (\Delta U_i) \rangle = \int (\Delta U_i) \sigma(u_{i^*}, u_{i^*}, u_{i_1}, u_{i_2}) d^* u_{i_1} d^* u_{i_1} d^* u_{i_2},
\]

(26)
and

\[ \langle (\Delta u_i)(\Delta u_j) \rangle = \int \sigma(u_i', u_j', u_k, u_\kappa) d^4 u_k' d^4 u_k. \]  

(27)

With these definitions, \( A_i \) and \( D_{ij} \) can be concisely written as

\[ A_i = \int \langle (\Delta u_i) \rangle (P^2 - 1) \mathcal{F}_i d^4 u_k. \]  

(28)

and

\[ D_{ij} = \int \langle (\Delta u_i)(\Delta u_j) \rangle (P^2 - 1) \mathcal{F}_i d^4 u_k. \]  

(29)

\( P \) is the four product \( u_i u_i' \).

Up to now the above formulation has been quite general and the nature of the colliding particles has not been specified. Since our main interest is in plasmas with Coulomb interactions we identify electrons with species 1 and ions with species 2 particles. The relativistic distribution function for electrons is denoted by \( \mathcal{F}_e(\chi_i, u_i) \) and that for ions by \( \mathcal{F}_i(\chi_i, u_i) \). Akama has given the forms (26) and (27) for electron-proton interaction. Generalising to electron-ion interactions by simply writing \( e^+ Z^1 \) for \( C^a \) in his expressions we can write the expressions (26)-(29) in the form

\[ A_i = \int \xi_i \mathcal{F}_i d^4 u_k, \]  

(30)

and

\[ D_{ij} = \int \xi_{ij} \mathcal{F}_i d^4 u_k, \]  

(31)

where the four-vector \( \xi_{ij} \) is given by
and the tensor $\xi_{ij}$ by

$$\xi_{ij} = \frac{\gamma_{ei}}{C^2} \frac{p^2}{(p^2)^{1/2} (\gamma + u_i u_j)} \left\{ \frac{m}{M} \left( U_i - U_j - \rho \left( U_i - \frac{m}{M} U_i \right) \right) \right\}, \tag{32}$$

In the above expressions $m$ is the mass of an electron and $M$ the mass of an ion. $\gamma_{ei}$ denotes the quantity $4\kappa \left( \frac{e^2}{\gamma + \lambda} \right) L N \Lambda$, where $\varepsilon_0$ is the permittivity of free space, $C$ the electronic charge and $\lambda$ the ionisation number for the ions. $LN\Lambda$ is the 'Coulomb logarithm' arising from the divergence nature of the scattering integrals for distant collisions. Apart from a numerical factor, the value of $\Lambda$ equals the number of particles in a Debye sphere and is much larger than unity and hence $LN\Lambda \gg 1$. Discussion of this term is deferred until §7.

The Boltzmann equation in its present form with the Fokker-Planck approximation for the collision term, involves delta functions in the distributions $f_e$ and $f_i$. Using the defining relation (11) and integrating out the delta functions in (23) yields a more useful transport equation in distributions $f(x,\lambda, u_e, t)$ (for electrons) and $F(x,\lambda, u_i, t)$ (for ions) on the 6-dimensional phase-space.

From (13) we have the more familiar form

$$\frac{\partial f}{\partial \varepsilon} + \frac{e u_e}{\sqrt{\lambda}} \frac{\partial f}{\partial x_e} + \frac{e}{m c} \frac{\partial f}{\partial u_e} = \int_{S_{ei}} (f_e) d\nu = C_{ei}(f), \tag{34}$$

where $u_e = \frac{1}{2} U_e$. From (23) $C_{ei}$ is given by
where the $A_{\lambda}$ and $D_{\lambda\nu}$ are the spatial components of the friction and diffusion terms as defined above. (34) represents the transport equation for the electronic distribution $f$ with electron-ion collision effects included. Electron-electron effects ($C_{ee}(f)$) may be deduced as a special case by writing $F=f$, $M=\infty$ and $\Xi=1$. These effects can be added on in (34) as a separate term.

The interesting feature of the Fokker-Planck term (35) is that it is valid for arbitrary mass ratio of colliding particles. Further, it is only the friction term $A_{\lambda}$ that distinguishes the masses of the interacting species, the diffusion term $D_{\lambda\nu}$ being independent of $M/M^\nu$.

The form of the interaction term given by (35) is the one used in the next chapter to derive the collision moment terms. The usefulness of this expression, as will be subsequently apparent, is that it involves derivatives only with respect to $U_{\lambda}$.

\section{Landau Equation for Electron-Ion Interaction}

In this section we show how the Fokker-Planck term (35) may be transformed into the Landau form and that in the non-relativistic limit it reduces to the well known classical result commonly found in the literature.

From (33) it is easy enough to show that the following tensor relations hold:

\[ \zeta_{ij} U_j = 0, \]

and

\[ \zeta_{ij} U_j^* = 0. \]
With the quantity \( P \) given by \( \mathcal{U}_i \mathcal{U}_i \), and remembering that \( \mathcal{U}_i = \mathcal{I} \mathcal{H}_i \) and similarly for \( \mathcal{U}_i \), then it follows from (36) and (37) that

\[
\xi_{\lambda} \frac{\partial \rho}{\partial \mathcal{U}_\lambda} = 0 \quad \xi_{\mu} \frac{\partial \rho}{\partial \mathcal{U}_\mu} = 0.
\] (38)

Our aim is to express the friction term \( \mathcal{A}_\lambda \) in terms of the diffusion coefficient \( D_\lambda \). To do this we first compute the quantities \( \frac{\partial \xi_{\lambda}}{\partial \mathcal{U}_\mu} \) and \( \frac{\partial \xi_{\mu}}{\partial \mathcal{U}_\rho} \). Writing \( g(\rho) = \frac{\xi_{\lambda}}{\xi_{\mu}} \frac{\mathcal{P}_\mu}{\mathcal{P}_\lambda} \), and by use of (38) there results the expressions

\[
\frac{\partial \xi_{\lambda}}{\partial \mathcal{U}_\mu} = \frac{-2g(\rho)}{\sqrt{1 + \mathcal{U}_i^2 (1 + \mathcal{U}_i^2)}} (\mathcal{U}_\lambda + \rho \mathcal{U}_\lambda),
\]

and

\[
\frac{\partial \xi_{\mu}}{\partial \mathcal{U}_\rho} = \frac{-2g(\rho)}{\sqrt{1 + \mathcal{U}_i^2 (1 + \mathcal{U}_i^2)}} (\rho \mathcal{U}_\lambda + \mathcal{U}_\lambda).
\] (39a-b)

Thus from (39a-b) and by comparison with (32) we have the relationship

\[
\frac{\partial \xi_{\lambda}}{\partial \mathcal{U}_\mu} - \frac{m}{M} \frac{\partial \xi_{\mu}}{\partial \mathcal{U}_\rho} = 2 \mathcal{F}_\lambda.
\] (40)

Substituting for the friction term \( \mathcal{A}_\lambda \) in (35) we have for the Fokker-Planck expression (35)

\[
C_{ei}(f) = -\frac{2}{\partial \mathcal{U}_\lambda} \int \frac{1}{2} \mathcal{U}_i \mathcal{U}_i f f d^3 \mathcal{U}_j + \frac{m}{M} \frac{\partial}{\partial \mathcal{U}_\lambda} \int \frac{1}{2} \mathcal{U}_i \mathcal{U}_i f f d^3 \mathcal{U}_j
\]

\[+ \frac{1}{2} \mathcal{U}_i \mathcal{U}_i \int \xi_{\lambda} f f d^3 \mathcal{U}_j.
\] (41)
An integration by parts of the second term and expansion of the third in (41) yields the Landau form of $C_{\varepsilon_i}$:

$$C_{\varepsilon_i}(f) = \frac{1}{2} \frac{\partial}{\partial Y_\mu} \int \xi_{\varepsilon i \mu} \left( \frac{\partial f}{\partial Y_\mu} - \frac{m}{M} \frac{\partial F}{\partial Y_\nu} f \right) d^3 Y.$$  (42)

This form of $C_{\varepsilon_i}$ now involves derivatives with respect to both $Y_\mu$ and $Y_\nu$ instead of just $Y_\mu$ as in (35). The three-tensor $\xi_{\varepsilon \mu}$ is as given by (33).

In the classical limit as $c$ (the velocity of light) tends to infinity, $U$ and $U_T$ tend to zero in such a way that $cU$ tends to $v$ (the electron velocity) and $cU_T$ tends to $v_T$ (the ion velocity). To complete the classical prescription, the distributions $f$ and $F$ are replaced by $c^3 f$ and $c^3 F$ (in order to make $\int f d^3 Y d^3 Y$ dimensionless). In the limit we have

$$\frac{\sqrt{p^2_\perp}}{\sqrt{1 + U^2}} \rightarrow |Y - Y'| = \gamma,$$

and

$$c^3 \xi_{\varepsilon i} \rightarrow \gamma \{ S_{\varepsilon i} \frac{v}{\gamma} - \frac{v_Y + v_T + v_Y + v_T}{\gamma} \}.$$

Hence (41) reduces to the classical Landau form

$$C_{\varepsilon_i}(f) = \frac{v_{\varepsilon_i}}{2} \frac{\partial}{\partial Y_\mu} \int d^3 Y \left[ \frac{\gamma}{\gamma_Y} - \frac{\gamma_Y}{\gamma^2} \right] \left[ F V_\mu f - \frac{\gamma}{\gamma_Y} f V_\mu f \right],$$  (43)

in which $\frac{1}{2}$ is the 3x3 unit matrix. For further discussion on (43) we refer the reader to Shkarofsky (Chapter 7).

§ 7 SCREENING TERM.

Up to now we have said very little about the Coulomb logarithm $\ln \Lambda$ appearing in the expression for the friction and diffusion
terms of the Fokker-Planck equation. The differential cross-section \( \sigma \) used in computing the averages \( \langle \Delta U_i \rangle \) and \( \langle \Delta U_i \Delta U_j \rangle \) is provided by quantum mechanics and may be written in the mass-centre system as (Akama 38)

\[
\sigma(\Omega) d\Omega = \frac{4\pi^2 e^2 (mM)^2}{(4\pi\varepsilon_0)^2 q^2} \left( A + B \cos \theta + C \cos^2 \theta \right) d\Omega,
\]

where the quantities \( A, B \) and \( C \) satisfy the relation

\[
A + B + C = \left( \frac{q^2 + \xi_{11}^2 \xi_{22}^2 / c^2}{m^2 M^2} \right).
\]

In the mass-centre system \( q \) represents the magnitude of the momentum of either of the colliding particles; \( m \) and \( \xi_m \) identify the mass and energy of an electron, and similarly \( M \) and \( \xi_M \) for the ions of charge \( Z \varepsilon / e \). The important point to note is that when the form (44) is used in computing the scattering averages of \( \Delta U_i \) etc the integrals diverge at \( \theta = 0 \). This divergent nature over scattering averages occurs in exactly the same way in the relativistic theory above as it does in the classical case. This is a direct consequence of the fact that the Coulomb force between charged particles decreases slowly with distance so that distant collisions with small deflections become important. The cross-section (44) exhibits a very strong forward peak for these distant collisions \( (\theta \sim 0) \) whereas in cross-sections for short range intermolecular forces this peak is not so pronounced.

In impact theory it is thus necessary to cut off the integrand over \( \theta \) at some minimum value. This limit corresponds to a cut off at a maximum impact parameter which is set equal to the Debye length \( (\kappa) \). Interactions at larger impact parameters are greatly reduced by the electron cloud shielding the ion (remember that we are assuming a
large number of particles in a Debye sphere for plasmas of interest to us).

In classical theory (p. 258 et seq) the value of the cut off parameter $\lambda$ is given as the ratio of the Debye length $\lambda_D$ to the minimum impact parameter $b$. Silin has shown from the wave theory approach based on the dielectric constant of the medium that, effectively, the same value of $\lambda$ holds in the relativistic case as well. The Debye shielding length ($\lambda_D$) for electrons is defined as the quantity $\sqrt{\frac{\varepsilon_0 K T}{m v^2}}$, while $b$ (the minimum impact parameter) is generally taken as that for $90^\circ$ deflections and is (for electron-ion collisions) given by ($\Sigma^2$, p. 206)

$$b = \left(\frac{1}{4\pi \varepsilon_0}\right) \frac{Ze^*}{m v^2}, \quad (45)$$

where $v$ is the electron velocity before collision. Since the logarithmic term $\ln \lambda$ changes only slowly, $m v^2$ in (45) is replaced by its average value $\frac{3}{2} K T$. Thus for $\lambda$ we can write, from (45)

$$\lambda = \frac{\lambda_D}{b} = \frac{12 \pi \varepsilon_0 K T}{Ze^*} \sqrt{\frac{K T e^*}{m v^2}}.$$

Since we have assumed many electrons in a Debye sphere one has

$$n_D = \frac{4}{3} \pi \lambda_D^3 = \frac{Z \lambda}{\alpha},$$

Typically for a temperature of $10^4 \text{ K}$ and electron density of $10^{11}$ cms$^{-3}$, $\ln \lambda$ has a value of 10.58.

To reiterate, the rarer violent collisions occur in the range $b < b'$, while the more frequent longer range interactions in which $b' < b < \lambda$ give rise to the bulk effects in Coulomb plasmas. For temperatures greater than $4 \times 10^5 \text{ K}$, Spitzer ($\Sigma^4$, p. 41) states that due to quantum-mechanical effects, the $\lambda$ factor must be reduced by the factor $2 \alpha C \frac{\varepsilon}{\sqrt{v}}$, where $C$ is the velocity of light and $\alpha$ is the fine structure constant.
\[ h_p = \frac{e^2}{2\xi h_F c} = \frac{1}{137}. \]

\( h_p \) is Planck's constant. Essentially whenever \( \nu_{h_F} = \frac{h_p}{m v} \) is greater than \( 26 \), the lower limit to the impact parameter is set at \( \nu_{h_F}/2 \) rather than zero. This is essentially a restatement of Heisenberg's uncertainty principle. If \( \Lambda_Q \) denotes the modified value of \( \Lambda \), we have

\[ \Lambda_Q = \frac{2 Z C}{137} \Lambda_{\text{CLASSICAL}} = Z \left( 4.2 \times 10^5 \right) \Lambda_{\text{CLASSICAL}}, \]

where the average value \( v = \sqrt{\frac{3kT}{m}} \) has been taken.

Finally it should be mentioned that effects such as ion screening, magnetic fields, plasma waves and streaming of high energy particles influence the value of \( \Lambda \). However, apart from the remarks made above, no further considerations will be given to the quantity \( \Lambda \).

\section{Invariance Relations for Collision Terms}

In this section some consideration is given to the behaviour of the collision terms under Lorentz transformations. We first make some general remarks concerning the relativistic collision term and then specialise to the form we shall subsequently be using (equation (35) or (41)).

From equation (13), the relativistic Boltzmann equation may be written as

\[ \sqrt{1 + u^2} \frac{df}{dt} = \sqrt{1 + u^2} \frac{df}{dt} + C(u, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial v}) + \sqrt{1 + u^2} F \frac{\partial f}{\partial v} = \sqrt{1 + u^2} C(f), \]

where \( F \) is the external force on a particle and \( C(f) \) represents the change in \( f \) due to collisions. Equation (46) defines the
generalised time derivative \( \frac{D}{Dt} \).

We consider two frames of reference \( S \) and \( S' \) where \( S' \) moves relative to \( S \) with velocity \( \mathbf{V} \) in the \( x \)-direction. The Lorentz transformation of any four-vector \( A_i \) reads:

\[
A_i' = \frac{1}{\alpha}(A_i + \frac{\mathbf{i} \mathbf{V}}{c} A_4), \quad A_1 = A_{1}', \quad A_2 = A_{2}', \quad A_4 = \frac{1}{\alpha}(A_{4} - \frac{\mathbf{i} \mathbf{V}}{c} A_{1}),
\]

where the Lorentz factor \( \alpha \) is given by

\[
\alpha = \sqrt{1 - \frac{V^2}{c^2}}.
\]

Now the quantity

\[
C(f) \, dt \, d^3 \mathbf{r} \, d^3 \mathbf{y},
\]

represents the number of collisions between particles occurring in the time interval \( dt \) with reduced velocities in the range \( d^3 \mathbf{y} \) about \( \mathbf{y} \) and positions in the range \( d^3 \mathbf{r} \) about \( \mathbf{r} \). This is a pure number and its expression in the frame \( S' \) is written as

\[
C'(f') \, dt' \, d^3 \mathbf{r}' \, d^3 \mathbf{y}',
\]

which therefore is equal to (48). As mentioned in § 3 of this chapter, the distribution \( f \) is invariant with respect to Lorentz transformations and so \( f = f' \). Further the four-dimensional volume element \( dt \, d^3 \mathbf{r} \) is also invariant.

From (47) it is straightforward to show that for the four-vectors \( \mathbf{U}_i \) and \( \mathbf{U}_i' \)
Clearly the Jacobian of the transformation yields

\[ \frac{\partial (u_1, u_2, u_3)}{\partial (u'_1, u'_2, u'_3)} = \frac{\sqrt{1 + u'^2}}{\sqrt{1 + u'^2}} \]

so that \( d^3\gamma \sqrt{1+u'^2} \) is an invariant volume element. Hence form the
identity of (48) and (49) we have the relation

\[ \sqrt{\text{h} u^2} C(f) = \sqrt{\text{h} u'^2} C'(f'), \tag{51} \]

or in other words the quantity \( \sqrt{\text{h} u^2} C(f) \) is an invariant.

Clemmow and Willson have shown that the terms represented by
\( \sqrt{h u^2} \frac{Df}{Dc} \) in (46) is also invariant. Thus we have established the
invariant form of Boltzmann's equation with collisions when it is
written in the form (46).

It is instructive to be able to show appropriate invariance when
the collision term is of the form (42) or (35). It is convenient to
write (42) in the form

\[ C_{ec}(f) = \frac{\partial}{\partial u_\lambda} \left( \frac{J_1}{\sqrt{h u^2}} \right), \tag{52} \]

where
\[
\mathcal{J}_\lambda = \int \xi_{\nu} \left\{ \frac{\partial F}{\partial u_{\nu}} - \frac{m}{\sqrt{1 + u_{\nu}^2}} \frac{\partial F}{\partial u_{\nu}} \right\} d^4u,
\]
and the tensor \( \xi_{\nu} \) is essentially

\[
\mathcal{G}(\epsilon) \left\{ \left( \epsilon^{\nu-1} \right) S_{\nu} - u_\lambda u_\nu - u_\lambda u_\nu \right. \\
- \left. \Gamma \left( u_\lambda u_\nu + u_\nu u_\lambda \right) \right\}.
\]

\( \mathcal{G}(\epsilon) \) is a function of the invariant product \( \epsilon = u_\lambda u_\lambda \) only.

It is a fairly straightforward matter to show (see Appendix this chapter) that the quantities

\[
\xi_{\nu} \frac{\partial \mathcal{G}}{\partial u_{\nu}} \quad \text{AND} \quad \xi_{\nu} \frac{\partial \mathcal{F}}{\partial u_{\nu}},
\]

represent four-vectors and transform according to Lorentz transformations. Accordingly, since \( F, \mathcal{G} \) and \( d^4u/\sqrt{1 + u^2} \) are invariants, it follows that the \( \mathcal{J}_\lambda \) form the first three components of a four-vector \( \mathcal{J}_\nu \). Further, since \( \xi_{\nu} u_\nu = 0 \) we have the result that

\[
\mathcal{J}_\nu = \frac{i \mathcal{J}_\nu u_\nu}{\sqrt{1 + u^2}}.
\]

Evidently \( \mathcal{J}_\nu \) behaves like a four-force. Using this fact, the transformation equations (47) and the relations (50) it follows straightforwardly that the quantity

\[
\sqrt{1 + u^2} \frac{\partial}{\partial u_\lambda} \left( \frac{\mathcal{J}_\lambda}{\sqrt{1 + u^2}} \right)
\]

is invariant, in agreement with the general result (51).

**APPENDIX.**

It is required to show that the quantities

\[
\xi_{\nu} \frac{\partial \mathcal{G}}{\partial u_{\nu}} \quad \text{AND} \quad \xi_{\nu} \frac{\partial \mathcal{F}}{\partial u_{\nu}},
\]

\( \xi_{\nu} \frac{\partial \mathcal{F}}{\partial u_{\nu}}, \)
where \( f = f(x, \lambda, \lambda, t) \), \( F = F(x, \lambda, \lambda, t) \), represent two four-vectors. \( f \) and \( F \) are invariant distributions and \( \xi_{ij}^* \) is the four tensor.

\[
\xi_{ij}^* = g(\rho) \left\{ \rho (\rho - 1) S_{ij} - U_i U_j - U_i U_j \right\},
\]

(2)

\( g(\rho) \) is a function of the four product \( \rho = U_i U_j \) only and is consequently invariant. Henceforth the \( g(\rho) \) term in \( \xi_{ij}^* \) will be ignored.

We consider to frames of reference \( S \) and \( S' \) where \( S' \) moves relative to \( S \) with velocity \( \mathbf{V} \) in the \( x \)-direction. The Lorentz transformation of any four vector is as given by (47). Firstly, for convenience, we define the quantities

\[
\beta_\lambda = \frac{1}{\sqrt{1 + U_i^2}}, \quad \beta_1 = \frac{1}{\sqrt{1 + U_i^2}}, \quad \gamma_2 = 1 - \frac{\mathbf{V}}{c} \beta_2 U_i, \quad \gamma_1 = 1 - \frac{\mathbf{V}}{c} \beta_1 U_i.
\]

Also from (47) it is easy to show that

\[
\beta'_{\lambda} = \frac{\lambda}{\gamma_2} \beta_{\lambda}, \quad \beta_1' = \frac{\lambda}{\gamma_1} \beta_1.
\]

(3)

In (3) \( \lambda \) is the Lorentz factor \( \sqrt{1 - \frac{\mathbf{V}^2}{c^2}} \). Further, we need to know how the derivatives with respect to reduced velocities transform. Firstly we note that similar relations to (50) hold for \( \mu \). Thus the terms \( \frac{\partial F}{\partial \mu} \) for example, are transformed according to the relations

\[
\frac{\partial F'}{\partial \mu'} = \left\{ \delta_{\mu'}^\mu \frac{\partial \beta'}{\partial \gamma_2} \beta_{\lambda} + \frac{\mathbf{V}}{c^2} \beta_2' U_i (1 - \delta_{\mu'}^\mu) \right\} \frac{\partial F}{\partial U_i} + (1 - \delta_{\mu'}^\mu) \left( \frac{\partial F'}{\partial \gamma_1} \right)_{\mu'},
\]

(4)

where summation over \( \mu \) is not implied. Similar expressions to (4) hold for the terms \( \frac{\partial F'}{\partial \mu} \).

Since the tensor \( \xi_{ij}^* \) is completely symmetric in \( U_i \) and \( U_i \), it is only necessary to consider one of the pair of vectors (1).
Subsequently, we will consider the four quantities $\xi_i \frac{\partial F}{\partial u_i}$ (to be denoted by $\tilde{J}_i$).

Let us look first at the term $\tilde{J}_i'$. Expressed in the $S'$ frame, this is given by

$$\tilde{J}_i' = \left\{ (\rho^{2-1}) \frac{\partial F}{\partial u_i} \right\} \left\{ \left[ (v'_2 u'_1 + u'_1 u'_2) + \rho' (u'_2 u'_1 + u'_1 u'_3) \right] \frac{\partial F}{\partial u'_2} \right\}.$$

With help from the transformation equations (47) and the differential relations (4) above, the representation (5) in the $S$ frame yields

$$\tilde{J}_i' = (\rho^{2-1}) \left\{ \frac{v_{i3}}{c_{i\beta}} \frac{\partial F}{\partial u_1} + \frac{\partial F}{\partial u_3} \right\}$$

From (6) we can group the terms not containing $\frac{\partial F}{\partial u_i}$, to give the expression

$$\left( \rho^{2-1} \right) \frac{\partial F}{\partial u_1} - (u_2 + \rho u_2) \left( \frac{\partial F}{\partial u_2} + \frac{\partial F}{\partial u_3} \right)$$

$$- (u_2 + \rho u_2) \left( \frac{\partial F}{\partial u_2} + \frac{\partial F}{\partial u_3} \right).$$

(7)
The terms with \( \frac{\partial F}{\partial \psi_1} \) have now only to be considered. The algebra is considerably reduced if we note that the following relations hold:

\[
U_1 \psi_1 + U_3 \psi_3 = \rho + \frac{1}{\beta_1 \beta_2} - \psi_1 \psi_3,
\]

and

\[
U_2^2 + U_3^2 = -1 + \frac{1}{\beta_2^2} - U_1^2.
\]

Hence the coefficient of \( \frac{\partial F}{\partial \psi_1} \) in (6) can be written as

\[
(p^2 - 1) \frac{\partial \psi_3}{\partial \psi_1} - (U_2 + \rho \psi_1) \left\{ \frac{U_3}{\beta_2} - \frac{U_3}{\beta_2} + \frac{U_3}{\beta_2} \left( \frac{1}{\beta_1} - \psi_1 \psi_3 \right) \right\}
\]

which after cancellation and use of the definition \( \psi_1 \) gives simply

\[
- U_1 U_1 - U_2^2 U_1 - \rho U_2 U_1 - \rho U_3 U_1.
\]

Hence (8) coupled with (7) gives the required result that

\[
\mathcal{J}_2' = \mathcal{J}.
\]

The calculation of \( \mathcal{J}_3' \) follows similar lines, (5) being replaced by

\[
\mathcal{J}_3' = (p^2 - 1) \frac{\partial F'}{\partial \psi_3} - (U_3' + \rho \psi_1') \psi_3' \frac{\partial F'}{\partial \psi_3'}
\]

\[
- (U_3' + \rho \psi_1') \psi_3' \frac{\partial F'}{\partial \psi_3'}.
\]

The \( \frac{\partial F'}{\partial \psi_3'} \) term is given by (4) with \( \mu = 3 \) while the terms \( \frac{\partial F'}{\partial \psi_1'} \) and \( \frac{\partial F'}{\partial \psi_2'} \) have been given in (6). The transformed quantities can be grouped according as to whether they contain the factor \( \frac{\partial F}{\partial \psi_1} \).
or not. Exactly the same procedure as above is used to reduce the coefficient of \( \frac{\partial F}{\partial U_1} \), which, in the final form yields (8) with the suffix '2' replaced by '3'. Hence the transformation law for the third component yields

\[
J'_3 = J_3. \tag{11}
\]

Thus, so far, we have shown that the space part of \( J_i \) perpendicular to the relative velocity of the two frames \( S \) and \( S' \) is invariant under a Lorentz transformation.

We now turn our attention to \( J_1 \) (\( \alpha \)-component). In the \( S' \) frame we have

\[
J'_1 = (\rho^2 - 1) \frac{\partial F'}{\partial U'_1} - (U'_1 + \rho U'_2) U'_2 \frac{\partial F'}{\partial U'_2} - (U'_1 + \rho U'_2) U'_3 \frac{\partial F'}{\partial U'_3}. \tag{12}
\]

Using the transformation laws (47) (§ 8) the representation of (12) in the \( S \) frame yields (remembering that \( \rho = \rho' \) and \( F = F' \))

\[
(\rho^2 - 1) \frac{\partial F}{\partial U_1} - \left\{ \frac{1}{\alpha} (U_1 - \frac{V}{c^2}) + \frac{\rho}{\alpha} (U_1 - \frac{V}{c^2}) \right\} \left\{ \frac{1}{\alpha} (U_1 - \frac{V}{c^2}) \frac{\partial F}{\partial U_1} \right\} + \frac{V U_2 U_3}{\alpha^2} \frac{\partial F}{\partial U_1} + \frac{V U_2 U_3}{C_2} \frac{\partial F}{\partial U_1} \right\} + \frac{V U_2}{C_2} \frac{\partial F}{\partial U_1} + \frac{V U_2}{C_2} \frac{\partial F}{\partial U_1} \right\} + \frac{V U_2}{C_2} \frac{\partial F}{\partial U_1} + \frac{V U_2}{C_2} \frac{\partial F}{\partial U_1} \right\}.
\]
From (13) we collect together all the terms not involving $\frac{\partial F}{\partial \psi_i}$. These give, after slight rearrangement,

$$-\frac{1}{\alpha}(U_1 + \psi U_1)(U_2 \frac{\partial F}{\partial U_2} + U_3 \frac{\partial F}{\partial U_3})$$

$$-\frac{1}{\alpha}(U_1 + \psi U_1)(U_3 \frac{\partial F}{\partial U_3} + U_3 \frac{\partial F}{\partial U_3})$$

$$-\frac{i\nu}{c\alpha}(\frac{i}{\beta_1} + \frac{i}{\beta_2})(U_2 \frac{\partial F}{\partial U_2} + U_3 \frac{\partial F}{\partial U_3})$$

$$-\frac{i\nu}{c\alpha}(\frac{i}{\beta_1} + \frac{i}{\beta_2})(U_3 \frac{\partial F}{\partial U_3} + U_3 \frac{\partial F}{\partial U_3})$$

(14)

Next we look at the coefficient of $\frac{\partial F}{\partial \psi_i}$ in (13). Using the fact that $U_1 \psi_i = -1$ and $U_1 \psi_i = \rho$, we can substitute for the terms $U_1 \psi_i + U_3 \psi_3$ and $U_3^2 + U_3^2$ to give the form

$$(\rho^2 - 1) \frac{\alpha}{\beta_1} - \frac{1}{\alpha} \left\{ U_1 - \frac{\nu}{c\beta_1} + \rho \left( U_1 - \frac{\nu}{c\beta_1} \right) \right\} \left\{ U_1 + \frac{\nu \beta_1 \rho}{c\gamma_1} \right\}$$

$$- \frac{1}{\alpha} \left\{ U_2 - \frac{\nu}{c\beta_1} + \rho \left( U_2 - \frac{\nu}{c\beta_1} \right) \right\} \left\{ U_2 - \frac{\nu \beta_1 \rho}{c\gamma_1} \right\}.$$  (15)

Remembering the definition of $\gamma_1$ and that $\gamma_2 = 1 - \frac{\nu}{c\beta_2} \psi_i$, a little algebra shows that (15) may be reduced to

$$\frac{1}{\alpha}(\rho^2 - 1) = -\frac{U_1^2}{\alpha} - \frac{U_2^2}{\alpha} - \rho U_1 U_1 - \rho U_1 U_1$$

$$+ \frac{i\nu}{c\alpha} \left\{ -\frac{U_1 i}{\beta_1} - \frac{U_2 i}{\beta_2} + \rho \frac{U_1 i}{\beta_1} + \rho \frac{U_1 i}{\beta_2} \right\}.$$  (16)
Combining (14) and (16) gives for $J_1'$ the expression

$$J_1' = \frac{1}{\alpha} (p^2 - 1) \sum \frac{\partial F}{\partial u_2} - \left\{ (u_1 + p u_2) u_2' + (u_1 + p u_2) u_2^{- \frac{2}{\beta}} \left( \frac{\partial F}{\partial u_2} \right) \right\},$$

and so clearly (17) is simply

$$J_1' = \frac{1}{\alpha} \left( J_1 + \frac{i c}{\sqrt{\gamma}} J_4 \right).$$

Lastly we look at the term $J_4'$. In the $S'$ frame this is written as

$$J_4' = -\left( \frac{\beta_1'}{\beta_1} + \frac{\beta_2'}{\beta_2} \right) u_2' \frac{\partial F'}{\partial u_2'} - \left( \frac{\beta_1'}{\beta_1} + \frac{\beta_2'}{\beta_2} \right) u_2' \frac{\partial F'}{\partial u_2'}.$$

In transforming to the $S$ frame we note that $\beta_1'$ and $\beta_2'$ are given by (3) as

$$\frac{1}{\beta_1'} = \frac{\beta_1}{\alpha \beta_1}, \quad \frac{1}{\beta_2'} = \frac{\beta_2}{\alpha \beta_2}.$$
\begin{equation}
\mathcal{J}_4' = -\frac{i}{\alpha} \left( \frac{\gamma_1}{\beta_1} + \frac{p_1}{\beta_1} \right) \left\{ \frac{i}{\alpha} \left( U_1 - \frac{\gamma}{c \beta_1} \right) \frac{\partial F}{\partial U_1} + \frac{U_2 \gamma_1}{c \beta_1} \frac{\partial F}{\partial U_1} \right. \\
+ U_2 \frac{\partial F}{\partial U_2} + \frac{U_3 \gamma_1 \gamma_2}{c \beta_1} \frac{\partial F}{\partial U_1} + U_3 \frac{\partial F}{\partial U_3} \right\} \\
- \frac{i}{\alpha} \left( \frac{\gamma_1}{\beta_2} + \frac{p_1}{\beta_1} \right) \left( U_1 - \frac{\gamma}{c \beta_1} \right) \frac{\partial F}{\partial U_1} + \frac{U_2 \gamma_1 \gamma_2}{c \beta_1} \frac{\partial F}{\partial U_1} \right. \\
+ U_2 \frac{\partial F}{\partial U_2} + \frac{U_3 \gamma_1 \gamma_2}{c \beta_1} \frac{\partial F}{\partial U_1} + U_3 \frac{\partial F}{\partial U_3} \right\}.
\end{equation}

In a manner reminiscent of the way we computed \( \mathcal{J}_4' \) above, we separate out the terms in (21) not involving \( \frac{\partial F}{\partial U_i} \) to give

\begin{equation}
\begin{align*}
- \frac{i}{\alpha} \left( \frac{1}{\beta_1} + \frac{p}{\beta_1} \right) &\left( U_1 \frac{\partial F}{\partial U_1} + U_3 \frac{\partial F}{\partial U_3} \right) \\
- \frac{i}{\alpha} \left( \frac{1}{\beta_2} + \frac{p}{\beta_1} \right) &\left( U_2 \frac{\partial F}{\partial U_2} + U_3 \frac{\partial F}{\partial U_3} \right) \\
+ \frac{i \gamma}{\alpha c} (U_1 + p U_1) &\left( U_2 \frac{\partial F}{\partial U_2} + U_3 \frac{\partial F}{\partial U_3} \right) \\
+ \frac{i \gamma}{\alpha c} (U_1 + p U_1) &\left( U_2 \frac{\partial F}{\partial U_2} + U_3 \frac{\partial F}{\partial U_3} \right).
\end{align*}
\end{equation}

In (22) we have used the fact that \( \gamma_1 \) and \( \gamma_2 \) are given by

\( \gamma_1 = 1 - \frac{\gamma}{c} \beta_1 U_1 \) AND \( \gamma_2 = 1 - \frac{\gamma}{c} \beta_2 U_1 \).

