PARAMETER REDUCTION IN DEFINITION BY

MULTI-SUCCESSOR RECURSION

by

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It is well known that in primitive recursive arithmetic with a single successor the number of parameters in a definition by recursion may be successively reduced. In this thesis I examine the possibility of effecting a similar reduction in the number of parameters in a definition by recursion in a multi-successor arithmetic.

The reduction process involves the discovery in multi-successor arithmetic of analogues of pairing functions and of functions which select the elements of an ordered pair. One of the difficulties in finding such functions is the construction within multi-successor arithmetic of suitable product and square root functions and establishing the properties of these functions, and the pairing functions, within a formalisation of multi-successor arithmetic. The reduction process involves of course an examination of what functions, if any, need to be adjoined to the initial functions to secure the generality of the reduction.
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CHAPTER I

INTRODUCTION

The subject of multiple successor arithmetic was first introduced in a paper by Vladeta Vučković in Mathematica Scandinavica in 1959 [5]. In single successor arithmetic every numeral has only one immediate successor. However in the paper mentioned above Vučković considered the possibility that every numeral has 'n' immediate successors. The successor of \( x \) in single successor arithmetic is written \( Sx \), and for the successors in multiple successor arithmetic we shall write \( S_u x \) where \( u \) will range from 1 to \( n \). In Vučković's paper the suffixes ranged from 0 to \( n-1 \), though in this thesis we have found it more convenient for the suffixes to range from 1 to \( n \). Rather than make an analysis of single successor arithmetic, we shall suppose that the reader has a working knowledge of this theory. For references see [1], [2] and [3].

Both single and multiple successor arithmetics are based on the theory of Recursive Functions, that is functions defined by a schema, which defines the function at a starting value and then at the successor of \( X \) in terms of the value of that function at \( X \). In the arithmetics that are considered in this thesis the starting value will be 0. By substitution the function is then defined successively for all values of \( X \).

From hereon in this thesis lower case letters will be used for variables and function symbols in single successor arithmetic, and higher case letters for variables and function symbols in multiple successor arithmetic. It is worth noting that we are in fact considering an infinite though countable set of arithmetics, that is for all values of \( n \geq 1 \). The theory that we shall consider is the commutative theory.
The numerals of single successor arithmetic are as follows:
0, 00, 000, 0000, 00000, ....... and clearly these form a complete ordering. The condition that \( x \geq y \) is given by the condition \( y \preceq x = 0 \).
This is a total ordering since for all \( x \) and \( y \) either \( x \preceq y = 0 \) or \( y \preceq x = 0 \), and if both are equal to zero then \( x = y \). Now in multiple successor arithmetic the numerals are as follows:
0, \( S_10 \), \( S_20 \), \( S_30 \), \( S_1S_10 \), \( S_1S_20 \), .... \( S_nS_n0 \), \( S_1S_1S_10 \), .... \( S_nS_nS_n0 \), ....
These numerals have a partial ordering which can be illustrated in '2' successor arithmetic with the aid of a lattice diagram.

\[
\begin{array}{c}
S_10 \quad S_1S_10 \\
S_2S_0 \quad S_2S_10 \quad S_1S_2 \quad S_2S_10 \\
S_1 \quad S_2S_0 \quad S_2 \quad S_2S_0 \\
0
\end{array}
\]

In commutative successor arithmetic the diagram becomes:

\[
\begin{array}{c}
S_1S_10 \quad S_1S_20 \quad S_1S_20 \quad S_2S_20 \\
S_1S_10 \quad S_1S_20 \quad S_2S_20 \\
S_1 \quad S_2 \quad S_2S_0 \\
0
\end{array}
\]
The theory of single successor arithmetic is built up from three initial functions, namely the Identity, Zero and Successor functions given by

$\text{I}_t^k(x_1,x_2,\ldots,x_k)$ written $x_t$
$Z(x)$ written $0$
$S(x)$ written $Sx$.

However in multiple successor arithmetic we obviously require the Identity and Zero functions though, in order to obtain $'n'$ successors for $X$ we require $'n'$ successor functions amongst the initial functions. The initial set of functions is therefore as follows:

$\text{I}_t^k(x_1,x_2,\ldots,x_k)$ written $x_t$
$Z(X)$ written $0$
$S_u(X)$ written $S_uX$ for $u = 1,\ldots,n$.

In order to define further functions in single successor arithmetic use is made of a defining schema called 'Definition by Primitive Recursion':

$f(x,y)$ is said to be defined by primitive recursion from $g(y), h(x,y,z)$ if

$f(0,y) = g(y)$
$f(Sx,y) = h(x,y,f(x,y))$

where $g(\ldots)$ and $h(\ldots)$ are initial functions or previously defined functions. Primitive recursive functions can also be defined by substitution from functions which have been previously defined by primitive recursion.

The above schema obviously defines $f(x,y)$ for all values of the variable $x$. That is, the function is defined at a starting value, namely $0$. Then from the second equation subsequent values of the function can be found by substituting $x = 0, S0, SS0, SSS0, SSSS0, \ldots$ in turn. In other words in multiple successor arithmetic with $'n'$
successors, the basic defining schema for a primitive recursive function would need to define the function at a starting value, namely 0, and then have further equations to evaluate the function for each different successor of X. Hence this schema will consist of 'n+1' equations as follows:-

A function F(X,Y) is defined by primitive recursion in multiple successor arithmetic by

\[ F(0,Y) = A(Y) \]
\[ F(S^X,Y) = B_u(X,Y,F(X,Y)) \quad u = 1, \ldots, n \]

where the functions A(\ldots) and B_u(\ldots) are initial functions or have been previously defined. A primitive recursive function can be defined by substitution as in single successor arithmetic.

In this thesis the successors of our arithmetic are commutative. Clearly this condition is not implicit in the equations for the definition of primitive recursion. It is therefore necessary to impose a restriction on the functions B_u(\ldots) u = 1, \ldots, n in order that the commutativity of the successors is maintained. We require that F(S^u S^v X,Y) = F(S^v S^u X,Y) for all u,v. Since

\[ F(S^u S^v X,Y) = B_u(S^v X,Y,F(S^v X,Y)) \]
\[ = B_u(S^v X,Y,B_v(X,Y,F(X,Y))) \]

and

\[ F(S^v S^u X,Y) = B_v(S^u X,Y,F(S^u X,Y)) \]
\[ = B_v(S^u X,Y,B_u(X,Y,F(X,Y))) \]

the condition is therefore that

\[ B_u(S^v X,Y,B_v(X,Y,Z)) = B_v(S^u X,Y,B_u(X,Y,Z)) \quad \text{for all } u,v. \]

For an illustration of this consider in '2' successor arithmetic the function defined by
\[ F(0, T) = Y \]
\[ F(S_1 X, Y) = F(X,Y) \]
\[ F(S_2 X, Y) = S_2 F(X,Y) \]

\( F(X,Y) \) is a primitive recursive function in commutative multiple successor arithmetic; since

\[ B_1(S_2 X, Y, B_2(X,Y,Z)) = B_1(S_2 X, Y, S_2 Z) \]
\[ = S_2 Z \]

and

\[ B_2(S_1 X, Y, B_1(X,Y,Z)) = B_2(S_1 X, Y, Z) \]
\[ = S_2 Z \]
\[ = B_1(S_2 X, Y, B_2(X,Y,Z)) \]

the commutativity condition is satisfied.

Now consider the function defined by

\[ F(0, Y) = Y \]
\[ F(S_1 X, Y) = S_1 F(X,Y) \]
\[ F(S_2 X, Y) = 0 \]

\( F(X,Y) \) is not a primitive recursive function in commutative recursive successor arithmetic since

\[ B_1(S_2 X, Y, B_2(X,Y,Z)) = B_1(S_2 X, Y, 0) \]
\[ = S_1 0 \]
\[ B_2(S_1 X, Y, B_1(X,Y,Z)) = B_2(S_1 X, Y, S_1 Z) \]
\[ = 0 \]
\[ \neq S_1 0 \]

the commutativity condition is not satisfied.

The uniqueness rule of inference for a primitive recursive function in single successor arithmetic is given by
\[
\begin{align*}
  f(0) &= g(0) \\
  f(Sx) &= h(x, f(x)) \\
  g(Sx) &= h(x, g(x)) \\
  f(x) &= g(x)
\end{align*}
\]

That is if two functions have the same defining equations in the definition by primitive recursion, then these two functions are identical, maintaining the uniqueness of definition by primitive recursion. Clearly in 'n' successor arithmetic we require the uniqueness rule of inference to be as follows:

\[
\begin{align*}
  F(0) &= G(0) \\
  F(S_x x) &= B_u (x, F(x)) & u = 1, \ldots, n \\
  G(S_x x) &= B_u (x, G(x)) & u = 1, \ldots, n \\
  F(x) &= G(x)
\end{align*}
\]

Parameters have been omitted from the above statements though these are implicit in the rules.

Let \( M_n \) be the set of numerals in 'n' successor arithmetic, then

\[
M_1 = 0, S0, SS0, SSS0, \ldots.
\]

Hence for any primitive recursive function the domain of the arguments cover all numerals that can be generated by repeated application of the successor functions to the zero function. That is letting \( D(X) \) represent the domain of \( X \), for a function \( F(X) \) we have

\[
D(F(X)) = M_n \text{ in 'n' successor arithmetic},
\]

and letting \( R(X) \) represent the range of \( X \), for a function \( F(X) \) we have

\[
R(F(X)) \subseteq M_n \text{ in 'n' successor arithmetic}.
\]

For an example of this consider the identity function \( I(X) \), in this instance,
D(I(X)) = M_n
R(I(X)) = M_n

and for the function 2.X (supposing a function such that \( F(X) = X + X \) exists in our arithmetic)

\[
D(2.X) = M_n
\]
\[
R(2.X) \subset M_n \quad ('\subset' \text{ meaning 'is a proper subset of'})
\]

and in single successor arithmetic we would have

\[
R(2.X) = 0, \, S0, \, SSS0, \, SSSSS0, \, SSSSSSS0, \ldots
\]

Also consider the functions \( S_u X \) \( u = 1, \ldots, n \). For this we have

\[
D(S_u X) = M_n
\]
\[
R(S_u X) \subset M_n \quad \text{since there is no } X \text{ such that } S_u X = 0.
\]

Equality in \( M_n \) is defined by means of an axiom due to Professor R. L. Goodstein, which is designed to avoid irregular models, given by

\[
S_a S_b S_c \ldots \ldots S_q 0 = S_{a'} S_{b'} S_{c'} \ldots \ldots S_{q'} 0
\]

with \( a \leq b \leq c \leq \ldots \leq q \) and \( a' \leq b' \leq c' \leq \ldots \leq q' \)

if and only if

\[
a = a', \quad b = b', \quad c = c', \ldots, \quad q = q'.
\]

However it has been shown by M. T. Partis in [4] that this axiom can be replaced by the condition \( S_1 0 \neq S_2 0 \).

We shall introduce several basic primitive recursive functions in \( 'n' \) successor arithmetic, and consider the properties of these functions.

First we shall define the function \( Y \sigma_v X, \, v = 0, \ldots, n-1 \) (the case \( v = n \) will be considered as the case \( v = 0 \)), as defined in Vučković's original paper.

The defining schema is

\[
Y \sigma_v 0 = Y
\]
\[
Y \sigma_v S_u X = S_{u+v} (Y \sigma_v X) \quad u = 1, \ldots, n.
\]
The suffix \( u+v \) is maintained within the range \([1, n]\) by taking the excess over \('n'. The commutativity condition for this function is given by

\[
R_u(S_w X, Y, R_w (X, Y, Z)) = R_w(S_w X, Y, R_w (X, Y, Z))
\]

\[
= S_{u+v} S_{w+v} Z
\]

\[
= S_{w+v} S_{u+v} Z
\]

\[
= R_w(S_u X, Y, R_u (X, Y, Z))
\]

In order to prove the properties of this function we require the uniqueness rule

\[
F(0) = G(0)
\]

\[
F(S_u X) = R_u(X, F(X)) \quad u = 1, \ldots, n
\]

\[
G(S_u X) = R_u(X, G(X)) \quad u = 1, \ldots, n
\]

\[
F(X) = G(X)
\]

A complete formal development of the arithmetic will be left to a later chapter; here we shall just prove simply certain properties of several linear functions in the arithmetic.

First we consider Vučković's function \( Y \sigma_v X \quad v = 0, \ldots, n-1 \).

**Lemma** \( (S_u Y) \sigma_v X = S_u (Y \sigma_v X) \quad u = 1, \ldots, n \)

**Proof** Denoting the right-hand side of the equation by \( R(X) \) and the left-hand side by \( L(X) \) we have that

\[
L(0) \neq (S_u Y) \sigma_v 0
\]

\[
= S_u Y \quad \text{Defn}
\]

\[
R(0) = S_u (Y \sigma_v 0)
\]

\[
= S_u Y \quad \text{Defn}
\]
\[ L(S^X) = (S^i_y) \sigma_v S^X \]
\[ = S_{w+v} (S^i_y) \sigma_v S^X \]
\[ = S_{w+v} L(X) \]
\[ R(S^X) = S_u (Y \sigma_v S^X) \]
\[ = S_{u+w+v} (Y \sigma_v S) \]
\[ = S_{w+v} S_u (Y \sigma_v S) \]
\[ = S_{w+v} R(X) . \]

Hence by the Uniqueness rule
\[ L(X) = R(X) \]

End of proof.

In general \( X \sigma_v (Y \sigma_v Z) \neq (X \sigma_v Y) \sigma_v Z \).

A simple counter-example serves to illustrate this; consider
\[ Y = S_{10}, \ Z = S_{20} \]
Then
\[ X \sigma_v (Y \sigma_v Z) = X \sigma_v (S_{10} \sigma_v S_{20}) \]
\[ = X \sigma_v S_{2+v}(S_{10} \sigma_v S_{20}) \]
\[ = X \sigma_v S_{2+v} S_{10} \]
\[ = S_{2+v+v} (X \sigma_v S_{10}) \]
\[ = S_{2+v+v} S_{1+v} (X \sigma_v S_{10}) \]
\[ = S_{2+v+v} S_{1+v} X \]
\[ (X \sigma_v Y) \sigma_v Z = (X \sigma_v S_{10}) \sigma_v S_{20} \]
\[ = S_{2+v} (X \sigma_v S_{10}) \sigma_v S_{20} \]
\[ = S_{2+v} (X \sigma_v S_{10}) \]
\[ = S_{2+v} S_{1+v} (X \sigma_v S_{10}) \]
\[ = S_{2+v} S_{1+v} X \]
and \( S_{2+v}S_{1+v}X \) is equal to \( S_{2+v}^{-1+v}X \) only when \( v = 0 \).

Also we find that in general

\[ X \sigma_v Y \neq Y \sigma_v X, \]

In order to prove this consider the particular instance where \( Y = 0 \):

\[ X \sigma_v 0 \neq 0 \sigma_v X. \]

If \( X = S_{20} \) then

\[ X \sigma_v 0 = S_{20} \sigma_v 0 \]

\[ = S_2(0 \sigma_v 0) \quad \text{Previous Lemma} \]

\[ = S_2 0 \quad \text{Defn} \]

and

\[ 0 \sigma_v X = 0 \sigma_v S_{20} \]

\[ = S_{2+v}(0 \sigma_v 0) \quad \text{Defn} \]

\[ = S_{2+v} 0 \quad \text{Defn} \]

and \( S_{2+v} 0 \) is only equal to \( S_{20} \) when \( v = 0 \)

Now we consider if this function is distributive over itself, that is if

\[ X \sigma_v(Y \sigma_u Z) = (X \sigma_v Y) \sigma_u(X \sigma_v Z) \quad (x \neq 0) \]

This condition is never true; put \( Y = S_{10} \), \( Z = S_{20} \)

\[ X \sigma_v(Y \sigma_u Z) = X \sigma_v(S_{10} \sigma_u S_{20}) \]

\[ = X \sigma_v S_{1+v} S_{2+u} 0 \quad \text{Defn} \]

\[ = S_{1+v} S_{2+u+v} X \quad \text{Defn} \]

\[ (X \sigma_v Y) \sigma_u(X \sigma_v Z) = (X \sigma_v S_{10}) \sigma_u(X \sigma_v S_{20}) \]

\[ = (S_{1+v} X) \sigma_u(S_{2+v} X) \quad \text{Defn} \]

\[ = S_{1+v} S_{2+v+u}(X \sigma_u X) \quad \text{Defn} \]
and $X \neq X \sigma_u X$ for any value of $u$ ($x \neq 0$).

Therefore we have proved that Vučković's function $Y \sigma^X_v$ is neither associative, nor commutative, except possibly in the case $v = 0$; this particular case is that of addition, that is the function $Y + X$ defined by the usual schema

$$Y + 0 = Y$$

$$Y + S_u X = S_u (Y + X) \quad u = 1, \ldots, n$$

**PROPOSITION**

$Y \sigma^X_0 = Y + X$

**Proof**

$L(0) = Y \sigma^0_0$

$$= Y \quad \text{Defn}$$

$R(0) = Y + 0$

$$= Y \quad \text{Defn}$$

$L(S^X) = Y \sigma^X_0 S^u\ X$

$$= S^u_0 (Y \sigma^X_0 X) \quad \text{Defn}$$

$$= S^u_0 (Y \sigma^X_0 X) \quad \text{Normal arithmetical rules of addition}$$

$$= S^u_0 L(X) \quad u = 1, \ldots, n$$

$R(S^X) = Y + S^u\ X$

$$= S^u (Y + X) \quad \text{Defn}$$

$$= S^u R(X) \quad u = 1, \ldots, n$$

Hence by the uniqueness rule of inference

$L(X) = R(X)$ as required.

It will be shown that addition is commutative and associative by a series of Lemmas.
LEMMA \[0 + X = X\]

Proof

\[L(0) = 0 + 0\]

\[= 0\] \hspace{1cm} \text{Defn}

\[R(0) = 0\]

\[L(S_uX) = 0 + S_uX\]

\[= S_u(0 + X)\] \hspace{1cm} \text{Defn}

\[= S_uL(X)\]

\[R(S_uX) = S_uX\]

\[= S_uR(X)\]

Hence by the uniqueness rule of inference

\[L(X) = R(X)\] as required.

LEMMA \[S_uY + X = S_u(Y + X)\]

Proof

\[L(0) = S_uY + 0\]

\[= S_uY\] \hspace{1cm} \text{Defn}

\[R(0) = S_u(Y + 0)\]

\[= S_uY\] \hspace{1cm} \text{Defn}

\[L(S_vX) = S_uY + S_vX\]

\[= S_v(S_uY + X)\] \hspace{1cm} \text{Defn}

\[= S_vL(X)\]

\[R(S_vX) = S_u(Y + S_vX)\]

\[= S_uS_v(Y + X)\] \hspace{1cm} \text{Defn}

\[= S_vS_u(Y + X)\] \hspace{1cm} \text{Commutativity of successor}

\[= S_vR(X)\]
Hence by the uniqueness rule of inference

\[ L(X) = R(X) \] as required

**PROPOSITION** \[ Y + X = X + Y \] Commutativity

**Proof**

\[ L(0) = Y + 0 \]
\[ = Y \]
\[ R(0) = 0 + Y \]
\[ = Y \]

\[ L(S^X) = Y + S^X \]
\[ = S^X(Y + X) \]
\[ = S^X L(X) \]

\[ R(S^X) = S^X + Y \]
\[ = S^X(Y + X) \]
\[ = S^X R(X) \]

Hence by the uniqueness rule of inference

\[ L(X) = R(X) \] as required.

**PROPOSITION** \[ (X + Y) + Z = X + (Y + Z) \] Associativity

**Proof**

\[ L(0) = (0 + Y) + Z \]
\[ = Y + Z \]
\[ R(0) = 0 + (Y + Z) \]
\[ = Y + Z \]

\[ L(S^X) = (S^X + Y) + Z \]
\[ = S^X(X + Y) + Z \]
\[ = S^X((X + Y) + Z) \]
\[ = S^X L(X) \]
\[ R(S^uX) = S^uX + (Y + Z) \]
\[ = S^u(X + (Y + Z)) \quad \text{Previous Lemma} \]
\[ = S^uR(X) \]

Hence by the uniqueness rule of inference

\[ L(X) = R(X). \]

Having considered addition functions in our arithmetic, we shall now proceed to consider difference functions.

The basic functions for the addition functions are the successor functions. For the difference functions the basic functions are the predecessor functions \( P^uX \ v = 1, \ldots, n \), defined by the schema

\[ P^u0 = 0 \]
\[ P^uS^wX = \begin{cases} X, & u = v \\ S^wP^vX, & u \neq v \end{cases} \]

the commutativity condition is

\[ B^w(S^wX, B^w(X,Y,Z)) = \begin{cases} S^wX, & u = v \\ S^uB^w(X,Y,Z) & u \neq v \end{cases} \]
\[ = \begin{cases} S^wX, & u = v \\ S^wX & u \neq v \land w = v \\ S^wS^uZ & u \neq v \land w \neq v \end{cases} \]
\[ = \begin{cases} S^uX & u = v \land w = v \\ S^wX & u = v \land w \neq v \\ S^uX & u \neq v \land w = v \\ S^wS^uZ & u \neq v \land w \neq v \end{cases} \quad \text{Commutativity of successors} \]
\[ = \begin{cases} S^uX & w = v \\ S^wX & w \neq v \land u = v \\ S^wS^uZ & w \neq v \land u \neq v \end{cases} \]
We are now able to introduce the difference function \( Y \hat{-} X \), defined by the schema.

\[
\begin{align*}
Y \hat{-} 0 &= Y \\
Y \hat{-} S_u X &= P_u (Y \hat{-} X) & u = 1, \ldots, n
\end{align*}
\]

In order to prove the commutativity condition for this function we first need to prove the following Lemma:

**Lemma** \( P_u P_v X = P_v P_u X \)

**Proof**

\[
\begin{align*}
L(0) &= P_u P_v 0 \\
&= P_u 0 \\
&= 0 & \text{Defn} \\
R(0) &= P_v P_u 0 \\
&= P_v 0 \\
&= 0 & \text{Defn} \\
L(S_w X) &= P_u P_v S_w X \\
&= \begin{cases} 
P_u X & v = w \\
P_v S_P X & v \neq w \\
\end{cases} & \text{Defn} \\
&= \begin{cases} 
P_u X & v = w \\
P_v X & v \neq w \& u = w \\
S_w P_u X & v \neq w \& u \neq w \\
\end{cases} & \text{Defn} \\
&= \begin{cases} 
P_u X & v = w \\
P_v X & v \neq w \& u = w \\
S_w L(X) & v \neq w \& u \neq w \\
\end{cases}
\end{align*}
\]
\[
\begin{align*}
R(S_w X) &= P_v P_u S_w X \\
&= \begin{cases} 
P_v X & \text{u=w} \\
P_v P_u S_w P_u X & \text{u\neq w} \\
\end{cases} \quad \text{Defn} \\
&= \begin{cases} 
P_v X & \text{u=w} \\
P_u X & \text{u\neq w \& v=w} \\
S_w P_v P_u X & \text{u\neq w \& v\neq w} \\
\end{cases} \quad \text{Defn} \\
&= \begin{cases} 
P_v X & \text{v=w} \\
P_u X & \text{v\neq w \& u=w} \\
S_w P_v P_u X & \text{v\neq w \& u\neq w} \\
\end{cases} \quad \text{Defn}
\end{align*}
\]

Now letting

\[
B_w(X,Y,Z) = \begin{cases} 
P_v X & \text{w=v} \\
P_u X & \text{w=u} \\
S_w Z & \text{w\neq v \& w\neq u} \\
\end{cases}
\]

then

\[
L(S_w X) = B_w(X,Y,L(X))
\]

\[
R(S_w X) = B_w(X,Y,R(X)).
\]

Hence by the uniqueness rule of inference

\[
L(X) = R(X) \quad \text{as required}
\]

The commutativity condition for \(Y - X\) follows by

\[
B_u(S_v X,Y,B_v(X,Y,Z)) = P_u B_v(X,Y,Z)
\]

\[
= P_u P_v Z
\]

\[
= P_v P_u Z \quad \text{Previous Lemma}
\]

\[
= P_v B_u(X,Y,Z)
\]

\[
= B_v(S_u X,Y,B_u(X,Y,Z)) \quad \text{as required}
\]
Corresponding to the Vušković's functions $\sigma_v$ for addition functions, for difference functions, we define functions $\tau_v$ by the schema

\[ Y \tau_v 0 = Y \]
\[ Y \tau_v S u X = P_{u+v}(Y \tau_v X) \quad v=0,\ldots,n-1, \]

again taking the excess over $n$ for $u+v$. The case of $v=n$ is considered as $v=0$.

For the commutativity condition we have

\[ B_u(S w X, Y, B_w(X, Y, Z)) = P_{u+v} B(X, Y, Z) \]
\[ = P_{u+v} P_{w+v} Z \]
\[ = P_{w+v} P_{u+v} Z \quad \text{Previous Lemma} \]
\[ = P_{w+v} B_u(X, Y, Z) \]
\[ = B_u(S w X, Y, B_u(X, Y, Z)) \quad \text{as required} \]

As for Vušković's functions $\sigma_v$, the particular case $v=0$ being addition, so for $\tau_v$, with $v=0$ we have the normal difference function $Y - X$.

**Proposition**

\[ Y \tau_0 X = Y - X \]

**Proof**

\[ L(0) = Y \tau_0 0 \]
\[ = Y \quad \text{Defn} \]
\[ R(0) = Y - 0 \]
\[ = Y \quad \text{Defn} \]
\[ L(S u X) = Y \tau_0 S u X \]
\[ = P_{u+0}(Y \tau_0 X) \quad \text{Defn} \]
\[ = P_u L(X) \]
\[ R(S^uX) = Y - S^uX \]
\[ = P_u(Y - x) \quad \text{Defn} \]
\[ = P_u R(X) \]

Hence by the uniqueness rule of inference
\[ L(X) = R(X) \]

The common properties of the addition function and the difference function in single successor arithmetic are
\[ y - x = Sy - Sx \]
\[ (y + x) - x = y \]
\[ x + (y - x) = y + (x - y) \]

For the first two of these we shall prove in multiple successor arithmetic, that is
\[ Y - X = S_y Y - S_y X \quad \text{for } v = 1, \ldots, n \]
\[ (Y + X) - x = Y \]

The third equation \( X + (Y - x) = Y + (X - Y) \), known as the Key Equation, will not be proved in this thesis. The proof in primitive recursive multiple successor arithmetic is due to Professor R.L. Goodstein, and can be found in [9].

**Proposition** \( Y - X = S_y Y - S_y X \)

**Proof**
\[ L(0) = Y - 0 \]
\[ = Y \quad \text{Defn} \]
\[ R(0) = S_y Y - S_y 0 \]
\[ = P_y(S_y Y - 0) \quad \text{Defn} \]
\[ = P_y S_y Y \quad \text{Defn} \]
\[ = Y \quad \text{Defn} \]
\[ L(S_u X) = Y \uparrow \uparrow S_u X \]
\[ = P_u (Y \uparrow \uparrow X) \quad \text{Defn} \]
\[ = P_u L(X) \]
\[ R(S_u X) = S_v Y \uparrow \uparrow S_v S_u X \]
\[ = S_v Y \uparrow \uparrow S_u S_v X \quad \text{Commutativity of successors} \]
\[ = P_u (S_v Y \uparrow \uparrow S_v X) \quad \text{Defn} \]
\[ = P_u R(X) \]

Hence by the uniqueness rule of inference

\[ (L(X) = R(X)) \text{ as required} \]

**PROPOSITION**

\[ (Y + X) \uparrow \uparrow X = Y \]

**Proof**

\[ L(0) = (Y + 0) \uparrow \uparrow 0 \]
\[ = Y + 0 \quad \text{Defn} \]
\[ = Y \quad \text{Defn} \]
\[ R(0) = Y \quad \text{Defn} \]
\[ L(S_u X) = (Y + S_u X) \uparrow \uparrow S_u X \]
\[ = S_u (Y + X) \uparrow \uparrow S_u X \quad \text{Defn} \]
\[ = (Y + X) \uparrow \uparrow X \quad \text{Previous Proposition} \]
\[ = L(X) \]
\[ R(S_u X) = Y \]
\[ = R(X) \]

Hence by the uniqueness rule of inference

\[ L(X) = R(X) \text{ as required.} \]
Let us now consider the following three equations, relating $\sigma_v$ and $\tau_v$.

