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I should like to thank Professor R. L. Goodstein for his valuable advice during the preparation of this thesis. Also Professor P. M. Cohn, now of Bedford College, London, but then of Manchester University, for originally introducing me to the general subject area of the problems under discussion.
The main purpose of this thesis is to study and compare various ordering properties of groups. We will principally be interested in ordered groups, right-ordered groups and what I choose to call generalized polyordered groups, but we will also bring other types of order-relation and other group theoretic properties into the discussion. An order on a group $G$ is an ordering of the elements of $G$ which is preserved under multiplication on both the left and the right by elements of $G$. A right-order on $G$ only needs to be preserved under multiplication on the right. A generalized polyorder on $G$ is a right-order with certain extra properties which make it approximate to an order.

The theory of ordered groups arose quite naturally out of the theory of ordered fields which in turn was a natural generalization of the algebraic theory of real numbers. Right-ordered groups are an obvious generalization of ordered groups and were first studied in depth by Paul Conrad in the late 1950's. He compared the properties of right-orders with those of orders and, in so doing, he introduced the concept of a generalized polyorder though without actually giving it a name.

The first three sections are devoted to presenting the basic concepts of the subject and to proving (sometimes rather sketchily) some of the standard results to be used in the later sections. For convenience in the applications, we have chosen to set these standard results in a more general context than is usual, dealing with right-orders on coset spaces instead of on the group itself. This is also of some interest in that it is related to the study of
groups of order-preserving permutations of ordered sets.

In Section 4 we ask whether the above-listed properties are of a local character. That is, whether they hold on a group $G$ if and only if they hold on all finitely generated subgroups of $G$. They are also compared with other local properties. For example, we are able to show that a locally indicable group can be right-ordered.

In recent years several authors have been producing results on the ordering properties of locally nilpotent groups. The most important of these are collected in Section 5 together with some new ones which have not yet appeared in the literature. They add up to a fairly complete picture of the situation in this case. In effect all the afore-mentioned properties are equivalent in torsion-free locally nilpotent groups.

In Section 6, we investigate the same problems for metabelian groups. Some positive results are achieved for groups $G$ which have a normal Abelian subgroup $A$ with finite Abelian factor group $G/A$, but the attempt to extend these to general metabelian groups is not very successful. Indeed we are usually able to produce counterexamples to prove them false.

Many questions remain open and we mention some of them at the appropriate points in the text.

Lemmas are numbered sequentially within each section, for example, Lemma 1.2, Lemma 2.3, etc. Theorems and Examples are numbered sequentially throughout, that is, no section number is attached to the theorem or example number. References to papers are made by giving the appropriate number from the list at the end, for example, Paul Conrad [4], and references to the two standard textbooks given in the list are made by giving merely the author's name and a page number, for example, Kurosh, Vol.1, p.121 or Fuchs, p.34.
1. Some Basic Results

This thesis is principally concerned with partial orderings on the set of elements of a group preserved under multiplication by elements of the group and most of the later work will be written in those terms. However it is of some interest to begin the discussion in a more general setting which helps to explain some of the methods used later.

Suppose that \( H \) is a subgroup of the group \( G \) and let \( G \) denote the set of all right-cosets of \( H \) in \( G \). We refer to this set as the right coset space of \( G \) over \( H \). The general setting mentioned in the first paragraph is that of transitive binary relations on the coset spaces \( G:H \). That is, subsets \( \mathcal{R} \) of the Cartesian product \( (G:H) \times (G:H) \) which satisfy the condition

\[
(1) \quad (Hg_1, Hg_2) \in \mathcal{R} \text{ and } (Hg_2, Hg_3) \in \mathcal{R} \Rightarrow (Hg_1, Hg_3) \in \mathcal{R}.
\]

We say that such a relation is compatible with \( G \) if it also satisfies the further condition

\[
(2) \quad (Hg_1, Hg_2) \in \mathcal{R} \Rightarrow (Hg_1, Hg_2g) \in \mathcal{R} \text{ for all } g \in G.
\]

In the special case when \( H \) is the identity subgroup of \( G \) relations satisfying conditions (1) and (2) are transitive relations on \( G \) preserved under multiplication on the right. Thus the present discussion includes for example partial right-orders on \( G \).

Another special example of interest is that when \( G \) is a group of permutations of a set \( \Omega \) (infinite for a non-trivial discussion) and \( H \) is the stabilizer of a symbol \( \alpha \in \Omega \). Relations on the right coset space then correspond to relations on \( \Omega \) preserved under the permutations from \( G \).
Associated with any relation $\mathcal{R}$ on $G:H$ which satisfies conditions (1) and (2) there is a subset $S$ of $G$ given by

$$S = \{ g \in G; (H,Hg) \in \mathcal{R} \},$$

This set is a subsemigroup of $G$ because, if $g_1 \in S$ and $g_2 \in S$, then $(H,Hg_1) \in \mathcal{R}$ and $(H,Hg_2) \in \mathcal{R}$. Hence, by (2), $(Hg_2,Hg_1g_2) \in \mathcal{R}$ and then, by (1), $(H,Hg_1g_2) \in \mathcal{R}$. Thus $g_1g_2 \in S$.

In addition $S$ also satisfies the condition

$$(4) \quad HS = SH = S$$

or, equivalently, $hS = S$ and $Sh = S$ for all $h \in H$. For, if $g \in S$ and $h \in H$, then, since $Hg = H(hg)$, $hg \in S$ and hence $hS \subseteq S$. Since also $h^{-1} \subseteq S$, we have $hS = S$ for all $h \in H$. Further, if $g \in S$ and $h \in H$, then $(H,Hg) \in \mathcal{R}$ implies, by (2),

$$(H,Hgh) = (H,Hgh) \in \mathcal{R}, \text{ i.e. } gh \in S. \text{ Hence } Sh \subseteq S \text{ and as above it follows that } Sh = S \text{ for all } h \in H.$$

We will say that any subsemigroup $S$ of $G$ which satisfies condition (4) is an $H$-subsemigroup of $G$. Thus we have shown that, associated with any transitive binary relation $\mathcal{R}$ on the right coset space $G:H$ which is compatible with $G$, there is a certain $H$-subsemigroup of $G$.

Conversely, if $S$ is an $H$-subsemigroup of $G$, we may define a binary relation $\mathcal{R}$ on $G:H$ which we then see satisfies conditions (1) and (2).

Given $S$ we define

$$(Hg_1,Hg_2) \in \mathcal{R} \iff g_2g_1^{-1} \in S.$$

First this definition does not depend on the choice of the representatives of the cosets $Hg_1$ and $Hg_2$. For, if $g'_1 \in Hg_1$ and $g'_2 \in Hg_2$, then $g'_1 = h_1g_1$ and $g'_2 = h_2g_2$ for some $h_1$ and $h_2$ in $H$ and we have
because $S$ is an $H$-subsemigroup.

Secondly it satisfies condition (1) because, if $(Hg_1, Hg_2) \in \mathcal{R}$ and $(Hg_2, Hg_3) \in \mathcal{R}$, then $g_2 g_1^{-1} \in S$ and $g_3 g_2^{-1} \in S$ and, since $S$ is a semigroup, $g_3 g_1^{-1} = g_3 g_2^{-1} g_2 g_1^{-1} \in S$, i.e. $(Hg_1, Hg_3) \in \mathcal{R}$.

Finally it satisfies condition (2) because, if $(Hg_1, Hg_2) \in \mathcal{R}$ and $g \in G$, then $g_2 g_1^{-1} \in S$ and hence $g_2 g (g_1 g)^{-1} = g_2 g_1^{-1} \in S$, i.e. $(Hg_1 g, Hg_2 g) \in \mathcal{R}$.

The correspondence thus established between binary relations on $G$ over $H$ and $H$-subsemigroups of $G$ is easily shown to be one-to-one and onto. Thus a study of the relations is equivalent to a study of the semigroups. Both formulations of the theory have their uses and we will use whichever is more convenient in any given context.

Note that a given relation $\mathcal{R}$ could also be represented by the set

$$S' = \{ g \in G; \ (Hg, H) \in \mathcal{R} \}$$

$S'$ is also an $H$-subsemigroup of $G$ and it is not difficult to see that it consists of the inverses of the elements of $S$, i.e. $S' = S^{-1}$.

To reconstitute $\mathcal{R}$ from $S'$ we must define

$$(Hg_1, Hg_2) \in \mathcal{R} \iff g_1 g_2^{-1} \in S'.$$

The relation $\mathcal{R}'$ obtained from $S'$ by defining

$$(Hg_1, Hg_2) \in \mathcal{R}' \iff g_2 g_1^{-1} \in S'$$

is the reflection of $\mathcal{R}$ in the diagonal, i.e.

$$(Hg_1, Hg_2) \in \mathcal{R}' \iff (Hg_2, Hg_1) \in \mathcal{R}.$$

Clearly we could give an analogous treatment to transitive binary relations on the left coset space of $G$ over $H$. In this case we would be interested in relations $\mathcal{L}$ which satisfy
This time the special case when H is the identity subgroup leads to relations on G which are preserved under multiplication on the left.

For a given relation $\mathcal{L}$ the set

$$S = \{ g \in G; (H, gH) \in \mathcal{L} \}$$

is an H-subsemigroup of G and we again have a one-to-one correspondence between the relations and the semigroups. From a given H-subsemigroup the corresponding relation $\mathcal{L}$ is defined by

$$(g_1 H, g_2 H) \in \mathcal{L} \iff g_1^{-1} g_2 \in S.$$}

When $\mathcal{R}$ is a transitive relation on $G:H$ compatible with $G$, then we could define a relation $\mathcal{L}$ on the left coset space by

$$(g_1 H, g_2 H) \in \mathcal{L} \iff (Hg_1, Hg_2) \in \mathcal{R}.$$}

Clearly $\mathcal{L}$ is transitive but in general it will not be compatible with $G$, i.e. condition (3) may not hold. If $\mathcal{L}$ does happen to satisfy condition (3), then we will say that $\mathcal{R}$ is two-sided. Otherwise $\mathcal{R}$ is one-sided or, in the usual terminology, a right relation. ($\mathcal{L}$ is a left relation.)

We note that $\mathcal{R}$ is two-sided if and only if the corresponding H-subsemigroup $S$ is normal in $G$. For, when $\mathcal{R}$ is two-sided, condition (3) for $\mathcal{L}$ may be interpreted as

$$(Hg_1, Hg_2) \in \mathcal{R} \iff (Hg g_1, Hg g_2) \in \mathcal{R}$$ for all $g \in G.$

Hence $g_2 (g_1 g_2)^{-1} = g_2 g_2^{-1} g_1^{-1} \in S$. By choosing $g_1 = 1$ we may make $g_2 g_1^{-1}$ equal to any element of $S$ and hence $S$ is normal in $G$. Conversely, when $S$ is normal in $G$ and $(Hg_1, Hg_2) \in \mathcal{R}$, then
Note that, in the special case when \( H \) is the identity subgroup, two-sided relations on \( G:H \) are two-sided relations on \( G \) in the usual sense. That is, relations on \( G \) preserved under multiplication on both the left and the right.

Since the relations on the left and right coset spaces are analogous (they are represented by the same semigroups), we need to consider only one type. Following the usual custom introduced by Conrad [4], we choose to interpret one-sided relations as right relations and the foregoing description has already been written with that in mind.

Transitive binary relations may generally be called "order-relations" and accordingly the different types are given names involving the word "order". We will add the prefix "right" when we wish to indicate that the relation is one-sided. Otherwise a relation without the prefix "right" may be assumed to be two-sided.

In this terminology the semigroup \( S \) consists of all the elements \( g \in G \) for which \( Hg \) is "greater than" \( H \) and is referred to as the positive cone of \( \mathcal{R} \).

In general a transitive relation \( \mathcal{R} \) is a total relation if for any pair of elements \( Hg_1 \) and \( Hg_2 \) of the basic set (here \( G:H \)) either \( Hg_1=Hg_2 \) or \( (Hg_1,Hg_2) \in \mathcal{R} \) or \( (Hg_2,Hg_1) \in \mathcal{R} \). In terms of the positive cone \( S \), this means that \( \mathcal{R} \) is total if, given any two elements \( g_1,g_2 \in G \), either \( g_2g_1^{-1} \in H \) or \( g_2g_1^{-1} \in S \) or \( g_2g_1^{-1} \in S \) (i.e. \( g_2g_1^{-1} \in S^{-1} \)). That is, \( \mathcal{R} \) is a total relation if \( H \cup S \cup S^{-1}=G \). Any relation which does not satisfy this condition is a partial relation.

A relation \( \mathcal{R} \) is said to be anti-symmetric if \( (Hg_1,Hg_2) \in \mathcal{R} \) implies \( (Hg_2,Hg_1) \notin \mathcal{R} \). Again in terms of the positive cone \( S \) this means that \( \mathcal{R} \) is anti-symmetric if \( g_2g_1^{-1} \in S \) implies \( g_1g_2^{-1} \notin S \) which, for \( S \neq \emptyset \), is equivalent to the condition \( S \cap S^{-1} = \emptyset \).
(the empty set). [The case \( S = \emptyset \) which corresponds to the trivial relation \( \mathcal{R} = \emptyset \) is also considered as being anti-symmetric.]

In the present context the condition \( S \cap S^{-1} = \emptyset \) may further be seen to be equivalent to the condition \( 1 \not\in S \). For, clearly, if \( 1 \in S \), then \( 1 \in S \cap S^{-1} \not\emptyset \). On the other hand, if \( s \in S \cap S^{-1} \not\emptyset \), then \( s = t^{-1} \) for some \( t \in S \) and \( st = 1 \in S \) because \( S \) is a semigroup. This is the form of the condition which we will normally use.

Note that, for an \( H \)-subsemigroup \( S \), if \( 1 \in S \) then \( H \subseteq S \) and so the condition \( 1 \not\in S \) is also equivalent to \( H \cap S = \emptyset \).

An anti-symmetric transitive relation \( \mathcal{R} \) on \( G:H \) is called a partial (right-)order on \( G:H \). Its positive cone \( S \) satisfies the condition \( 1 \not\in S \). [This terminology is used here for "strict" partial orderings only, i.e. those for which we would use the symbol < as opposed to \( \leq \).] The trivial relation \( \mathcal{R} = \emptyset \) is a partial order in this sense.

When \( \mathcal{R} \) is also a total relation it is referred to as a (right-)order on \( G:H \) — we do not need to use the prefix "total". The positive cone then satisfies \( 1 \not\in S \) and \( H \cup S \cup S^{-1} = G \).

A relation \( \mathcal{R} \) which contains all the pairs \((Hg, Hg)\), i.e. for which \( 1 \in S \), is said to be a partial (right-) preorder on \( G \times H \).

When \( \mathcal{R} \) is also a total relation we say it is a (right-) preorder on \( G:H \). In this case the positive cone satisfies the conditions \( 1 \in S \) and \( H \cup S \cup S^{-1} = G \) which may be simplified to \( 1 \in S \) and \( S \cup S^{-1} = G \). Clearly \( G \) itself satisfies these conditions and is an \( H \)-subsemigroup (for any \( H \)). The relation it represents is the one containing all pairs \((Hg_1, Hg_2)\). We will refer to this as the trivial preorder on \( G:H \) and to all others as non-trivial preorders.

Although it is convenient to continue to use the concept of partial preorders, there is a sense in which they are no different from the earlier
concept of partial orders on coset spaces. The connection is elucidated in the first lemma.

**Lemma 1.1** Let $S$ be the positive cone of a partial (right-) preorder $\mathcal{R}$ on the coset space $G:H$. Then $K = S \cap S^{-1}$ is a subgroup of $G$ containing $H$ and $P = S \setminus K$ is the positive cone of a partial right-order on $G:K$. If $\mathcal{R}$ is two-sided, then both $K$ and $P$ are normal in $G$.

**Proof** Since $S$ contains $H$, $S^{-1}$ also contains $H$ and hence $K$ contains $H$. In particular $K$ is not empty. As an intersection of semigroups, $K$ is closed under multiplication and, from its construction, it is clearly closed under inversion. It follows that $K$ is a subgroup of $G$.

When $\mathcal{R}$ is two-sided, $S$ is normal in $G$. Then $S^{-1}$ is also normal in $G$ and hence $K$ is normal in $G$.

We show that $P = S \setminus K$ is a $K$-subsemigroup of $G$.

Suppose that $p_1$ and $p_2$ are in $P$. Then certainly $p_1p_2 \in S = P \cup K$. If $p_1p_2 \in K$, then $p_1p_2 = s^{-1}$ for some $s \in S$ and hence $p_1 = s^{-1}p_2^{-1} \in S^{-1}$. But this contradicts the definition of $P$ and we deduce that $p_1p_2$ is in $P$. Thus $P$ is a semigroup.

Now suppose that $p \in P$ and $k \in K$. Then, as before, $kp \in S$ and $pk \in S$. If either $kp \in K$ or $pk \in K$, then $p \notin K$ would be a contradiction. We deduce that $kp \in P$ and $pk \in P$ for all $p \in P$ and all $k \in K$. It follows that $KP = PK = P$, that is, $P$ is a $K$-subsemigroup. Further, if $S$ is normal in $G$, then $P$, being the difference of normal subsets of $G$, is also normal in $G$.

Since $1 \notin P$, $P$ is the positive cone of a partial right-order on $G:K$ which is two-sided if the original partial (right-) preorder is two-sided.
Although the construction in the lemma is a very natural one, it does not produce a one-to-one correspondence between the partial (right-) preorders and partial (right-) orders on coset spaces of $G$. This is essentially because the semigroup $P$ represents a partial (right-) order not only on $G:K$ but also on many other coset spaces. For example on $G\#L$, where $L$ is any subgroup of $K$.

In general, if $P$ is a subsemigroup of $G$ not containing the identity, then $P$ represents a partial (right-) order on any coset space $G:H$ for which $P$ is an $H$-subsemigroup. All such subgroups $H$ must be contained in the set $F(P) = \{g \in G; \ gP = Pg = P\}$. In fact it is not difficult to show that $F(P)$ itself is a subgroup of $G$ and so is the largest possible subgroup of $G$ for which $P$ is an $F(P)$-subsemigroup. Thus $P$ is the positive cone of a partial (right-) order on the coset space $G:H$ if and only if $H$ is a subgroup of $F(P)$. In the special case dealt with in Lemma 1.1, we have shown that the subgroup $K$ is contained in $F(P)$.

It is of some interest to know under what conditions a subsemigroup $P$ of $G$ is the positive cone of a total (right-) order on some coset space $G:H$. For, in that case, the set $S = H \cup P$ will be the positive cone of a total (right-) preorder on $G$ which is non-trivial provided $P \neq \emptyset$.

**Lemma 1.2** The subsemigroup $P$ of $G$ is the positive cone of a total (right-) order on a coset space of $G$ if and only if the set $H = G \setminus (P \cup P^{-1})$ is a subgroup of $G$.

**Proof** If $P$ is a positive cone of a (right-) order on $G\#H$ say, then, from the definition, we have $P \cup P^{-1} \cup H = G$ and $G \setminus (P \cup P^{-1}) = H$ is obviously a subgroup.
On the other hand, if $H = G \setminus (P \cup P^{-1})$ is a subgroup, then we first observe that $1 \notin P$ because obviously $1 \in H$. It is therefore sufficient to show that $P$ is an $H$-semisubgroup. Suppose that $p \in P$ and $h \in H$. Since $G = H \cup P \cup P^{-1}$ and $hp \in G$, we need to show that $hp \notin H \cup P^{-1}$. But $hp \in H$ implies $p \in H$ ($H$ is a subgroup) and $hp \in P^{-1}$ implies $h \in P^{-1}P^{-1} \subseteq P^{-1}$ both of which are contradictions. Hence $hp \in P$ for all $h \in H$ and all $p \in P$ and it follows that $HP = P$. Similarly $PH = P$ and $P$ is an $H$-subsemigroup.

We note that in the case when $P$ is normal in $G$, then $H$ is a normal subgroup of $G$ and $P$ corresponds to an order on the factor group $G/H$.

The next lemma describes a method whereby partial right-orders may be combined to construct new partial right-orders. The technique is one which will be used on many occasions in the subsequent discussion.

**Lemma 1.3** Let $H$ and $K$ be subgroups of the group $G$ such that $H \leq K \leq G$ and let $P_1$ and $P_2$ be the positive cones of partial right-orders on $G:K$ and $K:H$ respectively. Then $P_1 \cup P_2$ is the positive cone of a partial right-order on $G:H$. In the case when $P_1$ and $P_2$ are two-sided partial orders on $G:K$ and $K:H$ respectively, $P_1 \cup P_2$ is a two-sided partial order on $G:H$ if and only if $P_2$ is normal in $G$.

**Proof** We have to show that $P_1 \cup P_2$ is an $H$-subsemigroup and that $1 \notin P_1 \cup P_2$.

Since $1 \notin P_1$ and $1 \notin P_2$ the second condition is quite obvious. Now suppose that $x$ and $y$ are in $P_1 \cup P_2$. If $x$ and $y$ are both in $P_1$, then $xy \in P_1$ and, if both are in $P_2$ then $xy \in P_2$.

If $x \in P_1$ and $y \in P_2$, then $xy \in P_1$ and $yx \in P_1$ because $P_1$ is a $K$-subsemigroup and $y \in P_2 \subseteq K$. 

Further we have

\[ H(P_1 \cup P_2) = HP_1 \cup HP_2 \]

because \( P_1 \) is a \( K \)-subsemigroup and \( P_2 \) is an \( H \)-subsemigroup.

Similarly \((P_1 \cup P_2)H = P_1 \cup P_2\) and so \( P_1 \cup P_2 \) is an \( H \)-subsemigroup.

If \( P_1 \) and \( P_2 \) are both two-sided, then \( P_1 \) is normal in \( G \)
but in general \( P_2 \) need only be normal in \( K \). If \( P_2 \) is actually
normal in \( G \), then clearly \( P_1 \cup P_2 \) is also normal in \( G \) and so it
is the positive cone of a partial order on \( G:H \).

Conversely, if \( P_1 \cup P_2 \) is normal in \( G \) and \( q \in P_2 \), then, for
any \( g \in G \), \( g^{-1}qg \in P_1 \cup P_2 \). If \( g^{-1}qg \in P_1 \), then \( q \in gP_1g^{-1} = P_1 \)
and this is a contradiction because \( P_1 \cap P_2 = \emptyset \). (In fact \( P_1 \cap K = \emptyset \).)
Thus \( g^{-1}qg \in P_2 \) for all \( g \in G \) and all \( q \in P_2 \) and \( P_2 \) is normal in \( G \).

Clearly this technique may be extended to any finite series of
subgroups \( H < K_1 < K_2 < \ldots < K_r < G \), so that we are able to combine
partial right-orders on the coset spaces \( G:K_r, K_r \triangleright K_{r-1}, \ldots, K_1 \triangleright H \)
to give a partial right-order on \( G:H \). For two-sided partial orders to
combine into a two-sided partial order it is necessary that the positive
cones should all be normal in \( G \).

Another concept fundamental to any discussion of partially (right-)
ordered groups is that of the convex subgroup. Roughly speaking
a subgroup \( K \) of \( G \) is convex if it contains all elements of \( G \) which
lie between any two elements of \( K \) in the partial (right-) order. The
concept may also be defined for partial (right-) orders on coset spaces.

In this context it is more natural and usually more convenient to
use the traditional method of denoting order relations by symbols such
as \( <, \prec, \preceq \), etc. So that, if \( \mathcal{R} \) is a partial (right-) order on
the coset space \( G:H \) then we would write \( Hg_1 < Hg_2 \) if and only if
Conditions (1) and (2) would then read

1. \( H_{g1} < H_{g2} \) and \( H_{g2} < H_{g3} \) \( \Rightarrow \) \( H_{g1} < H_{g3} \)

and

2. \( H_{g1} < H_{g2} \) \( \Rightarrow \) \( H_{g1} g < H_{g2} g \) for all \( g \in G \).

The positive cone of \( R \) then consists of all elements \( g \in G \) whose coset \( Hg \) is greater than the identity coset \( H \). We will make the convention that this notation will only be used for anti-symmetric relations, that is, ones whose positive cone does not contain the identity. Otherwise we would have to allow the possibility that, for some unequal cosets, \( H_{g1} < H_{g2} \) and \( H_{g2} < H_{g1} \). We prefer not to do this.

We say that a subgroup \( K \) of \( G \) is convex in a partial (right-) order \( \leq \) on \( G;H \) if, whenever \( Hk_1 \leq Hg \leq Hk_2 \) for some elements \( k_1,k_2 \in K \), then \( g \) is also in \( K \).

Note that, since \( Hh = H \) for all \( h \in H \), it follows from the definition that a convex subgroup \( K \) must contain \( H \). The subgroup \( H \) itself is convex as is the full group \( G \). Any other convex subgroup \((\neq G \text{ or } H)\) will be referred to as a non-trivial proper convex subgroup.

The smallest convex subgroup of \( G \) containing a given subset \( T \) of \( G \) is called the convex hull of \( T \) and is denoted by \( C(T) \). It is equal to the intersection of all the convex subgroups containing \( T \).