Finally we can collect together the terms in (21) that make up the coefficient of \( \frac{\partial F}{\partial U_1} \). The terms in the curly brackets of (21)
containing \( \frac{\partial F}{\partial \mathbf{u}_r} \) are simplified as previously by replacing \( \mathbf{u}_1 \mathbf{u}_r + \mathbf{u}_3 \mathbf{u}_r \) by \( \rho + \frac{1}{\beta_1 \beta_2} - \mathbf{u}_1 \mathbf{u}_r \) and \( \mathbf{u}_1 \mathbf{u}_r - \mathbf{u}_3 \mathbf{u}_r \) by \(-1 + \frac{1}{\beta_2^2} - \mathbf{u}_1 \mathbf{u}_r \). After some simplification, the terms may be written in the form

\[
\frac{1}{\alpha} \left( -\frac{i}{\beta_1} \mathbf{u}_1 - \frac{i}{\beta_2} \mathbf{u}_2 - \frac{i \rho}{\beta_1} \mathbf{u}_1 - \frac{i \rho}{\beta_2} \mathbf{u}_1 \right)
\]

\[
- \frac{i \nu}{c} \left( -\mathbf{u}_1 \mathbf{u}_1 - \mathbf{u}_2 \mathbf{u}_2 - \rho \mathbf{u}_1 \mathbf{u}_1 - \rho \mathbf{u}_2 \mathbf{u}_2 + (p^1 - i) \right). \tag{23}
\]

Combining (22) and (23) gives the representation of \( \mathbf{J}_4' \) in the \( S' \)-frame as

\[
\mathbf{J}_4' = \frac{1}{\alpha} \left\{ -\frac{i}{\beta_1} \mathbf{u}_1 - \frac{i}{\beta_2} \mathbf{u}_2 - \frac{i \rho}{\beta_1} \mathbf{u}_1 - \frac{i \rho}{\beta_2} \mathbf{u}_1 \right\} \frac{\partial F}{\partial \mathbf{u}_r}
\]

\[
- \frac{i \nu}{c} \left( (p^1 - i) \delta_{\mu \nu} - \mathbf{u}_1 \mathbf{u}_2 - \mathbf{u}_2 \mathbf{u}_1 - \mathbf{u}_2 \mathbf{u}_2 + (p^1 - i) \right) \frac{\partial F}{\partial \mathbf{u}_r}.
\tag{24}
\]

Again, noting that \( \mathbf{u}_4 = \frac{i}{\beta_1} \) and \( \mathbf{u}_4' = \frac{i}{\beta_2} \), we have from (24) that

\[
\mathbf{J}_4' = \frac{1}{\alpha} \left( \mathbf{J}_4' - \frac{i \nu}{c} \mathbf{J}_4' \right). \tag{25}
\]

Hence the quantities represented by \( \mathbf{J}_4 \) transform according to (9), (11), (18) and (25) which are exactly the Lorentz transformation equations. As already mentioned, since \( \xi^*_{ij} \) is symmetric in \( \mathbf{u}_i \) and \( \mathbf{u}_j \) we need only consider one of the pair of vectors (1). Hence we have shown that the quantities

\[
\xi^*_{ij} \frac{\partial F}{\partial \mathbf{u}_r} \text{ and } \xi^*_{ij} \frac{\partial f}{\partial \mathbf{u}_r},
\]

where \( F \) and \( f \) are invariant distributions (regarded as functions of \( \mathbf{u}_i \) and \( \mathbf{u}_a \) respectively) and \( \xi^*_{ij} \) is given by (2), represent two four-vectors.
INTRODUCTION.

This chapter is concerned with the construction of collision moments $C^{(l)}$ up to $l=1$ for electron-ion and electron-electron effects. The procedure is by no means easy and involves some tedious tensor manipulation. Because of its complexity we felt that development of these terms together with verification of the conservation laws, warranted a separate chapter.

In classical theory it is possible to write the friction ($A^A_\lambda$) and diffusion ($D^\tau_{\lambda\mu}$) terms in the Fokker-Planck collision term as derivatives of scalar functions of the velocity vector. It is a relatively easy matter to expand these functions in terms of spherical harmonics or the equivalent tensorial expansions and hence obtain the collision moments. However no such simplification afforded itself in the relativistic case. For this reason the analysis becomes much more involved although we have tried to relegate to the appendices some of the more laborious calculations which are nonetheless straightforward.

It is convenient to give a brief resume of the rest of this chapter. §2 is concerned with the expansion of the friction term to the first order in spherical harmonics which is then converted to a tensor form. §3 similarly deals with the diffusion term. In both of these sections much of the tedious calculations and tensorial manipulations are carried out in the appendices to this chapter. We hope that this will aid the reader to follow the derivations with the minimum of effect in the main text. For a more detailed analysis of a particular calculation, the appropriate appendix may be referred to. §4 is concerned with the calculation of the collision moments $C^{(0)}$ and $C^{(1)}$ to first order in relativistic terms (essentially retaining terms...
up to \( (\sqrt{v/c} )^3 \). We show that it is possible to formulate the terms in a similar way to the classical theory in terms of Allis integrals.

Much of the analysis of sections 2, 3 and 4 requires labouring through many tensor manipulations, expansions and differentiations. Our main purpose is to show these manipulations clearly and present the results at each stage of the calculation. This results in much formalism and absence of physical concepts until the latter half of section 4. On a first reading these derivations may be omitted and one may proceed directly to these particular cases beginning on page 146.

§ 5 discusses the classical limit for the collision moments. In § 6 a brief discussion of the momentum conserving properties of the collision terms is given.

§ 2 EXPANDED FRICTION TERM

At this stage we give a brief summary of the problem together with, for reference, the definitions of the Fokker-Planck friction and diffusion terms.

We can write Boltzmann's equation for the electronic function \( f \) in the symbolic form

\[
\frac{Df}{Dt} = C(f),
\]

where \( \frac{Df}{Dt} \) represents the rate of change of \( f \) due to its explicit dependence on time, advection, and external forces. The form of \( \frac{Df}{Dt} \) is given in equation (34) of Chapter 4. However, it is only the collision term \( C(f) \) that is of interest to us at present. Chapter 3 dealt with the tensor or equivalent spherical harmonic expansion of \( \frac{Df}{Dt} \) and assumed that \( C(f) \) was expandible in the spherical harmonic form.
\[ C(f) = \sum_{lms} C_{l ms} Y_{l ms}(\theta, \phi) \]
\[ = C^{(0)} + C^{(1)} \cdot \frac{\mathbf{U}}{U} + \ldots \quad (2) \]

( \( U, \theta, \phi \) ) are the polar co-ordinates of the reduced velocity vector \( \mathbf{U} \).

It is \( C^{(0)} \) and \( C^{(1)} \) that we wish to calculate for both electron-electron and electron-ion collisions. Our calculations in this and the subsequent section will proceed as for electron-ion interaction \( (C_{ei}) \) with the electron-electron \( (C_{ee}) \) effects coming out as a special case. For reference the Fokker-Planck term \( C_{ei}(f) \) is given by

\[ C_{ei}(f) = -\frac{\partial}{\partial U_\alpha}(A_\lambda f) + \frac{1}{2} \frac{\partial^2}{\partial U_\alpha \partial \nu_\mu}(D_{\lambda\mu} f), \quad (3) \]

where

\[ A_\lambda = \int \mathcal{L}_\lambda F d^3 \mathbf{U}, \]
\[ D_{\lambda\mu} = \int \mathcal{E}_{\lambda\mu} F d^3 \mathbf{U}, \quad (4a-b) \]

and \( \mathcal{L}_\lambda \) and \( \mathcal{E}_{\lambda\mu} \) are themselves given by:

\[ \mathcal{L}_\lambda = \frac{\gamma_{ei}}{c^2 (p_{e i})^{1/2} \sqrt{\nu_{\alpha} \nu_{\lambda}}} \left\{ \frac{\gamma_{\lambda}}{M} \nu_{\lambda} - U_\lambda - \rho (U_\lambda - \frac{m}{M} U_\lambda) \right\}, \]

and

\[ \mathcal{E}_{\lambda\mu} = \frac{\gamma_{ei}}{c^2 (p_{e i})^{1/2} \sqrt{\nu_{\alpha} \nu_{\lambda}}} \left\{ (p_{e i} - 1) \delta_{\mu\nu} - U_\lambda \delta_{\mu\nu} - U_\lambda U_\nu \right\} \]
\[ - \rho (U_\lambda U_\mu + U_\lambda U_\mu), \quad (5a-b) \]
\[ \rho, \text{ as we have seen, represents the term } \nabla_i V_i \text{ which can be written as} \]
\[ u_{\lambda} \overset{\perp}{\mathcal{U}} - \sqrt{\mathcal{U}} \frac{\partial}{\partial \mathcal{U}} \mathcal{U} \]

\( F \) is the ion distribution function, considered a function of \( \mathcal{U}_{\lambda}, \mathcal{X}_{\lambda} \) and \( t \), and is also assumed expandable in tensor form as

\[ F(\mathcal{U}_{\lambda}) = \sum_{\ell \neq 0} \mathcal{F}_{\ell\mathcal{M}}(\mathcal{U}) Y_{\ell\mathcal{M}}(\hat{\theta}', \hat{\phi}') \]
\[ = F^{(0)} + F^{(1)} \frac{\partial}{\partial \mathcal{U}} + \ldots \]

(6)

where \( (\mathcal{U}, \hat{\theta}', \hat{\phi}') \) are the polar co-ordinates of the vector \( \mathcal{U} \).

The aim is to substitute the expansion (6) into the definitions (4), compute \( A_{\lambda} \) and \( D_{\mathcal{X}_{\lambda}} \) and hence \( C_{\ell\mathcal{M}} \) from (3). In the process a linearisation technique will be employed in which products of first order terms will be ignored. This is justified provided disturbances arising from external fields and gradients in the plasma parameters are sufficiently small. It is easy to see that a headlong attack into the problem is fraught with difficulties because of the complex structure of the \( \lambda_{\lambda} \) and \( \mathcal{X}_{\lambda} \) terms appearing in (4). In classical theory it is possible to express these terms as derivatives of scalar functions (see for example \( \text{p. 269 et seq} \) known as Rosenbluth potentials. It is a relatively straightforward matter to perform calculations with these invariant scalar quantities. Such a simplification is not forthcoming in the relativistic case. As has been mentioned this leads to a somewhat more complicated formulation. Indeed, since relativistic potentials do not seem to exist, the formulation of the Fokker-Planck coefficients up to any order in the scatterer's (ion) distribution appears to be a very difficult task. We have not attempted any general analysis, but rather, restricted ourselves to a first order expansion only. This in itself involves some complicated formalism although, in principle, it could be extended.
to higher orders.

Let us turn our attention to a more detailed study of the friction term $A_{\lambda}$. It is obtained by integrating the quantity $\mathcal{L}_{\lambda} F$ over the whole of the ion velocity space with $\mathcal{L}_{\lambda}$ and $F$ given by (4a) and (5a) respectively. Our object in this section is to perform the integration over all directions of the vector $\mathcal{Y}$, i.e. over all angles $\theta^{'}, \phi^{'}$.

If we write $\mathcal{U}_{\lambda}$ and $\mathcal{U}_{\lambda}$ in polar co-ordinates with corresponding angles $\theta, \phi$ and $\theta^{'}, \phi^{'}$ then $\mathcal{U}_{\lambda} \mathcal{U}_{\lambda}$ may be replaced by $\mathcal{U} \mathcal{U} \mathcal{Z}$ where

$$\mathcal{Z} = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi').$$

Clearly $\mathcal{Z}$ represents the cosine of the angle between the vectors $\mathcal{Y}$ and $\mathcal{Y}$.

The integrand in equation (4a) contains trigonometric functions of the angles $\theta^{'}$ and $\phi^{'}$ through the form of $P$ and from the definition of $F$ given by equation (6). Firstly, our aim is to carry out the integrals over the angles $\theta^{'}$ and $\phi^{'}$. In this respect we note that the functions of $P$ in $\mathcal{L}_{\lambda}$ may be expanded as an infinite series in powers of $\mathcal{Z}$ in the form

$$\frac{P^1}{(P^3-1)^{3/2}} = \sum_{\lambda=0}^{\infty} D_{\lambda}^{(1)} P_{\lambda}(\mathcal{Z})$$

and

$$\frac{P^3}{(P^3-1)^{3/2}} = \sum_{\lambda=0}^{\infty} D_{\lambda}^{(3)} P_{\lambda}(\mathcal{Z}).$$

(8a-b)

In equations (8a-b), $P_{\lambda}(\mathcal{Z})$ represents the well known Legendre polynomial of order $\lambda$, and the expansion coefficients $D_{\lambda}^{(2)}$ and $D_{\lambda}^{(3)}$, considered as functions of $\mathcal{U}$ and $\mathcal{U}$ only, are given by
The $D$ functions are obtained using the orthogonality property of the Legendre polynomials $P_n(z)$ over the interval $[-1, 1]$, which can be written as

\[ D_n^{(2)} = (n + 1) \int_{-1}^{+1} \frac{\rho^2 P_n(z) \, dz}{(\rho^2 - 1)^{3/4}}, \]

and

\[ D_n^{(3)} = \int_{-1}^{+1} \frac{\rho^3 P_n(z) \, dz}{(\rho^2 - 1)^{3/4}}. \]

The superscript of the $D$'s indicates the appropriate power of $\rho$ in the numerator of the $P$ -expressions (8a-b).

From the definitions of $\zeta_\lambda$ and $\lambda_\lambda$, it follows that $A_\lambda$ can be written in the alternative form

\[ A_\lambda = \frac{Y_{e\xi}}{e} \sum_{n=0}^{\infty} \left\{ \int \frac{P_{\lambda}(z)}{\sqrt{1 + \rho^2}} \left[ \frac{m}{m} \lambda D^{(1)}_n - \lambda_\lambda D^{(3)}_n \right] F U \, d\Omega \right\} + \int \frac{P_{\lambda}(z)}{\sqrt{1 + \rho^2}} \left[ \frac{m}{m} \lambda D^{(1)}_n - \lambda_\lambda D^{(3)}_n \right] F U \, d\Omega, \]

where $d\Omega = \sin \Theta \, d\Theta \, d\Phi'$ is an element of solid angle in ion velocity space. Further, the terms in the integrands (apart from $P_n(z)$), can be rearranged into a spherical harmonic expansion in $Y_{\text{rms}}(\Theta', \Phi')$, with coefficients depending on $D_n^{(2)}$, $D_n^{(3)}$, $F(0)$.
and the components of $F^{(1)}$. Appendix 3 gives useful formulae to carry this out. If the following well known theorem for spherical harmonics is employed, namely

\[
\int_0^{2\pi} \int_0^{\pi} Y_{lms}^{(\theta', \phi')} P_{lm}(\theta) \sin \theta' \, d\theta' = \frac{4\pi}{(2l+1)} Y_{lms}(\theta, \phi) S_{lm},
\]

then only the first few values of $l$ in the infinite series (10) contribute non-zero values. Detailed examination shows that the first integrand yields a spherical harmonic expansion in $\theta', \phi'$ to $l=1$ only, so that the first series contributes to $A_\lambda$ from the $\lambda=0$ and $\lambda=1$ terms only. The second integrand contains the tensor $U^\lambda U_\lambda$ so that a second order ($l=2$) expansion is appropriate, yielding contributions from $l=0$, 1 and 2 only. The appropriate calculations to perform the angle integrations and convert to a cartesian tensor expansion are straightforward but rather tedious. For this reason the appropriate manipulation is carried out in Appendix 1. These derivations can be omitted on first reading if one so wishes and the reader can proceed directly to the final result of this section.

Although it was not clear at the outset, it is now apparent why we introduced the expansions (8a-b). With the aid of (11), and knowing the coefficients $D^{(x,y)}_{\lambda}$, the angle integrations in (10) can be performing with the minimum of effort yielding non-vanishing results for the first few terms of the series only. From Appendix 1 it is possible to write the friction term $A_\lambda$ in the cartesian tensor form:

\[
A_\lambda = A_{\lambda 0} + A_{\lambda \mu} \frac{\nu_\mu}{U} + A_{\lambda \nu} \frac{\nu_\nu}{U^2} + \ldots,
\]

where the tensor $A_{\lambda \mu \nu}$ is symmetric and irreducible (i.e. $A_{\lambda \mu \nu}=0$) in the subscripts $\mu$ and $\nu$. The elements of the various tensors
in the expansion (12) are exhibited in Appendix 1.

The friction term contribution to the electron-ion collision term \( C_{ei}(f) \) is, from (3),

\[
- \frac{2}{\partial \nu_\lambda} \left( A_\lambda f \right). \tag{13}
\]

In the spirit of the expansion for the ion distribution \( F \) given by (6), we similarly expand the electron function \( f \) as

\[
\hat{f} = f^{(0)} + f^{(1)} + \frac{0}{0} + \ldots \tag{14}
\]

Hence (13) becomes, with the aid of (12) and (14),

\[
- \frac{2}{\partial \nu_\lambda} \left( A_\lambda f \right) = - \frac{2}{\partial \nu} \left( A_{\lambda 0} f^{(0)} \right) \frac{\nu_\lambda}{\nu} - \frac{2}{\partial \nu} \left( A_{\mu \nu} f^{(1)} \right) \frac{\nu_\mu \nu_\nu}{\nu^2} - A_{\lambda 0} f^{(0)} \frac{\nu_\lambda}{\nu^2} - A_{\mu \nu} f^{(1)} \frac{\nu_\mu \nu_\nu}{\nu^2} - A_{\lambda 0} f^{(0)} \frac{\nu_\lambda}{\nu^2} - A_{\mu \nu} f^{(1)} \frac{\nu_\mu \nu_\nu}{\nu^2}.
\tag{15}
\]

In (16) we have linearised the expression by neglecting products of the first order terms \( f^{(1)} \) and \( F^{(0)} \). In this respect we note that \( A_{\lambda 0} \) and \( A_{\mu \nu} \) depend on \( F^{(1)} \) only, and \( A_{\mu} \) on \( F^{(0)} \) only.

From Appendix 1, the expansion tensors \( A_{\lambda 0} \) and \( A_{\mu \nu} \) in (12) can be written as

\[
A_{\lambda 0} = \frac{4\pi \gamma c_i}{3C^2} \int_0^\infty F^{(1)}_\lambda g_\lambda d\nu,
\]

and

\[
A_{\mu \nu} = \frac{4\pi \gamma c_i}{C} \int_0^\infty F^{(0)}_\mu g_1 d\nu. \tag{16a-b}
\]
where the kernels \( g_{\alpha, \alpha} \), considered as functions of \( U \) and \( \bar{U} \) only, can be explicitly written as

\[
g_{\alpha} = \frac{U^2}{\sqrt{1+u^2} \sqrt{1+\bar{u}^2}} \left\{ \frac{m}{M} \left( \frac{1}{3} U^{(3)}_{\alpha} + \frac{1}{3} \bar{U}^{(3)}_{\alpha} \right) - \frac{1}{3} U^{(3)}_{\alpha} - \frac{1}{3} \bar{U}^{(3)}_{\alpha} \right\} ,
\]

and

\[
g_{\beta} = \frac{U^2}{\sqrt{1+u^2} \sqrt{1+\bar{u}^2}} \left\{ \frac{m}{M} \left( \frac{1}{3} U^{(3)}_{\beta} + \frac{1}{3} \bar{U}^{(3)}_{\beta} \right) - \frac{1}{3} U^{(3)}_{\beta} - \frac{1}{3} \bar{U}^{(3)}_{\beta} \right\} .
\]

(17a-b)

For the tensor \( \lambda_{\mu, \gamma} \) we have the representation, in Kronecker deltas,

\[
\lambda_{\mu, \gamma} = -\frac{4}{3} \delta_{\lambda, \mu} \beta_{\lambda} + 2 \delta_{\lambda, \lambda} \beta_{\lambda} + 2 \delta_{\lambda, \gamma} \beta_{\mu},
\]

where

\[
\beta_{\lambda} = \frac{\pi Y e}{c^3} \int_{0}^{\infty} \mathbb{E}^{-1}_{\lambda, \mu} \, dU, \quad \lambda = 1, 2, \ldots, 3
\]

and the kernel \( g_{2} \) is given by

\[
g_{2} = \frac{U^2}{\sqrt{1+u^2} \sqrt{1+\bar{u}^2}} \left\{ \frac{m}{M} \left( \frac{1}{3} U^{(3)}_{2} + \frac{1}{3} \bar{U}^{(3)}_{2} \right) - \frac{1}{3} U^{(3)}_{2} - \frac{1}{3} \bar{U}^{(3)}_{2} \right\} .
\]

(18)

It is a straightforward matter to perform the tensor products in (15) with the definitions above. The result yields the following form for (13) namely.
As can be seen, the form of (19), even to first order, yields a complicated expression when one substitutes for the kernels with the D-functions given previously.

Equation (19) is the final result of this section. It represents the friction term contribution to the Fokker-Planck collision term for electron-ion interaction. In (19), the angle integration over all directions of ion velocity has been evaluated leaving a complicated mixed integrodifferential relation. The contribution to $C_{\text{el}}^{(0)}$ (the first collision moment) comes from the leading term in (19), the others contributing to $C_{\text{el}}^{(1)}$.

$\S 3$ EXPANDED DIFFUSION TERM.

The diffusion term is tackled in a similar manner to the friction term considered previously. However since now we are dealing with a second order tensor, the analysis becomes more involved. In classical
theory it is possible to write the diffusion tensor \( \mathcal{D}_{\lambda\mu} \) in the form (see e.g. \( \ell \), p. 272).

\[
\mathcal{D}_{\lambda\mu} = \frac{\partial^2}{\partial \chi \partial \chi'} G(\chi),
\]

where \( G(\chi) \) is a scalar function of velocity. It is then possible to expand \( G \) in terms of cartesian tensors and substitute into the Fokker-Planck equation. In the relativistic case it appears that no such scalar potential exists and some of the mathematical elegance is lost.

The diffusion tensor in the relativistic case is given by

\[
\mathcal{D}_{\lambda\mu} = \int \varepsilon_{\lambda\mu}' F \, d^3 \chi',
\]

where \( F \) is the ion distribution, \( \chi' \) the ion reduced velocity and \( \varepsilon_{\lambda\mu}' \) is given in terms of \( \chi' \) and \( \chi \) as

\[
\varepsilon_{\lambda\mu}' = \frac{Y_{ei}}{C(p' - 1)} \frac{p^2}{\gamma_{(m)}} \left\{ (p' - 1) \delta_{\lambda\mu} - U_{\lambda} U_{\mu} - U_{\lambda} U_{\mu} - \rho \left( U_{\lambda} U_{\mu} - U_{\mu} U_{\lambda} \right) \right\}.
\]

\( \rho \) has been given previously in \( \ell \).

As in the previous section, the ion-distribution is expanded to first order in spherical harmonics. The expression for \( F \) (equation (6)) is substituted into the defining equation (20) and the angular integrations over \( \theta' \) and \( \phi' \) carried out. The procedure is similar to that outlined in \( \ell \). In Appendix 2 we show how the tensor \( \mathcal{D}_{\lambda\mu} \) may be expressed in cartesian form to order three as

\[
\mathcal{D}_{\lambda\mu} = K_{\lambda\mu} - W_{\lambda\mu},
\]
where
\[ K_{\lambda\mu} = K_{\lambda\mu 0} + K_{\lambda\mu \varphi} \frac{U}{U^2}, \]
and
\[ W_{\lambda\mu} = W_{\lambda\mu 0} + W_{\lambda\mu \varphi} \frac{U}{U^2} + W_{\lambda\mu \varphi \phi} \frac{U^2}{U^2} + W_{\lambda\mu \varphi \phi \phi} \frac{U^2}{U^2}. \]

(21a-c)
The elements of the tensors on the right hand side of equations (21b-c) are functions of position, time and the electron reduced speed \( U \) only. If \( P_n(z) \) denotes the Legendre function of order \( n \), we define one further type of D-function (cf. 9a-b) by
\[ D_n^{(i)} = (n + \frac{1}{2}) \int_{-1}^{+1} \rho^2 P_n(z) \frac{dz}{\sqrt{\rho^2 - 1}}. \]

(22)
In terms of the D-functions, various combinations of these are defined as follows (see Appendix 2):
\[
\begin{align*}
    h_0 &= \left\{ \frac{1}{2} (U^2 + \bar{U}^2) \bar{U}^2 D_0^{(2)} + \frac{2}{3} \bar{U}^3 D_1^{(3)} \right\} \beta_1 \beta_2, \\
    h_i &= \left\{ \frac{1}{2} (U^2 + \bar{U}^2) \bar{U}^2 D_1^{(2)} + \frac{2}{3} \bar{U}^3 D_0^{(3)} + \frac{2}{3} \bar{U}^3 D_1^{(3)} \right\} \beta_1 \beta_2, \\
    h_i' &= \left\{ \frac{1}{15} (U^2 + \bar{U}^2) \bar{U}^2 D_1^{(2)} + \frac{2}{25} \bar{U}^3 D_0^{(3)} \right\} \beta_1 \beta_2, \\
    h_2 &= \left\{ \frac{1}{15} (U^2 + \bar{U}^2) \bar{U}^2 D_1^{(2)} + \frac{2}{25} \bar{U}^3 D_0^{(3)} \right\} \beta_1 \beta_2, \\
and \text{lastly} \\
    h_3 &= \left\{ \frac{1}{3} (U^2 D_1^{(2)} + \frac{1}{3} \bar{U}^3 D_2^{(2)}) \bar{U}^2 + \frac{2}{5} \bar{U}^3 D_1^{(3)} \right\} \beta_1 \beta_2. 
\end{align*}
\]

(23a-e)
In the definitions of the \( h \) terms we have put \( \beta_i = \sqrt{1 + \bar{U}^2} \) and
In Appendix 2 we give the forms of the various tensors appearing in (21b-c) together with the dependence of the elements on the \( h \) functions defined above. We will return to this in a moment.

The diffusion term contribution to the Fokker-Planck collision term is given by

\[
\frac{1}{2} \frac{\partial^2}{\partial t \partial \mu} \left( D_{\mu \rho} f \right).
\]

Again, the electron distribution \( f \) is expanded to first order as in (14). The next task is to evaluate (24) with the defining equations (21a-c). As for the friction term, we intend to linearise the expression by neglecting products of first order terms \( f^{(0)} \) and \( F^{(1)} \).

In this respect we note from Appendix 2 that \( K_{\lambda \nu}, W_{\lambda \nu} \) and \( W_{\lambda \nu} \) depend on the directional distribution \( F^{(1)} \), while \( K_{\lambda \nu} \), \( W_{\lambda \nu} \) and \( W_{\lambda \nu} \) depend on the isotropic part of the ion distribution \( F^{(0)} \).

The \( K_{\lambda \nu} \) contribution to \( D_{\lambda \mu} \) in (21a) is easily evaluated. From Appendix 2 we have for the definitions of \( K_{\lambda \nu} \) and \( K_{\lambda \nu} \), the expressions:

\[
K_{\lambda \nu} = \frac{e \epsilon_i e^2}{3} \delta_{\lambda \nu} 4\pi \int d_0 F^{(0)} d\mathbf{U},
\]

where

\[
d_0 = D^{(i)} \mathbf{U}^2 \beta_1 \beta_2,
\]

\[
K_{\lambda \nu} = \frac{e \epsilon_i e^2}{3} \delta_{\lambda \nu} 4\pi \int d_1 F^{(1)} d\mathbf{U},
\]

and

\[
d_1 = D^{(i)} \mathbf{U}^1 \beta_1 \beta_2.
\]

(25a-d)
Using the results that

\[ \nabla_v^2(f) = \frac{\partial^2 f}{\partial v^2} + \frac{2}{U} \frac{\partial f}{\partial U}, \]

for any function \( f = f(U) \), and

\[ \nabla^2(fg) = (\nabla^2 f)g + 2\nabla f \cdot \nabla g + (\nabla^2 g)f; \]

we can write

\[
\frac{1}{2} \frac{\partial^2}{\partial U \partial u} (K_{\nu} f) = 2\pi \frac{\gamma_e}{c^3} \frac{\partial}{\partial U} \left( \int_0^\infty d_\nu F^{(0)}(dU) \right) \\
+ \frac{4\pi \gamma_e}{c^3} \frac{U}{\partial U} \left( \int_0^\infty d_\nu F^{(0)}(dU) \right) \\
+ \left[ 2\pi \gamma_e \left[ D \right] \left\{ \int_0^\infty d_\nu F^{(0)}(dU) \right\} \right] \frac{U}{\partial U} \\
+ \left[ 2\pi \gamma_e \left[ D \right] \left\{ \int_0^\infty d_\nu F^{(0)}(dU) \right\} \right] \frac{U}{\partial U}.
\]

In (26), \( \left[ D \right] \) represents the diffusion operator

\[
\left[ D \right] = \frac{\partial^2}{\partial U^2} + \frac{2}{U} \frac{\partial}{\partial U} - \frac{2}{U^2}.
\]

It remains to evaluate the contribution

\[
\frac{1}{2} \frac{\partial^2}{\partial U \partial u} (W_{\mu} f),
\]

with \( W_{\mu} \) given by (21c) and the appropriate expansion elements given in Appendix 2. The electron distribution for \( f \) is given by (14) and substitution into (28) yields the linearised relation for \( W_{\mu} f \) as
The tensor elements (\(W\) terms) of the right hand side of (29) are functions of the magnitude of \(U\) only, so that, for example,

\[
\frac{\partial W_{\mu \nu}}{\partial U_\lambda} = \frac{\partial W_{\mu \nu}^*}{\partial U} \frac{U_\lambda}{U}.
\]

The \(W_{\mu \nu} f^{(0)}\) term in (29) is easy since \(W_{\mu \nu}^*\) is a Kronecker function given by

\[
W_{\mu \nu} = \delta_{\mu \nu} \frac{4\pi \gamma_{ei}}{C^3} \int_0^\infty \frac{h_0 F^{(0)}}{dU} dU.
\]

The final result yields a contribution to (28) in the form

\[
\frac{1}{2} \frac{\partial^2}{\partial U^2} \left( W_{\mu \nu} f^{(0)} \right) = \frac{2\pi \gamma_{ei}}{C^3} \left\{ \int_0^\infty \frac{h_0 F^{(0)}}{dU} dU \right\}^2 + \frac{4\pi \gamma_{ei}}{C^3 U} \left\{ \int_0^\infty h_0 F^{(0)} dU \right\}.
\]

The \(W_{\mu \nu} \frac{f^{(1)} U}{U}\) term is also easily evaluated, and has the form
where the operator \([\mathcal{D}]\), has been defined in equation (27) above.

For the remaining terms in (29) the procedure is as above. One first differentiates the appropriate term according to (28) and then evaluates the various tensor products so formed according to the method outlined in Appendix 2. Much simplification is afforded if one appeals to the symmetry properties of the tensors since most of the products generated are simply multiples of one basic type. For example, considering the third term of (29) we have

\[
\frac{1}{2} \frac{\partial^2}{\partial \omega \partial \nu} \left( W_{\lambda \mu \nu} f^{(e)} \frac{U_x U_y}{U^2} \right) = \frac{1}{2} \frac{\partial}{\partial \nu} \left\{ \frac{1}{U} \frac{\partial}{\partial \nu} \left( W_{\lambda \mu \nu} f^{(e)} \frac{U_x U_y}{U^2} \right) \right\} \frac{U_x U_y U_x}{U} \\
+ 2 \frac{\partial}{\partial \nu} \left( \frac{W_{\lambda \mu \nu}}{U^2} \right) \frac{U_x U_y}{U} + \frac{W_{\lambda \mu \nu}}{U^2} f^{(e)}
\]

(32)

where we have used the fact that \( W_{\lambda \mu \nu} \) is symmetrical in \( \lambda, \mu, \nu \) and also \( W_{\lambda \mu \nu} = 0 \). In terms of the quantity

\[
S = \frac{2\pi \gamma_e i}{C^3} \int_0^\infty h_x f^{(e)} dU,
\]

we can write the right hand side of (32) as
\[ \frac{2}{3} \left\{ \frac{\partial^2}{\partial \omega^2} (f^{(2)} S) + \frac{5}{\omega} \frac{\partial}{\partial \omega} (f^{(2)} S) + \frac{3}{\omega^2} f^{(2)} S \right\}. \]  

The fourth term of (29) contributes to (28) the expression

\[ \frac{1}{2} \frac{\partial^2}{\partial \omega \partial \nu} \left( W_{\nu \lambda} f^{(2)} (\frac{\nu \nu \omega \nu}{\omega^2}) \right) = \frac{2}{3} \left[ D \right] \left\{ f^{(2)} \right\} \frac{\nu}{\omega}, \]  

where \[ D \] represents the diffusion operator, similar to (27), given by

\[ \left[ D \right] = \frac{\partial^2}{\partial \omega^2} + \frac{5}{\omega} \frac{\partial}{\partial \omega} + \frac{4}{\omega^2}. \]  

The fifth term of (29) involves the tensor \( W_{\nu \lambda} \) which is given by equation (23) of Appendix 2 of this chapter. The elements of this tensor depend on two sets of quantities

\[ A_{\lambda} = \frac{4 \pi Y_{\lambda \nu}}{C^3} \int_0^\infty h_{i \nu} F_{\lambda}^{(1)} d\nu, \]

and

\[ E_{\lambda} = \frac{4 \pi Y_{\lambda \nu}}{C^3} \int_0^\infty h_{\nu \nu} F_{\lambda}^{(1)} d\nu. \]

Explicitly we have the tensor form

\[ W_{\nu \lambda} = \frac{1}{2} \left( A_{\lambda} - E_{\lambda} \right) \delta_{\nu \lambda} + \frac{1}{2} \left( A_{\nu} - E_{\nu} \right) \delta_{\lambda \nu} + E_{\nu} E_{\lambda}, \]

so that

\[ \frac{1}{2} \frac{\partial^2}{\partial \omega \partial \nu} \left( W_{\nu \lambda} f^{(2)} (\frac{\nu \nu \omega \nu}{\omega^2}) \right) = \frac{2 \pi Y_{\lambda \nu}}{C^3} \left[ D \right] \left\{ f^{(2)} \int_0^\infty h_{i \nu} F_{\lambda}^{(1)} d\nu \right\} \frac{\nu}{\omega}. \]
We note that in equation (36) the $E_\lambda$ terms do not appear, affording some simplification of the results.

Lastly we have the term

$$\frac{1}{2} \frac{\partial^2}{\partial u^\alpha \partial u^\beta} \left( W_{\gamma\mu \nu \xi \rho} f^{(0)} u_\gamma u_\mu u_\nu u_\xi u_\rho \right),$$

(37)
to compute. The Kronecker delta representation of the fifth order tensor is given in equation (32) of Appendix 2, and because of its rather lengthy appearance, we do not propose to repeat it here. The first point to note is that contraction of the first two suffices yields zero; that is

$$W_{\lambda \lambda \gamma \alpha \beta} = 0,$$

where, of course, summation over $\lambda$ is implied. The tensor $W_{\gamma \mu \nu \xi \rho}$ is symmetrical in interchange of $\lambda$ and $\mu$ and also in any two of the suffices $\gamma, \alpha, \beta$. Using these properties, (37) becomes

$$\frac{1}{2} \frac{\partial}{\partial u} \left\{ \frac{1}{U} \frac{\partial}{\partial u} \left[ W_{\gamma \mu \nu \xi \rho} f^{(0)} \right] \right\} u_\lambda u_\mu u_\nu u_\xi u_\rho$$

$$+ \frac{3}{U} \frac{\partial}{\partial u} \left\{ \frac{W_{\gamma \mu \lambda \xi \rho}}{U^3} f^{(0)} \right\} u_\mu u_\xi u_\rho + \frac{3}{U^3} W_{\gamma \mu \lambda \xi \rho} f^{(0)} u_\rho.$$

(38)

Evaluation of the remaining tensor products of (38) yields the final form

$$\frac{4}{5} \kappa \frac{Y_{\xi_i}}{2} \left[ D \right]_3 \left\{ f^{(0)} \int_{\infty}^\infty F^{(1)} h_3 dU \right\} \cdot \frac{U}{U},$$

(39)

where $\left[ D \right]_3$ represents the third type of diffusion operator defined as
This completes the rather lengthy discussion of the diffusion term. The contribution of the $K_{\lambda\mu}$ term to (24) is given by (26). The contribution of $W_{\lambda\mu}$ is more complicated and the separate contributions (without sign) are given by equations (30), (31), (33), (34), (36) and lastly (39).

Thus far we have expanded both the friction and diffusion terms of the relativistic Fokker-Planck equation to first order in cartesian tensors of the reduced velocity. The assumptions made at this stage are that both electron and ion distributions are expanded to first order and the contributions in the collision terms arising from products of first order terms can be neglected. This is justified providing disturbances from fields and from gradients in plasma parameters is sufficiently small. Also it is worth noting at this stage that no assumption has been made about the relative masses of the colliding particles, and the equations can in fact be applied to the case of comparable particle mass. In the next section, special cases of the collision term will be derived for electron-electron and electron-ion interaction.

