PULSATIONS FROM ROTARY POSITIVE DISPLACEMENT GAS METERS

Thesis Presented to the Faculty of Science of the University of Leicester for the Degree of Doctor of Philosophy.

Josué Marc Jeannon, M.Sc. (Lond.)

1975
CONTENTS

LIST OF FIGURES AND PLATES

NOTATION

ACKNOWLEDGMENTS

ABSTRACT

1. INTRODUCTION
   1.1 General
   1.2 Literature Survey

2. IMPELLER GEOMETRY
   2.1 Two-lobed Involute Profile
   2.2 Meter Geometry in Terms of Characteristic Parameters

3. ACOUSTIC MODEL OF UNDUCTED ROTARY POSITIVE DISPLACEMENT GAS METERS
   3.1 The Acoustic Near Field
   3.2 Computation of Near Field of Unducted Source
   3.3 Acoustic Impedance

4. ACOUSTIC FIELD OF DUCTED ROTARY POSITIVE DISPLACEMENT GAS METERS
   4.1 Propagation of Sound in Ducts
   4.2 Open-Ended Uniform Duct
   4.3 Uniform Duct with Side-Branch
   4.4 Complex Duct Configuration

5. EFFECTS OF PULSATIONS ON ROTARY POSITIVE DISPLACEMENT GAS METERS
   5.1 Supercharging Effects
   5.2 Effects on Slip
6. SUPPRESSION OF PULSATIONS FROM ROTARY POSITIVE DISPLACEMENT GAS METERS

6.1 Concentric Helmholtz Resonators
6.2 Design Considerations.

7. EXPERIMENTAL VERIFICATION
7.1 The Meter Experimental Model
7.2 Acoustic Pressure Measuring Instruments
7.3 Measurement of Acoustic Pressure of Unducted Meter
7.4 Measurement of Acoustic Pressure of Ducted Meter
7.5 Experimental Resonator

8. DISCUSSION OF RESULTS
8.1 Unducted Meter
8.2 Ducted Meter
8.3 Resonator

9. CONCLUSIONS

10. REFERENCES

11. APPENDICES
11.1 Calculation of Cross-section, $A_o(\theta)$ of outlet boundary between impellers
11.2 Fourier Coefficients of Source Volume Velocity, $U'(t)$
11.3 Series Combination of Ducted Impedances
11.4 Frequency Noise Spectrum of Unducted Meter
11.5 Frequency Noise Spectrum of Meter with Simple Uniform Duct
11.6 Frequency Noise Spectrum of Meter with Uniform Duct and Side-Branch
11.7 Frequency Noise Spectrum of Meter with Complex Duct
11.8 Frequency Noise Spectrum of Meter with Uniform Duct with Side-Branch and Concentric Helmholtz Resonator.
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Cross-section of Rotary Positive Displacement Meter</td>
</tr>
<tr>
<td>2</td>
<td>Error Curve: (a) for steady flows (b) for pulsating flows</td>
</tr>
<tr>
<td>3</td>
<td>Impeller Profile</td>
</tr>
<tr>
<td>4</td>
<td>The Involute Area</td>
</tr>
<tr>
<td>5</td>
<td>General Impeller Angular Position</td>
</tr>
<tr>
<td>6</td>
<td>Angular Co-ordinates of Points of Contact</td>
</tr>
<tr>
<td>7</td>
<td>Unducted Source</td>
</tr>
<tr>
<td>8</td>
<td>Variation of Acoustic Pressure of Unducted Meter with shaft Speed $n = 1$</td>
</tr>
<tr>
<td>9</td>
<td>Variation of Acoustic Pressure of Unducted Meter with Shaft Speed $n = 2$</td>
</tr>
<tr>
<td>10</td>
<td>Variation of Acoustic Pressure of Unducted Meter with Shaft Speed $n = 3$</td>
</tr>
<tr>
<td>11</td>
<td>Variation of Acoustic Pressure of Unducted Meter with Shaft Speed $n = 4$</td>
</tr>
<tr>
<td>12</td>
<td>Variation of Acoustic Pressure of Unducted Meter with Shaft Speed $n = 5$</td>
</tr>
<tr>
<td>13</td>
<td>Acoustic Impedance</td>
</tr>
<tr>
<td>14</td>
<td>Ducted Source</td>
</tr>
<tr>
<td>15</td>
<td>Variation of Acoustic Pressure of Ducted Meter with Shaft Speed $n = 1$ (Uniform Open-Ended Duct)</td>
</tr>
<tr>
<td>16</td>
<td>Variation of Acoustic Pressure of Ducted Meter with Shaft Speed $n = 2$ (Uniform Open-Ended Duct)</td>
</tr>
<tr>
<td>17</td>
<td>Variation of Acoustic Pressure of Ducted Meter with Shaft Speed $n = 3$ (Uniform Open-Ended Duct)</td>
</tr>
<tr>
<td>18</td>
<td>Variation of Acoustic Pressure of Ducted Meter with Shaft Speed $n = 4$ (Uniform Open-Ended Duct)</td>
</tr>
</tbody>
</table>
Figure 19  Variation of Acoustic Pressure of Ducted Meter with Shaft Speed \( n = 5 \) (Uniform Open-Ended Duct)

Figure 20  Uniform Duct with Side-Branch

Figure 21  Variation of Acoustic Pressure of Ducted Meter with Shaft Speed \( n = 1 \) (Uniform Open-Ended Duct with Side-Branch)

Figure 22  Variation of Acoustic Pressure of Ducted Meter with Shaft Speed \( n = 2 \) (Uniform Open-Ended Duct with Side-Branch)

Figure 23  Combination of Duct Impedances
(a) Parallel Combination
(b) Series Combination

Figure 24  Complex Duct

Figure 25  Algorithm for Calculating Acoustic Pressure of Ducted Meters

Figure 26  Variation of Acoustic Pressure of Ducted Meter with Shaft Speen \( n = 1 \) (Complex Duct)

Figure 27  Variation of Acoustic Pressure of Ducted Meter with Shaft Speed \( n = 2 \) (Complex Duct)

Figure 28  Variation of Acoustic Pressure of Ducted Meter with Shaft Speed \( n = 3 \) (Complex Duct)

Figure 29  Variation of Acoustic Pressure of Ducted Meter with Shaft Speed \( n = 4 \) (Complex Duct)

Figure 30  Variation of Acoustic Pressure of Ducted Meter with Shaft Speed \( n = 5 \) (Complex Duct)

Figure 31  Calculated Acoustic Excess Error Against Shaft Speed (Complex Duct)

Figure 32  Acoustic Excess Error Against Acoustic Pressure

Figure 33  Concentric Helmholtz Resonator

Figure 34  Variation of Transmission Loss with Attenuation Parameter

Figure 35  Performance of Experimental Helmholtz Resonator

Figure 36  Experimental Meter Model
Figure 37  Acoustic Pressure Measuring Instruments

(a) Simplified Circuit of Condenser Microphone and Preamplifier

(b) Equivalent Circuit of Condenser Microphone and Preamplifier

Figure 38  Experimental Set-up for Measurement of Acoustic Pressure of Unducted Meter

Figure 39  Experimental Set-up for Measurement of Acoustic Pressure of Ducted Meter

Plate 1  General Experimental Set-up
Plate 2  Meter Experimental Model and External Drive
Plate 3  Acoustic Measuring Instruments
Plate 4  Acoustic Measuring Instruments
Plate 5  Concentric Helmholtz Resonator
NOTATION

a - Base circle radius

a_o - Speed of sound

A,B,C,D - Geometric Meter Constants functions

A_m - Cross-section of measuring chamber

A_o - Cross-section of outlet boundary between impellers and
tip circles

A_i - Cross-section of inlet boundary between impellers and
tip circles

b - Involute outer radius

c - Impeller's centres distance

c_o - Conductivity

C_1, C_2 - Constants functions of gas and meter parameters

d - Root circle radius

e - Meter error

f - Frequency (Hz)

F - Force

g - Cylinder radius

h - \frac{1}{2}(Op - QR) - see Figure 5

k - Wave number

l - Pipe length

n - Harmonic order

N - Shaft speed (rev/s)

P_o - Line pressure

p(t) - Instantaneous acoustic pressure

\tilde{p} - Root-mean square acoustic pressure

\Delta P - Pressure loss

Q - Velocity flow rate

r - radial distance
- Waist radius displacement
- Equivalent Source radius
- Area
- Transmission loss
- Time
- Instantaneous source volume velocity per unit area
- Source volume velocity
- Volume
- Impeller width
- Axial distance
- Constants
- Impedance
- Cylindrical polar co-ordinates
- Distance parameter
- Involute angular parameters
- Impeller rotation
- Velocity potential
- Pressure angle
- Density
- Angular velocity
- Cylindrical polar co-ordinates
ACKNOWLEDGMENTS

The financial and moral support of the Gas Standards Branch of the Department of Energy is gratefully acknowledged. My grateful thanks go to Dr. David J. Cockrell, my supervisor, for his unfailing patience and counsel during this work. My thanks go to my colleagues at Gas Standards Branch and to the numerous people who, in one way or another, helped to make this thesis possible. My thanks are also extended to the Head of the Engineering Department of the University of Leicester.
ABSTRACT

An acoustic model is developed to predict the acoustic pressure waves generated by rotary positive displacement gas meters. The meter is approximated, in the long wavelength limit, to a small, rigid and circular piston, pulsating with harmonic time dependence inside an infinite baffle. The acoustic source volume velocity is first derived from the geometry of the measuring part of the meter. Rayleigh's formulation is then applied to derive the acoustic pressure fields of the meter for both the unducted and the ducted situations. The effects of the self-induced pressure waves on the registration of the meter are then considered. The performance of the meter is dependent on the configuration of the pipe network to which the meter is attached. The meter calibration curve can be seriously distorted by the self-induced acoustic pressure waves. The effectiveness of concentric Helmholtz type resonators to suppress undesired sound pressure waves in meter systems is discussed. Experimental verification shows that the acoustic model developed is adequate to predict the "a.c" response of rotary positive displacement gas meters for most of the common pipe elements. Linear plane wave theory is used throughout and the effects of flow and friction are assumed to be negligible.
1. INTRODUCTION

1.1 General

Rotary positive displacement machines were originally used in mines as ventilators [1, Carbutt, 1877]. Later versions with higher speeds and pressure differences were used as superchargers and scavenging blowers in aircraft, automotive engines and aircraft cabins. Present applications include pneumatic conveying, vapour compressing, vacuum creating and gas metering when the machine motors. The idea of using a rotary positive displacement machine as a gas meter first occurred to the American gas engineer, J.A. Brown in 1920, who wanted a meter to measure the volume of gas used in an experiment. He converted a Connersville exhauster to a gas meter by allowing the gas to drive the impellers and used a revolution counter to measure the volume of gas passed. He found the meter reliable and drew the idea to the attention of the Connersville Blower Company, Indiana, USA, where the meter was first developed. Since then, rotary positive displacement gas meters have played a major role in the metering of large volumes of fuel gas in the gas industry. They are generally considered reliable and accurate over a wide range of flows and are used as reference meters by government calibration authorities. The cross-section of the measuring part of a typical meter is shown in Figure 1. It essentially consists of two identical intermeshing impellers which can rotate freely inside a rigid cylindrical casing. A constant phase of $\pi/2$ is maintained between the impellers by means of a gearing system. Very fine clearances between the impellers and between the impellers and the casing form a near seal between the inlet and outlet ports. The quality of the meter is predominantly governed by its ability to maintain these clearances to a practicable minimum.
FIG. 1 Cross-section of Rotary Positive Displacement Meter
This is achieved by shaping the impellers carefully. During each complete revolution of the impeller shaft, finite volumes of gas from the inlet port are trapped between the impeller lobe and the casing and are displaced to the outlet port. Thus, the total volume of gas passed can be directly inferred from the number of shaft revolutions, except for the small volume of gas which flows through the clearances. This small clearance flow is commonly known as "slip".

The calibration characteristics of rotary positive displacement gas meters are predominantly determined by the clearance flows which are a mixture of Couette and orifice flows. Most research so far has concentrated on the effects of density and viscosity on these clearance flows. Alexander [2, 1971] has reviewed the published literature on the effects of pressures up to 1000 psi, on the calibration characteristics of rotary positive displacement gas meters. By using boundary layer techniques, Eujen [3, 1970] was able to obtain expressions for the mechanical and aerodynamical pressure losses and hence the total pressure loss, \( \Delta P \), across the meter. By assuming that the "slip" due to \( \Delta P \) is the only source of meter error, Eujen obtained an expression for the meter calibration curve of the form

\[
e = c_1 - \frac{c_2}{Q} \times 100,
\]

where \( e \) is the meter error, \( c_1 \) and \( c_2 \) are constant functions of meter and gas parameters and \( Q \) is the volume flow rate. Eujen's equation agrees well with experimentally observed error curves, except at a series of harmonically related discrete points where the experimental curves deviate from the rectangular hyperbolic law. (Figure 2). These deviations have been explained qualitatively as being due to resonant vibrations of the gas columns inside the pipe connected to the meter,
Fig. 2 Error Curve

a. For steady flows  
b. For pulsating flows
caused by the pulsating flow from the meter [4, Eujen 1963].