1. $Y \tau_v X = S_u Y \tau_v S_u X$

2. $(Y \sigma_v X) \tau_v X = Y$

3. $X \sigma_v (Y \tau_v X) = Y \sigma_v (X \tau_v Y)$

It would surely be remarkable if all three equations were true, especially 3., which would be a generalisation of the Key Equation, but unfortunately this equation is not true. Equation 1. needs a slight adjustment, and equation 2. is true. Equation 1. we adjust to

$$Y \tau_v X = S_{u+v} Y \tau_v S_u X$$

PROPOSITION

$$Y \tau_v X = S_{u+v} Y \tau_v S_u X$$

Proof

$L(0) = Y \tau_v 0$

$$= Y$$

$R(0) = S_{u+v} Y \tau_v S_u 0$

$$= P_{u+v} (S_{u+v} Y \tau_v 0)$$
$$= P_{u+v} S_{u+v} Y$$
$$= Y$$

$L(S_w X) = Y \tau_v S_w X$

$$= P_{w+v} (Y \tau_v X)$$
$$= P_{w+v} L(X)$
Hence by the uniqueness rule of inference

\[ L(x) = R(x) \]
as required

**PROPOSITION**

\[ (Y \sigma_v x) \tau_v x = Y \]

**Proof**

\[ L(0) = (Y \sigma_v 0) \tau_v 0 \]

\[ = Y \sigma_v 0 \quad \text{Defn} \]

\[ = Y \quad \text{Defn} \]

\[ R(0) = Y \]

\[ L(S_u x) = (Y \sigma_v S_u x) \tau_v S_u x \]

\[ = S_u \sigma_v (Y \sigma_v x) \tau_v S_u x \quad \text{Defn} \]

\[ = (Y \sigma_v x) \tau_v x \quad \text{Previous Proposition} \]

\[ = L(x) \]

\[ R(S_u x) = Y \]

\[ = R(x) \]

Hence by the uniqueness rule of inference

\[ L(x) = R(x) \]

Further, the equation \( Z \cdot (Y + X) = (Z \cdot Y) \cdot X \) is provable in both single and multiple successor arithmetic; using the \( \sigma_v \) and \( \tau_v \) functions we are able to make the following generalisation of this equation:
\[ Z \tau_u(Y \sigma_v X) = (Z \tau_u Y) \tau_{u+v} X \]

**PROPOSITION**

\[ Z \tau_u(Y \sigma_v X) = (Z \tau_u Y) \tau_{u+v} X \]

**Proof**

\[ L(0) = Z \tau_u(Y \sigma_v 0) \]

\[ = Z \tau_u Y \quad \text{Defn} \]

\[ R(0) = (Z \tau_u Y) \tau_{u+v} 0 \]

\[ = Z \tau_u Y \quad \text{Defn} \]

\[ L(S_w X) = Z \tau_u(Y \sigma_v S_w X) \]

\[ = Z \tau_u S_{w+v}(Y \sigma_v X) \quad \text{Defn} \]

\[ = P_{w+v+u}(Z \tau_u(Y \sigma_v X)) \quad \text{Defn} \]

\[ = P_{w+v+u} L(X) \]

\[ R(S_w X) = (Z \tau_u Y) \tau_{u+v} S_w X \]

\[ = P_{w+v+u} ((Z \tau_u Y) \tau_{u+v} X) \quad \text{Defn} \]

\[ = P_{w+v+u} R(X) \]

Hence by the uniqueness rule of inference,

\[ L(X) = R(X) \]

**PROPOSITION**

\[ Y \sigma_v (X \tau_v Y) \neq X \sigma_v (Y \tau_v X) \]

**Proof**

A simple counter-example will serve to illustrate this result.

Put \( Y = S_1 0 \), \( X = S_2 0 \), \( v = 1 \)

\[ L(X) = S_1 0 \sigma_2(S_2 0 \tau_1 S_1 0) \]

\[ = S_1 0 \sigma_2 P_2(S_2 0 \tau_1 0) \quad \text{Defn} \]

\[ = S_1 0 \sigma_1 P_2 S_2 0 \quad \text{Defn} \]

\[ = S_1 0 \sigma_1 0 \quad \text{Defn} \]

\[ = S_1 0 \quad \text{Defn} \]
\[ R(X) = S_20 \sigma_1 (S_10 \tau_1 S_20) \]
\[ = S_20 \sigma_1 P_3 (S_10 \tau_1 0) \]
\[ = S_20 \sigma_1 P_3 S_1 0 \]
\[ = S_20 \sigma_1 S_1 0 \]
\[ = S_2 (S_20 \sigma_1 0) \]
\[ = S_2 S_20 \]
\[ \neq L(X) \]

The multiplication functions in this arithmetic will be left until a later chapter.

A function particular to multiple successor arithmetic, which we have not yet considered, is the component function. This is a most valuable function which allows us to apply and utilise many of the results and principles of single successor arithmetic. The component function \( C_v X \) is defined by the schema

\[ C_v 0 = 0 \]

\[ C_v S_u X = \begin{cases}  
C_v X & u \neq v \\
S_u C_v X & u = v 
\end{cases} \]

the commutativity condition is

\[ R_u (S_w X, Y, R_w (X, Y, Z)) = \begin{cases}  
R_w (X, Y, Z) & u \neq v \\
S_u R_w (X, Y, Z) & u = v 
\end{cases} \]

\[ = \begin{cases}  
Z & u \neq v \land w \neq v \\
S_w Z & u \neq w \land w = v \\
S_u Z & u = v \land w \neq v \\
S_u S_u Z & u = v \land w = v 
\end{cases} \]
Clearly \( C^X \) only allows use of the successors \( S_v \) in \( X \), and thus reduces \( X \) to one type of successor. The reason the word component is used for this function is due to M.T.Partis in \([4]\), where it is proved that variables in multiple successor arithmetic can be regarded as 'n'tuples with only one successor. The function \( C^X \) reduces \( X \) to only the 'v'th component, that is if

\[
X = (X_1, X_2, \ldots, X_n)
\]

\[
C^X = (0, 0, \ldots, 0, X_v, 0, \ldots, 0)
\]

Further properties of the component function will be considered in the formal development of the arithmetic in chapter III.

**PROBLEMS ATTEMPTED**

The problems that have been attempted in this thesis are:

1. Reduction of the number of parameters in definition by primitive recursion.

An 'm+1'ary function \( F(X, Y_1, Y_2, \ldots, Y_m) \) is said to be defined by primitive recursion from \( A \) and \( B \) if

\[
F(0, Y_1, Y_2, \ldots, Y_m) = A(Y_1, Y_2, \ldots, Y_m)
\]

\[
F(S_u, Y_1, Y_2, \ldots, Y_m) = B_u(X, Y_1, Y_2, \ldots, Y_m, F(X, Y_1, Y_2, \ldots, Y_m))
\]

\[ u=1, \ldots, n \]
where \( A(Y_1, Y_2, \ldots, Y_m) \) and \( B_u(X, Y_1, Y_2, \ldots, Y_m, Z) \) \( u=1, \ldots, n \) are initial functions or previously defined functions, and further, in order to preserve commutativity of the successors

\[
B_u(S_v X, Y_1, Y_2, \ldots, Y_m, B_v(X, Y_1, Y_2, \ldots, Y_m, Z)) = B_v(S_u X, Y_1, Y_2, \ldots, Y_m, B_u(X, Y_1, Y_2, \ldots, Y_m, Z))
\]

for all \( u, v \).

The problem is to reduce this definition to the defining schemata.

(i) Define \( F(X, Y) \) by the schema, called \( R_1 \)

\[
F(0, Y) = A(Y) \\
F(S_u X, Y) = B_u(X, Y, F(X, Y)) \quad u=1, \ldots, n
\]

where \( B_u(S_v X, Y, B_v(X, Y, Z)) = B_v(S_u X, Y, B_u(X, Y, Z)) \) for all \( u, v \).

(ii) Define \( F(X, Y) \) by the schema, called \( R_1^* \)

\[
F(0, Y) = A(Y) \\
F(S_u X, Y) = B_u(X, F(X, Y)) \quad u=1, \ldots, n
\]

where \( B_u(S_v X, B_v(X, Y, Z)) = B_v(S_u X, B_u(X, Y, Z)) \) for all \( u, v \).

(iii) Define \( F(X, Y) \) by the schema, called \( R_1^{**} \)

\[
F(0, Y) = A(Y) \\
F(S_u X, Y) = B_u(Y, F(X, Y)) \quad u=1, \ldots, n
\]

where \( B_u(Y, B_v(Y, Z)) = B_v(Y, B_u(Y, Z)) \) for all \( u, v \).

(iv) Define \( F(X, Y) \) by the schema called \( R_1^{***} \)

\[
F(0, Y) = A(Y) \\
F(S_u X, Y) = B_u(F(X, Y)) \quad u=1, \ldots, n
\]

where \( B_u(B_v(Z)) = B_v(B_u(Z)) \) for all \( u, v \).
We consider further what functions, if any, are required to be added to the initial set of functions in order to produce the set of all primitive recursive functions, from each of these definitions.

In single successor arithmetic, the reduction of parameters in the definition of primitive recursion is to be found in [1] and [3].

2. A complete analysis of X.

In single successor arithmetic a numeral $x$ can be compared relatively simply to any other numeral with the aid of the difference function $y - x$. Using both $y - x$ and $x - y$ a complete comparison is made between $x$ and $y$. In multiple successor arithmetic, however, $X$ is not so easily analysed. The questions which we seek to answer are the following, in order to obtain a complete analysis of $X$.

(i) How many successor symbols in $X$?

(ii) How many different successor symbols?

(iii) What these different successor symbols are.

(iv) How many of each successor symbol?

With these four questions answered the analysis of $X$ is obviously complete and a comparison between $X$ and $Y$ can be effected.
CHAPTER II.

REDUCTION OF PARAMETERS IN THE DEFINITION BY PRIMITIVE RECURSION

In this chapter we will consider the problem of the reduction of parameters in the definition of primitive recursion in Commutative Multiple Successor Arithmetics.

The first task will be to consider the reduction, given that we have already in our arithmetic a set of functions with special properties. Different functions will then be considered for this set. In chapter III the arithmetic will be formalised and a proof of these properties will be given.

In single successor arithmetic the reduction is solved with the aid of three functions \( L(x), K(x) \) and \( J(u,v) \), such that

\[
L(J(u,v)) = u \\
K(J(u,v)) = v
\]

We shall suppose first of all that in commutative multiple successor arithmetic we have three functions \( L(x), K(x) \) & \( J(U,V) \) with the above properties. With these functions we are able to reduce the number of parameters in the definition of primitive recursion.

Consider therefore the definition of primitive recursion with 'm' parameters (m finite); this schema we will call \( R_m: \)

A function \( F(X,Y_1,Y_2,...,Y_m) \) is said to be defined by primitive recursion from \( A \) and \( B \) if

\[
F(0,Y_1,Y_2,...,Y_m) = A(Y_1,Y_2,...,Y_m) \\
F(S_uX,Y_1,Y_2,...,Y_m) = B_u(X,Y_1,Y_2,...,Y_m,F(X,Y_1,Y_2,...,Y_m)) \quad u = 1,...,n
\]

where \( B_u(S_vX,Y_1,Y_2,...,Y_m,B_v(X,Y_1,Y_2,...,Y_m)) = B_v(S_uX,Y_1,Y_2,...,Y_m,B_u(X,Y_1,Y_2,...,Y_m)) \) all \( u,v \)

and \( A(....) \) and \( B_u(....) \) \( u=1,...,n \) are previously defined functions.
Now let \( W = J(Y_1, Y_2) \)
and let \( G(X, W, Y_3, Y_4, \ldots, Y_m) = F(X, L(W), K(W), Y_3, Y_4, \ldots, Y_m) \)
\[ = F(X, Y_1, Y_2, \ldots, Y_m) \).

Then \( G(X, W, Y_3, Y_4, \ldots, Y_m) \) is defined by
\[
G(0, W, Y_3, Y_4, \ldots, Y_m) = A(L(W), K(W), Y_3, \ldots, Y_m)
\]
\[ = A'(W, Y_3, Y_4, \ldots, Y_m) \]
\[
G(S_u X, W, Y_3, Y_4, \ldots, Y_m) = B_u(X, L(W), K(W), Y_3, \ldots, Y_m, G(X, W, Y_3, \ldots, Y_m))
\]
\[ u = 1, \ldots, n \]
\[ = B'_u(X, W, Y_3, \ldots, Y_m, G(X, W, Y_3, \ldots, Y_m)) \]
\[ u = 1, \ldots, n \]

and the commutativity condition reduces to
\[
B_u(S_v X, L(W), K(W), Y_3, \ldots, Y_m, B_v(X, L(W), K(W), Y_3, \ldots, Y_m)) =
B_v(S_u X, L(W), K(W), Y_3, \ldots, Y_m, B_u(X, L(W), K(W), Y_3, \ldots, Y_m, Z))
\]
\[ \text{all } u, v \]

and therefore
\[
B'_u(S_v X, W, Y_3, \ldots, Y_m, B'_v(X, W, Y_3, \ldots, Y_m, Z)) = B'_v(S_u X, W, Y_3, \ldots, Y_m, B'_u(X, W, Y_3, \ldots, Y_m, Z))
\]
\[ \text{all } u, v \]

Hence the definition by primitive recursion is now reduced to a schema
\( R_{m-1} \) containing \( m-1 \) parameters, that is, \( F(X, Y_1, Y_2, \ldots, Y_{m-1}) \) is defined by
\[
F(0, Y_1, Y_2, \ldots, Y_{m-1}) = A(Y_1, Y_2, \ldots, Y_{m-1})
\]
\[
F(S_u X, Y_1, Y_2, \ldots, Y_{m-1}) = B_u(X, Y_1, \ldots, Y_{m-1}, F(X, Y_1, \ldots, Y_{m-1})) \quad u = 1, \ldots, n
\]
where
\[
B_u(S_v X, Y_1, \ldots, Y_{m-1}, B_v(X, Y_1, \ldots, Y_{m-1}, Z)) = B_v(S_u X, Y_1, \ldots, Y_{m-1}, B_u(X, Y_1, \ldots, Y_{m-1}, Z))
\]
\[ \text{all } u, v. \]
By repeated use of this method we can reduce the definition to a definition with only one parameter $R_1$, that is, $F(X,Y)$ is defined from $A$ and $B$ by:

$$F(0,Y) = A(Y)$$

$$F(S_u X, Y) = B_u(X, Y, F(X, Y)) \quad u = 1, \ldots, n$$

where $R_u(S_v X, Y, B_v(X, Y, Z)) = R_v(S_u X, Y, B_u(X, Y, Z)) \quad \text{all } u, v$.

The proof just given reduces $R_m$ to $R_1$ in $m-1$ steps.

A similar technique can be used to complete the proof in a single step. We proceed as follows:

Let $W = J(Y_1, J(Y_2, J(Y_3, \ldots, J(Y_{m-1}, Y_m)))) \ldots$

Let $K^2(W) = K(K(W))$, and $K^\tau(W) = \overline{K(K(\ldots, K(W))})\ldots$

We therefore have that

$$L(W) = Y_1$$

$$L(K(W)) = Y_2$$

$$L(K^2(W)) = Y_3$$

$$\vdots$$

$$L(K^{m-2}(W)) = Y_{m-1}$$

$$K^{m-1}(W) = Y_m$$

and let $G(X, W) = F(X, L(W), L(K(W)), \ldots, L(K^{m-2}(W)), K^{m-1}(W))$.

We therefore define $G(X, W)$ by

$$G(0, W) = A(L(W), L(K(W)), \ldots, L(K^{m-2}(W)), K^{m-1}(W))$$

$$= A'(W), \text{ say.}$$
\[ G(S_uX, W) = B_u(X, L(W), L(K(W)), \ldots, L(K^{m-2}(W)), K^{m-1}(W)), \]
\[ F(X, L(W), L(K(W)), \ldots, L(K^{m-1}(W)), K^{m-1}(W))) \]
\[ u = 1, \ldots, n \]
\[ = B'_u(X, W, G(X, W)), \text{ say } u = 1, \ldots, n \]

and the commutativity condition reduces to

\[ B_u(S_vX, L(W), L(K(W)), \ldots, L(K^{m-2}(W)), K^{m-1}(W)), B_v(X, L(W), L(K(W))), \ldots, L(K^{m-2}(W)), K^{m-1}(W))) = B_v(S_uX, L(W), L(K(W)), \ldots, L(K^{m-2}(W)), K^{m-1}(W)), B_u(X, L(W), L(K(W))), \ldots, L(K^{m-2}(W)), K^{m-1}(W))) \]

\[ \therefore B'_u(S_vX, W, B'_v(X, W, Z)) = B'_v(S_uX, W, B'_u(X, W, Z)) \quad \text{all } u, v. \]

Thus \( R_m \) is reduced to \( R_1 \).

We now require:

(i) to reduce \( R_1 \) to \( R_1^* \) where \( F(X, Y) \) is defined by the schema

\[ F(0, Y) = A(Y) \]
\[ F(S_uX, Y) = B_u(X, F(X, Y)) \quad u = 1, \ldots, n \]

where \( B_u(S_vX, B_v(Y, Z)) = B_v(S_uX, B_u(Y, Z)) \quad \text{all } u, v \)

(ii) to reduce \( R_1 \) to \( R_1^{**} \) where \( F(X, Y) \) is defined by the schema

\[ F(0, Y) = A(Y) \]
\[ F(S_uX, Y) = B_u(Y, F(X, Y)) \quad u = 1, \ldots, n \]

where \( B_u(Y, B_v(Y, Z)) = B_v(Y, B_u(Y, Z)) \quad \text{all } u, v. \)

(iii) to reduce \( R_1 \) to \( R_1^{***} \) where \( F(X, Y) \) is defined by the schema

\[ F(0, Y) = A(Y) \]
\[ F(S_uX, Y) = B_u(F(X, Y)) \quad u = 1, \ldots, n \]

where \( B_u(B_v(Z)) = B_v(B_u(Z)) \quad \text{all } u, v. \)
Reduction of \( R_1 \) to \( R_1^* \):

Let \( G(X,Y) = J(Y,F(X,Y)) \).

Then \( G(X,Y) \) is defined by the schema

\[
G(0,Y) = J(Y,A(Y)) = A(Y), \text{ say}
\]

\[
G(S_uX,Y) = J(Y,F(S_uX,Y)) = J(Y,B_u(X,Y,F(X,Y)))
\]

\[
= J(L(G(X,Y)),B_u(X,L(G(X,Y)),K(G(X,Y))))
\]

\[
= B_u'(X,G(X,Y)), \text{ say } u = 1, \ldots, n
\]

and for the commutativity condition we let

\[
Z' = J(Y,Z).
\]

Now \( B_v'(S_uX,B_v'(X,Z')) = B_u'(S_uX,J(L(Z'),B_v(X,L(Z'),K(Z')))) \)

\[
= B_u'(S_uX,J(Y,B_v'(X,Y,Z)))
\]

\[
= J(L(J(Y,B_v'(X,Y,Z))),B_u(S_uX,L(J(Y,B_v'(X,Y,Z))),K(J(Y,B_v'(X,Y,Z)))))
\]

\[
= J(Y,B_u(S_uX,B_v'(X,Y,Z)))
\]

\[
= J(Y,B_v(S_uX,B_u(X,Y,Z)))
\]

\[
= J(L(J(Y,B_u(X,Y,Z))),B_v(S_uX,L(J(Y,B_u(X,Y,Z))),K(J(Y,B_u(X,Y,Z)))))
\]

\[
= B_v'(S_uX,J(Y,B_u(X,Y,Z)))
\]

\[
= B_v'(S_uX,J(L(Z'),B_u(X,L(Z'),K(Z'))))
\]

\[
= B_v'(S_uX,B_u'(X,Z')) \quad \text{all } u, v,
\]

which is the required commutativity condition.

Thus \( F(X,Y) = K(G(X,Y)) \).
Reduction of $R_1$ to $R_1^{**}$:

Let $G(X,Y) = J(X,F(X,Y))$.

Then $G(X,Y)$ is defined by the schema

$$G(0,Y) = J(0,A(Y)) = A'(Y), \text{ say}$$

$$G(S_u X,Y) = J(S_u X,F(S_u X,Y)) \quad u = 1, \ldots, n$$

$$= J(S_u X, B_u (X,Y,F(X,Y)))$$

$$= J(S_u L(G(X,Y)), B_u (L(G(X,Y)), Y,K(G(X,Y))))$$

$$= B'_u (Y, G(X,Y)), \text{ say } u = 1, \ldots, n$$

and for the commutativity condition we let

$$Z' = J(X,Z).$$

Thus

$$B'_u (Y, B'_v (Y,Z')) = B'_u (J(S_v L(Z'), B_v (L(Z'), Y,K(Z'))))$$

$$= B'_u (Y, J(S_v X, B_v (X,Y,Z)))$$

$$= J(S_u L(J(S_v X, B_v (X,Y,Z))), B_u (L(J(S_v X, B_v (X,Y,Z))), Y,K(J(S_v X, B_v (X,Y,Z)))))$$

$$= J(S_u S_u X, B_u (S_v X, B_v (X,Y,Z)))$$

$$= J(S_v S_u X, B_v (S_u X, B_u (X,Y,Z))) \quad \text{(commutativity condition for } R_1 \text{ and commutativity of successors})$$

$$= J(S_v L(J(S_u X, B_u (X,Y,Z))), B_v (L(J(S_u X, B_u (X,Y,Z))), Y,K(J(S_u X, B_u (X,Y,Z)))))$$

$$= B'_v (Y, J(S_u X, B_u (X,Y,Z)))$$

$$= B'_v (J(S_u L(Z'), B_u L(Z'), Y,K(Z'))))$$

$$= B'_v (Y, B'_u (Y,Z')) \quad \text{ all } u,v,$$

which is the required commutativity condition.
Thus $F(X,Y) = K(G(X,Y))$.

Reduction of $R_1$ to $R_1^{***}$:

The reduction of $R_1$ to $R_1^{***}$ can be achieved in three ways, that is

1. $R_1$ to $R_1^*$ then $R_1^*$ to $R_1^{***}$
2. $R_1$ to $R_1^{**}$ then $R_1^{**}$ to $R_1^{***}$
3. $R_1$ direct to $R_1^{***}$.

Both the reductions $R_1^*$ to $R_1^{***}$ and $R_1^{**}$ to $R_1^{***}$ are similar to the reductions $R_1$ to $R_1^{**}$ and $R_1$ to $R_1^*$ respectively.

We shall consider the reductions $R_1^*$ to $R_1^{***}$ and $R_1$ to $R_1^{***}$.

Reduction $R_1^*$ to $R_1^{***}$:

$R_1^*$ is the schema

$$F(0,Y) = A(Y)$$
$$F(S^X,Y) = B_1(X,F(X,Y)) \quad u = 1, \ldots, n$$

where $B_u(S^X,B_v(X,Z)) = B_v(S^X,B_u(X,Z))$ all $u,v$.

First let $G(X,Y) = J(X,F(X,Y))$.

Then $G(X,Y)$ is defined by the schema

$$G(0,Y) = J(0,A(Y))$$
$$= A'(Y), \text{ say}$$
$$G(S^X,Y) = J(S^X,F(S^X,Y))$$
$$= J(S^X,B_u(X,F(X,Y))) \quad u = 1, \ldots, n$$
$$= J(S^X,L(G(X,Y)),B_u(L(G(X,Y)),K(G(X,Y))))$$
$$= B_u'(G(X,Y)), \text{ say,} \quad u = 1, \ldots, n.$$
and for the commutativity condition we let

\[ Z' = J(X, Z) \]

Then

\[
B'_u(B'_v(Z')) = B'_u(J(S_v L(Z'), B_v (L(Z'), K(Z'))))
\]

\[
= B'_u(J(S_v X, B_v (X, Z)))
\]

\[
= J(S_u L(J(S_v X, B_v (X, Z))), B_u (L(J(S_v X, B_v (X, Z))), K(J(S_v X, B_v (X, Z)))))
\]

\[
= J(S_u S_v X, B_u (S_v X, B_v (X, Z)))
\]

\[
= J(S_v S_u X, B_v (S_u X, B_u (X, Z)))
\]

(commutativity condition of \(R_1^*\) and commutativity of successors)

\[
= J(S_v L(J(S_u X, B_u (X, Z))), B_v (L(J(S_u X, B_u (X, Z))), K(J(S_u X, B_u (X, Z)))))
\]

\[
= B'_v(J(S_u X, B_u (X, Z)))
\]

\[
= B'_v(J(S_u L(Z'), B_u (L(Z'), K(Z'))))
\]

\[
= B'_v(B'_u(Z'))
\]

which is the required commutativity condition.

Thus \(F(X, Y) = K(G(X, Y))\).

Reduction of \(R_1\) to \(R_1^{***}\):

\(R_1\) is the schema

\[
F(0, Y) = A(Y)
\]

\[
F(S_u X, Y) = B_u(x, y, F(x, y)) \quad u = 1, \ldots, n
\]

where \(B_u(S_v X, Y, B_v (X, Y, Z)) = B_v(S_u X, Y, B_u (X, Y, Z))\) all \(u, v\).

First let \(G(X, Y) = J(X, J(Y, F(X, Y)))\).

Then \(G(X, Y)\) is defined by the schema

\[
G(0, Y) = J(0, J(Y, F(0, Y)))
\]

\[
= J(0, J(Y, A(Y)))
\]

\[
= A'(Y), \text{ say}
\]
\[ G(S^X, Y) = J(S^X, J(F(S^X, Y))) \quad u = 1, \ldots, n \]

\[ = J(S^X, J(Y, B_u(X, Y, F(X, Y)))) \]

\[ = J(S^X, J(L(G(X, Y)), J(L(K(G(X, Y))), B_u(L(G(X, Y)), L(K(G(X, Y))))) \]

\[ = B_u'(G(X, Y)), \text{ say } u = 1, \ldots, n \]

and for the commutativity condition we let

\[ Z' = J(X, J(Y, Z)). \]

Then

\[ B_u'(B_v'(Z')) = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

\[ = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

\[ = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

\[ = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

\[ = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

\[ = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

\[ = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

\[ = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

\[ = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

\[ = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

\[ = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

\[ = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

\[ = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

\[ = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

\[ = B_u'(J(S^X, J(Y, B_v(X, Y, Z)))) \]

which is the required commutativity condition.

Thus \( F(X, Y) = K(K(G(X, Y))) \).
In the previous sections the problem of reduction of parameters in the definition of primitive recursion with 'm' parameters has been successfully achieved with the aid of the set of functions $L(X)$, $K(X)$ & $J(U,V)$. Now we must consider what functions have the required properties

\begin{align*}
L(J(U,V)) &= U \\
K(J(U,V)) &= V.
\end{align*}

In order to discover these functions, we first examine the functions used in the reduction in single successor arithmetic. To do this we need to define the following.

**Defn.** $y + x$ Addition

- $y + 0 = y$
- $y + Sx = S(y + x)$

**Defn.** $Px$ Predecessor

- $P0 = 0$
- $PSx = x$

**Defn.** $y \cdot x$ Difference

- $y \cdot 0 = y$
- $y \cdot Sx = P(y \cdot x)$

**Defn.** $y \cdot x$ Multiplication

- $y \cdot 0 = 0$
- $y \cdot Sx = y \cdot x + y$

**Defn.** $x \cdot x = x^2$ Notational definition (explicit)

**Defn.** $Rx$ Square root

- $R0 = 0$
- $RSx = Rx + (S0 \cdot ((S(ex))^2 - Sx))$
Clearly, from the definition, $Rx$ is the integer part of the square root of $x$, or the square root of the greatest perfect square in $x$.

Now with the aid of these functions we can define $L(x)$, $K(x)$ & $J(u,v)$ as follows:

$$J(u,v) = (u + v)^2 + u$$
$$L(x) = x \div (Rx)^2$$
$$K(x) = Rx \div L(x)$$

Obviously, informally,

$$L(J(u,v)) = ((u + v)^2 + u) \div (R((u + v)^2 + u))^2$$

$$= ((u + v)^2 + u) \div (u + v)^2$$

$$= u$$

$$K(J(u,v)) = (u + v) \div u$$

$$= v \quad \text{as required.}$$

In multiple successor arithmetic however the corresponding functions are not quite so straightforward.