To prove a subgroup \( K \) of \( G \) containing \( H \) is convex in \( \leq \), it is sufficient to show that, if \( H \leq Hg \leq Hk \) for some \( k \in K \), then \( g \in K \). For then, if \( Hk_1 \leq Hg \leq Hk_2 \) for some \( k_1,k_2 \in K \), we have by condition (2) \( H = Hk_1k_1^{-1} \leq Hgk_1^{-1} \leq Hk_2k_1^{-1} \). Hence, since \( k_2k_1^{-1} \in K \), we have \( gk_1^{-1} \in K \) and it follows that \( g \in K \). Note that only condition (2) (and not condition (3)) has been used here so that this argument does apply to one-sided relations. Conrad in [4] seems to have overlooked this point.
be used quite often later, but another criterion will also be of considerable use.

Suppose that $P$ is the positive cone of the partial (right-) order $<$ on $G:H$. The condition for convexity in the last paragraph could then be restated as "$K$ is convex if and only if the set $H \cup (P \cap K)$ is convex."

A second criterion for convexity is obtained by considering, instead of $P \cap K$, its complement $P^* = P \setminus (P \cap K)$ in $P$.

**Lemma 1.4** The subgroup $K$ of $G$ containing $H$ is convex in the partial (right-) order $\leq$ on $G:H$ if and only if the set $P^* = P \setminus (P \cap K)$ is a semigroup.

**Proof** Suppose that $K$ is convex and that $p_1$ and $p_2$ are elements of $P^*$. Then certainly $p_1 \in P$, $p_2 \in P$ and hence $p_1 p_2 \in P$. Since $p_1 \in P$, we have $H < H p_1$ and hence $H p_2 < H p_1 p_2$, but also $p_2 \in P$ and hence altogether we have $H < H p_2 < H p_1 p_2$. If $p_1 p_2$ were in $K$, then by convexity, $p_2$ would also be in $K$. This contradicts the assumption that $p_2 \notin P^*$ and we deduce that $p_1 p_2 \notin K$. Hence $p_1 p_2 \in P^*$ and $P^*$ is a semigroup.

Now suppose that $P^*$ is a semigroup and that $H < H g < H k$ for some $g \in G$ and $k \in K$. If $H g = H$ or $H g = H k$, then $g \in H$ or $g k^{-1} \in H$ and, in both cases, it follows that $g \in K$. Thus we may suppose that in fact $H < H g < H k$. Then $g \in P$ and $k g^{-1} \in P$.

If $g \notin K$, then $g \in P^*$ and $k g^{-1} \in P^*$ and, since $P^*$ is a semigroup, $k = (k g^{-1}) g \in P^*$. This is a contradiction of the definition of $P^*$ and we deduce that $g$ must be in $K$. Hence $K$ is convex.

Referring back to Lemma 1.3, we can now see that the construction of the combined partial (right-) order on $G:H$ from partial (right-) orders on $G:K$ and $K:H$ is done in such a way that the subgroup $K$
is convex. For, in that case, \( P = P_1 \cup P_2 \) and 

\[ P^* = (P_1 \cup P_2) \setminus ((P_1 \cup P_2) \cap K) = (P_1 \cup P_2) \setminus P = P_1 \] is obviously 
a K-subsemigroup. In the light of this last remark, it is perhaps 
reasonable to ask if all convex subgroups arise in this way. That is, 
if \( K \) is convex in \( \prec \), does it follow that \( P^* \) is a K-subsemigroup 
and so represents a partial (right-)order on \( G:K \)? The answer in 
general is quite easily seen to be in the negative.

**Example 1** Let \( G \) be the group which is generated by two elements 
a and \( b \) with the relation \( b^{-1}ab = a^{-1} \). Then each element of \( G \) 
has a unique representation in the form \( a^m b^n \), where \( m \) and \( n \) are 
integers. Now let \( H = \{1\} \) and let \( \prec \) be the partial right-order 
whose positive cone is \( P = S(a) \). [Since \( G \) is torsion-free, \( 1 \notin P \).] 
Then the subgroup \( K \) of \( G \) generated by \( b \) is convex in \( \prec \), because 
\( P^* = P \setminus (P \cap K) = P \setminus \emptyset = P \) is a semigroup. But clearly in this case 
\( P^* \) is not a K-subsemigroup.

Nevertheless we may still be able to use \( \prec \) to "induce" a partial 
(right-) order on \( G:K \). For this we need the general concept of the 
"\( H \)-subsemigroup of \( G \) generated by a given subset \( T \) of \( G \)".

In general, if \( H \) is a subgroup of \( G \), and \( T \) is a subset of \( G \), 
then there is at least one \( H \)-subsemigroup which contains \( T \), viz. 
\( G \) itself. Also it is clear that the intersection of any number of 
\( H \)-subsemigroups containing \( T \) is also an \( H \)-subsemigroup containing \( T \). 
Thus we may define the \( H \)-subsemigroup of \( G \) generated by \( T \) to be the 
intersection of all \( H \)-subsemigroups which contain \( T \). We denote it by 
\( S_H(T) \). It is not difficult to see that \( S_H(T) \) consists of all elements 
of the form \( h_0 t_1 h_1 t_2 h_2 \ldots t_r h_r \), where \( r \geq 1 \), \( t_i \in T \) and \( h_i \in H \) for 
all \( i = 0, 1, \ldots, r \).
Similarly we define the normal $H$-subsemigroup generated by $T$ to be the intersection of all $H$-subsemigroups which contain $T$ and which are also normal in $G$. We will denote this by $\overline{S}_H(T)$.

In the special case when $H$ is the identity subgroup, we simplify the notation to write $S(T)$ instead of $S\{1\}(T)$ and $\overline{S}(T)$ instead of $\overline{S}\{1\}(T)$.

Now all we require for $<$ to induce a partial right-order on $G:K$ is that $1 \notin S^K(P*)$ for then $S^K(P*)$ will be the positive cone of such a partial right-order. [For a two-sided partial order, we require $1 \notin \overline{S}^K(P*)$.] Unfortunately the convexity of $K$ is not sufficient to ensure that this method works either. The same example as before suffices to show this. For there $1 = ab^{-1}ab \in S^K(P*)$.

In general, if $1 \notin S^K(P*)$, we will say that the partial right-order on $G:K$ whose positive cone is $S^K(P*)$ is the one induced by the partial right-order $<$. It is interesting to note that the condition $1 \notin S^K(P*)$ is sufficient to ensure the convexity of $K$ in $<$.

**Lemma 1.5** In the notation of Lemma 1.4, if $1 \notin S^K(P*)$, then $K$ is convex in $<$.

**Proof** By Lemma 1.4, we only need to show that $P^*$ is a semigroup.

Suppose therefore that $p_1$ and $p_2$ are elements of $P^*$. Then certainly $p_1p_2 \in P$. If $p_1p_2 \in K$, then $p_1p_2 \in K \cap S^K(P*)$ and hence $1 = (p_1p_2)^{-1}p_1p_2 \in S^K(P*)$ giving a contradiction. Thus $p_1p_2 \notin K$ and it follows that $p_1p_2 \in P^*$.

Another natural way in which we might try to induce a partial (right-) order $<$ on $G:K$ from a given partial (right-) order on $G:H$ would be to define

$$Kg_1 < Kg_2 \iff Kg_1 \neq Kg_2 \text{ and } k_1g_2 < k_2g_2 \text{ for some } k_1k_2 \in K.$$
This definition does not always produce a transitive relation (though it is compatible with $G$) but, when it does, $<$ happens to coincide with the partial right-order induced by $<$ through $S_K(P^*)$. In all the instances where we will want to use induced partial right-orders this present definition does work and will be the one which is used.

We close this section by enumerating some of the situations in which $<$ is transitive.

Lemma 1.6 The relation $<$ on $G:K$ defined as above is a partial right-order on $G:K$ if and only if $KP^*K$ is a semigroup. In that case $S_K(P^*) = KP^*K$ and $<$ coincides with the partial right-order on $G:K$ given by $S_K(P^*)$. In particular, $K$ is convex in $<$.

Proof We first observe that, from the definition of $<$, $Kg_1 < Kg_2$ if and only if $g_2g_1^{-1} \not\in K$ and there are elements $k_1,k_2 \in K$ such that $k_2g_2g_1^{-1}k_1^{-1} \in P$, i.e. if and only if $g_2g_1^{-1} \in KPK \setminus (KPK \cap K) = KP^*K$.

Note that, since $P^* \cap K = \emptyset$, $1 \not\in KP^*K$.

Thus, if $<$ is to be a partial (right-) order on $G:K$ its positive cone will have to be $KP^*K$. Clearly if $KP^*K$ is a semigroup, then it is a $K$-subsemigroup and we reach the conclusion of the lemma, i.e. $<$ is a partial right-order if and only if $KP^*K$ is a semigroup.

The rest of the statement of the lemma follows immediately with an application of Lemma 1.5.

We remark that for $<$ to be two-sided, $KP^*K$ would have to be normal in $G$ as well as being a semigroup.

Corollary Let $K$ be a subgroup of $G$ convex in the partial (right-) order $<$ on $G:H$. Then, in each of the following three cases, the relation $<$ is a partial right-order on $G:K$. 

(i) $<$ is two sided.
(ii) $K$ is normal in $G$.
(iii) $<$ is a total (right-) order on $G : H$.

**Proof** In each case we have merely to show that $KP^*K$ is a semigroup. Since $K$ is convex, $P^*$ itself is a semigroup.

(i) When $<$ is two-sided, $P$ is normal in $G$ and we show that $P^*$ is normalized by $K$. Suppose $p \in P^*$ and $k \in K$, then certainly $k^{-1}pk \in P$ and $k^{-1}pk \in K$ would imply $p \in K$ - a contradiction. Hence $k^{-1}pk \in P^*$ for all $p \in P^*$ and $k \in K$.

It follows that, if $k_1p_1k_2 \in KP^*K$ and $k_3p_3k_4 \in KP^*K$ then

$$k_1p_1k_2 \cdot k_3p_3k_4 = k_1 \cdot p_1 \cdot k_2k_3p_3^{-1}k_2^{-1} \cdot k_2k_3k_4 \in KP^*K$$

and $KP^*K$ is a semigroup.

(ii) When $K$ is normal in $G$, then we have

$$k_1p_1k_2 \cdot k_3p_3k_4 = k_1p_1p_2 \cdot p_2^{-1}k_2k_3p_2k_4 \in KP^*K;$$

and again $KP^*K$ is a semigroup.

(iii) When $<$ is a total (right-) order on $G:H$, then we have $H \cup P \cup P^{-1} = G$ and, since $P = P^* \cup (P \cap K)$, it follows that $K \cup P^* \cup P^{-1} = G$.

Now suppose that $p \in P^*$ and $k \in K$. If either $pk \in K$ or $kp \in K$, then $p \in K$ - a contradiction. If $pk \in P^{-1}$ or $kp \in P^{-1}$, then $pk = q^{-1}$ or $kp = q^{-1}$ for some $q \in P^*$. Hence $qp = k^{-1}$ or $pq = k^{-1}$ and, in both cases, there is a contradiction because $pq$ and $qp \in P^*$ cannot be in $K$. We deduce that both $pk$ and $kp$ are in $P^*$ for all $p \in P^*$ and $k \in K$. Hence $KP^*K = P^*$ is a semigroup.

[We observe that the last of these contradicts Conrad's remark in [4], p.27 that, for a right-order, $a \in P^*$ does not imply $Ka \subseteq P^*$.]
In the last of these three cases, that is when \(<\) is a total (right-) order, we see that the positive cone of \(<\) is essentially the same as that of \(<\) and it is quite clear that its convex subgroups are the same as those of \(<\) which contain \(K\). This observation coupled with Lemma 1.3 leads us to a technique for modifying a given right-order while keeping some of its convex subgroup structure unchanged.

More explicitly, if \(<\) is a right-order on \(G:H\) and \(K\) is a convex subgroup, then \(<\) induces a right-order \(\prec\) on \(G:K\) which may be combined with any right-order on \(K:H\) (Lemma 1.3) to give a new right-order \(\prec\) on \(G:H\). The convex subgroups of \(\prec\) above \(K\) will be exactly the same as those for \(<\). This technique will be used several times in subsequent sections. Note however that, even if \(<\) is two-sided, the technique cannot be expected to produce a two-sided order \(\prec\) in general. Indeed, even \(\prec\) may not be two-sided. For that, \(K\) would have to be normal in \(G\) and, as we shall see in Section 2, that may not be the case in general. However, if \(K\) is normal in \(G\), then \(P^*\) is also normal in \(G\) and \(<\) is two-sided. For \(\prec\) also to be two-sided, \(<\) must then be combined with an order on \(K\) whose positive cone is normal in \(G\). The large number of conditions involved in all this make the technique less useful for two-sided orders.

Finally we observe that every right-order on a coset space \(G:H\) is induced by a right-order on a suitable factor group of \(G\). Indeed, if \(H^*\) is the cone of \(H\), i.e. the largest normal subgroup of \(G\) contained in \(H\) and equal to \(\bigcap_{g \in G} g^{-1}Hg\), then there is a right-order on \(G/H^*\) which induces the given right-order on \(G:H\). The elements \(h\) of \(H^*\) are distinguished by the property that \(HgH = Hg\) for all \(g \in G\). That is, they are the elements of \(G\) which "fix" all the right cosets in \(G:H\) under multiplication on the right.
Lemma 1.7  If $\prec$ is a right-order on $G:H$ and $H^*$ is the cone of $H$, then there is a right-order on $G/H^*$ in which $H/H^*$ is convex and which agrees with $\prec$ on $G:H$.

Proof  We use a method first described by P.M. Cohn [3] for right-ordering a group of order preserving automorphisms of an ordered set. Here the ordered set is $G:H$ and the group is $G/H^*$ permuting the members of $G:H$ by multiplication on the right.

We start by choosing any well-order $\preceq$ on the set $G:H$, which, for the purposes of our proof, has first member $H$. Now, given any coset $H^*a \neq H^*$, there must be some cosets $Hg$ for which $Hga \neq Hg$. Let $Hg_a$ be the first of these in the well-order $\preceq$. Then we define a right-order $\prec$ on $G:H^*$ by

$$H^*a \succ H^* \iff Hg_a \succ Hg_a.$$  

It is quite straightforward to verify that $\prec$ is a right-order in which $H$ is convex and, because $H$ is the first element in $\preceq$, $\prec$ agrees with $\prec$ on $G:H$.

We will make use of this result in Section 5.
2. Total (Right-) orders and Generalized Polyorders

The purpose of this section is to present some of the standard results concerning the system of convex subgroups of a total (right-) order and to set them as far as possible in the context of right-orders on coset spaces as introduced in Section 1. We also introduce the concept of a generalized polyorder which is to be studied in more detail in later sections.

The existence (or non-existence) of proper non-trivial convex subgroups is closely related to the concept of Archimedean ordering and we start with a discussion of this relationship.

Let $<$ be a right-order on the coset space $G/H$. We say that a coset $Hx$ is infinitely greater than the coset $Hy$ and write $Hx >> Hy$ if and only if either $Hx > Hg$ for all elements $g$ in the subgroup $\langle H, y \rangle$ generated by $H$ and $y$ or $Hx^{-1} > Hg$ for all $g \in \langle H, y \rangle$. It is not difficult to show that this definition is independent of the choice of representatives for the cosets. For example, if $y_1 \in Hy$, then $\langle H, y_1 \rangle = \langle H, y \rangle$ and so the condition is the same for $y_1$ as for $y$.

When $H$ is normal in $G$, $\langle H, y \rangle$ is equal to the set of all elements of the form $hy^n$, where $h \in H$ and $n$ is an integer. In that case the definition simplifies to $Hx >> Hy$ if and only if $Hx > Hy^n$ for all integers $n$ or $Hx^{-1} > Hy^n$ for all $n$. In particular, when $H = \{1\}$, this becomes the usual definition for a right-order on $G$. That is, $x >> y$ if and only if $x > y^n$ for all $n$ or $x^{-1} > y^n$ for all $n$.

We say that $<$ is Archimedean if there are no cosets infinitely greater than any non-trivial coset.
Paul Conrad [4] has shown that an Archimedean right-order on a group $G$ ($H = \{1\}$) must actually be two-sided. As the next example shows, this is not true in general for right-orders on coset spaces.

**Example 2** Let $G$ be the set of all ordered pairs $(x,m)$ in which $x$ is rational and $m$ is an integer. Define multiplication in $G$ by

$$(x,m)(y,n) = (x + 2^m y, m + n).$$

Then $G$ is a group with identity element $(0,0)$. We define a right-order $<$ on $G$ by giving its positive cone $P$ as follows:

$$(x,m) \in P \iff x > 0 \text{ or } x = 0 \text{ and } m > 0.$$  

The set $P$ is a semigroup because, if $(x,m) \in P$ and $(y,n) \in P$, then $x > 0$ and $y > 0$ and it follows that $x + 2^m y > 0$ with equality only when $x = y = 0$. In the latter case, we then have $m > 0$ and $n > 0$ and hence $m + n > 0$. In all cases $(x,m)(y,n) \in P$. Clearly $(0,0) \notin P$ and, if $(x,m)$ is a non-identity element which is not in $P$, then $(x,m)^{-1} = (-2^{-m}x,-m)$ is in $P$. This shows that $P \cup P^{-1} \cup \{1\} = G$ and $<$ is a total right-order on $G$.

The set $H$ of all pairs of the form $(0,m)$ is a subgroup of $G$ and, as we shall see, it is convex in $<$. For, suppose $(0,0) < (x,m) < (0,n)$ for some $n$, then $x > 0$ and

$$(0,0) < (0,n)(x,m)^{-1} = (-2^{-m}x,n-m).$$  

But $-2^{-m}x < 0$ giving a contradiction unless $x = 0$, i.e. unless $(x,m) \in H$.

Thus, by Corollary (iii) of Lemma 1.6, $<$ induces a right-order $<$ on the coset space $G:H$. We show that $<$ is Archimedean but not two-sided.

First $<$ is not two-sided because, if it were, $H$ would have to be normal in $G$. However the conjugate $(1,0)^{-1}(0,m)(1,0) = (-1) + 2^m,m) \notin H$ shows that this is not so.
Now suppose that $H(x,m)$ and $H(y,n)$ are non-trivial cosets (i.e. $x \neq 0$ and $y \neq 0$) and, for the sake of argument, that $H \subset H(x,m)$ and $H \subset H(y,n)$. [Otherwise we replace $(x,m)$ and $(y,n)$ by their inverses as necessary.] We need to show that there is some element $(z,p) \in \langle H(x,m) \rangle$ such that $H(y,n) \prec H(z,p)$.

From the way in which $\prec$ is defined, this is the same as showing that $(y,n) < (z,p)$, for some $(z,p) \in \langle H(x,m) \rangle$.

In fact it turns out that we may choose $(z,p)$ to be $(x,0)^k$ for sufficiently large $k > 0$. For certainly

$$(x,0) = (x,m)(0,-m) \in \langle H(x,m) \rangle$$

and then

$$(x,0)^k(y,n)^{-1} = (kx,0)(-2^{-n}y,-n)$$

is positive provided $k > 2^{-n}y/x$. This completes the proof.

Thus the best possible generalization of Conrad's result is as follows.

**Lemma 2.1** If $\prec$ is an Archimedean right-order on the coset space $G:H$ and $H$ is a normal subgroup of $G$, then $\prec$ is two-sided.

**Proof** Let $P$ be the positive cone of $\prec$. Then we have that $P$ is an $H$-subsemigroup of $G$, $1 \notin P$, and $H \cup P \cup P^{-1} = G$.

We want to show that $P$ is normal in $G$, i.e. that $xyx^{-1} \in P$ for all $x \in G$ and $y \in P$. Since $H$ is normal in $G$ and $y \notin H$, in any case $xyx^{-1}$ cannot be in $H$. We therefore only need to eliminate the possibility that $xyx^{-1} \in P^{-1}$.

Suppose first that $x \in P$. Since $H$ is normal in $G$ and both $Hx$ and $Hy$ are positive, the Archimedean property is equivalent to the existence of positive integers $n$ such that $Hx < Hy^n$. Let $m$ be the least of these.
Now, if $xyx^{-1} \in P^-$, then we have $Hxyx^{-1} < H$ and hence $Hxy < Hx < Hy^m$. Multiplying on the right by $y^{-1}$ we then have $Hx < Hy^m$ which contradicts the choice of $m$. We conclude that $xyx^{-1} \notin P^-$ and hence $xyx^{-1} \in P$. In terms of $P$ itself this says that $xPx^{-1} \subseteq P$ for all $x \in P$.

Now consider $x^{-1}yx$, where $x \in P$. If $x^{-1}yx \in P^-$, then $x^{-1}y^{-1}x \in P$ and hence, from the first part of the proof, $x(x^{-1}y^{-1}x)x^{-1} = y^{-1} \in P$. This is again a contradiction because $y \in P$ and we deduce that $x^{-1}yx \notin P^-$ and we can now say that $xPx^{-1} \subseteq P$ for all $x \in P \cup P^-$.

The fact that $xPx^{-1} \subseteq P$ for all $x \in H$ as well follows because $P$ is an $H$-subsemigroup. All together we deduce that $xPx^{-1} = P$ for all $x \in G$ and hence $\prec$ is two-sided.

The main result on Archimedean two-sided orders is due to O. Hölzer (see Fuchs, pp. 45-46). As we shall need this result on several occasions later, we will state the theorem here and give a brief outline of its proof.

For this theorem and for use throughout the rest of this work, we make the convention that $\mathbb{R}$ denotes the additive group of real numbers ordered by the natural ordering. Two partially (right-) ordered groups $G_1$ and $G_2$ are said to be order-isomorphic if there is a group isomorphism $\phi: G_1 \to G_2$ such that, whenever $g_1 < g_2$ in $G_1$ then $g_1 \phi < g_2 \phi$ in $G_2$.

**Theorem 1 (O. Hölzer)** Any Archimedean ordered group $G$ must be order-isomorphic to a subgroup of $\mathbb{R}$.

**Proof** First, if there is a positive element $a$ in $G$ which is greater than no other positive element, then $G$ must actually be generated by $a$. For, if $g$ is a positive element not equal to a power of $a$, then, by the Archimedean property, there must be
a positive integer \( n \) such that \( a^n < g < a^{n+1} \). But then
\[ 1 < ga^{-n} < a \]
produces a positive element less than \( a \). In this case \( G \) is order-isomorphic to the additive group of integers.

Suppose therefore that, for every positive element \( x \), there is an element \( y \) with \( 1 < y < x \). Then there must actually be a \( y \) such that \( 1 < y < y^2 < x \). For, if not and \( 1 < y < x < y^2 \) for some \( y \), then \( 1 < xy^{-1} < y < x \) and hence \( x < (xy^{-1})^2 = xy^{-1}xy^{-1} \). But then \( y = yx^{-1} < xy^{-1}xy^{-1} = xy^{-1} \) gives a contradiction.

[This has used the two-sidedness of \( < \).]

Now suppose there is a commutator in \( G \) not equal to the identity - say \( x = a^{-1}b^{-1}ab > 1 \). We may assume without loss of generality that \( a \) and \( b \) are both positive - otherwise replace them by \( a^{-1} \) and \( b^{-1} \) and \( x \) by \( x^{-1} \) if necessary. Choose an element \( y \) as above with \( 1 < y < y^2 < x \). By the Archimedean property, there are integers \( m \) and \( n \) such that \( y^m < a < y^{m+1} \) and \( y^n < b < y^{n+1} \). Using the fact that \( < \) is two-sided a straightforward calculation then shows that
\[ x = a^{-1}b^{-1}ab < y^{-m}y^{-n}y^{m+1}y^{n+1} = y^2 \leq x \] giving a contradiction.
We deduce that all commutators are equal to the identity and hence that \( G \) is Abelian.

The proof is completed by setting up a correspondence between the elements of \( G \) and certain Dedekind sections of the rational numbers. We choose any positive element \( a \) to correspond to the number 1 and then, for any \( x \) in \( G \), we define \( L(x) \) to be the set of all rationals \( m/n \) for which \( a^m \leq x^n \) and \( U(x) \) the set of all \( m/n \) for which \( a^m > x^n \). The pair \((L(x), U(x))\) is a Dedekind section of the rationals and the mapping \( x \mapsto (L(x), U(x)) \) is an order-isomorphism from \( G \) into \( \mathbb{R} \). \[ \square \]
In conjunction with Lemma 2.1, this theorem shows that if $<$ is an Archimedean right-order on the coset space $G:H$ and $H$ is normal in $G$, then $G/H$ is order-isomorphic to a subgroup of $\mathbb{R}$.

Now how do these concepts relate to the existence of convex subgroups? We observe first that, if $<$ is a right-order on $G:H$, and $C$ is convex in $<$, and $x \notin C$, then $Hx > H$ for all $y \in C$. For when $y \in C$, $\langle H,y \rangle$ is contained in $C$ and hence, if $H < Hx < Hg$ or $H < Hx^{-1} < Hg$ for any $g \in \langle H,y \rangle$, then $x$ would be in $C$ - a contradiction. Thus, in particular, if $x \notin C(y)$ (the convex hull of $y$), then $Hx > H$. It follows that, if $<$ is Archimedean, then there can be no non-trivial proper convex subgroups. As the next example shows, the converse of this statement is not true in general.

**Example 3** (D.M. Smirnov [16]) Let $G$ be the group which consists of all $2 \times 2$ matrices of the form $\begin{pmatrix} k & a \\ 0 & 1 \end{pmatrix}$ where $k$ and $a$ are rationals with $k > 0$. We define a right-order on $G$ by setting $\begin{pmatrix} k & a \\ 0 & 1 \end{pmatrix} > \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \iff k + a > 1$,

where $\varepsilon$ is some chosen positive irrational real number.