Equation (1) is based on steady flows and can be said to describe the "d.c." response of the meter. It is shown in this work that the "peaks" exhibited by the experimental calibration curves are manifestations of the "a.c." response of the meter due to the "a.c." component of its flows. These fluctuating velocity components are of a highly distorted sine wave shape and introduce a modulation of about 10-15 per cent. [5, Bean, et al 1946; 6, Stein et al 1974].

Priede [7, 1966] has shown experimentally that the rapid formation of the V-shaped opening between the impellers is the major source of noise in these machines. So far, a theoretical investigation of that proposition has not yet been offered in the literature.

This major source of acoustic pressure waves in rotary positive displacement machines at low frequencies is investigated both theoretically and experimentally in the present work. A theoretical model is developed to predict the amplitude of the self-generated acoustic pressure waves. The meter is approximated to an equivalent flat and circular rigid piston source pulsating with harmonic time dependence inside an infinite baffle. The volume velocity of the source is first derived from the geometry of the measuring part of the meter. The axial acoustic pressure of the unducted source in the near-field is then derived by applying Rayleigh's formulation for the velocity potential of a piston source. The model is then verified in the more realistic situation when the source is attached to a pipe network characterised by input and output impedances. The calculation of the acoustic pressure field for the ducted case is based on the assumption that the meter acts as a constant acoustic velocity source. This means that for the same impeller angular speed,
the source acoustic velocity for both the unducted and the ducted situations is assumed to be the same. This is a physically sound assumption as [8, Cremer, 1971] the acoustic velocity in the vicinity of the impellers is solely determined by the impeller rotation. The rotational inertia reactions of the impellers are large compared with the fluctuating loading of the duct and so cannot appreciably affect the rotation of the impellers. Piston in baffle calculations are presented for comparison with measured results when the meter "output" port is actually placed in an infinite baffle. In turn, the accuracy of these results is then compared with that for situations in which the meter output port is terminated in a duct network. This comparison provides some check on the validity of the basic assumption that the machine is acting like a constant acoustic velocity piston source.

A supercharging effect of the acoustic pressure waves on the trapped volume on induction, is offered as the main explanation for the "peaks" that are observed on the observed calibration curve. This effect is maximum when the meter is operated at a frequency near the resonant frequency of the piping system.

The prediction of the "a.c." response of rotary positive displacement gas meters with their actual associated pipework is thus very desirable because the assessment procedure for these meters takes place on test rigs of different pipe configurations from that in which the meter is normally installed. The supercharging effect can be significant enough to cause serious distortion of the meter calibration curve. On grounds of minimising revenue losses in metering industrial gases, a model, so far lacking in the literature, which will predict the "actual" performance of the meter "in situ", is very desirable.
The application of Helmholtz type acoustic filters to suppress undesirable pulsations is also discussed. It is shown that this type of resonator can be successfully used to maintain acoustic pressures below a desired level.

1.2 Literature Survey

A search of the literature reveals that the majority of published research on rotary positive displacement machines are concerned with their application as blowers and superchargers [References 9-19]. This is probably because these two applications are the machines' most common applications in industry. Bean's paper [5] is the first in the literature to deal with these machines in the context of gas metering. This paper is concerned with the calibration of large capacity rotary gas meters and reports the so-called "Transfer Method" for the first time. References [20-23] mention the problem of pulsations from rotary positive displacement gas meters but contain no analysis of the problem. Stein et al [6] has used the water hammer theory to calculate transient phenomena in rotary meter systems, where a modulation of 10 per cent is assumed to be introduced by the meter. The approach adopted in this work to the solution of problems due to pulsations from rotary positive displacement gas meters is believed to be original. The essence of this thesis is published in references [24,25]
2. IMPELLER GEOMETRY

2.1 Two-lobed Involute Profile

The shape of the impeller is dictated by the requirement that line contact must always be made, without interference with the motion of the mating impeller. This is achieved by imposing certain conditions on the profile of the impeller. The required condition is that the common normal at the point of contact must always pass through a fixed point on the straight line joining the centres of rotation [26, Bevan 1965]. The fixed point divides the straight line into segments which are inversely proportional to the angular velocities of the rotating surfaces. In the case of rotary positive displacement gas meters, the impellers rotate with equal and opposite angular velocities and so the common normal at the point of contact must always pass through the middle point of the line of centres. The required conditions are satisfied if the sections of the impellers over which line contact is made are given the shape of an involute defined as the locus of a point on a straight line which rolls without slipping on the circumference of a circle called the base circle. The motion of the two surfaces at the point of contact then is one of pure rolling. Cycloidal profiles have also been used, but involute rotors are preferred because they give a higher throughput for a given impeller diameter [18, McDouguld, 1971]. Thus, section FD in Figures 1 and 3 are part of an involute developed from a base circle of radius, \( a \).

The tip radius of the involute, \( DO \), is denoted by, \( b \). The root region, \( FG \), which joins the extreme points of the involute at the base circle is designed to avoid interference with the mating impeller tip, \( AD \). In practice, an arbitrary design characterised by a matching root radius, \( d \), displaced by an amount, \( r_w \), from the pitch circle is adopted.

Figure 1. For interference free rotation of the impellers, the point
Fig. 3 Impeller Profile
of contact must always lie on the base circles' common tangent, which make an angle, $\psi$, called the pressure angle, with the common tangent to the pitch circles at the pitch point.

2.2 Meter Geometry in Terms of Characteristic Parameters.

The geometry of the measuring section of the meter can be fully described in terms of the non-dimensional quantities, $\frac{b}{a}$, $\frac{d}{a}$ and $\psi$, called the characteristic parameters.

(a) The involute profile angle, $\text{FOD}$.  

The involute profile angle, $\text{FOD}$, Figure 3, can be calculated from the equation of an involute,

$$r^2 = a^2 (1 + \theta^2)$$

where $r$ is the radius vector of a point $P$ on the involute and $\theta$ is the angle between the radius $OH$ and the $x$-axis in Figure 3. The angle $\text{FOE}'$ is given by

$$\text{FOE}' = \int_a^b \frac{d\theta}{dr} dr = \frac{1}{a^2} \int_a^b \frac{r}{\sqrt{\left(\frac{r}{a}\right)^2 - 1}} dr$$

$$= \sqrt{\left(\frac{b}{a}\right)^2 - 1}$$

(b) Inclination of involute tip and root radii.

The angle, $\nu$, of the involute root radius, $FO$, with the impeller diameter can be calculated from: $(f; 1)$

$$MN = MQ + QN$$
or 2a tan $\psi = a[\psi - \left(\frac{\pi}{2} - \nu\right)] + a(\psi + \nu)$

$$\nu = \tan \psi - \psi + \frac{\pi}{4}.$$  

The inclination, $\beta$, of the involute tip radius is given by

$$\beta = \nu - F \theta D$$

$$\beta = \tan \psi - \psi + \frac{\pi}{4} \sqrt{\left(\frac{D}{a}\right)^2 - 1 + \cos^{-1}\left(\frac{a}{D}\right)}.$$  

(c) Centres distance and root circle radius.

From $\triangle AOQP$ in Figure 1, the distance, $c$, between the centres is given by

$$\frac{c}{a} = 2 \sec \psi.$$  

By applying the sine rule twice to $\triangle AFOX$, the distance $OX$ is given by

$$OX = d \frac{\cos[\sin^{-1}\left(\frac{a}{d} \cos \nu\right) - \nu]}{\cos \nu}.$$  

The root circle radius, $e$, then is given by

$$e = \frac{d}{a} \left\{ \frac{\cos[\sin^{-1}\left(\frac{a}{d} \cos \nu\right) - \nu]}{\cos \nu} - 1 \right\}.$$  

(d) Cylinder radius and waist radius displacement.

The waist radius displacement, $r_w = \frac{c}{2} - OX$, is given by

$$r_w = \sec \psi - \frac{d}{a} \frac{\cos[\sin^{-1}\left(\frac{a}{d} \cos \nu\right) - \nu]}{\cos \nu}.$$  

The cylinder radius, $OA$, denoted by, $g$, is given by

$$g = \sec \psi + \frac{d + r_w}{a}.$$
(e) Cross-section of measuring space and cyclic volume.

The cross-section \( A_m \) of the measuring space - the shaded area in Figure 1 - is given by

\[
A_m = \text{Semi-circle of radius} \ g - 2 \ (\text{FOX} + \text{DOF} + \text{DOE})
\]

The area FOX = \( \frac{1}{2} a \ OX \cos \nu \)

\[
= \frac{1}{2} a \ d \cos \left[ \sin^{-1} \left( \frac{a}{d} \cos \nu \right) - \nu \right]
\]

The area DOE = \( \frac{1}{2} b \ OE \sin \beta \)

\[
= \frac{1}{2} b \ (a \sec \psi + r_w) \sin \beta
\]

The area of the involute section, DOF, is given by

\[
\text{DOF} = \frac{1}{2} a^2 \int_{0}^{	heta_1} \frac{r^2}{a^2} \ d\theta
\]

where \( \theta_1 \) - Figure 4 - is given by

\[
\theta_1 = \theta - \tan^{-1} \frac{a}{b}
\]

and

\[
\theta_2 = \sqrt{(\frac{a}{b})^2 - 1}
\]

By substituting from equations (2), (16) and (17), equation (15) gives

\[
\text{DOF} = \frac{1}{2} a^2 \int_{0}^{\theta_2} \theta^2 \ d\theta
\]
FIG. 4 The involute area
The cross-section, \( A_m \), is then given by

\[
\frac{A_m}{a^2} = \frac{\pi}{2} \left( \sec \psi + \frac{d + r_w}{a} \right)^2 - \frac{d}{a} \cos \left[ \sin^{-1} \left( \frac{a}{d} \cos \varphi \right) - \varphi \right]
\]

\[
- \frac{1}{3} \left( \left( \frac{b}{a} \right)^2 - 1 \right) + \frac{b}{a} \left( \sec \psi + \frac{r_w}{a} \right) \sin \beta .
\]

If, \( W \), is the width of the impeller, the meter cycle volume, \( v \),
is given by

\[
v = 4 A_m W,
\]
since a trapped volume is emptied four times during each impeller revolution.

(f) Variation of cross-section between impellers.

The major source of acoustic pressure waves from rotary positive displacement machines is the fluctuating manner in which each trapped volume is displaced. Figure 5 shows the impellers at a general angular position, \( \theta_r \). A trapped volume is displaced only when it is in the boundary included between the impellers' surface and the tip circles - the shaded area in Figure 5. This boundary corresponds to Priede's V-shaped openings between the impellers. The dependence of the cross-section, \( A_o(\theta_r) \), of the outlet boundary, on the angle of rotation, \( \theta_r \), is shown in Appendix (1) to be of the form:

\[
\frac{A_o(\theta_r)}{a^2} = A + B \theta_r + C \theta_r^2 + D \theta_r^3 \ldots
\]
FIG. 5 General impeller angular position
FIG. 6 Angular co-ordinates of point of contact
Where:

\[ A = \frac{1}{2}(\sec \psi + \frac{d + r_w}{a}) \cdot [\pi + \cos^{-1}(1 + \frac{d + r_w}{a} \cos \psi)] + \sec \psi \left( \frac{d + r_w}{a} \right) \]

\[ \times \cos \left[ \cos^{-1}(1 + \frac{d + r_w}{a} \cos \psi) \right] - \left( \frac{d}{a} \right) \cos^{-1}(\frac{a}{d} \cos \psi) \]

\[ - \left( \frac{b}{a} \right) (\sec \psi + \frac{r_w}{a}) \sin \beta - \frac{1}{3} \left[ \left( \frac{b}{a} \right)^2 - 1 \right]^2 \]

\[ - \frac{\pi}{4} \left[ \sec^2 \psi + \frac{\pi^2}{4} \right] \]

\[ B = \frac{\pi^2}{16} - 2 \sec \psi \frac{d + r_w}{a} - \left( \frac{d + r_w}{a} \right) \]

\[ C = -\frac{\pi}{4} \]

\[ D = \frac{1}{3} \]

Equation (21) is an important result in this work and will be used below to calculate the pressure waves generated by rotary positive displacement meters.

The displaced volume, \( v(\theta_r) \), during an impeller rotation of \( \theta_r \), is given by

\[ v(\theta_r) = W[A_o(\theta_r) - A_o(0)] \]

\[ 0 \leq \theta \leq \frac{\pi}{2} \]
The cross-section, $A_{i}(0_{r})$, of the corresponding inlet boundary is given by

\[
\frac{A_{i}(0_{r})}{a_{2}} = A + B \left( \frac{\pi}{2} - 0_{r} \right) + C \left( \frac{\pi}{2} - 0_{r} \right)^{2} + D \left( \frac{\pi}{2} - 0_{r} \right)^{3}
\]
3. ACOUSTIC MODEL OF UNDUCTED ROTARY POSITIVE DISPLACEMENT GAS METERS

3.1 The acoustic rear field

A description of the acoustic pressure field of a rotary positive displacement gas meter is possible in principle, if the distribution of the fluctuating velocity at the source is known accurately. The problem is difficult however, because of the complex nature of the flows in the vicinity of the impellers. The volume displacement takes place across the curved surface, QBR, in Figure 5, which acts as the source surface. The direction of the velocities across that surface cannot be specified accurately. In order to investigate the principal characteristics of the source, certain assumptions and simplifications have to be made. As an approximation, the meter is modelled by a flat, circular and rigid piston, vibrating with harmonic time dependence inside an infinite baffle. The flow is assumed normal to the piston surface. Such an approximation is valid as long as the dimensions of the source are small compared with the sound wavelengths concerned. This condition is always satisfied in the case of rotary positive displacement gas meters as they are always operated in the low frequency regime.

