CREEP OF STRUCTURES SUBJECTED
TO CYCLIC LOADING

A dissertation submitted to the University of Leicester
for the degree of Doctor of Philosophy.

by

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JULY 1972.
PREFACE

The work for this dissertation was carried out at Leicester University between May 1968 and April 1972. The author is indebted to Professor F.A. Leckie for suggesting the problem and his encouragement, guidance and criticism which have been greatly appreciated.

Thanks are due to Professor G.D.S. MacLellan for the provision of facilities in the Engineering Department of Leicester University and the Director of Leicester University Computer Laboratory for the use of a digital computer. The author also gratefully acknowledges financial support by the Science Research Council for the experimental work described in Chapter 2.

The author would like to thank Dr. A.R.S. Ponter for many helpful discussions on the mechanics of continua and Professor J.B. Martin for profitable discussions during his sabbatical leave at Leicester University in the academic year 1969-70.

Among many others to whom the author is grateful are: Mr. C.J. Morrison for assistance with the experimental work; Mrs. B.E. Knowles for typing a difficult manuscript; and Mrs. L.D. Aucott, Mrs. B. Shore and Mrs. J.J. Westerman who traced the figures.

This dissertation describes the original work of the author and no part of it has been submitted to any other University.
SUMMARY

The development over the last decade of the reference stress method for estimating the deformation of structures composed of time dependent Maxwell material is reviewed, together with the implications of recently derived energy theorems based on idealized material models.

Experiments are described which confirm predictions implicit in two energy theorems which extend the concepts of a plastic limit load and a plastic shakedown state to situations involving time hardening creep.

The influence of constitutive relationships on stress redistribution effects which in turn affect the deformation of structures subjected to both constant and cyclic histories of loading are considered, and it is argued that the two energy theorems derived for time hardening materials provide conservative bounds which permit the designer to estimate deformation of structures composed of a wider class of materials with related constitutive relationships.

An empirical method is proposed for estimating structural creep deformation due to cyclic loading. The method applies to structures composed of materials whose creep law for constant uniaxial stress is known, but knowledge of the form of the creep law for time varying stress is not required, as use is made of data obtained from a single cyclic creep test and results are obtained from a weighted time hardening calculation.
In order to check the proposed procedure calculations were performed for a two-bar structure in which stress redistribution effects were particularly severe. At worst the errors in the predicted deformation rate corresponded to a 2% error in the applied load. The results also suggest that in most practical situations the actual solution is likely to correspond to an optimal upper bound provided by one of the energy theorems. The method also permits this optimal bound to be applied to structures composed of a wider class of materials with related constitutive relationships.
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NOTATION

a  Inner radius of tube
A  Dimensionless parameter defined in text
A(t) Complementary strain energy function
b  Width of beam, outer radius of tube
B, B_o Dimensionless parameters defined in text which are measures of energy dissipation obtained from bounds
C_{ijkl} Elastic constants tensor
d  Depth of beam
D, D_o Dimensionless parameters defined in text which are measures of energy dissipation obtained from elastic-creep calculations
e, e_{ij} elastic strain, elastic strain tensor
E  Young's Modulus
E(\sigma_{ij}) Complementary strain energy density
f(\sigma_{ij}) Yield function
f(t) Function of time
F  Generalized Load
g(t) Function of time
H  Parameter defined in text
I  Second moment of area
k  Parameter in creep equation
K  Dimensionless curvature rate
\ell  Length of bar, superscript denoting lower bound
m  Andrade constant in creep law
M(t) Moment
n  Stress exponent in creep law
p, p_{ij} Plastic strain, plastic strain tensor
p(t) Pressure
\( P, P(t) \) \( P_i(t) \) \( P_L, P_S \) \( r \) \( s \) \( s_{ij} \) \( S \) \( t, t_o \) \( t_{10} \) \( T \) \( u \) \( U \) \( U^o \) \( U^S \) \( v, v_{ij} \) \( V \) \( W \) \( X \) \( y \) \( z \) \( \alpha \) \( \beta \) \( \gamma \) \( \delta \) \( \hat{\delta} \) 

Load

Limit load and shakedown load respectively

Number of cycles, radius, suffix denoting radial direction

Denotes stationary state

Deviatoric stress tensor

Surface to which loads are applied

Time, arbitrary time measure

Stress redistribution time

Period of loading cycle

Radial displacement, superscript denoting upper bound

Dimensionless radial displacement rate of bore of tube

Initial elastic complementary strain energy

Elastic complementary strain energy due to stationary stress field

Creep strain, creep strain tensor

Volume

Work

Dimensionless displacement rate of two-bar structure

Distance from neutral axis of beam

Distance along axis of tube, suffix denoting axial direction

Parameter quantifying reference stress

Parameter relating volume of uniaxial specimen to volume of structure

Parameter quantifying equivalent steady stress

Displacement of two-bar structure

Creep displacement accumulated during first cycle in absence of stress redistribution effects
\( \Delta \) Denotes a finite increment  
\( \varepsilon \) Total strain, creep strain  
\( \varepsilon_{ij} \) Total strain tensor, creep strain tensor  
\( \varepsilon_e \) Effective strain  
\( \varepsilon_o \) Arbitrary strain measure  
\( \bar{\varepsilon} \) Strain due to reference stress  
\( \dot{\varepsilon} \) Creep strain accumulated during first cycle  
\( \zeta_o \) Strain intercept  
\( \eta \) Shape parameter  
\( \theta \) Suffix denoting circumferential direction  
\( \kappa \) Curvature  
\( \lambda \) Loading parameter  
\( \mu \) Plastic multiplier, loading parameter  
\( \nu \) Poisson's ratio  
\( \bar{\rho} \) Constant residual stress field  
\( \rho(t) \) Residual stress field  
\( \sigma, \sigma_{ij}, \sigma_o \) Stress, stress tensor, arbitrary stress measure respectively  
\( \sigma_e \) Effective stress  
\( \bar{\sigma} \) Reference stress  
\( \sigma^\kappa \) Equivalent steady stress  
\( \sigma^\kappa_{ij} \) Arbitrary stress field  
\( \dot{\sigma}_{ij} \) Instantaneous linear elastic stress field  
\( \tau \) Time, period of stress cycle  
\( \tau_{10} \) Dimensionless stress redistribution time  
\( \phi \) Energy dissipation function  
\( \phi_o \) Dimensionless structural parameter  
\( \chi, \chi_o \) Normalized displacement of two-bar structure and associated intercept respectively  
\( \psi, \psi_o \) Normalized strain and associated intercept respectively
1.1

CHAPTER 1

DISCUSSION OF THE PROBLEM.

1.1 Introduction

The laws of thermodynamics are such that efficient operation of power plant requires that components are subjected to loads at high temperature, and the creep performance of components is therefore a limiting factor in operating efficiency. The high capital cost of power plant requires that the majority of the components have a design life of about 20 years, although 100,000 hours (11.5 years) rupture data is often used as a basis for design. Metallurgists have produced creep resistant steels which permit higher operating temperatures and pressures at the expense of ductility, and fears have been expressed that the use of such materials may lead to catastrophic failure.

Although some plant is run continuously except for periodic overhaul, other plant is operational for only part of the time, frequently being subjected to periodic operation, (i.e. several hours a day each day). The majority of component design is based on one or other of the codes of practice, which present a set of empirical rules, which if obeyed should result in a safe design. Unfortunately these codes of practice present a severe limitation to designers and can result in uneconomic designs, and in some areas present little guidance to the designer. There is a clear need for methods of analysis which are simple and give the designer help in making decisions, which are based on rational concepts but are not merely a set of empirical relationships. Once an initial design has been obtained then more complex and time consuming calculations can be performed if this is considered necessary.
1.2 Constitutive Relationships and Structural Performance.

The analysis of structures composed of time dependent materials has received considerable attention and the relevant literature is extensive. Two differing approaches to the problem have tended to emerge. On the one hand much attention has been paid to the acquisition of uniaxial tensile creep data and considerable effort has been expended in attempting to devise constitutive relationships which adequately predict the material behaviour for time varying stress and temperature. Van Leeuwen [1.1] reviews the majority of the relevant literature in this field.

The constitutive relations which most closely fit the observed material behaviour are such that the structural analysis consists essentially of the solution of a set of non-linear (integro-) differential equations with non-constant coefficients. The advent of high speed digital computers has made possible the step-wise integration (with respect to time) of these equations using techniques which are now well established, [1.2, 1.3, 1.4]. Elastic, creep and plastic components of the strain field can be incorporated together with temperature effects, and although the effort required to write and de-bug such a computer program is considerable, such computations have been performed [1.5]. However the material model which is adopted is of necessity limited in several respects. Firstly the plastic and creep effects can be expected to interact and initial prestraining can be expected to influence the subsequent creep performance of the material. Although more complex material models (e.g. a work hardening plastic component) can be included, they require a knowledge of the material behaviour which can only be gained from extensive testing. Secondly there is no hypothetical creep law which adequately predicts material behaviour due to time varying stress [1.6]. Furthermore many materials exhibit anisotropic
behaviour and the Bauschinger effect is frequently neglected in material models. The effects of welds can further influence the material behaviour locally.

It can be seen therefore that any such computer solutions would require an enormous testing programme to provide all the material data required, and this is clearly impractical. The inevitable conclusion is therefore that although adequate computing power is now available to permit the solution of such problems, the basic lack of realistic constitutive relations precludes such an approach from being justified and alternative methods of analysis have been evolved.

Some workers have continued to concentrate their efforts on an empirical description of uniaxial creep test data, and this has resulted in many hypothetical creep laws being proposed (1.1). The tendency has been for new proposed laws to become more complex and less suited to digital computer applications (1.6, 1.7, 1.8).

Another philosophy has centred on the premise that there is little future in the adoption and use of creep laws of increasing complexity which although descriptive are not predictive. Furthermore the increased precision with which certain aspects of material behaviour is described does not necessarily result in greater accuracy in the prediction of structural behaviour. Several workers (1.9, 1.10, 1.11, 1.12) have approached the problem from a different direction. Instead of attempting to describe the creep behaviour of materials from specific test data, a simple material model was initially adopted and structural behaviour for the hypothetical material then examined to determine what parameters are important and what creep tests are required to enable realistic but admittedly approximate predictions of structural behaviour to be made. A related field has been the derivation and application of energy theorems for structures composed of certain model materials (1.13, 1.14, 1.15, 1.16, 1.17). Some of these
energy theorems are basically extensions of the extremum principles for elastic continua \((1.18)\), and have been progressively extended to encompass a wider class of material models and more general classes of loading.

1.3 Deformation due to Secondary Creep.

Hoff \((1.19)\) considered secondary creep deformation in the absence of elastic effects and argued that this was justified for structures in which the creep strain was of the order of \(1\%\) since the maximum elastic strain was of the order \(0.1\%\). The stress distribution can then be considered invariant with time and the well known theorems of minimum total energy and minimum complementary energy can be applied as if to a non-linear elastic material to obtain bounds on the energy dissipation in the structure.

A uniaxial creep law of the form

\[ \dot{\varepsilon} = k \sigma^n \]  

\((1.1)\)

has much to commend it as it permits useful theoretical simplifications and in many situations a precise knowledge of the value of \(n\) is unimportant. The work of Calladine and Drucker \((1.13,1.20)\) on Nesting Surfaces showed that the energy dissipation in a minimum weight structure could be expressed in a form invariant with \(n\) and that little variation with \(n\) may occur "in spite of departure from the minimum weight condition." For structures containing stress concentrations, bounds on the energy dissipation could be obtained by considering geometrically similar structures composed of materials having values of \(n\) greater and less than that for the material in question. For example the linear case \(n = 1\) and the rigid-perfectly plastic case \(n \to \infty\) frequently permit analytical simplifications and provide bounds on the energy dissipation of the structure composed of a material for which \(1 < n < \infty\).
Anderson Gardner and Hodgkins (1.9) considered the deformation rate of beams composed of material where the stress index \( n \) was itself some function of stress, and suggested that the assumption of a creep law of the form given by equation (1.1) would allow sufficiently accurate estimation of the deformation rate providing the creep test was performed at a "representative stress". In other words providing the creep test data was obtained at a suitable stress level the variation of \( n \) with \( \sigma \) was not important in practice.

Mackenzie (1.11) considered the creep deformation of various structures for a range of values of \( n \) and showed that the deformation rate could be estimated with acceptable accuracy with the data obtained from a single creep test at a "reference" stress without precise knowledge of the value of \( n \). Leckie (1.21) has extended the concept of a reference stress based on energy dissipation in the stationary state and it is this procedure which is extended further in Chapter 5.

The outcome of these ideas has been a procedure for obtaining a reasonable estimate of structural creep deformation from a minimum of information about the material behaviour. These ideas will be considered further in Chapter 3.

1.4 Creep due to Variable Loading.

The lack of adequate constitutive equations for time varying stress reduce considerably the worth of any calculation for variable loading. If stress redistribution effects are known to be small or the number of changes in load throughout the life of the structure is known to be small, then the reference stress technique should provide a sufficiently accurate estimate of structural deformation.
1.6

Many situations involving variable loading are periodic and as such afford some simplification of the problem. Calculations have been performed for a tube subjected to cyclic histories of pressure and temperature, by Frederick, Chubb and Bromley (1.5). Although such work is impressive it is of limited value, without a satisfactory constitutive equation for time varying stress. Ponter (1.17) has recently derived work bounds for structures composed of time-hardening Maxwell material and subjected to variable loading. These bounds provide a measure of the severity of stress-redistribution occurring and are therefore indicative of situations where the reference stress method can be used with confidence. It has not been clear however what should be done when stress redistribution effects are significant. A method of estimating structural deformation in such circumstances is described in Chapter 5.

1.5 Creep due to Cyclic Loading.

The concept of plastic shakedown has been extended to include time hardening creep (1.16). Provided the loading is kept below \( \frac{n}{n+1} \) of the plastic shakedown load, the effects of local plastic strains upon the overall structural deformation can be safely neglected. An elastic-creep material model is therefore adequate. The great advantage of the reference stress technique is that it only requires a minimum of material test data and relatively simple structural calculations. If any extension of the reference stress method to cyclic histories of loading is to remain attractive to the design engineer it will be necessary to retain the essential simplicity of the method. In the following chapters an empirical method of estimating structural deformation due to cyclic loading is developed. In situations where the effects of stress redistribution are found to
be small or a design based on an upper bound remains acceptable, the time hardening result can be applied to structures composed of materials with related constitutive relationships whose creep law for time varying stress is unknown, by applying a weighting factor obtained from a single cyclic stress creep test.

When stress redistribution effects are known to be significant and an elastic-creep calculation is required, a weighted time hardening calculation is employed. As before, the weighting factors are obtained from a single cyclic creep test. The creep law for time varying stress is not required and the difficulties frequently experienced when other hypothetical creep laws are used are avoided.
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CHAPTER 2.

PLASTIC EFFECTS.

2.1. **Time Independent Plasticity.**

The analysis of structures composed of time-independent elastic-plastic material was developed on the basis of simple material models [2.1]. Although such drastic idealizations as the elastic-perfectly plastic and rigid-perfectly plastic models may limit the validity of any subsequent analysis in the strictest sense, the results obtained are usually conservative and permit the designer to make decisions and obtain a rapid feel for the problem.

While an elastic analysis for a highly redundant structure gives an exact solution, it may bear no more resemblance to reality than an assumed stress field used to obtain a lower bound to a limit load. The elastic solution is valid providing very restrictive (and in practice unrealistic) assumptions regarding the compatibility of components are met. Most structures will suffer from a lack of fit of components and from misalignment of supports. Providing the structure is composed of material sufficiently ductile to permit local plastic deformation without fracture, then the elastic analysis will be satisfactory since the elastic stress field is one of a set of statically admissible stress fields for the structure. However other
2.2.  

statically admissible stress fields are usually simpler to obtain, which give better lower bounds to the limit load. It can therefore be argued that intelligent use of a simple material model may permit more economical design with less effort on the part of the designer, and a solution which may be no further from the truth, than that obtained from other assumptions which are (for different reasons) equally invalid. The plastic design methods are acceptable because they are inherently conservative, and for ductile materials, safely applicable.

2.2.  Plastic Effects in the Creep Range.

Leckie and Ponter (2.2) and Ponter (2.3) have derived energy theorems which extend the concepts of a limit load and plastic shakedown to situations involving time hardening creep. The total strain rate is considered separable into an elastic, a time hardening creep and a perfectly plastic component i.e.

\[ \dot{\varepsilon} = \dot{\varepsilon} + \dot{\gamma} + \dot{\rho} \]

where \( \dot{\varepsilon} \) is the total strain rate, \( \dot{\varepsilon} \) is the elastic strain rate, \( \dot{\gamma} \) is the creep strain rate and \( \dot{\rho} \) is the plastic strain rate.

