VIBRATIONS OF PRELOADED CYLINDRICAL SHELLS

by

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The text is not legible due to the handwriting style.
The work for this dissertation was started at Cambridge University from October 1966 to January 1968. Between January 1968 and March 1970 the work was carried out at Leicester University. The change of University was due to the appointment of my supervisor, Professor F.A. Leckie, to a Chair of Engineering in the University of Leicester.

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This dissertation describes the original work of the author and no part of it has been submitted to any other University.

O.F. Palacios
SUMMARY

A theoretical and experimental investigation of the dynamic behaviour of preloaded cylindrical shells including the effects of meridional cracking has been carried out.

A Donnell-type equation is derived to study preloaded cylindrical shell vibrations. The solution is obtained using the Galerkin Method for the following initial external loads:

(i) Axial compression combined with torsion
(ii) Bending moment
(iii) Axial compression combined with bending moment
(iv) Periodic axial compression
(v) Periodic bending moment.

A simply supported cylindrical shell was tested under axial compression, bending moment and axial compression combined with bending moment. The results are in fair agreement with the present theoretical solution.

The analytical study of the vibrations of cracked shells is carried out by introducing the modifications to the strain and kinetic energy functions expressed in terms of normal co-ordinates. It is shown that cracks reduce the natural frequency and change the nodal configuration associated with the lowest natural frequency. The results of the experiments are in excellent agreement with the theoretical predictions.
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\[ k = \frac{k^2}{12a^2} \]

Changes in curvature in directions x and y

\[ k_1, k_2 \]

Twist

\[ l \]

Length of shell

\[ m \]

Number of waves in the axial direction

\[ M_x, M_\phi \]

Resultant bending moment per unit length

\[ n \]

Number of waves in the circumferential direction

\[ N \]

Resultant normal stress per unit length

\[ P_x \]

Axial stress factor, \( \sigma_x \frac{1-t^2}{E} \)

\[ P_\phi \]

Tangential stress factor, \( \sigma_\phi \frac{1-t^2}{E} \)

\[ P_l \]

Maximum stress factor in bending

\[ P_o \]

Constant axial stress factor

\[ P^* = \frac{k(\lambda^2 + \nu^2)^{1/2} + \lambda^3 (1-\nu^2)}{\lambda^2 (\lambda^2 + \nu^2)^2} \]

Critical buckling load of a cylindrical shell

\[ Q_x, Q_\phi \]

Shear stress resultants

\[ Q \]

Operator defined in (4.11)
S罢
\[ S \text{ Total strain energy at any displacement} \]

\( S_b \)
\[ \text{Strain energy due to bending} \]

\( S_s \)
\[ \text{Strain energy due to stretching} \]

\( t \)
\[ \text{Time} \]

\( T \)
\[ \text{Total kinetic energy at any displacement} \]

\( u,v,w \)
\[ \text{Component displacements of a point on the middle surface} \]

\( U,V,W \)
\[ \text{Dimensionless displacements} \]

\( x,y,z \)
\[ \text{Co-ordinate distances in axial, circumferential and radial directions.} \]

\( a \)
\[ \text{Subtended angle between two cracks} \]

\( Y \)
\[ \text{Shear strain in middle surface} \]

\[ \Delta = \frac{\rho(1-v^2)a^2}{E} \]
\[ \epsilon_1, \epsilon_2 \]
\[ \epsilon_X, \epsilon_\phi, \epsilon_{x\phi} \]
\[ \eta_b \]
\[ \eta_s \]
\[ \lambda = \frac{\pi \sigma a}{E} \]
\[ \mu = \frac{P_t}{2(P^* - P_0)} \]
\[ \rho \]
\[ v \]
\[ \sigma_x, \sigma_y \]
\[ \sigma_{xy} \]
\[ c^0 \]
\[ \text{Initial static membrane stresses due to the external static loading} \]
Stresses due to shell vibrations

θ  Frequency of external force

ϕ  Angular co-ordinate

ψ  Subtended angle between supports of an arch

τ  Shear stress factor, $\sigma_x \frac{1-\nu^2}{E}$

ω  Circular frequency, $2\pi f$

$\omega = \sqrt{\frac{P_o}{P^2}}$  Circular frequency of a cylinder under axial load $P_o$

$\Omega = \omega^2 \Delta$  Frequency factor

χ  Normal co-ordinate

$v^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \phi^2}$

$v^4 = \frac{3}{h_x} + 2 \frac{3}{h_x^2} \frac{\partial^2}{\partial x \partial \phi^2} + \frac{3}{h_\phi^2}$

$v^{-4} = \text{inverse of } v^4$  Defined by $v^{-4}v^4 = w$
CHAPTER I

VIBRATION OF CYLINDERS

I.1 Introduction

Interest in the present work was stimulated by the collapse of three cooling towers during gale force winds on the 1st November 1965 at Ferrybridge in Yorkshire. Although the report of the Committee of Enquiry stated that the primary cause of failure was due to "tensile failure within the shell fabric", the possible importance of vibration as a contributory cause of the failure remained uncertain. The natural frequencies of the shells were found to be significantly higher than the vortex shedding frequency at the time of failure, but preload and damage due to cracking were not included in the analysis. The present work is concerned with the influence of preload and meridional cracking on the natural frequencies of cylindrical shells.

I.2 Historical Review

Lord Rayleigh (1894) derived an approximate expression for the natural frequencies of vibration of cylindrical shells. He considered the vibration theory of thin shells to be of two separate types:

(a) The inextensional mode - bending was considered as the predominant straining action and stretching was ignored.

(b) The extensional mode for which only stretching was considered.

He presented the extensional vibrations of an infinitely long cylindrical shell in detail.
Love (1927) formulated a general dynamic theory of cylindrical shells which included both bending and extensional deformations. Even though Love described the method of attack, he only discussed the particular cases of extensional vibration for cylindrical shells having finite length.

Flügge (1934) derived a set of cylindrical shell equations which included bending terms up to the second order in shell thickness. He did not solve the problem in its most general form, but suggested a solution for a simply supported cylindrical shell, in the form of trigonometric functions which satisfied the boundary conditions. This general method outlined by Flügge requires numerical evaluation of an eighth-order determinant to find its eigenvalues. This is certainly the reason that this approach was not feasible before the advent of high-speed digital computers.

Timoshenko (1940) derived a similar set of equations but he was only concerned with the static analysis of thin cylindrical shells.

Arnold and Warburton (1948) investigated the vibrations of cylindrical shells both theoretically and experimentally. Their experimental results showed excellent agreement with the theory for the range of parameters considered. Arnold and Warburton assumed trigonometric displacement functions which satisfied the boundary conditions and derived expressions for the strain and kinetic energies in terms of the three displacements and their derivatives. They used Timoshenko strain-displacement relations. Lagrange's equations were written in terms of the three displacements and their derivatives and from these equations a cubic frequency equation was obtained.
Donnell (1933) derived a simple set of equations which he used to investigate the stability of thin-walled circular cylinders under torsion and obtained an acceptable agreement with experiment.

The solution of problems of equilibrium, stability and dynamics of cylindrical shells have been given by a number of authors using Donnell type equations. However, in view of the simplifications made by this author during the formulation the range of applicability is limited and the method can be considered an approximate one only.

Donnell's equations with the addition of inertia terms, yields a simplified method of analysis for vibrating cylindrical shells. Yu (1956) used this method and was able to obtain simple expressions for the radial frequencies of clamped and simply supported cylinders. He assumed the displacement components to be in the form of trigonometric functions and made the same important simplification used by Donnell (\(| \frac{1}{n^2} | << 1\)) i.e. he restricted the analysis to a large number of circumferential waves relative to the number of axial waves. By means of the simplification mentioned above the eighth-degree auxiliary equation is replaced by another of fourth-degree which can then be more readily solved.

Morley (1959) modified Donnell's equation by adding two extra terms to the eighth-order differential equation given by Donnell in the static analysis of thin cylindrical shells. The solution to this modified equation was very close to that of Flügge's equation.

During the past few years Donnell's equations have enjoyed wide usage because they can be manipulated easily but
as an approximate solution they were a subject of controversy. Forsberg (1966) was able to get the exact solution to the problem of cylindrical shell vibration and gave the range of applicability of the approximate solution given by Donnell using the exact solution as a base for comparison. The comparisons were made on the basis of natural frequency, mode shape and nodal force distribution. For the exact solution, Forsberg considered the equations of motion developed by Flügge and assumed the displacements to be given by trigonometric functions which satisfy the boundary conditions. Substitution of the assumed solution into the set of differential equations yielded eight simultaneous algebraic equations whose determinant will vanish for each natural frequency \( \omega \). Thus the natural frequency associated with a given shell geometry i.e., \( a/h \) and \( k/a \) and a chosen circumferential wave number \( n \), may be determined. He covered the entire range of problems of interest by varying the initial input to the determinant, i.e., by varying \( a/h, k/a, n, \) or the boundary conditions. The result of his work showed that the Donnell's equations are valid for all \( n \) and all \( a/h \) if attention is restricted to shells having \( \frac{E}{\rho a} < 20 \). The error incurred in the frequency rapidly diminishes as \( \frac{k}{\rho a} \) decreases.

Little attention has been given to the problem of vibrations of cylindrical shells subjected to initial stress although Flügge, Timoshenko and Donnell have considered in their work terms involving external forces in the radial, axial and circumferential directions.

Federhofer (1936) investigated the particular case of \( n = 0 \) for vibrations of a cylindrical shell subjected to an axial compressive load and showed that the natural frequency
decreases as the axial load increases. (Such behaviour where frequency decreases with increasing axial load is well known for beam-columns and plates.) Koval (1962) studied the vibrations of a cylinder subjected to an initial torque. He used three differential equations of the Donnell type under the same Donnell assumption used by Yu and showed that the frequency tends to zero as the torque tends to its critical value in buckling with a certain buckling pattern. This buckling pattern depends on the geometric characteristics of the cylinder.

Herrmann and Armenakas (1962), on the basis of the non-linear three dimensional theory of elasticity, formulated several linear theories of the motion of elastic cylindrical shells subjected to a general state of initial stress. These theories are applicable to a variety of problems involving initial stress (bending moment, transverse shear forces, twist moments, membrane forces) but no solutions have been given.

1.3 Discussion

Theories for the vibrations of cylinders have been formulated under different assumptions by the authors mentioned above. Love, Flügge and Timoshenko made different approximations as regards strain ($\epsilon$) and changes of curvature ($k$). The discrepancies occur only in terms which have little numerical significance. Love and Flügge both take into account the trapezoidal shape of the faces perpendicular to the cylinder axis (Fig. 2.1b). Timoshenko assumes a square instead of the trapezoidal shape. In view of the assumptions and approximations made by each of these three authors during the formulation the system of equations are not the same.
Donnell presents the formulation considering the expressions for the inextensional and extensional strain in the middle surface of the shell to be the same as those for the case of a flat plate with the addition of the term $\frac{\theta}{a}$ to the expression for the strain in the circumferential direction.

The main advantage of Donnell's equations is that they are more manageable than any other set, since the three simplified equations can be transformed into a single eighth-order partial differential equation in $w$ in which the effects of the displacements $u$ and $v$ are properly taken into account. Due to this advantage, the solution to the mathematical problem yields compact expressions, so, in practice, this formulation is preferable if their accuracy is satisfactory for the purposes of engineering applications.

Morley improved Donnell's equation for the static analysis of cylindrical shells. He removed the source of the error and yet retained the essential simplicity in the eighth-order differential equation.

So far, to the author's knowledge, no research has been done into the dynamic analysis of cylinders using Morley's equation.

I.4 Scope of the present work

To the author's knowledge no investigation of the vibrations of a finite cylindrical shell subjected to an initial axial load has been published. The first part of the present work considers this problem. It is intended to find the dynamic response of any thin cylindrical shell with respect to its geometrical characteristics $(\frac{\ell}{a}, \frac{t}{a})$ and nodal pattern $(m, n)$.

The method of solution consists of the two following phases:
(1) Solution of a Donnell type equation.

A Donnell type equation is first derived and solved in detail. It is possible to find a solution in closed form in the case of a simply supported cylinder. The advantages in this case are:
(i) The problem is represented by relatively simple equations.
(ii) The solution does not require laborious computations and accurate results are obtained if the cylinder satisfies the following condition: \( \frac{L}{ma} < 20 \).

(2) Solution of a Morley equation.

Bearing in mind that Donnell's equation is only valid for cylinders having \( \frac{L}{ma} < 20 \), Morley's equation, with the addition of the inertia terms is used to investigate the dynamic behaviour of cylinders for which \( \frac{L}{ma} > 20 \). It is shown that there are no limitations on Morley's equation, i.e. the solution of Morley's equation gives very good approximation for the whole range of parameters \( (\frac{L}{a}, \frac{L}{a}, m, n) \).

It may be mentioned that it is essential to present the solution of Donnell's equation since for certain cases this solution is better than that of Morley in the range \( 0 < \frac{L}{a} < 20 \).

In the second part of this work, the previous analysis is extended to cover several other types of initial stresses. These are axial compression combined with torsion, bending moment, axial compression combined with bending moment, periodic axial load and periodic bending moment.