As can be clearly seen the functions $L(x)$, $K(x)$ & $J(u,v)$ are based on the relations

$$R(u^2) = u$$

and

$$R((u + v)^2 + u) = u + v \quad .$$

In other words, we seek to define a suitable multiplication function with a suitable square root function.

Let us consider the functions we have for multiplication, their properties, and possible square roots.
Vušković's multiplication function defined by

\[ Y^*0 = 0 \]

\[ Y^*_u = Y^*X \sigma_u Y \quad u=1,\ldots,n \]

The commutativity condition requires the equality of

\[ R_u(S^uX,Y,B_v(X,Y,Z)) = (Z \sigma^v Y) \sigma_u Y \]

\[ R_v(S^uX,Y,B_u(X,Y,Z)) = (Z \sigma^u X) \sigma_v Y . \]

We shall prove these equal by the following lemma.

**Lemma**

\[ (Y \sigma^v X) \sigma_u X = (Y \sigma_u X) \sigma^v X \]

**Proof**

\[ L(0,Y) = (Y \sigma^v 0) \sigma_u 0 \]

\[ = (Y \sigma^v 0) \quad \text{Defn} \]

\[ = Y \quad \text{Defn} \]

\[ R(0,Y) = (Y \sigma_u 0) \sigma^v 0 \]

\[ = (Y \sigma_u 0) \quad \text{Defn} \]

\[ = Y \quad \text{Defn} \]

\[ L(S^uX,Y) = (Y \sigma^v S^uX) \sigma_u S^uX \]

\[ = S^v_{w+u}(Y \sigma^v S^uX) \sigma_u X \quad \text{Defn} \]

\[ = S^v_{w+u}S^v_{w+v}(Y \sigma^v X) \sigma_u X \quad \text{Defn} \]

\[ R(S^uX,Y) = (Y \sigma_u S^uX) \sigma^v S^uX \]

\[ = S^v_{w+v}(Y \sigma_u S^uX) \sigma_v X \quad \text{Defn} \]

\[ = S^v_{w+v}S^v_{w+u}(Y \sigma_u X) \sigma_v X \quad \text{Defn} \]

\[ = S^v_{w+u}S^v_{w+v}(Y \sigma_u X) \sigma_v X \quad \text{(commutativity of successors)} \]
\[ L(X,Y) = R(X,Y) \]

\[ R_u(S_u X, Y, R_v(X, Y, Z)) = R_v(S_u X, Y, R_u(X, Y, Z)) \]

The properties of this function include commutativity, associativity, and distribution over \( \sigma_u \) functions for all \( u \); the case of \( u = 0 \) is addition.

Commutativity is proved by the following lemmas:

**Lemma**

\( 0 \ast X = 0 \)

**Proof**

\[ L(0) = 0 \ast 0 = 0 \quad \text{Defn} \]
\[ R(0) = 0 \]

\[ L(S_u X) = 0 \ast S_u X \]
\[ = 0 \ast X \sigma_u 0 \quad \text{Defn} \]
\[ = 0 \ast X \quad \text{Defn} \]
\[ = L(X) \]

\[ R(S_u X) = 0 \]
\[ = R(X) \]

\[ \therefore \quad L(X) = R(X) \quad \text{by uniqueness} \]

**Lemma**

\( (Z \sigma_v Y) \sigma_u X = (Z \sigma_u X) \sigma_v Y \)

**Proof**

\[ L(0, Y, Z) = (Z \sigma_v Y) \sigma_u 0 \]
\[ = (Z \sigma_v Y) \quad \text{Defn} \]

\[ R(0, Y, Z) = (Z \sigma_u 0) \sigma_v Y \]
\[ = Z \sigma_v Y \quad \text{Defn} \]
\[ L(S^X Y, Z) = (Z \sigma_Y) \sigma_u S^X \]
\[ = S^X (Z \sigma_Y) \sigma_u X \]
\[ = S^X L(X, Y, Z) \]
\[ R(S^X Y, Z) = (Z \sigma_u S^X) \sigma_Y \]
\[ = (S^X (Z \sigma_u X)) \sigma_Y \]
\[ = S^X (Z \sigma_u X) \sigma_Y \]
\[ = S^X (R(X, Y, Z)) \]

\[ \therefore \quad L(X, Y, Z) = R(X, Y, Z) \quad \text{by uniqueness} \]

**Lemma**

\[ S_u Y^X = Y^X \sigma_u X \]

**Proof**

\[ L(0, Y) = S_u Y^0 \]
\[ = 0 \quad \text{Defn} \]
\[ R(0, Y) = Y^0 \sigma_u 0 \]
\[ = Y^0 \quad \text{Defn} \]
\[ = 0 \quad \text{Defn} \]

\[ L(S_v X, Y) = S_u Y^* S_v X \]
\[ = S_u Y^* \sigma_v S_v X \]
\[ = S_u Y^* \sigma_v S_v Y \]
\[ = S_u+V (S_u Y^* X) \sigma_v Y \]
\[ = S_u+V (L(X, Y)) \sigma_v Y \]
\[ R(S_v X, Y) = Y^* S_v X \sigma_u S_v X \]
\[ = (Y^* \sigma_v Y) \sigma_u S_v X \]
\[ = S_v^u (Y^* \sigma_v Y) \sigma_u X \]
\[ = S_v^u (Y^* \sigma_u X) \sigma_v Y \]
\[ = S_v^u (R(X, Y)) \sigma_v Y \]
\[ L(X, Y) = R(X, Y) \text{ by uniqueness} \]

**Proposition**

\[ Y^* X = X^* Y \]

**Commutativity**

**Proof**

\[ L(0, Y) = Y^* 0 \]

\[ = 0 \text{ Defn} \]

\[ R(0, Y) = 0^* Y \]

\[ = 0 \text{ Previous Lemma} \]

\[ L(S_u X, Y) = Y^* S_u X \]

\[ = Y^* X \sigma_u Y \text{ Defn} \]

\[ = L(X, Y) \sigma_u Y \]

\[ R(S_u X, Y) = S_u X^* Y \]

\[ = X^* Y \sigma_u Y \text{ Defn} \]

\[ = R(X, Y) \sigma_u Y \]

\[ L(X, Y) = R(X, Y) \text{ by uniqueness} \]

For the distributive property we need the following Lemmas:

**Lemma**

\[ Z \sigma_u (Y \sigma_v X) = (Z \sigma_u Y) \sigma_{v+u} X \]

**Proof**

\[ L(0, Y, Z) = Z \sigma_u (Y \sigma_v 0) \]

\[ = Z \sigma_u Y \text{ Defn} \]

\[ R(0, Y, Z) = (Z \sigma_u Y) \sigma_{v+u} 0 \]

\[ = Z \sigma_u Y \text{ Defn} \]

\[ L(S_w X, Y, Z) = Z \sigma_u (Y \sigma_v S_w X) \]

\[ = Z \sigma_u S_{w+v} (Y \sigma_v X) \text{ Defn} \]

\[ = S_{w+v+u} Z \sigma_u (Y \sigma_v X) \text{ Defn} \]
\[ E(S^X.Y.Z) = (Z \sigma_u Y) \sigma_{v+u} X \]

\[ R(S^X.Y.Z) = (Z \sigma_u Y) \sigma_{v+u} S^X \]

\[ = S^X.\sigma_u (Z \sigma_u Y) \sigma_{v+u} X \quad \text{Defn} \]

\[ = S^X R(X,Y,Z) \]

\[
\therefore \quad L(X,Y,Z) = R(X,Y,Z) \quad \text{by uniqueness}
\]

**Proposition**  
\[ Z^*(Y \sigma_u X) = Z^*Y \sigma_u Z^*X \quad \text{Distributive} \]

**Proof**  
\[ L(0,Y,Z) = Z^*(Y \sigma_u 0) \]

\[ = Z^*Y \quad \text{Defn} \]

\[ R(0,Y,Z) = Z^*Y \sigma_u Z^*0 \]

\[ = Z^*Y \sigma_u 0 \quad \text{Defn} \]

\[ = Z^*Y \quad \text{Defn} \]

\[ L(S_v X,Y,Z) = Z^*(Y \sigma_u S_v X) \]

\[ = Z^*S_{v+u} (Y \sigma_u X) \quad \text{Defn} \]

\[ = Z^*(Y \sigma_u X) \sigma_{v+u} Z \quad \text{Defn} \]

\[ = L(X,Y,Z) \sigma_{v+u} Z \]

\[ R(S_v X,Y,Z) = Z^*Y \sigma_u Z^*S_v X \]

\[ = Z^*Y \sigma_u (Z^*X \sigma_v Z) \quad \text{Defn} \]

\[ = (Z^*Y \sigma_u Z^*X) \sigma_{v+u} Z \quad \text{Previous Lemma} \]

\[ = R(X,Y,Z) \sigma_{v+u} Z \]

\[
\therefore \quad L(X,Y,Z) = R(X,Y,Z) \quad \text{by uniqueness}
\]
PROPOSITION \((Z*Y)*X = Z*(Y*X)\)  

**Proof**  
\[L(0,Y,Z) = (Z*Y)*0\]  
\[= 0\]  
Defn  
\[R(0,Y,Z) = Z*(Y*0)\]  
\[= Z*0\]  
Defn  
\[= 0\]  
Defn  
\[L(S_uX,Y,Z) = (Z*Y)*S_uX\]  
\[= (Z*Y)*X \sigma_u(Z*Y)\]  
Defn  
\[= L(X,Y,Z) \sigma_u(Z*Y)\]  
Defn  
\[R(S_uX,Y,Z) = Z*(Y*S_uX)\]  
\[= Z*(Y*X \sigma_u Y)\]  
Defn  
\[= Z*(Y*X) \sigma_u Z*Y\]  
Previous Proposition  
\[= R(X,Y,Z) \sigma_u(Z*Y)\]  
Defn  
\[.:\]  
\[L(X,Y,Z) = R(X,Y,Z)\]  
by uniqueness  

In general however \(Z^*(Y \tau_u X) \neq Z^*Y \tau_u Z^*X\).  

Consider the counter-example  
\[u=0, \ X=S_20, \ Y=S_10, \ Z=S_3S_10\]  
\[S_2S_10^*(S_10 \tau_0 S_20) = S_2S_10^*P_0(S_10 \tau_0 0)\]  
\[= S_2S_10^*P_2S_10\]  
Defn  
\[= S_2S_10^*S_10\]  
Defn  
\[= S_2S_10^*S_10\]  
Defn  
\[= 0 S_1S_2S_10\]  
Defn  
\[= S_3S_20\]  
Defn
\[ S_2S_10*S_10 \tau_0 S_2S_10*S_20 = (S_2S_10*0 \sigma_1 S_2S_10) \tau_0 (S_2S_10*0 \sigma_2 S_2S_10) \text{ Defn} \]
\[ = S_2S_20 \tau_0 S_2S_20 \text{ Defn} \]
\[ = P_4P_3S_3S_20 \text{ Defn} \]
\[ = S_20 \text{ Defn} \]

Thus in general \( Z*(Y \tau_0 X) \neq Z*Y \tau_0 Z*X \).

2. We shall now consider the multiplicative function defined by

\[ Y^X0 = 0 \]
\[ Y^Xu \times X = Y^X + Y \quad u = 1, \ldots, n \]

and the commutativity condition

\[ B_u(S vX, Y, B_v(X, Y, Z)) = B_v(X, Y, Z) + Y \]
\[ = (Z + Y) + Y \]
\[ = B_u(X, Y, Z) + Y \]
\[ = B_v(S vX, Y, B_u(X, Y, Z)). \]

Clearly the function does not distinguish between successors, but unfortunately when \( n > 1 \) the function is not commutative. We shall prove this by a simple counter-example.

Put \( Y = S_10, \quad X = S_20 \)

\[ Y^X = S_10^X + S_20 \]
\[ = S_10^0 + S_10 \text{ Defn} \]
\[ = 0 + S_10 \text{ Defn} \]
\[ = S_10 \text{ Defn} \]
\[ X^Y = S_2^0 X S_1^0 \]

\[ = S_2^0 X^0 + S_2^0 \quad \text{Defn} \]

\[ = 0 + S_2^0 \quad \text{Defn} \]

\[ = S_2^0 \quad \text{Previous Lemma} \]

**PROPOSITION** \[ Z^{(Y+X)} = Z^X Y + Z^X X \] **Distributive over addition**

**Proof**

\[ L(0,Y,Z) = Z^{X(Y+0)} \]

\[ = Z^X Y \quad \text{Defn} \]

\[ R(0,Y,Z) = Z^X Y + Z^X 0 \]

\[ = Z^X Y + 0 \quad \text{Defn} \]

\[ = Z^X Y \quad \text{Defn} \]

\[ L(S_u X, Y, Z) = Z^X (Y + S_u X) \]

\[ = Z^X S_u (Y + X) \quad \text{Defn} \]

\[ = Z^X (Y + X) + Z \quad \text{Defn} \]

\[ = L(X,Y,Z) + Z \]

\[ R(S_u X, Y, Z) = Z^X Y + Z^X S_u X \]

\[ = Z^X Y + (Z^X X + Z) \quad \text{Defn} \]

\[ = Z^X Y + Z^X X + Z \quad \text{Associative rule for addition} \]

\[ = R(X,Y,Z) + Z \]

\[ \therefore \quad L(X,Y,Z) = R(X,Y,Z) \quad \text{by uniqueness} \]

**PROPOSITION** \[ (Z^X)^Y X = Z^X (Y^X X) \] **Associative**

**Proof**

\[ L(0,Y,Z) = (Z^X)^0 Y \]

\[ = 0 \quad \text{Defn} \]
\[
R(0,Y,Z) = Z^X(Y^0)
= Z^0
= 0
\]

Defn

\[
L(S_u X,Y,Z) = (Z^Y)^X S_u X
= (Z^X)^X + (Z^X)
= L(X,Y,Z) + (Z^X)
\]

Defn

\[
R(S_u X,Y,Z) = Z^X(Y^S_u X)
= Z^X(Y^X + Y)
= Z^X(Y^X) + Z^X
\]

Previous Lemma

\[
= R(X,Y,Z) + (Z^X)
\]

Defn

\[
L(X,Y,Z) = R(X,Y,Z)
\]

by uniqueness

However, although this function is distributive over addition, it is not distributive over the \( \sigma_u \) functions.

That is \( Z^X(Y \sigma_u X) \neq Z^Y \sigma_u Z^X \)

Put \( Z = S_20 \), \( Y = S_10 \), \( X = S_20 \)

\[
S_20^X(S_10 \sigma_u S_20) = S_20^X(S_20^X S_10 \sigma_u)
\]

Defn

\[
= S_20^X S_20
\]

Defn

\[
= S_20^X + S_20
\]

Defn

\[
= S_20^X + S_20 + S_20
\]

Defn

\[
= S_2 S_20
\]

Defn

\[
S_20^X S_10 \sigma_u S_20^X S_20 = (S_20^X + S_20) \sigma_u (S_20^X + S_20)
\]

Defn

\[
= S_20 \sigma_u S_20
\]

Defn

\[
= S_20 S_0
\]

Defn

and \( S_{2+u} S_0 \) is only equal to \( S_2 S_20 \) when \( u = 0 \).
Also we find that in general

\[ Z^X(Y^uX) \neq Z^Y^uZ^X \]

Put \( u = 0, \ Z = S_10, \ Y = S_2S_10, \ X = S_1S_10 \)

\[
S_10^X (S_2S_10 \tau_u S_1S_10) = S_10^X P_1P_1S_2S_10 \\
= S_10^X S_20 \\
= S_10
\]

\[
S_10^X S_2S_10 \tau_u S_10^X S_1S_10 = (S_10^X S_10 + S_10) \tau_u (S_10^X S_10 + S_10) \text{ Defn}
\]

\[
= (S_10^X S_10 + S_10 + S_10) \tau_u (S_10^X S_10 + S_10 + S_10) \text{ Defn}
\]

3. For the last multiplication function we consider the function defined by

\[
Y \cdot 0 = 0 \\
Y \cdot S_u X = Y \cdot X + C_u Y \text{ \ \ \ \ \ u = 1, \ldots, n}
\]

and the commutativity condition is

\[
B_u(S_vX,Y,B_v(X,Y,Z)) = B_v(X,Y,Z) + C_u Y \\
= (Z + C_v Y) + C_u Y \\
= (Z + C_u Y) + C_v Y \text{ \ Associative rule of addition}
\]

\[
= B_u(X,Y,Z) + C_v Y \\
= B_v(S_uX,Y,B_u(X,Y,Z))
\]

This function is commutative, proved by the following Lemmas.
**Lemma**

\[ 0 \cdot X = 0 \]

**Proof**

\[ L(0) = 0.0 \]

\[ = 0 \quad \text{Defn} \]

\[ R(0) = 0 \]

\[ L(S^X X) = 0 \cdot S^X X \]

\[ = 0 \cdot X + C_u 0 \quad \text{Defn} \]

\[ = 0 \cdot X + 0 \quad \text{Defn} \]

\[ = 0 \cdot X \quad \text{Defn} \]

\[ = L(X) \]

\[ R(S^X X) = 0 \]

\[ = R(X) \]

\[ \therefore \quad L(X) = R(X) \quad \text{by uniqueness} \]

**Lemma**

\[ S_u Y \cdot X = Y \cdot X + C_X \]

**Proof**

\[ L(0, Y) = S_u Y \cdot 0 \]

\[ = 0 \quad \text{Defn} \]

\[ R(0, Y) = Y \cdot 0 + C_u 0 \]

\[ = 0 + 0 \quad \text{Defn} \]

\[ = 0 \quad \text{Defn} \]

\[ L(S_v X, Y) = S_u Y \cdot S_v X \]

\[ = S_u Y \cdot X + C_v S_v Y \quad \text{Defn} \]

\[ = \begin{cases} S_u Y \cdot X + S_v C_v Y & u=v \\ S_u Y \cdot X + C_v Y & u \neq v \end{cases} \quad \text{Defn} \]
\[ S(Y, X + C_Y) u=v \]
\[ L(X, Y) + C_Y u\neq v \]
\[ S(L(X, Y) + C_Y) u=v \]
\[ L(X, Y) + C_Y u\neq v \]

\[ R(S_X Y) = Y S_X + C_u S_X \]
\[ = Y X + C_Y + C_u S_X \quad \text{Defn} \]
\[ = Y X + C_Y + C_X \quad \text{Defn} \]
\[ \{ \begin{array}{l} S(L(X + C_Y) + C_Y) u=v \\ (Y X + C_u X) + C_Y \text{ Defn \\ &amp; Associative law of addition} \\ (Y X + C_u X) + C_Y \quad \text{Defn} \end{array} \]
\[ = \begin{array}{l} S(R(X, Y) + C_Y) u=v \\ R(X, Y) + C_Y \quad \text{Defn} \end{array} \]

\[ \therefore \quad L(X, Y) = R(X, Y) \quad \text{by uniqueness.} \]

**PROPOSITION**

\[ Y X = X Y \quad \text{Commutative} \]

**Proof**

\[ L(0, Y) = Y 0 \]
\[ = 0 \quad \text{Defn} \]

\[ R(0, Y) = 0 Y \]
\[ = 0 \quad \text{Previous Lemma} \]

\[ L(S_X Y) = Y S_X \]
\[ = Y X + C_Y \quad \text{Defn} \]
\[ = L(X, Y) + C_Y \]

\[ R(S_X Y) = S_X Y \]
\[ = X Y + C_u Y \quad \text{Previous Lemma} \]
\[ = R(X, Y) + C_u Y \]

\[ \therefore L(X, Y) = R(X, Y) \quad \text{by uniqueness} \]

Consider the distributive property of this function over addition.

**Proposition** \( Z(Y + X) = ZY + ZX \) **Distribution over addition**

**Proof**

\[ L(0, Y, Z) = Z(Y + 0) \]
\[ = ZY \quad \text{Defn} \]
\[ R(0, Y, Z) = ZY + Z0 \]
\[ = ZY + 0 \quad \text{Defn} \]
\[ = ZY \quad \text{Defn} \]
\[ L(S^X, Y, Z) = Z(Y + S^X) \]
\[ = ZS^X(Y + X) \quad \text{Defn} \]
\[ = Z(Y + X) + C_u Z \quad \text{Defn} \]
\[ = L(X, Y, Z) + C_u Z \]
\[ R(S^X, Y, Z) = ZY + ZS^X \]
\[ = ZY + ZX + C_u Z \quad \text{Defn \& Associative law of addition} \]
\[ = R(X, Y, Z) + C_u Z \]
\[ \therefore L(X, Y, Z) = R(X, Y, Z) \quad \text{by uniqueness}. \]

To prove associativity we require to prove the following Lemmas.

**Lemma** \( C_u C_v X = C_v C_u X \)

**Proof**

\[ L(0) = C_u C_v 0 \]
\[ = C_u 0 \quad \text{Defn} \]
\[ = 0 \quad \text{Defn} \]
\[
R(0) = C_{v \cdot u}^0 \\
= C_v^0 \\
= 0 \\
\]

Defn

\[
L(S_w X) = C_{v \cdot S_w^X}^X \\
= \left\{ \begin{array}{ll}
C_{v \cdot X}^u & \text{if } v \neq w \\
C_{S_w^X \cdot X}^u & \text{if } v = w \\
\end{array} \right. \\
\]

Defn

\[
= \left\{ \begin{array}{ll}
C_{v \cdot X}^u & \text{if } v \neq w \\
C_{S_w^X \cdot X}^u & \text{if } v = w & \text{or } u \neq w \\
\end{array} \right. \\
\]

Defn

\[
= \left\{ \begin{array}{ll}
L(X) & \text{or } u = w \\
S_w L(X) & \text{or } u = w \\
\end{array} \right. \\
\]

\[
R(S_w X) = C_{v \cdot S_w^X}^X \\
= \left\{ \begin{array}{ll}
C_{v \cdot X}^u & \text{if } u \neq w \\
C_{S_w^X \cdot X}^u & \text{if } u = w \\
\end{array} \right. \\
\]

Defn

\[
= \left\{ \begin{array}{ll}
C_{v \cdot X}^u & \text{if } u \neq w \\
C_{S_w^X \cdot X}^u & \text{if } u = w & \text{or } v \neq w \\
\end{array} \right. \\
\]

Defn

\[
= \left\{ \begin{array}{ll}
L(X) & \text{or } u \neq w \\
S_w L(X) & \text{or } u = w \\
\end{array} \right. \\
\]

\[
R(X) = C_{v \cdot X}^u \\
= \left\{ \begin{array}{ll}
L(X) & \text{if } u \neq w \\
S_w L(X) & \text{if } u = w \\
\end{array} \right. \\
\]

Defn

\[
\therefore \\
L(X) = R(X) \\
\text{by uniqueness.}
\]
**Lemma**  \( C_u(Y + X) = C_u Y + C_u X \)

**Proof**

\[
L(0, Y) = C_u(Y + 0) \\
= C_u Y \\
R(0, Y) = C_u Y + C_u 0 \\
= C_u Y + 0 \\
= C_u Y
\]

\[
L(S_v X, Y) = C_u(Y + S_v X) \\
= C_u S_v (Y + X) \\
= \begin{cases} 
S_v C_u (Y + X) & u = v \\
C_u (Y + X) & u \neq v 
\end{cases} \\
= \begin{cases} 
S_v L(X, Y) & u = v \\
L(X, Y) & u \neq v 
\end{cases}
\]

\[
R(S_v X, Y) = C_u Y + C_u S_v X \\
= \begin{cases} 
C_u Y + S_v C_u X & u = v \\
C_u Y + C_u X & u \neq v 
\end{cases} \\
= \begin{cases} 
S_v (C_u Y + C_u X) & u = v \\
C_u Y + C_u X & u \neq v 
\end{cases} \\
= \begin{cases} 
S_v R(X, Y) & u = v \\
R(X, Y) & u \neq v 
\end{cases}
\]

\[\therefore \quad L(X, Y) = R(X, Y) \]  \text{by uniqueness.}

**Lemma** \( (C_u Y)_X = C_u (Y_X) \)
Proof

\[
\begin{align*}
L(0,Y) &= (C^Y)_O \\
&= 0 \text{ Defn}
\end{align*}
\]

\[
\begin{align*}
R(0,Y) &= C_u(Y.0) \\
&= C_u \text{ Defn}
\end{align*}
\]

\[
\begin{align*}
&= 0 \text{ Defn}
\end{align*}
\]

\[
\begin{align*}
L(S^X,Y) &= (C^Y)_S X \\
&= (C^Y)_X + C^Y 	ext{ Defn}
\end{align*}
\]

\[
\begin{align*}
&= L(X,Y) + E(S^X,Y) \\
&= C^Y(X + C^Y) \text{ Defn}
\end{align*}
\]

\[
\begin{align*}
&= R(X,Y) + C^Y \text{ Previous Lemma}
\end{align*}
\]

\[
\begin{align*}
&= R(X,Y) \text{ by uniqueness}
\end{align*}
\]

and therefore by the commutativity property we can say \((C^Y)_X = X.(C^Y)\).

**Proposition** \(Z.(Y.X) = (Z.Y).X\) **Associative**

Proof

\[
\begin{align*}
L(0,Y,Z) &= Z.(Y.0) \\
&= Z.0 \text{ Defn}
\end{align*}
\]

\[
\begin{align*}
&= 0 \text{ Defn}
\end{align*}
\]

\[
\begin{align*}
R(0,Y,Z) &= (Z.Y)_0 \\
&= 0 \text{ Defn}
\end{align*}
\]

\[
\begin{align*}
L(S^X,Y,Z) &= Z.(Y.S^X) \\
&= Z.(Y.X + C^Y) \text{ Defn}
\end{align*}
\]

\[
\begin{align*}
&= Z.(Y.X) + Z.(C^Y) \text{ Previous Lemma}
\end{align*}
\]
= L(X,Y,Z) + C_u(Z,Y)

R(S_uX,Y,Z) = (Z.Y).S_uX

= (Z.Y).X + C_u(Z,Y)  \text{Defn}

= R(X,Y,Z) + C_u(Z,Y)

\therefore L(X,Y,Z) = R(X,Y,Z)

This function is not, however, distributive over \( \sigma_u \) functions, nor the \( t_u \) functions, for all \( u \). Though the distributive property is true in both cases with \( u = 0 \), the first we have proved, the second we shall state, and leave the complete proof until chapter III.

\[ Z.(Y - X) = Z.Y - Z.X \]  \text{Distributive over difference}

Having established the properties of these three multiplication functions, we are now able to consider their squares, and possible square roots.

1. Vučković's multiplication function.

\[ Y^0 = 0 \]

\[ Y^uX = Y^X \sigma_uY \quad u = 1, \ldots, n \quad (\sigma_n = \sigma_0) \]

In 4 successor arithmetic the squares of numerals will be as follows.

<table>
<thead>
<tr>
<th>Numeral</th>
<th>Square</th>
<th>Numeral</th>
<th>Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( S_1 )0</td>
<td>( S_2S_2S_2S_20 )</td>
</tr>
<tr>
<td>( S_1 )0</td>
<td>( S_20 )</td>
<td>( S_1 )0</td>
<td>( S_2S_2S_2S_40 )</td>
</tr>
<tr>
<td>( S_2 )0</td>
<td>( S_40 )</td>
<td>( S_1 )0</td>
<td>( S_2S_4S_4S_20 )</td>
</tr>
<tr>
<td>( S_3 )0</td>
<td>( S_20 )</td>
<td>( S_1 )40</td>
<td>( S_2S_1S_1S_40 )</td>
</tr>
<tr>
<td>( S_4 )0</td>
<td>( S_40 )</td>
<td>( S_2S_20 )</td>
<td>( S_2S_4S_4S_40 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( S_2S_20 )</td>
<td>( S_4S_1S_1S_20 )</td>
</tr>
</tbody>
</table>
From the above table it can be seen that obviously there cannot be a useful square root function for this multiplication function, as both S\textsubscript{1}0 and S\textsubscript{3}0 have the same square.

2. The function defined by

\[ Y^0 = 0 \]
\[ Y^X_0 = Y^X + Y \quad u = 1, \ldots, n \]

In 4 successor arithmetic the squares of numerals will be as follows.