The material is assumed to be stable, that is to say surfaces of constant complementary strain energy density, constant energy dissipation rate and the yield surface are all convex in stress space. The creep strain rate for uniaxial stress is given by

\[ \dot{\gamma}/\dot{\gamma}_0 = (\sigma/\sigma_0)^n g(t) \]  \hspace{1cm} (2.1)

The material model can be criticised on the grounds that it is not possible to distinguish between instantaneous plastic deformation and time dependent deformation experimentally. The material model assumes behaviour of the form shown by the
unbroken curve in the log $V \sim \log \sigma$ plane shown in Fig. 2.1. where there is a discontinuity in slope at the yield stress $\sigma_y$.

In practice materials exhibit a smooth transition as shown by the broken curve in Fig. 2.1. However it should be remembered that a stress $\sim$ strain curve is usually obtained from a test of a few minutes duration whereas structural creep deformation usually occurs over a period of years.

The first theorem is applicable to structures subjected to steady loading and implies that once stress-redistribution has taken place and a stationary state of stress has been achieved, then providing the applied load $P$ satisfies the inequality

$$\frac{P}{P_L} \leq \frac{n}{n+1}$$

where $n$ is the stress index and $P_L$ is the plastic limit load, then the influence of plastic strains on the overall deformation of the structure will be small.

The second theorem is concerned with structures subjected to cyclic histories of loading and implies that once a cyclic state of stress has been achieved, then incremental plastic deformation will be small, providing

$$\frac{P}{P_S} \leq \frac{n}{n+1}$$

where $P_S$ is the plastic shakedown load.

Both these theorems are based upon a material model which neglects such effects as work-hardening, the Bauschinger effect and local changes in the shape of the yield surface. Furthermore the time hardening law is known to be unrealistic for situations involving time varying stress.

As it is difficult to define the boundary between instantaneous plastic straining and creep behaviour (Fig. 2.1) as the limit state is approached, it can be argued that the yield
surface and energy dissipation surface will be similarly distorted as a result of creep and plastic straining. The material model assumed in the derivation of the theorems does not assume material isotropy, merely that the yield surface and energy dissipation surface be convex in stress space. The derivation of the theorems does not require that the surface of constant energy dissipation rate and the yield surface be the same shape, but it is unlikely that the two surfaces will differ appreciably in practice. At present there is insufficient experimental evidence on this matter. Analysis is greatly simplified if the two surfaces are assumed to be the same shape and errors due to this assumption are likely to be second order.

Consider a set of geometrically similar structures constructed of material whose uniaxial strain rate is given by equation (2.1). Each of these structures is subjected to a set of proportional steady loads $\lambda P_1$ (where $\lambda$ varies from structure to structure) and at a corresponding point on each structure the deformation rate $\dot{\delta}$ is measured at some time $t$ after application of the loads. A plot of $\log \dot{\delta} \sim \log \lambda$ (where $\lambda$ is unity at the limit state) produces a straight line of slope $n$ in the absence of any plastic deformation. The first theorem predicts that plastic deformation should become significant at loads above $n/n+1$ and the slope of the curve should begin to increase above this value of loading, as shown in Fig. 2.2. Similar tests on a set of minimum weight structures could reasonably be expected to produce a straight line, in the $\log \dot{\delta} \sim \log \lambda$ plane up to the limit state at which point the deformation rate would become infinite (see Fig. 2.3). Therefore a structure which is well removed from a minimum weight design presents a more stringent test of the theorems than would a minimum weight structure.
Limit and Shakedown Loading Tests on Portal Frames in the Creep Range.

The extension of the concepts of a plastic limit load and plastic shakedown to the creep range is clearly of use to the design engineer. The complexity of design calculations for time dependent structures in the plastic range need not be emphasized. The concepts of a plastic limit load and plastic shakedown are well established and widely understood and used in design. However the material model on which the energy theorems are based does not provide a complete description of the behaviour of metals at high temperature. Although it can be argued that the material model should be sufficiently representative to ensure that the theoretical predictions are sensibly accurate, experimental verification is clearly desirable. An experimental programme was therefore initiated.

The structure chosen was a portal frame with encastré supports. A portal frame was chosen as it met the following requirements:

1. It was well removed from a minimum weight structure, which would not have provided a sufficiently stringent test of the energy theorems.
2. The shakedown load was significantly below the limit load. ($P_L/P_s = 1.13$ for the geometry chosen).
3. The structure was easily machined from commercially pure half hard aluminium sheet, which was adopted as the test material as it creeps at room temperature. The stress-strain curve for the material is also very similar to that of typical structural steels at elevated temperature.
The portal frame is shown in Fig. 2.4. All the models were machined from the sheet in the same direction relative to the direction of rolling. This ensured that the influence of any anisotropy was the same for all models.

2.3.1. Steady Load Tests.

Determination of the Limit Load.

The limit load was determined by applying a monotonically increasing load, \( H = V = P \). The horizontal deflection \( \delta_H \) and the vertical deflection \( \delta_V \) of the points of application of the loads were measured for each increment of load. The whole test was only of a few minutes duration ensuring that the frame behaviour was sensibly time-independent. The load-horizontal deflection curve is shown in Fig. 2.5 from which it will be seen that the limit load, \( P_L \) for the frame is about 39Kg. The vertical deflection \( \delta_V \) was small and was not therefore considered further.

Creep Tests.

A series of tests at constant load \( H = V = P \) were performed for various values of \( P/P_L \). Test duration was typically 800 hours. The creep strain-time curves obtained are shown in Fig. 2.6. Creep displacement rates, \( \dot{\delta}_H \), were taken from 650 - 800 hours; a curve of \( \log \dot{\delta}_H \sim \log P/P_L \) is shown in Fig. 2.7.

2.3.2. Variable Load Tests.

Determination of the Shakedown Load.

The incremental collapse of portal frames subjected to cyclic loading well below the limit load was described by Neal and Symonds (2.4). The present tests employed a similar frame
geometry and followed the same cycle of loading.

The loading cycle employed was:

(a) $H = V = P$
(b) $H = V = 0$
(c) $H = P, V = 0$
(d) $H = V = 0$

In order to ascertain the approximate value of the shakedown load the frame was subjected to 3 cycles of loading for values of $P$ of 20, 25, 30, 35 and 37 Kg. The total plastic sidesway displacement, $\delta_H$, accumulated at the end of each cycle is shown graphically in Fig. 2.8. At the end of this test significant sidesway had occurred and another frame was therefore used for a more accurate determination of the shakedown load. The frame was subjected to roughly ten cycles of loading for values of $P$ of 31, 32 and 33 Kg. From these tests the shakedown load, $P_s$, was considered to be 33 Kg.

The duration of these tests was sufficiently short for time independent behaviour to be assumed.

Creep Tests.

The duration of each loading cycle was a week. The programme of loading was such that the loading periods (a), (b) and (c) were each of 24 hours duration while period (d) lasted 4 days. A typical plot of sidesway displacement against time is shown in Fig. 2.9. The total permanent sidesway displacement at the end of each cycle is shown graphically in Fig. 2.10. These results are referred to a false zero at the end of first loading cycle to permit a clearer presentation of results. The displacements during the first few loading cycles are not only dependent upon the applied loads but also upon the initial state
of stress in the unloaded structure, and this can lead to certain apparent inconsistencies. It will be noted that for certain tests presented in Fig. 2.10, the rate of deformation per cycle does not decrease monotonically. Interpretation of the results therefore requires some judgement and for most results the displacement rate has been taken between cycles 4 and 7. However the results for loads of 30 and 32.5 kg were taken from the fourth cycle. It is unlikely that there is great error due to this procedure as for these two tests the geometry changes become significant.

A curve of log $\delta$ vs log $P/P_S$ is shown in Fig. 2.11. The permanent deformation rate is in $\mu$m/cycle.

2.4. Discussion of Results.

The steady load results presented in Fig. 2.7 are remarkably free from scatter, only one point being significantly off the curve. The slope of the straight portion of the curve is 2.8 (i.e. $n = 2.8$) and the ratio $n/n+1 = 0.74$. The transition of the curve from a straight line to a more rapidly rising curve is seen to occur where predicted by the energy theorem i.e. at a value of $P/P_L = n/n+1$.

The variable load tests results shown in Fig. 2.11 have some scatter which reflects the difficulty of performing such tests and some lack of uniformity in the duration of certain loading cycles and variation in the temperature conditions which occurred during some tests. There is also thought to be some possibility that some of the portal frames were machined at an angle of $90^0$ relative to the remainder and the influence
of anisotropy is known to be significant for the material used. However the general trend in the results is encouraging and within the scatter present in the data the onset of detectable plastic straining is as predicted at a value of $P/P_s \leq n/n+1$.

The tests described tend to confirm the validity of the theorems, which, although based on a simple material model, permit the extension of the concepts of limit and shakedown loads to structures composed of time dependent materials. Plastic effects can therefore be neglected providing:

1. Steady loads are kept within the range
   $$P/P_L \leq n/n+1$$

2. Varying loads satisfy the inequality
   $$P/P_S \leq n/n+1$$

Where $P_L$ and $P_S$ are the plastic limit load and plastic shakedown load respectively.

Many power plant components are designed for a life of 100,000 hours. In the past the design criterion has been uniaxial rupture data, but recently there has been a move towards a criterion based upon 1% maximum strain. Such design requirements ensure that load levels are usually within the range below $n/n+1$ of the plastic limit load.
The information given in Table 2.1 is based on material data given by Odqvist (2.5). As a yield stress is not given, the 0.2% proof stress is used instead. It is further assumed that the reference stress associated with the stationary creep solution is 2/3 of the maximum stress in the limit state, i.e. the yield stress.

Table 2.1.

<table>
<thead>
<tr>
<th>Material</th>
<th>0.2% Proof Stress N/mm²</th>
<th>100,000 hour Rupture Stress N/mm²</th>
<th>Fraction of Limit Load on Rupture Basis</th>
<th>Stress for 1% Strain in 100,000 Hours N/mm²</th>
<th>Fraction of Limit Load on 1% Strain Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rolled Chromium Steel at 595°C</td>
<td>225</td>
<td>81.5</td>
<td>0.54</td>
<td>14.8</td>
<td>0.10</td>
</tr>
<tr>
<td>Rolled Stainless Steel at 815°C</td>
<td>106</td>
<td>10.8</td>
<td>0.15</td>
<td>2.55</td>
<td>0.036</td>
</tr>
</tbody>
</table>

The results given in Table 2.1 emphasize the difference in load levels which can occur for differing design criteria. At lower temperatures the load levels will be higher proportions of the limit load, and conversely smaller fractions of the limit load at higher temperatures.

Similar conclusions can be drawn for structures subjected to variable loading. In Chapter 4 the concept of an equivalent steady stress is introduced. This is the constant
stress which produces the same accumulation of strain as a cyclic history of stress. A typical value of the equivalent steady stress for a cycle consisting of a steady stress applied for half the time and zero stress for the remainder is 0.9. The maximum permitted stress levels for 100,000 hours life are thus increased by 11% compared with the corresponding values for steady loading. The plastic shakedown load is typically 0.9 of the plastic limit load, and this information is sufficient for the material data given in Table 2.1 to be used to give an indication of typical load levels as a fraction of the plastic shakedown load. These values are given in Table 2.2.

<table>
<thead>
<tr>
<th>Material</th>
<th>Fraction of Shakedown Load on Rupture Basis</th>
<th>Fraction of Shakedown Load on 1% Strain Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rolled Chromium Steel at 595°C</td>
<td>0.67</td>
<td>0.12</td>
</tr>
<tr>
<td>Rolled Stainless Steel at 815°C</td>
<td>0.19</td>
<td>0.044</td>
</tr>
</tbody>
</table>

It can be seen therefore that for most design situations, the influence of plastic strains can be neglected, and an elastic-creep material model will be adequate, as at worst the quotient \( \frac{n}{n+1} \) will be 0.75 for a value of \( n = 3 \).
<table>
<thead>
<tr>
<th>References</th>
<th>Authors</th>
<th>Title</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Prager, W.</td>
<td>An Introduction to Plasticity</td>
<td>Addison-Wesley, 1959.</td>
</tr>
</tbody>
</table>
FIG. 2.1  VARIATION OF CREEP RATE WITH STRESS
FIG. 22 CREEP OF TYPICAL STRUCTURE IN PLASTIC RANGE
FIG. 2.3 CREEP OF MINIMUM WEIGHT STRUCTURE IN PLASTIC RANGE
FIG. 2-4 PORTAL FRAME
FIG. 25  LIMIT LOAD TEST
Fig. 2.6 Steady Load Creep Deformation

SIDESWAY \( \delta_h \) in mm vs. TIME IN HOURS

- \( P = 36 \text{ Kg} \)
- \( P = 32.5 \text{ Kg} \)
- \( P = 30 \text{ Kg} \)
- \( P = 27.5 \text{ Kg} \)
- \( P = 26 \text{ Kg} \)
- \( P = 25 \text{ Kg} \)
- \( P = 22.5 \text{ Kg} \)
- \( P = 20 \text{ Kg} \)
Determination of Shakedown Load

Sidesway at end of cycle in mm

No. of cycles

37 Kg
35 Kg
32 Kg
30 Kg
25 Kg
20 Kg

Fig. 2.8 Determination of Shakedown Load
FIG. 2.10 TIME DEPENDENT SHAKEDOWN TEST

SIDESWAY, $\delta_H$, AT END OF CYCLE IN mm

NUMBER OF CYCLES

- 32.5 Kg
- 30 Kg
- 28.5 Kg
- 27.5 Kg
- 26 Kg
- 25 Kg
- 22.5 Kg
- 20 Kg
- 21.5 Kg
\begin{figure}[h]
    \centering
    \includegraphics[width=\textwidth]{figure211.png}
    \caption{Plot showing the relationship between \( \log \frac{P}{P_0} \) and \( \log \delta H \text{ (\textmu m/cycle)} \).}
    \label{fig:211}
\end{figure}
3.1.

CHAPTER 3

STRESS REDISTRIBUTION EFFECTS.


In Chapter 2 it was argued that providing steady loads were kept below \( n_{n+1} \) of the plastic limit load and varying loads within a range less than \( n_{n+1} \) of the plastic shakedown load, an elastic-creep material model was adequate. In the present Chapter the influence of stress redistribution effects on the deformation of structures composed of Maxwell material is considered for a particular class of constitutive relationships.

Constant stress uniaxial creep tests on many materials produce data which can be described reasonably well by constitutive equations of the form

\[
\frac{\varepsilon(t)}{\varepsilon_o} = \left(\frac{\sigma}{\sigma_o}\right)^n \left(\frac{t}{t_o}\right)^m
\]

(3.1)

where \( \varepsilon_o \) and \( \sigma_o \) are constants for the material and \( t_o \) is some time measure.

Several proposals have been made to generalize equation (3.1). In the present work three of these hypothetical material models are used for the purpose of illustration. They ignore such effects as creep hesitation and recovery and do not fit experimental data very well for some materials.
3.2. 

However it will be shown later that they do provide insight into the creep tests likely to be most useful. The laws chosen and their mathematical forms are:

(a) The time hardening law

$$\frac{\dot{\varepsilon}_{ij}(t)}{\varepsilon_o} = \frac{3}{2} \left( \frac{\sigma(t)}{\sigma_o} \right)^{n-1} \frac{\sigma(t)}{\sigma_o} \frac{t_m}{t_o}$$

(b) The strain hardening law

$$\frac{\dot{\varepsilon}_{ij}(t)}{\varepsilon_o} = \frac{3}{2} \left( \frac{\sigma(t)}{\sigma_o} \right)^{n-1} \frac{\sigma(t)}{\sigma_o} \frac{t_m}{t_o} \left( \frac{\varepsilon(t)}{\varepsilon_o} \right)^{l-m/m}$$

(c) Rabotnov's 3.1. work hardening law

$$\frac{\dot{\varepsilon}_{ij}(t)}{\varepsilon_o} = \frac{3}{2} \left( \frac{\sigma(t)}{\sigma_o} \right)^{n-1} \frac{\sigma(t)}{\sigma_o} \frac{t_m}{t_o} \left( \frac{\varepsilon(t)}{\varepsilon_o} \right)^{l-m/m}$$

where $s_{ij}$ is the stress deviator, and

$$\{\sigma(t)\}^2 = \frac{3}{2} s_{ij}(t) s_{ij}(t)$$

$$\{\varepsilon(t)\}^2 = \frac{2}{3} \varepsilon_{ij}(t) \varepsilon_{ij}(t)$$

3.2. Stress Redistribution due to Steady Loading.

Hoff 3.2. considered stress redistribution in a structure composed of Maxwell material undergoing secondary creep and showed that the stress field in the structure was asymptotic to a steady state. Hult 3.3. subsequently generalized Hoff's result to structures composed of Maxwell materials whose creep laws are of the form given by
3.3. equations (3.2, 3.3 or 3.4). He termed the stress field which is approached asymptotically the stationary state. Leckie and Martin (3.4) later showed that structures composed of time hardening Maxwell materials converge monotonically to the stationary state which is also that for which the energy dissipation rate is a minimum. This result is consistent with the theorem of minimum complementary strain energy which can be applied to structures in the stationary state as first noticed by Hoff (3.2).