Even though a different type of initial load changes the differential equation, it is proved that the Galerkin method can successfully be used for this type of problem.
The final part of this work is devoted to the investigation of the effects of cracks on vibrating cylinders. This has hitherto received no attention. Two techniques are suggested for solving the analytical problem:

(I) Use of Galerkin's method.

The method is applied in the usual manner. The most difficult and essential part of this method is to predetermine suitable displacement functions which satisfy the boundary conditions since a crack in the circumferential direction imposes the conditions that the bending moment and shear force should vanish ($M_\phi = Q_\phi = 0$). For a large number of cracks the solution to the problem becomes extremely laborious.

(II) Use of strain and kinetic energy expressions.

This technique makes use of Rayleigh's principle. The expressions for the strain and kinetic energy are those of a cracked cylinder expressed in normal coordinates. This method is shown to be very successful for a fairly large number of cracks.
II.1 Theoretical Background

II.1a Modes of free vibration

The modes of vibration of rotationally symmetric shells may be classified in terms of axial and circumferential wave numbers. For axisymmetric shells with homogeneous boundary conditions the variables (in the field equations) may be considered separable with respect to circumferential wave number $n$. This assumption has been confirmed by the experimental observation of various authors (Arnold and Warburton (1948) and (1951), Koval (1961), etc.).

The three component displacement vectors $u$, $v$ and $w$ along the axis $x$, $y$ and $z$ respectively (Fig. 2.1 a, b) may be considered to be of the form

$$u(\phi, x, t) = U_n(x) \cos n\phi \cos \omega t$$
$$v(\phi, x, t) = V_n(x) \sin n\phi \cos \omega t$$
$$w(\phi, x, t) = W(x) \cos n\phi \cos \omega t$$

The circumferential displacement shapes are always assumed to be a harmonic function of the circumferential coordinate $\phi$. Typical circumferential displacement shapes are shown in Fig. 2.2 a.

The form of the axial displacement functions $U_n(x)$, $V_n(x)$, $W_n(x)$ will be dependent upon the boundary restraints at the edges of the shell. Typical modes of axial vibration are shown in Fig. 2.2 b and 2.2 e.
For a simply supported cylindrical shell the axial displacements is frequently assumed to be of the form

\[ U_n(x) = \cos \frac{m \pi x}{L} \]
\[ V_n(x) = \sin \frac{m \pi x}{L} \]
\[ W_n(x) = \sin \frac{n \pi x}{L} \]

The mode of vibration is therefore seen to be characterised by the circumferential wave number \( n \) and the axial wave number \( m \). For each modal configuration \((n, m)\) there are found to be three orthogonal modes of vibration each of which is associated with a mode dominated by displacements in the \( u, v, w \) directions respectively. The present work is only concerned with modes of vibration associated with displacements which are largest in the \( w \) direction. The other two sets of modes have very high frequencies and are not considered relevant to the present problem, and furthermore the method of analysis employed is such that the frequency equation is reduced to one which only gives the modes associated with predominantly radial displacements.

It has been found (Arnold and Warburton 1948) that for a fixed number of circumferential waves \( n \) the frequency increases monotonically with an increasing number of axial waves; such a behaviour is similar to that of vibrations of beams and plates. This holds true for all the range of shell parameters \((r/a, \ell/a)\) and for all boundary conditions. In contrast to this, the value of \( n \) which corresponds to a mode shape having the minimum frequency depends strongly upon the length to radius ratio of the shell.
II.1 b Differential Equations

In the present study Donnell type equations will be used which are derived using Hamilton's principle. Donnell's equations have had wide usage in static and dynamic analysis of cylindrical shells but were often limited to ranges in which the number of circumferential waves (n) is large; however, Forsberg (1965) has shown that such equations are valid for all n and all values of $a/h$ if attention is restricted to shells having $\frac{a}{ma} < 20$.

It is shown in the present work that the Morley equations can be used when the values of $\frac{a}{ma}$ is greater than 20 and the results are excellent for engineering purposes. The advantage of using the Morley equations is that they are as manageable as those of Donnell.

II.1 c Effects of In-plane Inertia Terms

Forsberg (1965) also studied the significance of in-plane inertia terms in the equation of motion.

The effects of neglecting in-plane inertia terms are:

(i) The frequency equation reduces from a cubic to a first order expression. Two of the three frequencies for a given nodal pattern are omitted.

(ii) The remaining frequency which is associated with radial motion only increases in magnitude. The magnitude of the error in the frequency depends upon n, but it is insensitive to $a/h$ and $l/a$. For large values of n (say $n > 10$) there is no significant effect of in-plane inertia. The maximum error being of the order of 4% in general, arises when $n = 2$, and for practical purposes the minimum frequency occurs when $n > 2$. Aware of this fact, the in-plane inertia terms will be neglected.
II.2 Assumptions

Assumptions play an important role in this analysis and the following, which are normal in the theory of elasticity and the theory of static and dynamics of thin shells, will be made.

1. The material is perfectly elastic, homogeneous and isotropic.
2. The shell is exactly cylindrical. Thin shells, whether test specimens or part of real structures, can hardly be expected to be perfectly cylindrical. Imperfections of shape are not a matter of great concern with regard to free vibration, however it is of importance in the case of dynamic stability.
3. The shell is thin with constant thickness. A shell is said to be thin if the value of the ratio \( \frac{h}{a} \) can be neglected in comparison with unity; as a rule, the theory will be applied only to shells whose thickness is less than one tenth of the radius of the cylinder.
4. The deflections are small in comparison with the thickness of the shell. If this limitation is not imposed, the governing differential equations become non-linear.
5. The straight lines in the cylinder wall which are perpendicular to the middle surface, remain straight and perpendicular to the middle surface, i.e. the distortion due to transverse shear is neglected.
6. Rotatory inertia of the shell wall will be neglected.
7. The normal stresses acting on planes parallel to the generators of the cylinder are neglected in comparison with the other stresses.
8. The in-plane inertia quantities \( \frac{\partial^2 u}{\partial t^2} \), \( \frac{\partial^2 v}{\partial t^2} \) in the directions of u and v respectively are neglected. The effect of the corresponding inertia forces on the frequency is very small in comparison with the radial inertia force.
II.3 Derivation of the Differential Equations of Motion

II.3 a The Coordinate System

A cylindrical shell of length $l$, thickness $h$ and mean radius $a$, is considered. An element of the cylinder in Fig. 2.1 b shows the coordinate system adopted; the $x$-axis is directed along the generator of the cylinder, $s$ is measured clockwise in the circumferential direction, and the $z$-axis is directed inward along the positive normal to the middle surface of the shell.

II.3 b Displacements

The deformed configuration of the shell is characterised by the displacements of its points lying on the middle surface. The components of displacement in directions $x$, $s$ and $z$ defining the deformed state of the cylinder are designated by $u$, $v$, and $w$ respectively.

II.3 c Stresses

The behaviour of the shell will be investigated under the action of external initial static load which causes a membrane state of stress. If displacements of points in the middle surface due to the membrane state of stress are denoted by $u_o$, $v_o$ and $w_o$, and displacements due to shell vibrations are denoted by $u'$, $v'$, $w'$. The total displacement vectors are therefore given by

$$u = u_o + u'$$
$$v = v_o + v'$$
$$w = w_o + w'$$

(2.1)
and the final state of stress

\[
\sigma_x = \sigma_x^0 + \sigma_x', \\
\sigma_\phi = \sigma_\phi^0 + \sigma_\phi', \\
\sigma_{x\phi} = \sigma_{x\phi}^0 + \sigma_{x\phi}'
\]

(2.2)

where \(\sigma_x^0, \sigma_\phi^0\) and \(\sigma_{x\phi}^0\) are the stresses due to external initial static load and \(\sigma_x', \sigma_\phi', \sigma_{x\phi}'\) are stresses due to vibration.

According to expressions (2.1), the final resulting equations will be found in two similar groups. The first one will contain static terms and the second one dynamic terms. The first, will be a set of equations which determines the equilibrium state, while the second will describe the small disturbance about this equilibrium position. Since the problem of interest is shell vibrations, the second set only will be considered. In what follows the primes have been omitted.

II.3d Hamilton's Principle

In an elastic shell continuously changing its state between time \(t_0\) to \(t_1\), Hamilton's principle states that \(I = 0\). \(I\) is a line integral along a dynamical path defined in the following form

\[
I = \int_{t_0}^{t_1} (T - S) \, dt
\]

(2.3)

where \(T\) and \(S\) are the kinetic and strain energy respectively.
II.3 Strain Energy

The total strain energy* for the deformed cylinder neglecting the trapezoidal form of the faces perpendicular to the x-axis is:

\[ S = \int \int \int (\sigma_x \varepsilon_x + \sigma_\phi \varepsilon_\phi + \sigma_{x\phi} \varepsilon_{x\phi}) a \, d\phi \, dx \, dz \]  

(2.4)

Now from Hooke's law

\[ \sigma_x = \sigma_0^x + \frac{E}{1-\nu^2} (\varepsilon_x + \nu \varepsilon_\phi) \]

\[ \sigma_\phi = \sigma_0^\phi + \frac{E}{1-\nu^2} (\varepsilon_\phi + \nu \varepsilon_x) \]  

(2.5)

\[ \sigma_{x\phi} = \sigma_{x\phi}^0 + \frac{E}{2(1+\nu)} \varepsilon_{x\phi} \]

Substituting from equations (2.5) in the strain energy expression (2.4)

\[ S = \int \int \int (\sigma_0^x \varepsilon_x + \sigma_0^\phi \varepsilon_\phi + \sigma_{0x\phi} \varepsilon_{x\phi}) a \, d\phi \, dx \, dz \]

\[ + \frac{E}{2(1-\nu^2)} \int \int \int (\varepsilon_x^2 + \varepsilon_\phi^2 + 2\nu \varepsilon_x \varepsilon_\phi) a \, d\phi \, dx \, dz \]  

(2.5)

* The expression for the total strain energy has been discussed by several authors Langar (1949), Bleich (1953), etc. who have considered the significance of each term in such an expression. In this analysis use will be made of those items that all authorities agree cannot be neglected.
If \( \varepsilon_1 \) and \( \varepsilon_2 \) are the strains in directions \( X \) and \( Y \), \( k_1, k_2 \) changes of curvature in direction \( X \) and \( Y \), \( \gamma \) shear strain, \( k_{12} \) twist. It can be written (Love (1892), p. 529)

\[
\varepsilon_x = \varepsilon_1 - zk_1
\]

\[
\varepsilon_\phi = \varepsilon_2 - zk_2
\]

\[
\varepsilon_{x\phi} = \gamma - 2zk_{12}
\]

At any instant, the strains and changes of curvature are given (Love (1892), p. 543 and Timoshenko (1940), p. 512) in terms of the displacements and their derivatives by

\[
\varepsilon_1 = \frac{3u}{3x}
\]

\[
\varepsilon_2 = \frac{1}{a} \frac{3v}{3\phi} - \frac{w}{a}
\]

\[
\gamma = \frac{3v}{3x} + \frac{1}{a} \frac{3u}{3\phi}
\]

\[
k_1 = \frac{3^2w}{3x^2}
\]

\[
k_2 = \frac{1}{a^2} \frac{3^2w}{3\phi^2}
\]

\[
k_{12} = \frac{1}{a} \frac{3^2w}{3x3\phi}
\]

Substituting from equations (2.8) into (2.7)

\[
\varepsilon_x = \frac{3u}{3x} - z \frac{3^2w}{3x^2}
\]

\[
\varepsilon_\phi = \frac{1}{a} \frac{3v}{3\phi} - \frac{w}{a} - \frac{z \frac{3^2w}{3\phi^2}}{a^2}
\]

\[
\varepsilon_{x\phi} = \frac{3v}{3x} + \frac{1}{a} \frac{3u}{3\phi} - \frac{2z \frac{3^2w}{3x3\phi}}{a}
\]
upon substitution of the above into (2.6) and integrating over
the thickness of the shell, the total expression for the strain
energy can be shown to be

\[
S = h \int_0^{2\pi a} \int_0^l \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial s} \right)^2 \right] + \sigma \left[ 3 \frac{\partial v}{\partial s} - \frac{\partial w}{\partial s} + \frac{1}{2} \left( \frac{\partial w}{\partial s} \right)^2 \right] + \\
+ \sigma \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial s} + \frac{\partial w}{\partial s} \right] dxds + \frac{Eh}{2(1-v^2)} \int_0^{2\pi a} \int_0^l \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial s} - \frac{\partial w}{\partial a} \right)^2 \\
+ 2v \frac{\partial u}{\partial x} \frac{\partial v}{\partial a} + \frac{1-v}{2} \left( \frac{\partial u}{\partial s} + \frac{\partial v}{\partial s} \right)^2 \right] dxds + \frac{Eh^3}{24(1-v^2)} \int_0^{2\pi a} \int_0^l \left( \frac{\partial ^2 w}{\partial x^2} \right)^2 \\
+ \left( \frac{\partial ^2 w}{\partial s} \right)^2 + 2v \frac{\partial ^2 w}{\partial x^2} \frac{\partial ^2 w}{\partial s} + 2(1-v) \left( \frac{\partial ^2 w}{\partial s^3} \right)^2 \right] dxds \quad (2.10)
\]