It is straightforward to verify that the set so defined is the positive cone of a right-order on $G$. The irrationality of $\varepsilon$ is needed for the ordering to be total - every matrix is "decided" by the condition because $k + \varepsilon a \neq 1$ for any non-identity matrix.

We show that there are no proper non-trivial convex subgroups in $G$. (Here $H = \{1\}$.) Suppose that $C$ is a non-trivial convex subgroup and suppose that $A = \begin{pmatrix} k & a \\ 0 & 1 \end{pmatrix}$ is a non-identity matrix in $C$. Without loss of generality, we may further suppose that $A$ is positive in the right-order $<$. 
Then $C$ must contain a matrix of the form $B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b > 0$. Indeed, if $b$ is any rational such that $0 < b < \frac{1}{e^k} (k + ea - 1)$, then $B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ is in $C$. For $AB^{-1} = \begin{pmatrix} k & -kb+a \\ 0 & 1 \end{pmatrix}$

and 

$$k + e(-kb+a) = k + ea - ekb$$

$$> k + ea - ek\left(\frac{1}{ek} (k+ea-1)\right)$$

$$= 1$$

means that $AB^{-1} > I$. That is, $I < B < A$ and hence $B \in C$ as required.

But now, if $A$ is any positive element of $G$, there is a power of $B$ which is greater than $A$. Because

$$B^nA^{-1} = \begin{pmatrix} 1 & nb \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/k & -a/k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/k & -a/k + nb \\ 0 & 1 \end{pmatrix}$$

and, providing $n$ is chosen to be greater than $\frac{1}{eb} \left(1 - \frac{1}{k} + \frac{ea}{k}\right)$,

we have $$\frac{1}{k} + e\left(-\frac{a}{k} + nb\right) > 1.$$ It follows that all positive matrices $A$ are in $C$ and hence that $\mathcal{E} = G$.

Finally we show that $<$ is not Archimedean by finding matrices $X$ and $Y$ in $G$ such that $X > Y^n$ for all integers $n$. Let $a$ be any rational such that $a > 1/e$ and let $e$ be any positive rational not equal to 1. Then put $X = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$.

Now $XY^{-n} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k^{-n} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k^{-n} & a \\ 0 & 1 \end{pmatrix}$ and this is positive for all $n$ because $k^{-n} + ea > 1$ for all $n$. Hence $X > Y^n$ for all integers $n$. In other words, $X >> Y$ and $<$ is not Archimedean.

For a two-sided order however the property of being Archimedean is equivalent to the non-existence of non-trivial proper convex subgroups. This is an immediate consequence of the next lemma.
Lemma 2.2 Let $\prec$ be a two-sided order on the coset space $G:H$. Then the convex hull $C(a)$ of an element $a \in G$ is equal to the set of all elements $x \in G$ such that $H^m a H^x H^n$ for some $m$ and $n$.

Proof Note that, since $\prec$ is two-sided, $H$ is normal in $G$ and we are really dealing with an order on $G/H$. Thus the condition may equally well be written as $(Ha)^m \preceq Hx \preceq (Ha)^n$.

The proof is very straightforward and rests on the fact that, if $\prec$ is two-sided and $Hx < Ha$, $Hy \preceq Hb$, then $(Hx)(Hy) = Hxy < (Ha)(Hb) = Hab$ and $Ha^{-1} < Hx^{-1}$.

We will return to this theme a little later but for now we turn our attention to the problem mentioned at the beginning of the section. That is, the classic problem of characterizing the system of convex subgroups of a (right-) ordered group. The problem has been solved very satisfactorily for two-sided orders and, to a certain extent, also for right-orders. However before presenting those results, it is convenient to make a few general comments regarding the convex subgroups of a right-order $\prec$ on a coset space $G:H$.

First we observe trivially that $H$ and $G$ are convex in $\prec$. Secondly we note that, if $C_1$ and $C_2$ are convex then either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$. For suppose, for example, that $C_2 \not\subseteq C_1$, then there must be an element $c_2 \in C_2$ with $c_2 \not\in C_1$. We may further suppose that $Hc_2 > H$, otherwise the subsequent argument could be applied to $c_2^{-1}$.

Now, if for any $c_1 \in C_1$, $H < Hc_2 \leq Hc_1$, then, by the convexity of $C_1$, $c_2 \in C_1$ and this is a contradiction. We deduce that $H \leq Hc_1 < Hc_2$ for all positive elements $c_1$ of $C_1$. By the convexity of $C_2$, it follows that $c_1 \in C_2$ for all positive $c_1$ and hence that $C_1 \subseteq C_2$. 

Thirdly, an arbitrary intersection of convex subgroups is also convex. For, if \( C_\lambda (\lambda \in \Lambda) \) are all convex and \( H \leq Hg \leq Hc \) for some \( c \in \bigcap_{\lambda \in \Lambda} C_\lambda \), then \( c \in C_\lambda \) for all \( \lambda \in \Lambda \) and, by convexity, \( g \in C_\lambda \) for all \( \lambda \in \Lambda \). That is, \( g \in \bigcap_{\lambda \in \Lambda} C_\lambda \). Similarly an arbitrary union \( \bigcup_{\lambda \in \Lambda} C_\lambda \) of convex subgroups is also convex.

For, if \( H \leq Hg \leq Hc \) for some \( c \in \bigcup_{\lambda \in \Lambda} C_\lambda \), then \( c \in C_{\lambda_0} \) for some \( \lambda_0 \) and, by convexity, \( g \in C_{\lambda_0} \). Hence \( g \in \bigcup_{\lambda \in \Lambda} C_\lambda \).

Finally, we remark that, if \( < \) is two-sided, then any conjugate \( x^{-1}Cx \) of a convex subgroup \( C \) is also convex. For, if \( H \leq Hg \leq Hx^{-1}cx \) for some \( c \in C \), then from the two-sidedness of \( < \), we have \( H \leq Hxgx^{-1} \leq Hc \). Hence \( xgx^{-1} \in C \) and \( g \in x^{-1}Cx \) as required.

Of course, when \( < \) is two-sided as in the last remark, the subgroup \( H \) is normal in \( G \) and we are really dealing with an order on \( G/H \). In this case we may as well assume that \( H = \{1\} \) and, for the next part of the discussion we will indeed make that assumption.

The problem of characterizing the system of convex subgroups of an ordered group was first solved by K.Iwasawa [9] in 1948 and a similar solution was found by A.I.Malcev [12] in 1949. Malcev's solution is reproduced in Fuchs, p.53. Its main contribution is a condition ((6) in Fuchs' numbering) which ensures that the factors in the system must be Archimedean ordered. This condition goes slightly outside the realm of Group Theory involving as it does the quotient field of a certain integral domain of automorphisms and is somewhat technical. However for our present purposes we do not need the full power of this theorem.

Other authors (Rieger, Podderygin - Fuchs p.51) have given other conditions of a more group theoretic nature which ensure that a given system \( \mathcal{C} \) of subgroups of \( G \) may be used to define an order on \( G \). They do not ensure however that the system \( \mathcal{C} \) of convex subgroups
in the order is exactly equal to $\mathcal{D}$. In fact $\mathcal{C}$ includes $\mathcal{D}$ but may be larger than $\mathcal{D}$ in general.

We choose here to state a form of the theorem which is not so powerful as Malcev's and is not as group theoretic as Podderygin's but which does give necessary and sufficient conditions for a system of subgroups to be the convex subgroups in some order on $G$. This will be sufficient for our purposes.

If $A$ and $B$ are members of a system $\mathcal{D}$ of subgroups of $G$ such that $B$ is maximally contained in $A$, then we say that $A \supset B$ is a jump in $\mathcal{D}$. That is, $A \supset B$ is a jump if, whenever $A \supset D \supset B$ and $D \in \mathcal{D}$, then $D = A$ or $D = B$.

We note as well that, if $\phi : G_1 \rightarrow G_2$ is an isomorphism between the groups $G_1$ and $G_2$ and $\prec_2$ is an order on $G_2$ then we may define an order $\prec_1$ on $G_1$ by $g_1 \prec_1 g_2 \iff \phi(g_1) \prec_2 \phi(g_2)$. That is, $\prec_1$ is defined so as to make $\phi$ an order-isomorphism. The verification that $\prec_1$ is an order is quite straightforward and shows that $\prec_1$ is really the "same" order as $\prec_2$. In particular, if $\prec_2$ is Archimedean, then so is $\prec_1$.

We will use the notation $N_G(A)$ for the normalizer in $G$ of the subset $A$ of $G$ and $a^g$ for the conjugate of $a$ by $g$, i.e. $a^g = g^{-1}ag$.

**Theorem 2**  The system $\mathcal{D}$ of subgroups of the group $G$ is the system of convex subgroups in some order on $G$ if and only if it satisfies the following six conditions.

(i) $\{1\}$ and $G$ are in $\mathcal{D}$.

(ii) $\mathcal{D}$ is totally ordered by inclusion.

(iii) $\mathcal{D}$ is closed under arbitrary intersections and unions.
(iv) If \( A \supset B \) is a jump in \( \mathcal{J} \), then \( B \) is normal in \( A \) and \( A/B \) is isomorphic to a subgroup of \( \mathbb{R} \).

(v) \( \mathcal{J} \) is closed under conjugation.

(vi) If \( A \supset B \) is a jump in \( \mathcal{J} \), then the order induced on \( A/B \) by the natural order on \( \mathbb{R} \) under the isomorphism of condition (iv) is preserved under all the automorphisms of \( A/B \) which are induced by conjugation with elements of \( G \).

[More explicitly, if \( g \in N_G(A) \), then \( g \in N_G(B) \) and the order induced on \( A/B \) is such that, if \( B_{a_1} < B_{a_2} \), then \( B_{a_1}^g < B_{a_2}^g \).]

Proof  If \( \mathcal{D} \) is the system of convex subgroups in some order on \( G \), then we have already observed that it satisfies conditions (i), (ii), (iii) and (v). When \( A \supset B \) is a jump in \( \mathcal{D} \), the normality of \( B \) in \( A \) follows from conditions (ii) and (v) and the rest of condition (iv) and condition (vi) then follow from Theorem 1.

Conversely suppose \( \mathcal{D} \) is a system of subgroups satisfying (i),..., (vi). We show how to define an order \( \prec \) on \( G \) for which the system of convex subgroups is \( \mathcal{D} \).

First we observe that, if \( A \supset B \) is a jump in \( \mathcal{D} \) and \( g \in G \), then \( A^g \supset B^g \) is also a jump in \( \mathcal{D} \). For certainly by (v) \( A^g \) and \( B^g \) are in \( \mathcal{D} \) and, if \( A^g \supset D \supset B^g \) for some \( D \in \mathcal{D} \), then \( A \supset D^{-1} \supset B \) is a contradiction. Hence the jumps in \( \mathcal{D} \) are divided into equivalence classes under conjugation by the elements of \( G \). We choose a complete set of representatives for these classes.

If \( A_1 \supset A_2 \) is one of those chosen representatives then we let \( \prec_A \) be the order on \( A_1/A_2 \) which is induced by the natural order on \( \mathbb{R} \) under the isomorphism \( \phi_A : A_1/A_2 \to \mathbb{R} \) whose existence is
guaranteed by condition (iv). Condition (vi) then ensures that \( \prec_A \) is preserved under all the automorphisms of \( A_1/A_2 \) induced by conjugation with elements of \( N_G(A_1) \). Now if \( B_1 \supset B_2 \) is any jump equivalent to \( A_1 \supset A_2 \), say \( B_1^g = A_1 \) and \( B_2^g = A_2 \), then the mapping \( B_2b \rightarrow A_2b^g \) (\( b \in B_1 \)) is an isomorphism from \( B_1/B_2 \) onto \( A_1/A_2 \) and so may be used as above to induce an order \( \prec_B \) on \( B_1/B_2 \) from the order \( \prec_A \). Explicitly \( B_2 \prec_B B_2b \) if and only if \( A_2 \prec_A A_2b^g \). The verification that \( \prec_B \) does not depend on the choice of the conjugator \( g \) and that it is preserved under the automorphisms of \( B_1/B_2 \) induced by conjugation with elements of \( G \) depends on the known property (vi) for \( \prec_A \) and on the observation that in general \( N_G(A^g) = N_G(A)^g \). The object of this part of the exercise is to choose orders on all the jumps in \( \mathcal{D} \) which are "compatible" with each other under conjugation and which are all Archimedean.

Next, for any non-identity element \( x \in G \), we define \( X_1 \) to be the intersection of all those members of \( \mathcal{D} \) which contain \( x \) and \( X_2 \) to be the union of all members of \( \mathcal{D} \) which do not contain \( x \). By (i), \( X_1 \) and \( X_2 \) are non-empty, by (iii) they are both in \( \mathcal{D} \) and by (ii) \( X_1 \supset X_2 \). It is easy to see that \( X_1 \supset X_2 \) is actually a jump in \( \mathcal{D} \). Thus every non-identity element of \( G \) occurs in some jump of \( \mathcal{D} \).

The order \( \prec \) on \( G \) may now be defined by giving its positive cone as follows. We set \( 1 \prec x \) if and only if \( x \neq 1 \) and \( X_2 \prec_X X_2x \) in the chosen order \( \prec_X \) on \( X_1/X_2 \).

The verification that \( \prec \) is an order is fairly straightforward. For example, to show that the defined positive cone is a semigroup we suppose that \( 1 \prec x \) and \( 1 \prec y \) and observe, in that case,
that the jump in $\mathcal{D}$ where $xy$ occurs is either $X_1 \supset X_2$ (the same as $x$) or $Y_1 \supset Y_2$ (the same as $y$), depending on whether $Y_1 \subseteq X_1$ or $X_1 \subseteq Y_1$. If it is $X_1 \supset X_2$, then we have $x_2 \prec_X x_2 x$ and $x_2 \prec_X x_2 y$ and hence

$$x_2 \prec_X (x_2 x)(x_2 y) = x_2(xy),$$

i.e. $1 \prec xy$. A similar argument works if the jump is $Y_1 \supset Y_2$.

The fact that $\prec$ is a total order follows from the observation that the jump where $x^{-1}$ occurs is the same as that for $x$, and, if $x_2 \prec_X x_2 x$, then $x_2 \prec_X x_2 x^{-1}$.

The two-sidedness of $\prec$ is a consequence of the way in which the orders on the jumps were chosen. If the jump where a given positive element $x$ occurs is $X_1 \supset X_2$, then the jump where the conjugate $x^g$ occurs is $X_1^g \supset X_2^g$. It may happen that this is the same as $X_1 \supset X_2$, i.e. $X_1^g = X_1$, but then $g \in N_G(X_1)$ and $\prec_X$ is preserved under conjugation by $g$. That is, since $1 \prec x$, $x_2 \prec_X x_2 x$ and hence $x_2 \prec_X x_2 x^g$. If $X_1^g \neq X_1$, then the order $\prec_X$ on $X_1^g \supset X_2^g$ is compatible with that on $X_1 \supset X_2$ (from the construction) and hence, since $x_2 \prec_X x_2 x$, $x_2^g \prec_X x_2^g x_2 x^g$,

i.e. $1 \prec x^g$.

Now, if $C$ is a convex subgroup in $\prec$, then we may define $C_1$ to be the intersection of all members of $\mathcal{D}$ which contain $C$ and $C_2$ to be the union of all members of $\mathcal{D}$ which are contained in $C$. Then by (i) and (iii), $C_1$ and $C_2$ are members of $\mathcal{D}$ and, since all the members of $\mathcal{D}$ are convex in $\prec$, it follows that $C_1 \supset C_2$ is actually a jump in $\mathcal{D}$. Then, as remarked at the end of Section 1, $C/C_2$ is a convex subgroup of the order $\prec_C$ on $C_1/C_2$. But $\prec_C$ was chosen to be Archimedean and so $C = C_1$ or $C = C_2$ - in fact $C = C_1 = C_2$. 
There are many examples of ordered groups and we shall see some of them later (Section 5) but for now we will merely give one example which is of interest because it shows that convex subgroups need not be normal in $G$. (Cf. Section 1, p. 17). This example was listed by Iwasawa in [9].

**Example 4** Let $G$ be the group of all monotone-increasing real functions $f$ on the interval $[0,1]$ for which $f(0) = 0$ and $f(1) = 1$, where $fg$ is defined by $(fg)(x) = f(g(x))$. The identity function in $G$ is the function $e(x) = x$. We define an order on $G$ by $e < f$ if and only if $f(x) < x$ for all $x < y$ and $f(y) > y$, where $0 < y < 1$. Then it is easy to show that the sets $G_\lambda = \{ f; f(x) = x \text{ for } 0 \leq x \leq 1-\lambda \}$ are convex subgroups of $G$ and that they are not normal in $G$.

Returning now to the general case of a right-order on a coset space $G:H$, we remark first that there is no obvious equivalent to Theorem 1. There seems to be no reasonable group theoretic property which would ensure that a right-order on $G:H$ has no non-trivial proper convex subgroups. For instance, Examples 1 and 2 show that the non-existence of non-trivial proper convex subgroups does not necessarily imply that $G$ is Abelian. The best we can do towards an equivalent of Theorem 2 is little more than a generalization of Lemma 1.3.

**Theorem 3** The system $\mathcal{O}$ of subgroups of the group $G$ containing $H$ is the system of convex subgroups in some right-order on $G:H$ if and only if it satisfies the following four conditions.
(i) $\mathcal{D}$ and $G$ are in $\mathcal{O}$.
(ii) $\mathcal{D}$ is totally ordered by inclusion.
(iii) $\mathcal{D}$ is closed under arbitrary intersections and unions.
(vii) If $A \supset B$ is a jump in $\mathcal{D}$, then there is a right-order on $A:B$ which has no non-trivial proper convex subgroups.

**Proof** The convex subgroups of a right-order certainly satisfy these conditions.

Conversely to define a right-order on $G:H$ we use the same technique as in the proof of Theorem 2. That is, each element $x \in G \setminus H$ lies in some jump $X_1 \supset X_2$ of $\mathcal{D}$ and we define $1 < x$ if and only if $X_2 < X_\times X_2 x$ in the right-order $<_X$ on $X_1: X_2$ given by condition (vii). The verification goes in exactly the same way as that in the proof of Theorem 2.

Thus it is not to be expected in general that the convex subgroups of a right-order would satisfy conditions (iv), (v) and (vi) of Theorem 2. In Example 2 we saw that $H = \{(0,m)\}$ is a convex subgroup of $G$ and the induced order $< \mathcal{X} \times$ on $G:H$ is Archimedean.

It follows that $< \mathcal{X}$ has no non-trivial proper convex subgroups and hence that $< \mathcal{X}$ has no convex subgroups between $G$ and $H$. That is, $G \supset H$ is a jump. Since $H$ is isomorphic to the additive group of integers, it also is Archimedean ordered by $< \mathcal{X}$ and hence $H \supset \{1\}$ is also a jump. Since $H$ is not normal in $G$, none of the conditions (iv), (v) and (vi) is satisfied by this example.

Conditions (v) and (vi) relate particularly to the two-sidedness of the order and a right-order satisfying them will automatically be an order. However a right-order may satisfy condition (iv) without necessarily being an order. It is of interest to look for such right-orders. They were first studied by Paul Conrad [4] but were not given a special name by him. We will call them generalized polyorders.
Thus a generalized polyorder on a coset space $G:H$ is a right-order on $G:H$ which is such that, if $A \supset B$ is a jump in the system of convex subgroups, then $B$ is normal in $A$ and $A/B$ is order-isomorphic to a subgroup of $\mathbb{R}$. In particular this means that in a generalized polyorder there is a normal system of convex subgroups in which the factors are ordered. For a polyorder on $G:H$ we require in addition that the normal system be finite.

Thus a polyorder on $G:H$ is a right-order on $G:H$ which has a finite series of convex subgroups $H = C_0 < C_1 < \ldots < C_r = G$ such that the right-orders induced on the coset spaces $C_i / C_{i-1}$ ($i = 1, \ldots, r$) are all two-sided. [It then follows that the series is a normal series.] Clearly a polyorder is automatically a generalized polyorder.

In [2], I originally introduced the term polyorder for what is here called a generalized polyorder. Upon reflection I now think that the present terminology is more appropriate and accords better with the usual convention for the prefix "poly" - see for example the definition of polycyclic.

The significance of generalized polyorders is that, for them, as for orders, the property of being Archimedean is equivalent to the non-existence of non-trivial proper convex subgroups. For, if there are no non-trivial proper convex subgroups, then $G \supset \{1\}$ becomes a jump in the system of convex subgroups and so $G$ is order-isomorphic to a subgroup of $\mathbb{R}$. It follows that the order is Archimedean. We may also show for a generalized polyorder that, if $Hx \gg Hy$, then $x \notin C(y)$. For this we need a lemma like Lemma 2.2.
Lemma 2.3  Let \(<\) be a generalized polyorder on \(G:H\). Then the convex hull \(C(a)\) of an element \(a \in G\) consists of all those elements \(x \in G\) such that \(Ha^m \leq Hx \leq Ha^n\) for some integers \(m\) and \(n\).

Proof  Let \(A_1 \supset A_2\) be the jump in the system of convex subgroups where \(a\) occurs. \((A_1 = C(a))\). Then \(A_2\) is normal in \(A_1\) and \(A_1/A_2\) is order-isomorphic to a subgroup of \(\mathbb{R}\). Thus \(A_1/A_2\) is Archimedean ordered and, if \(x\) is any element of \(A_1\), then there are integers \(m\) and \(n\) such that \(A_2a^m \leq A_2x \leq A_2a^n\). But this is equivalent to saying that \(Ha^m \leq Hx \leq Ha^n\), and it follows that all elements of \(A_1\) are in the set described. Conversely it is clear, since \(A_1\) is convex, that every element in the set is in \(A_1\). Hence \(A_1 = C(a)\) is equal to the given set.

Corollary  If \(<\) is a generalized polyorder on \(G:H\) and \(Hx \gg Hy\), then \(x \notin C(y)\).

Proof  By the lemma, if \(x \in C(y)\), then there are integers \(m\) and \(n\) such that \(Hy^m \leq Hx \leq Hy^n\) and also integers \(m'\) and \(n'\) such that \(Hy^{m'} \leq Hx^{-1} \leq Hy^{n'}\). Thus neither \(Hx\) nor \(Hx^{-1}\) is greater than all cosets \(Hy^n\) for all \(n\) and so neither is greater than all cosets \(Hg\) for all \(g \in \langle H,y \rangle\). That is, \(Hx \not\gg Hy\) and we have a contradiction.

In Example 2 (p. 20) the induced right-order \(<\) on the given coset space \(G:H\) is Archimedean. It therefore follows, in that example, that, if \(Hx \gg Hy\), then \(Hy = H\) and certainly \(x \notin C(y)\). We note however that \(<\) is not a generalized polyorder. In other words, the condition "\(Hx \gg Hy\) implies \(x \notin C(y)\)" is not sufficient on its own to ensure that a right-order \(<\) is actually a generalized polyorder.
Whether it is sufficient in the case when \( H \) is normal in \( G \) or, more particularly, when \( H = \{1\} \) is not clear and that remains an open question. However it is sufficient when used in conjunction with another simple condition of the same type.

**Lemma 2.4** The right-order \( < \) on \( G:H \) is a generalized polyorder if and only if

(i) \( Hx \gg Hy \) implies \( x \notin C(y) \)

and (ii) for all \( x \) and \( y \) in \( G \) either \( Hx = Hy = H \) or \( Hx \gg H[x,y] \) or \( Hy \gg H[x,y] \),

where \( [x,y] = x^{-1}y^{-1}xy \).

**Proof** We have already seen that a generalized polyorder satisfies condition (i). For (ii), we look for the jump \( A \supset B \) in the system of convex subgroups where the largest of \( Hx, Hx^{-1}, Hy, Hy^{-1} \) occurs. If at least one of \( x \) and \( y \) is not in \( H \), then \( A \neq H \). In that case \( B \) is normal in \( A \) and \( A/B \) is Abelian. Hence \( [x,y] \) is in \( B \) and, whichever of \( x \) and \( y \) is not in \( B \), we have

\( Hx \gg H[x,y] \) or \( Hy \gg H[x,y] \).

Conversely, suppose \( A \supset B \) is a jump in the system of convex subgroups and \( x,y \in A \). Then by (ii) either \( Hx \gg H[x,y] \) or \( Hy \gg H[x,y] \) or \( Hx = Hy = H \). Hence by (i) either \( x \notin C([x,y]) \) or \( y \notin C([x,y]) \) or \( [x,y] \in H \). In all cases it follows that \( C([x,y]) \) is contained in \( B \) and, in particular, that \( [x,y] \in B \).

Hence \( B \) is normal in \( A \) and \( A/B \) is Abelian. The induced right-order on \( A/B \) is therefore two-sided and by (i) it is Archimedean. By Theorem 1, \( A/B \) is order-isomorphic to a subgroup of \( \mathbb{R} \) as required. ||
The conditions of this lemma are of interest in themselves and are of particular use when dealing with nilpotent groups where the commutator structure is especially relevant. However, for other applications, it is convenient to have a single condition necessary and sufficient for a right-order to be a generalized polyorder.

In [4] Paul Conrad was able to give such a condition in the case of right-orders on the group $G$. Explicitly he proved that a right-order $<$ on the group $G$ is a generalized polyorder if and only if, for any two positive elements $x$ and $y$, there is a positive integer $n$ such that $(xy)^n > yx$. This condition may be generalized to deal with right-orders on a coset space $G:H$ and we will state the theorem in this generalized form.