Structures composed of Maxwell material whose creep law is of the form

$$\frac{\epsilon(t)}{\epsilon_0} = \left(\frac{\sigma}{\sigma_0}\right)^n g(t), \quad (3.5)$$

and subjected to steady loading will exhibit an initial elastic deformation $\delta_e$ and then undergo stress redistribution towards the stationary state. A typical plot of deformation, $\delta \sim g(t)$, is shown in Fig. 3.1. Presentation of the deformation against a time base of $g(t)$ is helpful in that the stationary state is seen to be reached when the slope of the curve, $d\delta/dg(t)$, becomes constant. Without loss of generality $g(t)$ can be replaced by $(t/t_0)^m$ and the creep law for the material is then given by equation (3.1).

The structural deformation can be considered to consist of three components: an initial elastic component, a component due to stress redistribution, and that due to stationary creep. If the component due to stress redistribution is neglected, the initial elastic deformation together with the stationary creep deformation provide a lower bound to the true deformation.

Marriott and Leckie (3.5) investigated stress redistribution due to steady loading for several structures and compared the results of calculations in which the time hardening and strain
hardening laws were used. They noted that the overall creep deformation did not differ significantly and that the stress histories were also similar. Furthermore they observed that there was a point in the structure where the stress was almost invariant with time and termed this the "skeletal point."
They argued that since the stationary creep deformation rate in the structure was directly proportional to the stationary creep strain rate at any point in the structure, an acceptable estimate of the creep deformation would be obtainable from creep data derived from a creep test at the "skeletal stress."
This idea is very similar to the "representative stress" method described by Anderson, Gardner and Hodgkins (3.6) for estimating secondary creep deformation previously discussed in Chapter 1. In both cases the "skeletal stress" and the "representative stress" were obtained by inspection. For the structures considered by Marriott and Leckie the deformation due to stress redistribution was found to be small and the approximate deformation given by adding the elastic deformation and the stationary state deformation obtained from a test at the "skeletal stress" was clearly adequate for practical purposes.

Mackenzie (3.7) proposed a more rigorous method of obtaining the appropriate stress level for a uniaxial creep test from consideration of the normalized deformation rate in the stationary state. He compared the deformation rate for some value of the stress index \( n \) to that for the corresponding rate with \( n = 1 \). He was then able to find a "reference stress" such that the normalized deformation rate was equal to unity. Thus the "reference stress" obtained by Mackenzie's method is determined analytically rather than by inspection as was the "skeletal point" and "representative stress" of Marriott and Leckie, and Anderson et al.
Mackenzie also observed that the position of the "skeletal point" coincides with the position at which the stress in the stationary state is independent of the value of $n$, and therefore the "reference stress method", (as it has become known), should give acceptable estimations of structural deformation for structures composed of materials where $n$ is a function of stress. This is consistent with the observations of Calladine and Drucker concerning Nesting Surfaces (3.8), mentioned previously in Chapter 1.

Later Leckie (3.9) improved the method of obtaining the reference stress suggested by Mackenzie and showed that by equating the initial elastic strain energy and the energy dissipated in the stationary state for the structure in question with the corresponding quantities for a uniaxial tensile specimen the value of the reference stress can be determined. This method is described in detail in Chapter 5.

Leckie and Martin (3.4) obtained work bounds for structures composed of time hardening Maxwell material and showed that the energy dissipation due to stress redistribution could be bounded from above and that for many structures of engineering interest this quantity is of the same order of magnitude as the initial elastic strain energy.

A particularly useful upper bound on the energy dissipation at large times due to Leckie and Martin is given by the inequality

$$ D^T \leq (n+1) \{U^S - U^D\} + \int_0^T \int_V \sigma^S_{ij} \dot{\varepsilon}^S_{ij} \, dV \quad (3.5) $$
where $U^0$ is the initial complementary strain energy and $U^S$ is the complementary strain energy due to the stationary stress field $\sigma_{ij}^s$.

In many design situations the stationary state solution may not be known but an approximate solution such as the plastic limit state may well be available. Leckie and Ponter (3.10) have computed upper bounds by making use of limit state solutions, and shown that the bounds so obtained can be acceptable and of use to the designer.

By way of illustration consider the two-bar structure shown in Fig. 3.2. subjected to a constant load $P = 1$ and for the following values of the relevant parameters:

\[
\begin{align*}
\eta &= 3 \\
\eta &= 4 \\
A &= 1 \\
\frac{\varepsilon_0}{\sigma_0^n} &= 24 \\
\varepsilon/E &= 1 \\
m &= 1/3
\end{align*}
\]

(3.7)

The shape parameter, $\eta$, permits the elastic stress concentration and the degree of stress redistribution to be varied. These effects are quantified in Chapter 5.

For the values given in equation (3.7) the structure is identical with that discussed by Leckie and Martin. Fig. 3.3 shows the deformation $\sim (t/t_0)^{1/3}$ curves of the structure for the three constitutive equations (3.2, 3.3 and 3.4). In this structure the effects of stress redistribution are particularly severe and the quotient $(U^S - U^D)/U^0 = 0.5$. The upper bound given by inequality (3.6) and the lower bound obtained by adding the initial elastic deformation and the stationary creep deformation are also shown.
It can be argued heuristically that the bound given by inequality (3.6) is conservative since it is for structures composed of time hardening material. Structures composed of other materials, for example those conforming to the strain hardening or work hardening hypotheses, will be "stiffer" in the sense that stress redistribution effects will increase the state variables of the material at a faster rate than would occur in the stationary state. The strain rate of a time hardening material subjected to a given stress is, by definition, only dependent upon time and therefore stress redistribution effects do not affect a structure composed of time hardening material in this way. The structure composed of time hardening material will therefore have no "memory" of the stress redistribution process, unlike structures composed of more general materials. At large times the stationary state deformation will have a dominating effect and the deformation rate will be sensibly independent of the particular creep law for time varying stress. Thus in a particular sense the material will "forget" the stress redistribution process. This is seen to be the case for the deformation histories shown in Fig. 3.3, where the greatest deformation is that for the structure composed of time hardening material.

It can be concluded that providing the work bounds derived by Leckie and Martin do not differ significantly then the reference stress method (which is implicitly associated with a lower bound solution) will provide a sufficiently accurate solution. It is likely that this will be the case for most design situations. The lack of dependence of the reference stress on the stress index $n$, implied by the work of Calladine and Drucker and noted for particular structures by Anderson et al.
and Mackenzie, means that the reference stress method does not require precise knowledge of the value of \( n \). When stress redistribution effects are particularly severe and the work bounds differ significantly, a more precise knowledge of \( n \) is required and a conservative estimate of structural deformation can be obtained from a time hardening calculation. However, in most cases stress redistribution effects are small, the additional deformation being of the order of the initial elastic deformation, and can therefore be neglected by the designer in most circumstances.

### 3.3. Stress Redistribution Times for Steady Loading

Calladine (3.11) evolved an approximate method for predicting the time for stress redistribution to take place for time hardening creep in terms of the time for the creep strain to equal the initial elastic strain. He denoted this time by \( t^* \). Calladine suggested two possible stress levels appropriate to the definition of \( t^* \). The first was the stationary stress at the point of peak initial elastic stress, and the other was an average value determined from consideration of the energy dissipation in the stationary state, which is similar to the procedure suggested by Mackenzie (3.7) and Leckie (3.9) for determining a reference stress and would seem a preferable basis for defining \( t^* \). Calladine’s method requires knowledge of the initial elastic stress distribution together with the stress distribution in the stationary state and creep data from a test at the appropriate stress level.

Since the stationary state is approached asymptotically, Calladine proposed that the stress redistribution time be given
by the time required for

$$\sigma(t_{10}) - \sigma_s = (\hat{\sigma} - \sigma_s)/10$$

where $t_{10}$ is the time required $\sigma_s$ is the stationary stress and $\hat{\sigma}$ is the initial elastic stress.

The approximate relationship quantifying $t_{10}$ derived by Calladine is

$$t_{10}/t^* = 2.3/n$$

which, it is claimed, gives an overestimate of the stress redistribution time, especially for larger values of $n$ and the quotient $(\hat{\sigma} - \sigma_s)/\sigma_s$.

Sim (3.12) has obtained values of $nt_{10}/t^*$ for different structures and these are presented in Table 3.1. If these results are compared with the value of 2.3 suggested by Calladine it will be seen that agreement is good except for the thick shells subjected to internal pressure, where the value of $nt_{10}/t^*$ can rise to 7.9. This tendency has also been observed by Mackenzie (3.13), who was concerned with stress relieving techniques in thick components composed of creep resistant materials. Mackenzie noted that states of hydrostatic stress in the interior of thick components could require a very long time to decay to an acceptably low value. He therefore questioned the effectiveness of current procedures for relieving such states of stress.
Table 3.1.

<table>
<thead>
<tr>
<th>Structure</th>
<th>Outer Radius</th>
<th>n</th>
<th>$nt_{10}/t^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Inner Radius</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thick cylinder subjected to</td>
<td>10</td>
<td>3</td>
<td>6.9</td>
</tr>
<tr>
<td>internal pressure</td>
<td>9</td>
<td>3.6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.6</td>
<td>2.9</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.25</td>
<td>3</td>
<td>2.6</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>2.4</td>
<td></td>
</tr>
<tr>
<td>Thick sphere subjected to</td>
<td>5</td>
<td>3</td>
<td>7.9</td>
</tr>
<tr>
<td>internal pressure</td>
<td>9</td>
<td>4.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.6</td>
<td>3.6</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>2.8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.25</td>
<td>3</td>
<td>2.8</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>2.2</td>
<td></td>
</tr>
<tr>
<td>Rectangular beam in bending</td>
<td>3</td>
<td>2.9</td>
<td></td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>3.0</td>
<td></td>
</tr>
</tbody>
</table>

Bill and Mackenzie (3.14) expressed the stress redistribution time in terms of the ratio of creep strain at time $t_{10}$ due to some reference stress $\sigma_R$ to the corresponding elastic strain. They showed that by a suitable choice of $\sigma_R$ their measure of stress redistribution time could be made sensibly independent of the stress index $n$. They also observed that the values of $\sigma_R$ so obtained did not differ significantly from those of the reference stress obtained for estimating deformation due to stationary creep. Thus providing the time
for stress redistribution $t_{10}$ is known for one value of $n$ it
should be possible to make acceptable estimates of the stress
redistribution time for other values of $n$.

Both Calladine and Bill and Mackenzie were concerned
with stress redistribution in structures composed of time
hardening Maxwell material. Inspection of the curves of
stress vs. time for a beam subjected to constant moment and a
thick cylinder subjected to internal pressure presented by
Marriott and Leckie (3.5) reveals that the redistribution
times for structures composed of strain hardening material
are significantly longer than the corresponding values for
time hardening material.

Calculations were performed to determine the stress
redistribution times for the structure shown in Fig. 3.2 for
the three constitutive relationships (3.2, 3.3 and 3.4).
The results are presented in terms of a dimensionless time
measure $t_{10}$ similar to that considered by Bill and Mackenzie,
save that the reference stress $\bar{\sigma}$ is that used to estimate
stationary creep deformation. The determination of $\bar{\sigma}$ for
the structure shown in Fig. 3.2 is described in Chapter 5.
The dimensionless time measure is therefore given by

$$t_{10} = \frac{\text{creep strain due to reference stress } \bar{\sigma} \text{ at time } t_{10}}{\text{corresponding elastic strain due to } \bar{\sigma}}$$

(3.8)

Values of $t_{10}$ for $n = 3$, $n = 2.5$ and $4$ and for the
three different materials are given in Table 3.2. It will be
noted that the time hardening results are significantly shorter
than those for strain hardening and work hardening materials.
From these results and those given by Marriott and Leckie, it
is seen that the time for stress redistribution can be very
dependent upon the constitutive equation for time varying stress, and since no satisfactory generalization of equation (3.1) has been proposed, the only reliable method of obtaining stress redistribution times at present is by experiment. Although stress as such cannot be measured, the stationary state is attained when the deformation \( \{t/t_0\}^m \) curve becomes sensibly linear.

### Table 3.2.

<table>
<thead>
<tr>
<th>( n = 3 )</th>
<th>( \tau_{10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>2.5</td>
</tr>
<tr>
<td>Time Hardening</td>
<td>0.77</td>
</tr>
<tr>
<td>Strain Hardening</td>
<td>2.190</td>
</tr>
<tr>
<td>Work Hardening</td>
<td>3.371</td>
</tr>
</tbody>
</table>

### 3.4. Stress Redistribution due to Cyclic Loading.

Frederick and Armstrong (3.15) showed that structures composed of elastic-perfectly plastic or elastic-creeping material whose strain rate is given by an equation of the form

\[
\frac{\dot{\varepsilon}_{ij}(t)}{\varepsilon_0} = \frac{3}{2} \left( \frac{\sigma(t)}{\sigma_0} \right)^{n-1} s_{ij}(t)
\]

and subjected to cyclic histories of loading would converge asymptotically to a unique cyclic state of stress. More recently Martin and Williams (3.16) provided alternative arguments to
establish the same theorem, and subsequently Ponter (3.17) argued that the theorem of Frederick and Armstrong could be extended to structures composed of materials whose creep law is of the form

\[ \frac{\dot{e}_{ij}(t)}{\varepsilon_o} = \frac{3}{2} \frac{\sigma(t)^{n-1}}{\sigma_o^n} s_{ij}(t) g \left( \frac{t}{t_o} \right) \]  

(3.10)

provided such a state of stress exists.

Creep due to cyclic loading can therefore be considered to consist of two regimes. The first, the convergence to a cyclic state, is in a sense analogous to the stress redistribution process for steady loading. The second is the cyclic state which corresponds to the stationary state for steady loading. The cyclic state can be computed either from the initial stress free state or from some intermediate state which may have been previously computed or guessed. Such a procedure is discussed by Frederick and Armstrong. An unsuccessful attempt to obtain a method of determining the cyclic state directly was made by Martin and Williams (3.16). The stationary state for steady loading is associated with the minimum energy dissipation rate, and the investigator may be led to assume that the cyclic state will be one which similarly minimizes the energy dissipation per cycle. However, it was shown by Martin and Williams that such an assumption could be dismissed by a simple counter example, and that the functional which is minimized is in fact a weighted integral of the energy dissipation rate, over a cycle. Their attempt to generalize the energy theorem of Leckie and Martin (and by implication the theorem of minimum complementary strain energy) to include cyclic loading was unsuccessful, as no physical interpretation could be put
upon the functional which was minimized in the cyclic state.

The energy theorems of Leckie and Martin were extended by Leckie and Ponter (3.18) to include a plastic strain component and these results were subsequently generalized further to include variable loading by Ponter (3.19). The predictions of Leckie and Ponter that the concepts of a limit load and plastic shakedown could be extended to include time hardening creep were investigated experimentally as described in Chapter 2. The energy theorem derived by Ponter (3.19) effectively generalizes the upper bound theorem of Leckie and Martin (Result 7) to situations involving cyclic histories of loading providing the loading does not exceed $nP_s/n+1$ where $P_s$ is the plastic shakedown load. The optimal upper bound associated with this theorem provides a useful measure of the severity of stress redistribution effects due to cyclic loading.

3.5. **Energy Dissipation Bounds for Cyclic Loading.**

When the effects of stress redistribution are small the reference stress method will provide an acceptable estimate of structural deformation. However, when the energy dissipation bounds (3.17) differ significantly then some other approach is required. A solution to this problem is described in Chapter 5.

The Work Bounds.

The theorems described in this section are due to Ponter (3.17, 3.19, 3.20, 3.21), and are based on a material model which was described briefly in Chapter 2. The total strain field has three constituent parts:

$$
\varepsilon_{ij} = \varepsilon_{ij} + v_{ij} + p_{ij}
$$
3.15.

where \( e_{ij}, v_{ij} \) and \( p_{ij} \) denote the elastic, creep and plastic strain components respectively. The material is stable in the sense that surfaces of constant complementary strain energy density, surfaces of constant energy dissipation rate, and the yield surface are all convex in stress space (3.22).

The constitutive relations for the individual strain components are given by:

- **elastic:** 
  \[
  e_{ij} = C_{ijkl} \sigma_{kl} \tag{3.12}
  \]

where \( C_{ijkl} \) denotes the elastic constants tensor which possesses the usual symmetry properties and gives rise to a positive definite complementary energy density

\[
E(\sigma_{ij}) = \frac{1}{2} C_{ijkl} \sigma_{ij} \sigma_{kl} > 0 \ , \ \sigma_{ij} \notin \sigma
\tag{3.13}
\]

- **creep:** 
  \[
  \dot{\sigma}_{ij} = \frac{3}{\kappa \sigma_{ij}} \left( \frac{\phi^{n+1}(\sigma_{kl})}{n+1} \right) g(t) \tag{3.14}
  \]

where \( \phi \) is a homogeneous function of degree one in \( \sigma_{ij} \) and \( g(t) \) is some positive function of time,

- **perfectly plastic:** 
  \[
  \dot{\sigma}_{ij} = \dot{\mu} \frac{\partial \phi}{\partial \sigma_{ij}} , \quad f(\sigma_{ij}) = 0 \tag{3.15}
  \]

where \( f(\sigma_{ij}) = 0 \) is a convex yield function, which defines a surface enclosing the stress free state, and the plastic multiplier is denoted by \( \mu \).