The variation in strain energy will be considered to
be the variation of each term in (2.10)

\[
\delta \int_A \frac{3u}{\partial x} dxds \quad (2.11)
\]

The double integration can be replaced by simple integrals
if it is remembered that for any function of x and s the following
relations hold

\[
\int_A \frac{3F}{\partial x} dxds = \int_A F \cos \alpha \, d\ell \quad (2.12)
\]

\[
\int_A \frac{3F}{\partial s} dxds = \int_A F \sin \alpha \, d\ell
\]
so that, using the first of equations (2.12), expression (2.11)
can be represented as follows

\[
\delta \left[ \frac{\partial u}{\partial x} \right]_{A} dxds = \int \frac{\partial (\delta u)}{\partial x} dxds
\]

\[
= \int \delta u \cos \alpha \, dl \quad \text{(2.13)}
\]

Consider the second term of (2.10)

\[
\delta \left[ \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right]_{A} dxds = \int \left( \frac{\partial w}{\partial x} \right) \frac{\partial (\delta w)}{\partial x} dxds
\]

\[
= \int \left[ \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial x} \delta w \right) - \frac{\partial^2 w}{\partial x^2} \delta w \right] dxds
\]

\[
= \int \frac{\partial w}{\partial x} \delta w \cos \alpha \, dl - \int \frac{\partial^2 w}{\partial x^2} \delta w dxds \quad \text{(a)}
\]

Proceeding in the same fashion with the following terms in (2.10)

\[
\delta \left[ \frac{1}{2} \left( \frac{\partial v}{\partial s} \right)^2 \right]_{A} dxds = \int \frac{\partial v}{\partial s} \delta w \sin \alpha \, dl - \int \frac{\partial^2 w}{\partial s^2} \delta w dxds \quad \text{(b)}
\]

\[
\delta \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial s} \right)_{A} dxds = \int \frac{\partial w}{\partial x} \delta w \cos \alpha \, dl + \int \frac{\partial w}{\partial s} \delta w \sin \alpha \, dl - \int \frac{\partial^2 w}{\partial x \partial s} \delta w dxds
\]

\[
= \int \frac{\partial w}{\partial x} \delta w \cos \alpha \, dl + \int \frac{\partial w}{\partial s} \delta w \sin \alpha \, dl - \int \frac{\partial^2 w}{\partial x \partial s} \delta w dxds \quad \text{(c)}
\]

\[
\delta \left( \frac{w^2}{2} \right)_{A} dxds = 2 \int \omega \delta w dxds
\]

\[
= \int \omega \delta w dxds \quad \text{(d)}
\]

\[
\delta \left( \frac{3v}{\partial s} \right)_{A} dxds = \int \frac{3v}{\partial s} \delta w dxds + \int \omega \sin \alpha \, dl - \int \frac{\partial 3v}{\partial s} \delta v dxds
\]

\[
= \int \frac{3v}{\partial s} \delta w dxds + \int \omega \sin \alpha \, dl - \int \frac{\partial 3v}{\partial s} \delta v dxds \quad \text{(d)}
\]
\[ \delta \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right|_A^{dxds} = 2 \int_A \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 (\delta w)}{\partial x^2} dxds \]

\[ = 2 \int_A \left[ \frac{\partial}{\partial x} \left( \frac{\partial^2 w}{\partial x^2} \right) \frac{\partial (\delta w)}{\partial x} - \frac{\partial^3 w}{\partial x^3} \frac{\partial (\delta w)}{\partial x} \right] dxds \]

\[ = 2 \int_A \left[ \frac{\partial^4 w}{\partial x^4} \left( \frac{\partial^2 w}{\partial x^2} \right) \frac{\partial (\delta w)}{\partial x} - \frac{\partial^3 w}{\partial x^3} \frac{\partial (\delta w)}{\partial x} \right] dxds + 2 \int_A \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial (\delta w)}{\partial x} - \frac{\partial^3 w}{\partial x^3} \frac{\partial (\delta w)}{\partial x} \right] \cos \alpha \, dt \]

(e)

It may be shown that

\[ \frac{\partial (\delta w)}{\partial x} = \frac{\partial (\delta w)}{\partial n} \frac{dn}{dx} + \frac{\partial (\delta w)}{\partial l} \frac{dl}{dx} = \frac{\partial (\delta w)}{\partial n} \cos \alpha = \frac{\partial (\delta w)}{\partial l} \sin \alpha \]

(f)

therefore

\[ \delta \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right|_A^{dxds} = 2 \int_A \frac{\partial^4 w}{\partial x^4} \delta wdxds + 2 \int_A \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial (\delta w)}{\partial x} - \frac{\partial^3 w}{\partial x^3} \frac{\partial (\delta w)}{\partial x} \right] \cos \alpha \, dt \]

- 2 \int_A \left( \frac{\partial^3 w}{\partial x^3} \right) \delta w \cos \alpha \, dt \]

(g)

Integrating by parts:

\[ \frac{\partial^2 w}{\partial x^2} \sin \alpha \cos \alpha \frac{\partial (\delta w)}{\partial l} \, dt = \frac{\partial^2 w}{\partial x^2} \sin \alpha \cos \alpha \delta w - \frac{\partial}{\partial l} \frac{\partial^2 w}{\partial x^2} \sin \alpha \cos \alpha \delta w \, dt \]

(h)
Bearing in mind that the first term on the right-hand side
of (h) vanishes, expression (g) becomes:

\[ \delta \left( \int_A \left( \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}^2} \right)^2 \delta x \delta s \right) = 2 \int_A \left( \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}^1} \delta \mathbf{x} \delta s + 2 \cos^2 \left( \frac{\partial \mathbf{w}}{\partial \mathbf{n}} \right) \delta \mathbf{t} \right) \]

\[ + 2 \left[ \frac{\partial}{\partial \mathbf{x}} \left( \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}^1} \sin \mathbf{a} \cos \mathbf{a} \right) \right] \delta \mathbf{w} \delta \mathbf{t} \quad (i) \]

Transforming in a similar way the variations of the
next terms in expression (2.10)

\[ \delta \left( \int_A \left( \frac{\partial^2 \mathbf{w}}{\partial \mathbf{s}^2} \right)^2 \delta x \delta s \right) = 2 \int_A \left( \frac{\partial^2 \mathbf{w}}{\partial \mathbf{s}^1} \delta \mathbf{x} \delta s + 2 \sin \mathbf{a} \cos \mathbf{a} \frac{\partial \mathbf{w}}{\partial \mathbf{n}} \delta \mathbf{t} \right) \]

\[ - 2 \left[ \frac{\partial}{\partial \mathbf{s}} \left( \frac{\partial^2 \mathbf{w}}{\partial \mathbf{s}^1} \sin \mathbf{a} \cos \mathbf{a} \right) \right] \delta \mathbf{w} \delta \mathbf{t} \quad (j) \]

\[ \delta \left( \int_A \left( \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}^1 \partial \mathbf{s}^1} \right)^2 \delta x \delta s \right) = 2 \int_A \left( \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}^1 \partial \mathbf{s}^1} \delta \mathbf{x} \delta s + \left( \frac{\partial^2 \mathbf{w}}{\partial \mathbf{s}^1} \cos^2 \mathbf{a} + \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}^1} \sin^2 \mathbf{a} \right) \frac{\partial \mathbf{w}}{\partial \mathbf{n}} \delta \mathbf{t} \right) \]

\[ - \left\{ \frac{\partial^3 \mathbf{w}}{\partial \mathbf{x}^2 \partial \mathbf{s}^1} \sin \mathbf{a} + \frac{\partial^3 \mathbf{w}}{\partial \mathbf{s}^1 \partial \mathbf{s}^2} \cos \mathbf{a} + \frac{\partial}{\partial \mathbf{s}} \left[ \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}^1 \partial \mathbf{s}^2} - \frac{\partial^2 \mathbf{w}}{\partial \mathbf{s}^2} \right] \sin \mathbf{a} \cos \mathbf{a} \right\} \delta \mathbf{w} \delta \mathbf{t} \quad (k) \]
\[ \delta \left( \frac{\partial w}{\partial x} \right)^2 \delta x ds = 2 \int_{A} \frac{\partial w}{\partial x} \delta w dx ds + 2 \int_{A} \frac{\partial^2 w}{\partial x^2} \sin \alpha \cos \alpha \frac{\partial (\delta w)}{\partial \alpha} \, ds \]

\[ + \int \left\{ \frac{\partial}{\partial \beta} \left[ \frac{\partial^2 w}{\partial x \partial \beta} \right] (\sin^2 \alpha - \cos^2 \alpha) \right\} - \frac{\partial^3 w}{\partial x \partial \beta^2} \cos \alpha - \frac{\partial^3 w}{\partial x^2 \beta} \sin \alpha \right\} \delta w \, ds \]

(2)

II.3 f **Kinetic Energy**

The kinetic energy at any instant is given by

\[ T = \frac{\rho h}{2} \int_{A} (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) dA \]

(2.14)

In which the dot indicates the derivative with respect \( t \).

The variation in kinetic energy is expressed:

\[ \delta \int_{t_0}^{t_1} T dt = \delta \int_{t_0}^{t_1} \left\{ \frac{\rho h}{2} \int_{A} (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) dA \right\} dt \]

(2.15)

\[ = \int_{t_0}^{t_1} \left\{ \frac{\rho h}{2} \int_{A} \dot{u}^2 \delta u dA + \frac{\rho h}{2} \int_{A} \dot{v}^2 \delta v dA \right\} dt \]

(2.16)

Integrating by parts and bearing in mind that:

\[ \left. \delta u \right|_{t_0}^{t_1} = \left. \delta v \right|_{t_0}^{t_1} = \left. \delta w \right|_{t_0}^{t_1} = 0 \]

(2.17)
the variation in kinetic energy finally can be expressed in the following form:

\[
\delta \int_{t_0}^{t_1} T \, dt = -\rho \int_{t_0}^{t_1} \ddot{u} \, \delta u \, dA - \rho \int_{t_0}^{t_1} \ddot{\sigma} \, \delta \sigma \, dA - \int_{t_0}^{t_1} \ddot{\psi} \, \delta \psi \, dA
\]

Once the variation in strain and kinetic energy have been found, the final result is available from Hamilton's Principle given by equation (2.3):

\[
-\rho \int_{t_0}^{t_1} \ddot{u} \, \delta u \, dA - \rho \int_{t_0}^{t_1} \ddot{\sigma} \, \delta \sigma \, dA - \int_{t_0}^{t_1} \ddot{\psi} \, \delta \psi \, dA
\]

\[
-\frac{E}{2(1-\nu^2)} \left\{ -2 \int_{t_0}^{t_1} \frac{\partial \psi}{\partial x} \delta u \, dA - 2 \int_{t_0}^{t_1} \frac{\partial^2 \psi}{\partial x^2} \delta u \, dA - 2 \int_{t_0}^{t_1} \frac{3\psi}{3x} \delta u \, dA - 2 \int_{t_0}^{t_1} \frac{\partial^2 \psi}{\partial x^2} \delta u \, dA - 2 \int_{t_0}^{t_1} \frac{3\psi}{3x} \delta u \, dA \right\}
\]

(2.18)
The expression (2.19) provides the equilibrium equations and the boundary conditions. The equilibrium equations are:

\[
\begin{align*}
\frac{3^2 u}{2} + \frac{1-v}{2} & \frac{3^2 u}{\partial s^2} + \frac{1+v}{2} \frac{3^2 v}{\partial x^2} - \nu \frac{3^2 w}{\partial s^2} - \frac{3^2 w}{\partial x^2} = \frac{\rho (1-v^2)}{E} \frac{3^2 u}{\partial t^2} \\
\frac{1+v}{2} & \frac{3^2 u}{\partial x^2 s} + \frac{1-v}{2} \frac{3^2 v}{\partial x^2} + \frac{3^2 v}{\partial s^2} - \frac{1}{a} \frac{3^2 w}{\partial s^2} = \frac{\rho (1-v^2)}{E} \frac{3^2 v}{\partial t^2} \\
\frac{1-v^2}{E} & \phi \frac{3^2 w}{\partial x^2} + \frac{1-v^2}{E} \frac{3^2 w}{\partial s^2} + \frac{1-v^2}{E} \frac{3^2 w}{\partial x^2 s} + \frac{1}{a} \frac{3^2 u}{\partial s^2} + \frac{1}{a} \frac{3^2 v}{\partial x s} - \frac{v}{a^2} \frac{h^2}{12} \frac{3^2 w}{\partial t^2} = \frac{\rho (1-v^2)}{E} \frac{3^2 w}{\partial t^2}
\end{align*}
\]
These equations may be put into dimensionless form by means of the substitution $u = \frac{u}{a}$, $v = \frac{v}{a}$, $w = \frac{w}{a}$, $\psi = \frac{\psi}{a}$ and using the notation

$$\frac{1-\nu^2}{E} c_x^0 = P_x \quad \frac{1-\nu^2}{E} c_\phi^0 = P_\phi \quad \frac{1-\nu^2}{E} c_{x\phi}^0 = \tau$$

$$k = \frac{h^2}{12a^2} \quad \Delta = \frac{p(1-\nu^2)a^2}{E} \quad \nu^4 \left( \frac{\frac{\partial^2 w}{\partial x^2}}{\frac{\partial^2 u}{\partial x^2}} + \frac{\frac{\partial^2 w}{\partial \phi^2}}{\frac{\partial^2 \psi}{\partial \phi^2}} \right)$$

one obtains the following set of equations:

$$\frac{3^2 u}{3x^2} + \frac{1-\nu^2}{2} \frac{\frac{\partial^2 u}{\partial x^2}}{\frac{\partial^2 u}{\partial x^2}} + \frac{1+\nu}{2} \frac{\frac{\partial^2 v}{\partial x^2}}{\frac{\partial^2 \psi}{\partial x^2}} - \frac{3w}{3x} = \Delta \frac{\partial^2 u}{\partial t^2}$$

$$\frac{1+\nu}{2} \frac{\frac{\partial^2 u}{\partial x^2}}{\frac{\partial^2 x^2}} + \frac{1-\nu^2}{2} \frac{\frac{\partial^2 v}{\partial x^2}}{\frac{\partial^2 \psi}{\partial x^2}} + \frac{\frac{\partial^2 v}{\partial \phi^2}}{\frac{\partial^2 \psi}{\partial \phi^2}} - \frac{3w}{3\phi} = \frac{\partial^2 v}{\partial \phi^2}$$

$$P_x \frac{\frac{\partial^2 w}{\partial x^2}}{\frac{\partial^2 x^2}} + P_\phi \frac{\frac{\partial^2 w}{\partial \phi^2}}{\frac{\partial^2 \phi^2}} + 2\nu \frac{\frac{\partial^2 w}{\partial x^2}}{\frac{\partial^2 x^2}} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial \phi^2} - w - kv^4 = \frac{\partial^2 w}{\partial t^2}$$