**Theorem 4** Let $<$ be a right-order on the coset space $G:H$ with positive cone $P$. Then $<$ is a generalized polyorder if and only if, given any two elements $x, y$ in $P \cup H$, there is a positive integer $n$ such that $(xy)^n (yx)^{-1} \in P \cup H$.

**Proof** A careful analysis of Conrad's proof shows that the same method suitably modified will work in this more general situation. It is only necessary in the calculations to remember that $P$ is an $H$-subsemigroup and that, if $C$ is a convex subgroup, then $Q = P \cap (G \setminus C)$ is the positive cone of a right-order on $G:C$, i.e. in particular, $Q$ is a $C$-subsemigroup.

We will not reproduce these lengthy calculations here but will merely observe that the most significant step is the determination of the convex hull of an element $a \in P$. Under the condition of the theorem, it can be shown that the set
$S = \{x \in P \cup H : x^n x^{-1} \in P \cup H \text{ for some } n > 0\}$

is a semigroup and that then $T = S \cup S^{-1}$ is a convex subgroup.

In fact $T$ is the convex hull of $a$.

Finally in this section we have a theorem which has some bearing on the problem of deciding when a generalized polyorder is actually a polyorder. The lemmas leading up to this theorem are entirely group theoretic and deal with a few standard results concerning soluble groups.

A group $G$ is soluble if it has a finite normal series with Abelian factors. We will define the rank $r(G)$ of a soluble group $G$ to be the sum of the ranks of the factors in a normal series with Abelian factors. Of course to make sense this definition must be independent of the choice of normal series and the proof of that is the content of the first lemma.

**Lemma 2.5** If (A) $G > A_1 > \ldots > A_r > A_{r+1} = \{1\}$

and (B) $G > B_1 > \ldots > B_s > B_{s+1} = \{1\}$

are normal series for $G$ with Abelian factors, then the sum of the ranks of the factors in (A) is the same as the sum of the ranks of the factors in (B).

**Proof** For an Abelian group $G$ with subgroup $H$, it is a standard result that $r(G) = r(G/H) + r(H)$. Hence if we refine say (A) by inserting extra terms between some of the subgroups $A_i$, the sum of the ranks will remain unchanged. By the Schreier Refinement Theorem (Kurosch, Vol.1, p.111), the series (A) and (B) have isomorphic refinements and it follows that they have the same sum of ranks.
Corollary If $H$ is a normal subgroup of the soluble group $G$, then $r(G) = r(G/H) + r(H)$.

Proof The derived series of $G/H$ and $H$ may be combined to give a normal series for $G$ with Abelian factors in which $H$ is one of the terms. The corollary follows by application of the lemma.

Lemma 2.6 Suppose $H$ is a proper isolated subgroup of the soluble group $G$, then $r(H) < r(G)$.

Proof We use induction on the derived length of $G$. When $G$ has derived length 1, it is Abelian and the lemma is known in that case.

Assume then that the lemma is true for all soluble groups of derived length less than that of $G$. Let $G'$ be the derived group of $G$ and consider the subgroups $H \cap G'$ and $HG'$.

By the Corollary to Lemma 2.5, we have

$$r(H) = r(H/H \cap G') + r(H \cap G')$$

$$= r(HG'/G') + r(H \cap G')$$

(Using the Isomorphism Theorem.)

If $H \cap G' = G'$, then $H/G'$ is a proper isolated subgroup of $G/G'$ and, since $G/G'$ is Abelian, we have that $r(H/G') < r(G/G')$.

It follows that

$$r(H) < r(G/G') + r(G') = r(G).$$

If $H \cap G' \neq G'$, then $H \cap G'$ is a proper isolated subgroup of $G'$ and so, by induction hypothesis, $r(H \cap G') < r(G')$. Again it follows that

$$r(H) < r(HG'/G') + r(G') < r(G/G') + r(G') = r(G).$$
The condition that $H$ be isolated in $G$ is not absolutely necessary in this lemma - it would be sufficient for example to postulate the existence of an element $g \in G$ with $g^n \notin H$ for all $n$ - but, in the application envisaged here, the subgroups involved are actually isolated in $G$.

**Theorem 5** Suppose $G$ is a soluble group for which $r(G)$ is finite. Then any right-order on a coset space $G:H$ can have at most $r(G) - r(H)$ proper convex subgroups.

**Proof** The crux of the proof lies in the observation that a convex subgroup $C$ must be isolated in $G$. We can either see this as a consequence of the third Corollary to Lemma 1.6, or we can prove it more directly by observing that, if $g$ is a positive element, and $g^n \in C$ for some $n$, then $1 < g < g^n$ implies $g \in C$. By Lemma 2.6 it follows that $r(C) < r(G)$. At the same time, since $H$ is a subgroup of $C$, we have $r(H) \leq r(C)$. Thus the convex subgroups form a series with strictly decreasing ranks in the range $r(H)$ to $r(C)$. Since $r(G)$ is finite, it is clear there can be at most $r(G) - r(H)$ proper convex subgroups.

As a direct consequence of this theorem we have the following result.

**Corollary** If $G$ is a soluble group with finite rank, and $<$ is a generalized polyorder on $G$, then $<$ is actually a polyorder.

**Proof** The convex system is a normal series with Abelian and, therefore, ordered factors.
3. Extensions of Partial (Right-) Orders

In this section we consider the problem of extending partial (right-) orders on coset spaces. Our main objective is to derive the standard criterion for a given partial (right-) order to be extended to a total (right-) order. This will be needed in Sections 4, 5 and 6.

Naturally we say that a partial (right-) order $\mathcal{R}_1$ on a coset space $G:H$ is an extension of the partial (right-) order $\mathcal{R}_2$ on $G:H$ if $\mathcal{R}_1$ contains $\mathcal{R}_2$. Or, in terms of positive cones, $P_1$ contains $P_2$.

L. Fuchs [6] has given a criterion for a partial order on $G$ to be extendable to a total order on $G$ (see Fuchs, p. 34). This is easily generalized to our present situation by making use of the concept of the (normal) $H$-subsemigroup generated by a subset $T$ of $G$ introduced on p. 13.

**Theorem 6** A partial (right-) order on the coset space $G:H$ with positive cone $P$ can be extended to a total (right-) order on $G:H$ if, and only if, for every finite set of elements $a_1, \ldots, a_n$ from $G\setminus H$, there are exponents $\varepsilon_1, \ldots, \varepsilon_n$ ($= \pm 1$) such that

\[
1 \notin S_H(P, a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}) \quad \text{(for right-orders)}
\]

or

\[
1 \notin \overline{S}_H(P, a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}) \quad \text{(for orders)}.
\]

**Proof** We will set out the proof for the case of partial right-orders. The proof for two-sided relations is exactly the same except that the semigroups involved have to be normal in $G$ and we use $\overline{S}_H(T)$ in place of $S_H(T)$. 
If the partial right-order with positive cone $P$ can be extended to a total right-order on $G:H$ with positive cone $Q$ say, then $P \subseteq Q$, $Q$ is an $H$-subsemigroup, $1 \notin Q$ and $H \cup Q \cup Q^{-1} = G$.

Thus, for any element $a \in G \setminus H$, either $a \in Q$ or $a^{-1} \in Q$.

Given any finite set $a_1, a_2, \ldots, a_n \in G \setminus H$, we choose exponents $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ ($\neq \pm 1$) such that $a_i^{\varepsilon_i} \in Q$ for all $i = 1, 2, \ldots, n$.

Then $S_{H}(P, a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}) \subseteq Q$ and, since $1 \notin Q$, we have $1 \notin S_{H}(P, a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n})$.

Now suppose that $P$ is an $H$-subsemigroup which has the property that, for any finite set of elements $a_1, a_2, \ldots, a_n \in G \setminus H$, there is a choice of exponents $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ such that

$$1 \notin S_{H}(P, a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}). \quad (*)$$

We first show that, if $a \in G \setminus (H \cup P \cup P^{-1})$, then at least one of the two $H$-subsemigroups $S_{H}(P, a)$ and $S_{H}(P, a^{-1})$ also has property $(*)$. For, if not, then there would be elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ in $G \setminus H$ such that

$$1 \in S_{H}(P, a, a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n})$$

and

$$1 \in S_{H}(P, a^{-1}, b_1^{n_1}, \ldots, b_m^{n_m})$$

for all possible choices of the exponents $\varepsilon_1, \ldots, \varepsilon_n, n_1, \ldots, n_m$.

But then $1 \in S_{H}(P, a^{\varepsilon}, a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}, b_1^{n_1}, \ldots, b_m^{n_m})$ for all choices of the exponents $\varepsilon, \varepsilon_1, \ldots, \varepsilon_n, n_1, \ldots, n_m$ ($\neq \pm 1$), contradicting property $(*)$ for $P$.

Next we use Zorn's Lemma to establish the existence of maximal partial right-orders extending the given partial right-order and with positive cones still satisfying property $(*)$. We need to show that all ascending chains of $H$-subsemigroups $P \subseteq P_1 \subseteq \cdots \subseteq P_\lambda \subseteq \cdots$, where $1 \notin P_\lambda$ and $P_\lambda$ has property $(*)$ for every $\lambda$, are bounded.

We put $M = \bigcup_\lambda P_\lambda$ and verify that $M$ is the required bound.
It is an H-subsemigroup because, if \( m_1 \) and \( m_2 \) are in \( M \) and \( h \in H \), then \( m_1 \in P_{\lambda_1} \) and \( m_2 \in P_{\lambda_2} \) for some \( \lambda_1 \) and \( \lambda_2 \).

Putting \( \lambda = \max(\lambda_1, \lambda_2) \), \( m_1 \in P_{\lambda} \) and \( m_2 \in P_{\lambda} \) and hence

\[ m_1 m_2 \in P_{\lambda} \subseteq M, \quad hm_1 \in P_{\lambda} \quad \text{and} \quad m_1 h \in P_{\lambda} \quad \text{for all} \quad h \in H. \]

Thus \( M \) is a semigroup and \( hm \in M, \quad mh \in M \quad \text{for all} \quad h \in H, \)
i.e. \( M \) is an H-subsemigroup. \[ M \text{ is normal in } G \text{ if all } P_{\lambda} \text{'s are normal in } G. \]

Also, since \( 1 \notin P_{\lambda} \) for any \( \lambda, \quad 1 \notin M. \)

Now \( M \) has property (*) . For, if not, there would be a set of elements \( a_1, \ldots, a_n \in G \setminus H \) such that \( 1 \in S_H(N, a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n}) \) for all possible choices of the exponents \((\epsilon = \pm 1)\).

Corresponding to each of the \( 2^n \) choices for \( \epsilon_1, \ldots, \epsilon_n \), there would then be a finite number of factors from \( M \) which, together with factors from \( H \) and \( \{a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n}\} \), would be in a product equal to the identity. The union of these \( 2^n \) finite sets of elements from \( M \) is obviously still finite and so is contained in one of the semigroups, say \( P_{\lambda} \), in the chain. But then \( 1 \in S_H(P_{\lambda}, a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n}) \) for all choices of the exponents and we have a contradiction of property (*) for \( P_{\lambda} \).

By Zorn’s Lemma we deduce that there are maximal extensions of the given partial right-order \( P \) which satisfy property (*).

Suppose that \( M \) is the positive cone of one of these. We show that \( M \cup M^{-1} \cup H = G \).

If not, then there is an element \( a \in G \) not in \( M \cup M^{-1} \cup H \).

But then, by the same argument as above, either \( S_H(N, a) \) or \( S_H(N, a^{-1}) \) does not contain the identity and still satisfies property (*).

This contradicts the maximality of \( M \) and we deduce that \( M \cup M^{-1} \cup H = G \).

Thus \( M \) represents a total right-order on \( G:H \).

By starting with the trivial partial (right-) order in this theorem we obtain the standard criterion for the existence of a total (right-) order on \( G:H \) which, in the case when \( H \) is the identity subgroup,
reduces to the criterion for a total (right-) order on $G$.

**Corollary** There is a (right-) order on the coset space $G : H$ if and only if, for every finite set of elements $a_1, \ldots, a_n \in G \setminus H$, there is a choice of the exponents $\epsilon_1, \ldots, \epsilon_n (\neq \pm 1)$ such that

$$1 \not\in S_H(a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n}) \quad \text{(right-order)}$$

or

$$1 \not\in S_H(a_1^{\overline{\epsilon_1}}, \ldots, a_n^{\overline{\epsilon_n}}) \quad \text{(order)}$$

In particular, a necessary condition for the right-orderability of $G : H$ is that $H$ should be isolated in $G$. For, if $H$ is not isolated, then there is an element $a \in G \setminus H$ such that $a^k \in H$ for some integer $k$. It is then clear that $1 \in S_H(a)$ and $1 \in S_H(a^{k-1})$ and $G : H$ cannot be right-ordered. This means that a right-ordered group must be torsion-free. We will have more to say about this latter observation in later sections (4 and 5).

As remarked at the end of Lemma 1.2, if there is a two-sided order on $G : H$, then $H$ is actually normal in $G$ and we are really dealing with an order on the factor group $G/H$. In other words, for two-sided orderings, there is no significant difference between the theory for coset spaces and the theory for orders on $G$ itself.

Theorem 6 is of central importance in the next section, but before going on to that application we should first say something about the problem of extending partial (right-) preorders.

In one sense this is a trivial exercise since the trivial (right-) preorder, whose positive cone is $G$, is always an extension of any partial (right-) preorder on $G : H$. However the problem becomes less trivial if instead we ask for an extension which does not increase the intersection $K = S \cap S^{-1}$ of the positive cone $S$ with its inverses (see Lemma 1.1). But then, by Lemma 1.1, we are really
dealing with partial (right-) orders on $G:H$ and the criterion for extending these is already given by Theorem 5.

For (right-) preorders it is more relevant to look for "refinements" rather than extensions. A (right-) preorder $\mathcal{R}_1$ on $G:H$ is said to be a refinement of the (right-) preorder $\mathcal{R}_2$ on $G:K$ if $H \leq K$ and, whenever $(Hg_1, Hg_2) \in \mathcal{R}_1$, then $(Kg_1, Kg_2) \in \mathcal{R}_2$. More simply in terms of their positive cones $S_1$ and $S_2$, $\mathcal{R}_1$ is a refinement of $\mathcal{R}_2$ if $S_1$ is contained in $S_2$. Roughly speaking $\mathcal{R}_1$ distinguishes between more pairs of elements than $\mathcal{R}_2$ and is nearer to ordering all the elements of $G$.

We ask under what conditions can a given (right-) preorder on $G:H$ be refined to a total (right-) order on $G:H$. Theorem 6 provides the answer to this question also. For, again following the notation of Lemma 1.1, if $P = S \setminus K$, then $P$ is the positive cone of a partial (right-) order on $G:H$ and refining $S$ is equivalent to extending $P$ to a total (right-) order, $F$ say, on $G:H$. This is where Theorem 6 could be applied. However it is worth carrying the discussion just a little further before appealing to the theorem.

The intersection $Q = F \cap K$ is an $H$-subsemigroup and in fact it is the positive cone of a total (right-) order on $K:H$. We observe that $F = P \cup Q$. This observation refers us back to Lemma 1.4 and we see that the (right-) preorder with positive cone $S$ can be refined to a total (right-) order on $G:H$ if and only if there is a total (right-) order on $K:H$ which, for the two-sided case, must also be normal in $G$. The condition for this is again provided by Theorem 6.
4. **Local Properties**

The main consequence of Theorem 6 is that (right-) orderability of a group is a local property. That is, a group \( G \) can be (right-) ordered if and only if every finitely generated subgroup of \( G \) can be (right-) ordered. The purpose of this section is to present this result and to compare various other local conditions.

Specifically the conditions to be considered are as follows.

0. \( G \) can be ordered.

LO. \( G \) can be locally ordered, i.e. every finitely generated subgroup of \( G \) can be ordered.

L pre 0. \( G \) can be locally preordered, i.e. every non-trivial finitely generated subgroup of \( G \) has a non-trivial preorder.

LI. \( G \) is locally indicable, i.e. every non-trivial finitely generated subgroup has a normal subgroup with infinite cyclic factor group.

RO. \( G \) can be right-ordered.

LRO. \( G \) can be locally right-ordered.

LR pre 0. \( G \) can be locally right-preordered.

The last two are interpreted in the same way as the corresponding conditions for orders. The condition LI is a concept which was first introduced and studied by Graham Higman [8] and, as we shall see, it is of some relevance in the present context.

The first result is the one mentioned at the beginning of this section. It is attributed (by Fuchs) to B. H. Neumann [14] for the case of ordered groups, but the main part of the work had already been done.
by Los [11]. Lorenzen [10] had a similar theorem at the same time. Here we may generalize the result to include the case of (right-) orders on coset spaces.

**Theorem 7** There is a (right-) order on the coset space $G:H$ if and only if, for every finitely generated subgroup $G_1$ of $G$, there is a (right-) order on the coset space $G_1 \triangleleft H_1$, where $H_1 = H \cap G_1$.

**Proof** If there is a (right-) order on $G:H$ with positive cone $P$ say, and $G_1$ is a subgroup of $G$, then it is easy to see that $P_1 = P \cap G_1$ is the positive cone of a (right-) order on $G_1 \triangleleft H_1$.

Conversely suppose there is a (right-) order on every finitely generated coset space $G_1 \triangleleft H_1$ as in the theorem and suppose that $G:H$ cannot be (right-) ordered. Then there must be a finite set of elements $a_1, \ldots, a_n \in G \setminus H$ such that, for all choices of $\varepsilon_1, \ldots, \varepsilon_n$ ($= \pm 1$),

$$1 \in S_{H}(a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n})$$

or

$$1 \in S_{H}(a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n})$$

(two-sided).

In the case of right-orders, this means that, for each of the $2^n$ choices of $\varepsilon_1, \ldots, \varepsilon_n$, there is a product of the elements $a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}$ and elements of $H$ which is equal to the identity element. Let $G_1$ be the subgroup of $G$ which is generated by $a_1, \ldots, a_n$ and all those elements of $H$ which appear in any of these products. Then $G_1$ is certainly finitely generated and $H_1 = H \cap G_1$ contains all the elements of $H$ in the products. Hence we still have in $G_1$ that

$$1 \in S_{H_1}(a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n})$$

for all choices of $\varepsilon_1, \ldots, \varepsilon_n$. This contradicts the assumption that $G_1:H_1$ can be right-ordered and we conclude that $G:H$ can also be right-ordered.
In the case of two-sided orders, we have $2^n$ products of conjugates of $a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n}$ and elements of $H$ ($H$ is normal in $G$) each equal to the identity. This time we let $G_1$ be the subgroup generated by $a_1, \ldots, a_n$, all the elements of $H$ which occur in any of the products and all the elements of $G$ which appear as conjugators in the products. Again $G_1$ is finitely generated and in $G_1$ we have 
$$1 \in \mathcal{G}_{H_1}(a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n})$$
for all choices of $\epsilon_1, \ldots, \epsilon_n$. This gives the same contradiction as before and this time we deduce that $G:H$ can be ordered.

For the conditions in our list, it follows from this theorem that $0$ and $LO$ are equivalent as are $RO$ and $LRO$.

If the group $G$ can be right-ordered, then it is clear that $G$ can also be (right-) preordered. For, if $P$ is the positive cone of a (right-) order on $G$, then $P \cup \{1\}$ is the positive cone of a (right-) preorder on $G$. It follows that $0$ implies $L$ pre-$0$ and $RO$ implies $LR$ pre-$0$.

What about the converses to these two implications? First local preorderability does not imply orderability. Example 1, (p./13) provides an example to illustrate this. This group $G$ is generated by elements $a$ and $b$ with the relation $bab^{-1} = a^{-1}$. In any order on $G$, if $a$ were positive, then $bab^{-1} = a^{-1}$ would also have to be positive and, if $a$ were negative, $bab^{-1} = a^{-1}$ would have to be negative. Thus there can be no order on $G$.

However $G$ is locally preorderable. If $A$ is the subgroup of $G$ generated by the element $a$, then $A$ is a normal subgroup of $G$ and $G/A$ and $A$ are infinite cyclic groups. Now, if $H$ is any
non-trivial finitely generated subgroup of \( G \), then \( H / (H \cap A) \) is isomorphic to \( HA / A \). If \( H \) is not contained in \( A \), then \( H / (H \cap A) \) is an infinite cyclic group and any order on this will correspond to a non-trivial preorder on \( H \). (Lemma 1.1) If \( H \) is contained in \( A \), then \( H \) itself is an infinite cyclic group and so can actually be ordered.

The situation is different however when we come to consider right-orders and indeed it turns out that local right-preorderability is equivalent to right-orderability. In fact we can even generalize this to deal with the case of right-ordered coset spaces following the pattern of Theorem 6.

Theorem 8 There is a right-order on the coset space \( G: H \) if and only if, for every finitely generated subgroup \( G_1 \) of \( G \) with \( G_1 \) not contained in \( H \), there is a non-trivial right-preorder on \( G_1 : H_1 \), where \( H_1 = H \cap G_1 \).

Proof If there is a right-order on \( G: H \), then, as we saw in Theorem 6, there is a right-order on \( G_1 : H_1 \) for any subgroup \( G_1 \). By Lemma 1.1 and since \( G_1 \not\subseteq H_1 \), this right-order corresponds to a non-trivial right-preorder on \( G_1 : H_1 \).

Conversely suppose that, for every finitely generated subgroup \( G_1 \) of \( G \) with \( G_1 \not\subseteq H \), there is a non-trivial right-preorder on \( G_1 : H_1 \). We show by induction on \( n \) that, for every finite set of elements \( a_1, \ldots, a_n \in G \setminus H \), there is a choice of exponents \( \varepsilon_1, \ldots, \varepsilon_n \) (± 1) such that \( 1 \not\in S_\varepsilon(a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}) \).
Suppose that \( n \) is the least positive integer for which there is a set \( a_1, \ldots, a_n \) of elements in \( G \setminus H \) such that

\[ 1 \in S_H(a_1^{e_1}, \ldots, a_n^{e_n}) \]

for all choices of the \( e \)'s.

As in the proof of Theorem 7, we let \( G_1 \) be the subgroup of \( G \) generated by \( a_1, \ldots, a_n \) and all the elements of \( H \) which appear in the \( 2^n \) products equal to the identity element. Then, as before, \( H_1 \) also contains all those elements of \( H \) and in \( G_1 \) we have that

\[ 1 \in S_{H_1}(a_1^{e_1}, \ldots, a_n^{e_n}) \]

for all choices of the \( e \)'s.

Now by assumption there is a non-trivial right-preorder on \( G_1 : H_1 \).

By Lemma 1.1, this corresponds to a right-order on a coset space \( G_1 : K \) where \( H_1 \triangleleft K \) and \( K \) is a proper subgroup of \( G_1 \). Let \( P \) be the positive cone of this right-order. Then \( P \) is a \( K \)-subsemigroup of \( G_1 \) such that \( P \cup P^{-1} \cup K = G_1 \) and \( 1 \not\in P \).

Since \( K \not\triangleleft G_1 \), not all the elements \( a_1, \ldots, a_n \) can be in \( K \).

Those which are not in \( K \) are in \( P \cup P^{-1} \) and hence, by renumbering them if necessary, we may assume that \( a_1^{e_1}, \ldots, a_r^{e_r} \in P \) for some exponents \( e_1, \ldots, e_r \) and some \( r \geq 1 \) and \( a_{r+1}, \ldots, a_n \in K \).

By induction hypothesis, there is a choice of exponents

\( e_{r+1}, \ldots, e_n \) such that \( 1 \not\in S_{H_1}(a_{r+1}^{e_{r+1}}, \ldots, a_n^{e_n}) \). It follows, since \( H_1 \) is contained in \( H \), that \( 1 \not\in S_{H_1}(a_{r+1}^{e_{r+1}}, \ldots, a_n^{e_n}) \) also.

Now, because \( P \) is a \( K \)-subsemigroup, every element of \( S_{H_1}(a_1^{e_1}, \ldots, a_n^{e_n}) \)

must either be in \( P \) (if any of \( a_1^{e_1}, \ldots, a_r^{e_r} \) are involved) or in \( S_{H_1}(a_{r+1}^{e_{r+1}}, \ldots, a_n^{e_n}) \). Now, by assumption, \( 1 \in S_{H_1}(a_1^{e_1}, \ldots, a_n^{e_n}) \) and, by construction, \( 1 \not\in S_{H_1}(a_{r+1}^{e_{r+1}}, \ldots, a_n^{e_n}) \) and so it follows that

\[ 1 \in P \] - a contradiction.
Hence $G$ satisfies the condition of Theorem 6 and there is a right-order on $G:H$.

The condition LI of local indicability comes into the picture in connection with local preorderability. In fact these two conditions are equivalent.

**Theorem 9** The group $G$ is locally indicable if and only if every non-trivial finitely generated subgroup of $G$ has a non-trivial preorder.

**Proof** Suppose $G$ is locally indicable. Then every non-trivial finitely generated subgroup $G_1$ has a normal subgroup $H_1$ with $G_1/H_1$ isomorphic to the additive group of integers. Thus there is an order on $G_1/H_1$ and, by Lemma 1.1, this corresponds to a non-trivial preorder on $G_1$.

Conversely suppose that every non-trivial finitely generated subgroup of $G$ has a non-trivial preorder and let $G_1$ be the subgroup generated by the elements $a_1, \ldots, a_n$. By Lemma 1.1, any non-trivial preorder on $G_1$ corresponds to an order $<$ on a coset space $G_1:H$ of $G_1$, where $H$ is a proper normal subgroup of $G_1$. Replacing generators by their inverses and renumbering if necessary, we may assume that $Ha_1 \geq Ha_2 \geq \cdots \geq Ha_n \geq H$.