\section*{84 \textit{COLLISION TERMS FOR MOMENT EQUATIONS.}}

In Chapter 3 on the tensor expansion for the relativistic Boltzmann equation it was assumed that the collision term $C(\mathbf{f})$ could be expressed in the tensorial form

\[ C(\mathbf{f}) = \sum_{\lambda} C^{(\lambda)} \frac{\mathbf{U}^{(\lambda)}}{U^2}. \]

In the previous two sections we have derived the appropriate terms in an expansion of the Fokker-Planck collision term to first order

\[ [D]_3 = \frac{\partial^2}{\partial U^2} + \frac{7}{U} \frac{\partial}{\partial U} + \frac{8}{U^2}. \]
(i.e. \( \lambda = 1 \) in (41)). By comparison with (41) the appropriate forms for \( C^{(0)} \) and \( C^{(1)} \) can be deduced. Starting with the \( C^{(0)} \) moment we have the form

\[
\frac{C^3}{4 \epsilon^2} C^{(0)}(\mathbf{f}^{(0)}) = -\frac{4 \pi}{U^2} \left\{ U^2 \int_0^\infty \frac{F^{(0)}}{\partial \mathbf{f}} \frac{dU}{dU} \right\} \\
+ \frac{2 \pi}{U^2} \left\{ U^2 \frac{\partial}{\partial U} \left[ \int_0^\infty F^{(0)} dU \right] \right\} \\
- \frac{2 \pi}{U^2} \left\{ U^2 \frac{\partial}{\partial U} \left[ \int_0^\infty F^{(0)} h_0 dU \right] \right\} - \frac{4 \pi}{3} \left[ A \right] \left\{ \int_0^\infty F^{(0)} h_1 dU \right\},
\]

(42)

where the kernels \( g_1, h_0, h_2 \) and \( d_0 \) are given by equations (17b), (23a, d) and (25b) respectively, and the operator \( [A] \) is given by

\[
[A] = \frac{\partial^2}{\partial U^2} + \frac{5}{U} \frac{\partial}{\partial U} + \frac{3}{U^2}.
\]

(43)

In fact \( [A] \), can be written in the alternative form

\[
[A] = \frac{1}{U^2} \left\{ \frac{1}{U} \frac{\partial}{\partial U} \left[ U^3 \ldots \right] \right\},
\]

where dots indicate any function of \( U \) upon which it operates. As can be seen from (42), the functional form of \( C^{(0)}(\mathbf{f}^{(0)}) \) becomes quite complicated when one substitutes the defining relations for the various kernels under the integral signs. We note that from (42) with the alternative form of \( [A] \), given above, the collision term \( C^{(0)} \) can be written in the alternative form

\[
C^{(0)}(\mathbf{f}^{(0)}) = \frac{1}{U^2} \frac{\partial}{\partial U} S(\mathbf{f}^{(0)}),
\]

(44)
where the form of $\mathcal{S}$ can be written down by comparison with (42). Equations (44) play an important role in demonstrating particle conservation in the absence of any creation or loss mechanism. This will be considered later in § 6.

Little further analytical progress can be made with the various collision moments unless approximations are made at this stage. Essentially the aim we adopt is to expand each of the kernels appearing in (42) in a power series of $\beta_T = \frac{KT}{MC^2}$; where $T$ is the ion temperature (assumed equal to the electron temperature in near equilibrium conditions) and $M$ is the ion mass. We retain terms up to $\beta_T$; that is we include both classical and first order relativistic effects. The procedure is valid provided $KT \ll MC^2$.

Firstly we note that in the classical limit, the equilibrium Maxwellian distribution is proportional to
\[ C^{-\frac{1}{2}} \lambda_i U^2 \]
for ions, where $\lambda_i = \frac{MC^2}{KT}$, and for electrons is proportional to
\[ C^{-\frac{1}{2}} \lambda_e U^2 \]
where $\lambda_e = \frac{MC^2}{KT}$. We thus define two new "scaled" reduced velocities $q$ and $Q$ given by
\[ q^2 = \frac{1}{2} \lambda_e U^2, \quad Q^2 = \frac{1}{2} \lambda_i U^2, \quad (45) \]
respectively. We further define a quantity $\mu = \frac{\lambda_i}{\lambda_e}$.

The procedure is to substitute for $U$ and $U$ in terms of $q$ and $Q$ (through equations (45)) in the various kernels $g_i$ etc appearing in (42). These functions can then be expanded in powers of $\beta_T$ (or $\lambda_i^{-1}$) to desired order as required. We observe that this is equivalent to applying the method of steepest descents to the various integrals involved in (42). Since both $F^{(0)}$ and $f^{(o)}$ can be considered as
proportional to $e^{-\frac{1}{2}\lambda_i U^2}$ and $e^{-\frac{1}{2}\lambda_e U^2}$ respectively when

$\lambda_i, \lambda_e \gg 1$, then important contributions to the integrals come from small values of $U$ and $U'$ only. We note further that, for near equilibrium conditions in which $T_e \sim T_i (\equiv T)$

$$\frac{\lambda_i}{\lambda_e} \sim \frac{M}{m} \gg 1,$$  \hfill (46)

for heavy ions of mass $M$.

To begin with we make one or two general remarks. Firstly, if one refers back to the defining equations of the various $h$, $d$ and $g$ functions appearing in (42) it is apparent that, apart from their dependence on $U$ and $U'$, they depend on the quantity $P$ where

$$P = U \cdot U' - \sqrt{1+U^2} \sqrt{1+U'^2}.$$ 

Writing, as before, $Z$ equal to the cosine of the angle between the two vectors $U$ and $U'$,

$$P = U U' Z - \sqrt{1+U^2} \sqrt{1+U'^2}.$$ 

Going over to the new 'scaled' velocities $q$ and $Q$ we can write

$$P = \frac{2 \sqrt{\mu q Q Z}}{\lambda_i} - \sqrt{1+\frac{q Z}{\lambda_i^2}} \sqrt{1+\frac{Q Z}{\lambda_i^2}}.$$ 

Expanding to second order in $\beta_T$ gives the form

$$P = -1 - \frac{\alpha}{\lambda_i^2} + \frac{\beta}{2\lambda_i^2},$$  \hfill (47)

where the quantities $\alpha$ and $\beta$ are given respectively by

$$\alpha = q^2 \mu + Q^2 - 2Qq \sqrt{\mu} Z,$$

and

$$\beta = (q^2 \mu - Q^2)^2.$$  \hfill (48)
$P$ is expanded up to order $\beta^{-1}$ since quantities like $(p^{-1})'$ appear in the definitions of the $g$ functions etc. Further, derivatives with respect to $U$ are replaced by those with respect to $q$, taking into account appropriate multiplying factors. We redefine the various differential operators introduced above as, for example

$$[D]_i = \frac{\partial^2}{\partial q^2} + \frac{2}{q^2} \frac{\partial}{\partial q} - \frac{2}{q},$$

the extra multiplying factor $\lambda_i$ being absorbed into the $h$, $g$, or $d$ function concerned.

The various $h$, $g$ or $d$ kernels introduced in the cartesian expansion of the friction and diffusion terms all depend on three basic functions defined by equations (9a-b) and (22).

If we apply the transformations (45) to the integrands of the $D$-functions we obtain the following forms:

$$\frac{\rho^2}{(p^2 - 1)^{3/2}} = \left(\frac{\lambda_i}{2\lambda}\right)^{3/2} \left(1 + \frac{5\lambda}{4\lambda_i} + \frac{3\beta}{4\lambda_i} + \ldots\right),$$

$$\frac{\rho^5}{(p^2 - 1)^{5/2}} = -\left(\frac{\lambda_i}{2\lambda}\right)^{5/2} \left(1 + \frac{9\lambda}{4\lambda_i} + \frac{3\beta}{4\lambda_i} + \ldots\right),$$

and

$$\frac{\rho}{\sqrt{p^2 - 1}} = \left(\frac{\lambda_i}{2\lambda}\right)^{1/2} \left(1 + \frac{7\lambda}{4\lambda_i} + \frac{\beta}{4\lambda_i} + \ldots\right),$$

with $\lambda$ and $\beta$ defined as above. The expansions can be substituted into the defining equations for the $D$-functions above and the integration over $S$ performed. To do this we first note that $\lambda$ can be written in the form

$$\lambda = (q \sqrt{\lambda}) \left(1 - 2h^2 \pm h^2\right),$$

(49)
where \( h = \frac{Q<}{(q/\mu)^2} \) and \( Q< \) is the lesser and \((q/\mu)\), the greater of \( q/\mu \) and \( Q \). Employing the generating relation for Legendre functions in the form

\[
(q/\mu)\alpha^{-\nu} = \sum_{n=0}^{\infty} h^n P_n(z),
\]

it is a straightforward matter to show that

\[
\alpha^{-\nu} = (q/\mu)^{\nu} \sum_{n=0}^{\infty} \left( \frac{h^n}{2n+3} - \frac{1}{2n-1} \right) h^n P_n(z),
\]

\[
\alpha^{-3/\nu} = (q/\mu)^{-3/\nu} \sum_{n=0}^{\infty} \frac{h^n}{1-h^2} (2n+1) P_n(z),
\]

and finally

\[
\alpha^{-5/\nu} = (q/\mu)^{-5/\nu} \sum_{n=0}^{\infty} \frac{h^n}{(1-h^2)^2} \left\{ 1 + \frac{2n}{3} + \frac{4n^2 h^2}{3(1-h^2)} \right\} (2n+1) P_n(z).
\]

We insert the expansions (49) into the defining equations (9a-b) and (22) and integrate over \( z \) using the above formulae for \( \alpha^{-\nu} \) etc and the orthogonality property for Legendre polynomials. The general forms for the three D-functions are

\[
D_{\nu}^{(1)} = \left( \frac{\lambda}{l} \right)^{\nu} \left\{ \frac{h^n}{(q/\mu)^{\nu}} + \frac{1}{4\lambda^2} \left( \frac{h^n}{2n+3} - \frac{1}{2n-1} \right) (q/\mu)^{\nu} h^n \right. \\
+ \left. \frac{(2n+1)}{4\lambda^2} (1-h^2) h^n (q/\mu)^{\nu} \right\},
\]

(51)
Using the general results of (51), (52) and (53), we can evaluate the kernels \( g_1 \), \( h_0 \) etc appearing in the zero moment collision term given by (42). The kernels to be evaluated are merely combinations of the D-functions above for \( n = 0, 1 \) and 2. The appropriate forms can be found by referring to the equation (17b) (for \( g_1 \) ), equations (23a,d) (for \( h_0 \) and \( h_1 \) ) and lastly equation (25b) (for \( d_0 \) ). The forms required for the evaluation of \( C^{(0)} \) will be considered in a moment.

Firstly we observe that the form of the expansion for the various functions depends on whether \( q \sqrt{\mu} \) is greater or less than \( Q \). Thus the integration over \( Q \) in equation (42) (alternatively \( U \) in the unscaled system) has to be split into two parts, namely values of \( Q < q \sqrt{\mu} \) and \( Q > q \sqrt{\mu} \). This unfortunately adds to the complex nature of the integrodifferential form of \( C^{(0)} \).
Returning to the evaluation of $c^{(0)}$ to order $\beta \tau$, we have four terms to compute, namely $\mathcal{g}_1$, $d_0$, $h_0$, and $h_1$. The forms of these are given by equations (17b), (25b) and (25a,d) respectively. All of the terms include the factor

$$\beta_1 \beta_1 = \left\{ (1 + U^2)(1 + V^2) \right\}^{1 - \nu_2},$$

which in the scaled system $q$ and $Q$ yields to order $\beta \tau$

$$\beta_1 \beta_2 = 1 - \beta \tau (Q^2 + \mu q^2).$$

Let us start by considering the first term on the right hand side of equation (42). Converting to the scaled system, in which $2\mu q^2 \beta \tau$ replaces $U^2$ and $2Q^2 \beta \tau$ replaces $U^2$, introduces a multiplying factor $\mu^{-\nu_2}$; that is

$$\frac{4\pi}{U^4} \frac{\partial}{\partial U} \left\{ U^2 f^{(o)} \right\} \int_0^\infty F^{(o)} \mathcal{g}_d U \right\} = \frac{4\pi}{q^4} \frac{\partial}{\partial q} \left\{ q^2 f^{(o)} \right\} \int_0^\infty F^{(o)} \mathcal{g}_d Q \right\}.$$

Henceforth all multiplying factors introduced in this way will be absorbed into the definition of the appropriate kernel.

For the $\mathcal{g}_1$ function we refer to its definition in equation (17b) and apply the expansions (52) and (53) each for $n = 0$ and 1. The final result is (together with the $\mu^{-\nu_2}$ factor) correct to order $\beta \tau$:

$$\mathcal{g}_1 = -\frac{Q^4}{\mu^{3/2} q^2} \left( 1 + \frac{m}{M} \right) + \frac{\beta \tau}{\mu^{\nu_2}} \left\{ \frac{Q^4}{\mu q^2} \left( 2 + \frac{5m}{3m} \right) - Q^2 \left( 1 + \frac{2m}{M} \right) \right\}, \quad q \sqrt{\mu} > Q$$

and

$$\mathcal{g}_1 = -\frac{4\beta \tau}{3\sqrt{\mu}} \left( Q q \sqrt{\mu} \right) \frac{m}{M}, \quad q \sqrt{\mu} < Q.$$  

(54)

Next we consider the function $d_0$. This is somewhat simpler, being directly related to the term $D^{(1)}$ as given by equation (25b).
Converting the second term on the right hand side of (42) to the scaled variables $q$ and $Q$ yields the multiplying term $\frac{1}{\mu} \sqrt{\frac{A_c}{2}}$.

Applying the expansion (51) for $\eta = 0$ we have for $d_0$ the expression (together with the extra factor):

$$d_0 = \frac{Q^2}{\mu^* q^* \mu} - \frac{\beta_T}{\mu} \left( \frac{2}{3} Q^4 q^* \mu - Q^2 q^* \mu \right), \quad q^* \mu > Q,$$

and

$$d_0 = \frac{Q}{\mu} - \frac{\beta_T}{\mu} \left( \frac{2}{3} Q q^* \mu - Q^3 \right), \quad Q > q^* \mu. \quad (55)$$

We have finally two $h_\eta$-type functions to consider in developing the $C^{(0)}$ collision moment. For $h_0$ we refer to equation (25a). We apply the expansions (52) and (53) for $\eta = 0$ and 1 respectively and retain terms up to order $\beta_T$. After a little tedious manipulation the resulting expansion for $h_0$ yields (with a $\frac{1}{\mu} \sqrt{\frac{A_c}{2}}$ factor):

$$h_0 = \frac{Q^2}{3 \mu^* q^* \mu} - \frac{\beta_T}{3 \mu} \left\{ \frac{2}{3} Q^4 q^* \mu - Q^2 q^* \mu \right\}, \quad q^* \mu > Q,$$

and

$$h_0 = \frac{Q}{3 \mu} - \frac{\beta_T}{3 \mu} \left\{ 2 (Q q^* \mu - Q^3) \right\}, \quad Q > q^* \mu. \quad (56)$$

The remaining kernel to be evaluated in the $C^{(0)}$ collision term is $h_1$. Its definition is given by equation (23a) and involves the functions $D^{(2)}$ and $D^{(3)}$. We apply the expansions (52) for $\eta = 0$ and 2 and (53) for $\eta = 1$, retaining terms up to $\beta_T$. The result for $h_1$ yields the expression:
\[ h_2 = \frac{Q^2}{q \sqrt{3} \mu} - \frac{q}{\sqrt{3} \mu \sigma} + \frac{\beta_r}{\mu} \left\{ \frac{2Q^4}{q^2 \mu^2} - \frac{3Q^2}{q \sigma} + 2q \sqrt{\mu} \right\}, q \sqrt{\mu} > Q, \]

\[ h_2 = 0, \quad q \sqrt{\mu} < Q. \]  

(57)

As before we have made the transformation to the scaled variables \( q \) and \( Q \) and absorbed the extra factor \( \frac{1}{\sqrt{\mu}} \) (arising from the integrodifferential operator) into the definition of \( h_2 \).

The procedure is to substitute the expansions (54) and (57) into equation (42) and simplify where possible the result, yielding the zero order electron-ion collision moment correct to terms in \( \beta_T \).

We define two types of integral, similar to those introduced by Allis in the classical theory (see e.g. [27], p. 276) of the form

\[ \int_{j}^{i} = \frac{4\pi}{\langle q \sqrt{\mu} \rangle} \int \frac{F^{(i)}}{Q^{j+2}} dQ, \quad J_{j}^{i} = \frac{4\pi}{\langle q \sqrt{\mu} \rangle} \int \frac{F^{(i)}}{Q^{j+2}} dQ. \]  

(58)

These satisfy the differential relations

\[ \frac{\partial I_{j}^{i}}{\partial q} = 4\pi \frac{q}{\sqrt{3} \mu} q^{2} F^{(i)} \frac{j}{q} - \frac{j}{q} \frac{I_{j}^{i}}{q}, \]

\[ \frac{\partial J_{j}^{i}}{\partial q} = -4\pi \frac{q}{\sqrt{3} \mu} q^{2} F^{(i)} \frac{j}{q} - \frac{j}{q} \frac{J_{j}^{i}}{q}. \]  

(59)

In the above definitions the \( i \) index refers to the \( i \)th moment of the ion-distribution (for our case we consider \( i = 0 \) and 1 only).

**THE \( C^{(O)} \) RELATION:**

In the spirit of our expansion to order \( \beta_T \) of the various kernels appearing in (42) we expand the collision moment \( C^{(O)} \) ( \( f^{(O)} \)) in the form

\[ C^{(O)} = C^{(O)}_0 + \beta_T C^{(O)}_1. \]  

(60)
It is simplest to consider the classical ($\beta_T$) and the relativistic ($\beta_T$) terms separately. Firstly we combine the functions $d_o$ and $h_o$, since the same operator acts on both, we then re-express the $[A]$, operator in its alternative form and finally apply the definitions of the $I$ and $J$ integrals. Thus for $C^{(0)}_o$ we have

$$\frac{C^{(0)}_o}{\gamma_{ei}} = \frac{1}{q^2}\frac{\partial}{\partial q}\left\{f^{(0)}(1 + \frac{m}{m})I^{(0)}_o\right\}$$

$$+ \frac{1}{q^2}\frac{\partial}{\partial q}\left\{q^2\frac{\partial}{\partial q}\left[\frac{f^{(0)}_1}{3}I^{(0)}_o + \frac{f^{(0)}_2}{3}J^{(0)}_1\right]\right\}$$

$$- \frac{1}{q^2}\frac{\partial}{\partial q}\left\{\frac{1}{3}\frac{\partial}{\partial q}\left[\frac{f^{(0)}_1}{3}I^{(0)}_o - \frac{f^{(0)}_2}{3}J^{(0)}_2\right]\right\}.$$  

Application of the relations (59) yields the simplified form

$$\frac{C^{(0)}_o}{\gamma_{ei}} = \frac{1}{q^2}\frac{\partial}{\partial q}\left[m f^{(0)}_1 I^{(0)}_o + \frac{q}{3}(I^{(0)}_1 - J^{(0)}_1)\frac{\partial f^{(0)}_1}{\partial q}\right].$$  

Similarly for the $C^{(0)}_1$ term we have

$$\frac{C^{(0)}_1}{\gamma_{ei}} = \frac{1}{q^2}\frac{\partial}{\partial q}\left\{f^{(0)}\left[(-2 - \frac{5m}{3m})q^2I^{(0)}_2 + (1 + \frac{2m}{M})q^2I^{(0)}_1\right] + \frac{2m}{3m}q^2J^{(0)}_1\right\}$$

$$+ \frac{1}{q^2}\frac{\partial}{\partial q}\left\{q^2\frac{\partial}{\partial q}\left[\frac{q}{3}f^{(0)}_1 I^{(0)}_1 + q f^{(0)}_2 J^{(0)}_2\right]\right\}$$

$$- \frac{1}{q^2}\frac{\partial}{\partial q}\left\{q^2\frac{\partial}{\partial q}\left[\frac{2}{3}q f^{(0)}_1 I^{(0)}_2 - q f^{(0)}_1 I^{(0)}_2 + \frac{q}{3} f^{(0)}_2 I^{(0)}_2\right]\right\}. $$
which, again, after some simplification using the equations (59) becomes:

$$\left( \frac{e}{M_i} \right)^3 \frac{C_i^{(0)}}{Y_{ei}} (f^{(0)}) = \frac{1}{q^2} \frac{\partial}{\partial q} \left\{ q^3 \left[ \frac{2m_1}{m_i} I^0 - \frac{5m_{1/2}}{3m_i} J^0 + \frac{4m_{1/3}}{3m_i} J^0 \right] \right\}$$

$$+ q^3 \frac{\partial}{\partial q} \left\{ \frac{2}{3} I^0 - \frac{2}{3} J^0 + \frac{1}{3} J^0 \right\}.$$  \( (62) \)

Equations (61) and (62) are the final results for the zero order collision moment \( C^{(0)} \). They describe electron-ion interaction for arbitrary mass ratio correct to first order relativistic terms. Special cases will be derived later in this section.

**THE \( C^{(1)} \) RELATION:**

From equation (19) for the friction term and equations (26), (31), (34), (36) and (39) for the diffusion term we have to work with the complicated expression

$$\left( \frac{e}{M_i} \right)^3 \frac{C_i^{(1)}}{Y_{ei}} (f^{(1)}) = -\frac{4\pi}{3} \frac{\partial}{\partial \nu} \left\{ \int_{-\infty}^{\infty} F^{(1)} g^0 \, d\nu \right\} - 4\pi \frac{\partial}{\partial \nu} \left\{ \int_{-\infty}^{\infty} F^{(1)} g^0 \, d\nu \right\}$$

$$- \frac{8\pi}{3} \frac{\partial}{\partial \nu} \left\{ \int_{-\infty}^{\infty} F^{(1)} g^2 \, d\nu \right\} - \frac{8\pi}{3} \int_{-\infty}^{\infty} F^{(1)} g^2 \, d\nu$$

$$- \frac{8\pi}{3} \int_{-\infty}^{\infty} F^{(1)} g^2 \, d\nu$$

$$+ \frac{2\pi}{3} \left[ D \right] \left\{ \int_{-\infty}^{\infty} h^0 F^{(1)} \, d\nu \right\} + 2\pi \left[ D \right] \left\{ \int_{-\infty}^{\infty} h^0 F^{(1)} \, d\nu \right\}$$

$$+ 2\pi \left[ D \right] \left\{ \int_{-\infty}^{\infty} h^0 F^{(1)} \, d\nu \right\} + 4\pi \left[ D \right] \left\{ \int_{-\infty}^{\infty} h^0 F^{(1)} \, d\nu \right\}$$

$$- 4\pi \left[ D \right] \left\{ \int_{-\infty}^{\infty} h^0 F^{(1)} \, d\nu \right\} - \frac{4\pi}{3} \left[ D \right] \left\{ \int_{-\infty}^{\infty} F^{(1)} h^0 \, d\nu \right\}$$

$$- \frac{4\pi}{3} \left[ D \right] \left\{ \int_{-\infty}^{\infty} F^{(1)} h^0 \, d\nu \right\}.$$  \( (63) \)
On the right side of (63) the first five terms originate from the friction term and the remaining six from the diffusion term. The operators \([\hat{D}_r] (r=1,2,3)\) are given by equations (9a,b) and equation (22). Some of the relevant kernels appearing in (63) have already been given in the calculation of \(C^{(0)}\). The remaining terms we need are \(g_0, g_1, d_1, h_1\) and finally \(h_3\).

Starting with \(g_0\) we use the defining equation (17a) and employ the D-expansions (52) and (53) for \(\Lambda = 0\) and 1 in each case. Converting the integrodifferential operators to scaled variables yields an extra multiplying factor \(\mu^\nu_1\) for the friction terms and \(\frac{1}{\mu^\nu_1} \sqrt{\frac{k}{2}}\) for the diffusion terms in (63). As in the calculations for \(C^{(0)}\), these factors will be absorbed into the definition of the appropriate kernel so that in the various operators we can just write \(q\) for \(U\) and \(Q\) for \(\hat{U}\).

For \(g_0\), retaining terms correct to order \(\beta_T\) gives the expression

\[
\begin{align*}
g_0 & = \frac{1}{\sqrt{\mu}} \left(1 + \frac{m}{M}\right) - \beta_T \frac{q \mu}{\sqrt{\mu}} \left(\frac{5}{3} + \frac{2m}{M}\right) + \beta_T \frac{Q}{\sqrt{\mu}} \left(2 + \frac{m}{M}\right), \\
& \quad q \sqrt{\mu} < Q,
\end{align*}
\]

and

\[
\begin{align*}
g_0 & = \beta_T \frac{4}{3} \frac{Q^5}{q \mu}, \\
& \quad q \sqrt{\mu} > Q.
\end{align*}
\]

Similarly \(g_2\) is given by (18) and we apply the expansions (52) and (53) for \(\Lambda = 1\) and 2 in each case. After some simplification we obtain

\[
\begin{align*}
g_2 & = -\frac{Q^5}{q \mu^3} \left(1 + \frac{m}{M}\right) - \beta_T \frac{Q^5}{q \mu} \left(\frac{2}{3} + \frac{m}{M}\right) + \beta_T \frac{Q^5}{q \mu^3} \left(\frac{11}{5} + \frac{m}{M}\right), \\
& \quad q \sqrt{\mu} > Q,
\end{align*}
\]

and

\[
\begin{align*}
g_2 & = \beta_T \frac{2}{15} q \frac{2}{\sqrt{\mu}}, \\
& \quad Q > q \sqrt{\mu}.
\end{align*}
\]
Next we consider \( \ell_1 \). This is much simpler since it involves only one type of \( D \)-function. For its definition we refer to (25d) and apply (51) with \( \ell = 1 \) obtaining

\[
\ell_1 = \frac{Q^5}{\nu^2} \left\{ \frac{Q}{\nu^2} - \frac{7}{5} \frac{Q^3}{\nu^2} - 2\beta_T Q \right\}, \quad Q > \nu
\]

and

\[
\ell_1 = \frac{Q^5}{\nu^2} \left\{ \frac{Q^3}{3\nu^2} - \frac{7}{5} \frac{Q^3}{\nu^2} - 2\beta_T Q \sqrt{\nu} \right\}, \quad Q > \nu
\]

In conclusion we consider the \( \ell \)-functions \( h^0_1 \) and \( h^0_3 \). For \( h_1 \) we refer to its defining equation (23b) and apply the expansions (52) for \( \ell = 1 \) and (53) for \( \ell = 0 \) and 2 respectively. Correct to terms of order \( \beta_T \) we obtain

\[
h^0_1 = \frac{1}{15\nu^2} \left\{ \frac{Q^3}{\nu^2} - \frac{13}{5} \beta_T \frac{Q^5}{\nu^2} - \frac{14\beta_T}{3} Q^3 \right\}, \quad Q > \nu
\]

and

\[
h^0_1 = \frac{1}{15\nu^2} \left\{ -Q^3 - \frac{13}{5} \beta_T Q^3 \mu^2 - \frac{14\beta_T}{3} Q^3 \right\}, \quad Q > \nu
\]

For \( h_3 \) we have the defining relation (23c) and apply equations (52) for \( \ell = 1 \) and 3 and (53) for \( \ell = 2 \). Again correct to order \( \beta_T \) we have

\[
h_3 = \frac{2}{9\nu^2} \frac{Q^3}{\nu^2} - \frac{2}{9\nu^2} \frac{Q^5}{\nu^2} + \frac{4\beta_T Q^3}{\nu^2} - \frac{3\beta_T}{5\nu^2} \frac{Q^5}{\nu^2} + \frac{3\beta_T}{7\nu^2} \frac{Q^7}{\nu^2}, \quad Q > \nu
\]

and

\[
h_3 = -\frac{64}{105} \beta_T \mu^2 \nu^2 Q^3, \quad Q \sqrt{\nu} < Q
\]
We now have available all the expanded forms of the kernels appearing in (63). Before proceeding with the reduction of the terms, we expand $\zeta^{(1)}$ in the form

$$\zeta^{(1)} = \zeta^{(1)} + \beta_\tau \zeta^{(1)}.$$

The forms of the terms $\zeta^{(1)}$ and $\zeta^{(1)}$ can be found by inspection from the terms generated in the expansion of the right hand side of (63).

Since there are so many terms involved in (63) we will consider separately the friction contribution (first five terms) and the remaining diffusion contributions to $\zeta^{(1)}$. The friction terms are the easiest to consider since only one differentiation per term, at most, is required. Taking the classical contribution (terms independent of $\beta_\tau$) first, we use the leading terms in the expansions (64), (54) and (65) for the kernels $\varphi_0$, $\varphi_1$ and $\varphi_2$ respectively. Using the definitions of the I and J integrals given earlier we derive the form

$$-\frac{1}{3\lambda^2} \frac{m}{q^2} \frac{\partial f^{(e)}_1}{\partial q} \int_{-1}^1 - \frac{1}{3\lambda^2} \frac{m}{q^2} \frac{\partial f^{(e)}_1}{\partial q} \int_{-1}^1$$

$$+ \frac{2}{3\lambda^2} \frac{m}{q^2} \frac{1}{q^2} \frac{\partial f^{(e)}_1}{\partial q} \int_{-1}^1 (1 + \frac{m}{M})$$

$$+ \frac{1}{q^2} \left( 1 + \frac{m}{M} \right) \frac{\partial f^{(1)}}{\partial q} \int_{-1}^1 + 4\pi \left( 1 + \frac{m}{M} \right) \left( \int_{-1}^1 f^{(1)} + \int_{-1}^1 f^{(2)} \right),$$

for the classical friction contribution to $\zeta^{(1)}$. The relativistic contribution ($\beta_\tau$ terms) follow in much the same manner although it is slightly more tedious as there are now more terms to consider. After some simplification one obtains the result
We now consider the diffusion terms of (63). These are the most tedious to calculate since each of the \([D]\) operators involves second derivatives. The calculations are nonetheless straightforward in principle. One simply substitutes the appropriate expansion of the \(d\) and \(h\) function given earlier, expresses the resulting integrals in terms of I or J functions and performs the differentiations according to the type of \([D]\) operator using the relations (59). Taking the classical terms first we need the appropriate values for the functions \(h_0, h_0^*, h_1, h_3, d_1\) and \(d_0\) given earlier. The result, after some simplification yields

\[
- 4\pi q^2 \mu (1 + \frac{m}{M}) (\frac{f^{(o)}}{\gamma} \frac{f^{(1)}}{\gamma} + \frac{f^{(1)}}{2}) \\
+ \frac{1}{\sqrt{\mu}} \frac{\partial f^{(o)}}{\partial q} \left[ (\frac{2m}{3M} + \frac{1}{3}) \frac{I_{-1}}{2^{1/2}} + \frac{2m}{3M} J_{0}^{1} - \frac{2}{3} (\frac{1}{5} + \frac{2m}{M}) J_{-1}^{1} \right] \\
+ \frac{1}{\sqrt{\mu}} \frac{\partial f^{(o)}}{\partial q} \left[ \frac{4}{3} (1 + \frac{m}{M}) \frac{I_{-1}}{2^{1/2}} + (\frac{4m}{3M} - \frac{2}{3}) J_{-2}^{1} \right] \\
+ \frac{1}{\sqrt{\mu}} \frac{\partial f^{(n)}}{\partial q} \left[ (1 + \frac{2m}{M}) I^{0} - (2 + \frac{5m}{3M}) I_{-1}^{0} + \frac{4m}{3M} J_{-1}^{0} \right] \\
+ \frac{1}{\sqrt{\mu}} \frac{\partial f^{(1)}}{\partial q} \left[ 2 (1 + \frac{2m}{M}) I_{-1}^{0} + \frac{4m}{M} J_{-1}^{0} \right].
\]
Lastly we consider the relativistic $\beta_T$ terms. This is the most tedious of the present calculations. Again, one substitutes the $\beta_T$ terms of the kernels $h_o$, $h^o_1$, $h^o_2$, $h^o_3$, $d_1$ and $d_0$ into the last six integrands of (63), re-expresses the integrals in terms of $I$ and $J$ functions and applies the diffusion operators to each term successively. It is simplest to group similar terms under the same operator. Thus we group together terms 6 and 10 forming one kernel $\frac{2}{3}d_1 - 2h^o_1$ and also the terms 7 and 8 with the combined functions $2d_0 - 2h_o$.

After simplification one obtains the expression

$$
-4\pi \left( f^{(0)} \frac{r^{(1)}}{r} + f^{(1)} \frac{r^{(0)}}{r} \right)
+ \frac{1}{qv^2} \frac{\partial f^{(0)}}{\partial q} \left[ -\frac{1}{5} I^{(1)}_3 - \frac{1}{5} J^{(1)}_2 + \frac{4}{5} J^{(2)}_2 \right]
+ \frac{1}{qv^2} \frac{\partial f^{(1)}}{\partial q} \left[ \frac{1}{5} I^{(1)}_3 + \frac{1}{5} J^{(2)}_2 \right]
+ \frac{1}{qv^2} \frac{\partial f^{(0)}}{\partial q} \left[ \frac{2}{3} J^{(0)}_1 - I^{(0)}_0 - \frac{1}{3} I^{(0)}_2 \right]
+ \frac{1}{qv^2} \frac{\partial f^{(1)}}{\partial q} \left[ \frac{2}{3} J^{(0)}_1 + \frac{1}{3} I^{(0)}_2 \right]
+ \frac{1}{qv^2} \frac{\partial f^{(0)}}{\partial q} \left[ -\frac{2}{3} J^{(0)}_1 - \frac{2}{3} J^{(0)}_2 + \frac{1}{3} I^{(0)}_2 \right].
$$

(72)
That completes the derivation of the contributions to $\Sigma^{(1)}$. It only remains to collect together the terms appropriate to $\Sigma^{(1)}_{n}$ and $\Sigma^{(1)}_{i}$. For $\Sigma^{(1)}_{o}$ we add the friction (70) and diffusion (72) terms to obtain

$$
\frac{\Sigma^{(1)}_{o}}{\gamma_{ei}} = \frac{1}{3q_{i}} \frac{\partial f^{(1)}_{o}}{\partial q_{i}} \left( I_{1}^{o} + J_{1}^{o} \right) + \frac{1}{3q_{i}} \frac{\partial f^{(1)}_{o}}{\partial q_{i}} \left( \frac{3m}{M} I_{0}^{o} + 2 J_{1}^{o} - I_{2}^{o} \right)
$$

$$
+ \frac{1}{3q_{i}} f^{(1)}_{o} \left( -3 I_{1}^{o} + I_{2}^{o} - 2 J_{1}^{o} \right) + \frac{4}{\gamma_{ei}} \left( f^{(1)}_{o} F^{(o)} + f^{(1)}_{o} F^{(o)} \right) \frac{m}{M}
$$

$$
+ \frac{1}{5q_{i}} \frac{\partial f^{(o)}}{\partial q_{i}} \left( I_{3}^{1} + J_{1}^{2} \right) + \frac{1}{15q_{i}} \frac{\partial f^{(o)}}{\partial q_{i}} \left( -3 I_{3}^{1} + (7 - \frac{5m}{M}) J_{3}^{1} + (-5 + 10 \frac{m}{M}) I_{5}^{1} \right)
$$

The relativistic contribution $\Sigma^{(1)}_{i}$ comes from the addition of (71) and (73) and results in

$$
\frac{\Sigma^{(1)}_{i}}{\gamma_{ei}} = -4 \pi \mu q_{i} \left( f^{(0)} F^{(1)} + f^{(1)} F^{(o)} \right) \frac{m}{M}
$$

$$
+ \frac{q_{i}}{\gamma_{ei}} \frac{\partial f^{(0)}}{\partial q_{i}} \left( \frac{3}{15} I_{3}^{1} - \frac{17}{35} I_{5}^{1} - \frac{3}{35} J_{2}^{1} + \frac{2}{15} J_{0}^{1} \right)
$$

$$
+ \frac{1}{\gamma_{ei}} \frac{\partial f^{(0)}}{\partial q_{i}} \left( -\frac{2}{3} \left( 1 - \frac{m}{M} \right) I_{0}^{1} + \left( -\frac{4m}{3M} + \frac{13}{15} \right) I_{0}^{1} + \frac{17}{35} I_{5}^{1}
$$

$$
\left( - \frac{19}{105} + \frac{2m}{3M} \right) J_{2}^{1} - \left( \frac{2}{15} + \frac{m}{3M} \right) J_{0}^{1} \right)
$$
The above analysis completes the reduction of the Fokker-Planck equation for arbitrary mass ratio of colliding particles. We have given the forms of the $C^{(0)}$ moment (equations (61) and (62)) and the $C^{(1)}$ moment (equations (74) and (75)) correct to first order relativistic effects i.e. up to terms in $\beta/\gamma$. We can now proceed to discuss several particular cases of the general collision terms, namely the contributions from electron-ion and electron-electron scattering. In the following section some results are given which illustrate the conservation laws and provide an additional check on our expansions so far.