They describe the general case of vibration of a cylindrical shell under initial stresses. It may be observed that when $P_x$ and $P_\phi$ are zero and the inertia terms are neglected, the set of equations reduces to that used by Donnell in his study of the torsional buckling of thin cylinders. Many other investigators (Love (1892), Flügge (1934), Naghdi (1953), Timoshenko (1936), Novozhilov (1959), Vlasov (1944) etc.) have developed differential equations describing the behaviour of thin shells similar to equations (2.22). All these formulations were developed using different strain-displacement relationships, consequently the resulting equations are not the same. Equations (2.22) can be transformed into a still more useful form by means of the following manipulations that were first carried out by Donnell.
By applying the operators \( \frac{\partial^2}{\partial x^2} \) and \( \frac{\partial^2}{\partial \phi^2} \) to the first equation in (2.22) and solving for the term in \( v \) in each case, and substituting these expressions in the equation obtained by applying \( \frac{\partial^2}{\partial x^2} \) to the second equation in (2.22) one may obtain the following equation

\[
\nu^h u = \nu^3 v - \nu^3 x \frac{\partial^3 v}{\partial x^3} \frac{\partial^2 v}{\partial \phi^2} \partial x
\]

Similarly we may obtain

\[
\nu^h v = (2 + v) \frac{\partial^3 v}{\partial x^2} \frac{\partial \phi}{\partial \phi} + \frac{\partial^3 v}{\partial \phi^3} \quad (2.23)
\]

Now, applying \( \frac{\partial}{\partial x} \) to the first of these equations and \( \frac{\partial}{\partial \phi} \) to the second, and substituting in the equation obtained by applying \( \nu^h \) to the third equation of (2.22), an equation is obtained from which both \( u \) and \( v \) have been eliminated.

\[
k^2 \nu^h w + \nu^h \left[ \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x \partial \phi} \right] + (1 - \nu^2) \frac{\partial^4 w}{\partial x^4} + \Delta \nu^h \frac{\partial^2 w}{\partial t^2} = 0 \quad (2.24)
\]

**II.4 Modified Donnell's equation**

Following Batdorf (1947) and multiplying equation (2.24) by the operator \( \nu^{-h} \), a modified Donnell's equation is obtained which is very useful particularly in the case of a simply supported cylindrical shell.

\[
k^2 \nu^h w - \left[ \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x \partial \phi} \right] + (1 - \nu^2) \nu^{-h} \frac{\partial^4 w}{\partial x^4} + \Delta \nu^{-h} \frac{\partial^2 w}{\partial t^2} = 0 \quad (2.25)
\]
This modified equation retains the advantages of Donnell's equation but is easier to solve as the eight-order differential equation has been reduced to one of fourth order. The problem therefore is to find a function of $w$ which satisfies

$$v^h(v^{-h}w) = w$$

where $v^{-h}w$ is a function originally suggested by Batdorf (1947). A satisfactory form of the function $v^{-h}w$ is not always easy to find. Batdorf has used this equation in the analysis of stability of cylinders with clamped as well as simply supported edges assuming trigonometric series for the deflection function $w$.

II.5 Morley's Equation

Morley (1959) has proposed a differential equation for static analysis of cylindrical shells. He in fact merely modified the Donnell equation by adding two extra terms neglected by Donnell. Rewriting Donnell's equation given by (2.24) for an unloaded shell and considering static analysis only

$$kv^h w + (1-v^2) \frac{\partial^4 w}{\partial x^4} = 0 \quad (2.25a)$$

Following the procedure pointed out by Donnell (described above), Morley reduced the set of three second order differential equations given by Flügge (1934) to the following eighth-order differential equation

$$kv^h (v^2 + 1)^2 w + (1-v^2) \frac{\partial^4 w}{\partial x^4} - 2k(1-v) \frac{\partial^2 w}{\partial x^2} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) w$$

$$+ k \left[ 2(1-v) \frac{\partial^4 w}{\partial x^4} - \frac{\partial^4 w}{\partial x^4} \right] = 0 \quad (2.25b)$$
By comparison of equations (2.25a) and (2.25b) a modified form of Donnell's equation is seen to be

\[ kV^h(v^2+1)^2w + (1-\nu^2) \frac{\partial^4 w}{\partial x^4} = 0 \]  

(2.25c)

Morley presented no analytical derivation of the equation but discussed the significance of the terms \(v^h w\) and \(2\nu^6 w\) that he added to Donnell's equation (2.25a) to produce the proposed equation (2.25c). Equations for \(u\) and \(v\) in (2.23) remained the same.

He solved the three equations (2.25a), (2.25b) and (2.25c) by assuming a solution to be of the form

\[ w = e^{P\phi}\cos\lambda x \]

and solving the resulting characteristic equations in \(P\).

Morley found that the roots of the characteristic equation obtained from Donnell's equation (2.25a) were significantly in error compared with the corresponding values obtained from Flügge's equation (2.25b) for the lower values of \(\lambda\), but that the roots obtained from Morley's equation (2.25c) were excellent approximations of the roots of Flügge's equation (2.25b). Up to the present time in the literature concerning cylindrical shells, Flügge's theory has been considered as an exact theory.

For the present work of dynamic analysis of cylindrical shells under initial stresses Morley's equation may be written

\[ kV^h(v^2+1)^2w + \nu^h \left[ \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x^2 \partial \phi} \right] + (1-\nu^2) \frac{\partial^4 w}{\partial x^4} + \Delta \nu^h \frac{\partial^2 w}{\partial t^2} = 0 \]  

(2.26)

In the following chapter, expression (2.26) will be investigated taking the solution of Flügge's equation as a base for comparison.
II.6 **Boundary Conditions**

The process of derivation of the differential equations of motion of a thin elastic cylindrical shell using Hamilton's principle, also yields force-displacements relations at the edges of the shell. The form of the displacements field which will in general satisfy these relations has not to the present author's knowledge been described and the present work is therefore confined to two sets of boundary conditions

(a) simply supported edges

\[ v = \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = 0, \ell \]  

(2.29)

and (b) clamped edges

\[ v = \frac{\partial w}{\partial x} = 0 \text{ at } x = 0, \ell \]  

(2.30)

It is important to call attention to the fact that the solution is not unique under the imposition of the above two ordinary boundary conditions (2.29) and (2.30) as four are required at each end of the cylinder. It implies then that two more boundary conditions at each edge should be prescribed, for example, one condition for \( u \) and one for \( v \) which will in fact defined completely the physical problem. Batdorf (1947) studied the stability of simply supported cylindrical shells using the modified Donnell's equation. He considered a solution for \( w \) using one or more terms of a Fourier series, and satisfying the boundary conditions term by term of the edges of the shell. The circumferential displacement \( v \) was found to be zero but the edges of the shell were free to warp in the axial direction, i.e. \( u \neq 0 \).
More recently Forsberg (1964) studied the influence of boundary conditions on the modal characteristics of thin cylindrical shells and showed that the condition placed on the longitudinal displacement $u$ in some cases was more influential than restrictions on the quantities $\frac{\partial w}{\partial x}$ and $\frac{\partial^2 w}{\partial x^2}$ corresponding to the slope and bending moment respectively.

In the present work, Donnell type equations (2.24), (2.25) and (2.26) together with the boundary conditions (2.29) and (2.30) will be used to solve the problem of vibrations of pre-stressed cylindrical shells.

II.7 Initial External Loads

The approximated field equations describing the dynamic behaviour of thin cylindrical shells under initial stress have been derived. The initial stress field has been expressed in terms of the stresses resultants $\sigma_x$, $\sigma_x^\phi$, $\sigma_\phi$.

States of pre-stress due to the following classes of loading can be considered:

(a) Axial load
(b) Bending moment
(c) Twisting moments
(d) Internal or external pressure

These loadings can be considered acting separately or simultaneously, and when the initial loads are time dependent the problem of dynamic stability arises. Some of these situations are considered in the following chapters of this thesis.
Fig. 2.1a  SHELL CO-ORDINATES AND DISPLACEMENTS

Fig. 2.1b  SHELL ELEMENT
CIRCUMFERENTIAL VIBRATION FORMS

\[ n = 0 \quad n = 1 \quad n = 2 \quad n = 3 \]

AXIAL VIBRATION FORMS

SIMPLY SUPPORTED ENDS

Clamped ends

Nodal pattern

\( n = 3 \quad m = 4 \)

Axial node

Circumferential node

Fig 2.2 FORMS OF VIBRATION OF THIN CYLINDERS
CHAPTER III

SOLUTION FOR CYLINDRICAL SHELLS
UNDER TORSION AND DIRECT AXIAL LOAD

III.1 Introduction

The problem of vibrations of thin cylindrical shells subjected to an initial state of stress has been shown to constitute an eigenvalue problem given by a set of three second order partial differential equations or by one partial differential equation of eighth-order. In both instances the homogeneous field equations have constant coefficients with homogeneous boundary conditions. It should be emphasized that the initial stress resultants $\sigma_x$ and $\sigma_{x\phi}$ are independent of the $x$ and $\phi$ co-ordinates and appear in the equations as constant coefficients. In order to determine the influence of prestress on the frequencies of vibration of cylindrical shells the eigenvalue problem given by equations (2.22) or (2.24) must be solved. The modes of vibration which are of most practical significance are those associated with the lowest natural frequencies, for certain problems the mode of vibration associated with the lowest natural frequency is dependent upon the state of initial stress (see for example Koval (1961)).

For a limited number of cases solutions of closed form have been found. However for many problems of practical interest solutions of closed forms are not available and approximate methods of analysis have been employed.
III.2 Approximate Solutions

From an engineering point of view, a problem can be solved making two different types of approximations:

(a) Approximations in formulation of the equations
(b) Approximations in solving these equations

In the first group, the construction of a mathematical model to represent a physical situation always involves simplifications which in certain cases can become a matter for concern as is evidenced by the limited applicability of a simple theory to the problem of structural stability of imperfective sensitive structures.

The idealisations made in the present work have been described in part II.2 of the previous chapter.

The second type of approximations arises when the possibility of finding an exact analytical solution is remote. Such approximations are more reliable than those mentioned above since it is easier to assess the resulting error. It should be stressed that an approximate solution may be of considerable use for engineering purposes even if the exact solution exists. However it is necessary to determine the regions in which the solution is reasonable and whether or not this region coincides with that of practical interest. Secondly there are many problems for which it appears unlikely an exact solution will be found and approximate methods must be used.

In the first part of this chapter, the system of equations (2.22) will be considered. The method of characteristic functions will be used together with the assumption given by Donnell who restricted his attention to configurations for which

\[ \left| \frac{\lambda}{\pi} \right|^2 \lesssim 1 \]

(3.1)
This restriction limits the applicability of the resulting solution to long shells of small radius having a large number of circumferential waves and a small number of axial waves. In the second part, the exact solution for a simply supported cylindrical shell will be considered in order to show the significance of the assumption (3.1)

Rewriting expression (2.22),

\[ \frac{3^2 u}{2} + \frac{1 - \nu}{2} \frac{3^2 u}{\phi^2} + \frac{1 + \nu}{2} \frac{3^2 v}{\phi^2} - \frac{3^2 w}{\phi} = \frac{3^2 u}{\phi} \]

\[ \frac{1 + \nu}{2} \frac{3^2 u}{\phi^2} + \frac{1 - \nu}{2} \frac{3^2 v}{\phi^2} + \frac{3^2 v}{\phi^2} - \frac{3^2 w}{\phi} = \frac{3^2 w}{\phi} \]

\[ \frac{2^2 w}{\phi} + \frac{2^2 u}{\phi} + \frac{2}{2} \frac{2^2 v}{\phi} + \frac{2}{\phi} \frac{2}{\phi} + \frac{2}{\phi} - \phi - v - w = \frac{3^2 w}{\phi} \]

Bearing in mind that each of the normal modes for a complete cylinder executes a simple harmonic motion with an associated natural frequency \( \omega \), the general form for displacements are assumed to be in the following form:

\[ u = A e^{\lambda x} e^{in\phi} \cos \omega t \]
\[ v = B e^{\lambda x} e^{in\phi} \cos \omega t \]
\[ w = C e^{\lambda x} e^{in\phi} \cos \omega t \]

Here, \( A, B \) and \( C \) are amplitude constants, \( n \) is an arbitrarily chosen positive integer equal to the number of circumferential waves and \( \lambda \) is to be determined from the auxiliary equations of (2.22).