Clearly $Ha_1 \neq H$ for otherwise we would have all the generators $a_1, \ldots, a_n$ in $H$ and hence a contradiction that $H = G_1$. Equally clearly, the convex hull of $a_1$ in the order $<$ must be $G_1$.

Let $C$ be the union of all convex subgroups in $<$ which do not contain $a_1$. Then $G_1 \supset C$ is a jump in the system of convex subgroups and it follows that $C$ is normal in $G_1$ and $G_1/C$ is order-isomorphic to a subgroup of $\mathbb{R}$. Since $G_1$ is finitely generated, $G_1/C$ is also finitely generated. Hence $G_1/C$ is a finitely generated torsion free Abelian group and it follows that it is actually a free Abelian group.
Such a group clearly has a normal subgroup with factor group isomorphic to the additive group of integers.

Thus $G$ is locally indicable.

Trivially a group which is locally preorderable is also locally right-preorderable and so an interesting consequence of this theorem and Theorem 8 is that a locally indicable group can be right-ordered. It is not clear whether the converse of this is true but a closer examination of the second part of the proof of Theorem 9 shows that the condition

\[ \text{LGP} \quad \text{Every finitely generated subgroup of } G \text{ has a generalized polyorder.} \]

is sufficient to make the group locally indicable. For, if $<$ is a generalized polyorder on the subgroup $G_1$ generated by $a_1, \ldots, a_n$, then we may assume $a_1 \geq a_2 \geq \ldots \geq a_n \geq 1$, and the jump in the convex system where $a_1$ occurs will be $G_1 \supset C$ for some $C$. The argument then proceeds just as before. Indeed it is clear from this argument that it would be sufficient for every finitely generated subgroup $G_1$ to have a non-trivial generalized polyorder on some coset space $G_1 \supset H$. However this latter condition is easily seen to be equivalent to local preorderability and so we will not include it as a new concept.

Conrad's Theorem (4) is sufficient to show that a right-order on a group $G$ is a generalized polyorder if and only if its restriction to any finitely generated subgroup is a generalized polyorder. The criterion of the theorem deals with only two elements $x$ and $y$ at a time and, if satisfied in the whole group, it will still be satisfied in the restriction to any subgroup containing $x$ and $y$. Thus the condition \[ \text{LGP} \] is a consequence of the further condition

\[ \text{GP} \quad G \text{ has a generalized polyorder.} \]
The discussion and results of this section may be summarized in the form of an implication diagram as follows.

\[ \begin{array}{c}
\text{O} \leftrightarrow \text{LO} \\
\downarrow \\
\text{GP} \\
\downarrow \\
\text{LGP} \\
\leftrightarrow ? \end{array} \]

\[ \begin{array}{c}
\text{RO} \leftrightarrow \text{LRO} \leftrightarrow \text{LR pre O} \leftrightarrow ? \\
\text{L pre O} \leftrightarrow \text{LI} \\
\end{array} \]

In this a bar indicates that a counterexample has been found to show that the given implication cannot be reversed. A question mark indicates that the particular question is still open. It seems unlikely that RO will imply LGP or LI in general, but, as we shall see in later sections, there will be some positive results in this connection, examples may be hard to come by. There is a possibility I think that LGP may be equivalent to LI or to GP but no proofs are yet forthcoming. Some of these problems will be dealt with for particular classes of groups in the next two sections.
5. **Locally Nilpotent Groups**

A group $G$ is nilpotent if it has a finite upper central series

$$\{1\} = Z_0 \lhd Z_1 \lhd \ldots \lhd Z_r = G$$

in which each factor $Z_i/Z_{i-1}$ $(i=1,\ldots,r)$ is equal to the centre of $G/Z_{i-1}$.

Equivalently, it is nilpotent if it has a finite lower central series

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_s = \{1\}$$

in which $G_{i+1} = [G,G_i]$ for each $i = 0,\ldots,s-1$.

When $G$ is nilpotent and torsion free it is not difficult to show that the factors in the upper central series must all be torsion free. (Kurosh, Vol.2, p.245.) Being Abelian, all the factors can therefore be ordered and orders on the factors may be combined as in Lemma 1.3 to give an order on $G$. However this direct method of proving the orderability of a torsion free nilpotent group misses a lot of the information which may be exposed in this case and we prefer to prove this result by a more revealing method later.

The group $G$ is locally nilpotent if every finitely generated subgroup of $G$ is nilpotent. In view of Theorems 4, 6 and 7, most properties of nilpotent groups in our present context will carry over to locally nilpotent groups.

We start by proving that, if $G$ is a locally nilpotent group and $H$ is an isolated subgroup of $G$, then every partial right-order on $G:H$ can be extended to a right-order on $G:H$. For the case $H$ is the identity subgroup, this result was first proved by me [1] for nilpotent groups of class 2. It was then extended, independently,
by Rhentulla [15] and Formanek [5] to nilpotent groups of arbitrary class and thereby to locally nilpotent groups. Both these authors use much the same method, but Formanek has a clearer presentation of the proof. The proof presented here for the general case $G : H$ follows that of Formanek.

The proof depends on the following lemma.

**Lemma 5.1** Let $G$ be a nilpotent group with upper central series
\[ \{1\} = Z_0 < Z_1 < \ldots < Z_r = G \]
and let $H$ be an isolated subgroup of $G$. Further let $P$ be an $H$-subsemigroup of $G$ such that $1 \not\in P$ and $P \cup P^{-1} \cup H$ contains $Z_i$ for some $i = 0, 1, \ldots, r-1$. Then, if $x$ is an element of $Z_{i+1} \setminus H$ such that $1 \in S_H(P,x)$, then there is a positive integer $m$ such that $x^{-m} \in P$.

**Proof** First we note that the general element of $S_H(P,x)$ may be expressed in the form
\[ p_0^i x^1 p_1 x^2 p_2 \ldots x^k p_k \]  \hspace{1cm} (1)
where each $i_j > 0$ and $p_j$ is either an element of $P$ or is in $H$, i.e. $p_j \in P \cup H$ for all $j = 0, 1, \ldots, k$. This arises from the general formula for the elements of $S_H(T)$ given on p. 13 and from that we may also see that, in the present case, (1) must involve an $x$ factor or at least one factor from $P$, i.e. $k \geq 1$ or, if $k = 0$, then $p_0 \in P$.

Since, by supposition $1 \not\in P$, an expression of the form (1) can therefore only be equal to the identity if it actually involves an $x$ factor. Thus, if $1 \in S_H(P,x)$, then there are expressions (1), with $k \geq 1$, equal to the identity. By a cyclic permutation of the factors, it follows that there are expressions
\[ x^{i_1 p_1} x^{i_2 p_2} \cdots x^{i_k p_k} = 1, \quad (E) \]

where \( k \geq 1 \), \( i_j > 0 \) and \( p_j \in P \cup H \) for \( j = 1, 2, \ldots, k \).

The main part of the proof is to show that there is such an expression \((E)\) with \( k = 1 \).

We associate with each expression \((E)\) an \( r \)-tuple of non-negative integers \( d(E) = (n_1, \ldots, n_r) \), where \( n_1 \) is the number of the factors \( p_j \) in \((E)\) which lie in \( Z_1 \setminus Z_{1-1} \). This may be thought of as a measure of the "complication" in \((E)\). If all the \( p_j \)'s were in \( Z_1 \) for example, \((E)\) would be very uncomplicated. For then the factors would commute and we would have directly \( x^{i_1} \cdots x^{i_k p_1 p_2 \cdots p_k} = 1 \) as required.

We order the \( r \)-tuples lexicographically from the right. That is to say

\[(m_1, \ldots, m_r) < (n_1, \ldots, n_r) \text{ if and only if } m_1 = n_1 \text{ for all } i = s+1, \ldots, r \text{ and } m_s < n_s.\]

This corresponds to the feeling that the more factors there are from higher up the central series, the more complicated is the expression \((E)\).

Now let

\[ x^{i_1 p_1} x^{i_2 p_2} \cdots x^{i_k p_k} = 1 \quad (A) \]

be an expression for which \( d(A) \) takes the least value in the lexicographic order amongst all expressions \((E)\), and let \( s \) be the highest index for which \((A)\) has factors in \( Z_s \setminus Z_{s-1} \), i.e. \( d(A) \) takes the form \((n_1, \ldots, n_s, 0, \ldots, 0)\) with \( n_s \neq 0 \). We show that, if \( k \geq 2 \), then there is an expression \((E)\) with \( d(E) < d(A) \) thereby producing a contradiction.
Amongst all the expressions
\[ x^{j_1q_1} x^{j_2q_2} \ldots x^{j_kq_k} = 1 \]  \hspace{1cm} (B)
with \( d(B) = d(A) \), there will be some with \( q_k \in Z_s \setminus Z_{s-1} \).
For, if \( g_k \) is not already in \( Z_s \setminus Z_{s-1} \), a cyclic permutation of the factors will bring one of the \( n_s \) (\( > 0 \)) factors from \( Z_s \setminus Z_{s-1} \) into the last position. An expression \((B)\) with \( q_k \in Z_s \setminus Z_{s-1} \) is said to be in normal form.

Finally in this initial construction we choose \((B)\) to be an expression in normal form for which \( d(B) = d(A) \) and which amongst all such expressions takes the least value for the exponent \( j_2(>0) \).
Note that this is where the assumption \( k \geq 2 \) is necessary.
Now we try to reduce \( j_2 \) or \( d(B) \) by commuting factors in \((B)\) and introducing commutators where necessary. In particular we consider the commutator \([q_1, x]\).

Since \( x \in Z_{i+1} \), \([q_1, x] \in Z_1 \) and hence by hypothesis \([q_1, x] \in P \cup P^{-1} \cup H \). This leads to two cases.

(i) \([q_1, x] \in P \cup H \). Then, by commuting \( q_1 \) and \( x \) in \((B)\), we obtain an expression
\[ x^{j_1+1} q_1[q_1, x] x^{j_2-1} q_2 \ldots x^{j_k} q_k = 1 \]  \hspace{1cm} (C)
Here \( q_1[q_1, x] \) is an element of \( P \cup H \) which lies in the same set \( Z_t \setminus Z_{t-1} \) as \( q \) and hence, provided \( j_2 > \frac{1}{2} \), \( d(C) = d(B) \). Further, since \( q_k \) is unaffected by this operation, \((C)\) is still in normal form. But, for \( j_2 > 1 \), this contradicts the choice of \((B)\) as having the least possible second exponent. We conclude that \( j_2 = 1 \).

But in that case \((C)\) is an expression of shorter length than \((B)\).
The element \( q_1[q_1, x] q_2 \) of \( P \cup H \) is in a set \( Z_t \setminus Z_{t-1} \) which is no higher up the central series than the higher of \( q_1 \) and \( q_2 \).
In fact it can only appear at a lower level if \( q_1 \) and \( q_2 \) happen to lie in the same set \( Z_u \setminus Z_{u-1} \). Otherwise \( q_1[x,q_2] \) lies in the same set as the higher of \( q_1 \) and \( q_2 \). In the latter case, \( d(C) \) is equal to \( d(B) \) at all indices except the lower of \( q_1 \) and \( q_2 \) where it is one less. In the former case, although \( d(C) \) is one greater than \( d(B) \) at the index \( t \), it is two less than \( d(B) \) at the index \( u(> t) \). It follows that, in all cases, \( d(C) < d(B) = d(A) \).

(ii) \( [q_1, x] \in P^{-1} \). Then \( q_1 = [x, q_1] = [q_1, x]^{-1} \in P \).

Now \( x^{j_1} q_1 = q_1 x q x q ... x q \) (\( j_1 \) \( x \)-factors)

and hence we have from (B)

\[
q_1 x q x q ... x q^{j_2} q_2 ... x^{k_k} q_k = 1
\]

or, equivalently,

\[
x q x q ... x q^{j_2} q_2 ... x^{k_k} q_k q_1 = 1
\] (D)

If \( q_1 \in Z_u \setminus Z_{u-1} \), then \( [x, q_1] = q \in Z_t \setminus Z_{t-1} \), where \( t < u \). Thus \( d(B) \) is increased by \( j_1 \) at the index \( t \) in \( d(D) \), but, if \( q_k q_1 \in Z_s \setminus Z_{s-1} \), it is decreased by \( j_k \) at the index \( u > t \).

If \( q_k q_1 \notin Z_s \setminus Z_{s-1} \), it must actually appear at a lower level and, in that case, \( d(B) \) is decreased by \( j_k \) at the index \( s \). In all cases again, \( d(D) < d(B) = d(A) \).

Thus in both cases (i) and (ii) we have produced an expression (E) with \( d(E) < d(A) \). This contradicts the choice of (A) and we conclude that \( k \) must be equal to 1. We therefore have an expression

\[
x^{j_1} p_1 = 1,
\]

where \( p_1 \in P \cup H \). But \( H \) is supposed to be isolated in \( G \) and
hence \( p_1 \) cannot be in \( H \). It must therefore be in \( P \) and we have \( x^{-1} \in P \) with \( i_1 > 0 \).

Now we may state and prove the general theorem.

**Theorem 10** Let \( G \) be a locally nilpotent group and \( H \) an isolated subgroup of \( G \). Then every partial right-order on \( G:H \) can be extended to a right-order on \( G:H \).

**Proof** We first prove the theorem for the case when \( G \) is nilpotent.

As in the proof of Lemma 5.1, let \( \{1\} = Z_0 \lhd Z_1 \lhd \ldots \lhd Z_r = G \) be the upper central series of \( G \).

Further let \( P \) be the positive cone of a maximal partial right-order on \( G:H \), the existence of which is guaranteed by Zorn's Lemma. We show by induction on \( i \) that \( Z_{i+1} \) is contained in \( P \cup P^{-1} \cup H \) for all \( i = 0, \ldots, r \).

Clearly \( Z_0 = \{1\} \subseteq P \cup P^{-1} \cup H \). Suppose then that \( Z_i \) is contained in \( P \cup P^{-1} \cup H \) and let \( x \) be an element of \( Z_{i+1} \setminus Z_i \). If \( x \notin P \cup P^{-1} \cup H \), then, since \( P \) is maximal, we must have \( 1 \in S_H(P,x) \) and \( 1 \in S_H(P,x^{-1}) \). By Lemma 5.1, it follows that \( x^{-m_1} \in P \) and \( (x^{-1})^{-m_2} = x^{m_2} \in P \) for some positive integers \( m_1 \) and \( m_2 \). But then \( (x^{-m_1})^{m_2}(x^{m_2})^{m_1} = 1 \in P \), giving a contradiction.

We conclude that \( Z_{i+1} \) is contained in \( P \cup P^{-1} \cup H \).

In particular \( Z_r = G \subseteq P \cup P^{-1} \cup H \) and hence \( P \) represents a total right-order on \( G:H \).

Now consider the case when \( G \) is locally nilpotent and suppose that \( P \) is the positive cone of a partial right-order on \( G:H \) which cannot be extended to a right-order on \( G:H \).
By Theorem 6, there must be a finite set of elements
\[ a_1, \ldots, a_n \in G \setminus H \]
such that
\[ 1 \in S_n(P, a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n}) \]
for all possible choices of the exponents \( \epsilon_1, \ldots, \epsilon_n (\pm 1) \).
In other words, for each of the \( 2^n \) choices of the exponents,
there must be a product of factors from \( P, H \) and \( \{ a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n} \} \)
equal to the identity. The finite set of all those elements which
appear in any of these products will generate a nilpotent subgroup, \( G_1 \),
say, of \( G \).

The subgroup \( H_1 = G_1 \cap H \) is isolated in \( G_1 \) and it contains
all the elements of \( H \) which appear in the \( 2^n \) products. The
semigroup \( P_1 = G_1 \cap P \) is an \( H_1 \)-subsemigroup of \( G_1 \) because, for
any \( h \in H_1 \), \( hP_1 = hG_1 \cap hP = G_1 \cap P \) and \( P_1h = G_1h \cap Ph = G_1 \cap P \).
Also \( 1 \notin P_1 \) because \( 1 \notin P \) and hence \( P_1 \) is the positive cone of
a partial right-order on \( G_1:H_1 \).

But now \( a_1, \ldots, a_n \in G_1 \setminus H_1 \) and
\[ 1 \in S_{H_1}(P_1, a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n}) \]
for all choices of the exponents \( \epsilon_1, \ldots, \epsilon_n (\pm 1) \). That is to
say \( P_1 \) cannot be extended to a right-order on \( G_1 \). This
contradicts the theorem already proved for nilpotent groups and so
we deduce that the theorem also holds for locally nilpotent groups.

**Corollary** If \( G \) is a torsion free locally nilpotent group and
\( H \) is an isolated subgroup of \( G \), then there is a right-order on \( G \) in
which \( H \) is convex.

**Proof** Since \( G \) is torsion free the identity subgroup \( \{1 \} \) is
isolated in \( G \) and therefore, by the theorem, \( G \) can be right-ordered
or, more particularly, \( H \) can be right-ordered. Since \( H \) is
isolated in \( G \), again by the theorem, there is a right-order on \( G:H \). By Lemma 1,3, the combination of these right-orders gives a right-order on \( G \) in which \( H \) is convex.

The corresponding result for two-sided orderings on locally nilpotent groups was proved earlier by A.I. Malcev [13]. In this case there is not a great deal to be gained by dealing with coset spaces because, as remarked earlier, if there is an order on \( G:H \), then \( H \) must be normal in \( G \) and we are really dealing with partial orders on \( G/H \).

The proof of Malcev's Theorem depends on a lemma similar to Lemma 5.1.

**Lemma 5.2** Suppose that \( G \) is a finitely generated nilpotent group and \( A \) is a normal subsemigroup of \( G \) which contains the identity. Suppose further that \( H \) is a subgroup of \( G \) such that, for every element \( h \in H \), there is a positive integer \( m \) such that \( h^m \in A \) modulo \([G,H]\). Then, for every element \( h \in H \), there is a positive integer \( k \) such that \( h^k \in A \).

**Proof** Since \( G \) is nilpotent, we may form a central series for \( H \) by defining \( H_0 = H \), and \( H_{i+1} = [G,H_i] \) which will terminate after a finite number of steps. Suppose \( H_{s+1} = \{1\} \), then \( H_s \neq \{1\} \) will be in the centre of \( G \). We use induction on the length of this series.

When \( s = 1 \), then \([G,H] = \{1\}\) and the condition of the lemma becomes the same as the required conclusion.

Suppose then that \( s \geq 2 \) and consider the group \( G/H_s \). In this group \( AH_s/H_s \) and \( H/H_s \) are a subsemigroup and a subgroup which satisfy
the conditions of the lemma. By induction hypothesis, we deduce that, for every element \( h \in H \), there is a positive integer \( m \) such that \( h^m \in AH_s \). That is \( h^m = az \) for some \( a \in A \) and \( z \in H_s \).

We complete the proof by showing that some power of \( z \) is in \( A \).

Since \( G \) is finitely generated and is nilpotent, every subgroup of \( G \) is also finitely generated. In particular, \( H_{s-1} \) is finitely generated. If \( g_1, \ldots, g_\alpha \) are the generators of \( G \) and \( h_1, \ldots, h_\beta \) are generators of \( H_{s-1} \), then \( H_s \) is generated by the commutators \([g_i, h_j] ; i = 1, \ldots, \alpha, j = 1, \ldots, \beta\). Thus every element \( z \in H_s \) may be expressed in the form \( z = \prod [g_i, h_j]^{m_{ij}} \) or \( \prod [g_i, h_j]^{m_{ij}, h_j} \).

Now, for each of the generators \( h_j \), we have positive integers \( m_j \) and \( n_j \) such that \( h_j^{n_j} \in AH_s \) and \( h_j^{-n_j} \in AH_s \). If \( n \) is the least common multiple of all the \( n_j \)'s and \( n_j \)'s, then \( h_j^n \in AH_s \) and \( h_j^{-n} \in AH_s \) for all \( j = 1, \ldots, \beta \). More precisely, suppose \( h_j^n u_j \in A \) and \( h_j^{-n} v_j \in A \), where \( u_j \) and \( v_j \) are in \( H_s \).

Here \( u_j v_j = h_j^n u_j h_j^{-n} v_j \), because \( u_j \) is in the centre of \( G \) and hence \( u_j v_j \in A \) for all \( j \). Hence the element \( z_0 = (u_1 v_1 \ldots u_\beta v_\beta)^\alpha \) is also in \( A \).

For the general element \( z \in H_s \), we then have

\[
z^n z_0 = \prod_{i,j} [g_i, h_j]^{m_{ij}} (u_1 v_1 \ldots u_\beta v_\beta)^\alpha
\]

\[
= \prod_{i,j} [g_i, h_j]^{m_{ij}} u_k v_j
\]

\[
= \prod_{i,j} g_i^{-m_{ij}} h_j^{-n} u_j g_i^{m_{ij}} h_j^n v_j
\]

Now \( g_i^{-m_{ij}} h_j^{-n} u_j g_i^{m_{ij}} \) is in \( A \) because \( h_j^{-n} u_j \) is in \( A \) and \( A \) is normal in \( G \). [This is the only point where the normality of \( A \) is used.] Since \( h_j^n v_j \) is also in \( A \), it follows that \( z^n z_0 \in A \), and this is true for all elements \( z \) of \( H_s \).
In particular \((z_0^{-1})^n z_0 = z_0^{-n+1} \in A\) and hence, since 
\[z^{n(n-1)} z_0^{(n-1)} \in A,\]
it follows that \(z^n(1) \in A\).

Finally, going back to the original element \(h\), we have 
\[h^m = az\]
and hence \(h^{mn(n-1)} = a^n(n-1)z^n(n-1) \in A\).

**Theorem 11** Let \(G\) be a torsion free locally nilpotent group.

Then every partial order on \(G\) can be extended to a total order on \(G\).

**Proof** Suppose first that \(G\) is a finitely generated nilpotent group and suppose that \(P\) is the positive cone of a partial order on \(G\). Let \(A\) be the semigroup \(P \cup \{1\}\).

Consider the subgroup \(H\) of \(G\) which is generated by all those elements \(x\) for which \(1 \in \overline{S}(P,x) \cap \overline{S}(P,x^{-1})\). If \(x\) is one of these generators, then \(1 \in \overline{S}(P,x)\) and, because \(P\) is normal in \(G\), \(\overline{S}(P,x) = P \cdot \overline{S}(x)\) so that there is an expression of the form

\[1 = p \cdot g_1^{-1} x_1 g_1 \cdots g_T^{-1} x_T g_T\]

for some elements \(g_i \in G\) and positive integers \(k_i\). Now

\[g_1^{-1} x_1 g_1 \cdots g_T^{-1} x_T g_T = x_1^{k_1} [1, g_1] \cdots x_T^{k_T} [1, g_T] \]

\[= x_1^{k_1} \cdots x_T^{k_T} \mod [G, H].\]

Thus, for some \(k > 0\), \(x^{-k} \in P[G, H]\). Similarly, since \(1 \in \overline{S}(P, x^{-1})\), we have, for some \(\ell > 0\), that \(x^{\ell} \in P[G, H]\).

It follows that, for any \(h \in H\), there is a positive integer \(m\) such that \(h^m \in A[G, H]\). Hence Lemma 5.2 applies to \(H\) and \(A\). By that lemma, for any \(h \in H\), there is a positive integer \(k\) such that \(h^k \in A\). Since \(G\) is torsion free, \(h^k\) cannot be equal to the identity element and so \(h^k \in P\). Thus, for any non-identity element \(h \in H\), there are positive integers \(k\) and \(\ell\) such that \(h^k \in P\) and \(h^{-\ell} \in P\). But then \(1 = (h^{k})^{\ell}(h^{-\ell})^k \in P\) is a contradiction.
We deduce that \( H = \{1\} \) and there are no non-identity elements \( x \) such that \( 1 \in \mathcal{S}(P,x) \cap \mathcal{S}(P,x^{-1}) \). Applying this same argument to a partial order \( M \) which is a maximal extension of \( P \), we see that there can be no elements in the set \( G \setminus (M \cup M^{-1} \cup \{1\}) \), i.e. \( M \cup M^{-1} \cup \{1\} = G \). In other words, \( M \) is actually a total order on \( G \).

Now suppose that \( G \) is locally nilpotent and \( P \) is the positive cone of a partial order on \( G \). If this partial order cannot be extended to a total order, then, by Theorem 6, there is a finite set of non-identity elements \( a_1, \ldots, a_n \) such that

\[
1 \in \mathcal{S}(P, a_1^{e_1}, \ldots, a_n^{e_n})
\]

for all choices of the exponents \( e_1, \ldots, e_n \). We let \( G_1 \) be the subgroup of \( G \) which is generated by \( a_1, \ldots, a_n \) and all the elements of \( G \) which appear as conjugators in the \( 2^n \) products which are equal to the identity. Then, if \( P_1 = P \cap G_1 \), we have in \( G_1 \) that

\[
1 \in \mathcal{S}(P_1, a_1^{e_1}, \ldots, a_n^{e_n})
\]

for all choices of the exponents. Thus \( P_1 \) cannot be extended to a total order on \( G_1 \). This contradicts the theorem as already proved for finitely generated nilpotent groups.