Acoustic energy in a compressible and essentially inviscid fluid distributes itself spatially and temporally as a scalar field which can be fully described by a continuous function of space and time. Such a field function must satisfy the wave equation and the appropriate boundary conditions. A convenient field variable is the velocity potential, $\phi$, defined through

$$ u = -\nabla \phi $$

where $u$ is the acoustic particle velocity. The instantaneous
acoustic pressure, \( p(t) \), is related to the velocity potential, \( \phi \), through the equation of motion of the disturbed medium,

\[
\rho_o \frac{\partial u}{\partial t} = - \nabla p
\]

where \( \rho_o \) is the average density of the medium surrounding the source. Equations (28) and (29) give

\[
p(t) = \rho_o \frac{\partial \phi}{\partial t}
\]

For a velocity potential with harmonic time dependence, \( \exp(i\omega t) \), where \( \omega \) is the source radian frequency, equation (30) reduces to

\[
p(t) = i \rho_o \omega \phi
\]

Hence, once the velocity potential, \( \phi \), is known, the pressure field of an equivalent piston source in an infinite baffle in which all duct effects are neglected can be calculated.

For a circular and rigid piston of area, \( S \), in an infinite baffle, pulsating into the medium with harmonic time dependence, i.e. a source system in which all sources vibrate with the same frequency and amplitude, the velocity potential is given by [27, Rayleigh]

\[
\phi = \frac{i\omega t}{2\pi} \int \frac{e^{-ikr}}{r} dS
\]

where \( r \) is the distance between an elemental source area, \( dS \), and the point of observation, and \( k \) is the wave number given by

\[
k = \frac{\omega}{a_o}
\]

where \( a_o \) is the velocity of sound in the medium. The instantaneous acoustic pressure, \( p(t) \), is then given by
The source volume velocity, $U'(t)$, can be calculated from equation (26):

\[
U'(t) = \frac{3}{\partial t} \quad V(\theta_r)
\]

\[
= \omega \quad a^2 \quad [B + 2C(\omega t) + 3D(\omega t)^2] \quad W
\]

\[0 \leq \omega t \leq \frac{\pi}{2}\]

where the relationship

\[
\theta_r = \omega t
\]

has been used.

Expressed as a Fourier series, (Appendix 2), the source volume velocity takes the form

\[
U'(t) = \omega \quad a^2 \quad \sum_{n=1}^{\infty} \frac{\cos 4n\omega t}{4n^2}
\]

The meter fundamental radian frequency is $4\pi \omega$ because four complete pulses are formed for each complete revolution of the impeller shaft. Also, because of the figure of eight shape of the impellers, the radian frequency, $\omega$, is equal to $8\pi N$, where $N$ is the number of revolutions per second. The reason for representing the source volume velocity in a Fourier series is because it is convenient to express it in a form with explicit harmonic time dependence so that equation (31) can be applied. Also, this form will be used below to calculate the frequency spectrum of the acoustic pressure field. This frequency spectrum
is easy to measure. The frequency spectrum will also indicate the relative importance of the various harmonics present in the pressure field. Such information is always useful for sound attenuating purposes.

In order to calculate the source velocity amplitude, \( U_0 \), the source surface area, \( S_o \), is required. Since the dimensions of the source are small compared with the sound wavelength, the source area can be equated to the surface QBR in Figure 5. The source area, \( S_o \), is then given by

\[
S_o = W g \left[ \sin^{-1} \left( \frac{c}{2g} \right) - \sin^{-1} \left( \frac{h}{2g} \right) \right]
\]

where \( h = \frac{1}{4}(OP - QR) \) in Figure 5.

The source velocity or source volume velocity per unit source area, \( U(t) \), is then given by, in complex number representation,

\[
U(t) = \frac{U'(t)}{S_o} = \sum_{n=1}^{\infty} U_n(o) e^{i\omega nt}
\]

where \( U_n(o) \) is the source velocity amplitude of the \( n^{th} \) harmonic and is given by

\[
U_n(o) = \frac{a^2 \omega}{g[\sin^{-1} \left( \frac{c}{2g} \right) - \sin^{-1} \left( \frac{h}{2g} \right)]} \frac{1}{4n^2}
\]

\[
= \frac{a^2 W \omega}{S_o} \frac{1}{4n^2}
\]

The velocity potential, \( \phi \), of the source is then given by
and the instantaneous acoustic pressure is

\[ p(t) = \frac{i \rho \omega^2 A^2 W}{2\pi S_o} \int_{S} \frac{-ik_n r}{r} dS \frac{e^{i4n\omega t}}{4n^2} \]

where \( k_n = \frac{n \omega N}{a_o} \) and it is implied that the real part of the right hand side of equation (42) is to be taken.

In principle, equation (42) gives the acoustic pressure field of the source at any point. However, its application for calculating the pressure field at general points near the source plane, as derived by Stenzel and Brosze [28], is complicated, except for points along the piston axis. In the case of rotary positive displacement gas meters, since the dimensions of the source are small compared with the sound wavelengths, the transverse variation of the pressure field at axial distances of the order of an equivalent piston radius or so from the source plane is reasonably small [29, Dyer]. Also, such axial distances are small compared with the acoustic wavelengths. Hence, acoustic pressures calculated at such axial points near the source are adequate representatives of the acoustic near pressure field.

Equation (42) can be integrated for axial points by writing

\[ r = (x^2 + y^2)^{\frac{1}{2}} \]

\[ dS = y dy d\psi \]

where \((y, \psi)\) are the cylindrical polar co-ordinates of an elemental area, \(dS\), and \(x\), is the axial distance of the point of
Fig. 7 Unducted source
observation from the source - Figure 7. Hence,

\[ p(t) = \frac{i \rho_o \omega^2 a^2 W}{S_o} \int_{n=1}^{r_o} \frac{e^{-ik_n(x^2 + y^2)^{1/2}}}{(x^2 + y^2)^{1/2}} x y \ dy \ \text{d}y \frac{e^{i4\pi n t}}{4n^2} \]

where \( r_o \) is the source equivalent radius. The approximate integration of equation (44) gives the following expression for the instantaneous acoustic pressure:

\[ p(t) = \frac{i \rho_o \omega^2 a^2 W}{S_o} \sum_{n=1}^{r_o} \frac{e^{i4\pi n t}}{4n^2} x \left\{ \left[ 1 + \frac{r_o}{x} \right]^{1/2} - 1 \right\} - \frac{i \ k_n x}{2} \left( \frac{r_o}{x} \right)^{1/2} - \left( \frac{k_n x}{\epsilon} \right)^{1/2} \left[ \left[ 1 + \frac{r_o}{x} \right]^{1/2} - 1 \right] + \ldots \]

In the limits, \( r_o \) and \( x \) are of the same order of magnitude and \( k_n \to 0 \), equation (45) simplifies to

\[ p(t) = \frac{i \rho_o \omega^2 a^2 W}{S} \sum_{n=1}^{\infty} \frac{e^{i4\pi n t}}{4n^2} \alpha \]

where \( \alpha = (x^2 + r_o^2)^{1/2} - x \), is a distance parameter.

The root mean square acoustic pressure - the pressure which is usually measured - of the nth harmonic, \( \overset{\circ}{p}_n(x) \), is then given by

\[ \overset{\circ}{p}_n(x) = \frac{16}{\sqrt{2}} \pi^2 \frac{W}{S} \rho_o a^2 \left( \frac{n}{n} \right)^2 \alpha \]
3.2 Computation of Near Field of Unducted Source

The model developed in the previous section for the generation of acoustic pressure waves by rotary positive displacement gas meters in the low frequency regime will now be used to calculate the acoustic pressure fields of an experimental meter when the ports of the meter are in an infinite baffle. The acoustic pressures were calculated at a point 0.5 feet from the meter for impeller speeds ranging from 3 to 11 revolutions per second and for \( n = 1 \) to 5.

The geometrical dimensions of the experimental model were:

i. Base circle radius, \( a \) = 1.1796 inches
ii. Involute tip radius, \( b \) = 2.1894 inches
iii. Root radius, \( d \) = 0.8906 inches
iv. Pressure angle, \( \psi \) = 38.14°
v. Involute angular limits: \( \nu \) = 51.85°
   \( \beta \) = 19.65°
vi. Displacement of waist radius, \( r_w \) = 0.0621 inches
vii. Centres distance, \( c \) = 3.0000 inches
viii. Impeller width, \( W \) = 10.0000 inches
ix. Cylinder radius, \( g \) = 2.4380 inches
x. Source equivalent radius, \( r_o \) = 2.1790 inches

The variable, \( a = (x^2 + r_o^2)^{\frac{1}{2}} \quad - x \)

\[
= \left[ 0.5^2 + \left( \frac{2.179}{12} \right)^2 \right]^{\frac{1}{2}} - 0.5
\]

\[= 0.03 \text{ ft.} \]

From equation (47), the root mean square acoustic pressure at \( x = 0.5 \) ft is given by
\[ p_n(x) = 20 \log_{10} \left| \frac{16}{\sqrt{2}} \times \pi^2 \times \frac{10}{12} \times \frac{144}{\pi \times 2.179^2} \times 0.08 \right| \times \left( \frac{1.1796}{12} \right)^2 \times 0.03 \times \frac{14.87}{.0002} \times \left( \frac{N}{n} \right)^2 \] dB

The computed results are shown in Table 1 and are plotted in Figures 8 to 12.

### Table 1

<table>
<thead>
<tr>
<th>Impeller Speed N (rev/s)</th>
<th>( p_n(x) ) (dB)</th>
<th>( \tilde{p}_n(x) ) (dB)</th>
<th>( \tilde{p}_n(x) ) (dB)</th>
<th>( \tilde{p}_n(x) ) (dB)</th>
<th>( \tilde{p}_n(x) ) (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>82.8</td>
<td>70.7</td>
<td>63.7</td>
<td>58.7</td>
<td>54.8</td>
</tr>
<tr>
<td>4</td>
<td>87.8</td>
<td>75.7</td>
<td>68.7</td>
<td>63.7</td>
<td>59.8</td>
</tr>
<tr>
<td>5</td>
<td>91.6</td>
<td>79.6</td>
<td>72.6</td>
<td>67.6</td>
<td>63.7</td>
</tr>
<tr>
<td>6</td>
<td>94.8</td>
<td>82.8</td>
<td>75.7</td>
<td>70.7</td>
<td>66.9</td>
</tr>
<tr>
<td>7</td>
<td>97.5</td>
<td>85.4</td>
<td>78.4</td>
<td>73.4</td>
<td>69.5</td>
</tr>
<tr>
<td>8</td>
<td>99.8</td>
<td>87.8</td>
<td>80.7</td>
<td>75.4</td>
<td>71.1</td>
</tr>
<tr>
<td>9</td>
<td>101.9</td>
<td>89.8</td>
<td>82.8</td>
<td>77.8</td>
<td>73.9</td>
</tr>
<tr>
<td>10</td>
<td>103.7</td>
<td>91.6</td>
<td>84.6</td>
<td>79.6</td>
<td>75.7</td>
</tr>
<tr>
<td>11</td>
<td>105.3</td>
<td>93.3</td>
<td>86.3</td>
<td>81.3</td>
<td>77.4</td>
</tr>
</tbody>
</table>

### 3.3 Acoustic Impedance

The description of the acoustic source would be incomplete without discussing the impedance at the source, defined as the ratio of the force exerted by the source on the fluid in contact with it, to the acoustic velocity of the field. This ratio, a complex...
Fig. 9 Variation of acoustic pressure of unducted meter with shaft-speed $n=2$
Fig. 10 Variation of acoustic pressure of unducted meter with shaft-speed n = 3
Fig. 11 Variation of acoustic pressure of unducted meter with shaft-speed $n = 4$
Fig. 12 Variation of acoustic pressure of unducted meter with shaft-speed n = 5
quantity, determines the rate at which the source radiates sound energy. It can be calculated by using equations (31) and (32) for the unducted case.

The pressure, \( p \), exerted on an elemental fluid area, \( dS' \), adjacent to the piston, by an elemental source area, \( dS \), situated at \( (z, 0) \) (see Figure 13) is given by

\[
\frac{dp}{dz} = \frac{i \rho_o \omega}{2\pi} U_o e^{i(\omega t - kz)} dS
\]

where \( k = \omega/a_o \). The total force, \( F \), exerted on the fluid by the whole piston is given by

\[
F = 2 \iint p dS' = \frac{2i\rho_o \omega U_o}{2\pi} \int_\sigma \int_\psi \int_\sigma \int_\psi dS e^{-ikz dz} e^{i\omega t}
\]

\[
= \frac{2\pi \rho_o \omega U_o e^{i\omega t}}{k} \int_\sigma \int_\psi \left[ 1 - J_0(2k\sigma) + i S_0(2k\sigma) \right] \sigma d\sigma
\]

where

\[
dS' = \sigma d\sigma d\psi \text{ with } 0 \leq \sigma \leq r_o \text{ and } 0 \leq \psi \leq 2\pi,
\]

\[
dS = \sigma dz d\theta \text{ with } 0 \leq \sigma \leq 2\sigma \cos \theta \text{ and } 0 \leq \theta \leq \pi/2,
\]

\[
J_0(x) = \frac{2}{\pi} \int_{-\pi}^{\pi} \cos(x \cos \theta) \, d\theta = \text{zero-order Bessel function},
\]

\[
S_0(x) = \frac{2}{\pi} \int_{-\pi}^{\pi} \sin(x \sin \theta) \, d\theta = \text{zero-order Struve function}.
\]

Upon using the facts that

\[
\int x J_0(x) \, dx = x J_1(x), \quad \int x S_0(x) \, dx = x S_1(x),
\]

where \( J_1(x) \) and \( S_1(x) \) are, respectively, first-order Bessel and Struve functions, equation (50) reduces to

\[
F = \frac{2 \rho_o \omega U_o e^{+i\omega t}}{k^3} \frac{\pi 2k r_o}{4k^3} [k r_o - J_1(2k r_o) + i S_1(2k r_o)]
\]
Fig. 13 Acoustic impedance
The acoustic impedance, \( Z_n(o) \) at the source is then given by
\[
Z_n(o) = S \rho_o a_o \left[ 1 - \frac{J_1(2kr_o)}{kr_o} + i \frac{S_1(2kr_o)}{kr_o} \right]
\]
\[= R_o(2kr_o) + i X_o(2kr_o),\]

where \( R_o(2kr_o) \) and \( X_o(2kr_o) \) are, respectively, the acoustic resistance and reactance at the source.