<table>
<thead>
<tr>
<th>Numeral</th>
<th>Square</th>
<th>Numeral</th>
<th>Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>S\textsubscript{1}S\textsubscript{1}0</td>
<td>S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}0</td>
</tr>
<tr>
<td>S\textsubscript{1}0</td>
<td>S\textsubscript{1}0</td>
<td>S\textsubscript{1}S\textsubscript{2}0</td>
<td>S\textsubscript{1}S\textsubscript{2}S\textsubscript{2}0</td>
</tr>
<tr>
<td>S\textsubscript{2}0</td>
<td>S\textsubscript{2}0</td>
<td>S\textsubscript{2}S\textsubscript{2}0</td>
<td>S\textsubscript{2}S\textsubscript{2}S\textsubscript{2}S\textsubscript{2}0</td>
</tr>
<tr>
<td>S\textsubscript{3}0</td>
<td>S\textsubscript{3}0</td>
<td>S\textsubscript{1}S\textsubscript{4}0</td>
<td>S\textsubscript{1}S\textsubscript{4}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{3}S\textsubscript{3}0</td>
</tr>
<tr>
<td>S\textsubscript{4}0</td>
<td>S\textsubscript{4}0</td>
<td>S\textsubscript{1}S\textsubscript{4}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{1}S\textsubscript{3}S\textsubscript{3}0</td>
<td></td>
</tr>
</tbody>
</table>

This function does give a unique square, and further if a square root function could be found then a unique answer would be obtained for square roots of 'perfect squares' under this multiplication rule. It can be seen that the square of a numeral is that numeral repeated the same number of times as there are successor symbols in that numeral. Therefore in order to obtain the square root of a numeral we would first have to analyse the number of successor symbols in that numeral, take the square root of that number, then further either analyse the particular successor symbols or 'divide' the numeral by the square root of the number of successor symbols. The questions of the number of successor symbols in a numeral, and the particular successor symbols which make up that numeral have been fully analysed in chapter IV.
It can be seen from the above that a square root function of this multiplication function would not be a 'convenient' function and therefore we shall leave this function and consider the third multiplication function.

3. The function defined by

\[ Y \cdot 0 = 0 \]
\[ Y \cdot S \cdot X = Y \cdot X + C \cdot Y \quad u = 1, \ldots, n \]

Consider the squares of numerals in 4 successor arithmetic.

<table>
<thead>
<tr>
<th>Numeral</th>
<th>Square</th>
<th>Numeral</th>
<th>Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>S_1 S_0</td>
<td>S_1 S_0</td>
</tr>
<tr>
<td>S_1 O</td>
<td>S_1 O</td>
<td>S_2 S_0</td>
<td>S_2 S_0</td>
</tr>
<tr>
<td>S_2 O</td>
<td>S_2 O</td>
<td>S_3 S_0</td>
<td>S_3 S_0 S_0 S_0</td>
</tr>
<tr>
<td>S_3 O</td>
<td>S_3 O</td>
<td>S_1 S_1 S_4 O</td>
<td>S_1 S_1 S_1 S_4 O</td>
</tr>
<tr>
<td>S_4 O</td>
<td>S_4 O</td>
<td>S_4 S_4 S_4 S_4 S_4 S_4 S_4 S_4</td>
<td></td>
</tr>
</tbody>
</table>

This appears to have a very reasonable square, which is the numeral split into components, and each component squared. To obtain the square root therefore we would first have to take components and then take the square root as in single successor arithmetic. Thus we would have functions \( R_{t_u}(X) \) defined by:

\[
R_{t_u}(0) = 0
\]

\[
R_{t_u}(S \cdot X) = \begin{cases} 
R_{t_u}(C \cdot X) + C \cdot (S \cdot 0 \cdot ((S \cdot R_{t_u}(C \cdot X)) \cdot S \cdot V \cdot C \cdot X)) & u = v \\
R_{t_u}(C \cdot X) & u \neq v 
\end{cases}
\]

That is \( R_{t_u}(X) \) is the square root of the \( S_u \) successor symbols in \( X \), the complete square root of \( X \) would then be given by \( R_t(X) \) defined by

\[
R_t(X) = R_{t_1}(X) + R_{t_2}(X) + \ldots + R_{t_n}(X)
\]
However we can avoid the $Rt_u(x)$ functions by the definition

$$Rt(0) = 0$$

$$Rt(S_u x) = Rt(x) + (S_u 0 \circ ((S_u Rt(x))^2 - S_u x)) \quad u=1,\ldots,n$$

The components are taken by the property that

$$C_u(S_u 0 \circ A) = (S_u 0 \circ A) .$$

For this function to be the one required the following properties need to be satisfied:

$$Rt(x.x) = x$$

$$Rt(x.x + x) = x$$

$$Rt((y + x)^2 + x) = y + x \quad ( (y + x)^2 = (y + x) \cdot (y + x) )$$

Further if $y > x$ that is $x \circ y = 0$, then $Rt(y) > Rt(x)$, that is $Rt(x) \circ Rt(y) = 0$.

All these properties will be proved formally in chapter III.

Further, we need to prove the properties

$$L(J(U,V)) = U$$

$$K(J(U,V)) = V .$$
CHAPTER III

FORMAL DEVELOPMENT OF COMMUTATIVE MULTIPLE SUCCESSOR ARITHMETIC

In this chapter we will develop the arithmetic formally and prove the properties of some of the functions already introduced in previous chapters. The notation used will be to number all axioms, rules of inference, definitions, lemmas, propositions, and theorems. The steps in a proof will then by illustrated by a number corresponding to the statement used on the right-hand side of the page. For the substitution rule of inference and equality rules numbering will not be used.

The functional notation \( L(...) \) and \( R(...) \) will be used to denote the left- and right-hand sides of equations. Where a letter is quoted in the parenthesis of \( L(...) \) or \( R(...) \) and does not appear on the corresponding side of the equation, then the zero function \( Z(...) \) of that letter is supposed to have been taken.

\[ R(x) = 0 \] should be taken as \( R(x) = Z(x) \), which will not affect the proof.

A proof will conclude with the statement \( L(...) = R(...) \).

Statements 1 to 6 give the axiomatic statement of the arithmetic.

1. Initial functions

\[ Z(X) \] written 0
\[ S_u(X) \] written \( S_uX \) \( u=1,\ldots,n \)
\[ I^k(X_1,X_2,\ldots,X_k) \] written \( X_1^k \)

2. Commutative condition

\[ S_uS_vX = S_vS_uX \] \( u,v=1,\ldots,n \)
Definition of functions by explicit and primitive recursive definitions, the latter given by:–

If \( A(Y_1, Y_2, \ldots, Y_m) \) and \( B_u(X, Y_1, Y_2, \ldots, Y_m, Z) \) \( u=1, \ldots, n \), are designated functions (initial functions or previously defined) then \( F(X, Y_1, Y_2, \ldots, Y_m) \) is designated if

\[
F(0, Y_1, Y_2, \ldots, Y_m) = A(Y_1, Y_2, \ldots, Y_m)
\]

\[
F(S_X, Y_1, Y_2, \ldots, Y_m) = B_u(X, Y_1, Y_2, \ldots, Y_m, F(X, Y_1, Y_2, \ldots, Y_m)) \quad u=1, \ldots, n
\]

and

\[
B_v(S_X, Y_1, Y_2, \ldots, Y_m, B_v(X, Y_1, Y_2, \ldots, Y_m, Z)) = B_v(S_X, Y_1, Y_2, \ldots, Y_m, B_u(X, Y_1, Y_2, \ldots, Y_m, Z)) \quad \text{all} \; u, v,
\]

in order to preserve the commutativity of the successors.

3. Rule of inference (equality rule)

\[
A = B \quad \text{This statement reads: 'If } A \text{ is equal to } B \text{ and } A \text{ is equal to } C \text{ then } B \text{ is equal to } C'.
\]

\[
A = C
\]

\[
B = C
\]

4. Rule of inference (substitution)

\[
F(Y) = G(Y)
\]

\[
F(A) = G(A)
\]

5. Rule of inference (substitution)

\[
A = B
\]

\[
F(A) = F(B)
\]

6. Rule of inference (uniqueness rule)

\[
F(0) = G(0)
\]

\[
F(S_u X) = H_u(X, F(X))
\]

\[
G(S_u X) = H_u(X, G(X)) \quad u=1, \ldots, n
\]

\[
F(X) = G(X)
\]
7. Definition of $Y + X$

$$Y + 0 = Y$$

$$Y + S^u X = S^u (Y + X) \quad u = 1, \ldots, n$$

8. Proposition $A = A$

Proof

$Y + 0 = Y$
$Y + 0 = Y$
$Y = Y$
$A = A$

9. Rule of inference

$$A = B$$

$$B = A$$

Proof

$A = B$
$A = A$
$B = A$
$	ext{Premises}$

10. Rule of inference

$$A = B$$

$$B = C$$

$$A = C$$

Proof

$A = B$
$B = A$
$B = C$
$A = C$
$	ext{Premises}$

$	ext{Premises}$
11. Rule of inference

\[ B = A \]
\[ C = A \]
\[ \therefore B = C \]

Proof
\[ B = A \]
\[ A = B \]
\[ C = A \]
\[ A = C \]
\[ B = C \]

12. Rule of inference

\[ A = B \]
\[ A = C \]
\[ B = D \]
\[ \therefore C = D \]

Proof
\[ A = B \]
\[ A = C \]
\[ B = C \]
\[ B = D \]
\[ \therefore C = D \]

13. Proposition

\[ 0 + X = X \]

Proof
\[ L(0) = 0 + 0 \]
\[ = 0 \]
\[ R(0) = 0 \]
\[ L(S_u X) = 0 + S_u X \]
\[ = S_u (0 + X) \]
\[ = S_u L(X) \]
\[ R(S^uX) = S^uX = S^uR(X) \]
\[ L(X) = R(X) \]

14. Proposition

\[ S^uY + X = S^u(Y + X) \]

Proof

\[ L(0,Y) = S^uY + 0 = S^uY \]
\[ R(0,Y) = S^u(Y + 0) = S^uY \]
\[ L(S^vX,Y) = S^uY + S^vX = S^v(S^uY + X) = S^vL(X,Y) \]
\[ R(S^vX,Y) = S^u(Y + S^vX) = S^uS^v(Y + X) = S^vS^u(Y + X) = S^vR(X,Y) \]
\[ L(X,Y) = R(X,Y) \]

15. Proposition

\[ X + Y = Y + X \]

Proof

\[ L(0,Y) = O + Y = Y \]
\[ R(0,Y) = Y + 0 = Y \]
\[ L(S^uX,Y) = S^uX + Y = S^u(X + Y) = S^uL(X,Y) \]
\[ R(S_u X, Y) = Y + S_u X \]
\[ = S_u (Y + X) \]
\[ = S_u R(X, Y) \]

16. Proposition

\[(A + B) + C = A + (B + C)\]

Proof

\[ L(0, B, C) = (0 + B) + C \]
\[ = B + C \]
\[ R(0, B, C) = 0 + (B + C) \]
\[ = B + C \]
\[ L(S_u A, B, C) = (S_u A + B) + C \]
\[ = S_u (A + B) + C \]
\[ = S_u ((A + B) + C) \]
\[ = S_u L(A, B, C) \]
\[ R(S_u A, B, C) = S_u A + (B + C) \]
\[ = S_u (A + (B + C)) \]
\[ = S_u R(A, B, C) \]
\[ L(A, B, C) = R(A, B, C) \]

17. Definition of \( P_u X \)

\[ P_u 0 = 0 \]
\[ P_u S_v X = \begin{cases} X & u \neq v \\ S_v P_u X & u = v \end{cases} \]

18. Definition of \( Y \rightleftharpoons X \)

\[ Y \rightleftharpoons 0 = Y \]
\[ Y \rightleftharpoons S_u X = P_u (Y \rightleftharpoons X) \]
19. Proposition

\[ 0 \cdot X = 0 \]

Proof
\[
L(0) = 0 \cdot 0 = 0
\]
\[
R(0) = 0
\]
\[
L(S_uX) = 0 \cdot S_uX = P_u(0 \cdot X)
\]
\[
= P_uL(X)
\]
\[
R(S_uX) = 0 = P_uR(X)
\]
\[
L(X) = R(X)
\]

20. Proposition

\[ (C \cdot B) \cdot A = C \cdot (B + A) \]

Proof
\[
L(O,B,C) = (C \cdot B) \cdot 0 = C \cdot B
\]
\[
R(O,B,C) = C \cdot (B + 0) = C \cdot B
\]
\[
L(S_uA,B,C) = (C \cdot B) \cdot S_uA = P_u((C \cdot B) \cdot A)
\]
\[
= P_uL(A,B,C)
\]
\[
R(S_uA,B,C) = C \cdot (B + S_uA) = C \cdot S_u(B + A)
\]
\[
= P_u(C \cdot (B + A))
\]
\[
= P_uR(A,B,C)
\]
\[
L(A,B,C) = R(A,B,C)
\]
21. Proposition

\[ S_u B \preceq S_u A = B \preceq A \]

**Proof**

\[ L(0, B) = S_u B \preceq S_u 0 \]
\[ = P_u (S_u B \preceq 0) \]
\[ = P_u S_u B \]
\[ = B \]

\[ R(0, B) = B \preceq 0 \]
\[ = B \]

\[ L(S_v A, B) = S_u B \preceq S_u S_v A \]
\[ = S_u B \preceq S_v S_u A \]
\[ = P_v (S_u B \preceq S_u A) \]
\[ = P_v L(A, B) \]

\[ R(S_v A, B) = B \preceq S_v A \]
\[ = P_v (B \preceq A) \]
\[ = P_v R(A, B) \]

\[ L(A, B) = R(A, B) \]

22. \[ A \preceq A = 0 \]

**Proof**

\[ L(0) = 0 \preceq 0 \]
\[ = 0 \]

\[ R(0) = 0 \]

\[ L(S_u A) = S_u A \preceq S_u A \]
\[ = A \preceq A \]
\[ = L(A) \]

\[ R(S_u A) = 0 \]
\[ = R(A) \]

\[ L(A) = R(A) \]
23. \((B + A) 
prec A = B\)

Proof

\[
L(0, B) = (B + 0) \prec 0 = B + 0 = B
\]

\[
R(0, B) = B
\]

\[
L(S_u A, B) = (B + S_u A) \prec S_u A = S_u (B + A) \prec S_u A = (B + A) \prec A = L(A, B)
\]

\[
R(S_u A, B) = B = R(A, B)
\]

\[
L(A, B) = R(A, B) = 6.
\]

24. \((B + A) \prec (C + A) = B \prec C\)

Proof

\[
L(0, B, C) = (B + 0) \prec (C + 0) = B \prec C = 7.
\]

\[
R(0, B, C) = B \prec C
\]

\[
L(S_u A, B, C) = (B + S_u A) \prec (C + S_u A) = S_u (B + A) \prec S_u (C + A) = (B + A) \prec (C + A) = L(A, B, C)
\]

\[
R(S_u A, B, C) = B \prec C = R(A, B, C)
\]

\[
L(A, B, C) = R(A, B, C) = 6.
\]

25. \(B \prec (B + A) = 0\)

Proof

\[
L(0, B) = B \prec (B + 0) = B \prec B = 0 = 22.
\]
\[ R(O, B) = 0 \]
\[ L(S_uA, B) = B \oplus (B + S_uA) \]
\[ = B \oplus S_u(B + A) \]
\[ = P_u(B \oplus (B + A)) \]
\[ = P_uL(A, B) \]
\[ R(S_uA, B) = 0 \]
\[ = P_u0 \]
\[ = P_uR(A, B) \]
\[ L(A, B) = R(A, B) \]

26. Rule of inference

\[ F(0) = G(0) \]
\[ F(S_uX) = G(S_uX) \quad u = 1, 2, \ldots, r \]
\[ F(S_uX) = H_u(X, F(X)) \quad u = r + 1, \ldots, n \]
\[ G(S_uX) = H_u(X, G(X)) \quad u = r + 1, \ldots, n \]

Proof Define \( M_u(X, Y) \) by

\[ M_u(X, Y) = \begin{cases} F(S_uX) & u = 1, 2, \ldots, r \\ H_u(X, Y) & u = r + 1, \ldots, n \end{cases} \]

Now \( F(S_uX) = G(S_uX) \quad u = 1, 2, \ldots, r \)

Hence \( M_u(X, Y) = \begin{cases} G(S_uX) & u = 1, 2, \ldots, r \\ H_u(X, Y) & u = r + 1, \ldots, n \end{cases} \)

We now have \( F(0) = G(0) \)
\[ F(S_uX) = M_u(X, Y) \quad u = 1, \ldots, n \]
\[ G(S_uX) = M_u(X, Y) \quad u = 1, \ldots, n \]

Hence \( F(X) = G(X) \)
27. Definition \( C_u X \)

\[
C_u 0 = 0 \\
C_u S_v X = \begin{cases} 
S_v C_u X & u = v \\
C_u X & u \neq v 
\end{cases}
\]

28. \( C_u C_v A \) = \[
\begin{cases} 
0 & u \neq v \\
C_u A & u = v 
\end{cases}
\]

Proof

\[
L(0) = C_u C_v 0 \\
= C_u 0 \\
= 0
\]

\( R(0) = 0 \)

\[
L(S_w A) = C_u C_v S_w A \\
= \begin{cases} 
C S_v C_A & v = w \\
C_u C_v A & v \neq w 
\end{cases}
\]

\[
= \begin{cases} 
S_w C_u C_v A & u = v = w \\
C_u C_v A & v = w & u \neq w \\
C_u C_v A & v \neq w 
\end{cases}
\]

\[
= \begin{cases} 
S_w L(A) & u = v = w \\
L(A) & u \neq w \text{ or } v \neq w 
\end{cases}
\]

\( R(S_w A) = \begin{cases} 
0 & u \neq v \\
C_u S_w A & u = v \\
0 & u \neq v \\
S_w C_A & u = v = w \\
C_A & u = v & u \neq w \\
S_w C_A & u = v = w \\
0 & u \neq v \\
C_u A & u = v \\
0 & u \neq v \\
C_u A & u = v \\
0 & u \neq v \text{ or } v \neq w 
\end{cases} \)
\[ L(A) = R(A) \]

29. \( C_u C_A = C_v C_A \)

Proof

\[
\begin{aligned}
C_u C_A &= \begin{cases} 
0 & u \neq v \\
C_u A & u = v
\end{cases} \\
&= \begin{cases} 
0 & u \neq v \\
C_v A & u = v
\end{cases} \\
&= C_v C_A
\end{aligned}
\]

30. \( C_u (B + A) = C_u B + C_u A \)

Proof

\[
\begin{aligned}
L(0, B) &= C_u (B + 0) \\
&= C_u B \\
R(0, B) &= C_u B + C_u 0 \\
&= C_u B + 0 \\
&= C_u B \\
L(S_v A, B) &= C_u (B + S_v A) \\
&= C_u S_v (B + A) \\
&= \begin{cases} 
S_v C_u (B + A) & u = v \\
C_u (B + A) & u \neq v
\end{cases} \\
R(S_v A, B) &= C_u B + C_u S_v A \\
&= \begin{cases} 
C_u B + S_v C_A & u = v \\
C_u B + C_u A & u \neq v
\end{cases}
\]
\[ \begin{align*}
&= \begin{bmatrix} S_v(C_uB + C_uA) & u=v \\
                  C_uB + C_uA & u\neq v 
\end{bmatrix} \\
&= \begin{bmatrix} S_vR(A,B) & u=v \\
                  R(A,B) & u\neq v 
\end{bmatrix}
\]

**L(A,B) = R(A,B)**

31. \[ C_uP_vA = \begin{bmatrix} P_vC_uA & u=v \\
                               C_u & u\neq v 
\end{bmatrix} \]

**Proof**

**L(0) = C_uP_v0**

\[ = C_u0 \]
\[ = 0 \]

**R(0) = \begin{bmatrix} P_vC_u0 & u=v \\
                               C_u0 & u\neq v 
\end{bmatrix} \]

\[ = \begin{bmatrix} 0 & u=v \\
                         0 & u\neq v 
\end{bmatrix} \]
\[ = 0 \]

**L(S_wA) = C_uP_vS_wA**

\[ = \begin{bmatrix} S_wC_uP_vA & v\neq w \\
                  C_uA & v=w 
\end{bmatrix} \]

\[ = \begin{bmatrix} S_wC_uP_vA & v\neq w & u=w \\
                  C_uP_vA & v\neq w & u\neq w \\
                  C_uA & v=w 
\end{bmatrix} \]

\[ = \begin{bmatrix} S_wL(A) & v\neq w & u=w \\
                  L(A) & v\neq w & u\neq w \\
                  C_uA & v=w 
\end{bmatrix} \]

**R(S_wA) = \begin{bmatrix} P_vC_uS_wA & u=v \\
                               C_uS_wA & u\neq v 
\end{bmatrix} \]
\[ \begin{align*}
&= \{ P_{v,w}C_{u} \mid u=v \land u=w \} \\
&= \{ P_{v}C_{u} \mid u=v \land u=w \} \\
&= \{ S_{w}C_{u} \mid u=w \} \\
&= \{ C_{u} \mid u=w \} \\
&= \{ C_{u} \mid u=v \land v=w \land u=w \} \\
&= \{ P_{v}C_{u} \mid u=v \land v=w \land u=w \} \\
&= \{ S_{w}C_{u} \mid u=w \} \\
&= \{ C_{u} \mid u=v \land v=w \land u=w \} \\
&= \{ C_{u} \mid v=w \} \\
&= \{ S_{R(A)} \mid v=w \land u=w (u=v) \} \\
&= \{ R(A) \mid v=w \} \\
&= \{ C_{u} \mid v=w \} \\
\end{align*} \]

\[ L(A) = R(A) \]
\[ L(S^A, B) = C_u(B \ominus S^A) \]
\[ = C_u P_v(B \ominus A) \]
\[ = \begin{cases} P_v C_u(B \ominus A) & \text{if } u=v \\ C_u(B \ominus A) & \text{if } u \neq v \end{cases} \]
\[ = \begin{cases} P_v L(A, B) & \text{if } u=v \\ L(A, B) & \text{if } u \neq v \end{cases} \]

\[ R(S^A, B) = C_u B \ominus C_v S^A \]
\[ = \begin{cases} C_u B \ominus S_v C_u A & \text{if } u=v \\ C_u B \ominus C_u A & \text{if } u \neq v \end{cases} \]
\[ = \begin{cases} P_v (C_u B \ominus C_u A) & \text{if } u=v \\ C_u B \ominus C_u A & \text{if } u \neq v \end{cases} \]
\[ = \begin{cases} P_v R(A, B) & \text{if } u=v \\ R(A, B) & \text{if } u \neq v \end{cases} \]

\[ L(A, B) = R(A, B) \]

33. \[ C_1 A + C_2 A + \ldots + C_n A = A \]

Proof

\[ L(0) = C_1 0 + C_2 0 + \ldots + C_n 0 \]
\[ = 0 + 0 + \ldots + 0 \]
\[ = 0 \]
\[ R(0) = 0 \]

\[ L(S^A) = C_1 S_u A + C_2 S_u A + \ldots + C_n S_u A \]
\[ = C_1 A + C_2 A + \ldots + S_u C_u A + \ldots + C_n A \]
\[ = S_u (C_1 A + C_2 A + \ldots + C_n A) \]
\[ = S_u L(A) \]

\[ R(S^A) = S_u A \]
\[ = S_u R(A) \]

\[ L(A) = R(A) \]
34. \( S_vB \vdash C_uA = S_v(B \vdash C_uA) \quad u \neq v \)

Proof

\[
L(0,B) = S_vB \vdash C_u0
= S_vB \vdash 0
= S_vB
\]

\[
R(0,B) = S_v(B \vdash C_u0)
= S_v(B \vdash 0)
= S_vB
\]

\[
L(S_wA,B) = S_vB \vdash C_uS_wA
= \begin{cases} 
S_vB \vdash S_wC_uA & u = w \\
S_vB \vdash C_uA & u \neq w
\end{cases}
\]

\[
= \begin{cases} 
P_w(S_vB \vdash C_uA) & u = w \\
S_vB \vdash C_uA & u \neq w
\end{cases}
\]

\[
= \begin{cases} 
P_wL(A,B) & u = w \\
L(A,B) & u \neq w
\end{cases}
\]

\[
R(S_wA,B) = S_v(B \vdash C_uS_wA)
= \begin{cases} 
S_v(B \vdash S_wC_uA) & u = w \\
S_v(B \vdash C_uA) & u \neq w
\end{cases}
\]

\[
= \begin{cases} 
P_wS_v(B \vdash C_uA) & u = w \\
S_v(B \vdash C_uA) & u \neq w
\end{cases}
\]

\[
= \begin{cases} 
P_wR(A,B) & u = w \\
R(A,B) & u \neq w
\end{cases}
\]

\[
L(A,B) = R(A,B)
\]
\[ P_v B + C_u A = P_v (B + C_u A) \quad u \notin v \]

Proof

\[ L(0, B) = P_v B + C_u 0 \]
\[ = P_v B + 0 \quad 27. \]
\[ = P_v B \quad 7. \]

\[ R(0, B) = P_v (B + C_u 0) \]
\[ = P_v (B + 0) \quad 27. \]
\[ = P_v B \quad 7. \]

\[ L(S_w A, B) = P_v B + C_u S_w A \]
\[ = \begin{cases} P_v B + S_w C_u A & u = w \quad 27. \\ P_v B + C_u A & u \neq w \quad 27. \end{cases} \]
\[ \begin{cases} S_w (P_v B + C_u A) & u = w \quad 7. \\ P_v B + C_u A & u \neq w \end{cases} \]
\[ = \begin{cases} S_w L(A, B) & u = w \\ L(A, B) & u \neq w \end{cases} \]

\[ R(S_w A, B) = P_v (B + C_u S_w A) \]
\[ = \begin{cases} P_v (B + S_w C_u A) & u = w \quad 27. \\ P_v (B + C_u A) & u \neq w \quad 27. \end{cases} \]
\[ \begin{cases} P_v S_w (B + C_u A) & u = w \quad 7. \\ P_v B + C_u A & u \neq w \end{cases} \]
\[ = \begin{cases} S_w P_v (B + C_u A) & u = w \quad (u \neq v \text{ and } \therefore v \neq w) \quad 17. \\ P_v B + C_u A & u \neq w \end{cases} \]
\[ \begin{cases} S_w R(A, B) & u = w \\ R(A, B) & u \neq w \end{cases} \]

\[ L(A, B) = R(A, B) \quad 6. \]
36. \((B + C_A) \vdash C_vA = (B \vdash C_vA) + C_uA\) \(u \not\vdash v\)

Proof

\[
\begin{align*}
L(0, B) &= (B + C_0) \vdash C_v0 \\
&= (B + 0) \vdash 0 \\
&= B \\
R(0, B) &= (B \vdash C_v0) + C_u0 \\
&= (B \vdash 0) + 0 \\
&= B
\end{align*}
\]

\[
\begin{align*}
L(S_wA, B) &= (B + C_{u,w} A) \vdash C_{v,w}A \\
&= \begin{cases} 
(B + C_{u,w} A) \vdash S_{w,v}C_{v,w} & v = w \\
(B + C_{u,w} A) \vdash C_{v,w} & v \not= w
\end{cases} \\
&= \begin{cases} 
(B + C_{u} A) \vdash S_{w,v}C_{v,w} & v = w \ (u \not\vdash v \text{ and } \therefore u \not\vdash w) \\
(B + S_{w}C_{w,u} A) \vdash C_{v,w} & v \not= w \land u = w \\
(B + C_{u,w} A) \vdash C_{v,w} & v \not= w \land u \not= w
\end{cases} \\
&= \begin{cases} 
P_w((B + C_{u} A) \vdash C_{v}A) & v = w \\
S_w((B + C_{u} A) \vdash C_{v}A) & v \not= w \land u = w \\
(P_wL(A, B) & v = w \\
S_wL(A, B) & v \not= w \land u = w \land u \not= w
\end{cases} \\
&= \begin{cases} 
R(S_wA, B) = (B \vdash C_{v,w}A) + C_{u,w}A
\end{cases}
\]
\[ (B - S^C^A) + C^S^A v=w 27. \]

\[ (B - C^A) + C^S^A v/w 27. \]

\[ [(B - S^C^A) + C^A v=w (u/v and \therefore u/w) 27. \]

\[ (B - C^A) + C^A v/w \& u/w 27. \]

\[ P_w (B - C^A) + C^A v=w 18. \]

\[ S_w ((B - C^A) + C^A) v=w & u=w \]

\[ (B - C^A) + C^A v/w & u/w \]

\[ L(A,B) = R(A,B) \]

\[ 37. \]

\[ C_u B - C_v A = C_u B \]

Proof

\[ L(O,B) = C_u B - C_v 0 \]

\[ = C_u B \]

\[ = C_u B \]

\[ R(O,B) = C_u B \]

\[ L(S_u A,B) = C_u B - C_v S_u A \]

\[ = \begin{cases} C_u B - S_w C^A & v=w 27. \\ C_u B - C^A & v\neq w 27. \end{cases} \]

\[ = \begin{cases} P_w (C_u B - C^A) & v=w 18. \\ C_u B - C^A & v\neq w \end{cases} \]
\[ L(A, B) = R(A, B) \]

Proof

\[ L(0) = P_u^0_v \]
\[ = P_u^0 \]
\[ = 0 \]

\[ R(0) = P_v^0_u \]
\[ = P_v^0 \]
\[ = 0 \]

\[ L(S_{wA}) = P_u^0 v_w\]

\[ R(S_{wA}, B) = C_{uB} \]
\[ = C_{uB} + 0 \]
\[ = C_{uB} + P_w^0 \]
\[ = P_w^0 (C_{uB} + 0) \]

38.

\[ P_u^0 v v = P_v^0 u \]

7.