The proof given here for the finitely generated case is not quite the same as that originally given by Malcev. He constructed a central series \( G = H_0 \supset H_1 \supset \cdots \supset H_s = \{1\} \) in which the subgroups \( H_i \) are all convex in the partial order corresponding to \( P \). If \( H_i \) is already defined, then \( H_{i+1} \) is defined to be the set of all elements \( g \in G \) for which there are positive integers \( n \) and \( \mu \) such that \( g^n \in A_1[G,H_i] \) and \( g^{-\mu} \in A_1[G,H_i] \), where \( A_1 = (P \cap H_i) \cup \{1\} \). Lemma 5.2 is needed to show that \( H_{i+1} \neq H_i \).
for any $i$. The partial order $\mathcal{P}$ then induces partial orders on
each of the factors $\Pi_i/\Pi_{i+1}$ which can all be extended to total orders.
The combination of these total orders as in Lemma 1.3 then produces
an order on $G$ which is an extension of the original partial order.
This method introduces a concept - convexity - which, as we have
seen, is not strictly necessary for the proof of Theorem 11.
However it is of interest in that it shows as a corollary that,
in a finitely generated nilpotent group, the system of convex
subgroups in any order must be a central series. This is also
a consequence of Graham's Theorem to appear later.

In the light of Theorems 10 and 11, the various properties
discussed in Section 4 are all equivalent in the case of torsion free
locally nilpotent groups. The circuit of the diagram on page 53
is completed by the simple argument that, if $G$ is right-orderable,
then it is torsion free and therefore, by Theorem 11, it can be ordered.
This simple observation however still leaves several questions unanswered.
In particular, what can be said about the system of convex subgroups of
a right-order (or an order) on a locally nilpotent group? The first
obvious comment in this direction is that not every right-order is an
order. We illustrate this point by describing some right-orders on
a nilpotent group which are not two-sided.

Example 5 Let $G$ be the group generated by the elements $a$, $b$ and
c with the relations $[a,c] = 1$, $[b,c] = 1$ and $[a,b] = c^2$. Then
$G$ is nilpotent of class 2 with centre $Z$ equal to the subgroup generated
by $c$. It is easy to see that $G$ is torsion free and that the elements
of $G$ have unique expressions in the form $a^{r} b^{s} c^{t}$. Since, in this
example, $r(G) = 3$, the number of non-trivial proper convex subgroups
of $G$ in any right-order must be either 0, 1 or 2 (Theorem 5).
We give two examples of right-orders on $G$ in which the numbers of non-trivial proper convex subgroups are 1 and 2 respectively.

(i) We define $1 < a^s b^t c^r$ if and only if $s > 0$ or $s = 0$ and $t > 0$ or $s = t = 0$ and $r > 0$.

(ii) We define $1 < a^r b^s c^t$ if and only if $s > 0$ or $s = 0$ and $r + t/2 > 0$.

It is easy to verify that both of these are right-orders. The first one has convex subgroups $\{a^r\}$ and $\{a^r c^t\}$ and the second has only the one convex subgroup $\{a^r c^t\}$.

Neither of them is an order because, in both cases, $a^{-1} c > 1$ but $b^{-1} a^{-1} c b = a^{-1} c^{-1} < 1$.

As we shall soon see, it is not possible to find a right-order on $G$ which has no non-trivial proper convex subgroups. For our intention now is to prove that every right-order on a locally nilpotent group is a generalized polyorder. The proof presented here will be the one published by me in [2]. Rheintulla [15] has independently given a different proof of the same result.

Some of the preliminary work has already been done in Section 2. For example in Lemma 2.4 we found conditions for a right-order to be a generalized polyorder the second of which is that, for any two elements $x$ and $y \in G \setminus H$, either $Hx \gg H[x,y]$ or $Hy \gg H[x,y]$. The idea of the present method is essentially to show that this condition is satisfied by a right-ordered locally nilpotent group and that this is sufficient for the right-order to be a generalized polyorder.

If $S$ is an arbitrary subset of a right-ordered group $G$, then $C(S)$ will denote the convex hull of $S$ and $I(S)$ will denote the isolator of $S$ in $G$. That is, $I(S)$ is the intersection of all
isolated subgroups of $G$ which contain $S$. Since any convex subgroup is isolated in $G$, it follows that $I(S)$ is contained in $C(S)$ but in general it may not be equal to it. In a locally nilpotent group the isolator of a normal subgroup $N$ is normal in $G$. For in that case $I(N)$ consists of those elements $x \in G$ such that $x^m \in N$ for some integer $m \neq 0$. (See Kurosch, Vol. 2, p. 249.)

We start by dealing with a very special case.

Lemma 5.3 Let $G$ be a torsion free nilpotent group of class 2 generated by the elements $a$ and $b$ and let $c = [a, b]$. Then, in any right-order $\prec$ on $G$, either $a \gg c$ or $b \gg c$.

Proof We may assume that $c > 1$. Otherwise the subsequent argument may be applied to $c^{-1} = [b, a]$ with the roles of $a$ and $b$ reversed.

Let $H$ be the subgroup of $G$ generated by $b$ and $c$ and $K$ the subgroup generated by $c$. If $b$ is not infinitely greater than $c$, then $b$ or $b^{-1}$ lies between powers of $c$ and $K$ is not convex in the order induced on $H$ by $\prec$. We show in this case that $a \gg c$.

Since $K$ is in the centre of $G$ and the centre of a torsion free nilpotent group is isolated in $G$, $b^k \neq c^m$ for any non-zero integers $k$ and $m$, and hence $H$ is a free Abelian group of rank 2. (The element $b$ is assumed not to be in the centre of $G$, otherwise $c = 1$ and the lemma is trivially true.) Following H.H. Teh's results on the possible orders on Abelian groups [18], this means that any order $\prec$ on $H$ is given as follows.

We choose a set of four real numbers $\alpha, \beta, \gamma, \delta$ such that $\alpha \delta - \beta \gamma \neq 0$ and then define

$$b^k c^m > 1 \iff \alpha k + \beta m > 0 \text{ or } \alpha k + \beta m = 0 \text{ and } \gamma k + \delta m > 0.$$  

If $\beta = 0$ in this, then $K$ is convex. For then $\alpha \neq 0$ and, if $1 < b^k c^m < c^n$, then $\alpha k > 0$ and, from $1 < b^{-1} c^{n-m}$, $-\alpha k > 0$. 
It follows that \( k = 0 \) and hence that \( b^k c^m = c^m \in K \). Since in our case \( K \) is not convex, we must have \( \beta \neq 0 \). Further, since we are assuming that \( c > 1 \), we must actually have \( \beta > 0 \). This allows some simplification. Putting \( \varepsilon = \alpha / \beta \), we have, in particular, that \( b^k c^m > 1 \) if \( \varepsilon k + m > 0 \).

Now we note that
\[
a^s c^s = b^{-s+2} c^{1+\lfloor \varepsilon(s-2) \rfloor} a b^{s-2} c^{1-\lfloor \varepsilon(s-2) \rfloor}
\]
and
\[
a^{-s} c^{-s} = b^{s-2} c^{1-\lfloor -\varepsilon(s-2) \rfloor} a^{-1} b^{-s+2} c^{1-\lfloor -\varepsilon(s-2) \rfloor}
\]
for all \( s \) (positive or negative), where \( \lfloor p \rfloor \) denotes the integer part of the real number \( p \).

We also note that the elements
\[
b^{-s+2} c^{1+\lfloor \varepsilon(s-2) \rfloor}, b^{s-2} c^{1-\lfloor \varepsilon(s-2) \rfloor}
\]
\[
b^{s-2} c^{1-\lfloor -\varepsilon(s-2) \rfloor}, b^{-s+2} c^{1-\lfloor -\varepsilon(s-2) \rfloor}
\]
are all positive because
\[
\varepsilon(-s+2) + 1 + \lfloor \varepsilon(s-2) \rfloor > \varepsilon(-s+2) + \varepsilon(s-2) = 0,
\]
\[
\varepsilon(s-2) + 1 - \lfloor \varepsilon(s-2) \rfloor > \varepsilon(s-2) + 1 - \varepsilon(s-2) = 1 > 0,
\]
\[
\varepsilon(s-2) + 1 + \lfloor -\varepsilon(s-2) \rfloor > \varepsilon(s-2) - \varepsilon(s-2) = 0,
\]
and
\[
\varepsilon(-s+2) + 1 - \lfloor -\varepsilon(s-2) \rfloor > \varepsilon(-s+2) + 1 + \varepsilon(s-2) = 1 > 0.
\]

Hence, if \( a > 1 \), then \( a > c^s \) for all \( s \) and, if \( a^{-1} > 1 \), then \( a^{-1} > c^s \) for all \( s \), i.e. \( a >> c \). This completes the proof of the lemma.

In general in any right-ordered group it follows that, if \( a \) and \( b \) are elements which commute with their commutator \( c \), then either \( a >> c \) or \( b >> c \). Otherwise there would be a contradiction to
the lemma in the restriction of the right-order to the subgroup generated by \( a \) and \( b \). However, as we have already seen (Example 3), we cannot in general deduce from this the very desirable conclusion that \( a \not\in C(c) \) or \( b \not\in C(c) \). In our present context however it suffices to be able to draw this conclusion in the special case when \( c \) is in the centre of the group.

**Lemma 5.4** Let \( G \) be a right-ordered group and suppose there is a non-identity commutator in the centre of \( G \). Then \( G \) has a non-trivial proper convex subgroup.

**Proof** We show that, for an element \( c \) in the centre of \( G \), the convex hull \( C(c) \) is equal to the set \( H \) of all \( g \in G \) such that \( c^m \leq g \leq c^n \), for some integers \( m, n \). For, if \( g_1 \) and \( g_2 \) are in \( H \), then \( gc_1 \leq g_1 \leq c_1^n \) and \( c_2^n \leq g_2 \leq c_2^n \) and hence \( g_1g_2 \leq c_1^n g_2 = g_2 c_1^n \leq c_1^{n+1} \)

\[
\left(g_1g_2 - c_1^n g_2 = g_2 c_1^n - c_2^{n+1}\right)
\]

(using the fact that \( c \) is in the centre.) That is, \( g_1 g_2 \in H \).

Also, since \( g \leq c_1^n \), \( 1 \leq c_1^n g^{-1} = g^{-1} c_1^n \) and therefore \( c^{-n} \leq g^{-1} \).

Similarly \( g^{-1} \leq c^{-n} \) and hence \( g^{-1} \in H \). Thus \( H \) is a subgroup which clearly contains \( c \).

It is convex, because, if \( 1 \leq g \leq h \) then \( 1 \leq g \leq h \leq c^n \) for some \( n \) and therefore \( g \leq H \). It follows that \( C(c) \subseteq H \). However it is clear that all elements of \( H \) must be in \( C(c) \) and hence \( C(c) = H \).

If \( c \) happens to be a non-identity commutator, say \( c = [a,b] \), then, by Lemma 5.3 either \( a \gg c \) or \( b \gg c \). Thus either \( a \not\in H \) or \( b \not\in H \) and hence \( C(c) = H \) is a non-trivial convex subgroup of \( G \).
We remark that as a corollary to this lemma we may deduce Smirnov's result [17] that any right-ordered ZA-group without non-trivial proper convex subgroups must be Abelian. In particular this applies to nilpotent groups. Thus a right-ordered nilpotent group without non-trivial proper convex subgroups must be Abelian and therefore, by Theorem 1, order-isomorphic to a subgroup of \( \mathbb{R} \).

**Theorem 12** Suppose that \( G \) is a locally nilpotent group and \( H \) is an isolated subgroup of \( G \). Then the system \( \mathcal{D} \) of subgroups of \( G \) is the system of convex subgroups in some right-order on \( G:H \) if and only if it satisfies the following four conditions.

1. \( H \) and \( G \) are in \( \mathcal{D} \).
2. \( \mathcal{D} \) is totally ordered by inclusion.
3. \( \mathcal{D} \) is closed under arbitrary intersections and unions.
4. If \( A \supset B \) is a jump in \( \mathcal{D} \), then \( B \) is normal in \( A \) and \( A/B \) is order-isomorphic to a subgroup of \( \mathbb{R} \).

In particular, every right-order on \( G:H \) is a generalized polyorder.

**Proof** Clearly, if \( \mathcal{D} \) satisfies these conditions then it also satisfies the conditions (i), (ii), (iii) and (vii) of Theorem 3 and so it is the system of convex subgroups in some right-order on \( G:H \).

For the converse we first observe that it will be sufficient to prove the theorem for right-orders on locally nilpotent groups. For then, if \( \prec \) is a right-order on \( G:H \) it is induced by a right-order \( \prec \) on \( G/H \). (Lemma 1.7). Now \( G/H \) is torsion free and locally nilpotent and, if the theorem is proved for such groups, then \( \prec \) is a generalized polyorder and therefore \( \prec \) is also a generalized polyorder.
We start by assuming that $G$ is nilpotent and that $\langle < \rangle$ is a right-order on $G$. Conditions (i), (ii) and (iii) are already proved (Theorem 3) and it only remains to prove condition (iv). Note that, in the light of Lemma 5.4, it will be sufficient in our present case to prove that, for any jump $A \triangleright B$ in $\mathcal{D}$, $B$ is normal in $A$. For then $A/B$ will be a right-ordered nilpotent group with no non-trivial proper convex subgroups and so must be order-isomorphic to a subgroup of $\mathcal{D}$.

We use induction on the class of $G$. If the class is $0$, then $G$ is trivial and trivially satisfies (iv).

Now suppose the class of $G$ is at least $1$ and that (iv) is satisfied by all right-ordered groups of a lower class. Let $A \triangleright B$ be a jump in $\mathcal{D}$. If $A$ is Abelian, then $B$ is obviously normal in $A$ and there is nothing more to prove. Assume therefore that the class of $A$ is at least $2$ and let $Z_1$ and $Z_2$ be the first two terms in the upper central series of $A$, i.e. $Z_1$ is the centre of $A$ and $Z_2/Z_1$ is the centre of $A/Z_1$.

For elements $a \in A$ and $b \in Z_2$, the commutator $c = [a, b]$ is in $Z_1$ and it follows from Lemma 5.3 that either $a \not> c$ or $b \not> c$. Hence, as in Lemma 5.4, either $a \notin C(c)$ or $b \notin C(c)$. In either case, it follows that $C(c) \not= A$ and hence that $C(c)$ is contained in $B$. In particular $c \in B$.

It follows that the subgroup $[A, Z_2]$ is contained in $B$, and, since $B$ is isolated in $G$, the isolator $I$ of $[A, Z_2]$ is also contained in $B$. Moreover, since $[A, Z_2]$ is normal in $A$, $I$ is also normal in $A$. The factor group $B/I$ is a torsion free nilpotent group and can therefore be ordered.

Now the right-order induced on $A:B$ by $\langle < \rangle$ and any order on $B/I$ may be combined as in Lemma 1.3.
to give a right-order on $A/I$ in which $B$ is convex. This gives a right-order on $A/I$ in which $B/I$ is convex and $A/I \supset B/I$ is a jump in the system of convex subgroups.

Since $I \supset [A, Z_2]$, the class of $A/I$ is less than the class of $A$ which in turn is at most equal to the class of $G$. By induction hypothesis $B/I$ is normal in $A/I$ and it follows that $B$ is normal in $A$. This completes the proof for nilpotent groups.

Thus, when $<$ is a right-order on a locally nilpotent group $G$, its restriction to any finitely generated subgroup is a generalized polyorder and so, as remarked in Section 4 (p. 52), $<$ itself is a generalized polyorder. This completes the proof of the theorem.

When $G$ is a finitely generated nilpotent group, all the factors in the lower central series are also finitely generated and so they all have finite rank. It follows that the rank $r(G)$ is also finite. Hence, by Theorem 5, the number of convex subgroups in any right-order is also finite. In view of the present theorem, this means that every right-order is actually a polyorder. Further, combining this observation with the result of Theorem 10, we see that in a torsion free locally nilpotent group every partial right-order on every finitely generated subgroup extends to a polyorder on the subgroup. It is interesting that, among ordered groups, this latter property is characteristic of torsion free locally nilpotent groups.

This result answers a question raised in [1], where a similar property was found to be characteristic of torsion free Abelian groups.

The proof depends on the following lemma.

**Lemma 5.5** Let $G$ be a finitely generated ordered group and suppose that every right-order on every subgroup of $G$ is a polyorder. Then $G$ is nilpotent.
Proof. We note first that, if \( K \) is a minimal convex subgroup in an order on a group \( G \), i.e. if \( K \supseteq \{1\} \) is a jump in the system of convex subgroups, then \( K \) must be normal in \( G \). For, by Theorem 2, \( g^{-1}Kg \supseteq K \) or \( g^{-1}Kg \subseteq K \) for all \( g \in G \).

Now let \( \prec \) be an order on the group \( G \). We divide the proof into several steps.

1. Given any element \( a \in G \), then there is a non-trivial subgroup \( C \) of \( G \) which is convex under \( \prec \) and which is centralized by \( a \):

   for, suppose \( A \supseteq B \) is the jump in the system of convex subgroups of \( \prec \) where \( a \) occurs, i.e. \( a \in A \) but \( a \notin B \). If \( B = \{1\} \), then, by Theorem 2, \( A \) is isomorphic to a subgroup of \( \mathbb{R} \). In particular it is Abelian and it follows that \( a \) is in the centralizer of \( A \). In other words, we may choose \( C = A \) in this case.

   Suppose therefore that \( B \neq \{1\} \), and consider the subgroup \( H \) of \( A \) which is generated by \( B \) and the element \( a \). Since, by Theorem 2, \( B \) is normal and isolated in \( A \), the elements of \( H \) have unique representations in the form \( ba^i \), where \( b \in B \) and \( i \) is an integer.

   We define a relation \( \prec \) on \( H \) by

   \[ ba^i \prec 1 \iff b < 1 \text{ or } b = 1 \text{ and } i > 0. \]

   This is a right-order on \( H \) because, if \( b_1a_1^i \prec 1 \) and \( b_2a_2^{i_2} \prec 1 \), then

   \[ b_1a_1^ib_2a_2^{i_2} = b_1a_1^ib_2a_2^{-i_1}a_1^{i+i_2} \]

   and, if \( b_2 \geq 1 \), then \( a_1^ib_2a^{-i_1} \prec 1 \), because \( \prec \) is two-sided.

   Also, if \( ba^i \neq 1 \), then

   \[ (ba^i)^{-1} = a^{-1}b^{-1}a^ia^{-1} \]
and, if \( b \neq 1 \), then \( b^{-1} \not\in \) 1 and \( a^{-1}b^{-1}a^{-1} \not\in \) 1 again using the fact that \(<\) is two-sided.

Further, the subgroup \( K \) of \( H \) which is generated by \( a \) is convex under \(<\). For, if \( 1 < ba^i < a^j \), then \( ba^{i-j} < 1 \) and hence \( b \not\in \) 1. But, on the other hand, from \( ba^i > 1 \) it follows that \( b > 1 \). Hence \( b = 1 \) and \( ba^i = a^i \in K \). We note also that \( K \) is a minimal convex subgroup of \(<\).

By hypothesis, \(<\) is a polyorder on \( H \) and hence there is a normal series of convex subgroups

\[
\{1\} = K_0 < K_1 < \ldots < K_T = H
\]
in which each factor is ordered.

Since \( K \) is minimal, it must be contained in \( K_1 \) and is therefore normal in \( K_1 \). [\(<\) is two-sided on \( K_1 \).] It follows that there are convex subgroups of \( H \) under \(<\) which properly contain \( K \) as a normal subgroup. [If \( K = K_1 \), then \( K < K_2 \).] Let \( K^+ \) be the union of all such convex subgroups and let \( C = K^+ \cap \) \( B \).

We show that \( C \) is the subgroup required.

As a union of convex subgroups, \( K^+ \) is clearly convex under \(<\) and it is also clear that \( K < K^+ \).

Further, if \( ba^i \in K^+ \), then \( (ba^i)a^{-1} = b \in K^+ \), and hence \( b \in B \cap K^+ = C \). That is, \( K^+ = CK \).

Since \( B < H \), \( C = B \cap K^+ \) is normal in \( K^+ \) and, since \( C \cap K = \{1\} \) it follows that \( K^+ \sim C \times K \).

Since \( K^+ \neq K \), \( C \) is non-trivial and \( a \in K \) commutes with all elements of \( C \). Thus it only remains to show that \( C \) is convex under \(<\).
Suppose $1 < g < c$ for some $g \in G$ and $c \in C$. Then certainly $g \in B$ because $B$ is convex under $\prec$. But then, as elements of $H$, $1 < g < c$ and hence $g \in K^+$ because $K^+$ is convex under $\prec$. Hence $g \in K^+ \cap B = C$.

2. There is a non-trivial subgroup of $G$ which is convex under $\prec$ and which lies in the centre of $G$:

for, suppose $G$ is generated by the elements $a_1, a_2, \ldots, a_r$. Then for each $i$ there is a non-trivial convex subgroup $C_i$ which is centralized by $a_i$. Let $C = \bigcap_{i=1}^{r} C_i$. Since $r$ is finite, $C = C_i$ for some $i$ and hence $C$ is non-trivial. Further $C$ is centralized by each of the generators $a_i$ and hence $C$ is in the centre of $G$.

3. If $C$ is a normal convex subgroup of $G$ under $\prec$, then $G/C$ satisfies all the hypotheses of the lemma:

certainly $G/C$ is finitely generated and is ordered by $\prec$.

Suppose $H/C$ is a subgroup of $G/C$ and $\prec$ is a right-order on $H/C$. Then $\prec$ may be combined with $\prec$ on $C$ to give a right-order $\rightarrow$ on $H$ in which $C$ is convex. Now $\rightarrow$ is a polyorder and hence there are convex subgroups

$$\{1\} = K_0 \prec K_1 \prec \ldots \prec K_r = H$$

such that each factor $K_{i+1}/K_i$ is ordered.

Since $C$ is also convex, there is some $i$ such that $K_i \prec C \prec K_{i+1}$. The induced right-order on $K_{i+1}/C$ is two-sided because it is induced by an order on $K_{i+1}/K_i$ and $C$ is normal in $K_{i+1}$. Hence the series

$$\{C\} \prec K_{i+1}/C \prec \ldots \prec K_r/C = H/C$$
consists of convex subgroups under \( \prec \) and the induced right-orders on the factors are all two-sided. Thus \( \prec \) is a polyorder.

4. \( G \) is a ZA-group:

We construct an ascending central series for \( G \) by transfinite induction. We start by choosing \( C_1 \) to be any non-trivial convex subgroup in the centre of \( G \).

Then, for any ordinal \( \lambda > 1 \), we define \( C_\lambda \) as follows:

(i) if \( \lambda = \alpha + 1 \) for some \( \alpha \), then \( C_\lambda \) is any convex subgroup which is such that \( C_\lambda \cap C_\alpha \) is a non-trivial convex subgroup in the centre of \( G/C_\alpha \);

(ii) if \( \lambda \) is a limit ordinal, then \( C_\lambda = \bigcup_{\alpha<\lambda} C_\alpha \).

In all cases it is clear that \( C_\lambda \) is normal and convex in \( G \) so that the construction may be continued transfinitely. Eventually \( C_\lambda = G \) for some \( \lambda \). Hence \( G \) is a ZA-group.

5. \( G \) is nilpotent:

Here it suffices to refer to a theorem of A.I. Malcev, the proof of which is reproduced in Kurosch, Vol.II, p.223, to the effect that a finitely generated ZA-group is nilpotent. This completes the proof of the lemma.

We remark that the hypothesis that \( G \) be ordered cannot be dispensed with in this lemma. The group \( G \) which is generated by the elements \( a, b \) with the relation \( b^{-1}ab = a^{-1} \) [Example 1, p.13] has the property that every right-order on every subgroup is a polyorder, but clearly it is not nilpotent.
Theorem 15 Let $G$ be an ordered group. Then $G$ is locally nilpotent if and only if every partial right-order on every finitely generated subgroup $H$ of $G$ can be extended to a polyorder on $H$.

Proof We have already observed the validity of this theorem one way round. Here we only need to show that the stated condition is sufficient to imply local nilpotency.

Suppose therefore that $H$ is a finitely generated subgroup of $G$ and that $<_$ is a right-order on a subgroup $K$ of $H$. The positive cone of $<_$ represents a partial right-order on $H$ which can therefore be extended to a polyorder on $H$. The restriction of this polyorder to $K$ is still a polyorder and is clearly equal to $<_$. That is, every right-order on every subgroup of $H$ is a polyorder. By Lemma 5.5, $H$ is nilpotent and hence $G$ is locally nilpotent.

We remark that in this case it is not known whether the condition of orderability may be omitted. The example given at the end of Lemma 5.5 will not do as a counterexample in this case because, in that example, there are partial right-orders which cannot be extended. It is also not known whether the condition that every partial right-order is extendable to a generalized polyorder is sufficient to make an ordered group locally nilpotent.

The idea for the last theorem was based on the observation that in a finitely generated nilpotent group every right-order is a polyorder. We now observe that, as the next example shows, this is not true for all nilpotent groups.