By using the series
\[
J_1(x) = \frac{x}{2} \left[ 1 - \frac{(x/2)^2}{1.2} + \frac{(x/2)^4}{1.2.2.3} \ldots \right],
\]
\[
S_1(x) = \frac{2x^2}{3\pi} \left[ 1 - \frac{x^2}{3.5} + \frac{x^4}{3.5.5.7} \ldots \right],
\]

it can be shown [30] that in the limit of long wavelength, when \( kr_o \) is small, \( Z_n(o) \) is given approximately by
\[
Z_n(o) = \frac{\rho_o S^2 \omega^2}{2\pi a_o} + \frac{i \rho_o 8r_o S \omega}{3\pi} \}
\[
= .01(Nn)^2 + i .03 Nn
\]

Thus the piston acts as a simple source and radiates energy in proportion to the square of the frequency.
4. ACOUSTIC FIELD OF DUCTED ROTARY POSITIVE DISPLACEMENT GAS METERS

4.1 Propagation of Sound in Ducts

Having established the mechanism by which rotary positive displacement gas meters generate acoustic pressure waves, we now consider the more realistic situation when these meters are connected to a pipe network. When the source radiates inside a duct, the problem of determining its acoustic pressure field becomes more complicated. All the sound energy is confined within the space of duct where interference with reflected waves in both axial and radial directions takes place. The result is an acoustic field which is completely different to that of the unducted sources and which is dependent on the configuration of the duct. The acoustic field of the ducted source is basically an interference pattern which propagates inside the duct.

In the case of the low frequency regime when the sound wavelength is long compared with the circumference of the duct, the radial acoustic waves are not propagated because their "cut-on" frequency is much higher than that of the source. The acoustic field then consists only of plane waves moving forward and backward along the axis of the duct. The ducted source situations then can be characterised by a uniform duct of effective length, \( l \), of area of cross-section \( S_0 \), with a flat, rigid and circular piston of mechanical impedance, \( Z_n(0) \), pulsating with harmonic time dependence, \( \exp(i\omega nt) \), at one end of the duct where \( x = 0 \), and terminated by a mechanical impedance, \( Z_n(l) \), at the other end where \( x = l \), Figure 14. If frictional forces are neglected, the instantaneous acoustic velocity, \( U_n(x) \) and acoustic pressure, \( p_n(x) \) of the nth harmonic at a point \( x \), inside the duct are respectively given by
Fig. 14 Ducted source
where $X$, $Y$ are constants which are determined by boundary conditions at $x = 0$ and $x = \ell$:

$$F_n(0) = Z_n(0) U_n(0) + S_o p_n(0)$$

$$0 = Z_n(\ell) U_n(\ell) - S_o p_n(\ell)$$

where $F_n(o) e^{i4\omega t}$ is the external force exerted on the medium in contact with the piston surface, or, what is the same thing, the external force being exerted in the $x$-direction to maintain its motion. From equations (54), (55), the acoustic velocities and pressures of the ends of the duct are given by:

$$U_n(0) = X - Y$$

$$p_n(0) = \rho_o a_o (X + Y)$$

and

$$U_n(\ell) = X e^{-ik_n \ell} - Y e^{ik_n \ell}$$

$$p_n(\ell) = \rho_o a_o (X e^{-ik_n \ell} + Y e^{ik_n \ell})$$

On substituting from equations (58) to (61) into equations (54) and (55) and rearranging, the following simultaneous equations for $X$ and $Y$ are obtained:

$$[Z_n(0) + \rho_o a_o S_o] X - [Z_n(0) - \rho_o a_o S_o] Y = F_n(0)$$

$$[Z_n(\ell) - \rho_o a_o S_o] e^{-ik_n \ell} X - [Z_n(\ell) + \rho_o a_o S_o] e^{ik_n \ell} Y = 0$$
The constant, X and Y are therefore given by

\[
X = \frac{1}{\Delta} [Z_n(x) + \rho_o a_o S_o] F_n(o) e^{ik_n\ell}
\]

\[
Y = \frac{1}{\Delta} [Z_n(x) - \rho_o a_o S_o] F_n(o) e^{-ik_n\ell}
\]

where \( \Delta \) is the determinant given by

\[
\Delta = \begin{vmatrix}
[Z_n(o) + \rho_o a_o S_o] & [Z_n(o) - \rho_o a_o S_o] \\
[Z_n(x) - \rho_o a_o S_o] e^{-ik_n\ell} & [Z_n(x) + \rho_o a_o S_o] e^{ik_n\ell}
\end{vmatrix}
\]

On substituting for \( X, Y \) and \( \Delta \) in equations (54) and (55) and simplifying we have:

\[
U_n(x) = \frac{\rho_o a_o S_o + iZ_n(x) \tan k_n(\ell - x)}{\rho_o a_o S_o [Z_n(o) + Z_n(\ell)] + i[\rho_o a_o S_o^2 + Z_n(o) Z_n(\ell)] \tan k_n \ell}
\]

\[
x \frac{\cos k_n(\ell - x)}{\cos k_n \ell} F_n(o) e^{i\Delta nt} \]

\[
p_n(x) = \frac{Z_n(x) + i\rho_o a_o S_o \tan k_n(\ell - x)}{\rho_o a_o S_o [Z_n(o) + Z_n(\ell)] + i[\rho_o a_o S_o^2 + Z_n(o) Z_n(\ell)] \tan k_n \ell}
\]

\[
x \frac{\cos k_n(\ell - x)}{\cos k_n \ell} F_n(o) e^{i\Delta nt} .
\]

By writing

\[
F_n(o) = \frac{U_n(o)}{U_n(o)} U_n(o)
\]

\[
= \begin{bmatrix}
Z_n(o) + \rho_o a_o S_o \frac{Z_n(x) + i\rho_o a_o S_o \tan k_n \ell}{\rho_o a_o S_o [Z_n(o) + Z_n(\ell)] + i[\rho_o a_o S_o^2 + Z_n(o) Z_n(\ell)] \tan k_n \ell}
\end{bmatrix} U_n(o)
\]

[From equation (67)]

\[
= Z_n(oo) U_n(o)
\]
into equation (68), where \( Z_n(\omega) \) is the nth harmonic of the driving point impedance, the instantaneous acoustic pressure inside the duct is given by

\[
p_n(x) = \frac{Z_n(\omega) + i \rho_o a_o S_o \tan k_n(\omega - x)}{[Z_n(\omega) + Z_n(\omega)] \rho_o a_o S_o + i \left[ Z_n(\omega) Z_n(\omega) + \rho_o a_o S_o \right] \tan k_n x} \\
x \frac{\cos k_n(\omega - x)}{\cos k_n x} Z_n(\omega) U_n(\omega) e^{i \omega t}
\]

On substituting for \( U_n(\omega) \), the source velocity from equation (40), the root mean square acoustic pressure \( p_n'(x) \) for the ducted case is given by:

\[
p_n'(x) = \frac{\rho_o a_o a^2 W \pi \sqrt{2}}{S_o} \left[ \frac{N}{n} \right] \\
x \left| \frac{Z_n(\omega) + i \rho_o a_o S_o \tan k_n(\omega - x)}{[Z_n(\omega) + Z_n(\omega)] \rho_o a_o S_o + i \left( Z_n(\omega) Z_n(\omega) + \rho_o a_o S_o \right) \tan k_n x} \right| \\
x \frac{\cos k_n(\omega - x)}{\cos k_n x} Z_n(\omega)
\]

The acoustic pressure generated by the meter when the latter is connected to a duct of known terminal impedance, \( Z_n(\omega) \), can hence be calculated.

Equation (71) will now be applied to calculate the acoustic pressure field of rotary positive displacement gas meters for various values of \( Z_n(\omega) \) which characterise typical duct elements.
4.2 **Open-Ended Uniform Duct**

The simplest duct configuration consists of a uniform duct which is open to the atmosphere at one end. This configuration corresponds to the situation under which rotary positive displacement gas meters are tested in the laboratory. The terminal impedance in this case is, \( Z_n(z) = 0 \). Equation (71) then reduces to

\[
\frac{\nu_n'(x)}{p_n(x)} = \frac{\rho_o a_o a^2 W \pi \sqrt{2}}{S_o} \sin k_n \frac{(l - \lambda x)}{\cos k_n z} \left( \frac{N}{n} \right)
\]

The root mean square acoustic pressure at a point \( x = 0.5 \text{ft} \) inside the duct of effective length, \( l = 8.14 \text{ft} \), is then given by

\[
\frac{\nu_n'(x)}{p_n(x)} = 20 \log_{10} \left| 0.08 \times \frac{1118}{1118} \times \left( \frac{1.1796}{12} \right)^2 \times \frac{10}{12} \times \frac{144}{\pi \times 2.179^2} \times \frac{14.87}{.0002} \times \frac{\pi}{\sin (n \frac{8\pi N}{1118} \times 7.63)} \times \frac{n}{\cos (n \frac{8\pi N}{1118} \times 8.14)} \right| \text{ dB}
\]

The calculated results are shown in Table 2 and are plotted in Figures 15 to 19.

**Table 2**

<table>
<thead>
<tr>
<th>Impeller Speed N(Rev/s)</th>
<th>( \frac{\nu_n'(x)}{p_n(x)} ) n=1 (dB)</th>
<th>( \frac{\nu_n'(x)}{p_n(x)} ) n=2 (dB)</th>
<th>( \frac{\nu_n'(x)}{p_n(x)} ) n=3 (dB)</th>
<th>( \frac{\nu_n'(x)}{p_n(x)} ) n=4 (dB)</th>
<th>( \frac{\nu_n'(x)}{p_n(x)} ) n=5 (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>132.0</td>
<td>136.3</td>
<td>149.6</td>
<td>128.3</td>
<td>118.0</td>
</tr>
<tr>
<td>4</td>
<td>127.9</td>
<td>152.5</td>
<td>133.3</td>
<td>119.3</td>
<td>115.9</td>
</tr>
<tr>
<td>5</td>
<td>143.1</td>
<td>146.9</td>
<td>126.9</td>
<td>119.8</td>
<td>143.7</td>
</tr>
<tr>
<td>6</td>
<td>148.3</td>
<td>140.3</td>
<td>116.5</td>
<td>139.2</td>
<td>131.0</td>
</tr>
<tr>
<td>7</td>
<td>154.4</td>
<td>136.2</td>
<td>130.1</td>
<td>140.0</td>
<td>118.5</td>
</tr>
<tr>
<td>8</td>
<td>164.6</td>
<td>131.1</td>
<td>144.2</td>
<td>131.0</td>
<td>132.2</td>
</tr>
<tr>
<td>9</td>
<td>168.7</td>
<td>123.6</td>
<td>149.5</td>
<td>113.9</td>
<td>140.9</td>
</tr>
<tr>
<td>10</td>
<td>159.0</td>
<td>131.8</td>
<td>139.9</td>
<td>136.0</td>
<td>130.9</td>
</tr>
<tr>
<td>11</td>
<td>159.9</td>
<td>141.5</td>
<td>153.9</td>
<td>149.8</td>
<td>116.9</td>
</tr>
</tbody>
</table>
Fig. 15 Variation of acoustic pressure of ducted meter with shaft-speed $n=1$ (uniform open-ended duct
Fig. 16 Variation of acoustic pressure ofducted meter with shaft-speed n=2 (uniform open-ended duct)
Fig. 17 Variation of acoustic pressure of ducted meter with shaft-speed $n=3$ (uniform open-ended duct)
Fig. 18 Variation of acoustic pressure of ducted meter with shaft-speed n=11 (uniform open-ended duct)
Fig. 19 Variation of acoustic pressure of ducted meter with shaft-speed n=5 (uniform open-ended duct)
4.3 Uniform Duct with Side-Branch