**PARTICULAR CASES: ELECTRON-ION SCATTERING.**

This situation is described by letting the heavy ions obey a Maxwellian distribution with no directional term i.e. $F_{E}^{(1)} = 0$. This essentially means that we ignore the ion recoil in the $\tilde{f}^{(0)}$ equation. The true Maxwellian in the relativistic case is sometimes referred to as the Synge-Juttner distribution [45], [46] and is proportional to

$$F^{(0)} \propto \exp\left\{ -\lambda_{i} \sqrt{1 + \frac{2\tilde{Y}^{2}}{\lambda_{i}}} \right\},$$

in our Q units. We recall that $2\lambda_{i}^{2}Q^{2} = \tilde{Y}^{2}$ where $\tilde{Y}$ is the ion reduced velocity related to the normal velocity $\sqrt{1 - \beta^{2}}$ by $c\tilde{Y} = \sqrt{1 - \beta^{2}} \gamma \tilde{Y}$. 
Now for the situation in which we are interested \( \lambda_i \gg 1 \), (76) may be expanded in the form

\[
F^{(o)} = N \left( \frac{\lambda_i}{2\kappa} \right)^{3/2} \left( 1 - \frac{5}{\kappa \lambda_i} + \frac{\nu^4}{2\kappa \lambda_i} \right) e^{-\nu^2},
\]

(77)

where the constant of proportionality is fixed by requiring that

\[ N = \int F^{(o)} \, d^3 y, \]

\( N \), of course, being the ion number density.

Now for the I and J integrals occurring in the expansion for \( C^{(o)} \), we can express them in terms of the error function \( \phi(x) \) when \( F^{(o)} \) has the form (77). Tables 5-1 and 5-2 show some integrals of the form

\[
\int_0^\infty x^n e^{-x^2} \, dx
\]

with

\[
\phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} \, dy, \quad \phi'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}.
\]

(78)

The limits for large \( x \) are also given. Remember that the upper limits for the \( I^o \) integrals are \( \sqrt{\nu} \nu \) and also for the lower limits of the \( J^o \) functions as well. Since the ratio of the mean thermal electron to ion velocity is very large (i.e. \( \mu \gg 1 \)), \( \sqrt{\nu} \nu \) may indeed be taken as large. Thus as a first approximation, putting

\[
\sqrt{\nu} \nu \sim \infty
\]

in the \( I^o \) and \( J^o \) terms of (61) and (62), with \( F^{(o)} \) given by (77), yields the forms

\[
C^{(o)} = \frac{N \nu}{C^3} \frac{\lambda_i}{2\kappa} \frac{3}{4} \frac{1}{\nu} \frac{\partial}{\partial \nu} \left\{ \frac{m}{M} f^{(o)} + \frac{1}{2\mu \nu} \left( 1 - \frac{5}{2\kappa \lambda_i} \right) \frac{\partial f^{(o)}}{\partial \nu} \right\},
\]

\[
C^{(1)} = \frac{N \nu}{C^3} \frac{\lambda_i}{2\kappa} \frac{3}{4} \frac{1}{\nu} \frac{\partial}{\partial \nu} \left\{ \frac{m}{M} f^{(o)} \left( 2\nu^2 - \frac{5}{2} \right) \right. \]

\[ + \left. \frac{\partial f^{(o)}}{\partial \nu} \left( \frac{3\nu}{\nu^2} - \frac{5}{2\kappa \lambda_i} \right) \right\}. \]

(79)
**TABLE 5-1**

**ERROR TYPE INTEGRALS**

\[ \left\{ \text{OF THE FORM} \int_0^\infty y^n e^{-y} dy \right\} 

<table>
<thead>
<tr>
<th>( n )</th>
<th>EXACT</th>
<th>LIMIT FOR LARGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( \frac{\sqrt{\pi}}{4} \left[ \phi(x) - x \phi'(x) \right] )</td>
<td>( \frac{\sqrt{\pi}}{4} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{\sqrt{\pi}}{2} \left[ \frac{3}{2} \phi(x) - \phi(x)(\frac{3x^2}{4} + \frac{x^4}{2}) \right] )</td>
<td>( \frac{3}{8} \sqrt{\pi} )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{\sqrt{\pi}}{2} \left[ \frac{15}{8} \phi(x) - \phi(x)(\frac{15x^2}{4} + \frac{5x^4}{4} + \frac{3x^6}{2}) \right] )</td>
<td>( \frac{15}{16} \sqrt{\pi} )</td>
</tr>
<tr>
<td>8</td>
<td>( \frac{\sqrt{\pi}}{2} \left[ \frac{105}{16} \phi(x) - \phi(x)(\frac{105x^2}{16} + \frac{35x^4}{4} + \frac{3x^6}{2} + \frac{5x^8}{2}) \right] )</td>
<td>( \frac{105}{32} \sqrt{\pi} )</td>
</tr>
</tbody>
</table>

**TABLE 5-2**

**INTEGRALS OF THE FORM: \( \int_0^\infty y^n e^{-y^2} dy \).**

<table>
<thead>
<tr>
<th>( n )</th>
<th>EXACT</th>
<th>LIMIT FOR LARGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{\sqrt{\pi}}{4} \phi'(x) )</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{\sqrt{\pi}}{4} \phi'(x)(1 + x^2) )</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{\sqrt{\pi}}{2} \phi(x)(\frac{1}{2}x^4 + x^2 + 1) )</td>
<td>0</td>
</tr>
</tbody>
</table>

The zero moment collision term \( C^{(0)} \) to order \( \beta_T = \lambda^{-1}_c \) is

\[ C^{(0)} = C^{(0)} + \beta_T C^{(s)} \]

If we use the results (79) and also the fact that in equilibrium under no forces \( f^{(s)} \) must also be Maxwellian of the form

\[ f^{(s)} = n \left( \frac{\lambda_c}{2\pi} \right)^{3/2} (1 - \frac{15}{8} \lambda_c + \frac{9}{2} \lambda_c^2) e^{-\frac{y^2}{2\lambda_c^2}} \]

Then it is easy to show that \( C^{(0)} \) vanishes correct to order \( \beta_T \) as it
should. In fact it is not necessary to use the approximate forms (79). One can evaluate the $I^0$ and $J^0$ integrals of equations (61) and (62) exactly for the distribution (77) using Tables 5-1 and 5-2. Using also the form of $f^{(e)}$ above and retaining terms up to order $\beta_T$ makes $C^{(0)}$ vanish identically.

We can now turn to the $C^{(1)}$ moment. This is made up from the expressions (74) and (75). Again we make the approximation $\sqrt{\mu} \varphi \sim \infty$ in the $I$ and $J$ integrals with the additional results $\mu > 1$, $\frac{M}{M} \ll 1$, and $p^{(1)} \sim 0$. We observe that under these conditions all $J$ integrals can be neglected and that the dominant $I$ term is $I^0$. Terms in $\frac{m}{M}$ are also neglected. Thus we obtain the greatly simplified forms

$$
\frac{C^{3^{(1)}}}{\sqrt{\lambda}} \sim \frac{1}{q^3} I^0 \int f^{(1)} dV,
$$

and

$$
\frac{C^{3^{(1)}}}{\sqrt{\lambda}} \sim -\frac{1}{q^3} I^0 \int f^{(1)} dV.
$$

(81)

In the spirit of our approximations we also put

$$
I^0 \sim N \left( \frac{\lambda_i}{2} \right)^{3/2},
$$

which we note is correct to order $\beta_T$.

The forms of the collision terms (81) include only the direct collisional effect of ions on electrons. Ion recoil is not included since we have put $F^{(1)} = 0$.

**ELECTRON-ELECTRON COLLISIONS.**

So far we have given the collision terms for electron-ion interaction. Under the approximations previously stated these reduce to fairly simple forms, especially for the $C^{(1)}$ term. When electron-electron effects are added to complete the description, the expressions become quite complicated.

We let $M = m$, $F^{(0)} = f^{(0)}$, $\mu = 1$, and $F^{(1)} = f^{(1)}$ in the general
equations (61) and (62) for $C^{(0)}$ and (74) and (75) for $C^{(1)}$. Also we put $\alpha=1$ in the definition of $\gamma_{ei}$ and redefine it as

$$\gamma_{ee} = 4\pi \left( \frac{e^2}{4\pi \varepsilon_0} \right)^2 \ln \frac{\lambda}{\Lambda},$$

where $\ln \Lambda$ is the Coulomb Logarithm. The $I$ and $J$ integrals become

$$I^i_{j} = \frac{4\pi}{q^2} \int_{q_j}^{q_i} f^{(i)} q^{2i} dq^i, \quad J^i_{j} = \frac{4\pi}{q^2} \int_{q_j}^{q_i} f^{(i)} q^{2i} dq^i. \quad (82)$$

Equations (61) and (62) become

$$C^{(0)} = \frac{\gamma_{ee}}{C^2} \frac{1}{q^2} \frac{\partial}{\partial q} \left\{ f^{(0)} \frac{2I_0}{3} + \frac{q^0}{3} (I_2 + J_0) \frac{\partial f^{(0)}}{\partial q} \right\},$$

and

$$C^{(0)} = \frac{\gamma_{ee}}{C^2} \frac{1}{q^2} \frac{\partial}{\partial q} \left\{ q^0 f^{(0)} \left[ \frac{2I_0}{3} - \frac{5}{3} I_2 + \frac{4}{3} J_0 \right] 
+ q^3 \frac{\partial f^{(0)}}{\partial q} \left[ \frac{I_0}{3} - \frac{2}{3} I_2 + \frac{1}{3} J_0 \right] \right\}. \quad (83)$$

Remembering that $\lambda_e = \mu_e \lambda_e$, a comparison of (83) with (79) shows that the electron-electron contribution is larger by a factor of approximately $\frac{M}{\lambda_e}$. Again using the tables 5-1 and 5-2 we see that $\mathcal{E}-\mathcal{E}$ effects are identically zero to order $\beta_T$ when $f^{(0)}$ has the form (80).

For the $C^{(1)}$ term we similarly get from equations (74) and (75) the forms
Expressions (79) and (81) describe $C$-i collisions for the $f^{(1)}$ and $f^{(0)}$ moment equations correct to first order relativistic effects.

\[
\frac{c^3}{\gamma_{ei}} C^{(1)}_{i} = \frac{1}{3q^3} \frac{\partial f^{(0)}}{\partial q^1} (I_1^0 + J_{-1}^0) + \frac{1}{3q^3} \frac{\partial f^{(0)}}{\partial q^1} (3I_0^0 + 2J_{-1}^0 - I_2^0) + \frac{1}{3q^3} \frac{\partial f^{(0)}}{\partial q^2} (-3I_0^0 + I_2^0 - 2J_{-1}^0) + 8\pi \frac{\partial f^{(0)}}{\partial q^2} (I_1^1 + J_{-2}^1) + \frac{1}{5q^3} \frac{\partial f^{(0)}}{\partial q^2} (-3I_3^1 + 2J_{-1}^1 + 5I_2^1),
\]

and

\[
\frac{c^3}{\gamma_{ee}} C^{(1)}_{i} = -8\pi q^2 q^1 \frac{\partial f^{(0)}}{\partial q^1} + q q^1 \frac{\partial f^{(0)}}{\partial q^1} \left( -\frac{7}{15} I_2^1 + \frac{17}{35} J_{-2}^1 - \frac{17}{35} I_{-1}^1 + \frac{17}{35} J_{-1}^1 \right) + \frac{4}{3q} f^{(0)} \left( I_1^1 + J_{-2}^1 \right) + \frac{1}{q^2} f^{(0)} \left( 3I_0^0 + I_2^0 - \frac{2}{3} I_1^0 + 4J_{-1}^0 - \frac{2}{3} J_1^0 \right) + q q^1 \frac{\partial f^{(0)}}{\partial q^1} \left( \frac{1}{3} J_1^0 - \frac{2}{3} I_1^0 + I_2^0 \right). \]

(84)
Similarly (83) and (84) describe e-e effects to the same order. The total collisional effect is found by adding the individual contributions. We note that when the e-e terms are added to the formulation, the expressions become very complicated indeed. The contribution of heavy-particle recoil in the \( f^{(l)} \) equation has been neglected here, but will be included in Chapter 6 when the results are applied to transport problems.

The above set of equations are the final result of this section.

It is instructive to verify the basic conservation laws of number density, momentum and energy for the above collision terms, as it provides a useful check on the calculations. This will be done in §6.

§5 CLASSICAL LIMIT FOR COLLISION MOMENTS

Up to now our theory has been developed in terms of the reduced-velocities \( \tilde{\mathcal{U}} \) and \( \tilde{\mathcal{V}} \) for electrons and ions respectively. The electron velocity \( \mathcal{V} \) is related to \( \tilde{\mathcal{U}} \) by

\[
\mathcal{V} = \frac{c \tilde{\mathcal{U}}}{\sqrt{1 + \mathcal{U}^2}}
\]

and similarly for the ion velocity \( \mathcal{V}' \). Related scaled variables \( q \) and \( Q \) were defined in the following way:

\[
q^2 = \frac{1}{2} \lambda_e \mathcal{U}^2 \quad \text{AND} \quad Q^2 = \frac{1}{2} \lambda_i \mathcal{V}^2,
\]

with

\[
\lambda_e = \frac{m c^2}{k T} \quad \text{AND} \quad \lambda_i = \frac{M c^2}{k T}.
\]

An effective thermalised situation with more or less equal electron and ion temperatures is assumed throughout.

The classical limit for the equations is found by letting \( \tilde{\mathcal{U}} \), \( \tilde{\mathcal{V}} \) tend to zero and \( c \) tend to infinity, in such a way that \( c \tilde{\mathcal{U}} \) and \( c \tilde{\mathcal{V}} \) remain finite and in the limit become equal to the classical velocities \( \mathcal{V} \) and \( \mathcal{V}' \). Further, the scaled variables tend to a limit given by

\[
q^2 \to \frac{1}{2} \frac{m \mathcal{V}^2}{k T} \quad \text{AND} \quad Q^2 \to \frac{1}{2} \frac{M \mathcal{V}'^2}{k T},
\]
so that, for example, $\beta$ becomes the ratio of the electron kinetic energy to the Boltzmann energy $KT$. The I and J integrals can be redefined in the form

$$\frac{I}{J} = \frac{4\pi}{V^2} \int_V F^{(i)} V^{2+i} dV, \quad \frac{J}{J} = \frac{4\pi}{V^2} \int_V F^{(i)} V^{2+i} dV,$$

for electron-ion collisions and similarly for electron-electron effects. The terms in $\beta_T$ can be omitted from the collision terms since $\beta_T \to 0$ as $\gamma \to \infty$. For the $f^{(o)}$ equation we have the classical result

$$\frac{1}{Y_{ei}} C^{(o)} = \int_{V} \frac{\partial}{\partial V} \left\{ f^{(1)} \frac{3}{m} I^{(o)} + V \left( I_{-1}^{(o)} + J_{-1}^{(o)} \right) \right\},$$

and for the $f^{(i)}$ equation

$$\frac{1}{Y_{ei}} C^{(i)} = \frac{1}{3V} \frac{\partial^2 f^{(ii)}}{\partial V^2} \left( I_{-2}^{(i)} + J_{-2}^{(i)} \right) + \frac{1}{3V} \frac{\partial f^{(ii)}}{\partial V} \left( \frac{3m}{M} I^{(i)} - I_{-1}^{(i)} + 2J_{-1}^{(i)} \right)$$

$$+ \frac{1}{3V^3} \int f^{(i)} (-3 I_{-1}^{(i)} + I_{-1}^{(i)} - 2J_{-1}^{(i)}) + 4 \pi \left( \frac{1}{2} f^{(i)} + f^{(o)} \frac{f^{(i)}}{\rho} \right)$$

$$+ \frac{1}{5V} \frac{\partial f^{(o)}}{\partial V} \left( I_{-1}^{(i)} + J_{-1}^{(i)} \right)$$

$$+ \frac{1}{15V} \frac{\partial f^{(o)}}{\partial V} \left( -3 I_{-1}^{(i)} + \left( \frac{5m}{M} \right) J_{-2}^{(i)} + \left( -5 + \frac{10m}{M} \right) I_{-1}^{(i)} \right).$$

$e-e$ effects can be deduced in the way described in § 4.

The reader should compare the classical forms above with those
in the variables $q$, namely equations (61) and (74). Apart from the $\mu$-factor they are effectively equivalent if one identifies $V$ with $q$. However, this correspondence is not strictly true when one carries through the correct limiting procedure described above.

The classical expressions have been derived by Shkarofsky $\cite{47}$, following the exposition of Rosenbluth et al. $\cite{48}$, using an expansion in cartesian tensors. Similar results have also been given by Allis $\cite{49}$ who introduced the so-called 'Allis-integrals' $I$ and $J$. Also see $\cite{2}$, Chapter 7 where the classical collision moments are given for the $f^{(e)}$, $f^{(u)}$, and $f^{(q)}$ equations; i.e. one more than considered here. The $f^{(q)}$ equation is needed to investigate viscosity effects in a plasma, but is beyond the scope of our present work in investigating relativistic effects.

§ 6 CONSERVATION LAWS FOR COLLISION TERMS.

We can now apply the formalism developed in the previous sections to some practical problems. So far in this chapter we have developed general expressions for the Fokker-Planck coefficients ($\S 2$ and $\S 3$) and given the collision terms for the first two moment equations correct to first order in relativistic effects. Our purpose in this section is to demonstrate that the collision terms do indeed satisfy the basic conservation laws for number density, momentum and energy in elastic encounters. Evidently we would expect that electron-electron collisions cannot contribute to the momentum and energy integrals, and in the case of number density the electron-ion term vanishes as well.

From equations (8) of Chapter 3, the relativistic Boltzmann equation may be written in the form

$$\frac{\partial f}{\partial t} + U_\lambda \frac{\partial f}{\partial x_\lambda} + \frac{F_\lambda}{mc} \frac{\partial f}{\partial U_\lambda} = C(f) \quad (85)$$

where

$$\beta = \left(1 + U^2\right)^{-\frac{1}{2}},$$
and \( C(\tilde{f}) \) is of course the collision term representing the combined
effects of electron-electron and electron-ion interactions. If \( \phi(\gamma) \) represents some molecular property of the gas described by the
distribution \( \tilde{f} \), then we can define the velocity average by the statement
\[
\overline{\phi(\gamma)} = \frac{\int \phi(\gamma) f d^3\gamma}{\int f d^3\gamma},
\]
where the integration is over all reduced velocity space. By definition of \( f \), clearly \( n \), the number density, is given by
\[
n = \int f d^3\gamma.
\]
It is assumed that for realistic distributions, as the velocity tends
to infinity, \( f \) tends to zero faster than any velocity dependent
quantity of interest. Thus velocity averages are in general, well
defined finite quantities.

The general equation of change for \( \phi \) can be found from
Boltzmann's equation (85), simply by multiplying through by \( \phi \) and
integrating out over \( \gamma \). There results the general equation
\[
\frac{\partial}{\partial t} (n \overline{\phi}) + \nabla \cdot (n \overline{\phi} \gamma) = n \left\{ \frac{\partial \overline{\phi}}{\partial t} + \gamma \cdot \nabla \phi + \frac{F \cdot \nabla \phi}{mc} + \Delta \overline{\phi} \right\}, \tag{86}
\]
where
\[
n \Delta \overline{\phi} = \int \phi C(f) d^3\gamma
\]
and
\[
\gamma = c_\beta \gamma.
\]
The left hand side of (86) is the normal conservation equation in
configuration space and the right hand side are the source terms. We
have used the fact that \( \nabla \cdot \overline{E} = 0 \) for the Lorentz force \( \overline{E} \).

This section is concerned with the evaluation of \( n \Delta \overline{\phi} \) for the
cases of particle, momentum and energy conservation - the only three
meaningful quantities in elastic encounters.
PARTICLE CONSERVATION.

The number density equation is obtained by putting $\phi = 1$ in (86) and the left hand sides reduces to the well known form of the continuity equation. The first three terms on the right hand side vanish by constancy of $\phi$ and we are left with the term

$$\nabla \Delta \phi = \int C(f) \, d^3 \Omega \, .$$

In general the collision term is expanded as

$$C(f) = \sum_{\ell} C^{(\ell)} f \frac{U^{(\ell)}}{U^2} \, .$$

Substitution into (87) yields the form

$$\nabla \Delta \phi = 4\pi \int_0^\infty C^{(0)} U^2 dU \, ,$$

which can easily be seen when one converts to spherical harmonics and uses their orthogonality condition (equation (5), Chapter 1). From (44) we see that $C^{(0)}$ is expressible as

$$C^{(0)} = \frac{1}{U^2} \frac{\partial S}{\partial U} \, ,$$

where $S$ is some function of the zero order moment, $f^{(0)}$, of $f$.

As we have mentioned above, for well behaved distributions, as $U \to \infty$, $f \to 0$ faster than any velocity dependent quantities of interest. This must also be true of the various moments $f^{(\ell)}$ of $f$, so that in particular $f^{(0)} \to 0$ as $U \to \infty$. Thus we see from (44) that $S(U \to \infty) = 0$ and also $S(0) = 0$. Substitution of (89) into (88) yields the trivial result

$$\nabla \Delta \phi = 0 \, .$$

Note that this result is true for both $e-e$ and $e-i$ collisions. Thus we have verified that the electron-electron and electron-ion contributions vanish. Particles must be conserved since no creation
or loss mechanism is included in the Fokker-Planck collision term. The above result merely shows that our relativistic formulation does indeed preserve this.

**MOMENTUM CONSERVATION.**

The momentum equation is obtained by putting \( \Phi = m_u \) in equation (86). Then, since \( F \cdot \nabla \Phi = F \) and \( \nabla \cdot \Phi = 0 \), we have

\[
\frac{\partial}{\partial t} \left( m_u \Delta \Phi \right) + \nabla \cdot \mathbf{P}_u = \mathbf{n} \left\{ F + m_u \Delta \Phi \right\}, \tag{90}
\]

where

\[
m_u \Delta \Phi = \int m_u \frac{C(f)}{d^5 \Phi},
\]

and \( \mathbf{P}_u \) is the pressure tensor defined by

\[
\mathbf{P}_u = m_u \beta \nabla \Phi \nabla \Phi.
\]

Expanding \( C(f) \) in its usual cartesian tensor form gives the result

\[
m_u \Delta \Phi = \frac{4 \pi}{3} \int_0^\infty m_u C^{(1)} \Phi^3 d\Phi. \tag{91}
\]

Again, (91) is easily shown when one converts to spherical harmonics and uses the appropriate orthogonality condition.

For \( C^{(1)} \) we use the expression

\[
C^{(1)} = C^{(n)} + \beta \tau C^{(i)}
\]

with \( C^{(n)} \) and \( C^{(i)} \) given by equations (74) and (75) respectively.

(91) is rewritten in terms of the scaled variable \( q \) as

\[
m_u \Delta \Phi = \frac{16 \pi m_u}{3 \lambda_e} \int_0^\infty C^{(1)} q^3 dq. \tag{92}
\]

Substitution for \( C^{(1)} \) gives, after several integrations by parts using the recursion relations (59) for \( I \) and \( J \), the relation
In actual fact the computation of (91) may be considerably reduced by referring to the general form of $\mathcal{Q}^{(1)}$ given by (63). The last six terms are of the general form

$$\left[D_r\right]_r \phi(q)$$

where $[D_r]$ ($r=1, 2 \text{ or } 3$) are certain second order differential operators given in $\mathcal{S}^4$ and $\phi$ just represents some function of $q$ upon which they act. Now it is easy to show that, providing $\phi \rightarrow 0$ as $q \rightarrow 0$, and is well behaved at $q=0$,

$$\int_0^\infty \left[D_r\right]_r \{\phi\} q^r dq = 0, \quad r=1, 2 \text{ or } 3.$$  

Thus if one multiplies (63) by $q^r dq$ and integrates over $q$, the diffusion terms are removed and only the friction contribution need be considered. The resulting classical and relativistic terms are given by (70) and (71) respectively and are slightly easier to use than the full expressions (74) and (75). The classical term in (93) is essentially the result of Shkarofsky [2.7] (p. 288). The $I^i$ and $J^i$ terms represent the effect of scatterer recoil on the momentum balance, while the $I^0$ and $J^0$ terms give rise to the direct scattered-particle momentum loss. We note the complicated effects in the $\beta_r$ term of (93).
Equation (93) does not obviously show momentum conservation, and an explicit demonstration of this is of some practical interest. We split equation (93) into two parts and consider the classical term first. A straightforward reversal of the integration order gives

$$\int_0^\infty f^{(o)}_2 dQ = 4\pi \int_0^\infty f^{(o)}_2 dQ \int_0^\infty F^{(i)}_2 dQ = 4\pi \int_0^\infty F^{(i)}_2 dQ \int_0^\infty f^{(o)}_2 dQ.$$  

Hence

$$\left(\mu c \Delta U_{cl}\right)_e = \frac{4\pi}{3c} \frac{\lambda_e^2}{\lambda_i} m Y_{ei} \left(1 + \frac{m}{M}\right) \left\{\int_0^\infty F^{(i)}_2 dQ \int_0^\infty f^{(o)}_2 dQ \right\},$$

(94)

where we have used the fact that $\mu \lambda_e = \lambda_i$. We recall that for electron-ion collisions

$$Y_{ei} = 4\pi \left(\frac{e^2 Z_i^2}{4\pi e_0 m_e}\right)^2 \ln \Lambda,$$

so that

$$Y_{ei} = \frac{M^2}{m^2} Y_{ie}.$$  

Thus (94) clearly demonstrates, insofar as the classical contribution is concerned, that (a) there is no net momentum transfer between like particles, and (b) that collisions between particles of species $\alpha$ and $\beta$ give rise to the opposite effect as those between $\beta$ and $\alpha$.

We must now consider the relativistic contribution to (93). This is more complicated and care is needed. The first and fourth terms in the integrand are taken together as are the second and fifth
and the remaining two. One then transforms the $J$ and $I$ integrals in a similar manner as above and obtains

$$(\mathcal{M} C \Delta \mathcal{L})_r = \frac{64 \pi^2 m \gamma_e}{3 c^2 (\lambda_e \lambda_i)^2} \mathcal{M} \times$$

$$X \left\{ \int_0^{4M} \frac{4M}{3 \sqrt{\mu}} F^{(0)} Q^3 dQ \int_0^{Q} f^{(0)} Q dQ - \frac{4M}{3} \int_0^{Q} f^{(1)} Q^3 dQ \int_0^{Q} F^{(0)} Q dQ \right. \\
- \left( \frac{5M+2M}{3} \right) \int_0^{Q} f^{(1)} Q^4 dQ + \left( \frac{5M+2M}{3} \right) \int_0^{Q} F^{(0)} Q^4 dQ \right. \\
+ (2M+\Lambda) \int_0^{Q} f^{(1)} Q^3 dQ \int_0^{Q} f^{(0)} Q^4 dQ - (2M+\Lambda) \int_0^{Q} f^{(1)} Q^4 dQ \int_0^{Q} F^{(0)} Q^3 dQ \right\}.$$

Equation (96) represents the relativistic (suffix $\tau$) contribution to the momentum integral (92). Again using (95), (96) shows that there is no contribution from like particles and that collisions between particles of species $\Phi$ and $\Phi$ give rise to the opposite effect as those between $\Phi$ and $\Psi$. Equations (94) and (96) demonstrate that momentum is conserved as it should be.

**ENERGY CONSERVATION.**

The energy transport equation is obtained by putting $\phi = E = n C (\sqrt{1+\upsilon^2} - 1)$ in (86). Since $\nabla_r E = 0$ we have

$$\frac{\partial}{\partial E} (n \overline{E}) + \nabla_r (n \overline{E} \overline{\gamma}) = n \left\{ \frac{F_r \cdot \nabla_r E}{\frac{d}{dt} \overline{E}} + \Delta \overline{E} \right\},$$

where the collisional term is
Forming to the scaled variable \( q \) yields for the energy \( E \), to order \( \beta_T \),

\[
E = kT_e \left( q^2 - \frac{\alpha q^4}{2} \beta_T + \ldots \right).
\]  

Expressing the collision term \( C \) as an expansion in terms of its moments \( g^{(k)} \) gives, after integrating out over angles,

\[
\pi \Delta E = 4\pi mc^3 \int_0^\infty E C^{(o)}(f^{(o)}) \nu^* d\nu.
\]  

To the same order as \( E \) above, \( C^{(o)} \) is expanded as

\[
C^{(o)} = C^{(o)}_0 + \beta_T C^{(o)}_1,
\]

where \( C^{(o)}_0 \) and \( C^{(o)}_1 \) are given by equations (61) and (62) respectively. The expansions (101) and (99) can then be substituted into (100) retaining terms to order \( \beta_T \). The integration variable can be re-expressed in terms of \( q \) and some integration by parts performed to remove derivatives of \( f^{(o)} \). The final result yields the form

\[
\pi \Delta E = \frac{16\sqrt{2\pi} \lambda_i}{\lambda_i \lambda_i} \left\{ \int_0^\infty \left[ (M\bar{I}_1 - m\bar{I}_0) \right] q^{(o)} \frac{dq}{q} \right. \\
+ \beta_T \int_0^\infty q^{(o)} \frac{d\nu}{\nu} \left\{ \left( \frac{4}{3}M + \frac{5}{3}m \right) \frac{1}{2} \bar{J}_2 - \left( \frac{4}{3}M + \frac{4}{3}m \right) \bar{J}_1 \\
+ (M\bar{I}_1 - m\bar{I}_0) \right\} d\nu \right\}.
\]  

(102)
Equation (102) represents the energy loss expression for electrons colliding with ions for arbitrary mass ratio $\frac{n}{M}$ and arbitrary distribution functions for the two species. Again, $\Delta E$ does not obviously exhibit the energy conservation law and an explicit demonstration of this, although somewhat tedious, is of some interest. The procedure is similar to that for momentum conservation earlier.

For the $J_0^i$ integrals one has the result

$$\int_0^\infty f^{(0)}_q J_0^i dq = \int_0^\infty f^{(0)}_q \tilde{q} J_0^i dq = \int_0^\infty f^{(0)}_q \tilde{q} J_0^i dq = \int_0^\infty f^{(0)}_q \tilde{q} J_0^i dq,$$

and similarly for $J_1^i$. After transformation of the $J$ integrals, (102) may be written in the general form:

$$\Delta E = \frac{6\sqrt{\frac{\pi}{\gamma}} \gamma m \gamma c^2}{CM \gamma c^2} X \left\{ \sqrt{J_0^i} M \int_0^\infty f^{(0)}_q Q dq \int_0^\infty f^{(0)}_q \tilde{q} \tilde{q} dq - \sqrt{\frac{J_0^i}{m}} \int_0^\infty f^{(0)}_q dq \int_0^\infty f^{(0)}_q Q dq \right.$$

$$+ \left( \frac{4}{3} M + \frac{2}{3} m \right) \sqrt{\frac{J_0^i}{m}} \int_0^\infty f^{(0)}_q dq \int_0^\infty f^{(0)}_q Q^3 dq \right. - \left( \frac{4}{3} M + \frac{2}{3} m \right) \sqrt{\frac{J_0^i}{m}} \int_0^\infty f^{(0)}_q dq \int_0^\infty f^{(0)}_q Q^3 dq \right.$$}

$$- \frac{m}{\sqrt{J_0^i}} \int_0^\infty f^{(0)}_q \tilde{q} dq \int_0^\infty f^{(0)}_q \tilde{q} \tilde{q} dq + \frac{M}{\sqrt{J_0^i}} \int_0^\infty f^{(0)}_q \tilde{q} \tilde{q} dq \int_0^\infty f^{(0)}_q \tilde{q} \tilde{q} dq \right\} \right.$$

(103)

The last four integrals in (103) represent the contribution to $\Delta E$ of the relativistic terms. By interchange of the roles of $m, M, f^{(0)}, F^{(0)}$ etc, (103) clearly demonstrates that to order $\beta \gamma$ energy is conserved in elastic collisions between particles, and that

$$\Delta E = \Delta E_{\lambda \phi},$$
One can go further and integrate (103) directly for arbitrary mass ratio if \( f^{(0)} \) and \( F^{(\phi)} \) are Maxwellian distributions at unequal temperatures \( T_e \) and \( T_s \) respectively. It is slightly easier in fact to refer to equation (102) and use the distributions given by (77) and (80). For example, one has for the classical part

\[
M \mathcal{J}^0_1 - m \mathcal{I}^0_0 = \kappa^{3/2} \phi(x) \left( x^2 M + n \right) - m \kappa^{3/2} \phi(x)
\]

where \( x^2 = \frac{m}{Z} \alpha^2 \) and \( \phi \) is given by (78). We have put \( F^{(\phi)} \sim e^{-Q^2} \) in the \( \mathcal{I}^0 \) and \( \mathcal{J}^0 \) integrals omitting the factor \( \sqrt{\frac{n}{2\pi}} \) as understood. For the \( \beta_T \) term of (102) we obtain

\[
\left( 4M + 5n \right) \mathcal{I}^0_2 - (5M + 4n) \mathcal{J}^0_1 + 3M \mathcal{J}^0_0 - 3n I^0 \nabla\phi \left( \frac{3M + 4n}{2} \right) - \frac{m}{2} \phi(x) \left( \frac{3M}{2x} + 3x \right) - \frac{m}{2} \phi(x) \left( \frac{15}{4x} + 3x \right)
\]

Finally one needs to substitute the relativistic (order \( \beta_T \)) terms of the \( f^{(\phi)} \) and \( F^{(\phi)} \) distributions into the leading integral of (102) and neglect powers of \( \beta_T \) above the first. To compute the integrals over \( q \) one needs the relations

\[
\int_0^\infty e^{-t y^2} y^n dy = \frac{\Gamma(n/2)}{t^{n/2}} \quad \text{FOR} \quad n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2},
\]

\[
\int_0^\infty e^{-S y \phi(\sqrt{\frac{y}{\alpha}})} y^n dy = (-1)^n \frac{d^n}{dS^n} \left( \frac{1}{S(1 + \alpha S)^{1/2}} \right) \quad \text{FOR} \quad n = 0, 1, 2.
\]

The final result is
The classical result for the energy loss term has been given previously by Spitzer (p. 80) and Shkarofsky (p. 289). The coefficient of the relativistic term \( \lambda_e^{-1} \) in the brackets can be written in the more concise form as

\[
\lambda_e \left( \frac{\mu + \mu^{-1} + 5}{4 \lambda_e \lambda_i \left( \frac{2kT_i}{M} + \frac{2kTe}{m} \right)} \right) \]

where \( \mu = \frac{\lambda_i}{\lambda_e} = \frac{mTe}{M T_i} \). (105) again demonstrates that e-\( \nabla \) collisions have the opposite effect to \( \nabla - \epsilon \) collisions and that the e-\( \epsilon \) contribution \( (T_i \equiv T_e) \) to the energy balance equation vanishes identically.