17.

35. 15.
\[ \begin{align*}
\text{L}(A,B) &= \text{R}(A,B) \\
39. \quad P_u(B \triangleleft A) &= P_u B \triangleleft A \\
\text{Proof} \quad L(0,B) &= P_u (B \triangleleft 0) \\
&= P_u B
\end{align*} \]
\[ R(0, B) = P_u^B = 0 \]
\[ = P_u \]

\[ L(S_v A, B) = P_u (B \leq S_v A) \]
\[ = P_u P_v (B \leq A) \]
\[ = P_v P_u (B \leq A) \]
\[ = P_v L(A, B) \]

\[ R(S_v A, B) = P_u B \leq S_v A \]
\[ = P_v (P_u B \leq A) \]
\[ = P_v R(A, B) \]

\[ L(A, B) = R(A, B) \]

40.
\[ P_v C_u A = \begin{cases} C_u P_A & u=v \\ C_u P_v & u \neq v \end{cases} \]

**Proof**

\[ L(0) = P_v C_u O \]
\[ = P_v O \]
\[ = 0 \]

\[ R(0) = \begin{cases} C_u P_v O & u=v \\ C_u O & u \neq v \end{cases} \]
\[ = \begin{cases} C_u O & u=v \\ 0 & u \neq v \end{cases} \]
\[ = \begin{cases} 0 & u=v \\ 0 & u \neq v \end{cases} \]
\[ = 0 \]

\[ L(S_v A) = P_v C_u S_v A \]
\[ = \begin{cases} P_v S_v C_A & u=w \\ P_v C_u & u \neq w \end{cases} \]
\[
\begin{align*}
\{ C_u A \} & \quad u = w \land v = w \quad 17. \\
\{ S_w P C_u A \} & \quad u = w \land v \neq w \quad 17. \\
\{ P C_u A \} & \quad u \neq w \\
\{ C_u A \} & \quad u = v = w \\
\{ S_w L(A) \} & \quad u = w \land v \neq w \\
\{ L(A) \} & \quad u \neq w \\
\{ C_{u'v} S_w A \} & \quad u = v \\
\{ C_u S_w A \} & \quad u \neq v \\
\{ C_u A \} & \quad u = v = w \\
\{ C_{u'v} P A \} & \quad u = v \land v \neq w \quad (u \neq w) \quad 27. \\
\{ S_w C_u A \} & \quad u \neq v \land u = w \quad 27. \\
\{ C_u A \} & \quad u \neq v \land u \neq w \\
\{ C_u A \} & \quad u = v = w \\
\{ S_w C_u A \} & \quad u = w \land v \neq w \quad (u \neq v) \\
\{ C_u P A \} & \quad u \neq w \land u = v \\
\{ C_u A \} & \quad u \neq w \land u \neq v \\
\{ C_u A \} & \quad u = v = w \\
\{ S_w R(A) \} & \quad u = w \land v \neq w \\
\{ R(A) \} & \quad u \neq w \\
\end{align*}
\]

\[ L(A) = R(A) \quad 26. \]

41. \[ C_u B \in A = C_u B \in C_u A \]
Proof

\[ L(0, B) = C_u B \div 0 \]
\[ = C_u B \]

\[ R(0, B) = C_u B \div C_u 0 \]
\[ = C_u B \div 0 \]
\[ = C_u B \]

\[ L(S_v A, B) = C_u B \div S_v A \]
\[ = P_v (C_u B \div A) \]

\[ = \begin{cases} P_v (C_u B \div A) & u=v \\ P_v C_u B \div A & u \neq v \end{cases} \]

\[ = \begin{cases} P_v (C_u B \div A) & u=v \\ C_u B \div A & u \neq v \end{cases} \]
\[ = \begin{cases} P_v L(A, B) & u=v \\ L(A, B) & u \neq v \end{cases} \]

\[ R(S_v A, B) = C_u B \div C_u S_v A \]
\[ = \begin{cases} C_u B \div S_v C_u A & u=v \\ C_u B \div C_u A & u \neq v \end{cases} \]

\[ = \begin{cases} P_v (C_u B \div C_u A) & u=v \\ C_u B \div C_u A & u \neq v \end{cases} \]
\[ = \begin{cases} P_v R(A, B) & u=v \\ R(A, B) & u \neq v \end{cases} \]

\[ L(A, B) = R(A, B) \]

42. \[ S_u 0 \div A = S_u 0 \div C_u A \]

Proof

\[ C_u S_u 0 = S_u C_u 0 \]
\[ = S_u 0 \]
\[ S_u^0 \cdot A = C_u S_u^0 \cdot A \]
\[ = C_u S_u^0 \cdot C_u A \]
\[ = S_u^0 \cdot C_u A \]

43. Definition of \( Y \cdot X \)

\[ Y \cdot 0 = 0 \]

\[ Y_S^u X = Y \cdot X + C_u Y \quad u = 1, \ldots, n \]

44.

\[ 0 \cdot X = 0 \]

Proof

\[ L(0) = 0 \cdot 0 \]

\[ = 0 \]

\[ R(0) = 0 \]

\[ L(S_u X) = 0 \cdot S_u X \]

\[ = 0 \cdot X + C_u 0 \]

\[ = 0 \cdot X + 0 \]

\[ = 0 \cdot X \]

\[ = L(X) \]

\[ R(S_u X) = 0 \]

\[ = R(X) \]

\[ L(X) = R(X) \]

45. \[ S_u Y \cdot X = Y \cdot X + C_u X \]

Proof

\[ L(0, Y) = S_u Y \cdot 0 \]

\[ = 0 \]

\[ R(0, Y) = Y \cdot 0 + C_u 0 \]

\[ = 0 + 0 \]

\[ = 0 \]
\[ L(S_X,Y) = S_Y \cdot S_X \]
\[ = S_Y \cdot X + C \cdot S_Y \]
\[ = \begin{cases} S_Y \cdot X + S_U \cdot C_Y & u=v \\ S_Y \cdot X + C_Y & u \neq v \end{cases} \]
\[ = \begin{cases} S_U (S_Y \cdot X) + C_Y & u=v \\ S_Y \cdot X \cdot C_Y & u \neq v \end{cases} \]
\[ = \begin{cases} S_Y (L(X,Y) + C_Y) & u=v \\ L(X,Y) + C_Y & u \neq v \end{cases} \]

\[ R(S_X,Y) = Y \cdot S_X + C \cdot S_X \]
\[ = \begin{cases} Y \cdot X + C_Y + S_U \cdot C_X & u=v \\ Y \cdot X + C_Y + C_U & u \neq v \end{cases} \]
\[ = \begin{cases} S_Y (Y \cdot X + C_Y + C_U) & u=v \\ Y \cdot X + C_Y + C_U & u \neq v \end{cases} \]
\[ = \begin{cases} S_Y (R(X,Y) + C_Y) & u=v \\ R(X,Y) + C_Y & u \neq v \end{cases} \]

\[ L(X,Y) = R(X,Y) \]

\[ X \cdot Y = Y \cdot X \]

**Proof**

\[ L(0,Y) = 0 \cdot Y \]
\[ = 0 \]
\[ = 0 \]

\[ R(0,Y) = Y \cdot 0 \]
\[ = 0 \]
\[ = 0 \]
\[ L(S_u, X, Y) = S_u X Y \]
\[ = X Y + C_u Y \]
\[ = L(X, Y) + C_u Y \]

\[ R(S_u, X, Y) = Y S_u X \]
\[ = Y X + C_u Y \]
\[ = R(X, Y) + C_u Y \]

\[ L(X, Y) = R(X, Y) \]

47.
\[ C_u(B.A) = C_u B C_u A \]

Proof
\[ L(0, B) = C_u(B.0) \]
\[ = C_u 0 \]
\[ = 0 \]

\[ R(0, B) = C_u B C_u 0 \]
\[ = C_u B 0 \]
\[ = 0 \]

\[ L(S_v A, B) = C_u(B.S_v A) \]
\[ = C_u (B.A) + C_v C_u B \]
\[ = C_u (B.A) + C_u B \quad u=v \]
\[ = C_u (B.A) + 0 \quad u \neq v \]

\[ R(S_v A, B) = C_u B C_u S_v A \]
\[ = C_v S_v C_u A \quad u=v \]
\[ = C_u B C_u A \quad u \neq v \]
\[ = C_u B C_u A + C_v C_u B \quad u=v \]
\[ = C_u B C_u A \quad u \neq v \]
\[
\begin{aligned}
&= \begin{cases} 
C_uB\cdot C_uA + C_uB & u=v \\
C_uB\cdot C_uA & u \neq v 
\end{cases} \\
&= \begin{cases} 
R(A,B) + C_uB & u=v \\
R(A,B) & u \neq v 
\end{cases}
\end{aligned}
\]

\[L(A,B) = R(A,B)\]

48. \(C_uB\cdot A = C_uB\cdot C_uA\)

Proof

\[L(0,B) = C_uB\cdot 0\]

\[= 0\]

\[R(0,B) = C_uB\cdot 0\]

\[= C_uB\cdot 0\]

\[= 0\]

\[L(S_vA,B) = C_uB\cdot S_vA\]

\[= C_uB\cdot A + C_{C_uB} v u\]

\[= \begin{cases} 
C_uB\cdot A + C_uB & u=v \\
C_uB\cdot A + 0 & u \neq v 
\end{cases} \\
= \begin{cases} 
L(A,B) + C_uB & u=v \\
L(A,B) & u \neq v 
\end{cases}
\]

\[R(S_vA,B) = C_uB\cdot C_uS_vA\]

\[= \begin{cases} 
C_uB\cdot S_vC_uA & u=v \\
C_uB\cdot C_uA & u \neq v 
\end{cases} \\
= \begin{cases} 
C_uB\cdot C_uA + C_{C_uB} & u=v \\
C_uB\cdot C_uA & u \neq v 
\end{cases} \\
= \begin{cases} 
C_uB\cdot C_uA + C_uB & u=v \\
C_uB\cdot C_uA & u \neq v 
\end{cases}
\]
\[
\begin{align*}
\{ R(A, B) + C_u B & \quad u=v \\
\{ R(A, B) & \quad u\neq v
\end{align*}
\]

\[ L(A, B) = R(A, B) \]

49. \[ C_u A \cdot B = A \cdot C_u B \]

Proof
\[ C_u A \cdot B = C_u A \cdot C_u B \]
\[ = C_u B \cdot C_u A \]
\[ = C_u B \cdot A \]
\[ = A \cdot C_u B \]

50. \[ C \cdot (B + A) = C \cdot B + C \cdot A \]

Proof
\[ L(0, B, C) = C \cdot (B + 0) \]
\[ = C \cdot B \]
\[ R(0, B, C) = C \cdot B + C \cdot 0 \]
\[ = C \cdot B + 0 \]
\[ = C \cdot B \]
\[ L(S_u A, B, C) = C \cdot (B + S_u A) \]
\[ = C \cdot S_u (B + A) \]
\[ = C \cdot (B + A) + C_u C \]
\[ = L(A, B, C) + C_u C \]
\[ R(S_u A, B, C) = C \cdot B + C \cdot S_u A \]
\[ = C \cdot B + C \cdot A + C_u C \]
\[ = R(A, B, C) + C_u C \]
\[ L(A, B, C) = R(A, B, C) \]
51. \((C.B).A = C.(B.A)\)

Proof
\[
L(O,B,C) = (C.B).O
\]
\[
= 0
g 43.
\]
\[
R(O,B,C) = C.(B.O)
\]
\[
= C.O
g 43.
\]
\[
= 0
g 43.
\]
\[
L(S_u A,B,C) = (C.B).S_u A
\]
\[
= (C.B).A + C_u(C.B)\]
\[
= L(A,B,C) + C_u(C.B)\]
\[
R(S_u A,B,C) = C.(B.S_u A)
\]
\[
= C.(B.A + C_u B)\]
\[
= C.(B.A) + C.C_u B\]
\[
= C.(B.A) + C_u(C.B)\]
\[
= R(A,B,C) + C_u(C.B)\]
\[
L(A,B,C) = R(A,B,C)\]

52. \(B.P_u A = B.A + C_u B\)

Proof
\[
L(O,B) = B.P_u 0
\]
\[
= B.0
g 17.
\]
\[
= 0
g 43.
\]
\[
R(O,B) = B.0 + C_u B
\]
\[
= 0 + C_u B\]
\[
= 0
g 19.
\]
\[ L(S_v A, B) = B P_u S_v A \]
\[ = \begin{cases} B A & u=v \\ B S_v P A & u \not= v \end{cases} \]
\[ = \begin{cases} B A & u=v \\ B P_u A + C_v B & u \not= v \end{cases} \]
\[ = \begin{cases} B A & u=v \\ L(A,B) + C_v B & u \not= v \end{cases} \]

\[ R(S_v A, B) = B S_v A ^ C_u B \]
\[ = (B A + C_v B) ^ C_u B \]
\[ = \begin{cases} (B A + C_u B) ^ C_u B & u=v \\ (B A ^ C_u B) + C_v B & u \not= v \end{cases} \]
\[ = \begin{cases} B A & u=v \\ R(A,B) + C_v B & u \not= v \end{cases} \]

\[ L(A,B) = R(A,B) \]

53. \[ c.(B ^ A) = c.B ^ c.A \]

Proof
\[ L(0,B,C) = c.(B ^ 0) \]
\[ = c.B \]
\[ R(0,B,C) = c.B ^ c.0 \]
\[ = c.B ^ 0 \]
\[ = c.B \]
\[ L(S_v A,B,C) = c.(B ^ S_v A) \]
\[ = c.P_u (B ^ A) \]
\[ = c.(B ^ A) ^ c_u C \]
\[ = L(A,B,C) ^ c_u C \]
\[ R(S_uA,B,C) = C.B \iff C.S_uA \]
\[ = C.B \iff (C.A + C_uC) \]
\[ = (C.B \iff C.A) \iff C_uC \]
\[ = R(A,B,C) \iff C_uC \]
\[ L(A,B,C) = R(A,B,C) \]

54. **Definition of** \(|A,B|\)
\[ |A,B| = (A \iff B) + (B \iff A) \]

55. **Rule of inference**
\[ A + B = 0 \]
\[ \underline{A = 0} \]
\[ B = 0 \]

**Proof**
\[ A + B = 0 \]
\[ \underline{A = (A + B) \iff B} \]
\[ = 0 \iff B \]
\[ = 0 \]
\[ B = (A + B) \iff A \]
\[ = 0 \iff A \]
\[ = 0 \]

56. **Rule of inference**
\[ A = B \]
\[ A \iff B = 0 \]
\[ B \iff A = 0 \]
Proof

\[ A = B \]

\[ A = B = B = B \]

\[ = 0 \]

\[ B = A = B = B \]

\[ = 0 \]

Premises

5.

22.

57. Rule of inference

\[ F(S_uX) = F(X) \]

\[ F(X) = F(0) \]

Proof

\[ L(0) = F(0) \]

\[ R(0) = F(0) \]

\[ L(S_uX) = F(S_uX) \]

\[ = F(X) \]

\[ = L(X) \]

\[ R(S_uX) = F(0) \]

\[ = R(X) \]

\[ L(X) = R(X) \]

Premises

5.

6.

58. Rule of inference

\[ F(0,Y) = 0 \]

\[ F(X,0) = 0 \]

\[ \underbrace{F(C_{u\overline{u}}X,C_{u\overline{u}}Y)}_{\text{for one value of } u} = \underbrace{F(C_{u\overline{u}}X,C_{u\overline{u}}Y)}_{\text{for that value of } u} \]

Proof

Consider first

\[ (a) \quad F(C_{u\overline{u}}X,C_{u\overline{u}}Y) = F(C_{u\overline{u}}X,C_{u\overline{u}}(Y - S_u0)) \]

Proof

\[ L(X,0) = F(C_{u\overline{u}}S_uX,C_{u\overline{u}}0) \]
\[
R(X,0) = F(C_u, C_u(0 \leq S_u)) = F(C_u, C_u0) = F(C_u, 0) = 0
\]

Premises

\[
L(X, Y) = F(C_u, C_u(0 \leq S_u)) = 0
\]

Premises

Now consider

\[F(C_u, C_u Y) = F(C_u(X \leq S_u), C_u(Y \leq S_u))\]
Proof

$L(0, y) = F(C_u X, C_u 0)

= F(C_u X, 0)

= 0 \quad \text{Premises}

R(0, y) = F(C_u (x \oplus S_u 0), C_u (0 \oplus S_u 0))

= F(C_u (x \oplus S_u 0), C_u 0)

= F(C_u (x \oplus S_u 0), 0)

= 0 \quad \text{Premises}

L(S_v X, y) = F(C_u S_v X, C_u y)

= \begin{cases} 
F(C_u S_v X, C_u y) & u = v \\
F(C_u X, C_u y) & u \neq v 
\end{cases} \quad 27.

R(S_v X, y) = F(C_u (S_v X \oplus S_u 0), C_u (y \oplus S_u 0))

= \begin{cases} 
F(C_u (S_v X \oplus S_u 0), C_u (y \oplus S_u 0)) & u = v \\
F(C_u S_v X \oplus C_u S_u 0, C_u (y \oplus S_u 0)) & u \neq v 
\end{cases} \quad 32.

L(X, y) = R(X, y) \text{ for (b)} \quad 26.

Now let

$G(N) = F(C_u (x \oplus N), C_u (y \oplus N))$
Hence \( G(S_v N) = F(C_u X \leq S_v N), C_u (Y \leq S_v N) \)

\[
= F(C_u P_v (X \leq N), C_u P_v (Y \leq N))
\]

\[
= \begin{cases} 
F(C_u ((X \leq N) \leq S_u 0), C_u ((Y \leq N) \leq S_u 0) & \text{u=v} \\
F(C_u (X \leq N), C_u (Y \leq N)) & \text{u\neq v}
\end{cases}
\]

\[
= \begin{cases} 
F(C_u (X \leq N), C_u (Y \leq N)) & \text{u=v} \\
F(C_u (X \leq N), C_u (Y \leq N)) & \text{u\neq v}
\end{cases}
\]

\[= G(N)\]

\[G(N) = G(0)\]

\[F(C_u (X \leq N), C_u (Y \leq N)) = F(C_u (X \leq 0), C_u (Y \leq 0)) = F(C_u X, C_u Y)\]

**In the particular instance where N = X we have:**

\[F(C_u X, C_u Y) = F(C_u (X \leq X), C_u (Y \leq C))\]

\[
= F(C_u 0, C_u (Y \leq X))
\]

\[
= F(0, C_u (Y \leq X))
\]

\[= 0\]

**Premises**

59. \( C_v (S_u 0 \leq A) = \begin{cases} S_u 0 \leq A & \text{u=v} \\
0 & \text{u\neq v}\end{cases} \)

**Proof** \( C_v (S_u 0 \leq A) = C_v S_u 0 \leq C_v A \)

\[
= \begin{cases} S_u 0 \leq C_u A & \text{u=v} \\
0 \leq C_u A & \text{u\neq v}
\end{cases}
\]

\[
= \begin{cases} S_u 0 \leq A & \text{u=v} \\
0 & \text{u\neq v}
\end{cases}
\]

\[= \begin{cases} S_u 0 \leq A & \text{u=v} \\
0 & \text{u\neq v}
\end{cases}\]
60. \[ A \cdot S_u = C_u A \]

Proof

\[
A \cdot S_u = A \cdot 0 + C_u A
\]

\[
= 0 + C_u A
\]

\[
= C_u A
\]

61. \[ S_u \cdot S_v A = \begin{cases} S_u \cdot S_v A \\ 0 \end{cases} \]

Proof

\[
S_u \cdot S_v A = S_u \cdot C_u S_v A
\]

\[
= \begin{cases} S_u \cdot C_u A \\ S_u \cdot S_u C_u A \end{cases} \]

\[
= \begin{cases} S_u \cdot A \\ 0 \cdot C_u A \end{cases} \]

\[
= \begin{cases} S_u \cdot A \\ 0 \end{cases} \]

62. \[ A \cdot (S_u \cdot A) = 0 \]

Proof

\[
L(0) = 0 \cdot (S_u \cdot 0)
\]

\[
= 0
\]

\[
R(0) = 0
\]

\[
L(S_v A) = S_v A \cdot (S_u \cdot S_v A)
\]

\[
= \begin{cases} S_v A \cdot 0 \\ S_v A \cdot (S_u \cdot A) \end{cases} \]

\[
= \begin{cases} 0 \\ A \cdot (S_u \cdot A) + C_v (S_u \cdot A) \end{cases} \]

\[
= \begin{cases} 0 \\ L(A) + 0 \end{cases} \]
\[ R(S \cdot A) = 0 \]

\[ \begin{cases} 0 & u=v \\ L(A) & u \neq v \end{cases} \]

\[ R(A) = 0 \]

\[ \begin{cases} 0 & u=v \\ 0 & u \neq v \end{cases} \]

\[ \begin{cases} 0 & u=v \\ R(A) & u \neq v \end{cases} \]

\[ L(A) = R(A) \]

63. Rule of inference

\[ F(0) = 0 \]

\[ (S_u \cdot F(X) \cdot F(S \cdot X) = 0 \]

\[ F(X) = 0 \]

Proof

Consider first

(a) \((S_u \cdot F(X) \cdot F(S \cdot X) = S_u \cdot F(X)\)

Proof

\[ (S_u \cdot F(X) \cdot S_u \cdot F(S \cdot X) = (S_u \cdot F(X)) \cdot S_u \cdot (S_u \cdot F(X)) \cdot F(S \cdot X) \]

\[ = (S_u \cdot F(X)) \cdot S_u \cdot 0 \text{ Premises} \]

\[ = (S_u \cdot F(X)) \cdot S_u 0 \]

\[ = C_u (S_u \cdot F(X)) \]

\[ = S_u \cdot F(X) \text{ for (a)} \]

Now define \(G(X)\) by

\[ G(0) = S_u 0 \]

\[ G(S \cdot X) = G(X) \cdot (S_u \cdot F(X)) \]
Let \( H(x) = G(S_x X) \)

\[
H(0) = G(S_0)
\]

\[
= G(0) \cdot (S_0 \cdot F(0))
\]

Defn.

Premises

\[
= G(0) \cdot (S_0 \cdot 0)
\]

\[
= S_0 \cdot S_0
\]

18.

\[
= C_u S_0
\]

60.

\[
= S_0
\]

27.

\[
= G(0)
\]

Defn.

\[ H(S_w X) = G(S_v S_w X) \]

\[
= G(S_w X) \cdot (S_u \cdot F(S_w X))
\]

Defn.

\[
= G(X) \cdot (S_u \cdot F(X)) \cdot (S_w \cdot F(S_w X))
\]

Defn.

\[
= G(X) \cdot (S_u \cdot F(X))
\]

(a)

\[
= G(S_w X)
\]

Defn.

\[ H(X) = G(X) \]

\[ G(S_v X) = G(X) \]

\[ G(X) = G(0) \]

57.

\[
= S_0
\]

Defn.

\[
S_0 = S_0 \cdot (S_0 \cdot F(X))
\]

Defn.

\[
= C_u (S_0 \cdot F(X))
\]

60.

\[
= S_u \cdot F(X)
\]

59.

\[
F(X) \cdot S_u = F(X) \cdot (S_u \cdot F(X))
\]

5.

\[
c_u F(X) = 0
\]

62.
As no specification has been made on \( u \) we can therefore say

\[ P(x) = 0 \]

by 53.

64. Definition (explicit)

\[ (x)^a = x^a = x \times x \]

65. Definition of \( R_t(x) \)

\[ R_t(0) = 0 \]

\[ R_t(sX) = R_t(x) + (s_0 \cdot ((s_0 R_t(x))^a \cdot s_0 x)) \quad u=1, \ldots, n \]

66. \( (s_0 \cdot ((s_0 (A + (s_0 B)))^a \cdot s_0 s_v c)) = (s_0 \cdot ((s_a)^a \cdot s_u c)) \) \( u \neq v \)

Proof

\[
(s_0 \cdot ((s_0 (A + (s_0 B)))^a \cdot s_0 s_v c)) = (s_0 \cdot c_{u} ((s_0 (A + (s_0 B)))^a \cdot s_0 s_v c)) \quad 42.
= (s_0 \cdot (c_{u} (s_0 (A + (s_0 B)))^a \cdot c_{u} s_v s_c)) \quad 32.
= (s_0 \cdot ((c_s (A + (s_0 B)))^a \cdot s_u c_s c_c)) \quad 47.27.
= (s_0 \cdot ((s_0 ((c_A + c_u (s_0 B)))^a \cdot s_u c_c)) \quad 30.27.
= (s_0 \cdot ((s_0 (c_A + 0)^a \cdot c_{u} s_c)) \quad 59.27.
= (s_0 \cdot ((s_0 c_{u} c_{u} a)^a \cdot c_{u} s_c)) \quad 7.
= (s_0 \cdot ((c_{u} s_{u} a)^a \cdot c_{u} s_{u} c)) \quad 27.
= (s_0 \cdot (c_{u} (s_{u} a)^a \cdot c_{u} s_{u} c)) \quad 47.
= (s_0 \cdot c_{u} ((s_{u} a)^a \cdot s_{u} c)) \quad 32.
= (s_0 \cdot ((s_{u} a)^a \cdot s_{u} c)) \quad 42.

67. The commutativity condition for \( R(x) \)

If \( u \neq v \) then
$B_u(S_vX, B_v(X,Z)) = Z + (S_v0 \Delta ((S_vZ)^2 \Delta S_vX)) +$

$(S_u0 \Delta ((S_u(Z + (S_u0 \Delta ((S_uZ)^2 \Delta S_uX))))^2 \Delta S_uS_vX))$ Defn.

$= Z + (S_v0 \Delta ((S_v(Z + (S_u0 \Delta ((S_uZ)^2 \Delta S_uX))))^2 \Delta S_vS_uX)) + 66.$

$(S_u0 \Delta ((S_uZ)^2 \Delta S_uX)) 66.$

$= Z + (S_u0 \Delta ((S_uZ)^2 \Delta S_uX)) +$

$(S_v0 \Delta ((S_v(Z + (S_u0 \Delta ((S_uZ)^2 \Delta S_uX))))^2 \Delta S_vS_uX))$ 15.

$= B_v(S_uX, B_u(Y,Z))$ for $u \neq v$ Defn.

If $u = v$ then

$B_u(S_vX, B_v(X,Z)) = B_v(S_uX, B_u(X,Z))$ $u = v$

Hence

$B_u(S_vX, B_v(X,Z)) = B_v(S_uX, B_u(X,Z))$ all $u, v$.