The next simple duct configuration of interest consists of an open-ended uniform duct of cross-sectional area, \( S(1) = \frac{\pi}{36} \text{ft}^2 \) and a closed side-branch of effective length \( l(B1) = 2.14 \text{ft} \) and cross-sectional area, \( S(B1) = \frac{\pi}{36} \text{ft}^2 \) - Figure 20. The length, \( l \), in this case is the distance between the source and the side-branch (= 6.22 ft). The distance between the side-branch and the open end is denoted by, \( l(1) = 14.64 \text{ft} \). This duct configuration approximates conditions under normal meter installation where the side-branch corresponds to a closed by-pass connected across the meter. Sound energy arriving at the first duct impedance discontinuity at \( A \) in Figure 20 is transmitted partly along the side-branch and partly along the open-ended section and the remaining energy is reflected back to the meter. Considerations of continuity of pressure and velocity at the side-branch show [30] that the impedance \( Z_n(l) \) is given by

\[
Z_n(l) = \frac{Z_n(1) Z_n(B)}{Z_n(1) + Z_n(B)}
\]

where \( Z_n(1) \) is the impedance of the open-ended section at discontinuity (1), and is given by [Appendix 3]

\[
Z_n(1) = i \rho_o a_o S(1) \tan k_n l(1) \]

\[
= i 7.8 \tan (.33Nn)
\]

and \( Z_n(B) \) is the side-branch impedance and is given by [Appendix 3]

\[
Z_n(B) = - i \rho_o a_o S(B1) \cot k_n l(B1) \]

\[
= - i 7.8 \cot (.05 Nn)
\]

The impedance \( Z_n(l) \) is therefore given by
Fig. 20 Uniform duct with side branch

s(0) = \frac{W_0}{2} ft^2

l(0) = 22 ft

S_0

z_0(b)

A

1

2

z_0(\infty)

z_0(z)

\frac{1}{15} = \frac{W_0}{8} ft^2

l(0) = 4.84 ft

z_0(0) = 0
The driving-point impedance, \( Z_n(\infty) \) in this case is given by

\[
Z_n(\infty) = Z_n(o) + \frac{Z_n(\infty)}{\rho_o a_o s_o} \left[ \frac{1}{\rho_o a_o s_o} + i Z_n(l) \tan \left( k \frac{L}{l} \right) \right]
\]

By substitution into equation (71), the root mean square pressure at a distance \( x (= 0.5 \text{ ft}) \) inside the pipe is given by

\[
p_n'(x) = 20 \log_{10} \left\{ 2.3 \times 10^6 \times \frac{N}{n} \times \frac{\cos (0.13Nn)}{\cos (0.14Nn)} \right\}
\]

The computed results are shown in Table 3 and are plotted in Figures 21 and 22.
<table>
<thead>
<tr>
<th>Impeller shaft N(rev/s)</th>
<th>$P_n'(x)$ $n=1$ (dB)</th>
<th>$P_n'(x)$ $n=2$ (dB)</th>
<th>$P_n'(x)$ $n=3$ (dB)</th>
<th>$P_n'(x)$ $n=4$ (dB)</th>
<th>$P_n'(x)$ $n=5$ (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>163.2</td>
<td>114.4</td>
<td>132.7</td>
<td>112.9</td>
<td>137.4</td>
</tr>
<tr>
<td>3.5</td>
<td>150.8</td>
<td>120.3</td>
<td>139.2</td>
<td>129.7</td>
<td>129.0</td>
</tr>
<tr>
<td>4</td>
<td>144.6</td>
<td>130.9</td>
<td>117.9</td>
<td>145.1</td>
<td>99.3</td>
</tr>
<tr>
<td>4.5</td>
<td>140.9</td>
<td>139.7</td>
<td>130.0</td>
<td>130.9</td>
<td>124.5</td>
</tr>
<tr>
<td>5</td>
<td>137.6</td>
<td>187.3</td>
<td>146.3</td>
<td>103.2</td>
<td>112.1</td>
</tr>
<tr>
<td>5.5</td>
<td>133.6</td>
<td>139.3</td>
<td>143.9</td>
<td>130.6</td>
<td>119.2</td>
</tr>
<tr>
<td>6</td>
<td>126.4</td>
<td>125</td>
<td>135.8</td>
<td>122.3</td>
<td>129.8</td>
</tr>
<tr>
<td>6.5</td>
<td>117.8</td>
<td>131.1</td>
<td>126.2</td>
<td>97.5</td>
<td>147.9</td>
</tr>
<tr>
<td>7</td>
<td>132.3</td>
<td>141.8</td>
<td>148.8</td>
<td>126.3</td>
<td>138.6</td>
</tr>
<tr>
<td>7.5</td>
<td>138.4</td>
<td>153.4</td>
<td>133.4</td>
<td>133.7</td>
<td>153.6</td>
</tr>
<tr>
<td>8</td>
<td>142.9</td>
<td>157.1</td>
<td>127.3</td>
<td>145.9</td>
<td>129.3</td>
</tr>
<tr>
<td>8.5</td>
<td>147.1</td>
<td>147.5</td>
<td>114.1</td>
<td>148.6</td>
<td>119.3</td>
</tr>
<tr>
<td>9</td>
<td>151.8</td>
<td>142.8</td>
<td>124.2</td>
<td>138.4</td>
<td>138.8</td>
</tr>
<tr>
<td>9.5</td>
<td>158.5</td>
<td>137.9</td>
<td>135.0</td>
<td>139.8</td>
<td>116.3</td>
</tr>
<tr>
<td>10</td>
<td>199.4</td>
<td>115.2</td>
<td>138.7</td>
<td>133.1</td>
<td>139.6</td>
</tr>
<tr>
<td>10.5</td>
<td>158.3</td>
<td>155.9</td>
<td>147.0</td>
<td>126.6</td>
<td>138.8</td>
</tr>
<tr>
<td>11</td>
<td>151.3</td>
<td>142.7</td>
<td>175.2</td>
<td>125.6</td>
<td>108.2</td>
</tr>
</tbody>
</table>
4.4 Complex Duct Configuration

The impedance, $Z_n(\ell)$ for more complex configurations can, in principle, be calculated by similar procedures, but in practice the resulting expressions for $Z_n(\ell)$ and $p_n'(x)$ become very cumbersome as the number of duct discontinuities increases. A more systematic approach is required for calculating the acoustic pressure of more complex duct networks.

Suppose the duct impedance discontinuities are marked from 0 to $M$, starting from the meter. ($M =$ total number of discontinuities in the configuration.) The required impedance, $Z_n(\ell)$, then is equal to $Z_n(1)$. We now show how $Z_n(\ell)$ can be calculated for a duct network of any configuration.

Any two consecutive duct impedances of the network, $Z_n(k)$ and $Z_n(k-1)$ are related to each other, either by the "parallel" relation:

$$Z_n(k-1) = \frac{Z_n(B) Z_n(k)}{Z_n(B) + Z_n(k)}$$

or by the "series" relation in matrix form [see Appendix 3]:

$$Z_n(k-1) = \frac{Z_n(k)}{\frac{\cos[k_n \ell(k-1)]}{\rho_o a_o S(k-1)} + \frac{i \rho a_o S(k-1) \sin[k_n \ell(k-1)]}{\rho_o a_o S(k-1)}} + \frac{i \rho a_o S(k-1) \tan[k_n \ell(k-1)]}{\rho_o a_o S(k-1) + i Z_n(k) \tan[k_n \ell(k-1)]}$$
depending on whether they are separated by a closed side branch of effective length $l(k-1)$ and cross-sectional area, $S(k-1)$ or by a uniform duct of equal effective length and area. Figure 23. If the final duct impedance, $Z_n(k=m)$ is known, and also if the structure of the duct network is known, (i.e. if the locations of the side-branches are known), the impedance $Z_n(k=m-1)$ can be calculated and the procedure repeated for reducing values of $k$ until $Z_n(1)$ and hence $Z_n(1)$ is obtained. Once $Z_n(1)$ is known, equation (71) can then be used to calculate the acoustic pressure field for the particular duct configuration. A computer can be easily programmed to perform these calculations. The acoustic pressure field of an experimental meter model connected to the duct network shown in Figure 24 was calculated by the above procedures. The flow-chart of the computer programme is shown in Figure 25. The computed results are shown in Table 4 and are plotted in Figures 26 to 30. The computer programme can be easily extended to give an analysis of the "a.c." behaviour of a meter with a wide range of duct configurations. Such analysis will then give the optimum pipe configuration for any meter situation.
<table>
<thead>
<tr>
<th>Impeller shaft N (rev/s)</th>
<th>$\hat{p}_n'(x)$ n=1 (dB)</th>
<th>$\hat{p}_n'(x)$ n=2 (dB)</th>
<th>$\hat{p}_n'(x)$ n=3 (dB)</th>
<th>$\hat{p}_n'(x)$ n=4 (dB)</th>
<th>$\hat{p}_n'(x)$ n=5 (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>130.6</td>
<td>164.2</td>
<td>120.3</td>
<td>124.4</td>
<td>135.7</td>
</tr>
<tr>
<td>3.5</td>
<td>121.4</td>
<td>131.5</td>
<td>136.7</td>
<td>135.2</td>
<td>141.3</td>
</tr>
<tr>
<td>4</td>
<td>123.4</td>
<td>113.9</td>
<td>129.4</td>
<td>118.6</td>
<td>123.3</td>
</tr>
<tr>
<td>4.5</td>
<td>134.5</td>
<td>127.4</td>
<td>132.8</td>
<td>137.5</td>
<td>125.8</td>
</tr>
<tr>
<td>5</td>
<td>141.7</td>
<td>137.8</td>
<td>144.6</td>
<td>127.2</td>
<td>111.2</td>
</tr>
<tr>
<td>5.5</td>
<td>149.9</td>
<td>129.3</td>
<td>132.1</td>
<td>134.7</td>
<td>106.4</td>
</tr>
<tr>
<td>6</td>
<td>176.3</td>
<td>136.4</td>
<td>142.5</td>
<td>122.3</td>
<td>130.2</td>
</tr>
<tr>
<td>6.5</td>
<td>150.2</td>
<td>131.2</td>
<td>134.5</td>
<td>107.7</td>
<td>147.9</td>
</tr>
<tr>
<td>7</td>
<td>143.6</td>
<td>147.3</td>
<td>115.6</td>
<td>122.7</td>
<td>138.1</td>
</tr>
<tr>
<td>7.5</td>
<td>137.5</td>
<td>151.7</td>
<td>134.6</td>
<td>134.0</td>
<td>139.7</td>
</tr>
<tr>
<td>8</td>
<td>126.0</td>
<td>130.6</td>
<td>127.3</td>
<td>145.9</td>
<td>126.9</td>
</tr>
<tr>
<td>8.5</td>
<td>130.4</td>
<td>155.3</td>
<td>110.5</td>
<td>148.5</td>
<td>129.4</td>
</tr>
<tr>
<td>9</td>
<td>139.4</td>
<td>149.5</td>
<td>134.2</td>
<td>110.5</td>
<td>127.9</td>
</tr>
<tr>
<td>9.5</td>
<td>144.9</td>
<td>143.6</td>
<td>131.4</td>
<td>136.2</td>
<td>124.9</td>
</tr>
<tr>
<td>10</td>
<td>149.8</td>
<td>139.2</td>
<td>139.0</td>
<td>130.8</td>
<td>141.0</td>
</tr>
<tr>
<td>10.5</td>
<td>155.8</td>
<td>122.6</td>
<td>147.0</td>
<td>142.2</td>
<td>136.2</td>
</tr>
<tr>
<td>11</td>
<td>171.3</td>
<td>146.8</td>
<td>144.7</td>
<td>115.4</td>
<td>136.6</td>
</tr>
</tbody>
</table>
a. Parallel combination:

\[ Z_n(K-1) = -i\rho_0 a_0 S(K-1) \csc(K_n(K-1)) Z_n(K) \]

\[ -i\rho_0 a_0 S(K-1) \csc(K_n(K-1)) + Z_n(K) \]

b. Series combination:

\[ Z_n(K-1) = \begin{bmatrix} \cos(K_n(K-1)) \\ \sin(K_n(K-1)) \end{bmatrix} \begin{bmatrix} Z_n(K) \\ 1 \end{bmatrix} \]

\[ \begin{bmatrix} \cos(K_n(K-1)) \\ \sin(K_n(K-1)) \end{bmatrix} \begin{bmatrix} -i\rho_0 a_0 S(K-1) \csc(K_n(K-1)) \\ i\rho_0 a_0 S(K-1) \end{bmatrix} \begin{bmatrix} Z_n(K) \\ 1 \end{bmatrix} \]

Fig. 23 Combination of duct impedances
Fig. 24 Complex duct
Fig. 25. Algorithm for calculating acoustic pressure of ducted meters.