As in all problems of small amplitude vibrations, the absolute value of the coefficients \( A, B \) and \( C \) are indeterminate. However, the ratios between them can be determined.
Substituting (3.2) into (2.22) one obtains

\[
\begin{pmatrix}
\lambda^2 + \Delta \omega^2 - \frac{1-v}{2} \eta^2 & \text{in} \lambda \frac{1+v}{2} & -\Delta \omega^2 \\
\text{in} \lambda \frac{1+v}{2} & \frac{1-v}{2} \lambda^2 + \Delta \omega^2 - \eta^2 & -\text{in} \\
\text{in} \lambda & \text{in} & P_x \lambda^2 + 2 \text{in} \lambda \kappa - \kappa \lambda^2 + \Delta \omega^2 - 1
\end{pmatrix}
\begin{pmatrix}
A \\
B \\
C
\end{pmatrix} = 0
\]

(3.3)

For a non-trivial solution, the determinant of the matrix of equation (3.3) must vanish, hence writing \(\Delta \omega = \Omega\), the resulting frequency equation is:

\[
\frac{1+v}{2} \eta^2 \left[ -n \left( \frac{\lambda^2}{n^2} - 1 \right) - \text{in} \lambda \left( \frac{\lambda^2}{n^2} - 1 \right) + P_x \frac{1-v}{2} \lambda^2 \left( \frac{\lambda^2}{n^2} - 1 \right)^2 + (1-v) \text{in} \frac{\lambda^2}{n^2} - 1 \right]^2
\]

\[
+ \frac{1-v}{2} \lambda^2 \eta^2 (3-v) - \Omega^2 - \frac{h_0}{2} (3-v) \left( n \left[ \frac{\lambda^2}{n^2} - 1 \right] - 3n \lambda^2 \left( \frac{\lambda^2}{n^2} - 1 \right) \right) + \frac{\Omega^2}{2} (3-v) \left( \frac{\lambda^2}{n^2} - 1 \right)
\]

\[
+ \Omega^2 P_x \lambda^2 + 2 \text{in} \lambda \eta^2 \lambda^2 - \kappa \lambda^2 \left[ \frac{\lambda^2}{n^2} - 1 \right]^2 + \Omega^3 = 0
\]

(3.4)

The amplitude ratios are:

\[
A = \frac{2n+v \left[ 1+\nu \left( \frac{\lambda^2}{n^2} \right)^2 \right]}{2n^2 - \eta^2 \Omega (3-v) \left[ 1- \frac{\lambda^2}{n^2} \right] + (1-v) n^4 \left[ 1- \frac{\lambda^2}{n^2} \right]^2}
\]

(3.5)

\[
B = \frac{2n-\eta^2 \Omega (3-v) \left[ 1- \frac{\lambda^2}{n^2} \right] + (1-v) n^4 \left[ 1- \frac{\lambda^2}{n^2} \right]^2}{2n^2 - \eta^2 \Omega (3-v) \left[ 1- \frac{\lambda^2}{n^2} \right] + (1-v) n^4 \left[ 1- \frac{\lambda^2}{n^2} \right]^2}
\]
The imaginary form of the second of equations (3.5) implies that the displacement vector \( v \) leads the displacement vectors \( u \) and \( w \) by \( \frac{\pi}{2} \).

\[
\begin{align*}
u(\phi,x,t) &= U(x,t) \cos n\phi \\
v(\phi,x,t) &= V(x,t) \sin n\phi \quad \text{ (3.5i)} \\
w(\phi,x,t) &= W(x,t) \cos n\phi
\end{align*}
\]

The auxiliary equation (3.4) is of eighth degree in \( \lambda \).

Theoretically, it is possible to find eight roots for \( \lambda \) in terms of \( \Omega \) and then these may be eliminated when the boundary conditions are taken into account. This final relationship contains \( \Omega \) as the only unknown and constitutes the frequency equation. The equation is cubic in \( \Omega \), and its three roots determine the three natural frequencies of the shell. As this procedure involves laborious computation, the problem can be tackled using assumptions (3.1) and (8) of section II.2. The first permits an analytical simplification in \( \lambda \) reducing the equation of eighth degree to one of fourth. With the second assumption, the equation in \( \Omega \) will be reduced from a cubic equation to one of first degree.

By virtue of the first assumption, equation (3.4) becomes:

\[
\lambda^4 - \frac{n^4}{1-\nu^2} \left[ \left( \frac{\lambda}{n} \right)^2 - 1 \right]^2 \left[ \frac{\lambda}{n} \right]^2 \left[ \frac{\lambda}{n} \right]^2 \left[ \frac{\lambda}{n} \right]^2 - 1 \right] \right] = 0
\]

and the amplitude ratios (3.5) are not affected.
Using the second mentioned assumption, equations (3.6) and (3.5) reduce to the following simplified form:

\[ \lambda^4 - \frac{n^4}{1-v^2} P \lambda^2 - \frac{2n^5}{1-v^2} \tau \lambda - \frac{n^4 - kn^8}{1-v^2} = 0 \quad (3.7) \]

The amplitude ratios become:

\[ \frac{A}{C} = \lambda - \frac{2\nu n + (1-\nu)n^2}{2n^2-(3-\nu)n^2+2(1-\nu)n^4} \quad (3.8) \]

\[ \frac{B}{C} = \frac{n(1-\nu)n^2}{2n^2-(3-\nu)n^2+2(1-\nu)n^4} \]

In what follows, a cylinder will be considered under three distinct states of initial stress, axial compression and twist applied simultaneously, twist only, and axial compression only.

**III.2a Simply Supported cylinder under axial compression and twist**

In the determination of the response of a thin cylindrical shell under axial compression and twist applied simultaneously, it is necessary to solve equation (3.7) as it stands. Assuming two purely imaginary roots:

\[ \lambda_1 = i(\xi + \eta) \quad (3.9) \]

\[ \lambda_2 = i(\xi - \eta) \]

Substitution of (3.9) into (3.7) implies that:

\[ \frac{n^4 P^2}{1-v^2} = 2(\xi^2 + \eta^2) + \frac{n^5 \tau}{\xi(1-v^2)} \quad (3.10) \]

\[ \frac{kn - n^4}{1-v^2} = (\xi^2 - \eta^2)^2 - \frac{n^5 \tau}{\xi(1-v^2)} (\xi^2 - \eta^2) \]
The other two roots become:

\[ \lambda_3 = -i(\xi - n^2 + \frac{n^2}{\sqrt{\xi(1-\nu)}}) \]

\[ \lambda_4 = -i(\xi + n^2 + \frac{n^2}{\sqrt{\xi(1-\nu)}}) \]  

(3.11)

The frequency factor \( \Omega \) in (3.10) still cannot be calculated as there are only two equations and three unknowns. In order to obtain an extra equation, the boundary conditions will be considered. For a simply supported cylinder, the boundary conditions are given by

\[ w = \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = 0, \ell \]  

(3.12)

Substituting the value of \( w \) given by expression (3.2) into (3.12) one may write:

\[ w(0) = \sum_{j=1}^{k} c_j = 0 \]

\[ w(\ell) = \sum_{j=1}^{k} c_j e^{\lambda_j \ell} = 0 \]  

(3.13)

\[ \frac{\partial^2 w(0)}{\partial x^2} = \sum_{j=1}^{k} \lambda_j^2 c_j = 0 \]

\[ \frac{\partial^2 w(\ell)}{\partial x^2} = \sum_{j=1}^{k} \lambda_j^2 c_j e^{\lambda_j \ell} = 0 \]

Expanding these expressions and writing them in matrix form, equations (3.13) become:
The trivial solution of the above homogeneous linear system of equations is avoided by setting the determinant of the coefficients equal to zero. Thus, this determinant takes the following form:

$\begin{align*}
\lambda_4 \lambda_2 \lambda_3 \lambda_2 \lambda_4 \lambda_2 \lambda_3 \lambda_2 \lambda_3 \\
(\lambda_4 - e^{-\xi})(\lambda_2 - e^{-\xi})(\lambda_3 \lambda_2 \lambda_4 + \lambda_3 \lambda_2 \lambda_4)
\end{align*}$

Substituting the values of $\lambda$ given by (3.9) and (3.11), the third required equation is found:

$\begin{align*}
\cos 2\xi l \cos \eta \cos \tau = & \frac{\left[ \frac{n^5}{\xi(1-\nu^2)} \right]^2 - 4\xi^2 \left[ \frac{2n^2}{\xi(1-\nu^2)} \frac{n^5}{\xi(1-\nu^2)} \right]}{8\xi^2 \eta^2 \left[ \frac{n^5}{\xi(1-\nu^2)} \right]}
\sin \eta \sin \tau = & \frac{n^5}{\xi(1-\nu^2)} \frac{\xi}{l}
\end{align*}$

The final solution is given by equations (3.10) and (3.16), which can be written in the following simplified form:

$\Omega = \frac{k \frac{h}{n} - \frac{1-\nu^2}{2}}{1-\nu^2} \left\{ 2\xi^2 + \frac{\frac{h}{n}}{2(1-\nu^2)} \left[ \frac{\nu^2}{\xi} - P_x \right] \right\}^2 + \frac{\frac{h}{n}}{2(1-\nu^2)} \left[ \frac{\nu^2}{\xi} - P_x \right]$
The above equations constitute the approximate solution for vibration of a simply supported cylindrical shell under torsion and direct axial load. Given any particular cylinder, the natural frequency can be calculated by using (3.17) in which all the terms are known with the exception of $\xi$, which can be found using the transcendental equation (3.18). In what follows, equations (3.17) and (3.18) will be used to study the next two particular cases:

III.2b Simply supported cylinder under initial static torque

Since the axial load $P_x$ is taken to be equal to zero, the general solution (3.17) and (3.18) reduces to the form

$$\Omega = \frac{kn}{h} - \frac{h(1-v^2)}{n^2} \xi^2 + \frac{n^2 \xi}{2(1-v^2)}$$

(3.19)
\[ \tan \left\{ -\xi^2 - \frac{\pi n^2}{2 \xi (1-\nu^2)} \right\} = \frac{2 \xi^2 - \xi^2 - \frac{\pi n^5}{2 \xi (1-\nu^2)} \sqrt{\xi^2 - \frac{\pi n^5}{2 \xi (1-\nu^2)}}}{2 \xi^2 + \left[ \frac{\pi n^5}{2 \xi (1-\nu^2)} \right]} \] (3.20)

These two expressions agree with those given by Koval (1961).

III.2c Simply supported cylinder under axial compression only

In this case, the general solution is reduced to only one simple expression:

\[ \Omega = k \pi^4 + \left( \frac{\pi n}{\xi} \right)^2 \left[ \frac{1-\nu^2}{\pi^4} \left( \frac{\pi n}{\xi} \right)^2 - \pi n \right] \] (3.21)

From consideration of (3.21) one can observe that:

1. The frequency of vibration of the cylinder decreases when the axial compression increases. As \( P_X \) approaches the value of \( \frac{1-\nu^2}{\pi^4} \left( \frac{\pi n}{\xi} \right)^2 \) the frequency vanishes and lateral buckling takes place.

2. If the cylinder is subjected to axial tension (change of the sign of \( P_X \) in equation (3.21)) it is found that when the axial tension increases, the frequency also increases.

This behaviour is known to take place in columns and plates, and was first discovered by Euler (1744). If \( P_X \) is set to zero in equation (3.21) the expression for the frequency of simply supported cylindrical shells is obtained:

\[ \Omega = k \pi^4 + \frac{1-\nu^2}{\pi^4} \left( \frac{\pi n}{\xi} \right)^2 \] (3.22)
In order to determine the condition for which the natural frequency vanishes, \( \Omega \) is set to zero in equation (3.21). Using this condition in equation (2.21) the axial stress for this condition is

\[
\sigma_x^0 = D \left[ \frac{a^4 L^4}{2 \pi^2 m^2 n^2} + \frac{E \pi^2}{n^2 D} \frac{m^2 n^2}{L^4} \right]
\]  

(3.23)

when

\[ n = \frac{\pi m}{L} \]  

(3.24)

Equation (3.23) takes the form

\[
\sigma_x^0 = D \left[ \frac{a^4}{ht^2} + \frac{E \pi^2}{a^2 D} \frac{m^2}{m^2 \pi^2} \right]
\]  

(3.25)

This last expression gives the buckling stress of a simply supported cylindrical shell which was given by Timoshenko (1910).