66. $(S_u0 \Delta A).((S_u0 \Delta (S_u0 \Delta A)) = 0$

Proof

$L(0) = (S_u0 \Delta 0).((S_u0 \Delta (S_u0 \Delta 0))$

$= S_u0.(S_u0 \Delta S_u0) 18.$

$= S_u0.0 22.$

$= 0 43.$

$R(0) = 0$

$L(S_vA) = (S_u0 \Delta S_vA).((S_u0 \Delta (S_u0 \Delta S_vA))$

$= \begin{cases} 0 \Delta (S_u0 \Delta (S_u0 \Delta A)) & u = v \\ (S_u0 \Delta A).((S_u0 \Delta (S_u0 \Delta A)) & u \neq v \end{cases} 61.$

$= \begin{cases} 0.((S_u0 \Delta 0) & u = v \\ L(A) & u \neq v \end{cases} 19.$

$= \begin{cases} 0 & u = v \\ L(A) & u \neq v \end{cases} 44.$
\[ R(S, A) = 0 \]
\[ = \begin{cases} 0 & u = v \\ 0 & u \neq v \end{cases} \]
\[ = \begin{cases} 0 & u = v \\ R(A) & u \neq v \end{cases} \]

\[ L(A) = R(A) \]

69. \((S, 0 \preceq (S, A \preceq B)) \cdot (S, 0 \preceq (B \preceq A)) = 0\)

Proof

\[ L(A, 0) = (S, 0 \preceq (S, A \preceq 0)) \cdot (S, 0 \preceq (0 \preceq A)) \]

\[ = (S, 0 \preceq S, A) \cdot (S, 0 \preceq 0) \]

\[ = 0 \cdot S, 0 \]

\[ = 0 \]

\[ L(0, B) = (S, 0 \preceq (S, 0 \preceq B)) \cdot (S, 0 \preceq (B \preceq 0)) \]

\[ = (S, 0 \preceq (S, 0 \preceq B)) \cdot (S, 0 \preceq B) \]

\[ = 0 \]

\[ L(C, S, A, C, S, B) = (S, 0 \preceq (S, C, S, A \preceq C, S, B)) \cdot (S, 0 \preceq (C, S, B \preceq C, S, A)) \]

\[ = (S, 0 \preceq (S, S, C, S, A \preceq S, C, B)) \cdot (S, 0 \preceq (S, C, B \preceq S, C, A)) \]

\[ = (S, 0 \preceq (S, C, A \preceq C, B)) \cdot (S, 0 \preceq (C, B \preceq C, A)) \]

\[ = L(C, A, C, B) \]

\[ L(C, A, C, B) = 0 \]

\[ (S, 0 \preceq (S, A \preceq B)) \cdot (S, 0 \preceq (B \preceq A)) = (S, 0 \preceq C, (S, A \preceq B)) \cdot (S, 0 \preceq C, (B \preceq A)) \]

\[ = (S, 0 \preceq (C, S, A \preceq C, B)) \cdot (S, 0 \preceq (C, B \preceq C, A)) \]

\[ = (S, 0 \preceq (S, C, A \preceq C, B)) \cdot (S, 0 \preceq (C, B \preceq C, A)) \]

\[ = L(C, A, C, B) \]

\[ = 0 \]

From above.
70. Rule of inference

\[ A \cdot B = A \cdot C \]

\[ A \mid B, C \mid = 0 \]

Proof

\[ A \mid B, C \mid = A \cdot ((B \cdot C) + (C \cdot B)) \]

\[ = A \cdot (B \cdot C) + A \cdot (C \cdot B) \]

\[ = (A \cdot B \cdot A \cdot C) + (A \cdot C \cdot A \cdot B) \]

\[ = 0 + 0 \]

\[ = 0 \]

Premises 56.

It is now necessary to quote the Key Equation. This is proved formally using primitive recursion under the same set of axioms in [4].

Thus

71. Key Equation

\[ A + (B \cdot A) = B + (A \cdot B) \]

72. \((S^0 \cdot X) \cdot F(C^0(X + Y)) = (S^0 \cdot X) \cdot F(C^0(Y))\)

Proof

\[ L(0,Y) = (S^0 \cdot 0 \cdot 0) \cdot F(C^0(0 + Y)) \]

\[ = S^0 \cdot F(C^0) \]

\[ = C^0 \cdot F(C^0) \]

\[ R(0,Y) = (S^0 \cdot 0 \cdot 0) \cdot F(C^0) \]

\[ = S^0 \cdot F(C^0) \]

\[ = C^0 \cdot F(C^0) \]

\[ L(S^0, X, Y) = (S^0 \cdot S^0, X) \cdot F(C^0(S^0, X + Y)) \]

\[ = \begin{cases} 0 \cdot F(C^0, S^0, X + Y) & \text{u=v} \\ (S^0 \cdot X) \cdot F(C^0, S^0, X + Y) & \text{u} \neq \text{v} \end{cases} \]

61.
In fact the equation

\[(S_u \circ X).F(C_u X + Y) = (S_u \circ X).F(Y)\]

is proved by the above schema, though the equation proved is sufficient for subsequent proofs.

73. Rule of inference

\[|A, B| = 0\]

\[A = B\]

Proof \[|A, B| = 0\] Premises.

\[(A \circ B) + (B \circ A) = 0\]

\[A \circ B = 0 \quad \ldots \quad \text{①}\]

\[B \circ A = 0 \quad \ldots \quad \text{②}\]

\[A = A + 0\]

\[= A + (B \circ A)\] From above ①

\[= B + (A \circ B)\] Key Equation

\[= B + 0\] From above ②

\[= B\]
74. Rule of inference

\[ F(0, y) = G(0, y) \]
\[ F(x, 0) = G(x, 0) \]
\[ F(C_u S_u X, C_u S_u Y) = F(C_u X, C_u Y) \] for one value of \( u \)
\[ G(C_u S_u X, C_u S_u Y) = G(C_u X, C_u Y) \] for the same value of \( u \).
\[ F(C_u X, C_u Y) = G(C_u X, C_u Y) \] for that value of \( u \).

Proof

Let \( H(X, y) = |F(X, y), G(X, y)| \)

Then \( H(0, y) = |F(0, y), G(0, y)| \)

\[ = 0 \] Premises 56. 7.

\[ H(x, 0) = |F(x, 0), G(x, 0)| \]

\[ = 0 \] Premises 56. 7.

\[ H(C_u S_u X, C_u S_u Y) = |F(C_u S_u X, C_u S_u Y), G(C_u S_u X, C_u S_u Y)| \]

\[ = |F(C_u X, C_u Y), G(C_u X, C_u Y)| \] \( u=v \) Premises

\[ |F(C_u X, C_u Y), G(C_u X, C_u Y)| \] \( u\neq v \) 27.

\[ = H(C_u X, C_u Y) \]

\[ H(C_u X, C_u Y) = 0 \] 58.

\[ F(C_u X, C_u Y) = G(C_u X, C_u Y) \] 73.

75. \[ |S_u X, S_u Y| = |X, Y| \]

Proof \[ |S_u X, S_u Y| = (S_u X \vdash S_u Y) + (S_u Y \vdash S_u X) \] 54.

\[ = (X \vdash Y) + (Y \vdash X) \] 21.

\[ = |X, Y| \] 54.
76. \( |c_u x, c_u y| = c_u |x, y| \)

Proof \( |c_u x, c_u y| = (c_u x \circ c_u y) + (c_u y \circ c_u x) \) 54.

\[ = c_u (x \circ y) + c_u (y \circ x) \] 32.

\[ = c_u ((x \circ y) + (y \circ x)) \] 30.

\[ = c_u |x, y| \] 54.

77. \( |x, y| = |y, x| \)

Proof \( |x, y| = (x \circ y) + (y \circ x) \) 54.

\[ = (y \circ x) + (x \circ y) \] 15.

\[ = |y, x| \] 54.

78. \( |0, y| = |y, 0| = y \)

Proof \( |0, y| = |y, 0| \) 77.

\[ = (y \circ 0) + (0 \circ y) \] 54.

\[ = y + 0 \] 18.19.

\[ = y \] 7.

79. \( (s_u 0 \circ |x, y|) \circ (s_u 0 \circ (x \circ y)) = (s_u 0 \circ |x, y|) \)

Proof \( L(0, y) = (s_u 0 \circ |0, y|) \circ (s_u 0 \circ (0 \circ y)) \)

\[ = (s_u 0 \circ y) \circ (s_u 0 \circ 0) \] 81.19.

\[ = (s_u 0 \circ y) \circ s_u 0 \] 18.

\[ = c_u (s_u 0 \circ y) \] 60.

\[ = (s_u 0 \circ y) \] 59.
R(0, Y) = \( (s^u \odot |0, Y| ) \)  
\[ = (s^u \odot Y) \] 78.

L(X, 0) = \( (s^u \odot |X, 0| ) \cdot (s^u \odot (x \odot 0)) \)  
\[ = (s^u \odot X) \cdot (s^u \odot x) \] 78.18.
\[ = s^u \cdot (s^u \odot x) \odot x \cdot (s^u \odot x) \] 53.46.
\[ = c^u (s^u \odot x) \odot 0 \] 60.62.
\[ = c^u (s^u \odot x) \] 18.
\[ = (s^u \odot x) \] 59.

R(X, 0) = \( (s^u \odot |X, 0| ) \)  
\[ = (s^u \odot X) \] 78.

\[ L(c^u s^u x, c^u s^u y) = (s^u \odot |c^u s^u x, c^u s^u y|) \cdot (s^u \odot (c^u s^u x \odot c^u s^u y)) \] 27.
\[ = (s^u \odot |s^u c^u x, s^u c^u y|) \cdot (s^u \odot (s^u c^u x \odot s^u c^u y)) \] 75.21.
\[ = L(c^u x, c^u y) \]

R(c^u s^u x, c^u s^u y) = \( (s^u \odot |c^u s^u x, c^u s^u y|) \)  
\[ = (s^u \odot |c^u x, c^u y|) \] 27.
\[ = (s^u \odot |c^u x, c^u y|) \] 75.
\[ = R(c^u x, c^u y) \]

\[ L(c^u x, c^u y) = R(c^u x, c^u y) \] 74.

Hence
\[ (s^u \odot |c^u x, c^u y|) \cdot (s^u \odot (c^u x \odot c^u y)) = (s^u \odot |c^u x, c^u y|) \] 76.32
\[ (s^u \odot |c^u x, y|) \cdot (s^u \odot c^u (x \odot y)) = (s^u \odot |c^u x, y|) \]
\[ (s^u \odot |x, y|) \cdot (s^u \odot (x \odot y)) = (s^u \odot |x, y|) \] as required. 42.
80. \((S_u \circ |X,Y|)(S_u \circ F(C_uX))F(C_uY) = 0\)

Proof \((S_u \circ X)F(C_u(X + Y)) = (S_u \circ Y)F(C_uY)\) 72.

Put \(x = y\) for \(X,\) and we have
\[(S_u \circ (X \circ Y))F(C_u((X \circ Y) + Y)) = (S_u \circ (X \circ Y))F(C_uY).\]

Multiply both sides of the equation by \((S_u \circ |X,Y|)\) and therefore
\[(S_u \circ |X,Y|)F(C_u((X \circ Y) + Y)) = (S_u \circ |X,Y|)F(C_uY)\]
70-79.

Similarly one can prove the equation
\[(S_u \circ |Y,X|)F(C_u((Y \circ X) + X)) = (S_u \circ |Y,X|)F(C_uX)\]

Hence from the above two equations, the Key Equation and 77., we have
\[(S_u \circ |X,Y|)F(C_uY) = (S_u \circ |X,Y|)F(C_uX).\]

Multiply both sides of the equation by \((S_u \circ F(C_uX));\) the right-hand side then becomes zero by 62. and therefore we have
\[(S_u \circ |X,Y|)(S_u \circ F(C_uX))F(C_uY) = 0\] as required.

81. \((S_u \circ (S_u \circ A)) + (S_u \circ A) = S_u\)

Proof \[L(0) = (S_u \circ (S_u \circ 0)) + (S_u \circ 0)\]
\[= (S_u \circ S_u) + S_u0\]
\[= 0 + S_u0\]
\[= S_u0\]
13.

\[R(0) = S_u0\]

\[L(S_uA) = (S_u \circ (S_u \circ S_uA)) + (S_u \circ S_uA)\]
\[
\begin{align*}
R(S_u) &= S_u \\
L(A) &= R(A)
\end{align*}
\]

82. \((S_u \land (S_u \land B)) \lor (S_u \land |A,B|) = (S_u \land (A \land B))\)

**Proof**

\[
\begin{align*}
L(A,0) &= (S_u \land (S_u \land 0)) + (S_u \land |A,0|) \\
&= (S_u \land S_u A) + (S_u \land A) \\
&= 0 + (S_u \land A) \\
&= S_u \land A
\end{align*}
\]

\[
\begin{align*}
R(A,0) &= (S_u \land (A \land 0)) \\
&= S_u \land A
\end{align*}
\]

\[
\begin{align*}
L(0,B) &= (S_u \land (S_u \land B)) + (S_u \land |0,B|) \\
&= (S_u \land (S_u \land B)) + (S_u \land B) \\
&= S_u
\end{align*}
\]
\[ R(0, B) = (S_u 0 \preceq (0 \preceq B)) \]
\[ = (S_u 0 \preceq 0) \]
\[ = S_u \]
\[ L(C_u S_u A, C_u S_u B) = (S_u 0 \preceq (S_u u S_u A \preceq S_u S_u B)) + (S_u 0 \preceq |C_u S_u A, C_u S_u B|) \]
\[ = (S_u 0 \preceq (S_u S_u C_u A \preceq S_u S_u B)) + (S_u 0 \preceq |S_u C_u A, S_u C_u B|) \]
\[ = (S_u 0 \preceq (S_u C_u A \preceq C_u B)) + (S_u 0 \preceq |C_u A, C_u B|) \]
\[ = L(C_u A, C_u B) \]
\[ R(C_u S_u A, C_u S_u B) = (S_u 0 \preceq (C_u S_u A \preceq C_u S_u B)) \]
\[ = (S_u 0 \preceq (S_u C_u A \preceq S_u C_u B)) \]
\[ = (S_u 0 \preceq (C_u A \preceq C_u B)) \]
\[ = R(C_u A, C_u B) \]
\[ L(C_u A, C_u B) = R(C_u A, C_u B) \]

Hence
\[ (S_u 0 \preceq (S_u C_u A \preceq C_u B)) + (S_u 0 \preceq |C_u A, C_u B|) = (S_u 0 \preceq (C_u A \preceq C_u B)) \]
\[ (S_u 0 \preceq (C_u S_u A \preceq C_u B)) + (S_u 0 \preceq |C_u A, C_u B|) = (S_u 0 \preceq C_u (A \preceq B)) \]
\[ (S_u 0 \preceq C_u (S_u A \preceq B)) + (S_u 0 \preceq |C_u, A, B|) = (S_u 0 \preceq C_u (A \preceq B)) \]
\[ (S_u 0 \preceq (S_u A \preceq B)) + (S_u 0 \preceq |A, B|) = (S_u 0 \preceq (A \preceq B)) \]

83. Rule of inference

\[ (S_u 0 \preceq (S_u A \preceq B)).C = 0 \]
\[ (S_u 0 \preceq |A, B|).C = 0 \]
\[ (S_u 0 \preceq (A \preceq B)).C = 0 \]

Proof
\[ (S_u 0 \preceq (A \preceq B)).C = ((S_u 0 \preceq (S_u A \preceq B)) + (S_u 0 \preceq |A, B|)).C \]
\[ = (S_u 0 \preceq (S_u A \preceq B)).C + (S_u 0 \preceq |A, B|).C \]
\[ = 0 + 0 \]
\[ = 0 \]
84. \[ A + (S_u 0 \cdot A) = S_u 0 + P_u A \]

Proof

\[ L(0) = 0 + (S_u 0 \cdot 0) \]

\[ = S_u 0 \]

\[ R(0) = S_u 0 + P_u 0 \]

\[ = S_u 0 + 0 \]

\[ = S_u 0 \]

\[ L(S_vA) = S_vA + (S_u 0 \cdot S_vA) \]

\[ = \begin{cases} S_uA + 0 & u=v \\ S_vA + (S_u 0 \cdot A) & u\neq v \end{cases} \]

\[ = \begin{cases} S_uA & u=v \\ S_vL(A) & u\neq v \end{cases} \]

\[ R(S_vA) = S_u 0 + P_u S_vA \]

\[ = \begin{cases} S_u 0 + A & u=v \\ S_u 0 + S_v P_u A & u\neq v \end{cases} \]

\[ = \begin{cases} S_u(A) & u=v \\ S_v(S_u 0 + P_u A) & u\neq v \end{cases} \]

\[ = \begin{cases} S_uA & u=v \\ S_vR(A) & u\neq v \end{cases} \]

\[ L(A) = R(A) \]

85. Rule of inference

\[ C_u B . A = 0 \]

\[ (S_u 0 \cdot A) . C = 0 \]

\[ C_{u B} . C = 0 \]
Proof

\[ 0 = (S_u \cdot 0 \cdot A) \cdot C \]

\[ = (S_u \cdot 0 \cdot C_u \cdot A) \cdot C \quad 42. \]

\[ = (S_u \cdot 0 \cdot C_u \cdot A) \cdot C \cdot B \quad 44.73. \]

\[ = S_u \cdot C \cdot B \cdot C_u \cdot A \cdot B \quad 53.46. \]

\[ = C_u \cdot (C \cdot B) \cdot C_u \cdot A \cdot B \quad 60.46.51. \]

\[ = C \cdot C_u \cdot B \cdot C \cdot C_u \cdot B \cdot A \quad 47.48.49. \]

\[ = C_u \cdot B \cdot C \cdot 0 \quad 46. \text{Premises.} \]

\[ = C_u \cdot B \cdot C \cdot 0 \quad 27. \]

\[ = C_u \cdot B \cdot C. \quad 18. \]

86. Rule of inference

\[
\frac{(A \cdot B) \cdot C = 0}{(A \cdot S_u \cdot B) \cdot C = 0}
\]

Proof

\[ (A \cdot S_u \cdot B) \cdot C = (A \cdot (B + S_u \cdot 0)) \cdot C \quad 7. \]

\[ = ((A \cdot B) \cdot S_u \cdot 0) \cdot C \quad 20. \]

\[ = (A \cdot B) \cdot C \cdot S_u \cdot 0 \cdot C \quad 53.46. \]

\[ = 0 \cdot S_u \cdot 0 \cdot C \quad \text{Premises} \]

\[ = 0 \quad 19. \]

87. \((S_u \cdot 0 \cdot A) \cdot (S_u \cdot 0 \cdot A) = S_u \cdot 0 \cdot A \quad \)

Proof

\[ (S_u \cdot 0 \cdot A) \cdot (S_u \cdot 0 \cdot A) = (S_u \cdot 0 \cdot A) \cdot S_u \cdot 0 \cdot (S_u \cdot 0 \cdot A) \cdot A \quad 53.46. \]

\[ = C_u (S_u \cdot 0 \cdot A) \cdot 0 \quad 62.63. \]

\[ = C_u (S_u \cdot 0 \cdot A) \quad 18. \]

\[ = S_u \cdot 0 \cdot A \quad 59. \]
Rt(C_X) = C^{Rt(x)}

Proof

\[ L(0) = Rt(C_0^u) \]
\[ = Rt(0) \]
\[ = 0 \]
\[ R(0) = C^u_0 \]
\[ = C^0_0 \]
\[ = 0 \]
\[ L(S_X) = Rt(C^u S_X) \]
\[ = \begin{cases} Rt(S^u C_X) & u = v \\ Rt(C_X) & u \neq v \end{cases} \]
\[ = \begin{cases} Rt(C_X) + (S^0 u \leq ((S^u Rt(C_X)^2 \leq S^u C_X)) u = v \\ L(X) & u \neq v \end{cases} \]
\[ = \begin{cases} L(X) + (S^0 u \leq C^u ((S^u Rt(C_X)^2 \leq C^u C_X)) u = v \\ L(X) & u \neq v \end{cases} \]
\[ = \begin{cases} L(X) + (S^0 u \leq C^u ((S^u Rt(C_X)^2 \leq C^u C_X)) u = v \\ L(X) & u \neq v \end{cases} \]
\[ = \begin{cases} L(X) + (S^0 u \leq C^u ((S^u Rt(C_X)^2 \leq C^u C_X)) u = v \\ L(X) & u \neq v \end{cases} \]
\[ = \begin{cases} L(X) + (S^0 u \leq C^u ((S^u Rt(C_X)^2 \leq C^u S_X)) u = v \\ L(X) & u \neq v \end{cases} \]
\[ a \{ L(X) + (S^0 u \leq ((S^u L(X))^2 \leq S^u X)) u = v \)
\[ \{ L(X) \}
\[ u \neq v \]
\[ R(S, X) = C_u \text{Rt}(S, X) \]
\[ = C_u(\text{Rt}(X)) + (S, 0 \cdot ((S, \text{Rt}(X))^2 \cdot S, X)) \]
\[ = C_u(\text{Rt}(X)) + C_u(S, 0 \cdot ((S, \text{Rt}(X))^2 \cdot S, X)) \]
\[ = \begin{cases} R(X) + (S, 0 \cdot C_u((S, \text{Rt}(X))^2 \cdot S, X)) & u=v \\ R(X) + 0 & u \neq v \end{cases} \]
\[ = \begin{cases} R(X) + (S, 0 \cdot (C_u(S, \text{Rt}(X))^2 \cdot C_u S, X)) & u=v \\ R(X) & u \neq v \end{cases} \]
\[ = \begin{cases} R(X) + (S, 0 \cdot (C_u C_u(S, \text{Rt}(X))^2 \cdot C_u S, X)) & u=v \\ R(X) & u \neq v \end{cases} \]
\[ = \begin{cases} R(X) + (S, 0 \cdot (C_u C_u(S, \text{Rt}(X))^2 \cdot C_u S, X)) & u=v \\ R(X) & u \neq v \end{cases} \]
\[ = \begin{cases} R(X) + (S, 0 \cdot ((S, C_u \text{Rt}(X))^2 \cdot S, X)) & u=v \\ R(X) & u \neq v \end{cases} \]
\[ = \begin{cases} R(X) + (S, 0 \cdot ((S, C_u \text{Rt}(X))^2 \cdot S, X)) & u=v \\ R(X) & u \neq v \end{cases} \]
\[ L(X) = R(X) \]

89. \[ C_u \text{Rt}(S, X) = C_u \text{Rt}(X) \]
\[ u \neq v \]

Proof \[ C_u \text{Rt}(S, X) = \text{Rt}(C_u S, X) \]
\[ = \text{Rt}(C_u X) \]
\[ u \neq v \]
\[ = C_u \text{Rt}(X) \]

88.

90. \[ (S, 0 \cdot |A, B|) = (S, 0 \cdot (A \cdot B)) \cdot (B \cdot A) \]

Proof \[ (S, 0 \cdot |A, B|) = (S, 0 \cdot ((A \cdot B) + (B \cdot A))) \]
\[ = (S, 0 \cdot (A \cdot B)) \cdot (B \cdot A) \]
54.
91. Rule of inference

\[ A \cdot C \cdot D = 0 \]
\[ (B \cdot \neg A) \cdot C = 0 \]
\[ B \cdot C \cdot D = 0 \]

Proof

\[ 0 = (B \cdot \neg A) \cdot C \]
\[ = (B \cdot \neg A) \cdot C \cdot D \]
\[ = B \cdot C \cdot D \cdot \neg A \cdot C \cdot D \]
\[ = B \cdot C \cdot D \cdot 0 \]
\[ = B \cdot C \cdot D. \]

Premises

70.44.
53.46.
18.

92. Rule of inference

\[ C_u \cdot A = 0 \]
\[ u = 1, \ldots, n \]
\[ A = 0 \]

Proof

\[ A = C_1 \cdot A + C_2 \cdot A + \ldots + C_n \cdot A \]
\[ = 0 + 0 + \ldots + 0 \]
\[ = 0 \]

Premises

33.
7.

93. Rule of inference

\[ A \cdot B = 0 \]
\[ A \cdot (S_u \cdot \neg B) = 0 \quad \text{all } u \quad \text{OR if one value } u \text{ then } C_u \cdot A = 0 \]
\[ A = 0 \]

Proof

\[ 0 = A \cdot (S_u \cdot \neg B) \]
\[ = A \cdot S_u \cdot \neg A \cdot B \]
\[ = C_u \cdot A \cdot 0 \]
\[ = C_u \cdot A \quad \text{for that value of } u. \]
\[ A = 0 \quad \text{if for all } u. \]

Premises

53.
18.

92.
94. \( C_u (B \circ A) (S_u A \circ B) = 0 \)

Proof

\[ L(A,0) = C_u (0 \circ A) (S_u A \circ 0) \]
\[ = C_u \cdot S_u A \]
\[ = 0 \cdot S_u A \]
\[ = 0 \]

\[ L(0,B) = C_u (B \circ 0) (S_u 0 \circ B) \]
\[ = C_u B \cdot (S_u 0 \circ B) \]
\[ = C_u 0 \]
\[ = 0 \]

\[ L(C_u S_u A, C_u S_u B) = C_u (C_u S_u B \circ C_u S_u A) (S_u C_u S_u A \circ C_u S_u B) \]
\[ = C_u (S_u C_u B \circ S_u C_u A) (S_u S_u C_u A \circ S_u S_u B) \]
\[ = C_u (C_u B \circ C_u A) (S_u C_u A \circ C_u B) \]
\[ = L(C_u A, C_u B) \]
\[ L(C_u A, C_u B) = 0 \]

\[ C_u (B \circ A) (S_u A \circ B) = C_u C_u (B \circ A) (S_u A \circ B) \]
\[ = C_u (C_u B \circ C_u A) (S_u C_u A \circ C_u B) \]
\[ = C_u (C_u B \circ C_u A) (S_u C_u A \circ C_u B) \]
\[ = L(C_u A, C_u B) \]
\[ = 0 \]

We now have sufficient results to prove the two conditions

\[ S_u X \circ (S_u Rt(X))^2 = 0 \]
\[ (Rt(X))^2 \circ X = 0 \]
which will lead to the result
\[ \text{Rt}(X^2) = X. \]

From here on \( \text{Rt}(X) \) will be referred to as \( RX \).

95. \( S_u X \leq (S_u RX)^2 = 0 \)

Proof In this proof we shall dispense with the notation for small steps and only use the notation for major formulae. A labelling method with the number in parenthesis will be used for statements for use later in the proof.

Definition of \( RX \)

\[ RO = 0 \]

\[ RS_u X = RX + (S_u 0 \leq ((S_u RX)^2 \leq S_u X)) \quad u=1,\ldots,n \]

the last term being equal to zero from 69. with \( A = S_u X, \quad B = (S_u RX)^2 \).