START

1. READ:
   1. No of impedance discontinuities $M$
   2. No of side branches $NB$
   3. Location of side-branches
      $N(IB); IB = 1 \text{ to } NB$
   4. $L(k), S(k); k = 1 \text{ to } N-1$
   5. Meter parameters
   6. $Z_n(M)$
   7. Initial shaft speed $NI$
   8. Final shaft speed $NF$
   9. Shaft speed increment $\Delta$

2. $N = NI$

3. $n = 1$

4. $K = M$

5. $IB = NB$

6. IS $k-1 = (IB)$?

7. $Z_n(k-1) = -i p_o a_s S(k-1) \cot[k_n S(k-1)] \frac{Z_n(k)}{-i p_o a_s S(k-1) \cot[k_n S(k-1)] + Z_n(k)}$

8. No

9. Yes

68.
\[ Z_n(k-1) = \frac{p \alpha_o S(k-1) \tan \left[ k \beta(k-1) \right]}{p \alpha_o S(k-1) + i Z_n(k) \tan \left[ k \beta(k-1) \right]} + Z_n(k) \]

- **7**
  - **IB = IB - 1**

- **8**
  - **k = k - 1**

- **9**
  - **IS K = 0 ?**
  - **No**

- **10**
  - **Yes**
  - **Z_n(1) = Z_n(1)**

- **11**
  - **CALCULATE \( \hat{p}_n(x) \)**
  - **[From eq. (71)]**

- **12**
  - **IB = NB**
  - **n = n + 1**
  - **k = m**

- **13**
  - **IS n = 5 ?**
  - **Yes**
  - **N = NI + \Delta**

- **14**
  - **No**

- **15**

- **16**
Fig. 26 Variation of acoustic pressure field of ducted meter with shaft-speed n=1 (complex duct)
Fig. 27 Variation of acoustic pressure of ducted meter with shaft-speed n=2 (complex duct)
Fig. 29 Variation of acoustic pressure of ducted meter with shaft-speed n=4 (complex duct)
5. EFFECT OF PULSATIONS ON ROTARY POSITIVE DISPLACEMENT GAS METERS

5.1 Supercharging Effects

Rotary positive displacement gas meters operate on the principle that N complete revolutions of the impeller shaft displace a volume of gas from the meter inlet port equal to ""4Nv + slip", into the outlet port. In the absence of pulsations, this volume is trapped at the line pressure, \( P_o \) and the shape of the error curve is adequately explained by the "slip" term. The error curve then is a rectangular hyperbola. [See Figure 2(a) and reference (3)].

When pulsations are present however, the trapped volumes are at a pressure of \( P_o + p_e \), where \( p_e \) is the instantaneous acoustic pressure at the meter \( (x = 0) \), at the end of the gas volume induction, just before the impeller closes, i.e. when

\[
\omega t = m \frac{\pi}{2}; \quad m = 0, 1, 2 \ldots
\]

The meter then will pass more or less volume (depending on the sign of \( p_e \)) than indicated. This supercharging effect is inherent in all reciprocating machines and is used to advantage in automobile and aircraft superchargers. The resulting error, \( e_a \), due to acoustic pressure waves is then given by [From equations (39), (40) and (79)]:

\[
e_a = \frac{p_e}{P_o} \times 100
\]

\[
= \frac{25 \rho_o a_o^2 W}{P_o S_o} \sum_{n=1}^{m} \text{Re} \left[ \frac{Z_n(\omega) + i \rho_o a_o S_o \tan k \omega}{[Z_n(\omega) + Z_n(1) \rho_o a_o S_o + i (Z_n(\omega) Z_n(1) \rho_o a_o S_o)^2 \tan k \omega]^n} \right]
\]

\[
\left( \frac{N}{n} \right) Z_n(\omega)
\]
The complete error curve then is a superposition of equations (1) and (82) and is given by:

\[ e = C_1 - \frac{C_2}{Q} \times 100 + e_a \]  

Equation (82) was used to calculate the acoustic excess error, \( e_a \), for the configuration shown in Figure 24. The results are shown in Table 5 and are plotted in Figure 31.

If the rms acoustic pressure at the meter is expressed in decibels, the associated error at line atmospheric pressure is given by

\[
e_a = \frac{0.0002 \times 10^{-3} \times 100}{1013.25} \text{antilog} 10 \left( \frac{P(dB)}{20} \right)
\]

In Figure 32, the acoustic error, \( e_a \), is plotted for values of acoustic pressures (in decibels) from 120 dB to 160 dB. It shows that acoustic pressures above 134 dB will introduce errors greater than 0.1%.

5.2 Effects on slip

The supercharging effect mentioned above is the major effect of pulsations on the meter accuracy. A smaller instantaneous error is also introduced by the effect of pulsations on the pressure loss, \( \Delta P \), and hence on the slip. The true pressure loss is equal to \( \Delta P + \delta p \), where \( \delta p \) is the difference between the acoustic pressure at the meter inlet and outlet ports, i.e.

\[ \delta p = p_{in}^\infty - p_{out}^\infty \]
<table>
<thead>
<tr>
<th>Impeller Shaft N(rev/s)</th>
<th>$e_a$ n=1 %</th>
<th>$e_a$ n=2 %</th>
<th>$e_a$ n=3 %</th>
<th>$e_a$ n=4 %</th>
<th>$e_a$ n=5 %</th>
<th>$\Sigma e_a$ %</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-.028</td>
<td>.020</td>
<td>-.001</td>
<td>.043</td>
<td>.005</td>
<td>.039</td>
</tr>
<tr>
<td>3.5</td>
<td>-.071</td>
<td>.010</td>
<td>-.069</td>
<td>.010</td>
<td>-.002</td>
<td>-.122</td>
</tr>
<tr>
<td>4</td>
<td>.007</td>
<td>-.002</td>
<td>.077</td>
<td>.000</td>
<td>-.0001</td>
<td>.081</td>
</tr>
<tr>
<td>4.5</td>
<td>-.008</td>
<td>-.003</td>
<td>.021</td>
<td>-.008</td>
<td>.002</td>
<td>.005</td>
</tr>
<tr>
<td>5</td>
<td>.015</td>
<td>-.064</td>
<td>.015</td>
<td>-.002</td>
<td>.006</td>
<td>-.030</td>
</tr>
<tr>
<td>5.5</td>
<td>.053</td>
<td>-.901</td>
<td>-.002</td>
<td>.004</td>
<td>-.003</td>
<td>-.849</td>
</tr>
<tr>
<td>6</td>
<td>.079</td>
<td>.173</td>
<td>-.014</td>
<td>.001</td>
<td>.056</td>
<td>.296</td>
</tr>
<tr>
<td>6.5</td>
<td>.075</td>
<td>.050</td>
<td>-.009</td>
<td>.003</td>
<td>.000</td>
<td>.119</td>
</tr>
<tr>
<td>7</td>
<td>.041</td>
<td>.039</td>
<td>-.001</td>
<td>.021</td>
<td>.006</td>
<td>.106</td>
</tr>
<tr>
<td>7.5</td>
<td>-.031</td>
<td>.034</td>
<td>.007</td>
<td>.088</td>
<td>.088</td>
<td>.102</td>
</tr>
<tr>
<td>8</td>
<td>-.007</td>
<td>-.002</td>
<td>.002</td>
<td>-.000</td>
<td>.000</td>
<td>-.006</td>
</tr>
<tr>
<td>8.5</td>
<td>.043</td>
<td>-.000</td>
<td>.013</td>
<td>.003</td>
<td>.000</td>
<td>.058</td>
</tr>
<tr>
<td>9</td>
<td>-.012</td>
<td>-.030</td>
<td>.003</td>
<td>.000</td>
<td>.000</td>
<td>-.039</td>
</tr>
<tr>
<td>9.5</td>
<td>-.105</td>
<td>-.036</td>
<td>.100</td>
<td>.006</td>
<td>.004</td>
<td>-.031</td>
</tr>
<tr>
<td>10</td>
<td>-.256</td>
<td>-.008</td>
<td>.156</td>
<td>.001</td>
<td>.009</td>
<td>-.098</td>
</tr>
<tr>
<td>10.5</td>
<td>-.622</td>
<td>-.002</td>
<td>.001</td>
<td>.006</td>
<td>.008</td>
<td>-.609</td>
</tr>
<tr>
<td>11</td>
<td>-3.606</td>
<td>.016</td>
<td>.001</td>
<td>.000</td>
<td>.009</td>
<td>-3.58</td>
</tr>
</tbody>
</table>
Fig. 31 Calculated acoustic excess error against shaft-speed (complex duct)
The instantaneous acoustic pressure difference, $\delta p$, can be either positive or negative and greater than $\Delta P$, depending on the network configurations at the meter inlet and outlet. This suggests a possible negative pressure loss, contrary to common experience. However, the average value of $\delta p$ over a complete meter cycle is zero and so, the average effect of pulsation on pressure loss and on slip is negligible.
The behaviour of rotary positive displacement gas meters as potential sources of acoustic pressure waves and the effects of these waves on the meter registration have been investigated in the previous chapter. When the parameters of the meter and the associated pipe network are known, equations (71) and (82) can be used to calculate the magnitudes of the acoustic pressure waves and of their associated excess errors and correction factors can then be applied to compensate for these errors. This procedure might not always be a convenient and sufficient method of controlling the effects of pulsations in a meter system. In many applications, when gas burners are used in close proximity for example, it is desirable to control the magnitude of the pressure waves below a certain level. Concentric helical type resonators have been successfully used to maintain the acoustic pressure waves below the 130 dB level.

A review of helical type resonators is given in reference [31]. Figure 32 shows a diagram of a concentric helical type resonator mounted on a uniform duct of cross-section $S$. The resonator consists of a series of concentric, small side branches — of effective length $L$ and area $S_b$ connecting the main duct to a chamber of volume $V$. A disturbance at the junction will be propagated in part down the main duct and in part through the resonator into the chamber. When the frequency, $f$, of the pulsation in the main duct is approximately equal to that of the side-branch, determined by the impedance, $Z$, the disturbance in the side-branch will resonate in the chamber. The graph in Fig. 32 indicates the acoustic excess error against acoustic pressure.
6.

SUPPRESSION OF PULSATIONS FROM ROTARY POSITIVE DISPLACEMENT GAS METER SYSTEMS

6.1 Concentric Helmholtz Resonators

The behaviour of rotary positive displacement gas meters as potential sources of acoustic pressure waves and the effects of these waves on the meter registration have been investigated in the previous chapters. When the parameters of the meter and of the associated pipe network are known, equations (71) and (82) can be used to calculate the magnitudes of the acoustic pressure waves and of their associated excess errors and correction factors can then be applied to compensate for these errors. This procedure might not always be a convenient and sufficient method of controlling the effects of pulsations in a meter system. In many applications, when gas burners are used in close proximity for example, it is desirable to reduce the amplitude of the pressure waves below a certain level. Concentric Helmholtz type resonators have been successfully used to maintain the acoustic pressure waves below the 130 dB level.

A review of Helmholtz type resonators is given in reference [31]. Figure 33 shows a schematic diagram of a concentric Helmholtz resonator mounted on a uniform duct of cross-section $S$. The resonator consists of a series of connectors - small side branches - of effective length $l_c$ and area $S_c$, connecting the main duct to a chamber of volume $V$. A displacement at a junction will be propagated in part down the main duct and in part through the connector into the chamber. When the frequency, $f$, of the pulsation in the main duct is approximately equal to that of the side-branch, determined by its impedance, $Z_n(c)$, the disturbance in the side-branch will not only be large compared with that in the main duct, but it
will also be periodically reinforced, causing more energy to be abstracted from the main transmission duct. The energy remaining in the main duct will be further reduced at the next junction, so that finally, if there is sufficient number of connectors, the amplitude of the incident wave can be reduced to a desired level. The acoustic impedance $Z_n(c)$ of each side-branch is equal to the sum of the impedances of the connector and of the volume chamber and is given by

$$Z_n(c) = i \rho_o a_o \left[ \frac{1}{S_c} \tan k_n l_c - \frac{1}{S_V} \cot k_n l_V \right]$$

where $l_V$ and $S_V$ are respectively the effective length and cross-section of the chamber. When $(k_n l)$ is small, the side-branch impedance is given by

$$Z_n(c) = i \rho_o a_o \left[ \frac{1}{S_c} \frac{\omega l_c}{a_o} - \frac{1}{S_V} \frac{a_o}{\omega l_V} \right]$$

$$= i \left( \frac{\omega \rho_o a_o^2}{c_o} - \frac{\rho_o a_o^2}{\omega V} \right)$$

where $c_o = \frac{S_c}{l_c}$

= conductivity of individual connector.

By using the principle of continuity of pressure and of velocity at the junction, it can be easily shown that the ratio of the amplitude $p_{n1}$ of the incident wave to that of the transmitted wave, $p_{nt}$, is given by

$$\frac{p_{n1}}{p_{nt}} = 1 + \frac{Z_n(D)}{2 Z_n(c)}$$
where \( Z_n(D) \) is the main duct impedance and is given by
\[
Z_n(D) = \frac{\rho_0 a_o}{S}.
\]

The transmission loss, \( T \), of the resonator, defined as
\[
T = 10 \log_{10} \left( \frac{P_i}{P_t} \right)^2
\]
is therefore given by
\[
T = 10 \log_{10} \left[ 1 + \frac{Z_n^2(D)}{4 Z_n^2(B)} \right]
\]
\[
= 10 \log_{10} \left[ 1 + \left( \frac{\frac{a_o}{25}}{\frac{nw}{c_o} - \frac{a_o^2}{nwV}} \right)^2 \right]
\]
The transmission loss is a maximum when
\[
\frac{nw}{c_o} = \frac{a_o^2}{nwV}
\]
From equation (90) the resonant frequency \( f_r \) of each connector is therefore given by
\[
f_r = \frac{nw}{2\pi}
\]
\[
= \frac{a_o}{2\pi} \sqrt{\frac{c_o}{V}}
\]
By substituting for \( a_o = f_r \sqrt{\frac{V}{c_o}} \) into equation (89), the expression for the transmission loss simplifies to
\[
T = 10 \log_{10} \left[ 1 + \left( \frac{\sqrt{\frac{c_o V}{25}}}{\frac{f/\bar{f}}{f_r} - \frac{\bar{f}/f}{f_r}} \right)^2 \right]
\]
The performance of the resonator is governed by the two parameters:

(a) the resonance parameter \( \sqrt{\frac{c_o}{V}} \), which determines its resonance frequency, \( f_r \); and

(b) the attenuation parameter \( \sqrt{\frac{c_o V}{2s}} \), which determines the transmission loss at frequencies different to \( f_r \).