Consider a structure subjected to a general history of loading \( P_i(t) \). The energy dissipated due to the formation of inelastic strains within the time \( 0 < t < T \) is given by

\[
W^T_0 = \int_0^T \int_V \sigma_{ij} (\dot{\sigma}_{ij} + \dot{p}_{ij}) \, dV \, dt \tag{3.16}
\]

where the prime denotes a work quantity associated purely with inelastic strains.
Ponter (3.17) has shown that \( W_o^T \) may be bounded from above and below by

\[
\int_0^T \int_V \dot{\sigma}^V_{ij} \, dV \, dt \leq W_o^T \leq (n+1) \left( A(0) - A(T) \right) + \int_0^T \int_V \dot{\sigma}^V_{ij} \, dV \, dt
\]

(3.17)

The creep energy dissipation rate \( \dot{\sigma}^V_{ij} \) is given by

\[
\dot{\sigma}^V_{ij} = \sigma_{ij} \dot{\varepsilon}_{ij} = \phi^{n+1}(\sigma_{ij}) \, g(t)
\]

(3.18)

and the stress fields \( \sigma_{ij}^S \) and \( \sigma_{ij}^* \) are defined as follows:

\( \sigma_{ij}^S \) is the stationary state stress field, i.e. that which exists in the absence of elastic strains (\( e_{ij} = 0 \)) and is, of course, subject to the restriction \( f(\sigma_{ij}^S) \leq 0 \).

\( \sigma_{ij}^* \) is given by

\[
\sigma_{ij}^* = \sigma_{ij}^* + \rho_{ij}
\]

where \( \sigma_{ij}^* \) denotes the instantaneous linear elastic stress field and \( \rho_{ij} \) denotes an arbitrary constant residual stress field which is in equilibrium internally and with zero applied loads.

Furthermore \( \sigma_{ij}^* \) is subject to the restraint \( f(\frac{n+1}{n} \sigma_{ij}^*) \leq 0 \).

\( A(t) \) denotes the total complementary elastic strain energy quantity

\[
A(t) = \int_V \left( \frac{1}{2} \mathbf{C}_{ijkl} (\rho_{ij}(t) - \bar{\rho}_{ij}) (\rho_{kl}(t) - \bar{\rho}_{kl}) \right) \, dV
\]

(3.19)
where \( \rho_{ij}(t) = \sigma_{ij}(t) - \sigma_{ij}(t) \), the instantaneous residual stress field. The quantity \( A(0) \) is known if the structure is initially unstressed and \( A(T) \) may be removed without violating the inequality.

When the structure is subjected to cyclic loading, the stress distribution \( \sigma_{ij}(t) \) approaches a cyclic state asymptotically (3.15, 3.16, 3.17). If the time interval \( 0 \leq t \leq T \) is taken as a complete cycle when this stationary cyclic state has been achieved the bounds assume a simpler form (3.17)

\[
W^f = \int_0^T \int_0^T \mathcal{A}^V \left( \sigma_{ij}^s \right) \, dV \, dt \leq \int_0^T \int_0^T \mathcal{A}^V \left( \sigma_{ij}^* \right) \, dV \, dt = W^u \quad (3.20),
\]

where \( W_o^T \) denotes the total work done by the applied loads.

\[
W_o^T = \int_0^T \int_0^T \sigma_{ij} \varepsilon_{ij} \, dV \, dt = \int_0^T \int_0^T p_i \dot{u}_i \, dS \, dt
\]

As the work depends upon the magnitude of the stress and not the sign, the difference between the two bounds provides a measure of the average deviation of the stress from \( \sigma_{ij}^s \), the purely viscous solution which provides the lower bound.

Ponter (3.20) has shown that the residual stress field \( \bar{\sigma}_{ij} \) which minimises the upper bound \( W^u \) makes the creep strain accumulated over a complete cycle \( \Delta_{ij}^u \) kinematically admissible, and \( \bar{\sigma}_{ij} \) is uniquely defined by this condition. The optimal stress field \( \bar{\sigma}_{ij} \) so formed provides a displacement field \( \Delta_{ij}^u \) from \( \Delta_{ij}^u \) which is given by

\[
\Delta_{ij}^u = \int_0^T \dot{\varepsilon}_{ij} \, dt = \int_0^T \frac{\partial}{\partial \sigma_{ij}} \left( \frac{\psi^{n+1}(\sigma_{kj})}{n+1} \right) g(t) \, dt
\]
The displacement field associated with the optimal upper bound can be interpreted as the limiting case as the cycle time tends to zero. Under such circumstances the creep strains accumulated over a cycle are small compared with the excursions in elastic strain, and stress redistribution is therefore occurring continuously with the result that any consequent ratchetting effects are maximised, (3.20).

The lower bound solution \( \sigma_{ij}^S \) provides, by definition, a kinematically admissible, strain rate field \( \dot{\varepsilon}_{ij}(\sigma_{kk}^S) \), from which can be derived a displacement rate field \( \dot{u}_i^S \) and the corresponding displacement \( \Delta u_i^S \) accumulated over a cycle. This solution can be interpreted as the asymptotic state in the structure when the cycle time is very long and the creep strains are large compared with the excursions in elastic strain, so that stress redistribution is only occurring for a small part of the time and stress redistribution effects are correspondingly small. The two bounds and their associated displacement fields therefore provide extreme modes of behaviour of the structure and their deviation provides some measure of the degree of ratchetting which may occur due to stress redistribution.

The plastic strains enter into the bounds merely as yield restraints upon the stress fields and the analysis will indicate the range of loading for which plastic straining will not occur. In general a stress history of the form \( \dot{\sigma}_{ij} \) is possible providing the loads are less than \( n/n+1 \) of a plastic shakedown state (3.19) as the experimental investigation described in Chapter 2 tended to confirm. The optimal upper bound will remain valid for a range of loading which may be found
directly from the solution. Although plastic deformation may well occur during the first few cycles before the stationary state has been attained any additional deformations so formed are included in the quantity, \((n+1) (A(0) - A(t))\) in the upper bound (3.17).

Now consider a structure which is initially in a stress free state and is subsequently subjected to a history of cyclic loading. The upper bound (3.17) can be considered to consist of two components: the first that due to creep in the cyclic state and the second, the elastic component \((n+1) A(0)\) provides a measure of the work done in the transition to the stationary cyclic state.

In the following sections bounding solutions are obtained for a beam subjected to cyclic histories of bending moment and a thick cylinder subjected to histories of cyclic internal pressure together with corresponding solutions for the two-bar structure shown in Fig. 3.2.

3.6. Rectangular Beam Subjected to Cyclic Bending Moment.

Consider a beam of rectangular cross-section with width \(b\) and depth \(d\) which is subjected to a cyclic history of bending moment \(M(t)\) with period \(T\). The beam will suffer a rate of change of curvature \(\dot{k}(t)\) which is related to the axial strain rate at a distance \(y\) from the neutral axis by

\[
\dot{e}(y,t) = \dot{k}(t) y
\]

(3.21)

where \(\dot{e}(t) = \sigma^n(t) g(t)\) for a uniaxial state of stress.
3.6.1. The Upper Bound Solution.

The linear elastic axial stress distribution is given by
\[ \sigma(y,t) = M(t) \frac{y}{I} \]  
where
\[ I = bd^3/12 \]
and the upper bound stress distribution is given by
\[ \sigma^*(y,t) = \sigma(y,t) + \bar{\rho}(y) \]
where \( \bar{\rho}(y) \) denotes an arbitrary function of \( y \) which satisfies the condition that the resultant axial load and bending moment are zero.

\[ \int_{-d/2}^{d/2} \bar{\rho}(y) \, dy = \int_{-d/2}^{d/2} y \bar{\rho}(y) \, dy = 0 \]  
(3.24)

The optimal upper bound is given by the condition that the accumulated creep strain over a cycle
\[ \Delta^u v(y) = \int_0^T (\sigma(y,t) + \bar{\rho}(y))^n g(t) \, dt \]  
(3.25)
shall be compatible and therefore derivable from a curvature \( \Delta^u \kappa(0,T) \) by
\[ \Delta^u v(y) = \Delta^u \kappa(0,T) y = \int_0^T (\sigma(y,t) + \bar{\rho}(y))^n g(t) \, dt \]  
(3.26)

Equation (3.26) permits the determination of \( \bar{\rho}(y) \) for a particular choice of \( \Delta^u \kappa(0,T) \). The required \( \Delta^u \kappa(0,T) \) and the associated residual stress field \( \bar{\rho}(y) \) are made determinate by the further condition given by equation (3.24). It can easily be shown that the first of these conditions (zero axial load) is automatically satisfied if the moment is zero.
The procedure adopted was as follows. A particular value of \( A^k(0,T) \) was chosen and equation (3.26) was solved for \( \delta(y) \) at a series of discrete points distance \( y \) from the neutral axis by a method such as Newton-Raphson. The moment condition given by equation (3.26) was then computed. The value of \( A^u(0,T) \) which allowed equation (3.26) to be satisfied was found by computing \( \delta \) for a sequence of values of \( A^u(0,T) \) commencing with a value given by the lower bound solution which is described below.

3.6.2. The Lower Bound Solution.

The lower bound solution is that provided by the purely viscous strain field integrated over the cycle, and is given by

\[
\dot{\kappa}(t) = \left\{(2n+1)/n\right\}^{2n+1} \left\{ g(t)/ b^n d^{2n+1} \right\} N(t)
\]

(3.27)

\[
A^u \kappa = \int_0^T \dot{\kappa}^S(t) \, dt
\]

(3.28)

\[
W^\ell = \int_0^T \dot{\kappa}^S(t) M(t) \, dt
\]

(3.29)

3.7. Thick-Walled Tube Subjected to Cyclic Internal Pressure.

Consider a tube of internal radius \( a \) and external radius \( b \) subjected to a cyclic history of internal pressure \( p(t) \) with period \( T \). It is assumed that the material is isotropic in both its elastic and creep behaviour and that uniform conditions
of plane strain apply along the length of the tube.

The creep strain component is assumed to obey a Von Mises flow rule obtained by substituting

$$\phi = \{(3/2) s_{kk} s_{kk}^{1/2} = \sigma_e^{1/2}\}
$$

into (3.14) to obtain

$$\dot{\varepsilon}_{ij} = \{\sigma_e^{(n-1)/2} s_{ij} g(t) \tag{3.30}\}
$$

3.7.1. The Upper Bound Solution.

Equation (3.30) implies that there exists no creep volume change rate and therefore no volume change over a complete cycle. The accumulated principal creep strains over a complete cycle are denoted by $\Delta\varepsilon_{rr}$, $\Delta\varepsilon_{\theta\theta}$, and $\Delta\varepsilon_{zz}$. Over a complete cycle

$$\Delta\varepsilon_{rr} + \Delta\varepsilon_{\theta\theta} + \Delta\varepsilon_{zz} = 0 \tag{3.31}\$$

The optimal upper bound solution requires compatibility of these strains with an increment of radial displacement $\Delta u(r)$, given by

$$\Delta\varepsilon_{rr} = \frac{d\Delta u(r)}{dr}, \quad \Delta\varepsilon_{\theta\theta} = \Delta u(r)/r \tag{3.32}\$$

The addition of the plane strain assumption $\Delta\varepsilon_{zz} = 0$, and equations (3.31) and (3.32) provides a differential equation for $\Delta u$ which possesses the solution

$$\Delta u(r) = \Delta u(a) a/r \tag{3.33}\$$

The linear elastic solution for internal pressure $p$ and Poisson's ratio $\nu$ is given by

$$\hat{\sigma}_{rr} = -pa^2 \{b^2/r^2 - 1\}/\{b^2-a^2\}$$

$$\hat{\sigma}_{\theta\theta} = pa^2 \{b^2/r^2 + 1\}/\{b^2-a^2\}$$

$$\hat{\sigma}_{zz} = \nu(\hat{\sigma}_{rr} + \hat{\sigma}_{\theta\theta}) = 2va^2p/\{b^2-a^2\} \tag{3.34}$$
3.23.

The optimal upper bound requires a residual stress field, $\tilde{\sigma}_{rr}, \tilde{\sigma}_{\theta\theta}, \tilde{\sigma}_{zz}$, which satisfies the equilibrium conditions

$$\frac{d\tilde{\sigma}_{rr}}{dr} - \frac{\tilde{\sigma}_{\theta\theta} - \tilde{\sigma}_{rr}}{r} = 0$$

(3.35)

and

$$\rho_{rr}(a) = \rho_{rr}(b) = 0$$

(3.36)

Furthermore the accumulated creep strain over a cycle due to $\dot{\epsilon}_{ij} = \hat{\epsilon}_{ij} + \rho_{ij}$ must be compatible, and therefore derivable from a displacement field $\Delta^u(r)$ of the form of equation (3.33).

Since $\Delta u_{zz} = 0$ by assumption, then from equation (3.30)

$$\int_0^T \{\sigma_e\}^{(n-1)/2} \dot{\sigma}_{zz} g(t) \, dt = 0$$

i.e.

$$\int_0^T \{\sigma_e\}^{(n-1)/2} \sigma_{zz} g(t) \, dt = \int_0^T \{\sigma_e\}^{(n-1)/2} \{\sigma_{rr} + \sigma_{\theta\theta}\} g(t) \, dt$$

This relationship may be used to eliminate $\dot{\sigma}_{zz}$ from the expressions for $\Delta v_{rr}$ and $\Delta v_{\theta\theta}$ to obtain from (3.30), (3.32) and (3.33).

$$\Delta v_{rr} = (1/2) \int_0^T \{\sigma_e\}^{(n-1)/2} (\sigma_{rr} - \sigma_{\theta\theta}) g(t) \, dt = - \Delta v_{\theta\theta}$$

$$= - \Delta u(a) a/r^2$$

(3.37)

and

$$\sigma_{rr}^* = \hat{\sigma}_{rr} + \tilde{\sigma}_{rr}$$

$$\sigma_{\theta\theta}^* = \hat{\sigma}_{\theta\theta} + \tilde{\sigma}_{\theta\theta}$$

(3.38)
The problem now becomes that of solving equation (3.37) for \( \bar{\rho}_{rr} \) and \( \bar{\rho}_{\theta\theta} \), subject to the equilibrium conditions (3.35) and (3.36). From uniqueness considerations only a particular value of \( \Delta^u u(a) \) will allow all these conditions to be satisfied for a given history of internal pressure \( p(t) \). Unfortunately as the quantity \( \sigma_e \) in the integrand of (3.37) involves \( \rho_{zz} \) the equations may not be uncoupled from each other. However, if elastic incompressibility is assumed (\( v=1/2 \)) these equations may be uncoupled to permit a simple method of solution. With this condition \( \hat{\sigma}_{zz} = (\hat{\sigma}_{rr} + \hat{\sigma}_{\theta\theta})/2 \) and the unique solution is obtained by assuming that

\[
\rho_{zz} = (\rho_{rr} + \rho_{\theta\theta})/2
\]

so that

\[
\sigma_{zz} = (\sigma_{rr} + \sigma_{\theta\theta})/2
\]

Upon substituting into \( \sigma_e \) equation (3.37) becomes an explicit function of \( \sigma_{\theta\theta} - \sigma_{rr} \) and therefore of \( \bar{\rho}_{\theta\theta} - \bar{\rho}_{rr} \) and may be expressed in the form

\[
\Delta^u u(a) a/r^2 = \int_0^T \left\{ \frac{3}{4} \right\} \left\{ (\hat{\sigma}_{\theta\theta} - \hat{\sigma}_{rr}) \right. + (\bar{\rho}_{\theta\theta} - \bar{\rho}_{rr}) \right. \}^n g(t) \, dt \tag{3.39}
\]

For a prescribed history \( p(t) \) and a chosen value of \( \Delta^u u(a) \) equation (3.39) may be solved for \( \bar{\rho}_{\theta\theta} - \bar{\rho}_{rr} \). The equilibrium equation (3.35) may be used to find \( \bar{\rho}_{rr} \) and \( \bar{\rho}_{\theta\theta} \) separately

\[
\bar{\rho}_{rr}(r) = \int_a^r \frac{(\bar{\rho}_{\theta\theta} - \bar{\rho}_{rr})}{r} \, dr \tag{3.40}
\]
and the equilibrium condition at the external radius (3.36) becomes

\[ \int_{0}^{b} \frac{\bar{p}_{\theta \theta} - \bar{p}_{rr}}{r} \, dr = 0 \quad (3.41) \]

The procedure adopted was as follows. Equation (3.39) was solved for \( \bar{p}_{\theta \theta} - \bar{p}_{rr} \) using a method such as Newton-Raphson for a range of values of \( \Delta^u u(a) \) and the correct value was identified as that which satisfied equation (3.41).