In the same way that the modes of free vibrations of a cylindrical shell can be considered to consist of two infinite sets of normal displacement functions which can be classified in terms of waves along \( x \) direction (\( m \)) and waves in the circumferential direction (\( n \)), the classical buckling problem also constitutes an eigenvalue problem characterised by two infinite sets of harmonic deformation shapes, which can be classified in terms of \( m \) and \( n \).

The limiting values of \( P_x \) in equation (3.21) are those for which \( \Omega \) vanishes. For given values of wave numbers \( m \) and \( n \), \( P_x \) is seen to be a function of the shell geometry and the shell material only. In the following section examples for particular values of \( m \) and \( n \) will be considered in detail. Assumption (3.1) will not be involved and the results obtained can be considered exact.
Initially it is helpful to restrict the analysis to axisymmetric modes of deformation as these result in a considerable simplification of the problem.

Axisymmetric modes of deformation are characterised by a circumferential wave number \( n \) of zero and are an infinite set in axial waves \((m)\).

When terms in \( n \) vanish equation (3.6) reduces to

\[
\lambda^4 + \frac{1}{k} p_x \lambda^2 + \frac{1-\nu^2}{k} = 0
\]

By proceeding in the same fashion as in the previous cases, two equations can be found by assuming there exist two roots for (3.26). The third one is given by the boundary conditions. These three equations are:

\[
\frac{1}{k} p_x = 2(\zeta^2 + \eta^2)
\]

\[
\frac{1-\nu^2}{k} = (\xi^2 - \eta^2)
\]

\[
\eta = \eta + \frac{m}{k}
\]

from which the expression for the frequency can be found:

\[
\Omega = (1-\nu^2) + (\frac{\pi m}{k})^2 \left[ k(\frac{\pi m}{k})^2 p_x \right]
\]

When \( p_x \) is set to zero the frequency of free vibrations is obtained:

\[
\Omega = (1-\nu^2) + k(\frac{\pi m}{k})^4
\]

Again, setting the frequency to zero in (3.28) the expression for the initial axial stress becomes:

\[
\sigma^0_x = D \left[ \frac{m^2 \pi^2}{h^2} + \frac{E}{2D} \frac{k^2}{m^2 \pi^2} \right]
\]

(3.30)
This expression is the same as that of (3.25) concerning critical stress in the buckling of a simply supported cylindrical shell. In discussing this solution, it must be remembered that the assumption (3.1) was not used, hence in the axisymmetric vibrations of any cylinder, where the frequency approaches zero the initial axial stress approaches its critical value.

III.2d Cylindrical shells with both edges clamped

Similar methods to those used for cylinders with simply supported edges are applied to cylinders with clamped edges. The procedure is much the same, but different boundary conditions arise.

The boundary conditions are prescribed by

\[ \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = 0, L \] (3.31)

In a similar manner as before

\[ w(0) = \sum_{j=1}^{n} \lambda_j c_j = 0 \]

\[ w(L) = \sum_{j=1}^{n} \lambda_j c_j e^{\lambda_j L} = 0 \] (3.32)

\[ \frac{\partial^2 w(0)}{\partial x^2} = \sum_{j=1}^{n} \lambda_j c_j = 0 \]

\[ \frac{\partial^2 w(L)}{\partial x^2} = \sum_{j=1}^{n} \lambda_j c_j e^{\lambda_j L} = 0 \]

which lead to

\[ (e^1 - e^2)(e^1 - e^3)(\lambda_2 \lambda_4 + \lambda_1 \lambda_3) + (e^{1^2} - e^1)(e^3 - e^2)(\lambda_1 \lambda_2 + \lambda_2 \lambda_3) \]

\[ + (e^4 - e^3)(e^2 - e^1)(\lambda_3 \lambda_4 + \lambda_1 \lambda_2) = 0 \] (3.33)
Substituting the values of \( \lambda \) given by (3.9) and (3.11) and applying the boundary conditions (3.31) one can obtain

\[
\cos \left( \sqrt{\frac{n^3}{2(1-v^2)}} \left[ \frac{p-x}{\xi} \right] - \xi^2 \xi + 2\pi n \right) \cos \left( \sqrt{\frac{n^3}{2(1-v^2)}} \left[ \frac{p-x}{\xi} \right] - \xi^2 \xi + 2\pi m \right)
\]

\[
-\cos \{2\xi \xi + 2\pi \} = \frac{3\xi^2}{\sqrt{2(1-v^2)}} \left[ \frac{p-x}{\xi} \right] - \xi^2 \sqrt{\frac{n^3}{2(1-v^2)}} \left[ \frac{p-x}{\xi} \right] - \xi^2
\]

\[
\sin \left( \sqrt{\frac{n^3}{2(1-v^2)}} \left[ \frac{p-x}{\xi} \right] - \xi^2 \xi + 2\pi m \right) \sin \left( \sqrt{\frac{n^3}{2(1-v^2)}} \left[ \frac{p-x}{\xi} \right] - \xi^2 \xi + 2\pi m \right)
\]

(3.34)

The general solution for cylindrical shells clamped at both edges is given by equations (3.17) and (3.34). Again the value of \( \xi \) can be calculated from the transcendental equation (3.34) which should be substituted in expression (3.17) to obtain the frequency. Again particular cases of initial stresses may be studied.

III.3 Exact Solution

The exact solution can be obtained more easily when a Donnell type equation (2.24) is employed. Clearly the assumed form of the displacements field must satisfy the boundary conditions. For many cases of practical interest it is possible to find appropriate functions.

A simply supported cylindrical shell under axial load is now considered.
Equation (2.24) will be repeated below for convenience.

\[ \nu^2 w + \frac{1}{k} \gamma^4 \left[ P_x \frac{\partial^2 w}{\partial x^2} \right] + \frac{\nu^2}{k} \frac{\partial^4 w}{\partial x^4} + \frac{\Delta}{k} \nu^4 \frac{\partial^2 w}{\partial t^2} = 0 \]

Taking the solution of (2.24) in the following form:

\[ w = C \cos n\phi \sin \frac{mn\phi}{L} x \sin \omega t \]  

(3.35)

which assumes that during vibration the generators of the shell sub-divide into \( m \) half-waves and the circumference into \( 2n \) half-waves. At the ends:

\[ w = 0 \text{ and } \frac{\partial^2 w}{\partial x^2} = 0 \]

which are the conditions of simply supported edges.

Substituting expression (3.35) in Eq. (2.24) and using the notation

\[ \lambda = \frac{mn\phi}{L} \]  

(3.36)

the following equation is obtained:

\[ (\lambda^2 + n^2)^2 \lambda^4 = \frac{P_x}{k} \left( \lambda^2 + n^2 \right)^2 \lambda^2 + \frac{\nu^2}{k} \lambda^4 = \frac{\omega^2 \Delta}{k} \left( \lambda^2 + n^2 \right)^2 \]  

(3.37)

By using the following notation

\[ \Omega = \omega^2 \Delta \]  

(3.38)

then

\[ \Omega = k \left( \lambda^2 + n^2 \right)^2 + \lambda^2 \left[ \frac{1-\nu^2}{(\lambda^2 + n^2)^2} \lambda^2 - P_x \right] \]  

(3.39)
Even though this last equation is an explicit expression for the frequency, it is not possible to see at first glance the general behaviour of the frequency because of the parameters \( \frac{a}{h}, \frac{\ell}{a}, n, m \). The final conclusions are determined from a series of curves in which the whole range of such parameters is presented.

In fig. (3.1), the frequency factor \( \sqrt{n} \) is plotted against the ratio of length to radius, \( \frac{\ell}{a} \), when \( P_x \) is set to zero. Flügge's solution given by Forsberg (1966) is taken as the standard for comparison. The conclusion is reached that (3.39) can be used within engineering accuracy for the whole range of the parameters \( \frac{a}{h}, n, m \) for a cylindrical shell satisfying the following condition:

\[
0 < \frac{\ell}{a} < 20
\]  

(3.40)

In fig. (3.1) only the nodal pattern \( m=1, n=1 \) was considered for different values of \( \frac{a}{h} \) and \( \frac{\ell}{a} \). It is shown that for other nodal patterns, solution (3.39) comes closer to the Flügge's solution as the values of \( m \) and \( n \) increase. It is an important point to note that the region of practical interest occurs within the range given by (3.40).

Before proceeding any further, it is appropriate to discuss the accuracy of the approximate solution given by equation (3.21). By proceeding in much the same manner as in the previous case, it is found in fig. (3.1) that the approximate solution gives good results when the following restriction is satisfied:

\[
10 < \frac{\ell}{a}
\]  

(3.41)
It is interesting to note that the same restriction (3.41) is imposed in the theory of elastic analysis of long shells.

According to condition (3.40) it is concluded that the approximate solution is valid for all values of $a/h$, $n$ and $m$ within the following range:

$$10 < \frac{a}{h} < 20 \quad (3.42)$$

Turning attention now to the case when the frequency is taken to be zero in equation (3.39), the expression for the initial axial stress can be written:

$$\sigma_x = 0 \quad \frac{1}{a^2 h} \left[ \left( \frac{2\pi n}{\lambda} \right)^2 + n^2 \right]^2 \left( \frac{\lambda}{\pi a} \right)^2 + \frac{E}{D} \left( \frac{\lambda}{a} \right)^2 \frac{2}{\left( \frac{2\pi n}{\lambda} \right)^2 + n^2} \quad (3.43)$$

which is a more general expression for the critical stress in buckling. For the case of axially symmetric vibration ($n=0$), this last expression reduces to that given by (3.25).

Having found that the solution derived so far can lead to intolerable errors in the prediction of the frequency when the ratio length $l$ to radius $a$ is greater than 20, (see eq. (3.40)), the next paragraph will be devoted to removing this restriction.

III.4 Morley's Equation in Dynamics of Cylindrical Shells

Morley (1958), proposed a differential equation to investigate the elastic stability of cylindrical shells, equation (2.25c). In his analysis, the following conclusions are useful for the present problem:

a) the solution of the Morley equation is quite close to the solution of the Flügge equations for any value of $l/a$, the range of the parameter $l/a$ is then unbounded.
b) Morley's equation retains the same simplicity of the equation (2.25a).

Rewriting Morley's equation (2.26) in dynamic analysis as

\[ v^h (v^2+1)^2 v + \frac{1}{k} v^h \left| \frac{\partial^2 w}{\partial x^2} \right| + \frac{1-v^2}{k} \frac{\partial^4 w}{\partial x^4} + A_h \frac{\partial^2 w}{\partial t^2} = 0 \]  (2.26)

An assumed solution of the form (3.35) is again employed

\[ w = C \cos n \phi \sin \frac{\nu \pi a}{l} x \sin \omega t \]  (3.35)

This satisfies the boundary conditions of a simply supported cylindrical shell. Substituting this into Morley's equation (2.26) results in an eighth-order algebraic equation for :

\[ (\lambda^2+n^2)^4 + (\lambda^2+n^2)^2 + 2(\lambda^2+n^2)^3 - \frac{P_x \lambda^2}{k} (\lambda^2+n^2)^2 + \frac{1-v^2}{k} \lambda^4 = \Omega (\lambda^2+n^2)^2 \]

from which

\[ \Omega = k \left[ (\lambda^2+n^2)(2+\lambda^2+n^2)+1 \right] + \lambda^2 \left[ \frac{1-v^2}{(\lambda^2+n^2)^2} \lambda^2 - P_x \right] \]  (3.44)

Again, in the same manner, this last expression is plotted graphically in figs. (3.2) and (3.3). It is seen that by using Morley's equation in the dynamic analysis of cylindrical shells, the solution is quite good for the whole range of the parameters \((a/h, \frac{k}{a}, n, m)\). It is concluded also that, although solution (3.39) for the mode \(n=1, m=1\) gives a better approximation than that of Morley when \(0 < \frac{k}{a} < 20\), outside this range when \(\frac{k}{a} > 20\) the improvement obtained is very good since the error diminishes as \(\frac{k}{a}\) increases. In fig. (3.3) other nodal configurations are considered and the same behaviour is observed.

In order to illustrate the application of equations (3.39) and (3.44) a cylindrical shell previously considered by Flügge (1934)
in his study of elastic stability has been chosen as an example. This shell has an effective length between end supports of 10 ft., a mean diameter of 2 ft. and a thickness of 0.011 ft., Poisson's ratio \(\nu = 0.3\).

Since in this particular example, the ratio \(\frac{\ell}{a}\) is smaller than 20, any of the solutions (3.39) or (3.44) will give a good approximation. Curves of frequency factor \(\Omega^2\) versus axial thrust \(P_x\) are given in fig. (3.4), where equation (3.39) was used to plot a series of curves for different values of circumferential wave number \(n\) and for axial wave number \(m=1\).

From this family of curves it is seen that the general behaviour of the cylinder during vibration is affected in the following manner:

(a) The frequency factor decreases as the load factor increases.

(b) The lowest natural frequency in this example occurs at a circumferential wave number of 2.