In Figure 34, the transmission loss, \( T \), is plotted against \( f/f_r \) for various values of the attenuation parameter. It is seen from Figure 34 that the resonator remains effective over a large range of frequencies if \( \sqrt{\frac{c_o V}{2s}} > 32 \).

6.2 Design Considerations

The design of a suitable Helmholtz resonator can be reached from the following considerations. If the parameters of the meter and of the pipe network are known, equation (71) can be used to obtain the "a.c." response of the meter system. The frequency band, \( \Delta f \), over which attenuation is desired and the amount of attenuation, \( \Delta T \), required are then obtained from the response curve. The resonant frequency, \( f_r \), is then taken as the centre frequency of \( \Delta f \). Once, \( f_r \) is known, equation (91) is then used to calculate the value of the resonance parameter \( \sqrt{\frac{c_o}{V}} \). A knowledge of \( \Delta T \) and Figure 34 determine the correct value of the attenuation parameter, \( \sqrt{\frac{c_o V}{2s}} \), and hence the value of \( \sqrt{c_o V} \). The design values of the conductivity \( c_o \) and of the volume, \( V \), are then respectively given by

\[
c_o = \sqrt{\frac{c_o}{V}} \times \sqrt{c_o V}
\]

and

\[
V = \sqrt{c_o V} \div \sqrt{\frac{c_o}{V}}
\]
Any combination of length, diameter and number of connectors that will give this conductivity is permissible so long as the acoustic particle displacement in the connector does not exceed the length of the connector. The conductivity of a group of \( n \) orifices of length \( l_c \) and individual area, \( s_c \) is approximately given by

\[
C_o = \frac{n s_c}{l_c + 0.8 \sqrt{s_c}}
\]

The performance of a Helmholtz resonator designed for the pipe configuration shown in Figure 20 is shown in Figure 35. Its design dimensions are given in Chapter 7.
Fig. 34 Variation of transmission loss with attenuation parameter
Fig. 35 Performance of a concentric Helmholtz resonator
7. EXPERIMENTAL VERIFICATION

7.1 The Meter Experimental Model

The general experimental set-up is shown in Plate 1. A modified 3 inch x 10 inch rotary positive displacement gas meter was used in all experimental verification. The meter was provided with perspex end-plates and with extended impeller shafts as shown in Figure 36. The purpose of the transparent end-plates was to make the measuring chamber visible for future flow visualisation and Schlieren studies. The impellers were driven externally to ensure that all external sound sources are excluded except those due to impeller rotation. A \( \frac{1}{2} \) horse-power dc motor in conjunction with a variable speed control unit was used. Plate 2.

7.2 Acoustic Pressure Measuring Instruments. Plates 3 and 4

Brual and Kjaer acoustic measuring equipment were used in all acoustic measurements. The acoustic pressure was measured by means of a condenser microphone exposed to the acoustic field - Figure 37. The condenser microphone consists essentially of a moveable diaphragm which is one electrode of a capacitor. A constant polarisation voltage, \( E_0 \), applied through a large resistance, \( R_p \) - Figure 37, maintains a constant charge, \( Q \), on the plates of the capacitor. A movement, \( \Delta x \), of the diaphragm due to the incident sound pressure produces a change in capacitance, \( \Delta C \), and hence an analogue voltage signal, \( \Delta E \), given by

\[
\Delta E = -\frac{K Q_0 C_o \Delta x}{C_o + C_s (1 + \frac{\Delta x}{X_o})}
\]

where \( K \) is the calibration factor; \( C_o \) is the cartridge capacitance;
$C_s$ is the stray capacitance; $x_o$ is the distance between the capacitor plates at equilibrium. Good linearity and large sensitivity are obtained by keeping the stray capacitance small compared with $C_o$. This is achieved by having the preamplifier as near to the capacitor as possible and by using a guard shield.

The preamplifier output voltage, $V_o$, is given by

$$V_o = \frac{i_0 \omega R C}{1 + i_0 \omega R C}$$

where

$$R = \frac{R_i R_p}{R_i + R_p}$$

$$C = C_o + C_s + C_i$$

where $R_i$ is the preamplifier input resistance; $C_i$ is the preamplifier input capacitance; $\omega$ is the radian frequency of the input pressure.

The signal from the preamplifier was further amplified, "frequency analysed" and recorded on calibrated chart. The analyser and level recorder were synchronised to give the root-mean square values of the sound pressures at finite frequencies. A pistonphone was used to verify the calibration of the instruments.

7.3 Measurement of Acoustic Pressure of Unducted Meter. Figure 38

This measurement was made in an anechoic chamber to simulate the infinite baffle condition. A $\frac{1}{2}$ inch condenser microphone mounted axially at a distance of 0.5 feet from the impeller was used in conjunction with a preamplifier and measuring amplifier type 2606.

Only seven results were obtained because the meter impellers collapsed before more measurements could be obtained. The experimental results are shown in Appendix 11.4.
7.4 Measurement of Acoustic Pressure of Ducted Meter, Figure 39

All measurements on the ducted meter were made with a \( \frac{1}{4} \) inch condenser microphone type 4136, mounted flush to the duct wall at a distance of 0.5 feet from the impellers. A measuring amplifier was not used in this case. This means that an open-circuit correction factor of 30 dB was applied to each measurement on the ducted meter. The experimental spectrograms obtained on the ducted meter are shown in Appendices 11.5 to 11.7.

7.5 Experimental Resonator

The experimental resonator was tuned to a resonant frequency, \( f_r = 40 \text{ Hz} \) - Figure 21. An attenuation parameter of 4.0 was chosen. The conductivity \( c_o \), and the volume, \( V \), of the resonator we calculated from

\[
\sqrt{\frac{c_o}{V}} = \frac{6.0 \times \pi \times 2}{1118} \quad 100
\]

\[
\sqrt{\frac{c_o}{V}} = \frac{4.0 \times \pi \times 2}{36} \quad 101
\]

Hence, \( c_o = 0.157 \text{ feet} \)

\( V = 3.1 \text{ cu. feet} \)

Twelve connectors, each of length, \( l_c = 1 \) inch were used. The area of cross-section, \( s_c \), of each connector was calculated from

\[
0.157 = \frac{12 \times s_c}{0.083 \times 0.8\sqrt{s_c}} \quad 102
\]

which gave \( s_c = 1.48 \times 10^{-3} \text{ sq. feet} \).

The diameter \( d_c \) of each connector was

\[
d_c = \left( \frac{4 \times 1.48 \times 10^{-3}}{\pi} \right)^{\frac{1}{2}} \approx 0.5 \text{ inches}.
\]

The experimental results obtained with the resonator are shown in Appendix 11.8.
(a) Simplified circuit of condenser microphone and preamplifier

(b) Equivalent circuit of condenser microphone and preamplifier

Fig. 37
Fig. 38 Experimental set up for measurement of acoustic pressure of unducted meter
Fig. 39 Experimental setup for measurement of acoustic pressure of ducted meter
Plate 1  General Experimental Set-up
8. DISCUSSION OF RESULTS

8.1 Unducted Meter. Figures 8 to 12

The agreement between theoretical predictions and observations is excellent for $n = 1$. The agreement is less good for the harmonics where a systematic difference of about 5 dB is found to exist. On the whole it can be said that the model would predict the acoustic pressure of the unducted meter rather well.

8.2 Ducted Meter. Figures 15 to 19; 21; 22; 26 to 30

Qualitative agreement in the ducted meter case seems excellent. The shapes of the pressure curves, i.e. the duct effects are well predicted by the model. The systematic difference between calculated and observed results is about 10 dB, except at the resonance peaks where the difference is much higher. Also, the resonance peaks of the calculated and observed curves do not always coincide. This discrepancy which has been reported elsewhere [32] seems to suggest that our basic assumption that the meter is a constant velocity source is not entirely correct. The acoustic loading of the duct may be sufficient to affect the velocity of the impellers.

8.3 Resonator Performance. Figure 36

The performance of the experimental Helmholtz resonator suggests that this type of resonator can be used to maintain the sound pressure level of meter systems to below the 130 dB level. Such resonators are easy to design and can be easily incorporated in the meter system.
9. CONCLUSION

1. Rotary positive displacement machines have been used for the metering of fuel gas since 1920. So far their calibration have been considered on the assumption of steady flows which results into a rectangular hyperbolic "d.c." response [Equation (1)] which is independent of the meter associated pipe configuration. These machines do however have a fluctuating flow component which gives rise to an "a.c." response which is dependent on the meter inlet pipe configuration. A survey of the literature shows that the pulsation aspects of rotary positive displacement machines in the context of gas metering have been neglected.

2. The geometry of the measuring section of two lobe involute rotary positive displacement gas meters has been described. An expression for the meter volume displacement has been derived.

3. A model has been developed for the generation of acoustic pressure waves generated by rotary positive displacement meters. The meter was approximated to a small, rigid and circular piston pulsating with harmonic time dependence inside an infinite baffle. The fluid velocities were assumed to be normal to the piston surfaces and the effects of flows and friction were assumed to be negligible. The prediction of the fundamental pressure wave of the unducted meter is excellent. A systematic difference exists in the prediction of the harmonics.

4. The model has been extended to the more realistic situation when the meter is connected to a pipe network characterised by a terminal impedance, $Z_n(1)$. A method of computing the terminal impedance, $Z_n(1)$ for complex pipe network has been developed. The model would predict the fundamental acoustic pressure waves of the ducted meter to a systematic difference of about 5 dB.
5. The supercharging effect of the self-induced acoustic pressure waves on the performance of rotary positive displacement gas meters was discussed.

6. Concentric Helmholtz type resonators have been successfully used to maintain acoustic pressure level below 130 dB which corresponds to an excess error of about 0.1%

7. The experimental verification of the theoretical predictions of the acoustic model was described.

8. Future improvement of the model would regard the meter as two cylindrical sources, \( \pi/2 \) out of phase, separated by a distance, \( c \), and pulsating inside a finite baffle. If necessary, the effects of flow, friction, and finite impedance could be included.

9. The pressure fields of the ducted meter have been predicted with less success. This suggests that the basic assumption that the meter is a constant velocity source may not be entirely correct. Acoustic duct loading may be significant to affect impeller rotation.

10. A list of references on rotary positive displacement machines was given.

11. All frequency spectra obtained during experimental verification were shown in appendix.
10. REFERENCES


2. A.J. ALEXANDER 1971. Internal Report - Department of Mechanical Engineering, University of Loughborough. Theoretical consideration of the calibration and performance of turbine and rotary displacement meters at pressures up to 1000 psi.

   (Translation: The influence of type of gas and gas pressure on the accuracy of rotary positive displacement gas meters.) Verlag R. Oldenbourg, München.

   (Translation: Experimental determination of error inherent in rotary positive displacement gas meters at high pressures).


11. APPENDICES

11.1 Calculation of cross-section, \(A_0(\theta_r)\) of outlet boundary between impellers.

From Figure 5.

\[ A_0(\theta_r) = AGYY CBA \]

\[ = AOB + BOP + PBC + OGY - OGP - GPY - 2(IAK + OIK + OXK) \quad A(1) \]

The area \(AOB = \frac{1}{2} g^2 BOA\)

\[ = \frac{1}{2} a^2 (BOE - \theta_r) \]

\[ = \frac{1}{2} a^2 (sec \psi + \frac{d + r \omega}{a})^2 \left[ \sin^{-1} \left( \frac{c}{2E} \right) - \theta_r \right] \]

\[ = \frac{1}{2} a^2 (sec \psi + \frac{d + r \omega}{a})^2 \left[ \cosec^{-1} \left( 1 + \frac{d + r \omega}{a} \cos \psi \right) - \theta_r \right] \]

The area \(BOP = \frac{1}{2} OP OB \cos BOE\)

\[ = \frac{1}{2} a^2 c g \cos \left[ \sin^{-1} \left( \frac{c}{2E} \right) \right] \]

\[ = \frac{1}{2} a^2 \sec \psi \left( \sec \psi + \frac{d + r \omega}{a} \right) \cos \cosec^{-1} \left( 1 + \frac{d + r \omega}{a} \cos \psi \right) \]

The area \(PBC = \frac{1}{2} g^2 BPC\)

\[ = \frac{1}{2} a^2 \left( \sec \psi + \frac{d + r \omega}{a} \right)^2 \left[ \pi - \cos^{-1} \left( \frac{c}{2E} \right) - \theta_r \right] \quad A(4) \]

The area \(IAK = FXH\) in Figure 1

\[ = \frac{1}{2} d^2 FXH \]

\[ = \frac{1}{2} d^2 \sin^{-1} \left( \frac{a}{d} \cos \nu \right) \quad A(5) \]

The area \(IOK = DOE\) in Figure 1

\[ = \frac{1}{2} a^2 \left( \frac{h}{a} \right) \left( \sec \psi + \frac{r \omega}{a} \right) \sin \beta \]

from equation (14)
The area $OXK = DOF$ in Figure 1

$$= \frac{a^2}{6} \left( \frac{b}{a} \right)^2 - \frac{3}{2}$$

from equation (18).