3.7.2. **The Lower Bound Solution.**

The stationary state solution for a Von Mises material (3.30) is well known (3.23). The relationship between displacement rate \( \dot{u}(a) \) and internal pressure \( p(t) \) is given by

\[ \dot{u}^S(a) = \frac{(2/n)^n p^n a H}{2(1-(a/b)^{2/n})^n} \]

where \( H = 2 \left(\frac{3}{4}\right)^{n+1/2} g(t) \)

and the rate of energy dissipation per unit length of tube is given by

\[ \dot{W}(t) = p(t) \dot{u}(a) 2\pi a \]

The lower bound quantities for a history of internal pressure \( p(t) \) are therefore given by

\[ \Delta u^L(a) = \int_{0}^{T} \dot{u}^S(a) \, dt \]

\[ \dot{W}^L = \int_{0}^{T} \dot{S}(t) \, dt \]

3.8. **Computed Examples.**

Work bounds and the associated deformation rates in the cyclic state were computed for the beam, the tube and the two-bar structure shown in Fig. 3.2 for histories of loading of the form shown in Fig. 3.4, where the generalized load \( F \) corresponds to the maximum values of applied loads \( M(t) \), \( p(t) \), and \( P(t) \) for beam, tube and two-bar structure respectively. The histories of loading are defined in terms of the two loading parameters \( \lambda \) and \( \mu \), where \( \lambda \) is the ratio of the magnitudes of the applied loads and \( \mu \) provides a measure of the relative duration of the maximum load.

The solutions are presented in terms of the non-dimensional quantities

\[
B = \frac{\mathcal{W}^u}{\mathcal{W}^\delta} \quad \text{and} \quad A = \frac{(n+1) A(0)}{U^D}
\]

where \( U^D \) is the maximum value of \( \int_V (1/2) C_{ijkl} \tilde{\gamma}_{ij} \tilde{\gamma}_{kl} \, dV \).

\( B \) therefore provides a measure of the severity of ratchetting which occurs in the cyclic state. If \( B \) is significantly greater than unity then stress redistribution effects must be considered. \( A \) is a measure of the energy dissipated in achieving the cyclic state compared with the elastic complementary strain energy due to the maximum applied load. \( A \) therefore provides an indication of how the additional deformation due to this stress redistribution process compares with the initial elastic deformation of the structure.

Similarly the deformation rates of the structures in the cyclic state are presented in an analogous non-dimensional form given by

\[
K = \frac{\Delta^u x}{\Delta^\delta x}, \quad U = \frac{\Delta^u u(a)}{\Delta^\delta u(a)}, \quad \text{and} \quad X = \frac{\Delta^u \delta}{\Delta^\delta \delta}
\]
3.27.

This form of presentation permits the influence of stress redistribution on the deformation rates to be readily assessed.

When the loading parameter $\lambda = 1$ the quantity $A$ corresponds to the quotient

$$A = \frac{(n+1)(U^s - U^o)}{U^o}$$

where

$$U^s = \int_C \frac{1}{2} C_{ijkl} \sigma_{ij} \sigma_{kl} \, dV$$

previously discussed by Leckie and Martin (3.4). For the structures under consideration the quotient is given by:

Beam

$$A = \frac{(n+1)\left\{(2n+1)/3n\right\}^2 3 \{n/(2+n)\} - 1}{\{n + (2n)\}^{1/n}}$$

Tube

$$A = \frac{2(b^2-a^2) b^{4/n} \left( b^{2-4/n} - a^{2-4/n} \right)}{a^{2} b^{2} n^{2} \left( b/a \right)^{2/n} - 1} \left\{ 2 - (4/n) \right\}$$

Two-bar Structure

$$A = \frac{3\{(2n)^{2/n} + 2n^{1/n}\} - 1}{2\{n + (2n)^{1/n}\}^{2}}$$

The variation of the quantities $B$, $A$ and $K$ for the beam with $n$, $\lambda$ and $\mu$ are shown in Figs. 3.5, 3.6 and 3.7. From Fig. 3.5 the largest value of $B$ for the range of parameters considered was less than 1.07 which corresponds to an increase in load of about 1% (for $n = 7$). In terms of the additional loading which would be necessary to make the lower bound coincide with the upper bound, the worst case is that for $n = 3$, $\mu = 0.25$ and $\lambda = 0.35$. The load increase is however still less than 1%.
The deformation ratio $K$ shows similar trends in Fig. 3.7. Again in the worst case the difference between the displacement rate $\Delta^2 K$ and the value associated with $\Delta^u K$ corresponds to a change in load of less than 1% ($n = 7$) and the corresponding value for $n = 3$ is approximately 1.25% increase in load.

The complementary strain energy quantity $A$ is shown in Fig. 3.6. It can be seen that for all values of $n$, the maximum value of $A$ occurs when $\lambda = 1$, i.e. for constant load. In accordance with the observation of Leckie and Martin the quantity $A$ is seen to be small, and the associated deformation is of the order of the initial elastic deformation, and in most circumstances may be neglected.

In view of these low values the effects of stress redistribution can be safely neglected, and in view of the difficulties which can be experienced in performing cyclic uniaxial creep tests, these results suggest that cyclic moment beam tests may provide a better method of obtaining "uniaxial" data by utilizing the reference stress concept (3.5, 3.6, 3.7, 3.9).

For the case of a tube subjected to a cyclic history of internal pressure, $p(t)$ a ratio $b/a = 2$ was chosen as a fairly extreme case where stress redistribution effects could be expected to be significant. The quantities $B$, $A$ and $U$ are shown in Figs. 3.8, 3.9 and 3.10. The maximum values of $B$ are slightly larger than those for the beam for the values of $n$ considered, and correspond to a change in load of approximately 1%. The values of $U$ in Fig. 3.10 show similar maxima to those for $K$ in Fig. 3.7. They differ in the region of $\lambda = 0$ where $U$ becomes slightly less than unity, signifying that $\Delta^u u < \Delta^K u$ which indicates that a recovery effect is operating, although the effect is very small in this instance.
In Fig. 3.9 the maximum values of $A$ are again seen to occur when $\lambda = 1$ and this maximum value is in all cases less than corresponding values for the beam.

Results for the two-bar structure shown in Fig. 3.2 are presented in Figs. 3.11, 3.12 and 3.13 for $\eta = 2.5$ and Figs. 3.14, 3.15 and 3.16 for $\eta = 4$. It will be seen from Figs. 3.11 and 3.14 that as the effects of stress redistribution become more severe, i.e. for higher values of $\eta$ and $\mu$ and lower values of $\lambda$, the highest values of $B$ tend to be associated with $\lambda = 0$ rather than $\lambda = 0.5$. As for the beam and tube the highest values of $A$ are associated with $\lambda = 1$ (Figs. 3.12 and 3.15) and for higher values of $\eta$ and $\mu$ $A$ is, in a particular sense, less dependent upon $\mu$.

For values of $\lambda = 0$ a marked recovery mechanism is seen to be operating (Figs. 3.13 and 3.16) with values of the normalized displacement $X$ significantly less than unity especially for situations associated with the higher values of $B$. These results indicate that the two-bar structure provides a very useful model which can be made to exhibit large stress redistribution effects by increasing the shape parameter $\eta$, but which exhibits trends which mirror those of the beam and tube for lower values of $\eta$.

It can be argued heuristically that the bounds obtained for a time hardening material are in a particular sense conservative. Provided the bounds are considered in a normalized form such as $B$ the values obtained provide conservative bounds for materials obeying other creep laws for time varying stress. The arguments to justify this assertion are similar to those advanced in section 3.2. for creep due to steady loading, although there is no need to normalize the bounds derived by Leckie and Martin.
General materials may exhibit a greater mean strain rate when subjected to cyclic histories of stress as is shown in Chapter 4. Consider two geometrically similar structures, one composed of time hardening material and the other composed of material obeying some other creep law for time varying stress which involves other state variables than merely time. At large times once the cyclic stationary state has been established, the influence of the stress redistribution process towards the cyclic state will be "forgotten". Any difference between the normalized energy dissipation rates for the two structures will depend upon the stress redistribution process in the stationary state. The time hardening material will not be influenced by the stress-redistribution process, but the general material will be made "stiffer" than would be the case if stress redistribution effects were absent, and therefore the normalized values of the bounds are seen to be conservative.

The method of computing optimal upper bounds described in section 3.7 provides results which are obtained far more cheaply in terms of computer time and human effort than from conventional elastic-creep calculations. Furthermore the procedure described in Chapter 5 can be employed to apply the results to a wider class of materials. The only additional information necessary can be obtained from a uniaxial cyclic creep test at prescribed stress levels. As the normalized values of the bounds are conservative the results can be used with confidence for the wider class of materials.

However for many structures the optimal upper bound may only be obtained at the expense of considerable analytical effort which may not be justified. Leckie and Ponter (3.10) have shown that plastic shakedown solutions can be used to
obtain non-optimal bounds. Frequently such bounds will still provide information on which a design decision can be based. Furthermore in many cases severe stress redistribution effects are associated with a high value of the stress index, \( n \), and little economic penalty is incurred if the design is based on an upper bound solution since a decrease in stress from \( \sigma \) to \( \sigma - \delta \sigma \) will result in a corresponding decrease in creep rate from \( \dot{\gamma} \) to \( \dot{\gamma} - n \delta \dot{\gamma} \).
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FIG. 33

\( \eta = 4 \quad n = 3 \)

\[ \left( \frac{t}{t_0} \right)^{\frac{1}{3}} \]
FIG. 3.5.
FIG. 36.
FIG 3.8
FIG. 3.9
FIG. 3.12
FIG. 3-14.
FIG. 3.15

Phenomenological theories of creep deformation depend largely on empirical relationships which provide a limited description of material behaviour. The time hardening theory has been shown to provide an acceptable description of material behaviour for steady loading, in that structural deformation can be predicted with sufficient accuracy for most engineering situations. Other hypothetical creep laws tend to introduce difficulties into the computational process. This is especially true of hereditary theories. Any method which is to gain wide acceptance needs to be analytically simple and must provide the maximum amount of information on structural performance from the minimum of material test data. In this Chapter an empirical method of quantifying cyclic stress creep data is suggested which permits the creep strain accumulated during cyclic histories of stress to be related to that of a similarly stressed time hardening material. This provides a means of applying the results of time hardening calculations to structures composed of a wider class of materials with related constitutive relationships.
4.2.

The vast majority of creep tests performed are isothermal uniaxial tensile tests, but very limited data is available for variable stress histories (4.1, 4.2). It would therefore be extremely helpful to be able to relate the strain accumulated during a cyclic stress test to that of an equivalent constant stress test, such a procedure is proposed in this Chapter.

In Chapter 3 the creep law given by

\[
\frac{\varepsilon(t)}{\varepsilon_0} = \left(\frac{\sigma}{\sigma_0}\right)^n \left\{\frac{t}{t_0}\right\}^m
\]  

(4.1)

was introduced together with three hypothetical creep laws which have been proposed to generalize equation (4.1).

For situations involving uniaxial time varying states of stress equations (3.2), (3.3) and (3.4) can be written as:

(a) The time hardening law:

\[
\dot{\varepsilon}(t) = \frac{m}{\varepsilon_0} \frac{\sigma(t)}{\sigma_0} \left(\frac{t}{t_0}\right)^{m-1}
\]  

(4.2).

(b) The strain hardening law

\[
\dot{\varepsilon}(t) = \frac{m}{\varepsilon_0} \frac{\sigma(t)}{\sigma_0} \left(\frac{t}{t_0}\right) \left\{\frac{\varepsilon_0}{\varepsilon(t)}\right\}^{(1-m)/m}
\]  

(4.3)

(c) Rabotnov's work hardening law

\[
\dot{\varepsilon}(t) = \frac{m}{\varepsilon_0} \frac{\sigma(t)}{\sigma_0} \frac{(1+n-m)/m}{(1-m)/m} \left\{\int_0^t \sigma(\tau) d\varepsilon(\tau)\right\}^{(1-m)/m}
\]  

(4.4)

Equations (4.2), (4.3) and (4.4) reduce to equation (4.1) for constant stress situations.
4.2. Identification of an Equivalent Constant Stress.

Taira (4.1) studied the creep strains which occur when a strain hardening material is subjected to cyclic stresses, and was able to show that the same creep strains could be accumulated by subjecting the material to a constant stress which he termed the "Equivalent Static Stress." Życzkowski (4.3) introduced the concept of a "Reduced stress" and it is this concept that is considered and extended in this Chapter. A similar concept is also discussed by Ohji and Marin (4.2).

A typical creep strain vs time curve for a material subjected to a cyclic history of stress is shown in Fig. 4.1, together with the corresponding strains due to constant stresses $\sigma$ and $\lambda \sigma$. A convenient method of presentation of constant stress data is to plot the creep strain on a time base $(t/t_0)^m$ so that constant stress data appears as a straight line (Fig. 4.2). When the cyclic creep data is plotted on the same time base, the resulting curve has the form shown in Fig. 4.2 and can be approximated by a straight line $(a,b)$ defined in terms of its slope and intercept at the origin. A constant stress $\sigma^*$ must therefore exist such that $\lambda \sigma < \sigma^* < \sigma$ which will produce a straight line through the origin with the same slope as that of the line $a,b$. This stress can be determined from constant stress creep data and one cyclic creep test.

Consider materials whose constant stress creep behaviour is given by equation (4.1), but which obey laws given by equations (4.2), (4.3) and (4.4) when subjected to time varying uniaxial stress. If these hypothetical materials are subjected
to cyclic histories of stress of(t) where f(t) is some periodic function which completes one cycle in time \( t \), then it can be shown that an equivalent constant stress \( \sigma^* \) is dependent upon the form of \( f(t) \) and the creep law for the material but is independent of \( t \).

By way of illustration values of \( \sigma^* \) are derived for materials whose creep laws are given by equations (4.2), (4.3) and (4.4) when subjected to cyclic stress histories of the form shown in Fig. 4.3

(a) **Time Hardening Material.**

The uniaxial strain rate is given by

\[
\dot{\varepsilon}(t) = \frac{m}{\tau_o} \left( \frac{\sigma(t)}{\sigma_o} \right)^n \frac{t^{m-1}}{t_o^m}
\]  

(4.2)

Consider the cycle of loading from time \( t - \tau \) to time \( t \) where the creep strain \( \Delta \varepsilon \) is given by

\[
\frac{\Delta \varepsilon}{\varepsilon_o} = \left( \frac{\sigma}{\sigma_o} \right)^n \left[ \left( \frac{t - \tau}{2} \right)^m - \left( \frac{t - \tau}{2} \right)^m \right] + \lambda^n \left( t^m - (t - \tau)^m \right) / t_o^m \]  

(4.5)

A steady stress \( \sigma^* \) which produces the same creep strain from time \( t - \tau \) to time \( t \) is given by

\[
\frac{\Delta \varepsilon}{\varepsilon_o} = \left( \frac{\sigma^*}{\sigma_o} \right)^n \left[ \left( \frac{t - \tau}{2} \right)^m - \left( \frac{t - \tau}{2} \right)^m \right] / t_o^m
\]  

(4.6)

Hence from equations (4.5) and (4.6)

\[
\sigma^* = \sigma^n \left( \frac{\left( \frac{1 - \tau}{2} \right)^m - \left( \frac{1 - \tau}{2} \right)^m + \lambda^n \left( 1 - \left( \frac{1 - \tau}{2} \right)^m \right)}{1 - \left( \frac{1 - \tau}{2} \right)^m} \right)
\]

After several cycles when \( \tau \ll t \) (\( m \) is usually \( < 1 \)) the binomial expansion gives

\[
\sigma^* = \sigma^n \{ 1 + \lambda^n \} / 2
\]
Hence after several cycles

\[ \sigma^* = \sigma((1 + \lambda^n)/2)^{1/n} = \gamma_{th} \sigma \]  

(4.7)

The equivalent stress \( \sigma^* \) given by equation (4.7) is independent of the parameter \( m \). The expression for \( \sigma^* \) given for \( m = 1 \) by Zyczkowski is that given by equation (4.7).

For time hardening materials \( \sigma^* \) is dependent only upon the form of the loading cycle and the stress index \( n \).