This completes the rather lengthy section on conservation laws for the \( C^{(0)} \) and \( C^{(1)} \) moment collision terms. We have demonstrated that they are consistent with the basic laws for number density, momentum and energy in elastic encounters thus providing an additional check on the expansions carried out.

Thus far we have given the general forms of the relativistic Fokker-Planck coefficients and derived the appropriate \( C^{(0)} \) and \( C^{(1)} \) collision moments correct to first order in \( \beta_T (\lambda_e^{-1}) \) terms. It has necessitated a considerable amount of formalism to develop the expressions into a form suitable for application to practical problems. The reader who has proceeded directly to the particular cases discussed in §4 should find no difficulty in continuing on to Chapter 6 in which some applications to the transport theory of plasmas are given. The more difficult tensorial manipulations carried out in the appendices of this chapter can be returned to at a later stage to fill in any gaps in the derivations.
APPENDIX 1.

Because of the tedious analysis, we have relegated the calculation of the friction term to this appendix. The diffusion term of the Fokker-Planck equation is tackled in Appendix 2.

The problem that confronts us is the evaluation of $A_\lambda$ given by equation (4a) of this chapter, with the ion distribution function $F$ expanded to first order in spherical harmonics as given by (6). We have mentioned in §2 that in the classical theory the friction and diffusion terms are expressible in terms of derivatives of scalar functions known as Rosenbluth potentials. In particular the friction term can be written in the form

$$A_\lambda = \frac{\partial H(y)}{\partial \nu_\lambda},$$  \hspace{1cm} (1)

where $y$ is the classical velocity. It is a relatively straightforward matter to expand $H$ in spherical harmonics and convert to an equivalent fully symmetric tensor form. The problem, when formulated relativistically, yields no such simple solution even if one is only interested in retaining terms up to $(\nu \epsilon)^\lambda$. Thus in this case, while one is capable of handling classical terms with the simplification afforded by (1), the relativistic terms must be handled in a different way. We adopted the point of view that, rather than try to mix approaches, we would attempt to make a unified approach to the reduction of the Fokker-Planck terms. This we do in the general case and make suitable approximations thereafter.

The general approach has been outlined in §2 for the friction term and it remains to evaluate equation (10) by integrating over angles (d$\nu_\lambda$). With the $D$-functions as defined, $A_\lambda$ can be written in the form

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where $y$ is the classical velocity. It is a relatively straightforward matter to expand $H$ in spherical harmonics and convert to an equivalent fully symmetric tensor form. The problem, when formulated relativistically, yields no such simple solution even if one is only interested in retaining terms up to $(\nu \epsilon)^\lambda$. Thus in this case, while one is capable of handling classical terms with the simplification afforded by (1), the relativistic terms must be handled in a different way. We adopted the point of view that, rather than try to mix approaches, we would attempt to make a unified approach to the reduction of the Fokker-Planck terms. This we do in the general case and make suitable approximations thereafter.

The general approach has been outlined in §2 for the friction term and it remains to evaluate equation (10) by integrating over angles (d$\nu_\lambda$). With the $D$-functions as defined, $A_\lambda$ can be written in the form
The ion distribution $F$ is expanded as

$$F = F^{(0)} + F^{(1)}_{110} Y_{110}' + F^{(1)}_{111} Y_{111}' + F^{(1)}_{100} Y_{100}' ,$$

(3)

where summation over the appropriate values of $m$ and $\zeta$ is implied.

Explicitly

$$F^{(i)}_{1m\zeta} Y_{1m\zeta}' = F^{(i)}_{110} Y_{110}' + F^{(i)}_{111} Y_{111}' + F^{(i)}_{100} Y_{100}' ,$$

and we note that the three spherical harmonics of order 1 are the direction cosines of a unit vector in the direction of $\mathbf{U}$. Further we have the definition that $\mathbf{Z}$ is given by

$$\mathbf{Z} = \cos \Theta \cos \Theta' + \sin \Theta \sin \Theta' \cos (\phi - \phi') ,$$

(4)

where $(\Theta, \phi)$ are the polar directions for $\mathbf{U}$ and $(\Theta', \phi')$ those for $\mathbf{U}'$. To help us evaluate the angle integrations in (2) we need the help of the well known theorem for spherical harmonics

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \Theta d\Theta Y_{lm\zeta}(\Theta, \phi) \rho_n(\mathbf{z}) = \frac{4\pi}{(2n+1)} Y_{lm\zeta}(\Theta, \phi) \delta_{l,n} .$$

(5)

The procedure then is to substitute (3) into (2) and for each component of the friction term $A_\lambda$ we re-express products of spherical harmonics in terms of harmonics of higher order. Use of (5) then enables us to evaluate the non-zero terms (i.e. up to $n = 2$) and regroup
the series in the form

\[ A_\lambda = A_{\lambda 0} + A_{\lambda 1 M S} Y_{1M S} + A_{\lambda 2 M S} Y_{2M S}, \]  

(6)

where the coefficients depend on the D-functions. The expression for \( A_\lambda \) thus derived can be converted into an equivalent cartesian tensor form which is more readily differentiated than the form (6).

In order to carry out the manipulations we need to express such terms as \( Y'_{110} Y'_{110} \) in terms of harmonics up to second order. Some useful general formulae are given in Appendix 3 to enable this to be done.

We begin with the \( A_x \) component. From (2) this is given by

\[ A_x = \frac{Y_{ei}}{C^3} \sum_{n=0}^{\infty} \int \{ \beta_1 \beta_2 U^3 \left( \frac{n}{M} D_n^{(2)} - D_n^{(3)} \right) [F^{(0)} Y'_{110} + F^{(1)} Y'_{110} Y'_{110}] \]

\[ + \beta_1 \beta_2 U^3 \left( \frac{n}{M} D_n^{(2)} - D_n^{(3)} \right) [F^{(0)} + F^{(1)} Y'_{110}] Y_{110} \rho(x) d\nu d\nu', \]

(7)

where

\[ Y'_{i M S} = Y_{i M S}(\theta', \phi'), \quad Y_{i M S} = Y_{i M S}(\theta, \phi) \quad \text{and} \]

\[ d\Omega' = \sin \theta' d\theta' d\phi'. \]

Using the formulae given in Appendix 3, we can convert products of first order harmonics into linear sums of zero and second order harmonics. In the first term of (7) we must carry out this rearrangement before application of (5). For the second, products of harmonics in the angles \( \theta' \) and \( \phi' \) do not appear and (5) is directly applicable. Application of (5) essentially changes a \( Y' \) term into a \( Y \) term with an appropriate constant multiplier.

When the \( \Omega' \) integration has been carried out any remaining products of the harmonics \( Y_{i M S} \) must be converted into appropriate linear sums of higher order terms. As a general rule, the product of harmonics of orders \( M \) and \( \Lambda \) can be expressed as a linear sum of harmonics of order up to and including \( M + \Lambda \). As an example, from Appendix 3, we can write

\[ Y''_{110} = \frac{1}{3} \left( - \frac{1}{3} Y'_{110} + \frac{1}{6} Y'_{210} \right), \]
and similarly for unprimed quantities. It is clear that when the appropriate transformations have been made only terms up to \( n=2 \) in (7) contribute non-zero values. Similar equations to (7) hold for the other components of \( A \).

We define the quantities \( A_{\lambda 0} \), \( A \) and \( B_\lambda \) by the equations

\[
A_{\lambda 0} = \frac{4\pi Y e i}{3c^3} \int_0^{\infty} F^{(1)}_\lambda g_0 d\nu,
\]

\[
A = \frac{4\pi Y e i}{c^3} \int_0^{\infty} F^{(3)} g_1 d\nu,
\]

and

\[
B_\lambda = \frac{\pi Y e i}{c^3} \int_0^{\infty} F^{(1)}_\lambda g_3 d\nu.
\]

For convenience we have also defined \( F_0^{(1)} = F_1^{(1)} \), \( F_1^{(1)} = F_2^{(1)} \) and \( F_0^{(1)} = F_3^{(1)} \). The associated kernels \( g_r (r = 0, 1 \text{ and } 2) \) are given in terms of the D-functions by the relations

\[
g_0 = \nu^2 \beta_1 \beta_2 \left\{ \frac{m}{M} \left( \nu Y^{(2)}_0 + \nu Y_1^{(3)} \right) - \nu Y_0^{(2)} - \frac{1}{3} \nu Y_1^{(3)} \right\},
\]

\[
g_1 = \nu^2 \beta_1 \beta_2 \left\{ \frac{m}{M} \left( \nu Y_1^{(2)} + \frac{1}{3} \nu Y_0^{(3)} \right) - \nu Y_1^{(2)} - \frac{1}{3} \nu Y_0^{(3)} \right\},
\]

and

\[
g_2 = \nu^2 \beta_1 \beta_2 \left\{ \frac{m}{M} \left( \frac{1}{5} \nu Y_2^{(4)} + \frac{1}{3} \nu Y_0^{(3)} \right) - \frac{1}{3} \nu Y_1^{(4)} - \frac{1}{3} \nu Y_1^{(3)} \right\}.
\]

The suffix \( r \) denotes the order of the harmonic which multiplies \( g_r \).

In terms of (8), (7) can be written, after integration over \( N \)

as

\[
A_x = A_{\lambda 0} + A Y_{110} + \frac{4}{6} \beta_1 Y_{120} - \frac{4}{3} \beta_1 Y_{100}
\]

\[+ \frac{4}{6} \beta_1 Y_{120} + \frac{4}{3} \beta_3 Y_{100}.
\]
Similar calculations for the \( A_y \) and \( A_z \) components of \( A \) yield the expressions

\[
A_y = A_{y0} + A_y Y_{10} - \frac{4}{6} Y_{2\pm 0} B_2 + \frac{4}{6} Y_{2\pm 1} B_1 + \frac{4}{3} B_3 Y_{211} - \frac{4}{3} B_2 Y_{200},
\]

(10)

and

\[
A_z = A_{z0} + A_y Y_{100} + \frac{2}{3} B_3 Y_{200} + \frac{4}{3} B_1 Y_{110} + \frac{4}{3} B_2 Y_{200},
\]

(11)

The aim now is to convert the spherical harmonic expansion to an equivalent cartesian expression

\[
A_\lambda = A_{\lambda 0} + A_\lambda \mu \frac{U_\mu}{U} + A_{\lambda \mu \nu} \frac{U_\mu U_\nu}{U^2} .
\]

(12)

Comparison of (12) with equations (9)-(11) yield for \( A_{\lambda \mu} \) and \( A_{\lambda \mu \nu} \) (\( A_{\lambda 0} \) has already been defined) the expressions

\[
A_{\lambda \mu} = S_{\lambda \mu} A, \quad A_{\lambda \mu \nu} = \begin{bmatrix}
\frac{2}{3} B_3 & 2B_2 & 2B_3 \\
2B_2 & -\frac{4}{3} B_1 & 0 \\
2B_3 & 0 & -\frac{4}{3} B_1
\end{bmatrix}, \quad A_{y\gamma} = \begin{bmatrix}
-\frac{4}{3} B_2 & 2B_1 & 0 \\
2B_2 & \frac{2}{3} B_3 & 2B_3 \\
0 & 2B_3 & -\frac{4}{3} B_1
\end{bmatrix}
\]

and

\[
A_{z\gamma} = \begin{bmatrix}
-\frac{4}{3} B_3 & 0 & 2B_1 \\
0 & -\frac{4}{3} B_3 & 2B_2 \\
2B_1 & 2B_2 & \frac{2}{3} B_3
\end{bmatrix}.
\]

(14)

\( A_{\lambda \mu} \) is symmetric in the subscripts \( \lambda \) and \( \mu \). In terms of Kronecker delta functions

\[
A_{\lambda \mu, \nu} = 2(\delta_{\lambda \mu} B_\nu + \delta_{\lambda \nu} B_\mu) - \frac{4}{3} S_{\lambda \nu} B_\mu.
\]
For obtaining the tensor \( A_{\gamma\nu} \) we have used the results given in §3 Chapter 1.

**APPENDIX 2**

In this appendix we outline the procedure to reduce the diffusion term in the Fokker-Planck equation. The term we have to evaluate is

\[
D_{\gamma\nu} = \int \mathcal{E}_{\gamma\nu} F \, d^3 \mathbf{y},
\]

where the tensor \( \mathcal{E}_{\gamma\nu} \) is given by

\[
\mathcal{E}^3 (\rho^2 - 1)^{\frac{1}{4}} \mathcal{E}_{\gamma\nu} = \sum_i \rho_i \beta_i \left[ (\rho^2 - 1) \mathcal{E}_{\gamma\nu} - \mathcal{E}_{\gamma\nu} \right] - \rho \left( \mathcal{E}_{\gamma\nu} + \mathcal{E}_{\gamma\nu} \right).
\]

\( F \) is the ion distribution and to first order is expressed as

\[
F = F^{(0)} + F^{(1)} \frac{\mathbf{v}}{\mathbf{v}}.
\]

As can be seen from the form of \( \mathcal{E}_{\gamma\nu} \), the problem is quite complicated. However, the general approach is similar to that for the friction term, except that now more terms are involved.

As with the friction term, we expand the functions of \( \rho \) in a series of Legendre polynomials in the term \( \mathcal{E} \left( = \frac{\mathbf{v} \cdot \mathbf{v}}{U} \right) \). Explicitly we define three D-type functions by the relations

\[
D^{(1)} = \left( n + \frac{1}{2} \right) \int_{-1}^{1} \frac{\rho^2 P_n(z) \, dz}{\sqrt{\rho^2 - 1}},
\]

and

\[
D^{(r)} = \left( n + \frac{1}{2} \right) \int_{-1}^{1} \frac{\rho^r P_n(z) \, dz}{(\rho^2 - 1)^{\frac{r}{2}}}, \quad r = 2, 3.
\]

The functions enable us to write
and
\[
\frac{\rho^r}{\sqrt{\rho^2 - 1}} = \sum_{n=0}^{\infty} D_n^{(r)} P_n(z),
\]

Combining (1), (2) and (5a,b) gives the following expression for $D_{\lambda\mu}$:
\[
D_{\lambda\mu} = \frac{\gamma e_i}{c_s} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \int_{\Omega} P_n(z) \beta_\lambda \beta_\mu \left[ D_n^{(l)} S_\mu \nu - (U_\lambda U_{\mu\lambda} + U_{\lambda\mu} U_\nu) D^{(2)}_n \right] \frac{F - \frac{1}{2} \left( F_{\mu}^{(1)} + F_{\lambda}^{(1)} Y_{\nu}^{(1)} \right)}{U^2 dU d\Omega},
\]

(6)

Let us first consider the Kronecker delta term in (6). We denote this contribution to the diffusion tensor $D_{\lambda\mu}$ by $K_{\lambda\mu}$ and write
\[
K_{\lambda\mu} = \frac{\gamma e_i}{c_s} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \int_{\Omega'} P_n(z) \beta_\lambda \beta_\mu \left( D_n^{(l)} F^{(2)} + F_{\lambda}^{(1)} Y_{\nu}^{(1)} \right) U^2 dU d\Omega,'
\]

(7)

where in (7) we have substituted for the ion distribution $F$ in terms of its first order expansion. Using the integral theorem for spherical harmonics in the form given by equation (5) of Appendix 1 enables the $\Omega'$ integration in (7) to be carried out. If we define the functions $d_r$ by
\[
d_r = D_r^{(l)} U^2 \beta_\lambda \beta_\mu, \quad r = 0, 1,
\]

(8)

then $K_{\lambda\mu}$ may be written in the form
after recombining the spherical harmonic expansion in terms of an equivalent cartesian tensor form. Again, the suffix on the \( d \)-function relates to the order of the harmonic which it multiplies.

We now turn our attention to the remaining terms in the integrand of (6). We define the tensor \( W_{\gamma \nu} \) by the statement

\[
W_{\gamma \nu} = \frac{\gamma_{\nu}}{\gamma_1} \sum_{n=0}^{\infty} \int_0^\infty \int_0^{2\pi} \rho_n(\Omega) \beta_1 \beta_2 \left\{ (U_{\lambda} U_{\nu} + U_{\nu} U_{\lambda}) D_{n}^{(1)} + (U_{\lambda} U_{\nu} + U_{\nu} U_{\lambda}) D_{n}^{(3)} \right\} F \, d\Omega \, d\beta_1 \,
\]

so that the diffusion term is related to (7) and (10) by

\[
D_{\gamma \nu} = K_{\gamma \nu} - W_{\gamma \nu}.
\]

Since the \( W_{\gamma \nu} \) tensor is symmetrical, only six terms need be considered. For the \( W_{\nu} \) term, the \( U_1 \) and \( U_{\nu} \) components are written as

\[
U_1 = U Y_{\nu 0}(\theta, \phi) \quad \text{AND} \quad U_{\nu} = U Y_{\nu 0}(\theta', \varphi')
\]

and the ion distribution is expressed as in (7). Other components of \( W_{\gamma \nu} \) can be written down with the components \( U_{\lambda} \) and \( U_{\nu} \) expressed in terms of the appropriate first order harmonics. In equation (10), where products of such harmonics occur, these are first re-expressed in terms of harmonics of higher order according to Appendix 3. The integration over \( \Omega' \) can be performed using the theorem given in Appendix 1. After completion of the angle integration (which essentially
replaces primed quantities with unprimed ones) products of unprimed harmonics are converted into linear sums of higher order terms. Since we are now dealing with terms like $Y_{10}^3$ these can be replaced by a series of harmonics up to order three. Repeated application of the appropriate formula in Appendix 3 will yield the required expansion. For example we can write

$$Y_{10}^{3'} = Y_{10}^{'}\left( Y_{10}^{'} Y_{10}^{'} \right)$$

$$= Y_{10}^{'} \left( \frac{1}{6} Y_{210}^{'} + \frac{1}{3} Y_{220}^{'} \right)$$

$$= \frac{3}{5} Y_{110}^{'} + \frac{1}{60} Y_{330}^{'} - \frac{1}{10} Y_{510}^{'} \text{ etc.}$$

with similar results for unprimed quantities.

We introduce a set of scalar functions of $U$ and $\mathcal{U}$ in terms of those introduced in equations (4a,b) by:

$$h_0 = \left\{ \frac{1}{3} (U^3 + U^2) U^1 Y_{0}^{(0)} + \frac{2}{9} U^3 U Y_{0}^{(3)} \right\} \beta_1 \beta_2$$

$$h_1 = \left\{ \frac{1}{5} (U^5 + U^4) U^1 Y_{0}^{(0)} + \frac{2}{3} U^3 U Y_{1}^{(3)} + \frac{2}{75} U^3 U Y_{1}^{(1)} \right\} \beta_1 \beta_2$$

$$h_2 = \left\{ \frac{1}{15} (U^5 + U^4) U^1 Y_{0}^{(0)} + \frac{2}{25} U^3 U Y_{2}^{(1)} \right\} \beta_1 \beta_2$$

and lastly

$$h_3 = \left\{ \frac{1}{3} (U^2 Y_{1}^{(0)} + U^1 Y_{2}^{(1)}) U^1 + \frac{2}{3} U^3 U Y_{2}^{(3)} \right\} \beta_1 \beta_2$$

(12a-e)

The suffixes on the $h$ functions indicate the order of the associated harmonic, and it is necessary to introduce two functions for the order one term. With these functions we further define integrals over the ions speed $\mathcal{U}$ as follows
\[ W_o = \frac{4\pi Y_{ei}}{c^3} \int_0^\infty h_0 F^{(0)} d\mathbf{\Omega} \]
\[ A = \frac{4\pi Y_{ei}}{c^3} \int_0^\infty h_1 F^{(1)} d\mathbf{\Omega} \]
\[ E = \frac{4\pi Y_{ei}}{c^3} \int_0^\infty h_2 F^{(1)} d\mathbf{\Omega} \]
\[ S = \frac{2\pi Y_{ei}}{c^3} \int_0^\infty h_3 F^{(0)} d\mathbf{\Omega} \]

and finally
\[ P = \frac{\pi Y_{ei}}{c^3} \int_0^\infty h_3 F^{(1)} d\mathbf{\Omega} \]

Equations (13a,d) define scalar integrals over \( \mathbf{\Omega} \) of the zero order part of the ion distribution. The remaining integrals depend on the first order term \( F^{(1)} \) and the vectors \( \mathbf{A}, \mathbf{E} \) and \( \mathbf{P} \) are written in component form as

\[ \mathbf{A} = (A,B,C), \quad \mathbf{E} = (E,F,G) \quad \text{and} \quad \mathbf{P} = (P,Q,R). \]

The directional distribution \( F^{(1)} \) in component form is given by

\[ F^{(1)} = \left( F_x^{(1)}, F_y^{(1)}, F_z^{(1)} \right). \]

In terms of (13a-e), the \( W_{ij} \) element of the diffusion tensor (10) may be written, after performing the integration over \( d\mathbf{\Omega} \) as:

\[ W_{ij} = W_0 + A Y_{110} + F Y_{111} + G Y_{110} + \frac{1}{3} S Y_{210} - \frac{2}{3} S Y_{200} \]
\[ + \frac{1}{15} P Y_{330} - \frac{2}{5} P Y_{310} + \frac{1}{15} Q Y_{331} - \frac{2}{15} Q Y_{311} + \frac{2}{15} R Y_{320} - \frac{4}{5} R Y_{500}. \]

(14)

Similar results hold for the other components of \( W_{ij} \). To carry out the integrations in (10) one first expands products of primed
harmonics in terms of higher order ones according to Appendix 3. The integration over $\mathcal{N}$ can then be performed and again products of spherical harmonics are replaced by the appropriate expansion in higher order terms. For each element of the diffusion tensor the final result yields an expression up to the third order harmonic as in (14).

For the other diagonal terms $W_{11}$ and $W_{35}$ we obtain the results

$$W_{22} = W_0 + EY_{110} + \beta Y_{210} + \gamma Y_{120} - \frac{1}{3} SY_{210} - \frac{1}{3} SY_{200}$$

$$- \frac{1}{15} \rho Y_{330} - \frac{2}{15} \rho Y_{310} - \frac{1}{15} \Phi Y_{331} - \frac{2}{5} \Phi Y_{311}$$

$$- \frac{2}{15} R Y_{320} - \frac{4}{5} R Y_{300} ,$$

and

$$W_{35} = W_0 + EY_{110} + \beta Y_{110} + \gamma Y_{120} + \frac{4}{3} SY_{200}$$

$$+ \frac{8}{15} \rho Y_{310} + \frac{8}{15} \Phi Y_{311} + \frac{8}{5} R Y_{300} .$$

(15)

When considering the off diagonal terms, only three terms need be computed by symmetry of $W_{11}$. Further, the scalar or $Y_{000}$ term does not appear in the expansion of these elements, affording some simplification of the results. For the $W_{12}$ term we have

$$W_{12} = \frac{1}{2} \left(A - E\right) Y_{111} + \frac{1}{2} \left(B - F\right) Y_{110} + \frac{1}{3} SY_{211}$$

$$+ \frac{1}{15} \rho Y_{331} - \frac{2}{15} \rho Y_{311} - \frac{1}{15} \Phi Y_{330} - \frac{2}{15} \Phi Y_{310} + \frac{2}{15} R Y_{300} .$$

(17)

The $W_{13}$ element is similarly given as

$$W_{13} = \frac{1}{2} \left(A - E\right) Y_{100} + \frac{1}{2} \left(C - G\right) Y_{110} + \frac{3}{3} SY_{210}$$

$$+ \frac{2}{15} \rho Y_{310} - \frac{4}{5} \rho Y_{300} + \frac{2}{15} \Phi Y_{311} + \frac{2}{15} R Y_{310} .$$

(18)
Once more for the 2, 3 element of \( W_{\lambda\mu} \) yields

\[
W_{23} = \frac{1}{2} (8 - 5) Y_{100} + \frac{1}{3} (C - G) Y_{111} + \frac{2}{3} SY_{211} \\
+ \frac{2}{15} PY_{321} - \frac{2}{15} Q Y_{320} - \frac{4}{5} Q Y_{300} + \frac{2}{15} R Y_{311}.
\]  

(19)

The derivation of equations (14)-(19) from (10) is straightforward but requires tedious application of the spherical harmonic identities given in Appendix 3. In general form, the tensor \( W_{\lambda\mu} \) may be expressed as

\[
W_{\lambda\mu} = W_{\lambda\mu\sigma} + \sum_{l=1}^{3} W_{\lambda\mu l m s} Y_{l m s}(\Theta, \Phi)
\]  

(20)

where the appropriate terms may be found by comparison with the above equations. In (20) summation over \( m \) and \( s \) is implied subject to the usual conditions

\[
m \leq l, \quad s = 0 \text{ or } 1.
\]

In principle the diffusion term given by equation (1) is now completely determined to first order in the ion distribution. However since it is derivatives with respect to the components of \( \mathbf{J} \) that we must evaluate, it is more convenient to express the expansion for \( D_{\lambda} \) in terms of cartesian tensors. The \( K_{\lambda\mu} \) term has been given in this form (equation (9)). It only remains to convert \( W_{\lambda\mu} \) into the form

\[
W_{\lambda\mu} = W_{\lambda\mu 0} + W_{\lambda\mu 3} \frac{U_0}{U} + W_{\lambda\mu 5} \frac{U_2}{U^2} + W_{\lambda\mu 7} \frac{U_3}{U^3}.
\]  

(21)

The procedure for obtaining the cartesian tensors up to order two has already been given in Chapter 1. By inspection, the \( W_{\lambda\mu 0} \) term is, from equations (14-19),

\[
W_{\lambda\mu 0} = W_0 S_{\lambda\mu}.
\]  

(22)
The tensor \( W_{\mu \nu} \) may be written down in submatrix form as (with \( \lambda \) representing rows and \( \mu \) columns)

\[
W_{\lambda \mu} = \begin{bmatrix}
A & F & G \\
\frac{1}{2}(B-F) & \frac{1}{2}(A-E) & 0 \\
\frac{1}{2}(C-G) & 0 & \frac{1}{2}(A-E)
\end{bmatrix},
\]

\[
W_{\nu \mu} = \begin{bmatrix}
\frac{1}{2}(B-F) & \frac{1}{2}(A-E) & 0 \\
E & B & G \\
0 & \frac{1}{2}(C-G) & \frac{1}{2}(B-F)
\end{bmatrix},
\]

and

\[
W_{\lambda \nu \mu} = \begin{bmatrix}
\frac{1}{2}(C-G) & 0 & \frac{1}{2}(A-E) \\
0 & \frac{1}{2}(C-G) & \frac{1}{2}(B-F) \\
E & F & C
\end{bmatrix}.
\]

In terms of Kronecker deltas this may be neatly written as

\[
W_{\lambda \mu \nu} = \frac{1}{2} \left( S_{\nu \lambda} A_{\mu} + A_{\lambda} S_{\mu \nu} \right) + \delta_{\mu \nu} E_{\lambda} - \frac{1}{2} \delta_{\lambda \nu} E_{\mu} - \frac{1}{2} \delta_{\lambda \mu} E_{\nu}.
\]

Similarly the fourth order tensor \( W_{\mu \alpha \beta} \) may be written down in submatrix form but is much simpler since the elements depend on the quantity \( S \). In fact it is easy to show that \( W_{\mu \alpha \beta} \) may be expressed in the form

\[
W_{\mu \alpha \beta} = S \left\{ \delta_{\lambda \alpha} \delta_{\nu \beta} + \delta_{\nu \alpha} \delta_{\lambda \beta} \right\} - \frac{2}{3} \delta_{\nu \alpha} \delta_{\mu \beta} S.
\]

For the fifth order tensor we require to know how the elements \( W_{\mu \alpha \beta \gamma \delta} \) in (21) are related to the \( W_{\mu 3 \gamma 5} \) terms in (20). Johnston [17] has derived the required form, the method being very similar to that described in Chapter 1 for converting second order harmonics to equivalent cartesian tensors. Essentially one has the identity

\[
\sum_{\gamma, \beta = 1}^{3} \frac{W_{\mu \alpha \beta \gamma \delta} U_{\nu} U_{\alpha} U_{\beta}}{U^3} = \sum_{m=1, 2, 3} W_{\mu 3 \gamma 5} Y_3 m(\theta, \phi),
\]
for \( \lambda, \mu = 1, 2 \) or \( 3 \). If we put \( W_{\mu} = h \) for convenience, then from (17) we can write

\[
\begin{align*}
    h_{xxx} &= -\frac{3}{2} h_{310} + 15 h_{330}, & h_{yyy} &= -\frac{3}{2} h_{310} - 15 h_{331}, \\
    h_{zzz} &= h_{300}, & h_{xxy} &= -\frac{1}{2} h_{311} + 15 h_{331}, \\
    h_{xyz} &= -\frac{1}{2} h_{300} - 15 h_{330}, & h_{xzz} &= 2 h_{310}, \\
    h_{xzx} &= -\frac{1}{2} h_{300} + 5 h_{320}, & h_{zyz} &= -\frac{1}{2} h_{300} - 5 h_{320}, \\
    h_{yzz} &= 2 h_{311}, & h_{xyz} &= 5 h_{311}.
\end{align*}
\]  

(25)

In (25) the lettered suffices refer to the appropriate cartesian components and the numerical ones to the corresponding harmonic terms. We note that the tensor \( W_{\mu \nu \lambda \beta} \) is fully symmetric in the first two and last three suffices, and that the operation of contraction on the cartesian tensors in (25) yields zero. The procedure then is, for each of the third order harmonic terms in equations (14-19), we apply the transformation equations (25). From equation (14) we have for the \( W_{\mu \nu \lambda \rho} \) terms

\[
\begin{align*}
    W_{1xxx} &= \frac{2}{5} P, & W_{1xyy} &= \frac{16}{15} Q, & W_{1zzz} &= \frac{16}{15} R, \\
    W_{1yyz} &= -\frac{4}{5} P, & W_{1yyz} &= -\frac{4}{5} Q, & W_{1yzz} &= -\frac{4}{5} R, \\
    W_{1zzz} &= -\frac{4}{5} P, & W_{1zzz} &= -\frac{4}{5} Q, & W_{1zzz} &= -\frac{4}{5} R, \\
    \text{and} & & W_{1xyz} &= 0.
\end{align*}
\]  

(26)

For the \( W_{21, \nu \lambda \beta} \) terms we have, writing the tensor in the scheme given by (26)
The last remaining diagonal term of $W_{\nu, \lambda}$ is given formally by

\[-\frac{4}{5} \rho, \quad -\frac{4}{5} Q, \quad -\frac{4}{15} R,\]
\[\frac{16}{15} \rho, \quad \frac{2}{5} Q, \quad \frac{16}{15} R,\]
\[-\frac{4}{15} \rho, \quad -\frac{4}{5} Q, \quad -\frac{4}{5} R,\]

AND \quad 0 \quad . \quad (27)

For the off diagonal terms we can write down the appropriate values in a similar scheme. For the $W_{12, \lambda}$ elements we can write

\[-\frac{4}{5} Q, \quad \frac{16}{15} \rho, \quad 0,\]
\[\frac{16}{15} Q, \quad -\frac{4}{5} \rho, \quad 0,\]
\[-\frac{4}{15} Q, \quad -\frac{4}{15} \rho, \quad 0,\]

and

\[\frac{2}{3} R. \quad (29)\]

The $W_{13, \lambda}$ elements are

\[-\frac{4}{5} R, \quad 0, \quad \frac{16}{15} \rho,\]
\[-\frac{4}{15} R, \quad 0, \quad -\frac{4}{15} \rho,\]
\[\frac{16}{15} R, \quad 0, \quad -\frac{4}{5} \rho,\]

and

\[\frac{2}{3} Q. \quad (30)\]

Finally for the $W_{13, \lambda}$ elements we have the results
It is possible in fact to write $\omega_{\mu \nu \lambda \beta \gamma}$ in terms of Kronecker delta functions. Firstly we observe that the tensor is symmetric in $\gamma, \lambda$ and $\beta$, and $\lambda$ and $\mu$. A little consideration shows that the only possible fifth order tensor of this form is

$$
\begin{align*}
\lambda_1 \rho_\beta \left( \delta_{\lambda \nu} \delta_{\mu \kappa} + \delta_{\lambda \kappa} \delta_{\mu \nu} \right) \\
+ \lambda_1 \rho_\lambda \left( \delta_{\gamma \beta} \delta_{\mu \nu} + \delta_{\lambda \nu} \delta_{\gamma \mu} \right) \\
+ \lambda_1 \rho_\nu \left( \delta_{\gamma \nu} \delta_{\mu \kappa} + \delta_{\lambda \kappa} \delta_{\gamma \nu} \right) \\
+ \lambda_2 \left( \rho_\beta \delta_{\gamma \nu} + \rho_\lambda \delta_{\gamma \beta} + \rho_\nu \delta_{\gamma \lambda} \right) \delta_{\mu \kappa} \\
+ \lambda_3 \left( \rho_\lambda \delta_{\mu \kappa} \delta_{\gamma \beta} + \rho_\mu \delta_{\gamma \lambda} \delta_{\mu \beta} \right) \\
+ \lambda_3 \left( \rho_\beta \delta_{\mu \beta} \delta_{\gamma \nu} + \rho_\mu \delta_{\gamma \beta} \delta_{\mu \nu} \right) \\
+ \lambda_3 \left( \rho_\lambda \delta_{\mu \beta} \delta_{\gamma \nu} + \rho_\mu \delta_{\gamma \lambda} \delta_{\mu \nu} \right),
\end{align*}
$$

where $\lambda_1, \lambda_2$ and $\lambda_3$ are constants and $\rho_\lambda$ represents any component of the vector $\rho$. Simple comparison of (32) with (26-31) for specific elements yields the values $\lambda_1 = \frac{2}{3}$, $\lambda_2 = -\frac{4}{15}$ and $\lambda_3 = -\frac{4}{15}$.

This completes the results of this appendix. We have expressed the diffusion term (1) in terms of a cartesian tensor expansion to
order three. This can be substituted into the Fokker-Planck equation and the diffusion contribution to the collision terms deduced. After differentiation with respect to the components of \( \mathbf{U} \) one is essentially left with products like

\[
\frac{\partial}{\partial \mathbf{U}} \left( \mathbf{W}_{\mu \nu \lambda \beta} \right) U_{\lambda} U_{\mu} U_{\nu} U_{\beta},
\]

to evaluate and similar terms for lower order tensors. At first sight such an expression looks rather cumbersome. However it is to be noted that the elements comprising the tensors are functions of the magnitude of \( \mathbf{U} \) only and hence are invariant with respect to the rotation group. Hence one can simply rotate the frame of reference so that (for example)

\[
\mathbf{U} = (U_x, 0, 0).
\]

Scalar products, such as above, are drastically reduced to only a few terms. After evaluation it is a simple matter to convert to the original system of co-ordinates. For the example above we have in the new co-ordinate system

\[
\frac{\partial}{\partial \mathbf{U}} \left( \mathbf{W}_{\mu \nu \lambda \beta} \right) U_{\lambda} U_{\mu} U_{\nu} U_{\beta} = W_{\mu \nu \lambda \beta} U_\mu U_\nu U_\lambda U_\beta
\]

\[
= \frac{3}{5} \frac{\partial^2 \mathbf{P}}{\partial \mathbf{U}^2} U_x U_x
\]

\[
= \frac{3}{5} \left( \frac{\partial^2 \mathbf{P}}{\partial \mathbf{U}^2} \cdot \mathbf{U} \right) U^4,
\]

since \( \mathbf{P} \cdot \mathbf{U} = (\mathbf{P} U_x + \mathbf{Q} U_\mu ; \mathbf{R} \mathbf{U}_x) \) and \( \mathbf{U} \) are invariant under rotations. The same result can of course be obtained if one writes down, and sums, all the elements in the general frame - all 243 terms!