The area $YPY = 2 FOH$ in Figure 1

$$= 2 (FOX - FXH)$$

$$= a^2 \left[ \frac{d}{a} \cos \left( \sin^{-1} \left( \frac{d}{a} \cos \psi \right) - \psi \right) \right]$$

$$- \left( \frac{d}{a} \right)^2 \sin^{-1} \left( \frac{d}{a} \cos \psi \right)$$

from equations (13) and A(5).

The area $OGX = 2 a^2 \int_{0}^{\theta_1} \theta^2 \, d\theta$

where $\theta_1 = MOX$ in Figures 5 and 6

$$= \theta_1 + \tan \psi - \frac{\pi}{4}$$

$$\therefore \, OGX = \frac{1}{6} a^2 (\theta_1 + \tan \psi - \frac{\pi}{4})^3$$

The area $GPY = 2 a^2 \int_{0}^{\theta_2} \theta^2 \, d\theta$

where $\theta_2 = Y'P \cdot M_2$ in Figures 5 and 6

$$= \frac{\pi}{4} + \tan \psi - \theta_1$$

$$\therefore \, GPY = \frac{1}{6} a^2 \left( \frac{\pi}{4} + \tan \psi - \theta_1 \right)^3$$
The area $OGP = \frac{1}{2} \text{ OP GN}$

where GN is perpendicular to OP

Now, $GN = GM \cos \psi$

$$= (MM \perp - GM \perp) \cos \psi$$

$$= a(\tan \psi - \theta_1) \cos \psi$$

$$= a\left(\frac{\pi}{4} - \theta_1\right) \cos \psi$$

$$. OGP = \frac{1}{2} 2a \sec \psi \ a\left(\frac{\pi}{4} - \theta_1\right) \cos \psi$$

$$= a^2 \left(\frac{\pi}{4} - \theta_1\right)$$ \hspace{1cm} A(11)

By substituting from A(2) - A(11) and rearranging, equation A(1) gives

$$\frac{A_o\left(\theta_1\right)}{a^2} = \frac{1}{2} \left[ \sec \psi + \frac{d + r_\omega}{a} \right] \left[ \pi + \cosec^{-1}\left(1 + \frac{d + r_\omega}{a} \cos \psi\right) \right]$$

$$- \sec^{-1}\left(1 + \frac{d + r_\omega}{a} \cos \psi\right) + \sec \psi \left(\sec \psi + \frac{d + r_\omega}{a}\right)$$

$$x \cos \left[\cosec^{-1}\left(1 + \frac{d + r_\omega}{a} \cos \psi\right)\right] - \left(\frac{d}{a}\right) \cos \left[\sin^{-1}\left(\frac{a}{d} \cos \nu\right) - \nu\right]$$

$$- \left(\frac{d}{a}\right) \left[\sin^{-1}\left(\frac{a}{d} \cos \nu\right) + \sin^{-1}\left(\frac{a}{d}\right) \cos \nu\right]$$

$$- \left(\frac{b}{a}\right) \left(\sec \psi + \frac{r_\omega}{a}\right) \sin \beta - \frac{1}{3} \left[\left(\frac{b}{a}\right)^2 - 1\right]$$

$$- \left(\frac{\pi}{4}\right) \left(\sec^2 \psi + \frac{\pi^2}{48}\right) + \frac{\pi}{16} - 2 \sec \psi \frac{d + r_\omega}{a}$$

$$- \left(\frac{d + r_\omega}{a}\right)^2 \theta_1 - \frac{\pi}{4} \theta_1^2 + \frac{1}{3} \theta_1^3$$ \hspace{1cm} A(12)
11.2 Fourier coefficient of source volume velocity $U'(t)$.

From equation (35)

$$U'(t) = \omega a^2 W (B + 2C \theta_r + 3D \theta_r^2)$$

$$0 \leq \theta_r \leq \frac{\pi}{2}$$

$$= \omega a^2 W \left[ A_0 + \sum_{n=1}^{\infty} \left( A_n \sin 4n0_r + B_n \cos 4n0_r \right) r \right]$$

The constant term, $A_0$, does not contribute to the acoustic field and is therefore neglected. The constant $A_n$ is given by

$$A_n = \frac{4}{\pi} \int_{0}^{\pi/2} \left( B + 2C \theta_r + 3D \theta_r^2 \right) \sin 4n \theta_r d\theta_r$$

$$= \frac{4}{\pi} \left[ B \left( \frac{\cos 4n0_r}{4n} \right)_{0}^{\pi/2} + 2C \left( \frac{\cos 4n0_r}{4n} + \frac{\sin 4n0_r}{16n^2} \right)_{0}^{\pi/2} \right]$$

$$+ 3D \left[ \frac{\theta_r}{4n} \cos 4n0_r + \frac{\theta_r}{8n^2} \sin 4n0_r + \frac{\cos 4n0_r}{32n^3} \right]_{0}^{\pi/2} \right]$$

$$= - \frac{1}{2n} \left[ 2C + \frac{\pi}{2} \right] 3D$$

$$= - \frac{1}{2n} \left[ - \frac{\pi}{2} + \frac{\pi}{2} \right] \quad \text{[From equations (24), (25)]}$$

$$= 0$$

The constant $B_n$ is given by

$$B_n = \frac{4}{\pi} \int_{0}^{\pi/2} \left( B + 2C \theta_r + 3D \theta_r^2 \right) \cos 4n0_r d\theta_r$$
The source volume velocity is therefore given by

\[ U'(t) = \omega a^2 W \sum_{n=1}^{\infty} \frac{\cos 4nwt}{4n^2} \]

\[ A(16) \]
11.3 Series Combination of Duct Impedances. Figure 20.

The pressure and velocity, $p_n(x)$ and $u_n(x)$ at a point $x$ from discontinuity $(k-1)$ respectively are given by

\[ p_n(x) = X'e^{i(\omega t - knx)} + Y'e^{i(\omega t + knx)} \]  \hspace{1cm} A(17)

\[ u_n(x) = \frac{X'e^{i(\omega t - knx)}}{\rho_o a_o S(k-1)} - \frac{Y'e^{i(\omega t + knx)}}{\rho_o a_o S(k-1)} \]  \hspace{1cm} A(18)

or in matrix form,

\[
\begin{bmatrix}
    p_n(x) \\
    u_n(x)
\end{bmatrix} =
\begin{bmatrix}
    e^{-iknx} & e^{iknx} \\
    \frac{e^{-iknx}}{\rho_o a_o S(k-1)} & \frac{e^{iknx}}{\rho_o a_o S(k-1)}
\end{bmatrix}
\begin{bmatrix}
    X' \\
    Y'
\end{bmatrix} e^{i\omega t} \hspace{1cm} A(19)
\]

(\text{where } X' \text{ and } Y' \text{ are constants}).

The acoustic fields at discontinuities $(k-1)$ and $(k)$ are respectively given by:

\[
\begin{bmatrix}
    p_n(k-1) \\
    u_n(k-1) \\
    p_n(k) \\
    u_n(k)
\end{bmatrix} =
\begin{bmatrix}
    1 & 1 \\
    \frac{1}{\rho_o a_o S(k-1)} & \frac{1}{\rho_o a_o S(k-1)} \\
    e^{-ikn&(k-1)} & e^{ikn&(k-1)} \\
    \frac{e^{-ikn&(k-1)}}{\rho_o a_o S(k-1)} & \frac{e^{ikn&(k-1)}}{\rho_o a_o S(k-1)}
\end{bmatrix}
\begin{bmatrix}
    X' \\
    Y'
\end{bmatrix} e^{i\omega t} \hspace{1cm} A(20)
\]

From equation A(21)
\[
\begin{bmatrix}
    X' \\
    Y'
\end{bmatrix} e^{i\omega t} =
\begin{bmatrix}
    e^{ik_n l(k-1)} & e^{ik_n l(k-1)} \\
    e^{-ik_n l(k-1)} & e^{ik_n (k-1)} \\
\end{bmatrix}^{-1}
\begin{bmatrix}
    p_n(k) \\
    u_n(k)
\end{bmatrix}
\]

\[
= \frac{1}{2}
\begin{bmatrix}
    e^{ik_n l(k-1)} & \rho_o a_o S(k-1) e^{ik_n l(k-1)} \\
    e^{-ik_n (k-1)} & \rho_o a_o S(k-1) e^{-ik_n l(k-1)} \\
\end{bmatrix}
\begin{bmatrix}
    p_n(x) \\
    u_n(k)
\end{bmatrix}
\]

By substituting for \( \begin{bmatrix} X' \\ Y' \end{bmatrix} e^{i\omega t} \) in equation A(20), we get

\[
\begin{bmatrix}
    p_n(k-1) \\
    u_n(k-1)
\end{bmatrix} =
\begin{bmatrix}
    1 & 1 \\
    \frac{1}{\rho_o a_o S(k-1)} & \frac{1}{\rho_o a_o S(k-1)}
\end{bmatrix}^{-1}
\begin{bmatrix}
    e^{ik_n l(k-1)} \\
    e^{-ik_n l(k-1)}
\end{bmatrix}
\begin{bmatrix}
    p_n(k) \\
    u_n(k)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    \cos [k_n l(k-1)] & \rho_o a_o S(k-1) \sin [k_n l(k-1)] \\
    i \frac{\sin [k_n l(k-1)]}{\rho_o a_o S(k-1)} & \cos [k_n l(k-1)]
\end{bmatrix}
\begin{bmatrix}
    p_n(k) \\
    u_n(k)
\end{bmatrix}
\]

A(23)
The pressures and velocities are respectively given by:

\[
p_n(k-1) = \begin{bmatrix} \cos[k_n(k-1)] & i \rho_o a_o S(k-1) \sin[k_n(k-1)] \\ i \sin[k_n(k-1)] & \cos[k_n(k-1)] \end{bmatrix} \begin{bmatrix} p_n(k) \\ u_n(k) \end{bmatrix}
\]

\[
u_n(k-1) = \begin{bmatrix} i \sin[k_n(k-1)] \\ \rho_o a_o S(k-1) \end{bmatrix} \begin{bmatrix} p_n(k) \\ u_n(k) \end{bmatrix}
\]

By writing \(Z_n(k-1) = \frac{p_n(k-1)}{u_n(k-1)}\):

\[
Z_n(k) = \frac{p_n(k)}{u_n(k)} ; \quad \begin{bmatrix} p_n(k) \\ u_n(k) \end{bmatrix} = \begin{bmatrix} Z_n(k) \\ 1 \end{bmatrix} u_n(k) ,
\]

equations A(24) and A(25) give

\[
Z_n(k-1) = \begin{bmatrix} \cos[k_n(k-1)] & i \rho_o a_o S(k-1) \sin[k_n(k-1)] \\ i \sin[k_n(k-1)] & \cos[k_n(k-1)] \end{bmatrix} \begin{bmatrix} Z_n(k) \\ 1 \end{bmatrix}
\]

\[
= \rho_o a_o S(k-1) \frac{Z_n(k) + i \rho_o a_o S(k-1) \tan[k_n(k-1)]}{\rho_o a_o S(k-1) + i Z_n(k) \tan[k_n(k-1)]}
\]

In the case of an open-ended uniform duct, \(Z_n(k) = 0\) and

\[
Z_n(k-1) = i \rho_o a_o S(k-1) \tan[k_n(k-1)]
\]

In the case of a uniform duct with a closed end, \(Z_n(k) = \infty\) and

\[
Z_n(k-1) = - i \rho_o a_o S(k-1) \cot[k_n(k-1)]
\]
11.4 R.M.S. Frequency Spectra of Unducted Meter.

Figures 8 - 12.
Noise spectrum outside outlet with no duct
attached \( r = 0.5 \) ft ; shaft speed 9.1 rev/sec
11.5 R.M.S. Frequency Spectra of Meter with Simple Uniform Duct.

Figures 15 - 19.
11.6 R.M.S. Frequency Spectra of Meter with Uniform Duct and Side-branch.

Figures 21, 22.
10.2 rev/s
11.7 R.M.S. Frequency Spectra of Meter with Complex Duct.

Figures 26 - 30.
R.M.S. Frequency Spectra of Meter with Uniform Duct, Side-branch and Concentric Helmholtz Resonators.

Figure 35.
Without silencer

3.7 rev/s
... Without silencer

4.5 rev/s
... Without silencer  Attenuation: 8 dB
5.4 rev/s
... Without silencer

7.5 rev/s
... Without silencer

8.0 rev/s
... Without silencer
8.5 rev/s
An acoustic model is developed to predict the acoustic pressure waves generated by rotary positive displacement gas meters. The meter is approximated, in the long wavelength limit, to a small, rigid and circular piston, pulsating with harmonic time dependence inside an infinite baffle. The acoustic source volume velocity is first derived from the geometry of the measuring part of the meter. Rayleigh's formulation is then applied to derive the acoustic pressure fields of the meter for both the unducted and the ducted situations. The effects of the self-induced pressure waves on the registration of the meter are then considered. The performance of the meter is dependent on the configuration of the pipe network to which the meter is attached. The meter calibration curve can be seriously distorted by the self-induced acoustic pressure waves. The effectiveness of concentric Helmholtz type resonators to suppress undesired sound pressure waves in meter systems is discussed. Experimental verification shows that the acoustic model developed is adequate to predict the "a.c" response of rotary positive displacement gas meters for most of the common pipe elements. Linear plane wave theory is used throughout and the effects of flow and friction are assumed to be negligible.