(c) The lower natural frequency becomes zero when the axial load tends to the critical one. Flügge gives for this example \(5.6 \times 10^{-3} P_x\) as the critical load factor in buckling of a freely supported cylindrical shell, the value given in fig. (3.4) is \(7.03 \times 10^{-3} P_x\).

Morley's equation was also used but only for the circumferential mode \(n=2\), and the approximation is better since the critical load factor found is \(6.3 \times 10^{-3} P_x\) which is nearer to the exact solution given by Flügge.

**Example 2**

As a second example a cylindrical shell which roughly corresponds to a Ferrybridge cooling tower shell (fig. 3.4a) has been considered. The shell dimensions are \(\ell = 375\) ft., \(a = 108\) ft. and the shell thickness \(h = 0.417\) ft.
The solution is presented in figs. 3.5 and 3.6 from which similar conclusions to the previous example are drawn. In fig. 3.5 results are presented for several nodal configurations and it is seen that the lowest value of the frequency factor and the critical load factor occurs when \( m=1 \). Solutions in the region of the origin in an area bounded by ABC are presented on a larger scale in fig. 3.6 and it is seen that the lowest natural frequencies are associated with a circumferential wave number \( n \) of 5. The corresponding curve for \( n=5 \) obtained from Morley's equation is shown as an unbroken line in fig. 3.6. These results are a better approximation to the true solution than those given by Donnell's equation.

Williams (1967) found that the lowest natural frequency of a Ferrybridge cone-toroid cooling tower was associated with a circumferential wave number of 4. The lowest frequency for the corresponding hyperboloid shell was associated with a circumferential wave number of 5.

III.5 Galerkin's Method

Galerkin's method has been used successfully in a large number of problems in mechanics. In the present study, it becomes very useful, particularly when the formulation of the problem is given by equation (2.25) as suggested by Batdorf.

Consider the differential equation in free vibrations of thin cylinders in the following form:

\[ Q(w) = 0 \]  \hspace{1cm} (3.45)
where $Q$ is a linear differential operator in $x$ and $y$ defined by

$$Q = \left\{ k^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 2 \nu \frac{\partial^2}{\partial x \partial \phi} + (1-\nu^2) \frac{\partial^4}{\partial y^4} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \phi^2} \right\}$$

(3.46)

Solution of (3.45) is assumed in the form of the double series

$$w = \sum_{m=0}^{\infty} a_m U_m + \sum_{m=0}^{\infty} b_m V_m$$

(3.47)

where the arbitrary functions $U_m$ and $V_m$ satisfy the boundary conditions but not necessarily the differential equation (3.45), hence, if (3.47) is substituted into (3.45), in general a residual $Q(w) = \varepsilon$ is obtained. For the exact solution the residual is, of course, identically zero, so $\varepsilon$ is the error in the differential equation. The Galerkin method consists of minimizing the residual $\varepsilon$ by choosing appropriate coefficients $a_m$ and $b_m$ which together with $Q(w)$ should satisfy the following equations:

$$\int_0^a \int_0^{2\pi} Q(w) U_p \, dx \, d\phi = 0$$

(3.48)

where

$$\int_0^a \int_0^{2\pi} Q(w) V_p \, dx \, d\phi = 0$$

$p = 0, 1, 2, \ldots, j$

III.5a Simply supported cylindrical shell

Differential equation (2.25) will be considered in the present analysis:

$$k^2 \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 2 \nu \frac{\partial^2}{\partial x \partial \phi} + (1-\nu^2) \frac{\partial^4}{\partial y^4} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \phi^2} = 0$$
Consider the harmonic vibrations of the shell. Assuming

\[ w(x, \phi, t) = \hat{w}(x, \phi) \cos \omega t \]  

(3.49)

Then, upon substitution of eq. (3.49) into eq. (2.25)

\[ k^2 \hat{W} + \frac{P_x}{2} \frac{\partial^2 \hat{W}}{\partial x^2} - 2t \frac{\partial^2 \hat{W}}{\partial x^2} + (1-v^2) \nu \frac{\partial^4 \hat{W}}{\partial x^2} = \hat{\omega} \hat{W} = 0 \]  

(3.50)

A suitable displacement function of the type (3.47) for this case may be taken

\[ \hat{w}(x, \phi) = \cos n \phi \sum_{m=1}^{\infty} a_m \sin \frac{\pi m a}{l} x + \sin n \phi \sum_{m=1}^{\infty} b_m \sin \frac{\pi m a}{l} x \]  

(3.51)

which satisfies the boundary conditions

\[ \frac{\partial^2 \hat{W}}{\partial x^2} = 0 \quad \text{at} \quad x = 0, l \]

for a shell with freely supported edges.

Equation (3.51) is equivalent to equation (3.47) if

\[ \hat{U} = \cos n \phi \sin \frac{\pi m a}{l} x \]  

(3.52)

\[ \hat{V} = \sin n \phi \sin \frac{\pi m a}{l} x \]

Substitution of expressions (3.51) and (3.52) into equations (3.48) and performing the integrations over the limits indicated yields the following equations

\[ a_p M_p + \frac{8 \pi n}{l} \sum_{m=1}^{\infty} b_m \frac{\pi m p}{p^2 - n^2} = 0 \]  

(3.53)

\[ b_p M_p - \frac{8 \pi n}{l} \sum_{m=1}^{\infty} a_p \frac{\pi m p}{p^2 - n^2} = 0 \]
where \( m + p \) is an odd number.

The infinite set of equations (3.53) is homogeneous, so, in order to have a non-trivial solution, the determinant of the coefficients should vanish.
This determinant can be rearranged in the following form:

\[
\begin{array}{cccccccc}
  a_1 & a_2 & a_3 & \ldots & b_1 & b_2 & b_3 & b_4 \\
  \text{p=1} & M_1 & 0 & 0 & 0 & -\frac{2T}{3} & 0 & -\frac{4}{15}T \\
  \text{p=2} & 0 & M_2 & 0 & \frac{2T}{3} & 0 & -\frac{6}{5}T & 0 \\
  \text{p=3} & 0 & 0 & 0 & 0 & \frac{6T}{5} & 0 & -\frac{12}{7}T \\
  \end{array}
\]

where \( T = \frac{8\pi T}{k} \)
Since the two subdeterminants in (3.56) are equivalent to each other, one of them will be sufficient to yield the frequency equation

\[
\begin{vmatrix}
M_1 & \frac{2T}{3} & 0 & \frac{1}{15}T & \cdots \\
\frac{2T}{3} & M_2 & -\frac{6T}{5} & 0 & \cdots \\
0 & -\frac{6T}{5} & M_3 & \frac{12T}{7} & \cdots \\
\frac{4}{15} & 0 & \frac{12T}{7} & M_4 & \\
\end{vmatrix} = 0 \quad (3.57)
\]

The second order determinant in (3.57), gives the first approximation for \( \Omega \)

\[
\Omega = \frac{M_1 + M_2}{2} - \sqrt{\left( \frac{M_1 - M_2}{2} \right)^2 + \frac{M_1}{2} r^2} \quad (3.58)
\]
where
\[ \overline{M}_p = k \left[ \pi^2 + \frac{p^2 \pi^2}{L^2} \right] - \frac{2 \pi^2}{L} \left[ \frac{(1-\nu^2) \pi^2 \pi^2}{L^2} \right] - F \] (3.59)

\[ \overline{N}_1 = \frac{4}{9} \left( \frac{8n}{L} \right)^2 \] (3.60)

The result can be improved if a second approximation is considered which corresponds to the third order determinant in (3.57)

\[ \Omega^3 - R_1 \Omega^2 + R_2 \Omega - R_3 = 0 \] (3.61)

where
\[ R_1 = \overline{M}_1 + \overline{M}_2 + \overline{M}_3 \]
\[ R_2 = \overline{M}_1 \overline{M}_2 + (\overline{M}_1 + \overline{M}_2) \overline{M}_3 - \left( \frac{h}{9} \right) + \frac{36}{25} \left( \frac{8n}{L} \right)^2 \tau^2 \]
\[ R_3 = \overline{M}_1 \overline{M}_2 \overline{M}_3 - (\overline{N}_1 \overline{M}_2 + \overline{N}_2 \overline{M}_1) \tau^2 \]
\[ N_2 = \left( \frac{36}{25} \left( \frac{8n}{L} \right)^2 \right) \]

and \( N_1 \) has the same value as given in (3.60).

For purposes of comparison with results already found, the torque \( \tau \) and the axial load \( P_x \) are set to zero. By doing so, equation (3.61) becomes

\[ (\overline{M}_1 - \Omega)(\overline{M}_2 - \Omega)(\overline{M}_3 - \Omega) = 0 \] (3.62)

in which the smallest root is
\[ \Omega = k \left[ \pi^2 + \frac{p^2 \pi^2}{L^2} \right] + \left( \frac{1-\nu^2}{\pi^2 \pi^2} \right) \frac{h}{L} \left[ \left( \frac{8n}{L} \right)^2 \right] \] (3.63)
This last expression is the same as that given by (3.39) when the axial load \( P_x \) is taken to be zero and the axial wave number equal to unity. As far as the wave number \( m \) is concerned, it is found, as in the previous analysis, that the minimum value of the frequency occurs when \( m \) is equal to unity.

In order to consider the case of zero frequency, equation (3.61) is solved for \( \tau \)

\[
\tau = \frac{\lambda}{\delta \omega a} \sqrt{\frac{(M_1 - \eta)(M_2 - \eta)(M_3 - \eta)}{\frac{4}{9} (M_3 - \eta) + \frac{36}{25} (M_1 - \eta)}} \quad (3.64)
\]

setting now \( \Omega = 0 \), this expression becomes:

\[
\tau = \frac{\lambda}{\delta \omega a} \sqrt{\frac{M_1 M_2 M_3}{\frac{4}{9} M_3 + \frac{36}{25} M_1}} \quad (3.65)
\]

This is a relation between the two initial stresses, the axial load \( P_x \) appearing in (3.59) and the initial torque \( \tau \).

It should be noted that expression (3.65) reduces to the critical buckling torque given by Koval (1962) and Batdorf (1947) when the axial load \( P_x \) is set to zero.

By way of example the same cylinder as previously examined in example 2 will be considered.

The solution is presented in fig. (3.7). Bearing in mind that the minimum values of the frequency factor \( \sqrt{\eta} \) occur when the number of axial waves is equal to unity, equation (3.64) is plotted for \( m=1 \) and for different values of \( n \) against \( \sqrt{\eta} \) and \( P_x \).
The surface which is generated from the internal envelope of the curves \( n=5 \) and \( n=7 \) moving along the \( P_x \) axis, represents the minimum values for the frequency factor \( \sqrt{\eta} \). The limits of this surface along the \( \sqrt{\eta} \), \( P_x \) and \( r \) axis are the lowest values of the frequency factor when the cylinder vibrates without initial stresses and the critical axial load and the critical torque of the cylinder when buckling occurs.

It is interesting to see that the solution for this particular example is given by two surfaces \( n=5 \) and \( n=7 \), fig. (3.8), which means that there are two important modes in the vibration of the cylinder. It can be noted also that surface \( n=6 \) has little influence on the solution since it smooths the discontinuity in the intersection of the surfaces \( n=5 \) and \( n=7 \). Finally, it may be observed that the rate of change of curvature on the resulting surface, fig. (3.7), is faster in the \( r \) direction than in the \( P_x \) direction which means that torsion has a more important effect on the frequency of vibration.

III.5b Cylindrical shell with clamped edges

The appropriate function to express the deflected shape of the shell in this case is:

\[
W(x,\phi) = \cos n\phi \sum_{m=1}^{\infty} a_m \left\{ \cos \left(\frac{(m-1)\pi}{l} x \right) - \cos \left(\frac{(m+1)\pi}{l} x \right) \right\} \\
+ \sin n\phi \sum_{m=1}^{\infty} b_m \left\{ \cos \left(\frac{(m-1)\pi}{l} x \right) - \cos \left(\frac{(m+1)\pi}{l} x \right) \right\} 
\]

\[ (3.66) \]
Thus upon substitution of this function into (3.50) and applying the Galerkin method in the same manner as that for the simply supported cylinder, the first approximation yields the following expression for \( \Omega \):

\[
\Omega = 3(\overline{M}_1 + \overline{M}_3) + 2(\overline{M}_0 + \overline{M}_2) - \sqrt{\left[3(\overline{M}_1 + \overline{M}_3) - 2(2\overline{M}_0 + \overline{M}_2)\right]^2 + \frac{(32)^3 \pi^2}{675\xi^2}} \tag{3.67}
\]

where \( \overline{M}_p \) is given by (3.59).