(b) **Strain Hardening Material.**

The uniaxial strain rate is given by

\[ \dot{\varepsilon}(t) = \frac{m \sigma(t)^{n/m}}{\varepsilon_o^{n/m} t_o^{m}} \left( \frac{\varepsilon_o}{\varepsilon(t)} \right)^{(1-m)/m} \]  

(4.3)

For the cycle of loading from time \( t-\tau \) to time \( t \) the creep strain \( \Delta \varepsilon \) is given by

\[ \frac{\Delta \varepsilon}{\varepsilon_o} = \left( \frac{\sigma}{\sigma_o} \right)^n \left( \frac{t/2 + \lambda^n}{\tau} \right)^m - \left( \frac{(t-\tau)/2 + \lambda^n}{\tau} \right)^m / t_o^m, \]  

(4.8)

A steady stress \( \sigma^* \) which produces the same creep strain from time \( t-\tau \) to time \( t \) is given by

\[ \frac{\Delta \varepsilon}{\varepsilon_o} = \left( \frac{\sigma^*}{\sigma_o} \right)^n \left( \frac{t^m - (t-\tau)^m}{t_o^m} \right) \]  

(4.6)

and hence from equations (4.6) and (4.8)

\[ \sigma^* = \frac{\sigma^n (1 + \lambda^n/m)}{t^m - (t-\tau)^m} \]  

where \( \lambda = \lambda(t) \).
Thus \[ \sigma^* = \sigma \left( \frac{1+\lambda^{n/m}}{2} \right)^m \]

and \[ \sigma^* = \sigma \left( \frac{1+\lambda^{n/m}}{2} \right)^{m/n} = \gamma_{sh} \sigma \]  

(4.9)

The equivalent stress \( \sigma^* \) given by equation (4.9) is not dependent upon the ratio \( \tau : t \) and is therefore valid for all \( t \). Thus for strain hardening materials the value of \( \sigma^* \) is dependent only upon the parameters \( n \) and \( m \) in the creep law and the form of the loading cycle.

(c) Work Hardening Material.

The uniaxial strain rate is given by

\[ \dot{\varepsilon}(t) = \frac{m \sigma (1+n-m)/m \varepsilon_o (l-m)/m}{\sigma_o^{n/m} t_o \left\{ \int_0^t \sigma(t) \, dc(t) \right\}^{(1-m)/m}} \]  

(4.4)

Similar analysis to that given above is laborious and complex for equation (4.4) and has not been pursued. Instead a numerical investigation was carried out for \( m = 1/3 \) and \( n = 3, 5, \) and \( 7 \) and the results compared with the corresponding results for strain hardening materials for histories of stress of the form shown in Fig. 4.3. The results given in Table 4.1 are discussed more fully later, but a significant result is that \( \sigma_{sh}^* \) and \( \sigma_{wh}^* \) (and therefore \( \gamma_{sh} \) and \( \gamma_{wh} \)) would appear to be identical for all practical purposes.
4.7

4.3. Creep due to Cyclic Histories of Stress.

In this section a study is made of the creep strains accumulated during cyclic stress tests on the materials whose creep rates are given by equations (4.2), (4.3), (4.4) and the results compared with those of tests at the equivalent constant stress.

For all the model materials and stress histories considered the results conformed to the general form of Fig. 4.2. The strain due to the cyclic history of stress was asymptotic to a straight line \((a,b)\) having an intercept \(\zeta_0\) at the origin.

Hence the creep strain given by

\[
\varepsilon(t)/\varepsilon_0 = \gamma \left( \frac{\sigma/\sigma_0}{\sigma/\sigma_0} \right)^n \left( \frac{t}{t_0} \right)^m = \left( \frac{\sigma^*}{\sigma_0} \right)^n \left( \frac{t}{t_0} \right)^m
\]
due to a constant stress \(\sigma^* = \gamma \sigma\) will differ from the creep strain obtained by application of a cyclic history of stress by an amount equal to the intercept \(\zeta_0/\varepsilon_0\).

An approximate expression for the creep strain due to a cyclic history of stress is therefore given by

\[
\varepsilon(t)/\varepsilon_0 = \gamma \left( \frac{\sigma/\sigma_0}{\sigma/\sigma_0} \right)^n \left( \frac{t}{t_0} \right)^m + \frac{\zeta_0}{\varepsilon_0} = \left( \frac{\sigma^*}{\sigma_0} \right)^n \left( \frac{t}{t_0} \right)^m + \frac{\zeta_0}{\varepsilon_0}.
\]

Time can be expressed in the form

\[t = rT\]

where \(T\) is the cycle time and \(r\) is the number of cycles of stress.

During the \(r+1^{th}\) cycle the increase in creep strain is given by

\[
\Delta \varepsilon/\varepsilon_0 = \gamma \left( \frac{\sigma/\sigma_0}{\sigma/\sigma_0} \right)^n \left( \frac{(r+1)^m - r^m}{t/T} \right) \varepsilon_0^m
\]

\[
= \left( \frac{\sigma^*}{\sigma_0} \right)^n \left( \frac{(r+1)^m - r^m}{t/T} \right) \varepsilon_0^m
\]

Although there is little significant difference between creep strains accumulated by strain hardening and work hardening materials subjected to a given cyclic stress history, time hardening materials accumulate less strain when subjected to the same history. Consider for example creep tests with
cyclic stress histories of the form shown in Fig. 4.3, and
\[ \lambda = 0. \]
Equation (4.7) reduces to
\[ \dot{\sigma}_{th} = \sigma/2^{1/n} \]
and equation (4.9) becomes
\[ \dot{\sigma}_{sh} = \sigma/2^{m/n} \]

The corresponding creep strains accumulated during the \( r+1 \)th cycle will be
\[ \Delta \varepsilon_{th}/\varepsilon_o = (\sigma^n/2) \{(r+1)^m - r^m\} \frac{T}{T_o} \]
and
\[ \Delta \varepsilon_{sh}/\varepsilon_o = (\sigma^n/2^m) \{(r+1)^m - r^m\} \frac{T}{T_o} \]

The creep strain for a strain hardening material will be \( 2^{1-m} \) times greater than that for a time hardening material subjected to the same cyclic history of stress.

It is helpful to present the results in a particular form which is compact and facilitates comparisons between the material deformation discussed in this Chapter and the corresponding structural displacement considered in Chapter 5. Recall that time can be expressed by the equation
\[ t = rT \]
where \( T \) is the cycle time and \( r \) is the number of cycles of stress. A normalized measure of creep strain is given by
\[ \psi = \varepsilon(t)/\varepsilon_o = r^m + \varepsilon_o/\dot{\varepsilon} = r^m + \psi_o \]
(4.10)
where
\[ \dot{\varepsilon}/\varepsilon_o = (\sigma^n/\sigma_o) \{(T/T_o)^m \}

The results are then presented in the \( \psi \sim r^m \) plane.

This choice of non-dimensional strain and time measures means that in Figs. 4.4 to 4.8 the straight line of unit slope through
4.9.

the origin, is the normalized creep strain which results when a uniaxial test is performed at the equivalent constant stress \( \sigma^* \).

In order to compare the creep strains due to cyclic histories of stress with those resulting from the equivalent constant stress, strains were calculated for materials with \( m = 1/3 \) subjected to histories of stress of the form shown in Fig. 4.3 for \( \lambda = 0 \) and \( \lambda = 1.5 \). The materials considered were those obeying the time, strain and work hardening hypotheses.

The results presented graphically in Figs. 4.4 to 4.7 conform to a standard shape, so, that after a few cycles, the values of \( \psi \) lie on a straight line which has unit slope and an intercept \( \psi_0 \) (Fig. 4.8) which varies with the law used the value of \( n \), and the nature of the cycle. In general the value of the stress index \( n \) has little effect on the form of the results. This can be seen in Figs. 4.5 and 4.7 which show the effect to be small, while in Figs. 4.4 and 4.6 the effect is non-existent. Values of \( \sigma_{wh}^* \), the corresponding values of \( \sigma_{sh}^* \) and values of \( \psi_0 \) for work hardening material are given for \( n = 3, 5 \) and \( 7 \) and a range of values of \( \lambda \) in Table 4.1. For \( \lambda = 0 \) the strain hardening and work hardening results are indistinguishable (Fig. 4.6). The intercept \( \psi_0 \) is always zero for strain hardening material, whereas the work hardening results exhibit small values of \( \psi_0 \) which for most practical purposes can be neglected. The mean slope of the \( \psi \sim r^{1/3} \) curves for work hardening materials rapidly converge to a value equal to that for the strain hardening material. Since the work hardening and strain hardening results are so similar, work hardening results for \( \lambda = 1.5 \) are not presented in graphical form.
The mean slope of the $\psi \sim r^{1/3}$ curves for time hardening materials is asymptotic to unity as implied by the analysis presented in Section 4.2, and of the materials considered, time hardening materials exhibit the largest values of $\psi_0$.

In order to obtain some knowledge of the likely maximum value of $\psi_0$, two extreme cases of the class of loading shown in Fig. 4.3 characterised by $\lambda = \infty$ and $\lambda = 0$ were applied to a time hardening material. The results shown in Fig. 4.8, indicate a value of $\psi_0$ between plus and minus 0.617. It is of interest to note however, that a very good fit to all the curves is obtained by drawing a straight line with unit slope from the value of $\psi$ obtained after one cycle of loading (Fig. 4.8).

It is perhaps worthwhile attempting to quantify the intercept $\zeta_0$ with respect to the corresponding elastic strain, thereby giving some indication of error incurred if the normalized intercept $\psi_0$ is ignored. The following material data corresponds approximately to that given by Odqvist (4.4) for a rolled stainless steel at 700°C:

$$n = 3$$

$$E = 137 \text{ kN/mm}^2$$

Stress for 1% strain in 100,000 hours 14.7 N/mm².

It is assumed that the material obeys a creep law of the form

$$\varepsilon = k \sigma^3 t^{1/3}$$

and that the cycle time is 24 hours with the stress being applied for half this time.

If a stress of 14.7 N/mm² produces 1% strain in 100,000 hours

$$k = \frac{0.01}{(14.7^3)(10^{5/3})}$$
If the equivalent constant stress is to be $14.7 \text{ N/mm}^2$ then for a time hardening material the maximum stress will be $14.7 \times 2^{1/3} \text{ N/mm}^2$ (equation 4.7). From equation (4.10)

$$\psi_o = \zeta_o/\dot{\varepsilon} = 0.617$$

where

$$\dot{\varepsilon} = 2k(14.7^3) (24^{1/3})$$

from which

$$\zeta_o = 0.617\dot{\varepsilon} = 0.00076 \quad \frac{14.7}{137000} = 0.00011.$$ 

The corresponding elastic strain is $\frac{14.7}{137000} = 0.00011.$ In general the material will undergo secondary creep and thus the value of $\zeta_o$ will be of the order of the corresponding elastic strain, and in most circumstances can be safely neglected.

### 4.4. Experimental Determination of Equivalent Constant Stress.

In sections 4.2 and 4.3 it has been shown that the creep strains resulting from cyclic stress histories can be defined in terms of the equivalent steady stress $\sigma^* = \gamma \sigma$ and the intercept $\psi_o$. The values of $\sigma^*$ and $\psi_o$ are independent of the length of cycle time and depend only on the form of the stress history and the material.

A procedure is now described for determining $\sigma^*$ and $\psi_o$ from constant stress creep data and a single cyclic stress creep test.

A series of constant stress creep tests performed on a material whose creep strain is given by

$$\varepsilon(t)/\varepsilon_o = (\sigma/\sigma_o)^n (t/t_o)^m \quad (4.1)$$

will yield results which can be presented as a series of plots of $\varepsilon(t) \sim (t/t_o)^m$. These will be a series of straight lines through the origin as shown in Fig. 4.9. The constants $\varepsilon_o$, $\sigma_o$ and $t_o$ have dimensions of strain, stress and time.
respectively. The dimensionless constants $n$ and $m$ are determined from the test data, while $\sigma_0$ and $t_0$ can be chosen for convenience. Having made this choice the value for $\varepsilon_0$ is fixed from experimental data.

The slopes of the lines in the $\varepsilon \sim \{(t/t_0)^m\}$ plane (Fig. 4.9) are denoted by

$$\dot{\varepsilon} = \frac{d\varepsilon}{d\{(t/t_0)^m\}}$$

and a plot of $\log \dot{\varepsilon} \sim \log \sigma$ gives a straight line of slope $n$ as shown in Fig. 4.10. A creep test with a cyclic stress history such as that shown in Fig. 4.3 will (if the effects of creep hesitation and recovery are small) yield a curve of $\varepsilon \sim \{(t/t_0)^m\}$ of the form shown in Fig. 4.11 (for $\lambda = 0$).

The mean slope $\dot{\varepsilon}$ of the curve shown in Fig. 4.11 can be identified after a few cycles of stress and the equivalent steady stress $\sigma^*$ is then determined from Fig. 4.10.

The intercept $\varepsilon_0$ is dependent upon the material and the form and duration of the cycle of stress. We can transform the plot of Fig. 4.11 into a form which is independent of the duration of the stress cycle. The time $T$ for one cycle of stress is $T$ and since $\sigma^*$ is known

$$\dot{\varepsilon}/\varepsilon_0 = (\sigma^*/\sigma_0)^n \{(T/t_0)^m\}$$

hence

$$\psi = \varepsilon(t)/\dot{\varepsilon}$$

and the intercept $\psi_0 = t_0/\dot{\varepsilon}$ can therefore be determined.

4.5. **Practical Implications.**

The hypothetical material models considered indicate that creep strain data for cyclic histories of stress can be
quantified by two constants $\gamma$ and $\psi_0$ and that these are
independent of cycle time. This would suggest that it should
be possible to determine these constants by means of an accelerated
creep test, and it is possible that this may be the case for
certain materials over certain ranges of stress and temperature.
Aldén, Aronsson and Rohlin (4.5) have performed cyclic stress
creep tests for histories of stress of the form shown in Fig. 4.3
for cycle times of 48 hours and 1000 hours and have concluded
that the overall creep rate is greater for the longer cycle
time. However the difference in rate is of the order of the
scatter in their results and it is difficult to draw firm
conclusions from the results presented, but it is likely that
results obtained from accelerated cyclic creep tests could be
misleading in some circumstances, although the results presented
by Aldén et al. tentatively suggest that below a certain critical
value of cycle time any such error will be small. An intensive
testing programme would be necessary to establish when such a
procedure can be used with confidence.

The analysis has been applied to a restricted class
of materials and frequently the stress index $n$ is known to be
itself a function of stress, and the choice of a function of
time of the form $(t/t_o)^m$ implies a restriction to primary creep.
However it is likely that cyclic creep data can be quantified
in terms of two constants in the manner described for a wider
class of creep laws than that given by $(t/t_o)^m$, although $\psi_0$
will not be independent of the cycle time $T$, and $\gamma$ will also
be more sensitive to cycle time.

The experimental results presented by Taira (4.1),
Ohji and Marín (4.2), and Aldén et al. (4.5) indicate that the
concept of an equivalent steady stress also applies to secondary creep. This corresponds to \( m = 1 \) when equations (4.7) and (4.9) become identical, which implies that all materials would exhibit a similar value of \( \gamma \) in secondary creep. Thus the theoretical predictions obtained from hypothetical models are seen to be indicative only of a trend, but the value of \( \gamma \) in secondary creep can be obtained from a cyclic creep test, although it is reasonable to expect the value of \( \gamma \) to vary with the material as in primary creep.

Ponter (4.6) has extended the theorem of plastic shakedown to structures composed of time hardening material and it can be argued that a test specimen, although nominally possessing a homogeneous state of stress, (and therefore being a minimum weight structure) will have internal stress concentrations due to the random alignment of the individual grains and metallurgical imperfections. The plastic shakedown load can therefore be compared with the stress \( \sigma_B \) at which the Bauschinger effect first becomes apparent, and plastic effects can therefore be expected to be negligible for excursions in stress less than \( n \sigma_B / (n+1) \). This restriction should provide not only a guide to the range of stress over which the procedure described in this Chapter can reasonably be expected to be applicable but also indicate to the designer a stress level below which material hysteresis is unlikely to occur.
REFERENCES


4.3. Życzkowski, M. "Discussion of Reference 4.1. pp. 119-120.


Table 4.1.

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Cyclic history of stress

Creep strain due to cyclic stress

Fig. 4.1
Fig. 4.2
Fig. 4.3
TIME HARDENING

\[ \sigma^* = \frac{\sigma}{2^{1/n}} \]

\[ n = 8 \quad \sigma^* = 0.794\sigma \]

FIG. 4.4
TIME HARDENING

\[ \sigma^* = \sigma \left( \frac{1+1.5n}{2} \right)^{1/n} \]

n = 3  \quad \sigma^* = 1.299\sigma

**Fig. 4.5**
Strain hardening and work hardening

\[ \sigma^* = \frac{\sigma}{2^{1/3} n} \]

\[ n = 3 \quad \sigma^* = 0.925 \sigma \]

Fig. 4.6
STRAIN HARDENING

\[ \sigma^* = \sigma \left\{ \frac{1+1.5^3n}{2} \right\}^{\frac{1}{3n}} \]

\( n = 3 \quad \sigma^* = 1.392\sigma \)

FIG. 4.7
FIG. 4.9

FIG. 4.10
FIG. 4-11.
5.1. Deformation in the Absence of Stress Redistribution Effects.

In Chapter 4 a method was described for isolating the two constants required to estimate the creep strains caused by cyclic histories of stress. It was shown that the concept of an equivalent steady stress permits the results of cyclic stress creep tests to be related to constant stress creep data. In the present Chapter a method is proposed for estimating structural creep deformation due to proportional cyclic loading. The method relies on the use of an equivalent steady stress obtained from a cyclic stress creep test and the creep law for time varying stress is therefore not required.