\( S_u 0 \leq (S_u S_u X \leq (S_u RX)^2)) \quad RS_u X = (S_u 0 \leq (S_u S_u X \leq (S_u RX)^2)) \quad RX \]

By 70. we have

\[ (S_u 0 \leq (S_u S_u X \leq (S_u RX)^2)).|RS_u X| = 0 \quad u=1,\ldots,n \quad (1) \]

Now by 72. we have

\[ (S_u 0 \leq |RX,RS_u X|). (S_u 0 \leq (S_u S_u X \leq (S_u RC_u X)^2)) (S_u S_u X \leq (S_u RS_u C_u X)^2) = 0 \]

and obviously

\[ (S_u 0 \leq |RX,RS_u X|). (S_u 0 \leq (S_u S_u X \leq (S_u RX)^2)) (S_u S_u X \leq (S_u RS_u X)^2) = 0 \quad (2) \]
Multiplying (1) by \((S_u S_u X - (S_u R S_u X)^2)\) and using 93. we find
\[
(S_u 0 \cdot (S_u S_u X - (S_u R S_u X)^2)) \cdot (S_u S_u X - (S_u R S_u X)^2) = 0
\] (3)

Multiply the defining equation for RS_u X by \((S_u 0 \cdot |(S_u R S_u X)^2, S_u X|)\) and using 71. we find
\[
(S_u 0 \cdot |(S_u R S_u X)^2, S_u X|). RS_u X = (S_u 0 \cdot |(S_u R S_u X)^2, S_u X|). R X + (S_u 0 \cdot |(S_u R S_u X)^2, S_u X|)
\]

Hence
\[
(S_u 0 \cdot |(S_u R S_u X)^2, S_u X|). RS_u X = (S_u 0 \cdot |(S_u R S_u X)^2, S_u X|). S_u RX
\]

and so
\[
(S_u 0 \cdot |(S_u R S_u X)^2, S_u X|). RS_u X, S_u RX = 0
\] (4)

by 72.
\[
(S_u 0 \cdot |RS_u X, S_u RX|). (S_u 0 \cdot |(S_u R S_u X)^2, S_u X|). |(R S_u X)^2, S_u X| = 0
\] (5)

and so, multiplying (4) by \(|(R S_u X)^2, S_u X|\), adding (5), taking the common factor out, then using 93. we have
\[
(S_u 0 \cdot |(S_u R S_u X)^2, S_u X|). (R S_u X)^2, S_u X| = 0
\] (6)

It follows therefore by 55. that
\[
(S_u 0 \cdot |(S_u R S_u X)^2, S_u X|). (S_u X - (R S_u X)^2) = 0
\]

Hence
\[
(S_u 0 \cdot |(S_u R S_u X)^2, S_u X|). (S_u S_u X - S_u (R S_u X)^2) = 0
\]
\[
(S_u 0 \cdot |(S_u R S_u X)^2, S_u X|). (S_u S_u X - S_u (R S_u X)^2) = (S_u 0 \cdot |(S_u R S_u X)^2, S_u X|). (C_u R S_u X + C_u R S_u X) = 0
\]
\[
(S_u 0 \cdot |(S_u R S_u X)^2, S_u X|). (S_u S_u X - (S_u (R S_u X)^2 + C_u R S_u X + C_u R S_u X)) = 0
\]
\[
(S_u 0 \cdot |(S_u R S_u X)^2, S_u X|). (S_u S_u X - (S_u R S_u X)^2) = 0
\] (7)
From (3) and (7) we have by 82.

\[(S_u \circ (S_u \circ (S_u \circ \text{RX})^2)) \cdot (S_u \circ S_u \circ (S_u \circ \text{RX})^2) = 0 \quad (8)\]

Now

\[(S_{u \circ v} \circ (S_{u \circ v} \circ (S_{u \circ v} \circ \text{RX})^2)) \cdot (S_{u \circ v} \circ S_{u \circ v} \circ (S_{u \circ v} \circ \text{RX})^2)\]

\[= (S_{u \circ v} \circ (S_{u \circ v} \circ (S_{u \circ v} \circ \text{RX})^2)) \cdot C_u (S_{u \circ v} \circ S_{u \circ v} \circ (S_{u \circ v} \circ \text{RX})^2)\]

\[= (S_{u \circ v} \circ (S_{u \circ v} \circ (S_{u \circ v} \circ \text{RX})^2)) \cdot C_u (S_{u \circ v} \circ S_{u \circ v} \circ (S_{u \circ v} \circ \text{RX})^2)\]

\[= (S_{u \circ v} \circ (S_{u \circ v} \circ (S_{u \circ v} \circ \text{RX})^2)) \cdot C_u (S_{u \circ v} \circ S_{u \circ v} \circ (S_{u \circ v} \circ \text{RX})^2)\]

\[= (S_{u \circ v} \circ (S_{u \circ v} \circ (S_{u \circ v} \circ \text{RX})^2)) \cdot C_u (S_{u \circ v} \circ S_{u \circ v} \circ (S_{u \circ v} \circ \text{RX})^2)\]

\[= 0\]

We can therefore say

\[(S_{u \circ v} \circ (S_{u \circ v} \circ (S_{u \circ v} \circ \text{RX})^2)) \cdot (S_{u \circ v} \circ S_{u \circ v} \circ (S_{u \circ v} \circ \text{RX})^2) = 0 \quad \text{for all } u, v,\]

and \(S_{u \circ v} \circ (S_{u \circ v} \circ \text{RO})^2 = 0\).

Hence by rule of inference 63.

\[(S_u \circ (S_u \circ \text{RX})^2) \cdot (S_u \circ S_u \circ (S_u \circ \text{RX})^2) = 0 \quad \text{as required.}\]

96. \((\text{RX})^2 \cdot X = 0\)

Proof From (4) in 95. we have

\[(S_u \circ |(S_u \circ (S_u \circ \text{RX})^2) \cdot S_u \circ (S_u \circ (S_u \circ \text{RX})^2)| = 0 \quad (1)\]

and from the defining equation for \(R_u X\)

\[((S_u \circ (S_u \circ \text{RX})^2) \cdot R_u X = ((S_u \circ (S_u \circ \text{RX})^2) \cdot S_u \circ (S_u \circ \text{RX})\cdot X\]

\[\cdot R_u X \cdot R_u X | = 0\]

by 72.

\[(S_u \circ |R_u X \cdot R_u X| \cdot (S_u \circ ((\text{RX})^2 \cdot X)) \cdot ((R_u X \cdot (S_u \circ \text{RX})^2) \cdot X) = 0\]
by virtue of the fact that
\[ F(C^X) = C^F(X) \text{ for } F(X) = (RX)^2 \]

Hence by 93.

\[(S_uRX)^2 \cdot S_uX) \cdot (S_0 \cdot ((RX)^2 \cdot X)) \cdot ((RS_uX)^2 \cdot X) = 0\]

and therefore by 86.

\[(S_uRX)^2 \cdot S_uX) \cdot (S_0 \cdot ((RX)^2 \cdot X)) \cdot ((RS_uX)^2 \cdot S_uX) = 0 \quad (2)\]

From (1) and (2), by 91., splitting (1), using 90. and 55., in 91. substitute

\[ A = ((S_uRX)^2 \cdot S_uX) \]
\[ B = S_u0 \cdot (S_uX \cdot (S_uRX)^2) \]
\[ C = (RS_uX)^2 \cdot S_uX \]
\[ D = S_u0 \cdot ((RX)^2 \cdot X) \]

\[(S_0 \cdot (S_uX \cdot (S_uRX)^2)) \cdot (S_0 \cdot ((RX)^2 \cdot X)) \cdot ((RS_uX)^2 \cdot S_uX) = 0 \]

by 95.

\[(S_0 \cdot ((RX)^2 \cdot X)) \cdot ((RS_uX)^2 \cdot S_uX) = 0 \quad (3)\]

Now

\[(S_0 \cdot ((RX)^2 \cdot X)) \cdot ((RS_vX)^2 \cdot S_vX) = (S_0 \cdot ((RX)^2 \cdot X)) \cdot C_u(\text{RS}_vX)^2 \cdot S_vX)\]

\[= (S_0 \cdot ((RX)^2 \cdot X)) \cdot C_u(\text{RS}_vX)^2 \cdot C_uS_vX)\]
\[= (S_0 \cdot ((RX)^2 \cdot X)) \cdot (\text{RS}_vX)^2 \cdot C_uX\]
\[= (S_0 \cdot ((RX)^2 \cdot X)) \cdot ((RX)^2 \cdot X)\]
\[= 0\]
We can therefore say,
\[(S_u^0 \cdot ((RX)^2 \cdot X)).((RS_v^X)^2 \cdot S_v^X = 0 \quad \text{for all } u,v,
\]
and
\[(RO)^2 \cdot 0 = 0 .\]

Hence by rule of inference 63.
\[(RX)^2 \cdot X = 0\]

97. Rule of inference

\[
\begin{align*}
A &= B = 0 \\
B &= A = 0 \\
\hline
A &= B
\end{align*}
\]

Proof
\[
A + (B \cdot A) = B + (A \cdot B) \quad \text{Key equation}
\]
\[
A + 0 = B + 0 \quad \text{Premises}
\]
\[
A = B \quad \text{7.}
\]

98. Rule of inference

\[
\begin{align*}
A^2 &= B^2 = 0 \\
\hline
A &= B = 0
\end{align*}
\]

Proof
Consider first the result

(a) \[(S_u^0 \cdot (X \cdot Y).X). (X \cdot Y) = 0\]

Now obviously

\[(S_u^0 \cdot (X \cdot Y).X). (X \cdot Y).X = 0\]

44.

and further,
\[(S^0_u (X^2 Y) X) (X^2 Y) (S^0_v X) = (S^0_u (X^2 Y) X) ((S^0_v X) X^2 (S^0_v X) Y) \]

\[= (S^0_u (X^2 Y) X) (0 (S^0_v X) Y) \]

\[= (S^0_u (X^2 Y) X) 0 \]

\[= 0 \]

Hence by rule of inference 93.

\[(S^0 u (X^2 Y) X) (X^2 Y) = 0 \quad \text{for (a)} \]

Put

\[X = A + B \]

\[Y = B + B \]

\[(S^0 u ((A + B) (B + B)) (A + B) ((A + B) (B + B)) = 0 \]

\[(S^0 u (A + B) (A + B) (A + B) = 0 \quad 24. \]

\[(S^0 u (A + B) (A + B) (A + B) = 0 \quad 46. \]

\[(S^0 u ((A^2 + B A) (A B + B^2) (A + B) = 0 \quad 53. \]

\[(S^0 u (A^2 + B^2) (A + B) = 0 \quad 46.24. \]

Hence

\[(S^0 u 0) (A + B) = 0 \quad \text{Premises} \]

\[S^0 u (A + B) = 0 \quad 18. \]

\[C^u (A + B) = 0 \quad \text{for } u=1, \ldots, n \quad 60. \]

\[A + B = 0 \quad \text{as required} \quad 92. \]

99. Rule of inference

\[S^u A^2 + B^2 = 0 \]

\[C^u (S^u A + B) = 0 \]

Proof

\[C^u (B - A) (S^u A + B) = 0 \quad \text{from 94}. \]
Hence,

\[ C_u(B + A) \cdot (B \mathbin{\hat{+}} A) \cdot (S_u A \mathbin{\hat{-}} B) = 0 \]  
\[ C_u(B^2 \mathbin{\hat{+}} A^2) \cdot (S_u A \mathbin{\hat{-}} B) = 0 \text{ for all } u. \]

Now

\[ (S_u 0 \mathbin{\hat{+}} (B^2 \mathbin{\hat{+}} A^2)) \cdot (S_u 0 \mathbin{\hat{+}} (S_u A^2 \mathbin{\hat{-}} B^2)) = 0 \text{ for all } u, \]

and therefore

\[ (S_u 0 \mathbin{\hat{+}} (S_u A^2 \mathbin{\hat{-}} B^2)) \cdot C_u(S_u A \mathbin{\hat{-}} B) = 0 \]

Premises

\[ S_u 0 \cdot (S_u A \mathbin{\hat{-}} B) = 0 \]

\[ C_u(S_u A \mathbin{\hat{-}} B) = 0 \text{ for } u=1,\ldots,n \]

100. \[ RA^2 = A \]

From 95.96., we have

\[ S_u X \mathbin{\hat{+}} (S_u RX)^2 = 0 \]

\[ (RX)^2 \mathbin{\hat{+}} X = 0 \]

Put \[ X = A^2 \]

\[ S_u A^2 \mathbin{\hat{+}} (S_u RA^2)^2 = 0 \]

\[ (RA^2)^2 \mathbin{\hat{+}} A^2 = 0 \]

\[ C_u(S_u A \mathbin{\hat{+}} S_u RA^2) = 0 \]

\[ RA^2 \mathbin{\hat{+}} A = 0 \]

\[ A \mathbin{\hat{+}} RA^2 = 0 \]

Hence \[ RA^2 = A \]
Having proved the basic property of the square root function, we now have to prove the properties

\[ X > Y \Rightarrow RX > RY \]

\[ R(X^2 + X + X) = R(X^2) \]

and

\[ R((X + Y)^2 + X) = X + Y \]

101. \[ Rt(X) - Rt(X + Y) = 0 \] (equivalent to \( X > Y \Rightarrow RX > RY \))

**Proof**

\[ L(X,0) = Rt(X) - Rt(X + 0) \]

\[ = Rt(X) - Rt(X) \]

\[ = 0 \]

\[ R(X,0) = 0 \]

\[ L(X,S_uY) = Rt(X) - Rt(X + S_uY) \]

\[ = Rt(X) - Rt(S_u(X + Y)) \]

\[ = Rt(X) - (Rt(X + Y) + (S_u0 - (S_uR(X + Y)^2 - S_u(X + Y)))) \]

\[ = (Rt(X) - (Rt(X + Y)) - (S_u0 - ((S_uRt(X + Y)^2 - S_u(X + Y)))) \]

\[ = L(X,Y) - (S_u0 - ((S_uRt(X + Y)^2 - S_u(X + Y)))) \]

\[ R(X,S_uY) = 0 \]

\[ = 0 - (S_u0 - ((S_uRt(X + Y)^2 - S_u(X + Y)))) \]

\[ = R(X,Y) - (S_u0 - ((S_uRt(X + Y)^2 - S_u(X + Y)))) \]

\[ L(X,Y) = R(X,Y) \]

102. \[ Rt(A^2 + A + A) = A \]

**Proof**

\[ S_uX - (S_uRt(X))^2 = 0 \]
Put $X = C_u(A^2 + A + A)$

$S_u C_u (A^2 + A + A) = (S_u R_u (C_u (A^2 + A + A)))^2 = 0$

$(S_u C_u A)^2 = (S_u C_u R_u (A^2 + A + A))^2 = 0$  

$S_u C_u A = S_u C_u R_u (A^2 + A + A) = 0$  

$C_u A = C_u R_u (A^2 + A + A) = 0$  (1)  

$(R_u (X))^2 = X = 0$

$S_u (R_u (X))^2 = S_u X = 0$  

$S_u (R_u (C_u (A^2 + A + A)))^2 = S_u C_u (A^2 + A + A) = 0$

$S_u (C_u R_u (A^2 + A + A))^2 = (S_u C_u A)^2 = 0$  

$C_u (S_u C_u R_u (A^2 + A + A) = S_u C_u A) = 0$  

$C_u R_u (A^2 + A + A) = C_u A = 0$  (2)  

Therefore from 97 we have

$C_u R_u (A^2 + A + A) = C_u A$

Hence we can say $R_u (A^2 + A + A) = A$  

103. Rule of inference

$A \rightarrow B = 0$

$B \rightarrow C = 0$

$A \rightarrow C = 0$

Proof $B + (A \rightarrow B) = A + (B \rightarrow A)$

$B = A + (B \rightarrow A)$

$B \rightarrow (B \rightarrow A) = (A + (B \rightarrow A)) \rightarrow (B \rightarrow A)$

$= A$  

$C + (B \rightarrow C) = B + (C \rightarrow B)$

$C = B + (C \rightarrow B)$

Key equation

Premises, 7.

Premises, 7.
Hence by substitution

\[ A - C = (B - (B - A)) - (B + (C - B)) \quad \text{from above} \]

\[ = B - ((B - A) + B + (C - B)) \quad 20. \]

\[ = 0 - ((B - A) + (C - B)) \quad 23. \]

\[ = 0 \quad \text{as required.} \quad 19. \]

104. \text{Rt}(U + V)^2 + U) = U + V

Proof

\[ \text{Rt}((U + V)^2 + U) - \text{Rt}(U + V)^2 + U + V + U + V) = 0 \quad 101. \]

\[ \text{Rt}((U + V)^2 + U + V + U + V) - (U + V) = 0 \quad 102. \]

\[ \text{Rt}((U + V)^2 + U) - (U + V) = 0 \quad (1) \quad 103. \]

\[ \text{Rt}((U + V)^2) - \text{Rt}((U + V)^2 + U) = 0 \quad 101. \]

\[ (U + V) - \text{Rt}((U + V)^2 + U) = 0 \quad (2) \quad 100. \]

The results (1) and (2) together using 97. give

\[ \text{Rt}((U + V)^2 + U) = U + V \]

105. \text{L}(J(U,V)) = U

Proof

\[ \text{L}(J(U,V)) = ((U + V)^2 + U) - (\text{Rt}((U + V)^2 + U))^2 \]

\[ = ((U + V)^2 + U) - (U + V)^2 \quad 104. \]

\[ = U \quad 23. \]

106. \text{K}(J(U,V)) = V

Proof

\[ \text{K}(J(U,V)) = \text{Rt}((U + V)^2 + U) - \text{L}(J(U,V)) \]

\[ = (U + V) - U \quad 105. \]

\[ = V \quad 23. \]
We have therefore proved that a set of functions with the required properties to achieve the reduction in parameters in the definition of primitive recursion exists in commutative multiple successor arithmetic.
CHAPTER IV

INITIAL FUNCTIONS REQUIRED FOR THE REDUCED DEFINITIONS
BY PRIMITIVE RECURSIONS

In this chapter we will consider the four defining schemas which were derived in Chapter II, and examine these to see if we need to adjoin further functions to the initial set of all primitive recursive functions.

The functions used to reduce the parameters in the definitions of primitive recursive functions were \( K(x) \), \( L(x) \), & \( J(U,V) \), defined as follows:

\[
J(U,V) = (U + V)^2 + U \\
L(X) = X \odot (RX)^2 \\
K(X) = RX \odot L(X)
\]

In order therefore to define all primitive recursive functions in the reduced defining schemas, we may need to adjoin further functions to the set of initial functions in order that the functions \( K(x) \), \( L(x) \) & \( J(U,V) \) can be defined.

Now \( K(x) \), \( L(x) \) & \( J(U,V) \) are explicitly comprised of the functions \( Y + X \), \( Y \odot X \), \( XX \), \( RX \).

If we are able to define these functions in each of the schemas \( R_1, R_1^*, R_1^{**} \) and \( R_1^{***} \), then there would be no need to adjoin further functions to the initial functions. However we shall find that in the case of \( R_1^{***} \) we require to adjoin a function to the initial functions, \( Q(X) \), which will be defined later.

\( R_1 \) This is the defining schema given by

\[
F(0,Y) = A(Y) \\
F(S_uX,Y) = R_u(X,Y,F(X,Y)) \quad u=1,\ldots,n
\]
\( Y + X \) is defined by
\[
F(0,Y) = Y \\
F(S_uX,Y) = S_uF(X,Y)
\]

For \( Y - X \) we first define \( P_uX \) by
\[
F(0) = 0 \\
F(S_uX) = \begin{cases} 
X & \text{if } u=v \\
S_vF(X) & \text{if } u \neq v
\end{cases}
\]

Then \( Y \cdot X \) is defined by
\[
F(0,Y) = Y \\
F(S_uX,Y) = P_uF(X,Y)
\]

\( C_uX \) is defined by
\[
F(0) = 0 \\
F(S_vX) = \begin{cases} 
S_vF(X) & \text{if } u=v \\
F(X) & \text{if } u \neq v
\end{cases}
\]

and therefore \( X \cdot X \) is defined by
\[
F(0) = 0 \\
F(S_uX) = S_uF(X) + C_uX + C_uX
\]

\( RX \) is defined by
\[
F(0) = 0 \\
F(S_uX) = F(X) + (S_u0 \cdot ((S_uF(X))^2 \cdot S_uX))
\]

Thus the initial functions and the defining schema \( R_1 \) alone are sufficient to define all primitive recursive functions.
This is the defining schema given by

\[ F(0, Y) = A(Y) \]
\[ F(S_u X, Y) = B_u(X, F(X, Y)) \quad u = 1, \ldots, n \]

Defining the functions \( Y + X \), \( Y \cdot X \), \( X \cdot X \) and \( R X \) as above in \( R_1 \) in fact defines all these functions under the reduced schema \( R_1^* \). Hence we can say that 'The initial functions and the defining schema \( R_1^* \) alone are sufficient to define all primitive recursive functions'.

This is the defining schema given by

\[ F(0, Y) = A(Y) \]
\[ F(S_u X, Y) = B_u(Y, F(X, Y)) \quad u = 1, \ldots, n \]

\( Y + X \) is defined by

\[ F(0, Y) = Y \]
\[ F(S_u X, Y) = S_u F(X, Y) \]

Now for \( Y \cdot X \) we have to define \( P_u X \), and to achieve this we proceed as follows.

\( C_u X \) is defined by

\[ F(0) = 0 \]
\[ F(S_v X) = \begin{cases} S_v F(X) & \text{if } u = v \\ F(X) & \text{if } u \neq v \end{cases} \]

\( X.Y \) is defined by

\[ F(0, Y) = 0 \]
\[ F(S_u X, Y) = F(X, Y) + C_u Y \]

and \( X \cdot X = F(X, X) \).
$S_0 \cdot X$ is defined by

$$F(0) = S_0$$

$$F(S_X) = \begin{cases} 0 & u=v \\ F(X) & u \neq v \end{cases}$$

Define $D_k^X$ by

$$F(0) = 0$$

$$F(S_X) = \begin{cases} S_0 \cdot F(X) & u=k \\ F(X) & u \neq k \end{cases}$$

This takes the value $S_0$ if $'S_k'$ appears an odd number of times in $X$.

We can therefore define $T_k^X$ by

$$F(0) = 0$$

$$F(S_X) = \begin{cases} S_0 \cdot F(X) \cdot (S_0 \cdot F(X)) + F(X) \cdot S_0 \cdot F(X) \cdot (S_0 \cdot (S_0 \cdot F(X))) & u=k \\ F(X) & u \neq k \end{cases}$$

This function is such that when $X$ contains only one of $'S_k'$ successors then $T_k^X$ contains an odd number of $'S_k'$ successors, otherwise $T_k^X$ contains an even number of $'S_k'$ successors. Hence the function defined by

$$S_0 \cdot D_k T_k^X$$

takes the value 0 if $X$ contains only one $'S_k'$ successors, and $S_0$ otherwise.

Define $R_{k}(X, 2)$ by

$$R_{k}(X, 2) = D_k X$$

and $R_{k}(X, 3)$ by

$$F(0) = 0$$
This function is the remainder of the number of 'S_k' successors in X when that number is divided by 3, e.g.

\[ RM_k(S_k, 0, 3) = S_k S_0 \]
\[ RM_k(S_k S_0, 3) = S_0 \]

Consider now the function \( N^X_k \) defined by

\[
F(0) = 0 \\
F(S_u X) = \begin{cases} 
(F(X) + S_k 0) \cdot (S_k 0 - RM_k(F(X), 3)) + \\
(F(X) + S_k S_0 \cdot (S_k 0 - (S_0 - RM_k(F(X), 3)))) & u=k \\
(F(X) + S_k S_0 \cdot (S_k 0 - RM_k(F(X), 3))) & u\neq k
\end{cases}
\]

In other words

\[
F(0) = 0 \\
F(S_u X) = \begin{cases} 
F(X) + S_k 0 \text{ if } RM_k(F(X), 3) = 0 & u=k \\
F(X) + S_k S_0 \text{ if } RM_k(F(X), 3) \neq 0 & u\neq k
\end{cases}
\]

Clearly \( N^X_k = N_k(C^X) \); that is we are only considering the 'S_k' successors in X to define \( N^X_k \), if we consider that 1 = \( S_k 0 \), 2 = \( S_k S_0 \), ..., etc. and consider X as \( C^X \) which would then be a number: then in normal arithmetical terms

\[
N^X_k = \begin{cases} 
3X/2 & \text{if } X \text{ is even} \\
(3X - 1)/2 & \text{if } X \text{ is odd}
\end{cases}
\]

as there are no successor symbols other than 'S_k' in \( N^X_k \).
This result we shall prove by induction on $X$; suppose the above is true for $X$, then

$$N_k(X + 1) = \begin{cases} 
3 \cdot (X + 1)/2 & \text{if } X + 1 \text{ is even} \\
(3 \cdot (X + 1) - 1)/2 & \text{if } X + 1 \text{ is odd}
\end{cases}$$

$$N_k(X + 1) = \begin{cases} 
(3 \cdot X + 3)/2 & \text{if } X \text{ is odd} \\
(3 \cdot X + 2)/2 & \text{if } X \text{ is even}
\end{cases}$$

$$= \begin{cases} 
(3 \cdot X - 1)/2 + 2 & \text{if } X \text{ is odd} \\
3 \cdot X/2 + 1 & \text{if } X \text{ is even}
\end{cases}$$

$$= \begin{cases} 
3 \cdot X/2 + 1 & \text{if } X \text{ is even} \\
(3 \cdot X - 1)/2 + 2 & \text{if } X \text{ is odd}
\end{cases}$$

This is equivalent to

$$N_k(X + 1) = \begin{cases} 
N_kX + 1 & \text{if } X \text{ is even} \\
N_kX + 2 & \text{if } X \text{ is odd}
\end{cases}$$

by hypothesis.

Now if $X$ is even $N_kX = 3 \cdot X/2$ and therefore divisible by 3, if $X$ is odd then $N_kX = (3 \cdot X - 1)/2$ and therefore not divisible by 3.

Hence

$$N_k(X + 1) = \begin{cases} 
N_kX + 1 & \text{if } RM(N_kX,3) = 0 \\
N_kX + 2 & \text{otherwise}
\end{cases}$$

The hypothesis is true for $X = 0$; hence $N_kX$ is the function stated.

Define now $M_k(X,Y)$ by

$$F(0,Y) = Y$$

$$F(S_uX,Y) = \begin{cases} 
N_k(F(X,Y),u), (S_0 \cdot RM_k(F(X,Y),2)) & u \neq k \\
F(X,Y) & u = k
\end{cases}$$
In order to prove the properties of this function we need first to define the function \((Y^X)_{k}\) by

\[
F(0,Y) = S_k^0
\]

\[
F(S_uX,Y) = \begin{cases} \{Y.F(X,Y) & \text{if } u=k \\ F(X,Y) & \text{if } u\neq k \end{cases}
\]

In the notation defined \((Y^X)_{k}\) is the function \(Y^X\) in normal arithmetical notation for the 'S\(_k\)' component of both \(X\) and \(Y\), e.g.

\[
(S_1S_kS_0^02^kS_k^0)_{k} = S_kS_kS_kS_k^0
\]

which with the previous notation \(S_k^0 = 1\), \(S_kS_k = 2\), etc. ... ignoring all other successors, then

\[
(S_1S_kS_0^02^kS_k^0)_{k} = 2^a \quad \text{(normal exponent)}
\]

\[
= 4
\]

\[
= S_kS_kS_kS_k^0
\]

We analyse the function \(M_k(X,Y)\) as follows.

Let \(C_kY\) be non-zero, and let \(2^t\) be the highest power of 2 that divides it exactly; then \(M_k(X,Y)\) is such that

- if \(t < X\) then \(M_k(X,Y) = 0\)
- if \(t \geq X\) then \(M_k(X,Y)\) is non-zero, and \(2^{t-X}\) is the highest power of 2 that divides it exactly.

**Proof** The proof is by induction on \(X\). The case \(X = 0\) gives

\[
M_k(X,Y) = M_k(0,Y)
\]

\[
\times Y
\]

\[
= 2^tA \quad (A \neq 0)
\]

\[
= 2^{t-0}A \quad \text{(as } t \geq X)\]

The result is therefore true for \(X = 0\). Let us assume the result for \(0 \leq X \leq m\), and consider the case \(m+1\).
Suppose $t < m$. Then by the hypothesis $M_k(m,Y) = 0$, and from the definition we have

$$M_k(m+1,Y) = N_k(M_k(m,Y)).(S_k \cdot R_k(M_k(m,Y),2)) \quad \text{as } 1 = S_k$$

$$= N_k(M_k(m,Y)).S_k$$

$$= N_k(0).S_k$$

$$= 0$$

It is worth noting here that as we are only using the $k$'th components then $A.S_k = A$, and $C_kA = A$, as we shall only consider the answer in terms of $S_k$ successors.

Now suppose $t = m$. Then by the hypothesis $M_k(m,Y)$ is non-zero, and $2^{t-m} = 2^0$ is the highest power of 2 that divides it exactly, that is $M_k(m,Y)$ is odd. Hence

$$M_k(m+1,Y) = N_k(M_k(m,Y)).(S_k \cdot R_k(M_k(m,Y),2))$$

As $M_k(m,Y)$ is odd then $R_k(M_k(m,Y),2) = S_k$ and so

$$M_k(m+1,Y) = N_k(M_k(m,Y)).(S_k \cdot S_k)$$

$$= N_k(M_k(m,Y)).S_k$$

$$= 0$$

as required as $t < m + 1$.

Finally, suppose $t > m$. Then by the hypothesis $M_k(m,Y)$ is non-zero and $2^{t-m}$ is the highest power of 2 that divides it exactly. Hence $M_k(m,Y)$ is even. Therefore

$$M_k(m+1,Y) = N_k(M_k(m,Y)).(S_k \cdot R_k(M_k(m,Y),2))$$

$$= N_k(M_k(m,Y)).(S_k \cdot 0)$$

$$= N_k(M_k(m,Y)).S_k$$

$$= N_k(M_k(m,Y))$$

$$= 3.M_k(m,Y)/2 \quad \text{as } M_k(m,Y) \text{ is even.}$$
We therefore have

\[ M_k(m+1,Y) = \frac{3}{2} M_k(m,Y) \]

Hence if \( 2^{t-m} \) is the highest power of 2 that divides \( M_k(m,Y) \) exactly, then \( 2^{(t-m)-1} \) is the highest power of 2 that divides \( M_k(m+1,Y) \) exactly, that is \( 2^{t-(m+1)} \) is the highest power of 2 that divides \( M_k(m+1,Y) \) exactly.