**APPENDIX 3.**

In the evaluation of the Fokker-Planck terms various formulae are required expressing the product of spherical harmonics as linear sums of higher order ones. In this appendix we derive formulae for the following products

\[
Y_{l_0} Y_{l m s}, \quad Y_{l_1} Y_{l m s} \quad \text{AND} \quad Y_{l_0 l m s}.
\]
To begin with we define the spherical harmonic \( Y_{l m}(\theta, \phi) \) of order \( l \) as

\[
Y_{l m} = \mathbf{P}_{l}^{m}(\cos \theta) \phi_{(m)s},
\]

where we have set

\[
\phi_{(m)s} = S_{s} \cos m \phi + S_{s} \sin m \phi.
\]

\( \mathbf{P}_{l}^{m} \) is the well known associated Legendre function of order \( l \) (see for example (27)). If we write \( Z = \cos \theta \) we have the useful relations

\[
\sqrt{1-Z^2} \cdot \mathbf{P}_{l}^{m}(Z) = \frac{1}{(2l+1)} \left( \mathbf{P}_{l+1}^{m}(Z) - \mathbf{P}_{l-1}^{m}(Z) \right),
\]

\[
\sqrt{1-Z^2} \cdot \mathbf{P}_{l+1}^{m}(Z) = (m+1)Z \mathbf{P}_{l}^{m-1}(Z) - (l-m+2) \mathbf{P}_{l+1}^{m-1}(Z),
\]

and

\[
(l-m+1) \mathbf{P}_{l+1}^{m} - (2l+1)Z \mathbf{P}_{l}^{m} + (l+m) \mathbf{P}_{l-1}^{m} = 0.
\]

We also observe that

\[
2 \sin \phi \phi_{(m)s} = \phi_{(m+s)s} S_{s} - \phi_{(m+s)s} S_{s} - \phi_{(m-s)s} S_{s} \phi_{(m+s)s} + \phi_{(m-s)s} S_{s} \phi_{(m+s)s}
\]

that

\[
2 \cos \phi \phi_{(m)s} = \phi_{(m+s)s} + \phi_{(m-s)s},
\]

and that
\[ Y_{10} = \cos \phi \sin \theta, \quad Y_{11} = \sin \phi \sin \theta, \quad Y_{100} = \cos \theta. \]  \tag{7}

Using (6) and (7) we have

\[ Y_{10} Y_{\lambda-5} = \frac{\sqrt{1-z^2}}{2} \rho_{\lambda} \left( \phi_{(a)} + \phi_{(b)} \right). \]  \tag{8}

Our aim is to eliminate the square root factor in (8) and match the superscript of \( \rho_{\lambda} \) with the bracketed subscript of \( \phi \) for each term of (8). We apply (4a,b) to (8) to get rid of the \( \sqrt{1-z^2} \) factor and then apply (4c) to eliminate the \( \sqrt{z} \rho_{\lambda} \) term. The result is

\[ Y_{10} Y_{\lambda m} = \frac{1}{2(2\ell+1)} \left\{ Y_{\ell+1, m+1, s} - Y_{\ell-1, m+1, s} \right\} + \frac{(\ell+m)(\ell+m-1)}{2(2\ell+1)} Y_{\ell-1, m-1, s} \]

\[ + \frac{(\ell-m+2)(m-\ell-1)}{2(2\ell+1)} Y_{\ell+1, m-1, s}. \]  \tag{9}

For \( m=0 \) the result (9) does not hold. In this case \( \phi_{(0)} = S_{00} \) and (4a) may be applied directly. We can incorporate this into the general case (9) by writing

\[ Y_{10} Y_{\lambda m} = \frac{(1+S_{0m})}{2(2\ell+1)} \left\{ Y_{\ell+1, m+1, s} - Y_{\ell-1, m+1, s} \right\} + \frac{(\ell+m)(\ell+m-1)}{2(2\ell+1)} Y_{\ell-1, m-1, s} \]

\[ + \frac{(\ell-m+2)(m-\ell-1)}{2(2\ell+1)} Y_{\ell+1, m-1, s}. \]  \tag{10}

For the term \( Y_{11} Y_{\ell m} \) we have, using (5) and (7),
Repeated application of \((4a-c)\) yields the final form of \((11)\) which can be written in the form

\[
\begin{align*}
\psi_{11,3} &= \frac{\sqrt{-1}}{2} \rho^m (\phi_{(m+1)i} - \phi_{(m)i} \delta_{0s} - \phi_{(m-1)i} \delta_{is}) \\
&- \phi_{(m-1)i} \delta_{0s} + \phi_{(m)i} \delta_{is}.
\end{align*}
\]  

(11)

where the Kronecker delta term \(\delta_{om}\) has been introduced for exactly the same reason as in \((10)\).

Finally we require an expression for \(\psi_{10,3}\). This is easy since

\[
\psi_{10,3} = \mathcal{A} \phi_{(m)i}^m.
\]

and use of \((4c)\) gives the final result

\[
\psi_{10,3} = \frac{(l+m+1)}{2(l+1)} \psi_{l+1,m,i} + \frac{(l+m)}{2(l+1)} \psi_{l-1,m,i}.
\]  

(13)
APPLICATIONS TO TRANSPORT PROBLEMS OF THE
RELATIVISTIC PLASMA.

§1 INTRODUCTION.

In the previous chapter we obtained expressions for the first two moments \( C^{(0)} \) and \( C^{(1)} \) of the Fokker-Planck collision term, correct to first order in the relativistic parameter \( \beta \). However we made no attempt to relate them to their appropriate left hand sides of the expanded Boltzmann equation. We did in fact derive the three basic equations of number density, momentum and energy. However, these equations still contain averages over the distribution function and can only be simplified usually by some assumption.

In this chapter, the integrodifferential equation for \( f^{(l)} \) is solved by an expansion of the magnitude of reduced velocity functions in Laguerre polynomials. This procedure has been adopted in the solution of the classical equation i.e. terms independent of \( \beta \) (see for example § 2, Chapter 8) and, we have found, is fairly well suited for handling the relativistic terms as well. One equates the forcing terms in the \( f^{(l)} \) equation for electrons to the appropriate collisional moment \( C^{(1)} \), which, for an exact description should include both electron-electron and electron-ion interaction. As will be seen it is the electron-electron terms which considerably complicate the analysis - even more so with the relativistic terms included as well. The equation for electrons is given in § 2 together with the \( e-e \) and \( e-i \) interaction terms. For \( e-i \) collisions the recoil of the ions is taken into account. This gives rise to two new terms not previously derived in § 4 of Chapter 5. In § 3 the Laguerre expansion is substituted into the equation for \( f^{(0)} \) and the matrix interaction terms derived. This is where the complexity of \( e-e \) effects shows up, since the expansion coefficients in the collision term are not
isolated by the orthogonality conditions of the polynomials.

From the solutions one can obtain directly the transport coefficients. In §4 we relate the direct current flow due to impressed electric fields and gradients in number density to the conductivity and diffusion coefficients respectively. Effects of magnetic fields have not been considered here. In §5 the relation of current to gradients in electron temperature via the thermoelectric coefficient is discussed. Finally in §6 we derive expressions for the electron energy flow and relate these to the forcing terms in Boltzmann's equation. In §7 we derive transport coefficients for the so-called Lorentzian plasma. This is an idealised case in which e-e effects are ignored and one is thus able to derive results exactly. We also apply the expansion method in terms of Laguerre polynomials and a comparison yields an additional check on the accuracy of our results. Some remarks and suggestions for further work are given in §8.

As we have said, this chapter is devoted to calculating the transport coefficients for electron current and energy flow correct to first order in relativistic effects (i.e. up to terms of order $\beta^2$). Many writers have investigated the classical problem and we refer the reader to the results presented by Shkarofsky et al $\mathcal{L}^2$ (Chapter 8) and references therein. Shkarofsky applies the approach of the expansion in Laguerre polynomials, and gives results for a fourth order expansion of the $J^{(4)}$ equation. These polynomials are similar the the Sonine polynomials used by Chapman and Cowling $\mathcal{L}^4$ in their classic work. We give results for the transport coefficients derivable from the equation for a fourth-order expansion and also for an expansion of order five - one more than that of Shkarofsky. Results of each coefficient are tabulated for various degrees of ionization of the plasma, that is for various values of $Z$. 
It is only recently that relativistic effects in transport theory have come under scrutiny. Many writers over the past decade or so have developed relativistic collision terms for the Boltzmann equation; e.g., Belyaev and Budker \cite{37}, Silin \cite{39}, Simon and Harris \cite{43} and Krizan \cite{44}, but due to their complexity, applications to practical problems have so far been very few. The first major attempt at investigating transport theory for the relativistic gas appears to have been begun by Israel \cite{35}. Israel's development of the governing equations essentially follows the 'classical' Chapman-Cowling approach. He derives transport coefficients for his so-called 'Relativistic-Maxwellian Molecules' in which simplifying assumptions concerning the collision cross-section enabled him to solve the eigenvalue problem of the linearised collision operator. The classical limit of the coefficients yields the familiar results for the Maxwellian molecule discussed, for example, in \cite{4}. Israel himself states that the reason for studying these molecules is because of their mathematical simplicity in handling the collision terms. Exact analytical results can be computed for this model. However their properties in relation to the behaviour of particles at high temperatures remains an open question. In a series of more recent papers \cite{36a-f} de-Groot, and co-workers have essentially extended Israel's work to a binary gas mixture and obtained results for the first order relativistic effects in the transport coefficients. Again the approach is via the Chapman-Cowling method.

In this chapter we apply the relativistic tensor expansion developed previously to the transport theory of plasmas with Coulomb interactions. The expanded Fokker-Planck coefficients have been derived from the collision integral originally given by Belyaev and Budker \cite{37}. Recently Akama \cite{38} has rederived this collision term from impact theory using the known relativistic cross-sections.
for electron-electron and electron-ion interaction. The important point to note is that the limitations of this approach are known since the cross-sections used are well defined, instead of making assumptions as regards their nature.

We do not derive the coefficients of viscosity here since this involves tackling the equation for \( f^{(1)} \) and is beyond the scope of this present work. Indeed the handling of the \( f^{(2)} \) equation is a formidable task due to the complexity of the electron-electron collision terms. However, in principle the \( f^{(2)} \) equation could be expanded in much the same way.

\section{The Equation for Electrons.}

In this section we intend to give the full \( f^{(1)} \) equation for electrons including the ion recoil terms. From equation (14), Chapter 3, the general equation for \( f^{(1)} \), neglecting the \( f^{(2)} \) tensor and putting \( \omega = 0 \) (i.e. no magnetic fields) is

\[
\frac{\partial f^{(1)}}{\partial \mathbf{e}} + \frac{C_{\mathbf{e}}}{\sqrt{1+u^2}} \nabla + c \frac{e}{m c} \frac{\partial f^{(6)}}{\partial \mathbf{u}} = C^{(1)} \cdot \zeta, \tag{1}
\]

where \( \zeta \) is a constant electric field. \( C^{(1)} \) is the first moment collision term which includes \( e-e \) and \( e-i \) interactions.

First order relativistic effects are found by scaling \( \mathbf{u} \) by the relation

\[
\frac{1}{\lambda_e} \lambda_e \mathbf{u}^2 = \zeta, \quad \lambda_e \gg 1, \tag{2}
\]

where \( \lambda_e = \frac{m c^3}{k T} \) and retaining terms up to order \( \lambda_e^{-1} \). We have also defined \( \lambda = \frac{m c^3}{k T} \) and \( \mu = \lambda \lambda_e \). A thermalised situation is envisaged in which the electron and ion temperatures are assumed equal. This is valid providing disturbing forces are sufficiently small. Writing equation (1) to first order in \( \lambda_e^{-1} \) yields

\[
C \left(1- \mathbf{u} \mathbf{e} \right) \nabla + c \frac{e}{m c} \frac{\partial f^{(6)}}{\partial \mathbf{u}} = \left( C^{(1)} + C^{(i)} \right) \mathbf{e} + \mathbf{e} \mathbf{e}'. \tag{3}
\]
where $C_0^1$ and $C_1^1$ for $e-e$ ($\lambda=\lambda_e$) and $e-i$ ($\lambda=\lambda_i$) collisions are given by equations (84) and (81) of Chapter 5 respectively. The partial derivative with respect to time has been dropped since in transport theory the variations in plasma quantities are assumed to be affected implicitly through the variations in such macroscopic quantities as temperature and number density.

For reference the $e-e$ expressions are

$$
\frac{c_s^3}{\gamma_{ee}^1} C_0^{(1)} = \frac{1}{3q} \frac{\partial f_{ei}^{(i)}}{\partial q^3} \left( I_1^o + J_1^o \right) + \frac{1}{3q^2} \frac{\partial f_{ei}^{(o)}}{\partial q^3} \left( 3I_1^o + 2J_1^o - I_2^o \right) + \frac{1}{3q^3} f_{ei}^{(i)} \left( -3I_2^o + I_2^o - 2J_1^o \right) + \left( 8\pi q f_{ei}^{(1)}}{f_{ei}^{(i)}} \right) + \frac{2f_{ei}^{(o)}}{3q^3} \left( I_3^o + J_3^o \right) + \frac{2f_{ei}^{(o)}}{5q^3} \left( -3I_2^o + 2J_2^o + 5J_3^o \right),
$$

and

$$
\frac{c_s^3}{\gamma_{ee}^1} C_1^{(1)} = -\left( 8\pi q f_{ei}^{(i)}}{f_{ei}^{(i)}} \right) + q f_{ei}^{(i)} \frac{\partial f_{ei}^{(i)}}{\partial q^3} \left( \frac{2}{15} I_3^o - \frac{17}{35} I_2^o - \frac{3}{35} J_2^o + \frac{3}{15} J_3^o \right) + \frac{2f_{ei}^{(o)}}{3q^3} \left( I_1^o + J_1^o \right) + \frac{1}{q^3} \left( 3I_2^o + I_2^o - \frac{2}{3} I_4^o + 4J_4^o - \frac{2}{3} J_5^o \right) + \frac{2f_{ei}^{(o)}}{3q^3} \left( 2I_3^o - \frac{3}{2} I_2^o + \frac{3}{2} I_4^o + \frac{4}{3} J_4^o + \frac{2}{3} J_5^o \right) + q f_{ei}^{(i)} \left( \frac{1}{3} J_1^o - \frac{2}{3} I_4^o + I_6^o \right),
$$

where

$$
Y_{ee} = 4\pi \left( \frac{e^2}{4\pi \epsilon_0 c} \right)^2 L_N \lambda_L = Y_{ei} / \bar{Z}^2,
$$

and $\bar{Z}$ is the ionisation number of the plasma. The collision term for
$e-e$ effects becomes

$$
\left( C_{e}^{(1)} \right)_{ee} = \left( C_{e}^{(0)} + \lambda e^{-i \lambda} \right)_{ee}.
$$

For the electron-ion terms we refer to equation (81) of Chapter 5. These do not include the ion-recoil and for completeness we give the terms here. For the ions we put $\sqrt{\mu \nu} \sim \infty$ in the $I$ and $J$ integrals and note that the dominant $I$ terms are $I_0$ and $I_1$ (with $\mu \gg 1$) and that all the $J$ terms are negligible. The electron to ion mass ratio $\mu / M$ is also neglected. Applying these approximations to equations (74) and (75) of Chapter 5 yields

$$
\frac{C_{e}^{(1)}}{\gamma_{ee}} \left( C_{e}^{(0)} \right)_{ee} = -\frac{I_0}{q^2} \int f^{(i)} - \frac{1}{3q^2} \int \delta \frac{f^{(c)}}{\partial q},
$$

and

$$
\frac{C_{e}^{(1)}}{\gamma_{ee}} \left( C_{e}^{(0)} \right)_{ee} = -\frac{I_0}{q^2} \int f^{(i)} - \frac{2}{3} \int \delta \frac{f^{(c)}}{\partial q}.
$$

To a good approximation we can write

$$
I_0 \sim \left( \frac{\lambda_i}{2} \right)^{3/2} N,
$$

where $N$ is the ion number density, and

$$
I_1 \sim \frac{4\pi}{q^2} \int_0^\infty \int f^{(i)} Q^3 dQ \sim \frac{3\lambda_i}{4q^4} N \bar{U},
$$

where $M_{e} \bar{U}$ represents the average ion drift momentum. The collision term for $e-i$ effects may be written as
\[
\frac{\lambda^2}{\varrho^2} (C_{e}^{(i)})_{ei} = \left( C_{e}^{(i)} + \lambda i \nabla (C_{e}^{(i)})_{ei} \right)
\]
\[
= -N \left( \frac{\lambda^2}{2 \varrho^2} \left( \frac{1}{\varrho^3} + \frac{1}{\varrho \lambda e} \right) \right) f^{(i)}(Y)
\]
\[
- N \frac{\lambda^2}{4 \varrho \lambda e} \left( \frac{1}{\varrho^3} + \frac{2}{\varrho \lambda e} \right) \frac{\partial^2 f^{(i)}}{\partial \varrho^2}.
\]

(6)

The total collision term for the right hand side of (3) becomes

\[
(C_{e}^{(i)})_{ee + ei} = (C_{e}^{(i)})_{ee} + (C_{e}^{(i)})_{ei}
\]

with the appropriate contributions given by equations (4) and (6). When \( \varepsilon - \varepsilon \) effects (including the relativistic contribution) are added to the formalism the expressions become very complicated indeed and approximate methods of solution must be used. This is discussed in the next section when \( f^{(i)} \) is expanded in a series of Laguerre polynomials. If \( \varepsilon - \varepsilon \) effects are neglected an expansion is not necessary since the right hand side of (3) then does not contain derivatives of \( f^{(i)} \). The transport coefficients can be derived exactly for this case (known as the Lorentzian plasma). This is done in §7.

§3 Calculation of Interaction Terms.

In this section equation (3) is solved by expanding the electronic distribution \( f^{(i)} \) in a series of Laguerre polynomials. The approach is similar to that used by Shkarofsky [50] who applied the method to the classical Fokker-Planck term. Throughout this analysis it is assumed that the zero order (\( f^{(0)} \)) distribution is Maxwellian; this is correct providing the disturbing forces, such as
electric fields and gradients, are small. The final results of this section will be simply numerical coefficients for the interaction terms, so the reader who wishes to follow the derivation of the matrix form of the Fokker-Planck equation (50) later, can skip to it without missing anything but the actual derivation.

In the method employed in this section, we expand the reduced velocity dependence of \( f^{(i)} \) as a series of Laguerre polynomials (for definitions of these polynomials see Chapter 9) for each component. That is we write

\[
f^{(i)} = q_i \beta \sum_r \left( \frac{\varrho^2}{\lambda} \right)^r L_r^{3/2}(\varrho^2),
\]

where \( \varrho = \frac{1}{2} \lambda \beta \), \( \lambda = \frac{mc^2}{kT} \), and \( \beta = \frac{\lambda}{2\pi} \). \( T \) is the temperature and \( L_r^{3/2} \) is the Laguerre polynomial of order \( 3/2 \). Henceforth the superscript \( 3/2 \) is to be understood. These polynomials are closely related to the Sonine polynomials of Chapman and Cowling and are chosen since the corresponding expansion converges well for small deviations from the Maxwellian function. Furthermore these polynomials represent an orthogonal set in the interval \([0, \infty]\).

The generating function for these polynomials is

\[
(1 - \varrho^2) \exp \left[ \frac{-\varrho^2 W}{1 - \varrho^2} \right] = \sum_{r=0}^{\infty} \varrho^r L_r(W)
\]

together with the orthogonality condition

\[
\int_0^{\infty} W L_r(W) L_s(W) dW = \frac{\Gamma \left( \frac{r + 5/2}{2} \right)}{\Gamma \left( \frac{r + 1}{2} \right)} \delta_{rs},
\]

where \( \Gamma(x) \) is the Gamma function. For reference, the first three polynomials are
The procedure is to substitute the expansion (7) into the integrodifferential equation for \( f^{(0)} \) and use the above orthogonality condition of the polynomials to determine the \( \beta \) -coefficients. Rewriting (3) in terms of the scaled velocity \( q \) gives

\[
q \sigma (1 - \frac{q^2}{\lambda_e}) \nabla f^{(0)} + \frac{e E}{n \sigma} \frac{\partial f^{(0)}}{\partial q} = (C^{(0)}) \sigma + e \epsilon_i , \tag{10}
\]

where \( \sigma = \sqrt{\frac{2 k T}{m}} \). The zero order distribution \( f^{(0)} \) is given by

\[
f^{(0)} = n \beta^2 \left( 1 - \frac{15}{4 \lambda_e} + \frac{2}{2 \lambda_e} \right) e^{-q^2} . \tag{11}
\]

Now in transport theory both \( \Gamma \) and \( \Gamma \) are slowly varying functions of \( \Gamma \) and \( \Gamma \), so that

\[
q \sigma (1 - \frac{q^2}{\lambda_e}) \nabla f^{(0)} = n q \sigma \beta^2 e^{-q^2} \left\{ \frac{\nabla \rho}{1} \left[ -\frac{3}{2} + q^2 \right] + \lambda_e^{-1} \left( -\frac{15}{16} - \frac{3}{8} q^2 - \frac{9}{4} q^4 + \frac{1}{2} q^6 \right) \right\}
\]

\[
+ \frac{\nabla n}{n} \left[ 1 + \lambda_e^{-1} \left( -\frac{15}{8} - q^2 + \frac{1}{2} q^4 \right) \right] ,
\]

and

\[
\frac{e E}{n \sigma} \frac{\partial f^{(0)}}{\partial q} = \frac{e E}{n \sigma} n \beta^2 e^{-q^2} \left\{ -2 q + \lambda_e^{-1} \left[ \frac{15}{4} q + 2 q^2 - q^3 \right] \right\} .
\tag{12a,b}
\]

The sum of equations (12a,b) represents the left hand side of the
equation. Henceforth the vector symbols will be dropped on the understanding that the expressions represent any one of the three components appropriate to the vector equation (10). If we multiply by the expression

$$\frac{\mathbf{e} \cdot \mathbf{E}^s(\varepsilon) \, d\varepsilon}{\Gamma(s) \, \eta \, \beta^3}$$

(13)

where $\mathbf{E} = q^2$, and integrate over $\varepsilon$, applying the orthogonality condition (9), we can obtain the values of the left hand side for $s = 0, 1, 2 \ldots$ etc and re-express these as a column matrix. For example, consider the terms involving the electric field (12b), namely

$$C_s = \frac{e \mathbf{E}}{m \sigma \Gamma(s)} \int_0^\infty e^{-\varepsilon^{3/2}} \varepsilon^{s-1/2} \mathbf{E}^s(\varepsilon) \left[ -2 + \lambda_e^{-1}(\varepsilon + 2\varepsilon - \varepsilon^2) \right] d\varepsilon.$$

We multiply both sides by $\alpha^s$ and sum over $s$, applying the generating relation (8) with $\xi$ replaced by $\alpha$. The result is

$$\sum_{s=0}^\infty \alpha^s C_s = \frac{e \mathbf{E}}{m \sigma \Gamma(s)} \int_0^\infty \varepsilon^{3/2} e^{-\varepsilon/\alpha} \varepsilon^{s-1} \left[ -2 + \lambda_e^{-1}(\varepsilon + 2\varepsilon - \varepsilon^2) \right] d\varepsilon$$

$$= \frac{e \mathbf{E}}{m \sigma \Gamma(s)} \left\{ -2 + \lambda_e^{-1}\left(\frac{25\alpha}{2} - \frac{35}{4} \alpha^2\right) \right\}.$$  

By equating powers of $\alpha$, the appropriate values of $C_s$ may be found. For the above example, we may express the result in column matrix form as

$$\frac{e \mathbf{E}}{m \nu} \begin{bmatrix} -2 \\ 0 \\ \vdots \end{bmatrix} + \frac{e \mathbf{E}}{m \sigma \lambda_e} \begin{bmatrix} 25/2 \\ -35/4 \\ 0 \end{bmatrix}.$$  

(14)
A similar procedure for the gradient \( \nabla_r \) -terms yields the matrix representation

\[
\begin{bmatrix}
\frac{\sigma V}{\pi} + \frac{\sigma V}{\lambda c^2} & 0 \\
-\frac{25}{4} & 35/8 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

The sum of the vectors (14) and (15) is the equivalent matrix representation of the left hand side of the \( f^{(l)} \) equation after application of the orthogonality condition (9). The results are correct to first order in relativistic effects.

If we denote the integrodifferential operator which represents the collision term by \([\circ]_0\), then we may write

\[
(\mathbb{C}^{(l)})_{ee+\varepsilon l} = [0] f^{(l)}_0.
\]

We will consider only one component of \( f^{(l)} \) and drop the vector symbols on the \( p \)-coefficients. Let us consider the \( T \)-component of \( f^{(l)} \) and operate on it with \([\circ]_0\). We have

\[
[0] f^{(l)}_T = [0] \sqrt{\beta} \beta^3 e^{-\sqrt{\beta}} p_r L_r(q^2),
\]

where the suffix \( T \) on \( f^{(l)} \) indicates the \( T \)th term in the series (7). Next we multiply by (13) and integrate over \( \varepsilon (= q^2) \) obtaining

\[
\int_0^\infty [0] f^{(l)}_T \mathcal{E} L_r(\varepsilon) d\varepsilon = \int_0^\infty \frac{\mathcal{E} L_r(\varepsilon)}{\Gamma(\frac{3}{2})} \left[0\right] \{q^2 e^{-\sqrt{\beta}} L_r(q^2)\} d\varepsilon.
\]

We can denote the integral on the right hand side of (17) by \(-H_{rs}\) and hence we have
If the $l$'s are written in column matrix form, then the $H_{rs}$ form a matrix with $S$ representing rows and $T$ columns. The $H_{rs}$ are made up of individual contributions from electron-electron and electron-ion collisions; that is we can split the $H_{rs}$ into two parts namely

$$H_{rs} = H_{rs}^{ee} + H_{rs}^{ie}.$$  (19)

To solve for the $H_{rs}$ coefficients we introduce the mathematical device used by Chapman and Cowling \(^4\), p. 150 which allows for all $H_{rs}$ terms by performing one integration. Essentially one forms the sum

$$\sum_{r} \sum_{s} \alpha^S \xi^r \xi^s H_{rs} = -\sum_{r} \sum_{s} \alpha^r \xi^s \int_{0}^{\infty} \frac{\xi L_{S}(\xi)}{\Gamma(\xi)} \left[ O \right] \left\{ \int \xi e^{-\xi} d\xi \right\} d\xi,$$

and then applies the generating relation (8). Remember that the operator is linear so that $\alpha^S$ may be taken with $L_{S}$ and $\xi^r$ with $L_{r}$. (8) is then applied first for $\sum L_{r} L_{r}$ and then for $\sum \alpha^r L_{r}$. The result is

$$-\sum_{r} \sum_{s} \alpha^S \xi^r H_{rs} = \left( L_{0} - \sum \xi^r L_{r} \right) \int_{0}^{\infty} \frac{\alpha \xi}{\Gamma(\xi)} \left[ O \right] \left\{ \int \xi e^{-\xi} d\xi \right\} d\xi.$$  (20)

For a given operator $\left[ O \right]$ , the integral on the right hand side of (20) can be performed to yield a function of $\alpha$ and $\xi$. A corresponding power series expansion will yield the $H_{rs}$ values since
this must be identical to the left hand side. The classical computations have been carried out by Shkarofsky et al. For completeness they will be briefly given alongside the relativistic calculations.

**INTERACTION TERMS: (1) ION CONTRIBUTION.**

This is the simplest to consider since no derivatives of \( f^{(i)} \) are involved. The form of the collision term is given by (6) and consists of two parts: the first involving \( f^{(i)} \) due to direct collision, and the second due to ion-recoil which involves only \( f^{(e)} \). For the \( f^{(o)} \) term we have

\[
-\frac{\nu_{ei}}{c^2 m_i} \frac{\lambda_e^2}{4 \sqrt{\mu}} \bar{\varphi}^0 \left( \frac{1}{\varphi^3} + \frac{2}{\nu_e \varphi^4} \right) \frac{\partial f^{(o)}}{\partial \varphi},
\]

as the ion-recoil contribution to \( e-i \) collisions. Again, we have dropped the vector symbol on \( \bar{\varphi} \) on the understanding that this refers to any one of the components. The expression for \( f^{(e)} \), (11), is substituted into (21) and terms up to order \( \lambda_e^{-1} \) retained. Next we multiply by (13) and integrate over \( \varepsilon \) obtaining the result

\[
\frac{2c \bar{U}}{\sigma} \left\langle \nu_{ei} \right\rangle \int_0^\infty \left( 1 - \frac{15}{8 \lambda_e} + \frac{\varepsilon}{\lambda_e} + \frac{\varepsilon^2}{2 \lambda_e} \right) e^{-\varepsilon L_i(\varepsilon)} d\varepsilon,
\]

where we have set \( \sigma^2 = \frac{2kT}{m} \) and defined

\[
\left\langle \nu_{ei} \right\rangle = \frac{4\pi N \nu_{ei} \beta^3}{3c^3}, \quad \beta^2 = \left( \frac{\lambda_e}{2k} \right).
\]

\( \left\langle \nu_{ei} \right\rangle \) is an effective electron-ion collision frequency as defined by Shkarofsky, p. 289. (22) can be written in the form

\[
\frac{2c \bar{U}}{\sigma} \bar{H}^{ei}_S
\]
which defines the elements \( H_{s}^{e} \). To compute these coefficients we form the sum

\[
\sum_{s} \alpha^{s} H_{s}^{e} = \langle \gamma_{e}^{s} \rangle \int_{0}^{\infty} \left( 1 - \frac{15}{8\lambda e} + \frac{\epsilon}{\lambda e} + \frac{\xi^{2}}{2\lambda e} \right) e^{-\frac{\epsilon}{\lambda e}} d\epsilon,
\]

where we have applied the generating relation (8) with \( \epsilon \) replaced by \( \xi \). Using the result

\[
\int_{0}^{\infty} e^{-\xi^{n} w} dw = \frac{\Gamma(n+1)}{\xi^{n+1}},
\]

for \( n = 0, 1, 2 \); (24) becomes

\[
\sum_{s} \alpha^{s} H_{s}^{e} = \langle \gamma_{e}^{s} \rangle \left\{ (1 - \xi)^{-\frac{3}{2}} + \frac{(1 - \xi)^{-\frac{1}{2}}}{8\lambda e} \right\} \left( 1 - 24\xi + 8\xi^{2} \right).
\]

In column matrix form up to \( S = 4 \) we have

\[
H_{s}^{e} = \langle \gamma_{e}^{s} \rangle \begin{bmatrix}
1 \\
\frac{3}{2} \\
\frac{15}{4} \\
\frac{35}{16} \\
\frac{315}{128}
\end{bmatrix}
+ \frac{\langle \gamma_{e}^{s} \rangle}{\lambda e} \begin{bmatrix}
\frac{1}{2} \\
-45/4 \\
-209/64 \\
-493/128 \\
-4435/1024
\end{bmatrix}.
\]

The ion-recoil part of the electron-ion interaction term may be written as

\[
\frac{2C U}{\sigma} H_{s}^{e},
\]

with the \( H_{s}^{e} \) given by (26) above, correct to first order relativistic effects.

Next the term corresponding to direct ion-electron interaction must be considered. It reads, from equation (6),
Using the expression (23) for $\langle \gamma e_i \rangle$ and the expansion (7) for $\int_0^\infty$, we multiply (27) by (13) and again integrate over $E$ to obtain

$$-\sum_r H_{rs}^e e_i p_r,$$

where

$$H_{rs}^e = \langle \gamma e_i \rangle \int_0^\infty (1 + \frac{E}{\lambda e}) e^{-E} \lambda_r(E) \lambda_s(E) dE.$$  

To evaluate the $H_{rs}^e$ it is simplest to form the sum

$$\sum_r \sum_s \alpha^s \gamma^r H_{rs}^e,$$

and apply the generating relation (8). Straightforward integration over $E$ using (25) for $\lambda = 0, 1$ yields the result

$$\sum_r \sum_s \alpha^s \gamma^r H_{rs}^e = \langle \gamma e_i \rangle \left\{ (1 - \xi)^{-\frac{1}{2}}(1 - \alpha)^{-\frac{1}{2}}(1 - \alpha_1)^{-1} \right.$$  

$$+ \lambda e \left[ (1 - \xi)^{-1}(1 - \alpha)^{-1}(1 - \alpha_1)^{-1} \right] \right\},$$

To find the $H_{rs}^e$, one can expand (29) in a Taylor or Binomial series and compare terms in $\alpha^s \gamma^r$ etc. The coefficients are put into matrix form with $S$ representing rows and $T$ columns. Explicitly up to $T = S = 4$, we have

$$-\frac{\gamma e_i}{\epsilon^{\frac{1}{2}}} \left( \frac{\lambda e}{2} \right)^{\frac{1}{2}} N \left( \frac{1}{q^3} + \frac{1}{q^3 \lambda e} \right) f_1^{(1)}.$$
\[ \mathbf{A}^e = \begin{bmatrix} 1 & 3/2 & 15/4 & 35/16 & 31/128 \\ 1/2 & 13/4 & 69/16 & 165/32 & 150/256 \\ 15/2 & 69/16 & 433/64 & 1077/128 & 10005/1024 \\ 35/16 & 1077/128 & 2957/256 & 2257/2048 & 2257/16384 \\ 315/128 & 15005/256 & 10005/1024 & 2257/2048 & 2257/16384 \end{bmatrix} \]

and


where

\[ H_{1s}^e = A_{1s}^e + \lambda_e^{-1} D_{1s}^e. \]

The complete e-i collision term in matrix form becomes

\[ \left( \sum_{\Gamma} \rho_\Gamma H_{\Gamma S}^e + \frac{2e}{c} \mathbf{H}_S^e \right), \]

with the matrix \( H_{1s}^e \) given by (30c) to \( \Gamma = S = 4 \), and the column vector \( H_S^e \) given by (26) again up to \( S = 4 \).

(2). ELECTRON CONTRIBUTION.

This is the most complicated of the interaction terms to compute since it consists of the expressions (4a,b). The classical operator has 14 terms and the relativistic part some 24 terms! In the classical expression one writes \( f^{(c)} = n \beta e^{-x^2} \) and uses this value in the \( I^0 \) and \( J^0 \) integrals. Terms of order \( \lambda_e^{-1} \) arise both from the classical operator (4a) and the relativistic expression (4b).
In the former one substitutes the $\lambda^{-1}$ terms of $f^{(c)}$ given in (11). In the latter the classical expression for $f^{(c)}$ is sufficient, correct to order $\lambda^{-1}$. The various contributions will be taken separately starting with the classical terms.

**CLASSICAL $\mathcal{C}-\mathcal{C}$ INTERACTION TERMS:**

The expressions we are about to derive have been given by Shkarofsky et al (2, p. 320 et seq) but we will briefly rederive them here for completeness. The same procedure is needed to develop the relativistic terms later, although the computations are much more involved owing to the lengthy nature of the integrodifferential operator.