In order to understand the creep behaviour of structures subjected to proportional cyclic loading it is helpful to study the behaviour when stress redistribution effects are neglected. The structure shown in Fig. 3.2. has been chosen to illustrate the procedure, since it is analytically simple, and is representative of structures with local stress concentration. The structure consists of two bars of Maxwell material so chosen that the degree of stress concentration increases with the shape parameter \( n \).
5.2.

Without loss of generality the form of the creep law is chosen to be

\[ \epsilon / \epsilon_0 = \left( \sigma / \sigma_0 \right)^n \left( t / t_0 \right)^{1/3} \]  (5.1)

When the structure is subjected to a constant load \( P \) the initial elastic displacement is

\[ \delta_e / \lambda = P_1 / AE = 2P_2 / AE \]  (5.2)

and the forces in the two members are

\[
\begin{align*}
P_1 &= 2P/3 \\
P_2 &= P/3
\end{align*}
\]  (5.3)

From equations (5.2) and (5.3)

\[ \delta_e / \lambda = 2P/3AE \]  (5.4)

When \( t \) is large and the elastic strains can be neglected in comparison with the creep strains the stationary creep displacement is

\[ \delta_s / \epsilon_0 = \left( P_1 / \sigma_o A \right)^n \left( t / t_0 \right)^{1/3} = 2n \left( P_2 / n \sigma_o A \right)^n \left( t / t_0 \right)^{1/3} \]  (5.5)

and the stationary forces are

\[
\begin{align*}
P_1 &= P(2n)^{1/n} / \left( n + (2n)^{1/n} \right) \\
P_2 &= Pn / \left( n + (2n)^{1/n} \right)
\end{align*}
\]  (5.6)

Equation (5.5) can therefore be rewritten as

\[ \delta_s / \epsilon_0 = \left( P / \sigma_o A \right)^n \left( 2n \left( n + (2n)^{1/n} \right)^n \right) \left( t / t_0 \right)^{1/3} \]  (5.7)

The ratio of the initial elastic stresses in the two members is given by \( \sigma_1 / \sigma_2 = 2n \) and the corresponding ratio for the stationary stresses is \( (2n)^{1/n} \). A measure of the stress redistribution is given by \( (2n)^{(n-1)/n} \), and stress redistribution effects can therefore be expected to increase with the shape parameter \( n \).
Neglecting stress redistribution effects the displacement of the applied load \( P \) is therefore

\[
\delta / k = 2P / 3AE + \varepsilon_o \left( 2n / \left( \eta + (2n)^{1/n} \right) \right) \left( P / \sigma_o A \right)^n \left( t / t_o \right)^{1/3} \tag{5.8}
\]

while the stored elastic energy is \( P^2 / 3AE \) and the stationary creep energy dissipation is

\[
\left( \varepsilon_o k P 2n / \left( \eta + (2n)^{1/n} \right) \right) \left( P / \sigma_o A \right)^n \left( t / t_o \right)^{1/3}
\]

Leckie (5.1) has shown that it is convenient to express such results in terms of a simple uniaxial specimen which has a volume \( \beta A l \left( 1 + 2n^2 \right) \) and is subjected to an average stress \( \bar{\sigma} = \alpha P / A \). The corresponding strain for this uniaxial specimen is

\[
\varepsilon = \bar{\sigma} / E + \varepsilon_o \left( \bar{\sigma} / \sigma_o \right)^n \left( t / t_o \right)^{1/3}
\]

Equating the elastic and creep energy components of the structure and uniaxial specimen yields the following expressions for \( \alpha \) and \( \beta \)

\[
P^2 / 3AE = \beta A l \left( 1 + 2n^2 \right) \alpha^2 / 2E = \beta A l \left( 1 + 2n^2 \right) \alpha^2 P^2 / 2A^2 E
\]

and

\[
\varepsilon_o k P \left( P / \sigma_o A \right)^n \left( 2n / \left( \eta + (2n)^{1/n} \right) \right) \left( t / t_o \right)^{1/3}
\]

\[
= \beta A l \left( 1 + 2n^2 \right) \varepsilon_o \left( \bar{\sigma}^{n+1} / \sigma_o^n \right) \left( t / t_o \right)^{1/3}
\]

from which

\[
\alpha = \left( 3n \right)^{1/(n-1)} / \left( \eta + (2n)^{1/n} \right) n / (n-1)
\]

\[
\beta = 2 / 3 \left( 1 + 2n^2 \right) \alpha^2 \tag{5.9}
\]
Since the creep displacement satisfies the relationship

\[ P \delta_C = 6\alpha l \{1+2n^2\} \bar{\varepsilon}_C \]

then

\[ \delta_C = \alpha 6 l \{1+2n^2\} \bar{\varepsilon}_C \]

(5.10)

where \( \bar{\varepsilon}_C \) is the creep strain in the uniaxial specimen subjected to a stress \( \bar{\sigma} \).

Consequently the displacement \( \delta_C \) can be determined from the raw tensile test data directly by applying equation (5.10) without the necessity of determining the constants \( \varepsilon_o \) and \( \sigma_o \).

The value of the stress index \( n \) must be known in order to calculate \( \alpha \) and \( \beta \) although it has been shown by Mackenzie (5.2) that in many structures the values of \( \alpha \) and, by implication, \( \beta \) are almost invariant over a large range of values of \( n \). However for many materials uniaxial tensile creep data is available for a range of stresses and \( n \) can be determined.

(The value of \( n \) is also required to evaluate values of the material parameter \( \gamma \) introduced in Chapter 4).

The reference stress for constant load is \( \bar{\sigma} \) and for variable loading \( P(t) \) the reference stress is simply

\[ \bar{\sigma}(t) = \alpha P(t)/A. \]

The corresponding uniaxial strain \( \varepsilon_C(t) \) can be determined experimentally from a cyclic stress creep test. In Chapter 4 it was shown that the strain \( \bar{\varepsilon}_C(t) \) accumulated during cyclic histories of stress can be expressed approximately in terms of an equivalent constant stress \( \bar{\sigma}_* \) and an intercept \( \xi_o \). If the maximum load is \( P_{max} \) then

\[ \bar{\sigma}_* = \alpha \gamma P_{max}/A = \alpha P_{max}/A \]

(5.11)
where $\gamma$ is a constant which is a property of the material and the form of the loading cycle and $P^* = \gamma P_{\text{max}}$ is defined as the equivalent constant load.

Hence in the absence of stress redistribution effects the displacement due to the creep strain is

$$\delta_c = a\beta l \{1+2n^2\} \varepsilon_c(t) \quad (5.12)$$

and in the case of cyclic loading for this material an approximate expression for $\varepsilon_c(t)$ is given by

$$\varepsilon_c(t) = \varepsilon_o \left(\frac{\bar{\sigma}}{\sigma_o}\right)^n \left\{\frac{t}{t_o}\right\}^{1/3} + \bar{\varepsilon}_o$$

as shown in Chapter 4.

Equation (5.12) can therefore be rewritten as

$$\delta_c = a\beta l \{1+2n^2\} \left[\varepsilon_o \left(\frac{\bar{\sigma}}{\sigma_o}\right)^n \left\{\frac{t}{t_o}\right\}^{1/3} + \bar{\varepsilon}_o\right]$$

and on substituting for $\beta$ (equation 5.9) and $\bar{\sigma}^*$ (equation 5.11) as

$$\delta_c = \frac{2}{3}a\varepsilon_o \gamma^{-1} \left(\frac{P_{\text{max}}}{\sigma_o A}\right)^n \left\{\frac{t}{t_o}\right\}^{1/3} + 2\varepsilon_o \bar{\varepsilon}_o/3 \quad (5.13)$$

Equation (5.13) gives an approximate expression for the displacement of the structure in the absence of stress redistribution effects. It should be borne in mind that $a$ depends upon the geometry of the structure and the positions of applied loads together with the material parameter $n$ (i.e. the stationary stress distribution). Whereas $\gamma$ and $\bar{\varepsilon}_o$ are material properties obtained from a cyclic creep test.

It is convenient for reasons which will become apparent later in this Chapter to present equation (5.13) in a slightly different form. Time can be expressed by means of an equation of the form

$$t = rT$$

where $T$ is the duration of one loading cycle and $r$ the number of cycles. The displacement can then be expressed in a
normalized form
\[ \chi = \delta_c / \delta = r^{1/3} + 2 \alpha \psi_0 / 3 \alpha \hat{\delta} \] (5.14)
where
\[ \hat{\delta} = (2/3) \epsilon_0 \alpha^{n-1} \gamma \left( \frac{P_{\text{max}}}{\sigma_0 A} \right)^n \left( \frac{T}{t_0} \right)^{1/3} \]

The line a, b shown in the \( \chi \sim r^{1/3} \) plane in Fig. 4.2 is the asymptote to which the actual creep displacement curve converges as the number of cycles increases. For some materials, e.g. strain hardening, this convergence occurs at the end of the first cycle as shown in Figs. (4.6, 4.7). This representation is particularly convenient since any form of creep law, which for constant uniaxial tension conforms to the law given by equation (5.1), will yield the same form of graph, the only difference being that different materials will give different intercepts. However, since the importance of the intercept diminishes with increasing number of cycles the plots for differing materials will be almost identical.

5.2. Deformation in the Presence of Stress Redistribution Effects.

In the absence of stress-redistribution effects the method described in the previous section provides a means of determining deformations. However the deformation of a structure composed of Maxwell material will have a component due to stress-redistribution effects. The importance of these effects has been studied in a general way by Ponter (5.3, 5.4) and is discussed in detail in Chapter 3, where it was argued that the ratio of the optimal upper bound to the corresponding lower bound, which was denoted by \( B \), provided a useful measure of the severity of
stress-redistribution effects. If this ratio is close to unity then stress-redistribution effects are small. However if the ratio $B$ is significantly greater than unity then stress-redistribution effects must be taken into account. In this section a procedure is suggested which can be used to calculate deformation when stress redistribution effects are significant.

5.2.1. A Computational Procedure.

The basis of the procedure is to predict the behaviour from computed results obtained assuming a time hardening law. These results are weighted by a factor dependent upon the equivalent steady stress for the material. In effect two geometrically similar structures are considered, which are composed of materials having the same elastic properties and the same creep law for constant stress

$$\varepsilon/\varepsilon_o = (\sigma/\sigma_o)^n \{t/t_o\}^{1/3}$$

(5.1)

but when subjected to cyclic histories of stress, one material obeys the time hardening law and the creep law of the general material is unknown. The two material constants $\gamma_g$ and $\zeta_{og}$ for the general material can be determined from a suitable cyclic creep test as described in Chapter 4, and the corresponding values of $\gamma_{th}$ and $\zeta_{oth}$ for the time hardening material can be easily obtained with a knowledge of the stress index $n$ and the form of the stress cycle. The stress history for the cyclic creep test can be obtained from the procedure described in section 5.1.

The structures are subjected to histories of cyclic loading of the same magnitude but the cycle times are $T_{th}$ and $T_g$ for the two structures (Fig. 5.1). The value of $T_g$ will be
known from the design requirements. A suitable value of $T_{th}$ is therefore required for which the stress redistribution effects in the cyclic state will be the same in the two structures.

In the absence of stress-redistribution effects the corresponding creep displacements for the two structures are given by equation (5.13)

$$
\delta_{th} = \frac{(2/3)\varepsilon_0 a^{n-1} \gamma_{th} (P_{\text{max}}/\sigma_0 A)^n (t/t_o)^{1/3} + 2\bar{\varepsilon}_{oth} / 3a}{3a} \tag{5.15}
$$

$$
\delta_g = \frac{(2/3)\varepsilon_0 a^{n-1} \gamma_g (P_{\text{max}}/\sigma_0 A)^n (t/t_o)^{1/3} + 2\bar{\varepsilon}_{og} / 3a}{3a} \tag{5.15}
$$

During the $r+1$th cycle the corresponding increases in creep deformation are given by

$$
\Delta \delta_{th} = \frac{(2/3)\varepsilon_0 a^{n-1} \gamma_{th} (P_{\text{max}}/\sigma_0 A)^n \{(r+1)^{1/3} - r^{1/3}\}}{T_{th}^{1/3}} \tag{5.16}
$$

and

$$
\Delta \delta_g = \frac{(2/3)\varepsilon_0 a^{n-1} \gamma_g (P_{\text{max}}/\sigma_0 A)^n \{(r+1)^{1/3} - r^{1/3}\}}{T_g^{1/3}} \tag{5.16}
$$

For structures subjected to steady loading several workers [5.5, 5.6, 5.7, 5.8] have considered the ratio of the creep deformations at some time $t$ to the corresponding initial elastic deformation to be a useful measure of the stress redistribution process.

In the cyclic state it is postulated that if the ratio of the creep deformation in a given cycle to the maximum excursion in elastic deformation during the cycle is the same, then the stress redistribution effects should be the same. Since the creep deformation with stress redistribution effects is not known, it is assumed that the values without stress redistribution, which are known, can be used to give an approximate ratio.
Since the excursion in elastic deformation will be the same for the two structures the requirement is that

\[ \Delta \delta_{th} = \Delta \delta_g \]

or (from equations (5.16))

\[ \gamma_{th} T_{th}^{1/3} = \gamma_g T_g^{1/3} \quad (5.17) \]

thus \( T_{th} \), the cycle time to be used in the time hardening calculation is given by

\[ T_{th} = T_g \left( \frac{\gamma_g}{\gamma_{th}} \right)^{3n} \quad (5.18) \]

With this relationship satisfied it is postulated that the displacement at the end of each cycle should be the same in both structures. This is not unreasonable since the cyclic state is in a sense analogous to the stationary state for steady loading. It was pointed out in Chapter 3 that at large times the deformation rate in the stationary state was independent of the creep law for time varying stress as the influence of the initial stress redistribution process was effectively "forgotten". For creep due to cyclic loading the deformation rate in the cyclic state is dependent upon the particular creep law for time varying stress, but by normalizing the deformation rate with respect to that which occurs in the absence of stress redistribution effects, the influence of stress redistribution in the cyclic state can be more readily assessed. Such a normalized measure of deformation is given by equation (5.14). At large times the cyclic state deformation rate should be independent of the influence of the stress redistribution process towards a cyclic state and with equation (5.18) satisfied the normalized displacement rates in both structures should be the
same. Furthermore satisfying equation (5.18) means that the expression for $\dot{\delta}$ of equation (5.14) is the same for both structures so that the curve in the $\chi \sim r^{1/3}$ plane can be used as a means of expressing the displacement $\sim$ time of both structures. When stress redistribution effects are non-existent the displacement $\sim$ time curves will have the form shown in Fig. 4.2 the only difference being one of intercept as has already been discussed. In order to give a quantitative measure of stress redistribution a dimensionless parameter is introduced which is expressed in terms of the strains associated with the reference stress. Hence, the dimensionless parameter is

$$\phi = \left( \frac{E\varepsilon_o}{\Delta \varepsilon} \right) \left\{ \frac{\sigma}{\sigma_o} \right\}^n \left\{ \frac{T}{t_o} \right\}^{1/3} = \left( \frac{E\varepsilon_o}{\Delta \varepsilon} \right) \left\{ \frac{\sigma_{\max}}{\sigma_o} \right\}^n \left\{ \frac{T}{t_o} \right\}^{1/3}$$

which is the ratio of the creep strain accumulated during the first cycle over the excursion of the elastic strain during the cycle, $\Delta \varepsilon/E$.

Equating $\phi$ and $\phi$ yields the same relationship between cycle times given by equation (5.18).

5.2.2. Typical Value of $\phi$.

In order to evaluate a typical value of the dimensionless parameter $\phi$ the following assumptions are made. The structure is in a cyclic state of stress throughout the design life of 100,000 hours during which the creep strain reaches a value of 1%. These are typical figures for design purposes. The yield stress of the material is assumed to be 200N/mm$^2$ and
Young's modulus to be 170 kN/mm\(^2\). The maximum variation in stress is assumed to be 2/3 of the yield stress, and to occur during a cycle of 24 hours duration. If the creep law is of the form given by equation (5.1) then the creep strain accumulated during the first cycle is given by

\[
\varepsilon_c = \frac{1}{1610}
\]

The excursion in elastic strain during the cycle is

\[
\Delta e_e = \frac{2}{3} \frac{200}{(170)(10^3)} = \frac{1}{1275}
\]

Thus the dimensionless parameter \( \phi = \frac{\varepsilon_c}{\Delta e_e} = \frac{1275}{1610} < 1 \).

Another estimate of \( \phi \) can be obtained by employing material data given by Odqvist (5.9). For convenience the rolled stainless steel at 700\(^\circ\)C considered in Chapter 4 is again chosen. It is assumed that the stress is applied for half the cycle time. The relevant data is:

\[
n = 3
\]
\[
E = 137 \text{ kN/mm}^2
\]

Stress for 1\% strain in 100,000 hours 14.7 N/mm\(^2\).