Hence the result.

From this result we can say that the function

\[ M_k(X,(S_k^Y)_{S_k^Y})_k \]

is non-zero in '\( S_k^Y \)' components, if and only if \( 2^X \) divides \( 2^Y \) exactly (in the previous notation), that is if and only if \( X \leq Y \), in '\( S_k^Y \)' components.

Hence

\[ S_k^0 \leq M_k(X,(S_k^Y)_{S_k^Y})_k = \begin{cases} 0 & \text{if } C_k(X \leq Y) = 0 \\ S_k^0 & \text{otherwise} \end{cases} \]

Define now the function \( I_k(X,Y) \) by

\[ F(0,Y) = Y \]

\[ F(S_uX,Y) = \begin{cases} (F(X,Y) + S_k^0).\{S_k^0 \leq M_k(Y,(S_k^Y)_{S_k^Y})_k\}_u^k \\ S_u^0 F(X,Y) & \text{if } u < k \end{cases} \]

which is such that

\[ F(S_k^X,Y) = \begin{cases} 0 & \text{if } F(X,Y) \geq Y \text{ (in '} S_k^Y \text{' components)} \\ S_k^0 F(X,Y) & \text{otherwise} \end{cases} \]

The function \( I_k(X,X) \) is therefore defined by

\[ F(0) = 0 \]

\[ F(S_uX) = \begin{cases} 0 & \text{if } F(X) \geq X \text{ (in '} S_k^X \text{' components)} \\ S_u^0 F(X) & \text{if } u = k \\ S_u^0 F(X) & \text{otherwise} \end{cases} \]
\( P^X_k \) is defined by

\[
F(0) = 0
\]

\[
F(S_uX) = \begin{cases} 
X & u = k \\
S_uF(X) & u \neq k
\end{cases}
\]

which is the same as

\[
F(0) = 0
\]

\[
F(S_uX) = \begin{cases} 
0 & \text{if } F(X) \geq X \text{ (in } S_k \text{ components)} \\
S_kF(X) & \text{otherwise} \\
S_uF(X) & u \neq k
\end{cases}
\]

Hence by uniqueness \( P^X_k = L^k(X,X) \), and \( L^k(X,Y) \) is defined in \( R_{1^{**}} \) by substituting \( X \) for \( Y \) in the function \( L^k(X,Y) \) which is defined under the schema \( R_{1^{**}} \). For the proof of this in single successor arithmetic see [8].

However we can obtain the result in the following manner.

\( DX \) or \( RM(X,2) \) is defined by

\[
F(0) = 0
\]

\[
F(S_uX) = S_u0 \circ F(X)
\]

\( TX \) is defined by

\[
F(0) = 0
\]

\[
F(S_uX) = S_uF(X) \cdot (S_u0 \circ F(X)) + F(X) \cdot S_uF(X) \cdot (S_u0 \circ (S_u0 \circ F(X)))
\]

\( RM(X,3) \) is defined by

\[
F(0) = 0
\]

\[
F(S_uX) = (S_u0 \circ F(X)) + S_uS_u0 \cdot (S_u0 \circ D(T(F(X))))
\]

These functions are such that

\[
DX = D_1X + D_2X + \ldots + D_nX
\]

\[
RM(X,2) = RM_1(X,2) + RM_2(X,2) + \ldots + RM_n(X,2)
\]

\[
TX = T_1X + T_2X + \ldots + T_nX
\]

\[
RM(X,3) = RM_1(X,3) + RM_2(X,3) + \ldots + RM_n(X,3)
\]
NX is defined by

\[ F(0) = 0 \]
\[ F(S_u \dot{x}) = S_u F(x) \cdot (S_u 0 \triangleq \text{RM}(F(x), 3)) + S_u S_u F(x) \cdot (S_u 0 \triangleq (S_u 0 \triangleq \text{RM}(F(x), 3))) \]

and \( NX = N_1 X + N_2 X + \ldots + N_n X \)

M(X,Y) is defined by

\[ F(0, Y) = Y \]
\[ F(S_u X, Y) = N(F(X,Y)) \cdot (S_u 0 \triangleq \text{RM}(F(X,Y), 2)) \]

and \( M(X,Y) = C_1 M_1(X,Y) + C_2 M_2(X,Y) + \ldots + C_n M_n(X,Y) \)

\((Y)^k\) is defined as before.

L(X,Y) is defined by

\[ F(0, Y) = Y \]
\[ F(S_u X, Y) = S_u F(X,Y) \cdot (S_u 0 \triangleq M(Y, (S_u S_u)^P(X,Y)_u)) \]

and \( L(X,Y) = C_1 L_1(X,Y) + C_2 L_2(X,Y) + \ldots + C_n L_n(X,Y) \)

Define \( \overset{\sim}{C}_k X \) by

\[ F(0) = 0 \]
\[ F(S_u X) = \begin{cases} F(X) & \text{if } u = k \\ S_u F(X) & \text{if } u \neq k \end{cases} \]

\( \overset{\sim}{C}_k X \) is therefore all the components of \( X \) except the 'k'th component; hence

\[ C_k X + \overset{\sim}{C}_k X = X \]

From this \( P_k X = C_k L(X,X) + \overset{\sim}{C}_k X \).
It is interesting to note that in multiple successor arithmetic, if it were necessary to adjoin the functions \( P_u X \ u=1,\ldots,n \) to the initial functions, it would have been sufficient to adjoin the function \( PX \) defined by

\[
F(0) = 0 \\
F(S_u X) = X
\]
as clearly \( P_k X = C_k PX + \bar{C}_k X \), and \( PX = L(X,X) \).

We now proceed to define \( Q_k X \), which is the function

\[
Q_k X = S_0 - (X - (RX)^2)
\]

where \( RX \) standing for \( R_t(X) \) as defined in Chapter III, Lemma 65.

Define \( Y \subseteq X \) by

\[
F(0,Y) = Y \\
F(S_u X,Y) = P_u F(X,Y)
\]

Consider now the function \( T(X,Y) \) by

\[
F(0,Y) = 0 \\
F(S_u X,Y) = S_u F(X,Y) + (S_u 0 - ((F(X,Y)^2 - Y) + (Y - F(X,Y)^2))).
\]

This function is such that \( T(S_u X,Y) \) is obtained from \( T(X,Y) \) by adding \( S_u 0 \) unless \( (T(X,Y))^2 = Y \), in which case we add \( S_u S_u 0 \). This can happen only if \( X^2 = Y \). It is clear therefore that

\[
T(X,Y) = \begin{cases} 
X & \text{if } X^2 \leq Y \\
X + QY & \text{if } X^2 > Y
\end{cases}
\]

where \( QX = Q_1 X + Q_2 X + \ldots + Q_n X \).

Hence

\[
Q_u X = T(S_u X,X) - S_u X
\]
From Q^X we obtain RX as follows:

RM(X,3) was defined by

\[ F(0) = 0 \]

\[ F(S_uX) = (S_u0 \circ F(X)) + S_uS_u0.\!(S_u0 \circ (S_u0 \circ DX(F(X)))) \]

where DX and TX are defined as previous.

We now define UX by

\[ F(0) = 0 \]

\[ F(S_uX) = S_uF(X) + EM_u(F(X),3) \]

Clearly

\[ UX = X + \text{Div}(X,2) \]

where Div(X,2) is the function defined by

\[ F(0) = 0 \]

\[ F(S_uX) = F(X) + EM_u(X,2) \]

We therefore obtain Div(X,2) in R_{**} by the explicit definition

\[ \text{Div}(X,2) = UX \cdot X. \]

Consider the function

\[ \nu_kX = X + S_kS_0.RX. \]

This can be defined by the schema

\[ F(0) = 0 \]

\[ F(S_uX) = \begin{cases} S_kF(X) + S_kS_0.Q_k(S_kX) & \text{if } u=k \\ S_uF(X) & \text{if } u\neq k \end{cases} \]

Now the following are equivalent, S_kX is square (that is (RS_{kX})^2 = S_kX), X is of the form N^2 + C_kN + C_kN. Hence from the definition of \nu_kX we can say that \nu_kX is of the form N^2 + C_kN + C_kN + C_kN + C_kN, and therefore \nu_kX + S_kS_kS_0 is square.
Our definition of $V^X_k$ now becomes

$$F(0) = 0$$

$$F(SuX) = \begin{cases} S_kF(X) + S_kS_kO_k(F(X) + S_kS_kS_kS_kO) & u=k \\ SuF(X) & u\neq k \end{cases}$$

and is therefore defined by schema $R_{1^{**}}$.

From this

$$RX = Div(V_1X \div X, 2) + Div(V_2X \div X, 2) + \ldots + Div(V_nX \div X, 2)$$

as

$$S_kS_kO_RX = S_kS_kO_RX$$

However we can define $V^X_k$ by

$$F(0) = 0$$

$$F(SuX) = SuF(X) + SuSuO_u(F(X) + SuSuS_uS_uO)$$

and

$$RX = Div(VX \div X, 2).$$

For the proof of this result in single successor arithmetic see [7].

However, having defined $Y \div X$ in $R_{1^{**}}$ we can define the function $W(X, Y)$ by

$$F(0, Y) = 0$$

$$F(SuX, Y) = F(X, Y) + (SuO \div ((SuF(X,Y))^2 \div Y))$$

and

$$RX = W(X, X)$$

We have now defined the functions

$Y + X$, $Y \div X$, $X \times X$, $RX$,

in $R_{1^{**}}$ without introducing any further functions to the set of initial functions. We can therefore say 'The initial functions and the defining schema $R_{1^{**}}$ alone are sufficient to define all primitive recursive functions'.
This is the defining schema given by

\[
F(0,Y) = A(Y)
\]

\[
F(S_u X, Y) = B_u (F(X, Y)) \quad u=1, \ldots, n
\]

$Y + X$ is defined by

\[
F(0,Y) = Y
\]

\[
F(S_u X, Y) = S_u F(X, Y)
\]

In this schema we now suppose that the function $Q_X$ is adjoined to the initial set of functions.

$C_k X$ is defined by

\[
F(0) = 0
\]

\[
F(S_u X) = \begin{cases} S_k F(X) & u = k \\ F(X) & u \neq k \end{cases}
\]

and therefore $Q_k X$ is obtained by

\[
Q_k X = C_k Q X
\]

Next define $S_k^0 - X$ by

\[
F(0) = S_k 0
\]

\[
F(S_u X) = \begin{cases} 0 & u = k \\ F(X) & u \neq k \end{cases}
\]

and therefore $Y . (S_k^0 - X)$ is defined by

\[
F(0,Y) = Y
\]

\[
F(S_u X, Y) = \begin{cases} 0 & u = k \\ F(X,Y) & u \neq k \end{cases}
\]

Before we can proceed it is necessary to prove the following Lemma.
Lemma \((S_v 0 \div X) = (S_v 0 \div (S_v 0 \div (S_v 0 \div X)))\)

Proof

\[ L(0) = S_v 0 \div 0 \]
\[ = S_v 0 \]
\[ = S_v 0 \]

\[ R(0) = (S_v 0 \div (S_v 0 \div (S_v 0 \div 0))) \]
\[ = (S_v 0 \div (S_v 0 \div S_v 0)) \]
\[ = S_v 0 \div 0 \]
\[ = S_v 0 \]

\[ L(S_u X) = S_v 0 \div S_u X \]
\[ = \begin{cases} 0 & \text{u=v} \\ S_v 0 \div X & \text{u\neq v} \end{cases} \]
\[ = \begin{cases} 0 & \text{u=v} \\ L(X) & \text{u\neq v} \end{cases} \]

\[ R(S_u X) = (S_v 0 \div (S_v 0 \div (S_v 0 \div S_u X))) \]
\[ = \begin{cases} (S_v 0 \div (S_v 0 \div 0)) & \text{u=v} \\ (S_v 0 \div (S_v 0 \div (S_v 0 \div X))) & \text{u\neq v} \end{cases} \]
\[ = \begin{cases} S_v 0 \div S_v 0 & \text{u=v} \\ R(X) & \text{u\neq v} \end{cases} \]
\[ = \begin{cases} 0 & \text{u=v} \\ R(X) & \text{u\neq v} \end{cases} \]

\[ L(X) = R(X) \]

Now in the function \(Y,S_k 0 \div X\) substitute \(S_k 0 \div X\) for \(Y\) and define \(D_X\) by

\[ F(0) = 0 \]

\[ F(S_u X) = \begin{cases} S_k 0 \div F(X) & \text{u=k} \\ F(X) & \text{u\neq k} \end{cases} \]
Now in the function $Y.(S_k 0 : X)$ substitute $D_k X$ for $Y$ and $S_k 0 : Q_k X$ for $X$; therefore as $Q_k X = S_k 0 : (X' (RX)^2)$ we have the function

$$D_k X.(S_k 0 : (X' (RX)^2)) = D_k X.Q_k X$$

and clearly from III.96, which proves that $(RX)^2 : X = 0$, we have that

$$Q_k X = (S_k 0 : |X, (RX)^2|)$$

and therefore

$$Q_k X = \begin{cases} S_k 0 & \text{if } X = (RX)^2 \text{ in } 'S_k' \text{ successors} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} S_k 0 & \text{if } X \text{ is square in } 'S_k' \text{ successors} \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$D_k X.Q_k X = \begin{cases} S_k 0 & \text{if } X \text{ is an odd square in } 'S_k' \text{ successors} \\ 0 & \text{otherwise} \end{cases}$$

Define therefore the function $E_k X$ by

$$E_k X = S_k S_k X.(S_k 0 : D_k X.Q_k X)$$

which is obtained in $R_1^{***}$ by substitution in $Y.(S_k 0 : X)$.

We can then say that

$$E_k X = \begin{cases} 0 & \text{if } X \text{ is an odd square in } 'S_k' \text{ successors} \\ S_k X + S_k S_k 0 & \text{otherwise} \end{cases}$$

Before we can proceed now we will have to define the function $X.X$ in $R_1^{***}$.

We are able to define $V_k X$ as before by

$$F(0) = 0$$

$$F(S_u X) = \begin{cases} S_k F(X) + S_k S_k 0.Q_k (F(X) + S_k S_k S_k 0) & u=k \\ S_u F(X) & u\neq k \end{cases}$$

and

$$V_k X = X + S_k S_k 0 . RX$$
From this clearly $X.X$ is defined by

\[
\begin{align*}
F(0) &= 0 \\
F(S_uX) &= S_uV_u(F(X))
\end{align*}
\]

Having now defined $X.X$ in $\mathbb{R}_1^{***}$ consider the function $H_k(X,Y)$, defined by

\[
\begin{align*}
F(0,Y) &= Y.Y + C_kY + S_k0 \\
F(S_uX,Y) &= \left\{ \begin{array}{ll}
E_k(F(X,Y)) & \text{u}\neq k \\
F(X,Y) & \text{u= k}
\end{array} \right.
\end{align*}
\]

Clearly

\[
H_k(X,X) = E_k^X(X.X + C_kX + S_k0)
\]

where $E_k^X$ is defined by

\[
\begin{align*}
F(0,Y) &= Y \\
F(S_uX,Y) &= \left\{ \begin{array}{ll}
E_k(F(X,Y)) & \text{u}\neq k \\
F(X,Y) & \text{u= k}
\end{array} \right.
\end{align*}
\]

$E_k^X$ is therefore $C_kX$ operations of $E_k'$ on $Y$.

Let $X$ contain an even number of $S_k'$ successors, then clearly, for $H_k(X,X)$, after $\frac{1}{2}X$ (normal arithmetical expression) operations of $E_k'$, as the argument is never an odd square in $S_k'$ successors, and every operation of $E_k'$ adds $S_kS_k0$ to the argument

\[
H_k(X,X) = E_k^X(X.X + C_kX + C_kX + S_k0)
\]

At this time the argument is now an odd square in $S_k'$ successors and thus,

\[
H_k(X,X) = E_k^X\cdot S_k0(0) = C_kX \cdot S_kS_k0
\]

Define $C_kX$ by

\[
\begin{align*}
F(0) &= 0 \\
F(S_uX) &= \left\{ \begin{array}{ll}
F(X) & \text{u}\neq k \\
S_uF(X) & \text{u= k}
\end{array} \right.
\end{align*}
\]
This function is all components of $X$ except the '$k$'th component, that is the inverse of the function $C_k X$

$$C_k X + \overline{C}_k X = X$$

From $H_k(X, X)$ we obtain $C_k P_k X$ by

$$C_k P_k X = (S_k 0 \circ (S_k 0 \circ X)) \circ (S_k 0 \circ DX) \circ S_k H_k(X, X) + DX \circ H_k(S_k X, S_k X)$$

which is obtained by repeated substitution into the function

$$Y \circ (S_k 0 \circ X)$$ (from the definition this is obviously the same as $(S_k 0 \circ X).Y$)

and $Y + X$.

From this

$$P_k X = C_k P_k X + \overline{C}_k X$$

and now we can define $Y \circ X$ by

$$F(0, Y) = Y$$

$$F(S_u X, Y) = P_u F(X, Y)$$

$TX$ is defined by

$$F(0) = 0$$

$$F(S_u X) = S_u F(X) \circ (S_u 0 \circ F(X)) + F(X) \circ S_u F(X). (S_u 0 \circ (S_u 0 \circ F(X)))$$

$RM(X, 3)$ is defined by

$$F(0) = 0$$

$$F(S_u X) = (S_u 0 \circ F(X)) + S_u S_u 0. (S_u 0 \circ (S_u 0 \circ D(T(F(X))))$$

$UX$ is defined by

$$F(0) = 0$$

$$F(S_u X) = S_u F(X) + RM_u (F(X), 3)$$

and $UX = X + \text{Div}(X, 2)$
We therefore have $\text{Div}(X,2)$ in $R_{1}^{***}$ by

$$\text{Div}(X,2) = UX \div X$$

$VX$ is defined by

$$p(0) = 0$$

$$p(S^X) = S^X F(X) + S^S^O Q_u(F(X) + S^S^S^O)$$

$$VX = X + S_1 S_2 S_3 \ldots S_n S_n 0.RX \text{ explicitly,}$$

and

$$RX = \text{Div}(VX \div X,2)$$

Hence by adjoining the function $Q_X$ to the initial functions we have defined the functions

$$Y + X, \ Y \div X, \ X.X, \ RX,$$

with the schema $R_{1}^{***}$.

We can therefore say that 'The initial functions with the function $Q_X$ and the defining schema $R_{1}^{***}$ are sufficient to define all primitive recursive functions'. We have not proved, though, that it is necessary to adjoin the function $Q_X$ to the initial functions.

For the proof of the above result in single successor arithmetic see [7].

In the normal definition of the initial functions we have the 'n' successor functions $S_u(X)$ $u=1,\ldots,n$. This set of n functions can in fact be reduced to 2 functions, that is $S_1 0$ and a function to generate further successor functions; for this we could use $0 \sigma_1 X$.

Then

$$S_2 0 = 0 \sigma_1 S_1 0 \text{ etc. ....}$$

In the next section in this chapter we analyse the successor symbols in $X$, as mentioned in Chapter I.
ANALYSIS OF 'X'

In single successor arithmetic, the successors in X are all the same and thus require no distinction. Further, the number of successor symbols in X can be compared with those in Y simply by use of the functions $X - Y$ and $Y - X$, which indicate out of X and Y which has the most successor symbols and by how many. In fact, in single successor arithmetic X by itself informs us how many successor symbols it contains.

The problem of how many and which successor symbols X contains is not so easily answered. A question that immediately arises is how does one count the successor symbols. There are $'n'$ different counting systems; that is one for each successor, but no common counting system.

To achieve a homogeneous counting system, we consider the property

$$S_1S_2\ldots S_n X = X \cdot S_1S_2\ldots S_n = X$$

and further

$$S_1S_1S_2\ldots S_n S_0X = X \cdot S_1S_1S_2\ldots S_n S_0 = X + X$$

etc..

This property seems to suggest that we consider

$$S_1S_2\ldots S_n \quad \text{as 1}$$

and

$$S_1S_1S_2\ldots S_n S_0 \quad \text{as 2}$$

etc.

We shall therefore use a dummy successor 'S' to count as follows:

- $S0$ to represent $S_1S_2\ldots S_n 0$
- $SS0$ to represent $S_1S_1S_2\ldots S_n S_0$

etc.

In order to analyse X in multiple successor arithmetic we wish to answer the following questions:
(i) How many successor symbols are there in X?

(ii) How many different successor symbols?

(iii) What are these different successor symbols?

(iv) How many are there of each different successor symbol?

(i) How many successor symbols in X?

The answer to this question would obviously partition the set \( M_n \) of all numerals in multiple successor arithmetic into the sets \( U_n^0, U_n^1, U_n^2, \ldots, U_n^r, \ldots \) where

\[
U_n^0 = \{0\}
\]

\[
U_n^1 = \{S_v^0 : v=1,\ldots,n\}
\]

\[
U_n^2 = \{S_u S_v^0 : u,v=1,\ldots,n, \text{ and } v \geq u\}
\]

etc.

The number of successor symbols in X is determined quite simply by the function \( \phi(x) \) defined by

\[
\phi(0) = 0
\]

\[
\phi(S_u X) = S_1 S_2 \cdots S_n \phi(X) \quad u=1,\ldots,n
\]

or in the counting notation

\[
\phi(0) = 0
\]

\[
\phi(S_u X) = S \phi(X)
\]

We can therefore say

\[ X \in U_n^r \text{ if and only if } |r \phi(X)| = 0. \]
(ii) How many different successor symbols in $X$?

The answer to this question obviously gives the sets

$$D_n^O, D_n^1, D_n^2, \ldots, D_n^r, \ldots$$

where

$$D_n^O = \{0\}$$

$$D_n^1 = \{S_u S_u \ldots S_u 0 : u=1,\ldots,n, \text{ and } p=1,2,\ldots\}$$

$$D_n^2 = \{S_u S_v \ldots S_v S_v \ldots S_v 0 : u,v=1,\ldots,n, \text{ and } u \neq v \text{ and } p,q=1,2,\ldots\}$$

etc.

We first define the function introduced in Chapter I:

$$Y^n X^0 = 0$$

$$Y^n X^u X = Y^n X + Y$$ \quad \text{for } u=1,\ldots,n.

This function is obviously not commutative, as notice is taken of the successor symbols in $Y$ but not those in $X$, and this particular property makes this function extremely useful.

Now define the function $\Phi(X)$ by

$$\Phi(0) = 0$$

$$\Phi(S_u X) = S_u \Phi(X)^X (S_u 0 - \Phi(X)) + \Phi(X)^X (S_u X - (S_u 0 - \Phi(X))) \quad \text{for } u=1,\ldots,n,$$

this function is such that

$$\Phi(S_u X) = \begin{cases} \Phi(X) & \text{if } \Phi(X) \text{ does not contain } 'S_u' \\ \Phi(X) & \text{if } \Phi(X) \text{ contains } 'S_u' \end{cases}$$

The important property of the function $Y^n X$ which is used in this definition is

$$Y = Y^n X^u 0 \quad \text{for all } u$$

and so $\Phi(\Phi(X))$ gives the number of different successors in $X$. We can therefore say

$$X \in D_n^r \text{ if and only if } |r_\Phi(\Phi(X))| = 0$$
and \( \psi(x) \in U^R_n \) if and only if \( x \in D^R_n \).

(iii) What are these different successor symbols in \( X \)?

For this we need to distinguish between one successor symbol and another, so we define \( \Sigma(X) \) by

\[
\Sigma(0) = 0
\]
\[
\Sigma(S^uX) = u + \Sigma(X) \quad u=1,\ldots,n.
\]

This definition is possible as \( n \) is finite.

\( \Sigma(X) \) is the sum of the suffixes of the successor symbols in \( X \).

Hence \( S^10 \) can be distinguished from \( S^20 \) by

\[
\Sigma(S^20) \neq \Sigma(S^10) = S0
\]
\[
\Sigma(S^10) \neq \Sigma(S^20) = 0
\]

and from \( S^30 \) by

\[
\Sigma(S^30) \neq \Sigma(S^10) = S30
\]
\[
\Sigma(S^20) \neq \Sigma(S^30) = 0
\]

We can now define \( L^1(X) \) and \( H^1(X) \) the lowest(suffix) and highest (suffix) of successor symbols of \( X \), by

\[
L^1(0) = 0
\]
\[
L^1(S^uX) = L^1(X).(1 \odot (\Sigma(L^1(X)) \odot \Sigma(S^u0))) +
\]
\[
S^u0.(1 \odot (1 \odot (\Sigma(L^1(X)) \odot \Sigma(S^u0)))) \quad u=1,\ldots,n,
\]

and

\[
H^1(0) = 0
\]
\[
H^1(S^uX) = H^1(X).(1 \odot (\Sigma(S^u0) \odot \Sigma(H^1(X)))) +
\]
\[
S^u0.(1 \odot (1 \odot (\Sigma(S^u0) \odot \Sigma(H^1(X))))) \quad u=1,\ldots,n.
\]
That is \( L^1(X) \) is defined by
\[
L^1(0) = 0
\]
\[
L^1(S_uX) = \begin{cases} 
L^1(X) & \text{if the suffix of } S_u \geq \text{ the suffix of } L^1(X) \\
S_u0 & \text{if the suffix of } S_u \lessdot \text{ the suffix of } L^1(X) 
\end{cases}
\]

and \( H^1(X) \) is defined by
\[
H^1(0) = 0
\]
\[
H^1(S_uX) = \begin{cases} 
H^1(X) & \text{if the suffix of } S_u \lessdot \text{ the suffix of } H^1(X) \\
S_u0 & \text{if the suffix of } S_u \geq \text{ the suffix of } H^1(X) 
\end{cases}
\]

Thus \( \Sigma(L^1(X)) \) gives the suffix of the lowest successor symbol and \( \Sigma(H^1(X)) \) gives the suffix of the highest successor symbol.

Now \( L^2(X) \), the second lowest successor symbol, is obviously given by
\[
L^2(X) = L^1(X - L^1(X) \phi(X))
\]
that is
\[
X - L^1(X) \phi(X)
\]
gives \( X \) without any of its lowest successor symbols, and therefore the lowest successor symbol of this is the second lowest successor symbol of \( X \).
Hence \( L^3(X) \) is given by
\[
L^3(X) = L^2(X - L^1(X) \phi(X))
\]
etc. ...

The highest successor symbol is therefore given by
\[
H^1(X) = \begin{cases} 
L^1(X) & \text{if } \phi(X) = 1 \\
\phi(\phi(X))(X) & \text{otherwise}
\end{cases}
\]

Similarly we have
\[
H^2(X) = H^1(X - H^1(X) \phi(X))
\]
and
\[ L^1(X) = \begin{cases} \{H^1(X) \} & \text{if } \phi(X) = 1 \\ H^\phi(\psi(X))(x) & \text{otherwise} \end{cases} \]

Thus

\[ \Sigma(L^1(X)), \Sigma(L^2(X)), \ldots \Sigma(L^\phi(\psi(X))(x)) \]

and

\[ \Sigma(H^1(X)), \Sigma(H^2(X)), \ldots \Sigma(H^\phi(\psi(X))(x)) \]

enumerate the suffixes of all the different successor symbols in \( X \),
the first series from lowest to highest and the second series from
highest to lowest.

(iv) How many are there of each different successor symbol?

We could answer this question by

\[ \phi(C_k X) = \text{the number of } 'S_k' \text{ successor symbols in } X \]

though we count the particular successor symbols that are in \( X \) by the
use of the function defined by

\[ \phi(X) = (1 - L^k(X))^X \phi(X) \]

which gives the number of the \( 'k' \)th lowest successor symbols in \( X \).

This is obtained by subtracting \( \phi(X) \) of all the other successor symbols
in \( X \) from \( X \) and counting the remainder, as there obviously cannot be
more than \( \phi(X) \) of any successor symbol in \( X \). Similarly we could have

\[ \phi(X) = (1 - H^k(X))^X \phi(X) \]

which gives the number of the \( 'k' \)th highest successor symbols in \( X \).
BIBLIOGRAPHY