Example 6 Let $G$ be the free nilpotent group of class 2 which is generated by infinitely many generators $a_1, a_2, a_3, \ldots$ (one for each positive integer). For each pair $i < j$, let $b_{ij}$ denote the
commutator \([a_i, a_j]\), then every element of \(G\) has a unique representation
in the form \(Ha^\alpha_iNb^\beta_{ij}\). For each \(i = 1, 2, \ldots\), let \(G_i\) be the
subgroup of \(G\) generated by \(a_i, a_{i+1}, \ldots\) and \(H_i\) the subgroup of \(G_i\)
generated by \(a_{i+1}, a_{i+2}, \ldots\) and the commutators \(b_i, b_{i+1}, b_{i+2}, \ldots\).
Each \(H_i\) is a normal subgroup of the corresponding \(G_i\) and \(G_{i+1}\) is
a normal subgroup of \(H_i\). Thus we have a normal system
\(G = G_1 \supset H_1 \supset G_2 \supset H_2 \supset \ldots\) for \(G\) in which the factors \(G_i/H_i\)
and \(H_i/G_{i+1}\) are all Abelian and torsion free. Also every \(G_i\) is
isomorphic to \(G\).

We order \(G_i/H_i\) by setting \(H_i a_i^\alpha > H_i\) if and only if \(\alpha_i > 0\)
and \(H_i/G_{i+1}\) by setting \(G_{i+1} b_{ij}^\beta > G_{i+1}\) if and only if \(\beta_{ij} = 0\)
for all \(j < k\) and \(\beta_{ik} > 0\) for some \(k\). These orders on the factors
may be combined to give a right-order \(<\) on \(G\).

Now, if \(C\) is a non-trivial normal convex subgroup of \(G\), then,
since each \(G_i\) is convex and \(\bigcap_i G_i = \{1\}\), \(C\) must contain some one
of the \(G_i\)'s. In particular \(C\) must contain an element \(a_1\).
Hence it must also contain \(a_1^{-1}a_1a_1 = a_1[a_1, a_1] = a_1b_{ij}^{-1}\). Therefore
\(b_{ij} \in C\). Hence \(C\) contains \(G_2\) and, by the same argument, \(C\) must
contain \(b_{12}\). From the way in which \(<\) is defined this means that
\(C = H_1\).

Similarly we may see that, if \(C\) is a non-trivial normal convex
subgroup of \(H_1\), then \(C \supseteq H_2\). These arguments clearly apply all down
the chain.

For \(<\) to be a polyorder there has to be a finite normal series
of convex subgroups with ordered factors. In the present example any
such series must start \(G = G_1 \supset H_1\), and the restriction to \(G_i\) must
also start \(G_i \supset H_i\). It follows that every \(H_i\) would have to be in
the series and so it cannot be finite. Hence \(<\) is not a polyorder.
However it is interesting to note that, in a torsion free nilpotent group, every isolated subgroup is convex in some polyorder. We have already seen from the Corollary to Theorem 10 and Theorem 12 that, in any case, it is convex in some generalized polyorder. The fact that this can actually be chosen to be a polyorder depends on the following theorem.

**Theorem 13** Let $G$ be a nilpotent group and suppose that $H$ is an isolated subgroup of $G$. Then there is a polyorder on $G:H$.

**Proof** We observe first that it will be sufficient to prove this theorem in the case when $G$ is torsion free. For in general, if $H$ is isolated in $G$, then, by Theorem 10 and Lemma 1.7, the core $H^*$ of $H$ is a normal isolated subgroup of $G$ and $H/H^*$ is an isolated subgroup of the torsion free nilpotent group $G/H^*$. Any polyorder on $G/H^*:H/H^*$ naturally corresponds to a polyorder on $G:H$. For the rest of the proof therefore we assume that $G$ is torsion free.

We use a double induction on the classes $m$ of $G$ and $n(m)$ of $H$, where the class pairs $(m,n)$ are ordered lexicographically from the left. That is, $(p,q)$ precedes $(m,n)$ if and only if $p < m$ or $p = m$ and $q < n$.

The theorem is trivially true when $G$ has class 0. Suppose therefore that $m > 1$ and that the theorem is true for all pairs of groups with class pair preceding $(m,n)$.

Let $K$ be the centralizer in $G$ of the centre $Z(H)$ of $H$. Then clearly $K$ contains $H$ and also the centre $Z$ of $G$. Since $G$ is torsion free and nilpotent, $K$, being a centralizer, is isolated in $G$. (See Kurosch, Vol.2, p.243).

Now the class of $G/Z$ is $m - 1$ and, if the class of $K/Z$ is $q$, say, then the class pair $(m - 1, q)$ precedes $(m,n)$. Hence, by induction hypothesis, there is a polyorder on $G/Z:K/Z$ which naturally
corresponds to a polyorder on $G:K$.

The class of $K/Z(H)$ is $p$, say, where $p \leq m$, and the class of $H/Z(H)$ is $n-1$ and hence the class pair $(p,n-1)$ again precedes $(m,n)$.

By induction hypothesis, there is a polyorder on $K/Z(H):H/Z(H)$ which naturally corresponds to a polyorder on $K:H$.

The combination of polyorders on $G:K$ and $K:H$ as in Lemma 1.3 clearly produces a polyorder on $G:H$.

**Corollary** In a torsion free nilpotent group, every isolated subgroup is convex in some polyorder.

**Proof** If $H$ is isolated in $G$, then, by the theorem, there is a polyorder on $G:H$. Since $H$ is torsion free and nilpotent, it follows from Theorem 11 that there is an order on $H$. The combination of these gives a polyorder on $G$ in which $H$ is convex.

Finally, can anything more be said about the convex subgroups of an ordered torsion free nilpotent group? George P. Graham [7] has proved the following result.

**Theorem 15** (George P. Graham) Suppose that $<$ is an order on the torsion free locally nilpotent group $G$. Then the system of convex subgroups of $G$ in $<$ is a central system of $G$. That is, each convex subgroup $C$ is normal in $G$ and, if $C_1 \supset C_2$ is a jump in the system, then $C_1/C_2$ is in the centre of $G/C_2$.

**Proof** Graham's proof involves a commutator calculation which relies on the fact that in a locally nilpotent group a repeated commutator of the form $[[[a,g],g],...,g]$ must eventually become equal to the identity.
However, as we remarked earlier, Malcev's method for proving Theorem 11 has already established the present theorem in the case when $G$ is finitely generated and nilpotent. We now see that that result may fairly easily be extended to the present more general case.

First, if $C$ is a convex subgroup of $G$ and $g$ is any element of $G$, we need to show that, for any $c \in C$, $g^{-1}cg \in C$. Consider the subgroup $H$ of $G$ generated by $g$ and $c$. The subgroup $H \cap C$ is convex in the restriction of $\prec$ to $H$ and so, by Malcev's Theorem, is normal in $H$. It follows that $g^{-1}cg \in H \cap C$, i.e. $g^{-1}cg \in C$ as required.

Now, suppose $C_1 \supset C_2$ is a jump in the system of convex subgroups, $x \in C_1 \setminus C_2$ and $g \in G$, and consider the subgroup $H$ of $G$ generated by $x$ and $g$. The subgroups $C_1' = H \cap C_1$ and $C_2' = H \cap C_2$ are convex in the restriction of $\prec$ to $H$ and, since $x \in H$, $C_1' \neq C_2'$. By the isomorphism theorem, $C_1'/C_1' \cong C_2/C_2'$ which is a subgroup of $C_1/C_2$. This isomorphism is actually an order-isomorphism and, since $C_1/C_2$ is Archimedean ordered, so is $C_1'/C_1'$. Hence $C_1' \supset C_2'$ is a jump in the system of convex subgroups of $H$. Again by Malcev's Theorem, $C_1'/C_1'$ is in the centre of $H/C_1'$ and, in particular, $[x,g] \in C_2'$.

Thus $[x,g] \in C_2$ for all $x \in C_1 \setminus C_2$ and for all $g \in G \setminus C_2$ and hence $G_1/C_2$ is in the centre of $G/C_2$.

As a consequence of this theorem we may now prove a result analogous to that of the Corollary to Theorem 10.

**Corollary** In a torsion free locally nilpotent group $G$ a subgroup $H$ of $G$ is convex in some order on $G$ if and only if it is normal and isolated in $G$. 
Proof. If $H$ is convex in some order, then, by the theorem, it is normal and isolated in $G$.

Conversely, if $H$ is normal and isolated in $G$, then there is an order on $G$. There is also an order on $H$ which may be chosen to be the restriction of an order on $G$. The combination of these will give an order on $G$ in which $H$ is convex.

To summarize the main results in this section, we have shown that, in a torsion free locally nilpotent group, every partial (right-) order can be extended to a total (right-) order, every right-order is a generalized polyorder and the system of convex subgroups in an order is a central system.
6. **Metabelian Groups**

In the previous section we saw how all the conditions listed in Section 4 (O, GP, etc.) are equivalent in torsion free locally nilpotent groups. Here we consider the same problems but this time for the class of metabelian groups. We obtain some positive results but many reasonably hoped for implications are shown to be untrue.

Of the conditions in Section 4, the particular ones to be studied now are the following.

- O. G can be ordered.
- RO. G can be right-ordered.
- GP. G can be generalized polyordered.
- P. G can be polyordered.

In addition to these we will also consider two purely group theoretic properties which are naturally related to those above. First we have

- TF. G is torsion free.

As we have already seen, every right-ordered group is torsion free and we will be interested to know whether the converse is true.

The other group theoretic property is

- R. G is an R-group (see Kurosch, Vol.2, p.242)

i.e. G has unique extraction of roots, or, in other words, if \( a^n = b^n \) for any integer \( n \), then \( a = b \).

The connection here is that any ordered group must be an R-group. For, if \( a \) and \( b \) are elements of G with \( a < b \), then \( a^2 < ab \) (multiplying on the left) and \( ab < b^2 \) (multiplying on the right) and hence \( a^2 < b^2 \). Similarly \( a^n < b^n \) for all \( n > 0 \) and it follows that \( a^n \neq b^n \) for any \( n \) unless \( a = b \).
In general the six conditions listed above are connected by the following implication diagram.

\[ \begin{array}{ccc}
\text{P} & \longrightarrow & \text{GP} \\
\text{O} & & \text{RO} \\
\text{R} & \longrightarrow & \text{TF}
\end{array} \]

Our intention is to investigate for metabelian groups the extent to which any or all of these may be reversed.

The first result concerns the relationship between \( P \) and \( R \).

**Theorem 16** A metabelian \( R \)-group can be polyordered.

**Proof** Let \( G' \) be the commutator subgroup of the group \( G \) and let \( K \) be the centralizer of \( G' \) in \( G \). Since \( G \) is metabelian, \( G' \) is Abelian and is therefore contained in \( K \) and indeed it is contained in the centre \( Z \) of \( K \). We show that the Abelian groups \( G/K, K/Z \) and \( Z \) are all torsion free.

Suppose first that, for some element \( g \) in \( G \), and for some \( n > 0 \), \( g^n \in K \). Then, for all \( h \in G' \), we have

\[ h^{-1}g^n h = (h^{-1}gh)^n = g^n. \]

But \( G \) is supposed to be an \( R \)-group and hence we have that \( h^{-1}gh = g \) for all \( h \in G' \). That is, \( g \in K \) and \( G/K \) is torsion free.

Now, being a subgroup of an \( R \)-group, \( K \) is also an \( R \)-group and it is well-known in that case that \( K/Z \) is an \( R \)-group (see Kurosch, Vol. 2, p. 243). In particular \( K/Z \) is torsion free.

Clearly \( Z \) also is torsion free.
Finally, using Lemma 1.3, orders on $G/K$, $K/Z$ and $Z$ may be combined to give a right-order on $G$ which, from its construction, is clearly a polyorder.

[We remark that it is still an open question as to whether a soluble $R$-group of derived length greater than 2 can be polyordered - or even right-ordered. There seems to be no obvious generalization of the above method which works.]

Thus for metabelian groups in general the implication diagram condenses to

$$
\begin{array}{c}
O & \iff & R & \iff & P & \iff & GP & \iff & RO & \iff & TF
\end{array}
$$

What about the possibility of reversing these arrows?

We show first that $R \Rightarrow 0$.

Example 7 Let $G$ be the group of all ordered pairs $(x,m)$ in which $x$ is rational and $m$ is an integer. We define multiplication in $G$ by

$$(x,m)(y,n) = (x + (-2)^m y, m + n).$$

This group has a normal Abelian subgroup consisting of the pairs $(x,0)$. The factor group is isomorphic to the additive group of integers and thus $G$ is metabelian. It cannot be ordered because

$$
(0,1)(x,0)(0,1)^{-1} = (0,1)(x,0)(0,-1) = (-2x,0) = (x,0)^{-2},
$$

and, if $(x,0) > (0,0)$ in any order on the group, then $(x,0)^{-2}$ would also have to be positive and this gives a contradiction.
Now suppose that \((x,m)^k = (y,n)^k\) for some elements \((x,m)\) and \((y,n)\) and \(k > 0\). Then
\[
(x + (-2)^m x + \ldots + (-2)^{(k-1)m} x, km) = (y + (-2)^n y + \ldots + (-2)^{(k-1)n} y, kn).
\]
From this we have, in particular, \(km = kn\) and therefore, since \(k \neq 0\), \(m = n\). Then \(ax = ay\), where \(a = 1 + (-2)^m + \ldots + (-2)^{(k-1)m}\).

Now \(a \neq 0\) for any \(k\) and it follows that \(x = y\). Hence \((x,m) = (y,n)\) and we see that \(G\) is an \(R\)-group.

Next \(P \not
arrow R\). An example has already been introduced as Example 1 (p. 13). That is, \(G\) is the group generated by elements \(a\) and \(b\) with the relation \(b^{-1}ab = a^{-1}\). This group has a normal Abelian subgroup - the subgroup generated by \(a\) - with factor group isomorphic to the additive group of integers. Thus \(G\) is metabelian and can clearly be polyordered. However it is not an \(R\)-group because \(b^2 = (ba)^2\) and obviously \(b \neq ba\).

The relationships between \(P\), \(GP\) and \(RO\) have not yet been worked out in full detail. Certainly there are examples of right-ordered metabelian groups in which the right-order is not a generalized polyorder. Examples 2 (p. 20) and 3 (p. 24) are both examples of this type. However, in all the examples known to me, there is always another right-order on the group which is a polyorder. Examples 2 and 3 can actually be ordered. What is really needed (assuming \(RO \not
arrow GP\)) is a right-ordered metabelian group which cannot be generalized polyordered in any way. As we shall see later such an example will have to be fairly complicated.

Lastly we show that \(TF \not
arrow RO\) in general.
Example 8 This example will be constructed as a normal extension of a suitable Abelian group $A$ by a non-cyclic group of order 4.

As we shall show later, there is a sense in which this will give a "minimal" example.

Let $F$ be the free Abelian group on the 9 generators $w_{ij} (i = 1,2,3; \ j = 1,2,3)$ written additively and let $B$ be the non-cyclic group of order 4 whose elements are $\{0,1,2,3\}$. We denote the group operation in $B$ by $*$ and note that its multiplication table is

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
1 & 0 & 3 & 2 \\
2 & 3 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}
\]

Corresponding to each element $k \in B$, we define a mapping $\alpha_k$ from $F$ into itself by setting

$$w_{ij}^{\alpha_k} = w_{ki} - w_{ki,j} * w_{k^*i,j}$$

for all $i$ and $j$ and extending by linearity to the other elements of $F$. (We make the convention here that $w_{io} = w_{oj} = 0$ for all $i$ and $j$.)

From its definition, each $\alpha_k$ is an endomorphism of $F$ and it is straightforward to show that

$$\alpha_k \alpha_\ell = \alpha_{k^*\ell} = \alpha_{\ell^*k}$$

for all $k$ and $\ell$. It follows that each $\alpha_k$ is actually an automorphism of $F$. The automorphisms have been chosen so that the set of elements $w_{ij}$ becomes a (normalized) factor system for a normal extension $G$ of $F$ by $B$. (See Kurosch, Vol.1, p.21 et seq.)
The group $G$ may be thought of as consisting of ordered pairs
$(f, i)$, where $f \in F$ and $i \in B$ and the multiplication in $G$ is
given by
$$(f_1, i)(f_2, j) = (f_1 + f_2 a_i + w_{ij}, i^*j).$$

This group $G$ can actually be right-ordered (see the remark
following Lemma 6.3 on p. 98), but we obtain an example of the type
presently required by factoring out the centre $Z$ of $G$. The first
task is to identify the elements of $Z$.

The element $(f_2, j)$ is in $Z$ if and only if, for all $i = 0, 1, 2, 3$,
$$(f_2, j)^{-1}(f_1, i)(f_2, j) = (-f_2 a_j - w_{jj}, j)(f_1 + f_2 a_i + w_{ij}, i^*j)$$
$$= (-f_2 a_j - f_1 a_j + f_2 a_i j + w_{ij} a_j + w_{ij}, i^*j, i)$$
$$= (f_1, i).$$

In particular, for $i = 0$, this reduces to $f_1 a_j = f_1$ for all $f_1 \in F$.

But, if $k \neq j$, then $w_{kk} a_j = w_{j, k} + w_{j, k} a_j$ unless $j = 0$.

Hence we must have $j = 0$ and the condition then reduces to
$f_2 a_i = f_2$ for all $i = 0, 1, 2, 3$.

Now, if $f_2 = \sum_{i, j \neq 0} \lambda_{ij} w_{ij}$, then
$$f_2 a_k = \sum_{i, j \neq 0} \lambda_{ij} \left\{ w_{k, i} - w_{k, i^*j} + w_{k, i^*j, i} \right\}$$
$$= \sum_{s \neq 0} \left\{ \frac{\lambda}{k - _1s} x, s \right\} w_{rs}$$
$$+ \sum_{s \neq 0} \left\{ \frac{\lambda}{s^t} - \frac{\lambda}{s^t - _1t} \right\} w_{ks}$$
$$= f_2.$$
Hence we have, for \( r \neq k \),
\[
\lambda_{Ts} = \lambda_{k^{-1}r,s}
\]
and, for \( r = k \),
\[
\lambda_{ks} = \sum_{t \neq 0} \left\{ \lambda_{st} - \lambda_{s^{-1}t, t} \right\}
\]
Since these have to hold for all \( k \), we deduce that \( \lambda_{1s} = \lambda_{2s} = \lambda_{3s} \) for all \( s = 1,2,3 \) and hence that \( f_2 \) is of the form
\[
f_2 = \lambda_1(w_{11} + w_{21} + w_{31}) + \lambda_2(w_{12} + w_{22} + w_{32}) + \lambda_3(w_{13} + w_{23} + w_{33})
\]
Thus the centre of \( G \) corresponds to the subgroup \( R \) of \( F \) which is generated by the elements \( w_{11} + w_{21} + w_{31}, w_{12} + w_{22} + w_{32} \) and \( w_{13} + w_{23} + w_{33} \). This is clearly an isolated subgroup of \( F \).
To show that \( G/Z \) is torsion-free we only need to show that the square of any non-central element \((f,k)\) cannot be in \( Z \). (The square of any element of \( G \) is, in any case, in \( F \)).
Now \((f,k)^2 = (f + f_{nk} + w_{kk}, 0)\) and, if \( f = \sum_{i,j \neq 0} \lambda_{ij}w_{ij} \), then
\[
f + f_{nk} + w_{kk} = w_{kk} + \sum_{i,j \neq 0} \lambda_{ij}(w_{ij} + w_{ki} - w_{k,i} + w_{k,j}).
\]
If \( k = 0 \), this all reduces to \( 2f \) and is in \( R \) if and only if \( f \in R \), i.e. \((f,k) = (f,0)\) was already a central element.
If \( k = 1 \), then to be in \( R \) the coefficients of \( w_{1s}, w_{2s}, w_{3s} \) must all be equal for each \( s = 1,2,3 \). From \( s = 1 \), we obtain
\[
1 + 2\lambda_{11} + \lambda_{12} + \lambda_{13} - \lambda_{23} - \lambda_{32} = \lambda_{21} + \lambda_{31},
\]
i.e.
\[
2\lambda_{11} + \lambda_{12} + \lambda_{13} - \lambda_{21} - \lambda_{23} - \lambda_{31} - \lambda_{32} = -1. \quad (1)
\]
From $s = 2$, we find
\[\lambda_{12} + \lambda_{21} + \lambda_{22} + \lambda_{23} - \lambda_{13} - \lambda_{12} = \lambda_{22} + \lambda_{32}\]
i.e.,
\[\lambda_{12} - \lambda_{13} + \lambda_{21} + \lambda_{23} - \lambda_{31} - \lambda_{32} = 0.\] (2)

Adding equations (1) and (2) we obtain a contradiction
\[2(\lambda_{11} + \lambda_{12} - \lambda_{31} - \lambda_{32}) = -1.\]

This shows that, if $(f,k)^2$ is in $Z$, then $k \neq 1$.

Similar calculations will show that also $k \neq 2$ or 3 and we deduce that, if $(f,k)^2$ is in $Z$, then $(f,k) = (f,0)$ is also in $Z$ and $G/Z$ is torsion free.

Now, if $g_1 = (0,1)$ and $g_2 = (0,2)$, straightforward calculations show that
\[g_1^2g_2^2g_1g_2^2 \in Z\]
and $g_1^2g_2^{-2}g_1g_2^{-1}g_1^2g_2^{-1} \in Z$
so that, modulo $Z$, the identity is in both the semigroups $S(g_1, g_2)$ and $S(g_1, g_2^{-1})$. It follows by Theorem 6 that $G/Z$ cannot be right-ordered.

Thus we may now fill in some of the gaps in the implication diagram for general metabelian groups to obtain

\[0 \not\overset{R}{\longrightarrow} P \not\overset{GP}{\rightarrow} RO \not\overset{TF}{\rightarrow}\]

That is the present situation for metabelian groups in general, but perhaps something more can be said if we restrict our attention to certain special types of metabelian group. In particular it is of interest to look at those groups $G$ which have a normal Abelian subgroup $A$ with finite Abelian factor group $G/A$. For these we shall see that it is possible to obtain more positive results.

The main theorem depends on three technical lemmas.
Lemma 6.1 Suppose that $A$ is a torsion free additive Abelian group and that $B$ is a finite group of automorphisms of $A$.

Say $|B| = n$. Let $\theta$ be the endomorphism of $A$ given by $\theta = \sum_{\beta \in B} \beta$ and let $K$ be the kernel of $\theta$ and $I$ the image of $\theta$.

Then

(i) $K$ is isolated in $A$,
(ii) $xa = x$ for all $x \in I$ and $a \in B$,
(iii) $K \cap I = \{0\}$,
(iv) $na \in K + I$ for all $a \in A$, and
(v) if $K \neq A$, then $I \neq \{0\}$.

Proof

(i) If $ma \in K$ for some $a \in A$ and $m > 0$, then $0 = (ma)\theta = m(a\theta)$. ($\theta$ is an endomorphism.) Since $A$ is torsion free, it follows that $a\theta = 0$ and hence that $a \in K$. Thus $K$ is isolated in $A$.

(ii) If $x = a\theta$ for some $a \in A$, then $x$

$$x = (a\theta)a = a(\theta a) = a\theta = x$$

because $\theta a = \sum_{\beta \in B} \beta a = \theta a$ as $\beta$ runs through all the elements of the group $B$, so does $\beta a$.

(iii) If $x \in K \cap I$, then $x = a\theta$ for some $a \in A$ and $0 = x\theta = (a\theta)\theta = n(a\theta) = nx$. Since $A$ is torsion free, it follows that $x = 0$ and hence that $K \cap I = \{0\}$

(iv) Clearly, for any $a \in A$,

$$na - a\theta = \sum_{\beta \in B} a(1 - \beta)$$

and

$$\sum_{\beta \in B} a(1 - \beta)\theta = \sum_{\beta \in B} a(\theta - \beta\theta) = 0 \quad (\theta \theta = \theta).$$

Hence $na \in K + I$.

(v) Now, if $K \neq A$, then, for any $a \notin K$, $na \in K + I$.

If $I \neq \{0\}$, then $na \in K$ and, by (i), $a \in K$. This is a contradiction and we deduce that $I \neq \{0\}$.  

Lemma 6.2 Suppose that $G$ is a torsion free group which has a normal Abelian subgroup $A$ with finite Abelian factor group $G/A$. Then $G$ induces a finite group $B$ of automorphisms of $A$ by conjugation and, in the notation of Lemma 6.1, the kernel $K$ of the endomorphism $\theta = \sum_{\beta \in B} \beta$ contains the commutator subgroup $G'$ of $G$.

Proof Suppose that $G/A$ is isomorphic to the Abelian group $D$. Then $G$ is a normal extension of $A$ by $D$ and its elements may be expressed as ordered pairs $(a,d)$, where $a \in A$ and $d \in D$. For this we will assume that $A$ is written additively and $D$ multiplicatively. The group operation in $G$ may then be expressed in the form

$$(a_1,d_1)(a_2,d_2) = (a_1 + a_2 \cdot \alpha_d, \omega(d_1,d_2), d_1d_2),$$

where $\alpha_d$ is an automorphism of $A$ corresponding to the element $d \in D$ and $\omega(d_1,d_2)$ is a factor system of $D$ over $A$ which satisfies the conditions

(i) $\alpha_d \cdot \alpha_{d_1} = \alpha_{d_2} \cdot \alpha_{d_1}$,

(ii) $\omega(1,d_2) = \omega(d_1,1) = 0$ for all $d_1$ and $d_2$ in $D$,

and (iii) $\omega(d_1,d_2) \cdot \omega = \omega(d,d_1) - \omega(d,d_1d_2) + \omega(dd_1,d_2)$ for all $d,d_1,d_2 \in D$.

Different elements $d_1$ and $d_2$ of $D$ may correspond to the same automorphism of $A$ and indeed this happens whenever $d_1d_2^{-1}$ corresponds to a coset of $A$ which is contained in the centralizer of $A$.