If the equivalent constant stress is to be 14.7 N/mm\(^2\) then for a time hardening material the maximum stress will be 14.7 \times 2^{1/3} N/mm\(^2\) (equation 4.7).

The excursion in elastic strain during the cycle is then

\[
\Delta e_e = \frac{(14.7)(2^{1/3})}{137000} = \frac{1}{7400}
\]

Thus \( \phi = \frac{\varepsilon_c}{\Delta e_e} = \frac{7400}{1610} = 4.6 \).
In general materials spend a large proportion of the time in secondary creep, and the values of $\phi$ obtained here are therefore upper bounds. For materials which also exhibit secondary creep $\phi$ can be expected to be $< 1$ and frequently $<< 1$.

5.2.3. Choice of Examples.

The structure considered is that shown in Fig. 3.2 and the form of the loading history is that shown in Fig. 3.4. The ratio $B$ obtained from the optimal upper bound due to Ponter (5.3) is used as a measure of the severity of stress redistribution.

The variation of $B$ with $\lambda$, $\mu$ and $n$ is shown in Figs. 3.11 and 3.14 for values of $\eta$ of 2.5 and 4 respectively. In general for the values of $\lambda$, $\mu$, $\eta$ and $n$ considered the maximum values of $B$ occur when $\lambda = 0$, although it would appear likely that for lower values of $\eta$ the maximum value of $B$ could well occur when $\lambda = 0.5$ as is the case for the beam and tube (Figs. 3.5 and 3.8).

The variation of $B$ with $\eta$ is shown in Fig. 5.2 for $\lambda = 0$, $\mu = 0.5$ and $n = 3, 5, 7$. For values of $\eta$ greater than 2, $B$ is seen to increase rapidly especially for higher values of $n$. If $B$ does not deviate significantly from unity it is not necessary to perform a time hardening calculation as very little error will be incurred if it is assumed that (equation 5.14)

$$\frac{dx}{d(r^{1/3})} = x' = 1$$

However, when $B$ is significantly greater than unity stress redistribution effects are significant and more detailed calculations become necessary.

From section 5.2.2. it is unlikely that the maximum value of the dimensionless parameter $\phi$ will exceed unity, and will usually be very much less. On this basis and from the discussion
presented above and in section 8 of Chapter 3 it would appear that the procedure will be rigourously tested if the behaviour of the structure is studied for the following range of parameters

\[ n = 3, 5 \]
\[ P = 1, \lambda = 0, \mu = 0.5 \]
\[ \eta = 2.5, 4 \]
\[ \phi = 0.5 \]

Two hypothetical creep laws were chosen as model general materials, the strain hardening law equation (4.3), and the work hardening law equation (4.4). If \( \lambda = 0 \) and \( \mu = 0.5 \), the equivalent steady stress \( \sigma^* \) for both these laws is given by

\[ \sigma^* = \gamma \sigma = \frac{\sigma}{2^{m/n}} = \frac{\sigma}{2^{1/3n}} \]

5.2.4. Results of Computing Procedure.

Elastic-creep stress redistribution calculations were performed for the structure shown in Fig. 3.2 for histories of loading of the form shown in Fig. 5.1 using well established methods (5.6, 5.10). A fairly general computer program permitted the parameters \( n, \lambda, \eta \) and \( \phi \) together with a choice of creep law for time varying stress to be read in as data.

Although the algorithms required are simple in principle, considerable time and effort were required to obtain stable solutions for strain hardening and work hardening materials. (An attempt to include hereditary creep laws was abandoned.) The comparative ease with which time hardening solutions can be obtained emphasizes the practical significance of the proposed procedure.
The displacement $v$ time results were plotted in the $v \sim r^{1/3}$ plane and the mean normalized deformation rate $\frac{dx}{d(r^{1/3})} = \chi'$ was found to be asymptotic to a constant value in all cases (Fig. 5.3). The intercept $\chi_o$ and the slope $\chi'$ determine the line $a,b$. Values of $\chi_o$ and $\chi'$ for the one thousandth cycle of loading are given in Table 5.1.

In the absence of stress redistribution effects $\chi'$ is equal to unity (equation 5.14) and the $v \sim r^{1/3}$ plot for the structural deformation is indistinguishable from the corresponding $\psi \sim r^{1/3}$ curve for the material (Fig. 4.2). When stress redistribution effects are present the values of $\chi'$ can be greater or less than unity depending upon the form of the loading cycle and the structural intercept $\chi_o$ will differ from the material intercept $\psi_o$ introduced in Chapter 4.

If the arguments advanced in Section 5.2.1. are correct, then for given values of $n$, $\eta$ and $\phi$ the values of $\chi'$ should ideally be identical. It can be seen from Table 5.1 that the slopes compare well, the maximum difference in the slope $\chi'$ being about 8%, when $n = 5$ and $\eta = 2.5$. This difference is equivalent to a difference in load level of less than 2%.

The intercept $\chi_o$ can be considered to contain two components, the material intercept $\psi_o$ and an additional term due to stress redistribution effects. For $\lambda = 0$, $\mu = 0.5$ the value of $\psi_o$ for time hardening material is 0.617 and for strain hardening and work hardening materials $\psi_o = 0$.

In order to evaluate the validity of the procedure more readily, comparison is made between the displacement $\sim \{t/t_o\}^{1/3}$ curves obtained for the two model general materials, and the corrected time hardening calculations used to predict the deformation of the structure composed of a general material.
Recall that the slope of the asymptote $a$, $b$ in the $\chi \sim r^{1/3}$ plane was denoted by $\chi'$ and given by

$$\chi' = \frac{dy}{d(r^{1/3})}$$

The corresponding slope $\dot{\delta}_c$ in the $\delta_c \sim (t/t_o)^{1/3}$ plane is given by

$$\dot{\delta}_c = \frac{d\delta_c}{d((t/t_o)^{1/3})}$$

$\delta_c$ is obtained by applying the transformation (c.f. equation 5.14)

$$\dot{\delta}_c = \chi' (2/3) \lambda \varepsilon_0 a^{n-1} \gamma_g \frac{\epsilon_{\max}}{\sigma_o A} \lambda^n$$

In order to predict the intercept for the structure composed of a general material, $\chi_{og}$, from the corresponding value $\chi_{oth}$ obtained from a time hardening calculation, the difference in the material intercepts $\psi_{og}$ and $\psi_{oth}$ should be taken into account. Thus an approximate value of the intercept for the structure composed of general material is given by

$$\chi_{og} = \chi_{oth} - \psi_{oth} + \psi_{og}$$

The approximate intercept $\delta_{og}$ is then obtained from the transformation (equation 5.14)

$$\delta_{og} = \chi_{og} (2/3) \lambda \varepsilon_0 a^{n-1} \gamma_g \frac{\epsilon_{\max}}{\sigma_o A} \left(\frac{T_g}{t_o}\right)^{1/3}$$

For convenience the following values of the relevant parameters were chosen

$$A = 1$$

$$\varepsilon_0/\sigma_o = 24$$

$$\varepsilon/E = 1$$

With these values and $n = 4$ and $n = 3$ the structure shown in Fig. 3.2 is the same as that considered by Leckie and Martin (5.11).
The $\delta_c \sim (t/t_o)^{1/3}$ curves for the strain hardening calculations together with the actual asymptotes and those predicted by the corresponding time hardening calculations are shown in Figs. 5.4 to 5.7. These results together with the corresponding work hardening results are also given in Table 5.2.

The suggested procedure has been based on arguments related to displacement, but if comparison is made so that displacements match then it is reasonable to expect that stress histories should also coincide. In Table 5.1 the force $P_1/P$ occurring at the end of each cycle when the structure has attained a cyclic state, can be seen to compare extremely favourably.

The energy dissipation bound derived by Ponter (5.3) for structures composed of time hardening materials is independent of the cycle time. The ratio of the optimal upper bound to the corresponding lower bound which was introduced in Chapter 3 is given in Table 5.1. together with the corresponding value, D, obtained from the actual structural calculation. The normalized deformation rate $X$ associated with the optimal upper bound and the corresponding value of $\chi'$ from the structural calculation are also given. It can be seen that the corresponding values of $B$ and $D$ and $X$ and $\chi'$ are extremely close which suggests that stress redistribution effects are sensibly independent of $\phi$ for values of $\phi$ likely to occur in practice. Calculations performed over a range of values of $\phi$ confirm that this is so. Curves of $D \sim \phi$ for $n = 3$, $\eta = 4$ and $\lambda = 0$, $\mu = 0.5$ are shown in Fig. 5.8. Similar curves were obtained for the other values of $n$ and $\eta$ considered but were very similar and are not therefore shown. Consequently, provided a reasonable value of $\phi$ has been chosen the predicted deformation rate from a single calculation is valid over a large range of cycle time. However the intercept $\delta_o$
5.17.

and the stresses predicted will be prone to some error. The insensitivity of the results to changes in $\phi$ is very helpful in practice because when $\phi$ is too large, instabilities arise in the quasi-steady state, and if $\phi$ is very short a prohibitively long computation will be necessary to reach the cyclic state.

An upper bound on the energy dissipation due to stress redistribution to achieve a stationary cyclic state is given by

$$B_o = (n+1) A(0)/P_0^3$$

where $A(0)$ is defined by equation (3.19) and $\delta$ is given by equation (5.14). The corresponding value $D_o$ obtained from the structural calculation is given by the intercept in the $D \sim r^{1/3}$ plane minus $\psi_o$.

It has been argued in Chapter 3 that the values of $B_o$ and $B_o$ obtained for a time hardening material are conservative and comparison of the values of $B$ and $B_o$ with the corresponding values of $D$ and $D_o$ obtained for strain hardening and work hardening materials confirms that this is so. This observation is of use in practice as it permits the safe use of upper bound solutions to a more general class of materials.

The results indicate that an upper bound solution should accurately reflect the actual structural behaviour in most practical situations. The deformation rate associated with the time hardening solution $\dot{\delta}_{th}$ can be simply weighted to give the corresponding deformation rate for a general material

$$\dot{\delta} = \dot{\delta}_{th} \{\gamma_g/\gamma_{th}\}^n$$

(c.f. equation 5.14)
If an upper bound solution is used the expression for the parameter \( a \) becomes

\[
a = \frac{(3\eta B)^{1/(n-1)}}{\eta^{1/n_1} \eta^{n/(n-1)}}
\]

(c.f. equation (5.9))

5.2.5. Application to Secondary Creep.

Many structures spend the majority of their lives in a regime analogous to secondary creep for steady loading, and for the majority of the design life the value of \( m = 1 \). In such circumstances the primary part of the creep curve can either be considered separately or ignored if the primary creep strain is sufficiently small. If it is necessary to include a primary creep calculation the two parts of the calculation can be joined by a discontinuity in the value of \( m \), but this need not involve a discontinuity in the slope of the deformation curve since the secondary part is chosen tangential to the primary curve.
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Table 5.1.
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Fig. 5.1

Loading cycles
Fig. 5.2
Strain hardening

$\phi = 0.5 \quad n = 3 \quad \gamma = 2.5 \quad P = 1 \quad \lambda = 0$

Fig. 5.4
Strain hardening

$\Phi = 0.5 \quad n = 5 \quad \eta = 2.5 \quad p = 1 \quad \lambda = 0$
Strain hardening

\( \Phi = 0.5 \quad n = 3 \quad \eta = 4 \quad P = 1 \quad \lambda = 0 \)

Fig. 5.6
Strain hardening
\[ \Phi = 0.5 \quad n = 5 \quad \eta = 4 \quad P = 1 \quad \lambda = 0 \]

Asymptote predicted from corrected time hardening calculation
Fig. 5.8
6.1 Design Implications.

A design procedure based on deformation now emerges which permits the designer to make rapid decisions based on comparatively simple calculations, thereby reducing the likelihood of errors which occur when complex computer calculations are employed, and the engineer's perspective is obscured by the complexities of the computational process.

A rough design can therefore be arrived at more readily and more complex calculations can be performed subsequently if required. The theorems discussed in previous Chapters are based on idealized material models which only provide a limited description of material behaviour, but indicate the most important features of structural behaviour and the material tests which are likely to be most useful in providing an estimate of structural deformation.

6.1.1 Steady Loading.

The concept of a plastic limit load has been extended to time hardening creep, and the tests on portal frames indicate that the theorem is valid for a wider class of materials. From the designer's standpoint a range of loading below n/n+1 of the plastic limit load is not restrictive as other considerations will usually result in a load level well below this. An elastic-creep material model is therefore sufficient. The importance of stress redistribution effects can be assessed by obtaining bounds on the
energy dissipation. If the stationary state stress distribution is not known and is not readily computed, an upper bound can be obtained from a guessed stationary state, such as the plastic limit state which is frequently computed as part of the design process. In many engineering situations the bounds do not differ significantly and therefore a lower bound (reference stress) analysis will suffice. If an upper bound on the dissipation rate has been obtained this may still provide a basis for design with little economic penalty.

When the bounds differ significantly a time hardening calculation will provide a conservative estimate of structural deformation, although the stress redistribution time is likely to be significantly less than the true value.

6.1.2. Cyclic Loading.

The concept of plastic shakedown has been extended to time hardening creep and tests on portal frames suggest that the theorem is valid for a wider class of materials. Thus providing variable loads are kept below $n/n+1$ of the plastic shakedown load an elastic-creep model is adequate as the influence of local plastic strains on the overall structural deformation can be neglected. For most design situations this requirement is unlikely to be restrictive.

Optimal bounds on the energy dissipation obtained for a rectangular section beam subjected to cyclic moment and a thick tube subjected to cyclic histories of internal pressure indicate that ratchetting effects can be expected to be small in most engineering situations involving isothermal creep. (Providing the loads remain below $n/n+1$ of the plastic shakedown load.) It has been argued that the work bounds obtained for time hardening material are in a particular sense conservative. The deformation rates associated with the work bounds can be very simply weighted to give the corresponding deformation rates for a more general class of materials.
6.3

The information required to calculate the weighting factor can be obtained from a single cyclic creep test at prescribed stress levels.

In most practical situations the cycle time is very short in comparison to the total life of the structure and the upper bound solution is likely to correspond to actual structural behaviour in the cyclic state. For many structures the determination of the optimal bounds may require considerable time and effort. However in such circumstances a plastic shakedown solution can be employed to give a non-optimal bound, which may still provide a satisfactory basis for design. In many cases the economic penalty incurred in basing the design upon an upper bound solution will be small, especially as the more extreme cases are usually associated with higher values of the stress index n, when a small reduction in stress results in an n-fold reduction in deformation rate.

In circumstances where the bounds differ considerably or when further information (such as actual stress histories) is required a weighted time hardening calculation is necessary.

A hierarchy of methods of increasing complexity are therefore available to the designer, and it is probable that only the most extreme designs with large stress concentrations will justify the use of complete elastic-creep calculations. The energy theorems derived for time hardening materials are in a particular sense conservative and an empirical procedure permits their application to structures composed of a wider class of materials with related constitutive relationships.

6.2. **Proposals for Future Work.**

The extension of the present work to non-proportional loading is not difficult in principle, at least from the structural standpoint, but time dependent material behaviour under non-proportional loading is not
well understood. Recent work by Blass and Findley (6.1) indicates that a relatively complex material model may be necessary for certain histories of loading. However it is thought likely that providing the stresses are kept below \( n/n+1 \) of that stress at which the Bauschinger effect first becomes discernable, an isotropically hardening material model should be adequate. This conjecture requires a considerable programme of material testing if it is to be verified.

Recent work by Ponte and Leckie (6.2) on ratchetting effects due to cyclic temperature shows that extension of the present work to include cyclic histories of temperature would also be valuable.
REFERENCES


SUMMARY

The development over the last decade of the reference stress method for estimating the deformation of structures composed of time dependent Maxwell material is reviewed, together with the implications of recently derived energy theorems based on idealized material models.

Experiments are described which confirm predictions implicit in two energy theorems which extend the concepts of a plastic limit load and a plastic shakedown state to situations involving time hardening creep.

The influence of constitutive relationships on stress redistribution effects which in turn affect the deformation of structures subjected to both constant and cyclic histories of loading are considered, and it is argued that the two energy theorems derived for time hardening materials provide conservative bounds which permit the designer to estimate deformation of structures composed of a wider class of materials with related constitutive relationships.

An empirical method is proposed for estimating structural creep deformation due to cyclic loading. The method applies to structures composed of materials whose creep law for constant uniaxial stress is known, but knowledge of the form of the creep law for time varying stress is not required, as use is made of data obtained from a single cyclic creep test and results are obtained from a weighted time hardening calculation.
In order to check the proposed procedure calculations were performed for a two-bar structure in which stress redistribution effects were particularly severe. At worst the errors in the predicted deformation rate corresponded to a 2% error in the applied load. The results also suggest that in most practical situations the actual solution is likely to correspond to an optimal upper bound provided by one of the energy theorems. The method also permits this optimal bound to be applied to structures composed of a wider class of materials with related constitutive relationships.