The classical operator is given by equation (4a) with $f^{(c)} = n \beta^3 e^{-\gamma^2}$. According to (20), rather than work with the quantity $\varphi e^{-\gamma^2} (\psi')$ (that is the $r$th component of the expansion of $f^{(c)}$), we use the simpler quantity $\sqrt{\lambda} e^{-\frac{\gamma^2}{\lambda}}$, after summation over $r$. Hence in (4a) we place

$$f^{(l)} = \sqrt{\lambda} e^{-\frac{\gamma^2}{\lambda}} , \quad f^{(c)} = n \beta^3 e^{-\gamma^2}.$$

For the $I'$ and $J'$ integrals we use the Tables 5-1 and 5-2. All the $I^o$ terms contribute an error function term of the form $\Phi (\frac{e^{\gamma^2}}{\xi})$. For the $I'$ and $J'$ integrals one can use Tables 5-1 and 5-2 with the substitution $\vartheta = \frac{e^{\gamma^2}}{\xi}$. The error function associated with these terms has the form $\Phi [\frac{e^{\gamma^2}}{\xi} (1 - \xi)^{\frac{1}{2}}]$. For example we have

$$I'_3 + J'_2 = \pi^{\frac{3}{2}} \left\{ \frac{-5}{\sqrt{\lambda}} e^{-\frac{\xi^2}{\lambda}} (1 - \xi)^{\frac{1}{2}} \left[ 1 + \frac{3}{2} \xi (1 - \xi) \right] \right\} + \frac{15}{4 \sqrt{\xi^2}} (1 - \xi)^{\frac{3}{2}} \Phi \left( \frac{\xi^2}{1 - \xi^2} \right),$$

and
For the $I^o$ and $J^o$ terms we can derive terms like

$$I_{1}^{o} + J_{1}^{o} = \kappa \sum_{n} \beta \left\{ \frac{3}{\sqrt{\kappa}} \phi(\epsilon) - \frac{3}{\sqrt{\kappa}} e^{-\epsilon^2} \right\},$$

involving the error function $\phi(\epsilon)$. The electron-electron contribution to the classical operator can be grouped into three terms with respective multipliers $e^\epsilon e^{-\epsilon^2}$, $\phi(\epsilon)$ and $\phi\left(\frac{\epsilon}{1-\xi}\right)$, as follows:

$$[0] \left( \frac{\epsilon}{\sqrt{\kappa}} e^{-\epsilon^2} \right) = \kappa \sum_{n} \beta \left\{ \frac{e^{-\epsilon} e^{-\frac{\epsilon}{1-\xi}}}{\sqrt{\kappa}} \left[ \frac{4(1-\xi e^\epsilon)}{\xi e(1-\xi)} \right] + 2 \right\}$$

$$+ \frac{2(1-\xi)}{e} \left( 1 - 2x - 3(1-\xi) \right)$$

$$+ \phi(\epsilon) \left[ \frac{e^{-\epsilon}}{\sqrt{\kappa}} \right] \left[ \frac{-2(1-\xi) + 2\xi e^\epsilon}{\xi e(1-\xi)} \right]$$

$$+ e^{-\epsilon} \phi\left[\frac{\epsilon}{1-\xi}\right] \left[ \frac{3(1-\xi)}{\xi} \right].$$

(33)

where $\kappa$ is the electron density and $\beta = \frac{\alpha \sqrt{\kappa}}{2\kappa}$. To complete the calculation we multiply (33) by

$$\frac{-(1-\xi)^{-\frac{5}{2}}}{\Gamma\left(\frac{5}{2}\right)} e^{-\frac{\alpha \epsilon}{1-\xi}} d\epsilon,$$
and integrate over $\mathcal{E}$. It is convenient at this stage to define an effective electron-electron collision frequency by

$$\langle \nu_{ee} \rangle = \frac{\kappa^{3/2} \beta^{3} \nu_{ei}}{\Gamma(\xi) \xi^{3} c^{3}} = \frac{\langle \nu_{ei} \rangle \kappa}{\xi^{2} N}.$$  

To perform the integrations, the results of (104a,b) (Chapter 5) are required. The result, in order of the terms in (33) is

$$-\frac{1}{\langle \nu_{ee} \rangle} \sum_{r} \sum_{s} \alpha^{s} \beta^{r} \beta_{rs}^{ee} = 3(2-\alpha-\xi)^{-\frac{3}{2}} \left[ - (1-\xi)^{2} + 2 - (1-\xi)^{2} \right]$$

$$+ (1-\alpha)^{3}(1-\xi)^{4}(2-\alpha-\xi)^{3} \left[ 2(1-\xi)^{2} + (1-\xi)^{2} - 3(1-\xi)^{2} \right]$$

$$- 2(1-\xi)^{2}(1-\alpha)^{4}(1-\xi)^{2}(2-\alpha-\xi)^{3} \left[ \frac{1}{2} \right]$$

$$+ 2 \alpha^{3}(1-\alpha)^{3}(1-\xi)^{4}(2-\alpha-\xi)^{3/2}$$

$$+ 3 \alpha^{5}(1-\alpha)^{6}(1-\xi)^{4}(2-\alpha-\xi)^{3/2}$$

$$+ (2-3\xi)^{4}(1-\alpha)^{2}(2-\alpha-\xi)^{-3/2}.$$  

In (35) $\beta_{rs}^{ee}$ represent the classical $\mathcal{E}$-$\mathcal{E}$ interaction coefficients corresponding to $\beta_{rs}^{ei}$ for $\mathcal{E}$-$\mathcal{E}$ effects given previously. To combine the terms algebraically it is simplest to add terms 7, 4 and 8, terms 1 and 9 and terms 10, 5, 6, 2 and 3 and then add the three groups. The final result gives the symmetric function

$$\frac{1}{\langle \nu_{ee} \rangle} \sum_{r} \sum_{s} \alpha^{s} \beta^{r} \beta_{rs}^{ee} = \frac{\xi^{3}(8-4\alpha-4\xi-2\alpha^{2}+2\alpha^{2}+2\alpha^{2}+3\xi^{3} \alpha^{2})}{(1-\xi)^{2}(2-\alpha-\xi)^{3/2}}.$$
Expanding (36) up to \( r = s = 4 \) gives a 5 \( \times \) 5 matrix for the \( A_{rs}^{ee} \), which can be explicitly written as

\[
A_{rs}^{ee} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & \frac{3}{4} & \frac{15}{32} & \frac{35}{128} \\
0 & \frac{3}{4} & \frac{45}{16} & \frac{309}{128} & \frac{885}{512} \\
0 & \frac{15}{32} & \frac{309}{128} & \frac{5657}{1024} & \frac{20349}{4096} \\
0 & \frac{35}{128} & \frac{865}{512} & \frac{20349}{4096} & \frac{144799}{16384}
\end{bmatrix}.
\] (37)

The classical contribution to the electron-electron terms becomes

\[
\sum_r P_r A_{rs}^{ee}.
\] (38)

The interesting feature of (37) are the zeroes in the first row and column of the matrix. As will be shown later, this is a direct consequence of the fact that electron-electron collisions do not contribute to the overall momentum balance.

**Relativistic e-e interaction terms:**

These come from two sources. Firstly equation (4a) with

\[
f^{(4)} = n \beta^3 E \left( -\frac{15}{8\alpha_e} + \frac{E^2}{2\lambda_e} \right) e^{-E}
\]

and secondly from (4b) with

\[
f^{(4)} = n \beta^3 E e^{-E}.
\]

For equation (4a) the term \(-n \beta^3 \frac{15}{8\alpha_e} e^{-E}\) is easily taken into account, the matrix elements being equal to \(-\frac{15}{8\alpha_e}\) times those given in (37). The contribution to the collision operator becomes

\[
\frac{15}{8\alpha_e} \sum_r P_r A_{rs}^{ee}.
\] (39)

Next we must consider the term \( n \beta^3 \frac{E^2}{2\lambda_e} e^{-E} \) for \( f^{(4)} \) in equation (4a). Again we use equation (20) to generate the matrix elements. With this form for \( f^{(4)} \) the I° and J° terms become more complex. For the I° integrals Table 5-1 is needed for \( n = 6 \) and 8, and for the J° integrals Table 5-2 is needed for \( n = 5 \). Hence in (4a) we substitute the forms
\[ f^{(i)} = \sqrt{\varepsilon} e^{-\frac{\varepsilon}{\lambda e}}, \quad f^{(j)} = n \beta^3 \varepsilon^2 e^{-\varepsilon/2\lambda e}. \tag{40} \]

For the \( I^i \) and \( J^j \) integrals these results have been given in the previous sub-section on the classical \( e-e \) interaction terms. Remember that it is only the \( f^{(j)} \) term we have altered. The respective multipliers are

\[ \frac{1}{5 \varepsilon^3} \frac{\partial^2}{\partial \varepsilon^2} \left( \frac{n \beta^3 \varepsilon^2 e^{-\frac{\varepsilon}{2\lambda e}}}{2 \lambda e} \right) = \frac{n \beta^3}{5 \lambda e^5} \left( 6 \varepsilon - 9 \varepsilon^2 + 2 \varepsilon^3 \right) e^{-\varepsilon} \]

for the term \( I_3^i + J_{-1}^i \), and

\[ \frac{1}{15 \varepsilon} \frac{\partial^2}{\partial \varepsilon^2} \left( \frac{n \beta^3 \varepsilon^2 e^{-\frac{\varepsilon}{2\lambda e}}}{2 \lambda e} \right) = \frac{n \beta^3}{15 \lambda e} \left( 2 \varepsilon^3 - \varepsilon \varepsilon^2 \right) e^{-\varepsilon} \]

for the term \(-3 I_3^i + 2 J_{-1}^i + 5 I_1^i \), with \( \varepsilon^2 = \varepsilon \).

The \( I^i \) and \( J^j \) terms involve \( \varepsilon^2 e^{-\varepsilon} \) and must be recalculated.

The first term of (4a) yields the expression

\[ I_2^o + J_{-1}^o = \frac{\pi^2 \lambda^3}{\lambda e} \left\{ \frac{105}{8 \varepsilon^2} \phi(\varepsilon^2_0) - e^{-\varepsilon} \left[ \frac{105}{8 \varepsilon^2} + \frac{27 \varepsilon^2}{4} + \frac{3 \varepsilon^3}{2} \right] \right\} \]

and the second

\[ 3 I_0^o - I_2^o + 2 J_{-1}^o = \frac{8 \pi^3 \lambda^3}{\lambda e} \left\{ \frac{45}{8} \phi(\varepsilon^2_0) - \frac{105}{16 \varepsilon^2} \phi(\varepsilon^2_0) + \frac{e^{-\varepsilon}}{\sqrt{\varepsilon^2}} \left[ \frac{105}{8 \varepsilon^2} + \frac{27 \varepsilon^2}{4} + \frac{3 \varepsilon^3}{2} \right] \right\} \]

After substituting these results into (4a) we can group the contributions into three groups with respective multipliers

\[ e^{-\varepsilon} e^{-\frac{\varepsilon}{\lambda e}}, \quad \phi(\varepsilon^2_0) \quad \text{and} \quad \phi \left[ \left( \frac{\varepsilon}{\lambda} \right)^{\varepsilon_1} \right] \]

as follows:
The next task is to multiply \((41)\) by the factor
\[
\frac{- (1 - \alpha)^{- \alpha} (1 - \zeta)^{- \alpha}}{\Gamma(\alpha)}
\]
and integrate over \(\zeta\). To perform the integrations the relations
\((104a, b)\) of Chapter 5 are required, \((104a)\) is needed for the values
\(n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \quad \text{and} \quad \frac{7}{2}\). The result \((104b)\) is required for \(n = 0, 1, 2\) with
\(s = (1 - \alpha \zeta)(1 - \alpha \zeta^2) \quad \text{and for} \quad n = 0, 1, 2\) with
\(s = (1 - \alpha \zeta)^{-1}\). The result in the order of the terms in \((41)\) can be written as
\[
- \frac{1}{\sqrt{\beta \epsilon}} \sum \sum \frac{\xi^{\alpha} \nu^{\alpha}}{\beta \epsilon} \eta_{rs} =
\]
\[
(2 - \alpha - \xi)^{3/4} \left[ \frac{35}{8} (1 - \zeta)^3 (1 - \zeta)^{-1} - 3 (1 - \zeta)^2 (1 - \zeta)^{-1} \right]
+ (2 - \alpha - \xi)^{7/4} \left[ \frac{135}{8} (1 - \zeta)^3 (1 - \zeta) - \frac{45}{8} (1 - \zeta)^2 (1 - \zeta)^{-1} \right]
+ (2 - \alpha - \xi)^{5/4} \left[ \frac{105}{8} (1 - \zeta)^2 (1 - \zeta)^{3/2} \right]
+ (1 - \alpha)^{1/2} (2 - \alpha - \xi)^{-1/2} \left[ \frac{35}{8} (1 - \zeta)^{-1} (1 - \zeta) - \frac{15}{8} (1 - \zeta)^{-1} \right]
\]
Equation (42) defines the matrix elements $\beta_{ee}^{15}$. To combine the terms algebraically it is probably easiest to first combine the terms 1, 14 and 18, terms 2, 3, 19 and 20, terms 4, 21 and 25, terms 9 and 17 with part of 5, terms 6, 16 and 13 with part of 10, terms 24 and 8 and finally the remaining parts of 5 and 10 (with numerical factor $\frac{30\pi}{7}$). These operations have the effect of removing all the factors save $(1 - \xi_1^2)$ and $(2 - \alpha_1 - \xi_1^2)$ from the denominators. One can then recombine the remaining terms, after some tedious but straightforward manipulation, to obtain

$$
-\sum_\Gamma \sum_s \frac{\alpha_s \beta_{ee}^{15}}{<\gamma_{ee}>} = \frac{1}{8} (1 - \xi_1^2)(2 - \alpha_1 - \xi_1^2)^{\frac{5}{2}} \chi_1(\alpha_1, \xi_1),
$$

(43)

where the function $\chi_1$ is given by the form

$$
\chi_1(\alpha_1, \rho) = -448\alpha^4 + 648\alpha^6 + 800\alpha^8 - 836\alpha^2 \xi_1 - 132\xi_1^3 - 477\alpha^3 \xi_1^2

- 528\alpha^3 + 152\alpha^3 \xi_1^2 + 1456\alpha^3 \xi_1 - 160\alpha^3 + 800\alpha^3 \xi_1^2 - 191\alpha^3 \xi_1

+ 152\alpha^4 + 60\alpha^4 \xi_1^2 + 185\alpha^4 \xi_1^3 - 94\alpha^4 \xi_1^2 - 324\alpha^4 \xi_1 - 126\alpha^4 \xi_1^2

+ 192\alpha^4 \xi_1 + 831\alpha^4 \xi_1^2 + 16\alpha^5 - 200\alpha^5 \xi_1 + 246\alpha^5 \xi_1^2 + 621\alpha^5 \xi_1

- 720\alpha^5 \xi_1^2 - 30\alpha^5 \xi_1^3 + 205\alpha^5 \xi_1^4 + 364\alpha^5 \xi_1^5 - 16\alpha^7 \xi_1^4 - 16\alpha^7 \xi_1^3.

+ 32\alpha^6 \xi_1 + 64\alpha^6 \xi_1^2 - 252\alpha^6 \xi_1^3 + 108\alpha^6 \xi_1^4 - 16\alpha^7 \xi_1^3 + 24\alpha^7 \xi_1^2.
We could expand (43) as it stands to obtain the \( B_{rr}^{\text{ee}} \), but these will have to be added to the elements of (39) and also to the contribution from equation (4b) which we have yet to derive. Rather than do this, we will derive the matrix coefficients for the collision operator (4b) (that is, an equivalent relation to (43)) and add all three results before expanding to obtain the total relativistic operator.

Lastly the operator (4b) has to be considered. This is the most awkward and tedious to handle since it consists of some 24 terms! The procedure is exactly as for the previous two calculations of e-e effects above. Again the essential aim is to perform the integral (20) with the operator \([O]\) given by (4b) and

\[
\phi^{(l)} = \sqrt{\epsilon} C^{-\frac{\epsilon}{1-\epsilon}}, \quad \phi^{(o)} = \pi \beta^3 e^{-\epsilon}.
\]

Note that it is sufficient to use the classical form of \( \phi^{(o)} \) to compute terms correct to order \( \lambda^{-1}_e \). Tables 5-1 and 5-2 are useful in evaluating the \( i \) and \( J \) integrals. For \( I^0 \) we require results from Table 5-1 with \( n = 2, 4 \) and 6. For \( J^0 \) Table 5-2 is useful with \( n = 1 \) and 3. Similarly the \( I^1 \) and \( J^1 \) terms may be computed incorporating the factor \( (1-\epsilon)^{-1} \). Consider the awkward fifth bracketed terms of (4b), we have the result

\[
3 I_0^0 - \frac{2}{3} I_4^0 + I_2^0 + 4 J_{-1}^0 - \frac{2}{3} J_1^0
= \pi^{3/2} \pi \beta^3 \left\{ \phi(\epsilon) \left[ 3 + \frac{3}{2\epsilon} - \frac{\epsilon}{2\epsilon^2} \right] + \frac{e^{-\epsilon}}{\sqrt{\epsilon}} \left[ \frac{\epsilon}{\epsilon^2} - \frac{1}{\epsilon^3} \right] \right\}.
\]

Similarly we can compute the sixth bracketed term to give

\[
2 I_0^0 + \frac{2}{3} I_4^0 - \frac{2}{3} I_2^0 + \frac{4}{3} J_{-1}^0 + \frac{2}{3} J_1^0
= \pi^{3/2} \pi \beta^3 \left\{ \phi(\epsilon) \left[ 2 - \epsilon + \frac{\epsilon}{2\epsilon^2} \right] + \frac{e^{-\epsilon}}{\sqrt{\epsilon}} \left[ 1 - \frac{5}{\epsilon^3} \right] \right\}.
\]
Next consider the second and third bracketed terms of equation (4b). We have, using the Tables 5-1 and 5-2 for \( n = 6 \) and \( 8 \) and \( n = 1 \) and \( 3 \) respectively, the results

\[
\frac{8}{15} I_3 - \frac{17}{35} I_5 - \frac{2}{35} J_{-2} + \frac{2}{15} J_1
\]

\[
= \pi^{3/4} \left\{ \phi \left[ \left( \frac{\xi}{1-\xi} \right)^{n/4} \right] \left[ \frac{2(1-\xi)^n}{\xi^{3/4}} - \frac{51}{8} (1-\xi)^n \right] \right. \\
- \frac{e^{-\frac{\xi}{\sqrt{\xi}}}}{\sqrt{\pi}} \left[ \frac{51}{4\xi^3} (1-\xi)^4 - \frac{9}{2\xi} (1-\xi)^3 - (1-\xi)^2 \right] \left\}.
\]

and

\[
- \frac{7}{15} I_3 + \frac{17}{35} I_5 + \frac{17}{35} J_{-2} - \frac{7}{15} J_1
\]

\[
= \pi^{3/4} \left\{ \phi \left[ \left( \frac{\xi}{1-\xi} \right)^{n/4} \right] \left[ \frac{7}{4\xi^{3/4}} (1-\xi)^n + \frac{51}{\xi^{n/4}} (1-\xi)^n \right] \right. \\
- \frac{e^{-\frac{\xi}{\sqrt{\xi}}}}{\sqrt{\pi}} \left[ \frac{51}{4\xi^3} (1-\xi)^4 + \frac{\xi}{\xi} (1-\xi)^3 + 2(1-\xi)^2 \right] \left\}.
\]

The first, fourth and last terms of (4b) can also be computed but are much easier. The final result can be grouped into three contributions (as for previous \( \epsilon - \epsilon \) calculations) with respective multipliers \( e^{-\xi} \), \( \phi(\xi^m) \) and \( \phi \left[ (\epsilon / (1-\xi))^{\nu_2} \right] \) as follows:

\[
[0] \left( \sqrt{\xi} e^{-\xi^{1-\xi}} \right)
\]

\[
= \frac{\pi^{3/4} \lambda^3}{\lambda e} \left\{ \frac{e^{-\xi} \xi^{-\nu_2} \nu_2}{\sqrt{\pi}} \left[ -8 \xi - \frac{1}{\xi} - \frac{20}{\xi(1-\xi)} - \frac{6}{(1-\xi)^3} + \\
+ \frac{20}{(1-\xi)^3} + \frac{51}{\xi} (1-\xi)^4 + \frac{(1-\xi)^3}{\xi} \right] \\
+ \frac{4 \xi}{(1-\xi)^3} + \frac{51}{\xi} (1-\xi)^4 + \frac{(1-\xi)^3}{\xi} \right\}.
\]
Again according to (20) we multiply by
\[
- \frac{\frac{1}{2}(1-\xi)^2 + 18(1-\xi)^3 + 2(1-\xi)^2 + 4\xi(1-\xi)^2}{\xi} \]
\[
+ \exp \left( -\frac{\xi}{1-\xi} \right) \left[ \frac{1}{2} - \frac{(1-\xi)}{1-\xi} + 8\xi(1-\xi) - \frac{5}{2}(1-\xi)^2 \right] \]
\[
+ \exp \left( \frac{\xi}{1-\xi} \right) \left[ \frac{5}{2} + \frac{1}{2\xi} - \frac{4\xi}{(1-\xi)^2} - \frac{7}{\xi(1-\xi)} + \frac{10}{(1-\xi)^3} - \frac{10}{(1-\xi)^4} \right],
\]
and integrate over \( \xi \). The integrations are performed using the relations (104a,b) of Chapter 5. (104a) is needed with \( n = \frac{1}{2}, \frac{3}{2} \) and \( \frac{5}{2} \). Relation (104b) is needed to handle the error function \( \phi \).

It is required for \( n = 0, 1 \) and \( 2 \) with \( S = (1-\alpha)^{-1} \) and for \( n = 0 \) and \( 1 \) with \( S = (1-\alpha)^{-1} \). The result, in the order of the terms in (44), can be written as

\[
\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \alpha^s \xi^r C_{r,s}^{ee}
\]

\[
= (2-\alpha-\xi)^{-\frac{7}{2}} \left[ \frac{15}{2}(1-\alpha)(1-\xi)^{3} + \frac{15}{2}(1-\alpha)(1-\xi)^{-1} - 15(1-\alpha)(1-\xi) \right]
\]
\[
+ (2-\alpha-\xi)^{-\frac{3}{2}} \left[ \frac{3}{2}(1-\xi)^{2} + \frac{21}{2}(1-\xi)^{3} + 15(1-\xi)^{2} - \frac{9}{2}(1-\xi) \right]
\]
\[
+ (2-\alpha-\xi)^{-\frac{1}{2}} \left[ \frac{1}{2}(1-\xi)^{2}(1-\alpha)^{-1} + \frac{51}{2}(1-\alpha)^{(1-\xi)^{3}} - 10(1-\alpha)(1-\xi)^{-1} \right]
\]
\[
- \frac{1}{2}(1-\alpha)(1-\xi)^{-1} - 2(1-\xi)(1-\alpha)^{-1}
\]
\[ + (1 - \alpha \xi) (2 - \alpha \xi) [\frac{1}{2} (1 - \alpha) (1 - \xi) + 10 (1 - \xi) (1 - \alpha)^{-1}] \]
\[ + (1 - \alpha \xi)(2 - \alpha \xi)^{3} [\frac{15}{2} - \frac{11}{2} (1 - \xi) + 15 (1 - \xi)^{-2}] \]
\[ + (1 - \alpha \xi)^{2} (2 - \alpha \xi)^{3} [5 (1 - \alpha)(1 - \xi) - 7 (1 - \alpha) - 10 (1 - \alpha)(1 - \xi)^{-1}] \]
\[ + (1 - \alpha \xi)^{4} (2 - \alpha \xi)^{3} [\frac{45}{2} (1 - \alpha)(1 - \xi)^{-1} - 15 (1 - \alpha)] \]
\[ + (1 - \alpha \xi)^{2} (2 - \alpha \xi)^{3} [30 (1 - \alpha)^{2} - 20 (1 - \alpha)^{2}(1 - \xi)] \]
\[ + (1 - \alpha \xi)^{3} (2 - \alpha \xi)^{3} [12 (1 - \alpha)^{3}(1 - \xi) - 8 (1 - \alpha)^{3}(1 - \xi)^{2}] \]
\[ + (2 - \alpha - \xi)^{3} [2 (1 - \alpha)^{-1} - \frac{1}{2} (1 - \xi)(1 - \xi)^{-1} - \frac{5}{2} (1 - \xi)^{3}(1 - \xi)^{-1}] \]
\[ + (2 - \alpha - \xi)^{3} [12 (1 - \xi)^{2} - 8 (1 - \alpha)(1 - \xi)] \]  

\[ \text{(45)} \]

Equation (45) defines the matrix elements \( C_{rs}^{ee} \). To combine the terms algebraically it is easiest to first combine the terms 17 and 12, terms 10, 14 and 20, terms 13 and 11, terms 28 and 8, and terms 29 and 9. Further one can combine the terms 6 and 17 and add to 21. Part of 16 can then be added to the resultant term 21 (with numerical factor \( \frac{15}{2} \)). Part of 2 (with numerical factor \( \frac{4}{1} \)) can be combined with 7 and the remainders of 2 and 16 recombined. These operations again have the effect of removing all the factors except \((1 - \alpha \xi)^{3}\) and \((2 - \alpha - \xi)^{3}\) from the denominators. Recombining the remaining terms gives the final form

\[ \frac{-1}{\langle \text{ee} \rangle} \sum_{r} \sum_{s} \alpha^{s} \beta^{r} C_{rs}^{ee} \chi_{2}(\alpha, \xi) \]

\[ \chi_{2}(\alpha, \xi) = \frac{1}{8} \left[ \frac{1}{(1 - \alpha)(1 - \xi)} \left( 2 - \alpha - \xi \right) \right] \]

\[ \text{(46)} \]
where \( \chi_1(x, t) \) is given by

\[
\chi_1(x, t) = -256x - 704x^2 + 832x^3 + 2336x^4 + 1184x^5 - 3192x^6 - 768x^7
\]

\[
-1840x^6 - 2128x^7 + 2020x^8 + 508x^2 + 2100x^3 + 704x^4 + 420x^5 + 1616x^6 + 508x^7 + 2100x^8
\]

\[
+ 272x^4 + 1256x^5 + 808x^6 + 420x^7 + 1616x^8
\]

\[
- 2716x^4 + 1232x^5 + 112x^6 + 32x^7 - 272x^8
\]

\[
- 112x^5 - 840x^6 - 780x^7 - 112x^8
\]

\[
- 112x^6 - 840x^7 - 780x^8
\]

\[
+ 112x^6 + 780x^7 - 780x^8
\]

\[
+ 1620x^6 + 780x^7 - 840x^8 + 192x^9 + 72x^10
\]

\[
- 1620x^6 + 712x^7 + 868x^8 - 480x^9 + 84x^10
\]

\[
+ 156x^6 - 144x^7 + 64x^8 + 136x^9 + 64x^{10}
\]

From equations (39), (43) and (46) the total relativistic contribution to e-e interaction is

\[
\sum_r \frac{p_r}{\lambda_e} \left\{ \frac{15}{8} A_{rs}^{ee} - B_{rs}^{ee} - C_{rs}^{ee} \right\}.
\]

The generating relations for the A, B and C elements are given by equations (36), (43) and (46) respectively. We can add the relations to obtain

\[
\frac{15}{8} A_{rs}^{ee} - B_{rs}^{ee} - C_{rs}^{ee} = -D_{rs}^{ee},
\]

and

\[
- \frac{1}{2}\langle V_{ee} \rangle \sum_r \sum_s \alpha^s \gamma^r D_{rs}^{ee}
\]

\[
= \frac{15}{8} (1 - 4\alpha)^2 (2 - \alpha - \xi) \epsilon_x \left[ 8 - 4 \epsilon_x - 4 \alpha - 2 \alpha^2 + 2 \epsilon_x - \epsilon_x \right]
\]
and the \( \chi_d \) function is given by

\[
\chi_d(\alpha, \xi) =
\]

\[
-704\xi + 424\xi^2 + 4632\xi^3 + 1632\xi^4 + 760\xi^5 + 1484\xi^6 - 4437\xi^7 + 7296\xi^8
\]

\[
-2128\xi^9 - 1928\xi^{10} + 2160\xi^{11} + 5032\xi^{12} + 276\xi^{13} + 424\xi^{14} + 2048\xi^{15}
\]

\[
+808\xi^{16} + 1096\xi^{17} - 1488\xi^{18} - 3704\xi^{19} + 2256\xi^{20} + 2494\xi^{21}
\]

\[
-48\xi^{22} - 628\xi^{23} - 112\xi^{24} - 828\xi^{25} - 780\xi^{26} - 264\xi^{27} + 2764\xi^{28}
\]

\[
-408\xi^{29} + 1080\xi^{30} - 1332\xi^{31} + 488\xi^{32} + 456\xi^{33} + 72\xi^{34}
\]

\[
+712\xi^{35} - 368\xi^{36} - 948\xi^{37} - 1028\xi^{38} + 520\xi^{39}
\]

\[
+ 804\xi^{40} - 189\xi^{41} - 48\xi^{42} - 104\xi^{43} - 14\xi^{44} + 316\xi^{45}
\]

\[
+ 136\xi^{46} - 108\xi^{47} + 16\xi^{48} - 24\xi^{49}
\]

Expanding up to \( r = s = 4 \) yields a 5 \times 5 matrix representation for \( D_{rs}^{ee} \) with \( s \) corresponding to columns and \( r \) to rows. In fact \( D_{rs}^{ee} \) takes the form

\[
D_{rs}^{ee} = \frac{\sqrt{2} \langle \phi_{ee} \rangle}{256} \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
704 & 1160 & -260 & -1453 & -8645/32 \\
-48 & 1766 & 3513 & 6465/16 & -46479/128 \\
-198 & 1591 & 60710 & 985917 & 2244957 \\
-345 & -101 & 99030 & 4485855 & 61147333/4096
\end{bmatrix}
\]

(49)

We note the completely unsymmetrical character of (49) as compared with its classical counterpart (37), and also the zeroes in the first
row. This form of the first row is a direct consequence of the fact that electron-electron collisions do not contribute to the overall momentum balance. This will be demonstrated a little later.

Having computed all the interaction terms in the Boltzmann equation for \( \varepsilon^{(i)} \), we are now able to express this in matrix form as

\[
- \left\{ \Lambda_{e}^{e} + \Lambda_{e}^{i} + \lambda_{e} \left[ \begin{array}{c} \varepsilon_{e}^{e} \\ \varepsilon_{e}^{i} \end{array} \right] \right\} \mathbf{P} + \frac{2e}{\sigma} \mathbf{H} = \mathbf{H} + \lambda_{e} \mathbf{H}.
\]

(50)

For the reader who has proceeded directly to equation (50) from page 192 some detailed explanation is needed. Equation (50) is the matrix form of the equation for \( \varepsilon^{(i)} \) (3). This is obtained by expanding \( \varepsilon^{(i)} \) in a series of Laguerre polynomials with indetermined coefficients \( P_{r} \) \((r = 0, 1, 2 \ldots)\) as in equation (7). One substitutes this expansion into the \( \varepsilon^{(i)} \) equation and uses the orthogonality property of the polynomials (9) to determine the \( P_{r} \) values. The result can be put into matrix form with the \( P_{r} \) represented by a column vector \( \mathbf{P} \). The left hand side of (50) is the equivalent matrix form of the collision terms, expanded correct to terms of order \( \lambda_{e} \) \((= \kappa T / m e^{2})\). The electron-ion interaction matrices \( \Lambda_{e}^{e} \) and \( \Lambda_{e}^{i} \) are exhibited in equations (30a-b) respectively for an expansion of order five. The ion recoil term in the \( \varepsilon^{(i)} \) equation is represented by the \( H_{e}^{e} \) column vector and is displayed in (26). Finally the electron-electron interaction matrices \( \Lambda_{e}^{e} \) and \( \Lambda_{e}^{i} \) are given by equations (37) and (49).

The right hand side of (50) represents the matrix form of the forcing terms in the Boltzmann equation for \( \varepsilon^{(i)} \) due to electric fields and gradients in plasma parameters. The appropriate \( \mathbf{H} \) and \( \mathbf{H} \) column vectors are given by equations (14) and (15).
By defining

\[ \Delta_{cl} \sim A_{e}^{e} + \Delta_{e}^{e} \sim, \]

and

\[ \Delta_{r} \sim D_{e}^{e} + \Delta_{e}^{e}, \]

(50) may be written in the form

\[ \Delta_{cl} \sim P \sim -H_{\sim} - \lambda \lambda_{e}^{e} \hbar_{\sim} + \frac{2c\Omega}{\sigma} H_{\sim}^{e} - \lambda \lambda_{e}^{e} A_{r} \sim P \sim. \]  \hspace{1cm} (51)

We recall that \( H_{e}^{e} \) is given, correct to order \( \lambda \lambda_{e}^{e} \), by equation (26). To solve (51) for the \( P \) coefficients correct to order \( \lambda \lambda_{e}^{e} \) we first need to solve the classical equation

\[ \Delta_{cl} \sim P \sim -H_{\sim} + \frac{2c\Omega}{\sigma} C_{\sim} \sim \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \],

obtained by omitting the terms in \( \lambda \lambda_{e}^{e} \) in (51). We have used the fact that

\[ \Delta_{cl} \sim \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3\pi}{2} \\ \frac{15}{16} \\ \frac{5}{32} \end{bmatrix} \]

and is the classical contribution to \( H_{e}^{e} \). We can then substitute the classical expression for \( P \) into the right hand side of (51) giving, correct to order \( \lambda \lambda_{e}^{e} \)

\[ \Delta_{cl} \sim P \sim -H_{\sim} - \lambda \lambda_{e}^{e} \hbar_{\sim} + \lambda \lambda_{e}^{e} A_{r} \sim \frac{\lambda}{\lambda_{e}^{e}} H_{\sim} \]

\[ + \frac{2c\Omega}{\sigma} H_{\sim}^{e} - \frac{2c\Omega}{\sigma} \lambda \lambda_{e}^{e} A_{r} \sim \frac{1}{\lambda_{e}^{e}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \]. \hspace{1cm} (52)
By appealing to the definitions of the matrices $\Delta_{cl}$ and $\Delta_{cr}$ and the column vector $\mathbf{n}^{e, i}$, it is a straightforward matter to show the identity

$$
\Delta_{cl} \left[ \begin{array}{c}
\frac{2eU}{\sigma}(1+\frac{\mathbf{n}^{e, i}}{2e\lambda_e}) \\
-\frac{7eU}{\sigma\lambda_e} \\
\frac{2eU}{\sigma\lambda_e} \\
0 \\
0 \\
0
\end{array} \right]
= \frac{2eU}{\sigma} \mathbf{n}^{e, i} - \frac{2eU}{\sigma} \lambda_e^{-1} \Delta_{cr} \left[ \begin{array}{c}
1 \\
0 \\
0 \\
0 \\
0
\end{array} \right],
$$

so that (52) may be written in the final, more compact form, as

$$
\Delta_{cl} \mathbf{P}' = -\mathbf{H} - \lambda_e^{-1} \mathbf{H} + \lambda_e^{-1} \Delta_{cr} \Delta_{cl}^{-1} \mathbf{H},
$$

(53)

where the modified $\mathbf{P}$ vector $\mathbf{P}'$ is given by

$$
\left[ \begin{array}{c}
\rho_0 - \frac{2eU}{\sigma}(1+\frac{\mathbf{n}^{e, i}}{2e\lambda_e}) \\
\rho_1 - \frac{7eU}{\sigma\lambda_e} \\
\rho_2 - \frac{2eU}{\sigma\lambda_e} \\
\rho_3 \\
\rho_4 \\
\rho_5 \\
\vdots
\end{array} \right].
$$

This is a unique feature for a two-component plasma in that the ion-recoil terms combine so neatly to order $\lambda_e^{-1}$. The result was found when we originally solved equation (52) and computed the $\mathbf{U}$ contribution to the P-terms. For the 4 x 4 matrix approximation to $\Delta_{cl}$ we found the numerical factors independent of the ionisation to a remarkable degree. This led us to investigate expressing (52) in the form (53) and up came the identity above.
Equation (53) is the basic result of this analysis. It gives the equivalent matrix representation of the \( j^{(i)} \) equation in terms of the expansion coefficients \( \rho_{\tau} \). In addition it shows how the individual interaction matrices can be added to give the complete collision operator. Neglect of the relativistic terms gives the classical limit of this equation as given by Shkarofsky et al. (equation 8-96) providing one puts the cyclotron frequency \( \omega_c = 0 \) since we have not included steady magnetic fields in this analysis.