For this reason, the group $B$ of automorphisms of $A$ induced by $G$ is only isomorphic to a factor group of $D$ in general. Consequently, the endomorphism $\tau = \sum_{d \in D} \omega_d$ of $A$ may not be equal to $\theta = \sum_{\beta \in B} \beta$.

However it is true that $\tau = m\theta$, where $m = |D|/|B|$.
Now, if \( a \) is in the kernel of \( \tau \), then

\[
0 = a\tau = a(m\theta) = m(a\theta)
\]

and, since \( A \) is torsion free, it follows that \( a\theta = 0 \) and hence that \( a \in K \). Thus, to prove the lemma, it will be sufficient to show that every commutator is in the kernel of \( \tau \).

Let \( g_1 = (a_1, d_1) \) and \( g_2 = (a_2, d_2) \), then

\[
[g_1, g_2] = (-a_1d_1^{-1} - w(d_1^{-1}, d_1), d_1^{-1})(-a_2d_2^{-1} - w(d_2^{-1}, d_2), d_2^{-1})(a_1, d_1)(a_2, d_2)
\]

\[
= (-a_1d_1^{-1} - w(d_1^{-1}, d_1) - a_2d_2^{-1} - w(d_2^{-1}, d_2)a_d_1^{-1} +
\]

\[
+ w(d_1^{-1}, d_2^{-1}, d_1^{-1}, d_2^{-1}) \cdot (a_1 + a_2d_1^{-1} + w(d_1, d_2), d_1d_2)
\]

\[
= (-a_1d_1^{-1} - w(d_1^{-1}, d_1) - a_2d_1^{-1} - w(d_2^{-1}, d_2)a_d_1^{-1} + w(d_1^{-1}, d_2^{-1})
\]

\[
+ a_2d_1^{-1} - a_1^{-1} + a_2d_1^{-1} + a_2d_1^{-1} - w(d_1^{-1}, d_1)
\]

\[
- w(d_1^{-1}, d_2^{-1}) + w(d_1^{-1}, d_2^{-1}) + w(d_1^{-1}, d_2^{-1}) + w(d_1^{-1}, d_2^{-1}, d_1d_2)
\]

\[
+ w(d_1^{-1}, d_2^{-1}, d_1) + w(d_1^{-1}, d_2^{-1}, d_1d_2) + w(d_1^{-1}, d_2^{-1}, d_1d_2, 1)
\]

\[
= (-a_1d_1^{-1} + a_1d_2^{-1} - a_1d_1^{-1} - a_2d_1^{-1} - w(d_1^{-1}, d_1)
\]

\[
- w(d_1^{-1}, d_2^{-1}) + w(d_1^{-1}, d_2^{-1}) + w(d_1^{-1}, d_2^{-1}, d_1d_2) + w(d_1^{-1}, d_2^{-1}, d_1d_2, 1) .
\]

To show that this is in \( K \), we need to show that the \( A \)-component is in the kernel of \( \tau \). For this we note that \( a_d\tau = \tau \) for all \( d \in D \).

Now

\[
(-a_1d_1^{-1} + a_1d_2^{-1} - a_1d_1^{-1})\tau = -a_1\tau + a_1\tau = 0
\]

and

\[
(-a_2d_1^{-1}d_2^{-1} + a_2d_1^{-1}a_d_1^{-1}d_2^{-1})\tau = -a_2\tau + a_2\tau = 0
\]
Furthermore

\[ w(d_1^{-1}, d_1) \tau = \sum_{d \in D} \left\{ w(d, d_1^{-1}) - w(d, 1) + w(dd_1^{-1}, d_1) \right\} \]

\[ = \sum_{d \in D} \left\{ w(d, d_1^{-1}) + w(d, d_1) \right\} \]

\[ [dd_1^{-1} \text{ runs through all the elements of } D \text{ as } d \text{ does.}] \]

\[ w(d_1^{-1}d_2^{-1}, d_1) \tau = \sum_{d \in D} \left\{ w(d, d_1^{-1}d_2^{-1}) - w(d, d_2^{-1}) + w(dd_1^{-1}d_2^{-1}, d_1) \right\} \]

\[ = \sum_{d \in D} \left\{ w(d, d_1^{-1}d_2^{-1}) - w(d, d_2^{-1}) + w(d, d_1) \right\} \]

\[ [dd_1^{-1}d_2^{-1} \text{ also runs through all the elements of } D \text{ as } d \text{ does.}] \]

\[ w(d_1^{-1}d_2^{-1}, d_2) \tau = \sum_{d \in D} \left\{ w(d, d_1^{-1}d_2^{-1}) - w(d, d_1^{-1}) + w(d, d_2) \right\} \]

and \[ w(d_2^{-1}, d_2) \tau = \sum_{d \in D} \left\{ w(d, d_2^{-1}) + w(d, d_2) \right\} \]

Hence

\[ \left\{ -w(d_1^{-1}, d_1) - w(d_1^{-1}d_2^{-1}, d_2) + w(d_1^{-1}d_2^{-1}, d_1) + w(d_2^{-1}, d_2) \right\} \tau = 0 \]

and this completes the proof.

Lemma 6.3 Suppose that \( G \) is a torsion free group which has a normal Abelian subgroup \( A \) with finite Abelian factor group \( G/A \) and suppose that, in the notation of Lemmas 6.1 and 6.2, the kernel \( K \) of \( \theta \) is equal to \( A \). Then \( G \) cannot be right-ordered.

Proof For this proof it is more convenient to treat the group \( A \) directly as a subgroup of \( G \). That is, \( A \) is to be written as a multiplicative group. In that case the endomorphism \( \tau \) (see the proof of Lemma 6.2) must be interpreted as follows.
We choose a complete set of coset representatives \( d_1 = 1, d_2, \ldots, d_n \) for \( G \) over \( A \) and then we may define a mapping \( \tau \) from \( G \) into \( G \) by setting \( g\tau = d_1gd_1^{-1} \cdots d_ngd_n^{-1} \). When restricted to \( A \), this becomes the endomorphism \( \tau \) as before. Now, if the kernel of \( \theta \) is equal to \( A \), then it is clear that the kernel of \( \tau \) is also equal to \( A \). That is, we have \( a\tau = 1 \) for all \( a \in A \).

Since \( G/A \) is supposed to be finite and Abelian, it is isomorphic to a direct product of finite cyclic groups. We may suppose therefore that there are cosets \( Ac_1, \ldots, Ac_r \) which are independent generators of \( G/A \) and that they have finite orders \( m_1, \ldots, m_r \), say. That is, in particular, \( c_i^{m_i} \in A \) for each \( i = 1, \ldots, r \). We will show that \( 1 \in S(c_1^{\varepsilon_1}, \ldots, c_r^{\varepsilon_r}) \) for all possible choices of the exponents \( \varepsilon_i (\pm 1) \) and hence by Theorem 6 that \( G \) cannot be right-ordered. The proof requires a certain amount of preliminary construction.

For any \( c \in G \), we define a mapping \( \theta_c : G \to G \) by
\[
g^\theta_c = g \cdot cgc^{-1} \cdot c^2gc^{-2} \cdot \cdot \cdot \cdot c^{m-1}gc^{-(m-1)},
\]
where \( m \) is the order of \( Ac \) as an element of \( G/A \). (i.e. \( c^m \in A \)).

When considered as mappings on \( A \), these are all endomorphisms of \( A \) and we may observe the following relationships amongst them.

(i) \( \theta_{ac} = \theta_{ca} = \theta_c \) for all \( a \in A \)
(ii) \( \theta_{c^{-1}} = \theta_c \)
(iii) \( \theta_{cd} = \theta_{dc} \) for all \( c \) and \( d \in G \)
(iv) \( \theta_{c_1} \cdots \theta_{c_r} = \theta_c \)

For (i) we merely note that \( (ac)^t = a^tc \) where \( a \in A \) and then
\[ (ac)^t g(ac)^{-t} = a^tc gc^{-t} c^{-t} a^{-1} = c gc^{-t}. \]
For (ii), (iii) and (iv), we note that, modulo $A$, $c^{-1}$ generates all the powers of $c$, $c$ and $d$ commute and $c_1, \ldots, c_r$ generate all the coset representatives $d_1, \ldots, d_n$.

Now we set up a sequence of words $w_i(x_1, \ldots, x_i)$ in variables $x_1, \ldots, x_r$, where $x_1$ is to take its values in the set $Ac_1 \cup Ac_1^{-1}$. In view of (i) and (ii), this will mean that all the possible values of $x_1$ will produce the same endomorphism $\theta_i$ of $A$. For the sake of convenience we will denote this endomorphism by $\theta_i$ and it will be understood in the general formulae which follow that this is to be interpreted as $\theta_{x_1}$. We note also that $x_1^{m_1} \in A$ and that for all $a \in A$, $(ax_1)^{m_1} = a\theta_1 x_1^{m_1}$.

The words $w_i$ are defined inductively by setting

$$w_1(x_1) = x_1^{m_1}$$

and then

$$w_{i+1}(x_1, \ldots, x_{i+1}) = w_i(x_1, \ldots, x_i)^{\theta_{i+1}} \cdot (x_{i+1} \theta_{i+1} \ldots \theta_i)^{2^{i-1}}.$$  

We remark that the words could be defined directly in terms of the $x_1$'s without reference to the $\theta_i$'s, but this seems to be the best formulation for the subsequent calculations. We proceed in several steps using induction on $i$.

1. The words $w_i$ take all their values in $A$.

   Clearly $w_1(x_1) = x_1^{m_1} \in A$ for all $x_1 \in Ac_1 \cup Ac_1^{-1}$.

   Then, if $w_i(x_1, \ldots, x_i)$ is known to take its values in $A$, $w_i(x_1, \ldots, x_i)^{\theta_{i+1}} \in A$. Since $x_{i+1}^{m_{i+1}} \in A$, $x_{i+1}^{m_{i+1}} \theta_1 \ldots \theta_i$ is also in $A$ and hence $w_{i+1}(x_1, \ldots, x_{i+1})$ takes its values in $A$.

2. For any $a \in A$, $w_i(ax_1, \ldots, ax_i) = (a\theta_1 \ldots \theta_i)^{2^{i-1}} w_i(x_1, \ldots, x_i)$.

   First $w_1(ax_1) = (ax_1)^{m_1} = a\theta_1 x_1^{m_1} = a\theta_1 w_1(x_1)$ and so the formula is correct for $i = 1$. 

Suppose it also true for \( i \) and consider \( w_{i+1} \). Now by definition,

\[
w_{i+1}(ax_1, \ldots, ax_{i+1}) = w_i(ax_1, \ldots, ax_i) \theta_{i+1} ((ax_{i+1})^{m_i+1} \theta_1 \ldots \theta_i)^{2i-1}
\]

\[
= ((a\theta_1 \ldots \theta_i)^{2i-1} w_i(x_1, \ldots, x_i) \theta_{i+1} ((a\theta_1 x_{i+1})^{m_i+1} \theta_1 \ldots \theta_i)^{2i-1}
\]

(Induction hypothesis)

\[
= ((a\theta_1 \ldots \theta_{i+1})^{2i-1} w_i(x_1, \ldots, x_i) \theta_{i+1} ((a\theta_1 \ldots \theta_{i+1}) x_{i+1} \theta_1 \ldots \theta_i)^{2i-1}
\]

(using (iii))

\[
= (a\theta_1 \ldots \theta_{i+1})^{2i} w_{i+1}(x_1, \ldots, x_{i+1})
\]

Thus the formula is proved by induction on \( i \).

3. The \( w_j \) are positive words in the sense that, when written out in full, they involve only positive powers of the variables.

This is clearly true for \( w_1 \) and consider \( w_{i+1} \). Applying the formula from 2, we have

\[
w_{i+1}(x_1, \ldots, x_{i+1}) = w_i(x_1, \ldots, x_i) \theta_{i+1} w_i(x_{i+1}, x_1, \ldots, x_{i+1} x_i) w_i(x_1, \ldots, x_i)^{-1}
\]

\[
= (w_i(x_1, \ldots, x_i) x_{i+1})^{m_i+1} \cdot x_{i+1}^{m_i+1} \cdot w_i(x_{i+1}, x_1, \ldots, x_{i+1} x_i) \cdot w_i(x_1, \ldots, x_i)^{-1}
\]

\[
= x_{i+1}(w_i(x_1, \ldots, x_i) x_{i+1})^{m_i+1} \cdot x_{i+1}^{m_i+1} \cdot w_i(x_1, \ldots, x_i^{m_i+1})
\]

Now, since \( w_i \) is a positive word, its first factor when written out as a product of \( x_1 \)'s must also have a factor \( x_{i+1}^{m_i+1} \) on the left and this will cancel the factor \( x_{i+1}^{m_i+1} \) in \( w_{i+1} \). The nett result is a positive word for \( w_{i+1} \).

4. For all \( i = 1, \ldots, r \), \( w_i(x_1, \ldots, x_i)^{m_1 \ldots m_i} \) is in the image of \( \theta_1 \ldots \theta_i \).
First \( w_1(x_1)^m_1 = (x_1^m_1)^m_1 = x_1^{m_1 m_1} \), because \( x_1 \) commutes with \( x_1^{m_1} \). That is, \( w_1(x_1)^m_1 \in \text{Im}\theta_1 \).

Suppose that \( w_1(x_1, \ldots, x_i)^m_1 \cdots m_i \in \text{Im}\theta_1 \cdots \theta_i \). Then
\[
\begin{align*}
w_{i+1}(x_1, \ldots, x_{i+1})^{m_1 \cdots m_{i+1}} &= \left\{ w_1(x_1, \ldots, x_i)^{\theta_1 \cdots \theta_{i+1}} \right\}^{m_1 \cdots m_{i+1}} \\
&= \left\{ x_{i+1}^{m_{i+1}} \theta_1 \cdots \theta_{i+1} \right\}^{2^{i-1} m_1 \cdots m_{i+1}}.
\end{align*}
\]

By induction hypothesis, \( w_1(x_1, \ldots, x_i)^m_1 \cdots m_{i+1} \) is clearly in \( \text{Im}\theta_1 \cdots \theta_{i+1} \). Also \( (x_{i+1}^{m_{i+1}})^{m_{i+1}} = x_{i+1}^{m_{i+1} \theta_{i+1}} \) and hence
\[
\left\{ x_{i+1}^{m_{i+1}} \theta_1 \cdots \theta_{i+1} \right\}^{m_{i+1}} \in \text{Im}\theta_1 \cdots \theta_{i+1} \quad \text{(again using (iii)).}
\]
Thus altogether \( w_{i+1}(x_1, \ldots, x_{i+1})^{m_1 \cdots m_{i+1}} \in \text{Im}\theta_1 \cdots \theta_{i+1} \).

Finally applying this last result to the case \( r \), we have
\( w_r(x_1, \ldots, x_r)^m_1 \cdots m_r = 1 \) and, assuming \( A \) is torsion free,
\( w_r(x_1, \ldots, x_r) = 1 \).

In particular, \( w_r(c_1^{\epsilon_1}, \ldots, c_r^{\epsilon_r}) = 1 \) for all possible choices of the exponents \( \epsilon_1 (= \pm 1) \) and, since \( w_r \) is a positive word, \( 1 \in S(c_1^{\epsilon_1}, \ldots, c_r^{\epsilon_r}) \) for all choices of the \( \epsilon_i \). Hence \( G \) cannot be right-ordered.

We remark that Example 3 (p. 87) is an example of a torsion free metabelian group in which "\( K = A \)" and that is another reason why it cannot be right-ordered. The intermediate group \( G \) appearing in the construction of the example does not have this property and the subgroup \( K \), which is actually isolated in \( G \) in this case, may be used in constructing a right-order on that group. (see the remark on p. 88.)
Corollary  Suppose that $G$ is an $R$-group which has a normal Abelian subgroup $A$ with finite Abelian factor group $G/A$. Then $G$ can be ordered.

Proof  Certainly $G$ can be polyordered (Theorem 15) and therefore, by the lemma, the kernel $K$ of $\theta$ is not equal to $A$. Hence, by (v) of Lemma 6.1, the image $I$ of $\theta$ is not equal to the identity. But, by (ii) of Lemma 6.1, $I$ is contained in the centre of $G$. In other words, $G$ has a non-trivial centre, $Z$.

Now, if $G$ is an $R$-group, then so is $G/Z$. The conditions of the corollary apply equally to $G/Z$ and hence $G/Z$ also has a non-trivial centre. By repeating this argument (transfinitely if necessary) we see that $G$ must be a $ZA$-group. Since any $ZA$-group is locally nilpotent (see Kurosch, Vol.II, p.223) $G$ can be ordered.

Theorem 17  Suppose that $G$ is a group which has a normal Abelian subgroup $A$ with finite Abelian factor group $G/A$. Then, if $G$ can be right-ordered, it can also be polyordered.

Proof  We use induction on the order of $G/A$. When $|G/A| = 1$, $G$ is Abelian and the theorem is trivially true. Suppose therefore that $|G/A| > 1$ and that the theorem is true for all metabelian groups which have a normal Abelian subgroup with Abelian factor group of order less than $|G/A|$.

If $G$ can be right-ordered, then, by Lemma 6.3, the kernel $K$ of $\theta$ is not equal to $A$ and, by Lemma 6.2, it contains the commutator subgroup $G'$. Hence $G/K$ is Abelian.
Clearly, if $G/K$ were torsion free, then any order on $G/K$ could be combined with an order on the Abelian group $K$ to give a polyorder on $G$ and the proof would be complete. However, $G/K$ may not be torsion free in general.

Suppose therefore that $T/K$ is the torsion subgroup of $G/K$. Again it is clear that $G$ can be polyordered if and only if $T$ can be polyordered. Any order on $G/T$ combined with a polyorder on $T$ would give a polyorder on $G$.

Since, by (i) of Lemma 6.1, $K$ is isolated in $A$, we must have $A \cap T = K$. Hence, by the Isomorphism Theorem, $T/K \cong TA/A$ - a subgroup of $G/A$. If $|T/K| < |G/A|$, then, by induction hypothesis, $T$ can be polyordered. It only remains to deal with the possibility that $|T/K| = |G/A|$, i.e. that $T/K$ is isomorphic to $G/A$.

In the natural isomorphism between $T/K$ and $G/A$ the coset $Kd$ corresponds to the coset $Ad$ ($d \in T$). The automorphism of $K$ induced by $d \in T$ maps $a$ to $dad^{-1}$ and is equal to the restriction of the automorphism of $A$ induced by $d$. Thus $T$ induces the same automorphisms of $K$ as $G$ does. It follows that the endomorphism $\theta_T$ of $K$ as a subgroup of $T$ is the same as $\theta$ on $K$. That is $K\theta_T = \{1\}$. But, by Lemma 6.3, this means that $T$ cannot be right-ordered and we have a contradiction. We deduce that $|T/K|$ cannot be equal to $|G/A|$ and this completes the proof.

Note that, if we were trying to prove a similar theorem for general metabelian groups - say $G$ has a normal Abelian subgroup $A$ with Abelian factor group $G/A$ - then it would be sufficient to deal with the case when $G/A$ is a torsion group. For then, if $T/A$ were the torsion subgroup of $G/A$ in general, then an order on $G/T$ could
be combined with a (generalized) polyorder on $T$ to give a (generalized) polyorder on $G$. Now, when $G/A$ is a torsion group, every finitely generated subgroup of $G$ will be of the type just dealt with in the theorem. In other words, we have actually shown that a metabelian group which can be right-ordered can be locally polyordered. In terms of the implication diagram in Section 4 (p.53), this means that, for metabelian groups, $RO \Rightarrow LGP$ and incidentally proves that $RO \iff LGP$. [That is, a metabelian group can be right-ordered if and only if it is locally indicable.] Thus the problem of establishing the status of Theorem 17 for general metabelian groups is closely associated with the more general problem mentioned in Section 4 of whether a locally generalized polyorderable group is generalized polyorderable.

Lastly we note that Example 8 falls into the category of groups presently being dealt with and so we still have $TF \iff RO$. Similarly Example 1 is also of this type. The group there has an Abelian normal subgroup generated by $a$ and $b^2$ (since $b^{-1}ab = a^{-1}$, $b^{-1}a^{-1}b = a$ and hence $b^{-2}ab^2 = a$) and its factor group is cyclic of order 2. Thus we also still have $P \iff R$.

Alternatively, the implication diagram for metabelian groups which have normal Abelian subgroups with finite Abelian factor groups becomes

$$O \iff R \iff P \iff GP \iff RO \iff TF$$

Amongst the groups for which $G/A$ is finite there are those for which $G/A$ is actually cyclic. For this type of group, there is just one more positive result. Example 1 still shows that $P \iff R$ even in this case.
Theorem 18 Suppose that the group $G$ has a normal Abelian subgroup $A$ with finite cyclic factor group $G/A$. Then $G$ can be right-ordered if and only if it is torsion free.

Proof A careful look at the proof of Theorem 17 shows that the crucial point which makes the induction work is that $K \neq A$.

In Theorem 17, this may be deduced from the fact that $G$ is assumed to be right-ordered. Here we show that $G$, being torsion free, is sufficient to ensure that $K \neq A$ and the induction may proceed as before noting only that $T/K$, being isomorphic to a subgroup of $G/A$, will still be cyclic.

Suppose that $Ac$ is a generator of $G/A$ and that $c^m \in A$ ($n$ is the order of $G/A$). We show that this element $c^m$ cannot be in $K$. For

$$c^m \cdot c \cdot c^{-1} \cdot c^2 \cdot c^{-2} \cdot \ldots \cdot c^{m-1} \cdot c^{-(m-1)}$$

$$= c^m$$

and $c^{n/2}$ cannot be equal to the identity because $G$ is torsion free. It follows that $K \neq A$ as required.

Of course, in this situation, any right-order on $G$ must be an extension of an order on $A$ and it may be thought that perhaps every order on $A$ could be extended in this way, thus providing an alternative proof of the theorem. However we now give an example to show that this is not true in general.

Example 9 Let $G$ be the set of all ordered pairs of the form $(x,r)$ where $x = (x_1, \ldots, x_5)$ is a vector with rational components and $r$ is an integer. Define multiplication in $G$ by
\[(x, r)(y, s) = (x + y^M, r + s)\]

where \( M \) is the matrix

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & -\frac{1}{2} & -\frac{3}{2} & 0 \\
0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\
0 & -\frac{3}{2} & 0 & 0 & -\frac{1}{2}
\end{pmatrix}
\]

Note that \( M^6 = I \). It is straightforward to verify that \( G \) is torsion free and metabelian. [In fact it is a split extension of \( Q \times Q \times Q \times Q \times Q \) by \( Z \), where \( Q \) stands for the additive group of rationals and \( Z \) for the integers.]

Now \( G \) has a normal Abelian subgroup \( A \) consisting of all pairs of the form \((x, 6k)\). For

\[
(x, 6k)(y, 6\ell) = (x + y^M 6k, 6(k + \ell))
\]

\[
= (x + y, 6(k + \ell))
\]

\[
= (y, 6\ell)(x, 6k)
\]

and

\[
(y, s)^{-1}(x, 6k)(y, s) = (-y^M -s, -s)(x + y^M 6k, 6k + s)
\]

\[
= (-y^M -s + x^M -s + y^M -s, 6k)
\]

\[
= (x^M -s, 6k)
\]

The factor group \( G/A \) is cyclic of order 6.

The order \(<\) on \( A \), which is defined as follows

\[(x, 6k) > (0, 0) \iff x_1 = ... = x_i = 0 \quad \text{and} \quad x_{i+1} > 0\]

for some \( i = 1, ..., 4 \)

or \( x = 0 \) and \( k > 0 \)

cannot be extended to a right-order on \( G \).
For, if \( \mathbf{x} = (-1, 0, 0, 0, 0) \) and \( \mathbf{y} = (0, -1, 0, 0, 0) \), then

\[(\mathbf{x}, 2)^2 = (3\mathbf{x}, 6) \text{ is in } A \text{ and } (\mathbf{y}, 3)^2 = (2\mathbf{y}, 6) \text{ is also in } A.\]

From the definition of \( < \) on \( A \), both \((\mathbf{x}, 2)^3\) and \((\mathbf{y}, 3)^2\) are negative and hence, in any extension of \( < \), \((\mathbf{x}, 2)\) and \((\mathbf{y}, 3)\) must also be negative. But \((\mathbf{x}, 2)(\mathbf{y}, 3) = (x + y, 3)^2, 5)\) and, by calculation, \((x + y, 3)^2, 5)^6 = (0, 30)\) which is positive in \( < \). Hence \((\mathbf{x}, 2)(\mathbf{y}, 3)\) would have to be positive in any extension of \( < \) and we have a contradiction. The order \( < \) cannot be extended.

Thus, for metabelian groups which have a finite cyclic factor group \( G/A \), the implication diagram becomes

\[O \leftrightarrow R \leftrightarrow P \leftrightarrow GP \leftrightarrow RO \leftrightarrow TF.\]

There still remains the question, in this and the previous case, of whether every right-order is a (generalized) polyorder. We have seen that, for general metabelian groups, this is not so, but both the examples given (Examples 2 and 3) do not have normal Abelian subgroups with finite Abelian factor groups. We note finally that a positive answer to this question would imply a positive answer to the question of whether, in a general metabelian group, right-orderability is sufficient to imply generalized polyorderability. For then the argument proposed on p. 100 would go through.
References

Textbooks


Papers


17. Smirnov, D.M. "On One-sided Orders in Groups with Ascending Central Series" (Russian), Algebra i Logika, 6 (1968), 77-78.