Finally we give some consideration to the interesting property that the \( \mathcal{E} - \mathcal{E} \) interaction matrices have zeroes in the first row and column (classical term) and zeroes in the first row (relativistic term). Firstly we denote the classical or relativistic operator by \( [0] \) (the argument follows through the same whichever term is chosen). Then from equation (20), the corresponding elements are given by

\[
H_{\mathcal{E} \mathcal{E}} = \int_0^{\infty} \frac{\mathcal{E}}{T(\mathcal{E})} [0] (j^{(i)}_{\tau}) d\mathcal{E},
\]

and

\[
j^{(i)}_{\tau} = n \beta^3 \mathcal{E} e^{-\mathcal{E}} L_{\tau}(\mathcal{E}).
\]

By comparison with equation (91) (Chapter 5) with \( \mathcal{C}^{(1)} \) replaced by \( [0] j^{(i)}_{\tau} \), and use of the results (94) or (96) for classical or relativistic interactions, it is obvious that \( H_{\mathcal{E} \mathcal{E}} = 0 \) in both cases. The first column of the classical matrix is also zero because of the symmetry of the generating relation (36). This result is just a restatement of the fact that \( \mathcal{E} - \mathcal{E} \) effects do not contribute to the overall momentum balance.

\section{Transport Coefficients of Conductivity and Diffusion}

In this section we shall investigate results for the direct electric-current flow in a plasma due to an impressed electric field and gradients in electron density. It is assumed that the forcing
terms are sufficiently small for linear theory to apply (that is the
linearised form of the Fokker-Planck collision term).

Since the electrons become relativistic at a much lower temperature
than the ions (\( \lambda_i \gg \lambda_e \)) we will use the average ion velocity \( \overline{\nu} \)
and \( C \) times the average reduced velocity \( \overline{\nu} \) interchangeably; that is
\[
\overline{\nu} \sim c \overline{\nu}.
\] (54)

The net direct current flow becomes
\[
\overline{J} = n_e \overline{\nu} - N\overline{\nu} e c \overline{\nu},
\] (55)
where \( e \) is the electronic charge (\(<0\)) and \( \overline{\nu} \) the average electron
velocity. If charge neutrality is assumed (this is not necessarily
a restriction since any other charge distribution can easily be
incorporated) then \( N = n \), so that (55) may be rewritten as
\[
\overline{J} = n_e (\overline{\nu} - c \overline{\nu}).
\] (56)

By definition of \( \overline{\nu} \) we have
\[
\overline{\nu} = \frac{4\pi}{3n} \int_0^\infty C \frac{f^{\nu} u^3 du}{\sqrt{1+u^2}}.
\]

\[
\overline{\nu} = \frac{4\pi}{3n} \int_0^\infty \frac{4}{\lambda_e^2 \gamma} f^{\nu} (1 - q^2/\lambda_e^2) q^3 dq,
\]
where \( \frac{1}{2} \lambda_e U^2 = q^2 \). The expansion for \( f^{\nu} \), as given by (7), is
substituted into the expression for \( \overline{\nu} \) and the integration over \( q \)
performed using the orthogonality relation (9) and the expressions for
\( L_0 \) and \( L_1 \). With \( S^2 = \frac{2\pi T}{m} \), one obtains the result
\[
\overline{J} = \frac{e n_0}{2} \left\{ \frac{1}{\lambda_e^2} \left( \frac{p}{\lambda_e^2} - \frac{5}{2} \frac{r}{\lambda_e} \right) \right\} - n_e C \overline{\nu},
\] (57)
so that \( J \) is related to the first two coefficients of the Laguerre
expansion of \( \mathcal{f}^{(1)} \) correct to terms of order \( \lambda_e^{-1} \).

If we are interested in current flow due to electric fields and gradients in \( \nabla \), we must solve (53) with only the \( \mathcal{E} \) and \( \nabla \nabla \) terms retained. Thus for the \( \rho 's \) we have the equation

\[
\Delta_{cl} \rho \sim = -H - \lambda_e^{-1} h + \lambda_e^{-1} \Delta_{cl}^{-1} \nabla \nabla \sim h,
\]

where

\[
\begin{bmatrix}
-\frac{2e\xi}{m\omega} + \frac{e\xi}{m^\omega} & 0 & 0 & \ldots \\
0 & 25/2 & 0 & \ldots \\
0 & 0 & -35/4 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

Since charge neutrality has been assumed, we have from equation (34) the result that

\[
\mathbb{Z} \langle \nu_{ee} \rangle = \langle \nu_{ee} \rangle.
\]

Furthermore the matrix \( \Delta_{cl} \) is given by

\[
\Delta_{cl} = A_{ei} + A_{ee},
\]

with \( A_{ei} \) and \( A_{ee} \) given by (30a) and (37) respectively. The coefficients in (58) can be evaluated correct to order \( \lambda_e^{-1} \) by inverting the matrix \( \Delta_{cl} \) and results tabulated for various values of \( \xi \).

We denote the co-factor of the \( r, s \) element of \( \Delta_{cl} \) by \( \Delta_{rs} \) and the determinant of \( \Delta_{cl} \) by \( |\Delta| \). Also we let

\[
\Delta^+ = \Delta_{cl}^{-1} \Delta_{rs} \Delta_{cl}^{-1}
\]

Hence from (58), with \( s \) denoting rows (\( s = 0, 1, \ldots \)) we have
\[ \rho_s = \frac{\Delta_{50}}{|\Delta|} \left( \frac{2e\varepsilon}{m\sigma} - \frac{\sigma \nabla n}{n} \right) + \frac{e\varepsilon}{m\sigma\lambda_e} \left( -\frac{25}{2} \frac{\Delta_{51}}{|\Delta|} + \frac{35}{4} \frac{\Delta_{52}}{|\Delta|} - 2 \Delta_{50}^r \right) - \frac{\sigma \nabla n}{n\lambda_e} \left( -\frac{25}{4} \frac{\Delta_{51}}{|\Delta|} + \frac{35}{8} \frac{\Delta_{52}}{|\Delta|} - \Delta_{50}^r \right). \]

\[ \Delta_{50}^r \] represents the \(s,0\) element of the matrix \( \Delta^r \) (\(s\) - rows, \(0\) - first column).

The direct current equation can be expressed in the form

\[ \overline{\mathbf{j}} = \varepsilon \left\{ \frac{\mathbf{E}}{\varepsilon_0} + \frac{kT}{n\lambda_e} \nabla n \right\}, \]

where \(\varepsilon\) is the conductivity, which, in the absence of magnetic fields is a scalar. Explicitly we have for \(\varepsilon\), correct to first order relativistic effects, the form

\[ \varepsilon = \frac{n e^2}{m \langle \mathcal{E}_e \rangle} \varepsilon_1 (1 + \varepsilon_2 / \lambda_e). \] (59)

The coefficients \(\varepsilon_1\) and \(\varepsilon_2\) are given in terms of the matrix elements as

\[ \varepsilon_1 = \frac{\Delta_{50}}{|\Delta|} \langle \mathcal{E}_e \rangle, \]

and

\[ \varepsilon_2 = -\frac{|\Delta|}{\Delta_{50}} \left[ -\frac{25}{4} \frac{\Delta_{51}}{|\Delta|} + \frac{35}{8} \frac{\Delta_{52}}{|\Delta|} - \Delta_{50}^r - \frac{5}{2} \left( \frac{\Delta_{50}}{|\Delta|} - \frac{\Delta_{50}}{|\Delta|} \right) \right]. \]
The expressions for $\xi_1$ and $\xi_2$ (which are functions of $S$) have been tabulated in Table 6-1 above for $S$-values of 1, 2, ..., 10, and for the 4 x 4 and 5 x 5 matrix approximations. There is little significant difference between the two sets of results, the fifth order approximation being, of course, the more accurate. The limiting values for large $S$ (i.e., the Lorentzian gas) are given later in § 7.

In the expression for $\mathcal{J}$, the coefficient multiplying $\nabla n$ can be written as

$$|e| D \nabla n$$

where $D$ is defined as the diffusion coefficient $\mathcal{S}_2$, equation 3-99b $\mathcal{J}$. Comparison with (59) gives the value

$$D = \frac{KT}{ne^2} \xi.$$

§ 5  THERMOELECTRIC COEFFICIENT.

We now consider the contribution to direct current flow in a
plasma due to gradients in electron temperature. The same equation as (58) applies, except that now the column vectors \( \mathbf{H} \) and \( \mathbf{h} \) are given by

\[
\mathbf{H} = \frac{\sigma V T}{T} \begin{bmatrix}
1 \\
-\frac{5}{2} \\
0 \\
0 \\
\vdots
\end{bmatrix}, \quad \mathbf{h} = \frac{\sigma V T}{T} \begin{bmatrix}
0 \\
-75/4 \\
315/8 \\
-315/16 \\
0 \\
\vdots
\end{bmatrix}
\]

(60)

For various values of \( Z \) one can evaluate the inverse matrix \( \mathbf{A}^{-1} \) and hence the \( P \) coefficients correct to order \( \lambda_e^{-1} \). Substitution into equation (57) gives the thermoelectric contribution to direct current flow as a multiplying factor times \( \nabla T \). Explicitly we write

\[
\mathbf{J} = \gamma \nabla T
\]

and

\[
\gamma = \frac{n|e|K}{m<\gamma_e>} \gamma_1 \left( 1 + \gamma_2/\lambda_e \right).
\]

(61)

The coefficients \( P_s \) are given in terms of the matrix elements as

\[
P_s = \frac{\sigma V T}{T} \left\{ -\frac{A_{s0}}{|A|} + \frac{5}{2} \frac{A_{s1}}{|A|} \right\} \\
- \frac{\sigma V T}{\lambda_e T} \left\{ -\frac{75}{4} \frac{A_{s1}}{|A|} + \frac{315}{8} \frac{A_{s2}}{|A|} - \frac{315}{16} \frac{A_{s3}}{|A|} \right\} \\
+ \frac{\sigma V T}{\lambda_e T} \left\{ A_{s0}^{\dagger} - \frac{5}{2} A_{s1}^{\dagger} \right\}
\]
with the notation as used in Equation 4. The terms \( \gamma_1 \) and \( \gamma_2 \) can then be computed via equation (57) and result in the expressions

\[
\gamma_1 = \left( \gamma_2 e_i \right) \left( \frac{\Delta_{00}}{|\Delta|} - \frac{5}{2} \frac{\Delta_{01}}{|\Delta|} \right),
\]

and

\[
\gamma_2 = \left\{ \left( -\frac{75}{4} \Delta_{00} + \frac{315}{8} \Delta_{02} - \frac{315}{16} \Delta_{03} - |\Delta| \left( \Delta_{oo}^+ - \frac{5}{2} \Delta_{01}^+ \right) \right) \frac{1}{|\Delta|} +\frac{5}{2} \left( -\Delta_{oo} + \Delta_{10} + \frac{5}{2} \Delta_{01} + \frac{5}{2} \Delta_{11} \right) \frac{1}{|\Delta|} \right\} \frac{1}{\gamma_1}.
\]

<table>
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<th>( 5 \times 5 \text{ MATRIX} )</th>
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</tr>
<tr>
<td>10</td>
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</table>

Again, the coefficients \( \gamma_1 \) and \( \gamma_2 \) are functions of the ionisation \( Z \) of the plasma and their values are listed in Table 6-2 for \( Z = 1, 2, \ldots, 10 \). Both the fourth and fifth order approximations are given. For the \( \gamma_1 \) term there is very little difference between the two sets of results. For the \( \gamma_2 \) term agreement is reasonable
except for the values corresponding to \( J = 1 \) and \( 2 \), indicating that the Laguerre expansion converges less well here. In §7 we discuss the limiting values of the coefficients for large \( S \) and apply the approximate method above to compare the accuracy with the exact solution.

From the current equation (57) and the equation for the \( P \)'s (53), we observe that, exactly, to order \( \lambda_e^{-1} \), the terms in \( \overline{U} \) cancel. Hence the complete expression for \( J \) can be written as

\[
J = \varepsilon \left( E + \frac{kT}{e} \nabla n \right) + \gamma \nabla T,
\]

with the conductivity \( \varepsilon \) given (59) and the thermoelectric coefficient \( \gamma \) given by equation (61).

§ 6 COEFFICIENTS FOR ELECTRON ENERGY FLOW.

In this section we relate the electron energy flow to the impressed electric field and gradients in electron number density and temperature by two further coefficients. In plasma work it is often convenient to measure quantities relative to the average plasma velocity \( \overline{V} \). One is then able to divide the effects of a gas into thermal or random effects and those due to the average velocity. If \( \overline{V} \) is the average ion velocity and \( \overline{V} \) the average electron velocity, then the mass flow velocity \( \overline{Q} \) can be defined as

\[
\overline{Q} (N_i n_i + n_e) = M_n \overline{V} + m_e \overline{V}_e
\]

which to order \( \lambda_e^{-1} \) is

\[
\overline{Q} \sim \overline{V} \sim C \overline{U}.
\]

That is the average ion-velocity is equal to the average plasma velocity as far as mass flow effects are concerned.

The intrinsic electron energy flow vector \( q_e \) is defined as

\[
q_e = \overline{C} \varepsilon \overline{V}.
\]
where $\xi$ is the energy of an electron measured relative to a frame moving with velocity $\underline{C}$, and the overbar denotes an average over the electron distribution. Explicitly for $\xi$, we have the usual form

$$\xi = mc^3 \left( \sqrt{1+\omega^2} - 1 \right),$$

where $c$ is the speed of light and $\omega$ represents the electron reduced velocity in the moving frame. $\omega$ is related by a Lorentz transformation to $U$ in the rest frame. The appropriate relations for any four-vector are given in Chapter 4, equation (46). They apply only for two frames moving relative to each other along the $X$-axis. For any general direction one can apply a rotation to the coordinate axes.

Firstly we require $\omega$ in terms of $U$. Throughout this analysis we neglect quadratic terms of $\underline{C}$ which essentially reduces the Lorentz factor $\alpha$ to 1. Thus we have

$$\omega \simeq U - \frac{\sqrt{1+U^2}}{\sqrt{1+\frac{U^2}{c^2}}},$$

and

$$\left(1+\omega^2\right)^{1/2} \simeq \left(1+U^2\right)^{1/2} \left(1+\frac{U^2}{c^2}\right).$$

Expanding $\xi \omega$ in terms of the scaled variable $q$ gives, correct to order $\lambda_e^{-1}$,

$$\xi \omega = mc^3 \left\{ \frac{q^2}{\lambda_e} - \omega (U \cdot U) - \frac{\sqrt{1+U^2}}{\sqrt{1+\frac{U^2}{c^2}}} \right\},$$

where $\frac{1}{2} \lambda_e U^2 = q^2$.

We must now take the average of $\xi \omega$ over the electron distribution function $f$. If we omit the $f^{(2)}$ moment, then
\[ \mathcal{U}(\mathbf{u} \cdot \mathbf{u}) = \frac{u^2}{3} \mathcal{U} = \frac{2}{3 \lambda_e} \overline{q^2} \mathcal{U} , \]

and

\[ \mathcal{U}(\mathbf{u} \cdot \mathbf{u}) \frac{q^2}{\lambda_e} = \frac{2}{3 \lambda_e^2} \overline{q^4} \mathcal{U} . \]

With these results the final general expression for \( \overline{q_e} \), correct to terms of order \( \lambda_e^{-1} \), becomes

\[ \overline{q_e} = n m c^2 \left\{ \frac{\mathcal{U} q^2}{\lambda_e} - \frac{5}{3 \lambda_e} \overline{q^2} \mathcal{U} - \frac{3}{2 \lambda_e} \overline{q^4} + \frac{11}{6} \mathcal{U} \overline{q^4} \right\}. \]  

(62)

\( \mathcal{U} \) is of course the average ion reduced-velocity.

Now, for any scalar function of reduced-velocity, say \( \phi \), the average over the electron distribution becomes

\[ \bar{\phi} = \frac{4 \pi}{n} \int_0^\infty \phi \cdot \mathbf{f}^{(e)} \mathbf{u} \, d\mathbf{u} . \]

That is \( \bar{\phi} \) is related only to \( \mathbf{f}^{(e)} \), the zero moment distribution of \( \mathbf{f} \). The appropriate form for \( \mathbf{f}^{(e)} \) is taken as

\[ \mathbf{f}^{(e)} = n \beta^2 \left( 1 - \frac{15}{8 \lambda_e} + \frac{q^4}{2 \lambda_e} \right) e^{-q^2} , \]  

(63)

with \( \beta^2 = \left( \frac{\lambda_e}{2 \pi} \right) \). (63) is the correct form for the Maxwellian with first order relativistic effects included. Its use is justified provided the disturbing forces in the plasma are small.

For equation (62) we require averages of \( \overline{q^2} \) and \( \overline{q^4} \), the former correct to terms in \( \lambda_e^{-1} \), the latter to classical accuracy only. Using (63) we have

\[ \overline{q^2} = \frac{3}{2} + \frac{15}{4 \lambda_e} . \]
and
\[ \overline{q^4} = \frac{15}{4}. \]

Thus \( q_e \) becomes
\[
q_e = \Pi \text{mc}^3 \left\{ \frac{\gamma q^2}{\lambda_e} - \frac{3}{2\lambda_e} \overline{q^2} - \frac{5}{2\lambda_e} \overline{q}^4 + \frac{5}{8\lambda_e} \overline{q}^6 \right\}. \tag{64}
\]

To evaluate the final averages over \( U \) in (64) we use the result that the average of a vector \( A \) is
\[
\overline{A} = \frac{4\pi}{3n} \int_0^\infty \mathcal{F}^{(1)}_0 A U^2 dU,
\]
where \( \mathcal{F}^{(1)}_0 \) is the first moment of the distribution \( f \). For \( \mathcal{F}^{(1)}_0 \) we also apply the expansion (7) in terms of Laguerre polynomials. The \( U \)-averages give the results
\[
\overline{U q^2 / \lambda_e} = \frac{5\sqrt{2}}{4\lambda_e^{3/2}} (p - \overline{\omega}),
\]
and
\[
\overline{U q^4 / \lambda_e^4} = \frac{35\sqrt{2}}{8\lambda_e^{3}} (p_1^2 - 2\overline{\omega}^2 + p_\omega^2).
\]

With these results, (64) can be expressed in terms of the \( p \) coefficients as
\[
q_e = \Pi \text{mc}^3 \left\{ \frac{5\sqrt{2}}{4\lambda_e^{3/2}} (p - \overline{\omega}) - \frac{105\sqrt{2}}{16\lambda_e^{3/2}} (p_1^2 - 2\overline{\omega}^2 + p_\omega^2) - \frac{5\overline{q}}{2\lambda_e} + \frac{5}{8\lambda_e^{3/2}} \right\}. \tag{65}
\]

For the \( p \)'s we refer to the full equation (53). Explicitly, the energy flow vector \( q_e \) may be expressed in the form
\[ q_e = -\mu \left\{ \frac{e}{\varepsilon} + \frac{kT}{n|e|} \nabla n \right\} - K_e \nabla T, \]

where

\[ \mu = \frac{e}{2} \sum_{m < \beta_{ei}} \mu_c \left( 1 + \frac{\mu_c}{\lambda_e} \right), \]

and

\[ K_e = \frac{e}{2} \sum_{m < \beta_{ei}} \left( 1 + \frac{\mu_c}{\lambda_e} \right). \]

One can express the \( \mu \) and \( K_e \) coefficients in terms of the co-factors of the inverse matrix \( \Delta_{ei}^{-1} \) and also the elements of \( \Delta_{ei}^{T} \), and tabulate the results for various values of the ionisation \( \beta \). With the notation of Eq. 4 for the various matrix elements, we find the following forms:

for the classical coefficients,

\[ \mu_1 = \langle V_{ei} \rangle \left( \Delta_{00} - \Delta_{10} \right) / |\Delta|, \]

and

\[ K_1 = \langle V_{ei} \rangle \left( \Delta_{00} - \frac{7}{2} \Delta_{01} + \frac{5}{2} \Delta_{11} \right); \]

for the relativistic coefficients,

\[ \mu_2 = \frac{1}{\mu_1} \left\{ \left[ -2\Delta_{10} + \frac{7}{2} \Delta_{02} + \frac{25}{2} \Delta_{11} - \frac{35}{14} \Delta_{12} - \frac{21}{4} \Delta_{00} \right] / |\Delta| \right. \]

\[ -2 \Delta_{00} + 2 \Delta_{10} \right\}, \]

and
\[ K_2 = \frac{1}{K_1} \left\{ -\frac{39}{8} \Delta_0 - \frac{273}{8} \Delta_{02} + \frac{15}{4} \Delta_n + \frac{105}{4} \Delta_{11} \right. \]
\[ + \frac{315}{16} (\Delta_{03} - \Delta_{13}) + \frac{21}{4} \Delta_{00} \} \frac{1}{|A|} \]
\[ + \Delta_{00} - \frac{5}{2} \Delta_{01} - \Delta_{10} + \frac{5}{2} \Delta_{11} \} \].

The coefficients \( \mu_1, \mu_2 \) and \( K_1, K_2 \) have been tabulated in Tables 6-3 and 6-4 respectively for \( g \)-values 1, 2, ..., 10. Both the 4 x 4 and 5 x 5 matrix approximations are given. There is little difference between the two expansions except for the \( K_2 \) coefficient at \( g = 1 \) indicating that the expansion converges less well for low \( g \)-values.

**TABLE 6-3**

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From (53) we can calculate the contribution of the \( \bar{U} \) terms to \( q_k \).

Simple substitution into (65) gives
\[ q_e = \frac{5}{2} \eta K T c U / \lambda_e , \]

with the classical term vanishing exactly. The complete energy flow vector may thus be written in the form

\[ q_e = \frac{5}{2} \eta K T c U / \lambda_e , \]

the last term arising from the relativistic ion-recoil terms in the collision expressions.

<table>
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§7 TRANSPORT COEFFICIENTS FOR THE LORENTZIAN PLASMA.

As we have seen, when the full collision term is specified to include both e-e and e-i effects, the analysis becomes extremely complex. If charge neutrality is assumed the average e-e and e-i collision frequencies are related by

\[ \langle \nu_{ee} \rangle = \frac{\langle \nu_{ei} \rangle}{Z} . \]
Thus for plasmas with a high ionisation number \( \langle \gamma_{ee} \rangle \ll \langle \gamma_{ei} \rangle \). That is, \( e-e \) effects become small in comparison with the \( e-i \) interactions. This situation pertains in the so-called Lorentzian gas—a hypothetically fully ionised gas in which the electrons do not interact with one another and the positive ions are at rest.

If one neglects \( e-e \) effects \( (Z = \infty) \), the collisional processes are governed solely by \( e-i \) interactions, and the \( f^{(0)} \) equation becomes

\[
\frac{c \mu}{\sqrt{\mu u^*}} \nabla \cdot f^{(0)} + \frac{e E}{m e} \frac{\partial f^{(0)}}{\partial u} = \left( C^{(u)} \right)_{ei},
\]

where the collision term is given by (6). (6) may be written in the form

\[
C^{(u)} = -\frac{3}{4} \frac{\langle \gamma_{ei} \rangle}{q^2} \left( \frac{1}{q^3} + \frac{1}{q e} \right) f^{(0)} + \frac{3}{4} \frac{\langle \gamma_{ei} \rangle}{q^2} \left( \frac{1}{q^3} + \frac{2}{q e} \right) \frac{\partial f^{(0)}}{\partial q}.
\]

For the \( \nabla \cdot f^{(0)} \) and \( \frac{\partial f^{(0)}}{\partial q} \) terms we refer the reader to equations (12a,b). Since the \( C^{(u)} \) term depends only on \( f^{(0)} \) and not on any derivatives, the solution may be expressed exactly. Our aim in this section is to evaluate the transport coefficients exactly for this case and compare the results with those derived from the expansion procedure discussed earlier.

**CONDUCTIVITY AND DIFFUSION.**

The expansion in Laguerre polynomials (7) can be substituted into (66) to give the following equation, correct to terms in \( \lambda_e^{-1} \) for the \( \rho \) coefficients

\[
-\frac{e E}{m e} \frac{\mu}{\sqrt{\mu u^*}} \beta q^3 -2q + \frac{15 q^3}{4 \lambda_e} + \frac{4 q^1}{\lambda_e} - \frac{q^5}{\lambda_e^2} = \sum_i P_i \Sigma_i (q_i^r).
\]
In (67) only the electric field has been retained in the forcing terms since we are interested in the value of the conductivity alone. To obtain the $P$'s we multiply (67) by

$$E^\tau \int \left( \frac{\varepsilon}{\varepsilon} \right) e^{-\varepsilon} d\varepsilon,$$

where $E = q^2$, and integrate over the range $[0, \infty]$. Further, one can multiply by $\alpha^s$ and sum over $s$, applying the generating relation (8) to the left hand side of (67). The resulting equation is

$$\sum_s \alpha^s \frac{P^s}{s!} \frac{\Gamma(s + \frac{1}{2})}{s!} = \frac{e \varepsilon^4}{\sqrt{2 \pi m}} \sigma \langle \varepsilon \rangle \int_0^\infty \left[ 2 \varepsilon^3 + \frac{1}{\lambda e} \varepsilon^2 \left( -\frac{15}{4} \varepsilon^3 - 5 \varepsilon^4 + \varepsilon^5 \right) - \frac{2}{\lambda e} \right] e^{-\varepsilon} d\varepsilon$$

$$= \frac{4 e \varepsilon^4}{3 \pi m \sigma \langle \varepsilon \rangle} \left\{ 12 (1 - \alpha)^3 + \frac{1}{\lambda e} \left[ \frac{45}{2} (1 - \alpha)^2 - 9 \alpha (1 - \alpha)^2 \right] + 120 (1 - \alpha)^2 \right\}.$$

By equating coefficients of $\alpha^s$, the corresponding $P_s$ can be found. The relevant values we need to compute the conductivity, and also the energy flow coefficient later on, are $P_s$ and $P_1$, to order $\lambda e^{-1}$ and $P_2$ to classical accuracy. From above, we find, with $\chi = \frac{e \varepsilon^4}{\sqrt{2 \pi m}}$,

$$\sigma \langle \chi \rangle P = \frac{64}{3 \pi} \chi + \frac{8}{3 \pi \lambda e} \chi,$$

$$\sigma \langle \chi \rangle P_1 = -\frac{64}{5 \pi} \chi - \frac{104}{\pi \lambda e} \chi,$$

and

$$\sigma \langle \chi \rangle P_2 = \frac{64}{3 \pi} \chi.$$
For the conductivity we substitute the appropriate \( P \) values above into (57), obtaining for the coefficient of the electric field term \( E \), the value

\[
E = \frac{\alpha e^2}{m \langle \gamma e \rangle} \cdot \frac{32}{3\pi} \left( 1 - \frac{31}{8\lambda_e} \right).
\]

The diffusion coefficient is related to \( \gamma \) as given in \( \Theta^4 \).

**Thermoelectric Coefficient.**

This is the coefficient multiplying the \( \nabla_x T \) term in the current equation. The equivalent equation to (67) retaining only the \( \nabla_x T \) terms, reads

\[
- \frac{\nabla_x T}{T} \frac{4\sigma q^2}{3\pi \langle \gamma e \rangle} \left\{ -\frac{3}{2} + q^2 + \lambda_e^{-1} \left[ \frac{15}{16} + \frac{9}{8} q^4 + \frac{13}{8} q^4 + \frac{1}{2} q^6 \right] \right\} = \sum P_{1f}(x)
\]

To solve for the \( P_{1f} \) coefficients we adopt the same procedure as for the conductivity above, obtaining

\[
\sum \alpha^{i} \tilde{P}_{s} \frac{\langle x + \zeta \rangle}{S!} = -\frac{\nabla_x T}{T} \frac{4\sigma}{3\pi \langle \gamma e \rangle} \left\{ -q(1-x)^{3/2} + 24(1-x)^{5/2}
\]

\[
+ \lambda_e^{-1} \left[ \frac{45}{8} (1-x)^{3/2} + 27(1-x)^{5/2}
\]

\[
- 390(1-x)^{7/2} + 360(1-x)^{9/2} \right\}.
\]

Again, we can equate coefficients of \( \alpha^{i} \) and obtain the appropriate \( \tilde{P}_{s} \) value. To compute the thermoelectric coefficient, and also the energy-flow terms later on, we require \( \lambda_e^{-1} \) and \( \tilde{P}_{s} \) to order \( \lambda_e^{-1} \) and \( \tilde{P}_{s} \) to classical accuracy. From above we have
\[
\langle \nu_{ei} \rangle \rho_0 = -\frac{80}{3\pi} \frac{\Delta T}{T} - 14 \frac{\Delta T}{T} \lambda_e^{-1} \]
\[
\langle \nu_{ei} \rangle \rho_1 = \frac{496}{15\pi} \frac{\Delta T}{T} + \frac{706}{3\pi} \frac{\Delta T}{T} \lambda_e^{-1} \]

and

\[
\langle \nu_{ei} \rangle \rho_2 = -\frac{592}{35\pi} \frac{\Delta T}{T} \]

To obtain the thermoelectric coefficient, we substitute the \( \rho \) values above into (57), retaining terms up to order \( \lambda_e^{-1} \). For the multiplying factor of \( \frac{\nu_i T}{\Delta T} \), there results the value

\[
\gamma = \frac{n|e|k}{m\langle \nu_{ei} \rangle} \frac{80}{3\pi} \left( 1 - \frac{217}{40\lambda_e} \right)
\]

**COEFFICIENTS OF ELECTRON ENERGY FLOW.**

The \( \rho \) values above can be used in equation (65) to derive the Lorentzian coefficients for electron-energy flow. Simple calculation yields the values, to order \( \lambda_e^{-1} \),

\[
\mu = \frac{5}{2} \frac{n|e|kT}{m\langle \nu_{ei} \rangle} \frac{256}{15\pi} \left( 1 - \frac{35}{8\lambda_e} \right)
\]

and

\[
K_e = \frac{5}{2} \frac{nK^2T}{m\langle \nu_{ei} \rangle} \frac{894}{15\pi} \left( 1 - \frac{315}{56\lambda_e} \right)
\]

In Table 6-5 we have compared the exact values of the various transport coefficients with those obtained from the approximate method described earlier. Both the fourth and fifth order approximations are given.
As can be seen from Table 6-5, there is good agreement between exact and approximate values except for the fourth order value of $\gamma_2$. However in this case the fifth order approximation produces a much closer answer - as it should.

8 CONCLUDING REMARKS.

In this chapter we have applied the relativistic tensor expansion derived in Chapter 3 to some transport problems for ionised plasmas. Coefficients for conductivity and the thermoelectric effect together with those for electron energy flow have been derived. Results for both the classical and first order relativistic effects have been given for various ionisations of the plasma. Exact expressions for the Lorentzian gas are also given. Our intention has been to apply the $\mathbf{n}(0)$ equation to calculate transport coefficients for current and energy flow due to electric fields and gradients in number density and temperature. We have not attempted to calculate the coefficient of viscosity here, since this involves tackling the equation for $\mathbf{j}^{(2)}$ and is beyond the scope of our present work. In principle, however, it could be attempted in much the same way.

In our analysis of first order relativistic effects we have kept

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the equations as simple as possible by not considering any time
dependence or magnetic fields. In fact an investigation of either or
both effects could be followed up since this requires little alteration
to the interaction matrices. Essentially for time variations one
could split up the $f^{(0)}$ distribution into D.C. and A.C. parts with
$e^{i\omega t}$ variations imposed. However our hope is that, while limiting
ourselves to some of the more simpler transport problems of the
relativistic gas, we have at least indicated the feasibility of such
a task for the Coulomb Plasma. Up to now, as we have mentioned, only
the relativistic 'Maxwellian-model' has been considered which permits
of exact analytical results for the first order relativistic term
(see $C_{35}$ and $C_{36a-f}$). While our results are approximate,
comparison with the special case Lorentzian gas indicate the variation
of the transport coefficients with the plasma ionisation number $Z$. 
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FIGURE 1

COMPLEX $\omega$-PLANE

$\mathcal{I}(\omega)$

$\Re(\omega)$

STABLE

0

UNSTABLE

"SEMI-CIRCLE AT INFINITY"
COMPLEX \( \zeta \)-PLANE
(for electrons)

**Figure 2**
COMPLEX $\zeta_i$-PLANE
(FOR IONS)

$\frac{d}{d(\zeta_i)}$

$\zeta_i$

$0$

UNSTABLE
STABLE

$k > 0$

$R(\zeta_i)$

UNSTABLE
STABLE

$k < 0$

FIGURE 3
ELECTRON MOTION ONLY

NYQUIST PLOT FOR \( \frac{\nu_0}{\omega_0 R_{ie}} = 0.1, \frac{T^*}{T} = 3, k < 0 \)

FIGURE 5
ELECTRON MOTION ONLY

NYQUIST PLOT FOR \( \frac{\gamma_{en}}{\omega L |k|} = 10, \frac{L}{T} = 3, k < 0 \)

\[ \Im \left( R(k_h^2) \right) \]

\[ -0.5 \quad -0.3 \quad 0 \quad 0.1 \quad 0.2 \]

\[ R(k_h^2) \]

**Figure 6.**
ELECTRON & ION MOTION

NYQUIST PLOT FOR $\frac{\omega_{pe}}{\omega_{ci}} = 0.01$, $I^*_T = 20$, $k > 0$

$A(k^2)$

$R(k^2)$

FIGURE 7
ELECTRON & ION MOTION

> NYQUIST PLOT FOR \( \frac{\beta}{\omega} = 0.1, \frac{T}{T_0} = 50, \; R > 0 \)

\[ \Re \{ \frac{1}{\omega} \} \]

\[ \Im \{ \frac{1}{\omega} \} \]

**Figure 8**
ELECTRON & ION MOTION

Nyquist plot for \( \frac{Y_{na}}{\omega e |\kappa_{eh}|} = 3 \), \( \frac{T^{*}}{T} = 50 \), \( R > 0 \)

**Figure 9**
ELECTRON & ION MOTION

NYQUIST PLOT FOR \( \frac{\gamma_{\infty}}{\omega_0 | \text{Re} \nu |} = 0.1 \), \( \frac{1}{\omega_0} = 100 \), \( k > 0 \)

FIGURE 10
ELECTRON & ION MOTION

Nyquist plot for
\[ \frac{\tau_{re}}{\omega_{e} |k_e|} = 10, \quad \frac{1}{\tau_0} = 100, \quad k > 0. \]

\[ \Im \left( \sum_{k} e^{i k x} \right) \]

\[ R(k_{x}, z) \]

**FIGURE 11**
ELECTRON & ION MOTION

NYQUIST PLOT FOR $\frac{\rho_0}{\omega |\mathbf{K} \cdot \mathbf{E}|} = 0.1, \frac{1}{T} = 1000, R > 0$.

FIGURE 12
This thesis covers essentially two problems in the kinetic theory of plasmas. The first concerns the investigation of plasma oscillations in a constant electric field - a topic investigated by Akheizer and Sitenko as early as 1956 \cite{1}. More recently Stenflo \cite{2} has considered the problem in which he replaces the collision integral of Boltzmann's equation by a Fokker-Planck term and a B.G.K. term. The dispersion relations derived by Stenflo contained a number of parameters the relative importance of which he did not clearly define. We have undertaken here a stability study of longitudinal oscillations of a weakly ionised gas permeated by a uniform electric field. A dispersion relation is formulated in terms of error-type functions and some computational studies are carried out for various plasma parameters of interest. The results are exhibited graphically in the form of Nyquist plots. The conclusions made by Stenflo and others regarding possible instabilities of the plasma needs modification, certainly in the context of a weakly ionised electron-ion gas.

The second topic covered here concerns the transport theory of relativistic gases. This has received increasing attention in recent years \cite{3,4}. Much attention has been devoted to calculating the first order relativistic effects on the transport coefficients. Up to now only the 'Maxwellian' model, investigated by Israel \cite{3}, has been considered. The method of attack is via the Chapman-Enskog approach. In this second topic we develop a more general approach to the problem by generalising the classical spherical harmonic solution of the Boltzmann equation to the relativistic case. The theory is applied to transport problems of fully ionised plasmas in the Coulomb field.
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