Again from the third order determinant, the second approximation yields a cubic equation in \( \Omega \):

\[
R_o \Omega^3 - R_1 \Omega^2 + R_2 \Omega - R_3 = 0 \tag{3.68}
\]

where

\[
R_o = 10
\]

\[
R_1 = 8\overline{M}_0 + 5\overline{M}_1 + 6\overline{M}_2 + 5\overline{M}_3 + 6\overline{M}_4
\]

\[
R_2 = (\overline{M}_1 + \overline{M}_3)(\overline{M}_0 + 3\overline{M}_2 + 3\overline{M}_4) + 4\overline{M}_0(\overline{M}_2 + \overline{M}_4) + 2\overline{M}_2 \overline{M}_4
\]

\[
= \frac{3N_3 + 2N_2 - (32)^3 \pi^2}{675\xi^2} \tau^2
\]

\[
R_3 = (\overline{M}_1 + \overline{M}_3) \left[ 2\overline{M}_0(\overline{M}_2 + \overline{M}_4) + \overline{M}_2 \overline{M}_4 \right] - \left[ N_2(\overline{M}_2 + \overline{M}_4) + N_3(2\overline{M}_0 + \overline{M}_2) - N_1 \overline{M}_2 \right] \tau^2
\]

\[
N_1 = \frac{(60)}{105} \left( \frac{352}{105} \right) \frac{6\mu n^2}{\xi^2}
\]

\[
N_2 = \left( \frac{32}{15} \right)^2 \frac{6\mu n^2}{\xi^2}
\]

\[
N_3 = \left( \frac{352}{105} \right)^2 \frac{6\mu n^2}{\xi^2}
\]

Solving equation (3.68) for \( \tau \),
\[ \tau = \frac{q}{\delta \alpha a} \sqrt{\frac{(M_1 + M_3)(2M_0 + M_2 + M_4 - 3\alpha)(M_2 + M_4 - 2\alpha) - (M_2 - \alpha)^2}{(32/15)(M_2 + M_4 - 2\alpha) - (64/15)(352/105)(M_2 - \alpha) + (352/105)^2 (2M_0 + M_2 - 3\alpha)}} \]  

(3.69)

The critical buckling torque is obtained in (3.69) when \( \Omega = 0 \) and \( P_x = 0 \), which is in agreement with that given by Batdorf and Koval:

\[ \tau = \frac{q}{\delta \alpha a} \sqrt{\frac{(M_1 + M_3)(2M_0 + M_2 + M_4) - (M_2)^2}{(32/15)(M_2 + M_4) - (64/15)(352/105)M_2 + (352/105)^2 (2M_0 + M_2)}} \]  

(3.70)

This completes the outline of the Galerkin Method which turns out to be very useful in this problem, particularly when equation (2.25) given by Batdorf is used. The results obtained from the second approximation are adequate for engineering purposes and the accuracy can be improved by taking higher order determinants from the resulting infinite order determinant.
Fig 3.1 COMPARISON OF RESULTS FROM FLÜGGE SOLUTION AND EQUATIONS (3.21) AND (3.39)
**Simply Supported Cylindrical Shell**

\[ \omega = 0 \quad M_x = 0 \]

At \( \chi = 0, \frac{b}{a} \)

\[ m = 1, n = 1, \quad \nu = 0.3 \]

All values of \( \frac{a}{h} \)

---

**Fig. 3.2** Comparison of results from Flügge solution and Equations (3.39) and (3.44)
**Fig 3.3** COMPARISON OF RESULTS FROM FLÜGGE SOLUTION AND EQUATION (3.44) FOR $m = 1$ and 7
EXAMPLE 1

SIMPPLY SUPPORTED CYLINDRICAL SHELL

\[ \omega = 0 \quad M_x = 0 \]

AT \[ \chi = 0, \quad \frac{L}{a} \]

\[ \frac{L}{a} = 10 \quad \frac{a}{h} = 942 \quad k = 10^{-5} \]

\[ U = 0.3 \quad m = 1 \]

EQUATION (3.44)

--- EQUATION (3.39)

Fig 3.4 EXAMPLE 1 USING EQUATIONS (3.39) an (3.44) FOR m = 1
Fig 3.4a  CYLINDER OF EXAMPLE 2.
EXAMPLE 2.
SIMPLY SUPPORTED CYLINDRICAL SHELL

\[ \omega = 0 \quad M_x = 0 \]
AT \( x = 0 \), \( \frac{L}{a} \)

\[ \frac{L}{a} = 3.5 \quad \frac{a}{h} = 260 \quad \nu = 0.3 \]

----- EQUATION (3.39)

Fig. 3.5 EXAPMLE 2 USING EQUATION (3.39) FOR DIFFERENT VALUES OF \( m \)
EXAMPLE 2
SIMPLY SUPPORTED CYLINDRICAL SHELL

\[ \omega = 0 \quad M_x = 0 \quad \text{AT} \quad x = 0, \quad \frac{l_d}{a} \]

\[ \frac{l_d}{a} = 3.5 \quad a/h = 260 \quad \nu = 0.3 \]

\[ m = 1 \]

---

EQUATION (3.39)
EQUATION (3.44)

---

Fig 3.6  EXAMPLE 2 USING EQUATIONS (3.39) and (3.44) FOR \( m = 1 \)
SIMPLY SUPPORTED
CYLINDRICAL SHELL

\[ \omega = 0 \quad M_k = 0 \]

At \( x = 0, \quad \frac{h}{a} = 3.5 \)

\[ \frac{\delta}{h} = 2.60 \]

\[ v = 0.3 \]

\[ m = 1 \]

Fig. 3.7 CYLINDER OF EXAMPLE 2 UNDER AXIAL LOAD AND TORSION. SOLUTION OF EQUATION (3.64) FOR \( m = 1 \)
SIMPLY SUPPORTED CYLINDRICAL SHELL

$\omega = 0 \quad M_x = 0$

AT $x = 0$, $\lambda/\alpha$

$\lambda/\alpha = 3.5$

$\alpha/h = 260$

$\nu = 0.3$

$m = 1$

Fig. 3.8. INTERSECTION OF SURFACES $n = 5$ and $n = 7$
CHAPTER IV

VIBRATIONS OF CYLINDRICAL SHELLS UNDER DIRECT AXIAL LOAD AND BENDING MOMENT.

DYNAMIC STABILITY ANALYSIS

IV.1 Introduction

So far, the vibrational problem of cylindrical shells has been the main concern in the present study, but the stability problem is by no means less important. It was seen in the previous chapter that the dynamic analysis of cylindrical shells, under initial stresses, leads to the problem of buckling when the frequency is set to zero. It is relevant for the present problem to consider the stability analysis of cylinders under bending.

Flügge (1934) first outlined the general problem of buckling of a thin cylinder under bending moments at its ends, and solved the particular case:

\[ \frac{h^2}{12R^2} = 10^{-6} \]

and \[ \frac{wha}{t} = 1 \]

He showed that the ratio between the maximum critical stress for bending alone is 1.3 times the critical stress for pure compression.

At the same time, Donnell (1934) carried out numerous experiments on specimens under axial compression and bending moment. The values that he found for the critical stress in bending were about 1.4 times those found in axial compression for all values of a/h. These results confirmed the theoretical
analysis given by Flügge. Finally, Timoshenko (1936), with the results of Flügge and Donnell, pointed out that it is possible to get satisfactory approximations which are on the safe side by assuming that buckling occurs in bending when the maximum compressive stress becomes equal to the critical stress calculated for symmetrical buckling. Since then, this statement has been used as a general rule.

In the following analysis the first part is concerned with the problem of vibrations of cylindrical shells subjected to external moment according to the theory of Flügge.

IV.2 Bending Moment on a Simply Supported Cylindrical Shell

Since the case of constant axial stresses has already been discussed, further analysis can be extended to linear variation of the initial axial stresses. Fig. 4.1.

Denoting by $\phi$ the angle that an axial plane makes with the plane in which bending occurs, the axial force $P_x$ is:

$$P_x = P_1 \cos\phi$$  \hspace{1cm} (4.1)

Thus, upon substitution of expression (4.1) into the set of equations (2.22) the vibration problem under bending in a cylindrical shell is presented:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-v}{2} \frac{\partial^2 v}{\partial \phi^2} + \frac{1+v}{2} \frac{\partial^2 v}{\partial x^2} - \frac{3v}{2} \frac{\partial^2 w}{\partial x^2} = 0$$

$$\frac{1+v}{2} \frac{\partial^2 u}{\partial x^2} + \frac{1-v}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial \phi^2} - \frac{3v}{2} \frac{\partial^2 w}{\partial \phi^2} = 0$$  \hspace{1cm} (4.2)

$$P_1 \cos\phi \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial \phi^2} - v - kV^4 w = \frac{\Delta^2 v}{2}$$
This simple procedure is valid since no derivations of the axial load $P_x$ with respect to $\phi$ occur in the derivation of equation (2.22).

The set of equations (4.2) is still homogeneous in $u$, $v$ and $w$, but now has variable coefficients in $\phi$. In view of this, the assumed solution should be in the general form:

\begin{align*}
  u &= \cos \lambda x \cos \omega t \sum_{n=0}^{\infty} A_n \cos n\phi \\
  v &= \sin \lambda x \cos \omega t \sum_{n=0}^{\infty} B_n \sin n\phi \\
  w &= \sin \lambda x \cos \omega t \sum_{n=0}^{\infty} C_n \cos n\phi
\end{align*}

(4.3)

This permits the representation of more complex displacement functions in the circumferential direction than simple sinusoidal waves. Solution (4.3) also fulfils the boundary conditions for simply supported cylindrical shells

\begin{align*}
  v &= w = \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = 0, \frac{a}{2}
\end{align*}

Substituting expressions (4.3) in equations (4.2) the following equations are obtained:

\begin{align*}
  \sum_{n=0}^{\infty} \left\{ A_n \left[ \lambda^2 + \frac{1-\nu}{2} n^2 \right] + B_n \left[ - \frac{1+\nu}{2} \lambda n \right] + C_n [\nu\lambda] \right\} \cos n\phi &= 0 \\
  \sum_{n=0}^{\infty} \left\{ A_n \left[ \frac{1+\nu}{2} \lambda n \right] + B_n \left[ - \frac{1-\nu}{2} \lambda^2 - n^2 \right] + C_n [n] \right\} \sin n\phi &= 0 \\
  \sum_{n=0}^{\infty} \left\{ A_n [-\nu\lambda] + B_n [n] + C_n \left[ -1-k(\lambda^2+n^2)^2 + \Omega \right] \right\} \cos n\phi &= P_1 \lambda^2 \sum_{n=0}^{\infty} C_n \cos n\phi \cos\phi \\
  &= \frac{1}{2} P_1 \lambda^2 \left[ \sum_{n=0}^{\infty} (C_{n-1} + C_{n+1}) \cos n\phi + C_0 \cos\phi \right]
\end{align*}

(4.4)
In order to transform the last expression of (4.4) into a homogeneous equation similar to the first two, the right hand side can be re-written as shown. If \( C_0 \) is zero the equation becomes homogeneous in \( \cos \, \pi \). Flügge showed that the irregularities that arise in a more general set for \( n = 0 \) and \( n = 1 \) are of little importance and that \( C_0 \) may be taken to be zero.

Equations (4.4) represent an infinite number of linear, algebraic equations for the coefficients \( A_n, B_n \) and \( C_n \). The corresponding equations for each value of \( n \) are

\[
A_n \left[ \lambda^2 + \frac{1-\nu}{2} n^2 \right] + B_n \left[ - \frac{1-\nu}{2} \lambda n \right] + C_n [\nu \lambda] = 0
\]

\[
A_n \left[ \frac{1-\nu}{2} \lambda n \right] + B_n \left[ - \frac{1-\nu}{2} \lambda^2 - n^2 \right] + C_n = 0
\]

(4.5)

\[
C_{n-1} \left[ \frac{1}{2} P_1 \lambda^2 \right] + A_n [-\nu \lambda] + B_n [n] + C_n \left[ -1-k(\lambda^2+n^2)^2+n \right] + C_{n+1} \left[ -\frac{1}{2} P_1 \lambda^2 \right] = 0
\]

For convenience, the following abbreviated notation is used:

\[
a_{11,n} A_n + a_{12,n} B_n + a_{13,n} C_n = 0
\]

\[
a_{21,n} A_n + a_{22,n} B_n + a_{23,n} C_n = 0
\]

(4.6)

\[
a_{31,n} A_n + a_{32,n} B_n + a_{33,n} C_n = \frac{1}{2} P_1 \lambda^2 \left[ C_{n-1} + C_{n+1} \right]
\]

From the first two equations of (4.6) \( A_n \) and \( B_n \) is obtained in terms of \( C_n \)

\[
A_n = \frac{(a_{12} a_{33} - a_{13} a_{22})}{(a_{11} a_{22} - a_{12} a_{21})} C_n
\]

(4.7)

\[
B_n = \frac{(a_{21} a_{13} - a_{11} a_{23})}{(a_{11} a_{22} - a_{12} a_{21})} C_n
\]
Substitution of \((4,7)\) into the last expression of \((4,6)\) yields the following expression:

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix} - \frac{i \pi \lambda^2}{P_1} C_n = 0 \quad (4.8)
\]

This equation may now be written for every integer \(n=1, 2, \ldots\) to obtain an infinite set of linear equations for an infinite number of unknowns \(C_1, C_2, C_3, \ldots\). The infinite determinant of the coefficients of this system of equations should vanish in order to have a non-trivial solution. If the solution is to be non-trivial, the infinite determinant of the coefficients of this system of equations must tend to zero. This yields an equation of the \(n\)th degree in \(\lambda\). Solution of this equation yields \(n\) values of \(\lambda\) of which only the smallest is of interest.

As an example of the application of equation \((4.8)\), the cylinder under axial uniform load studied in the previous chapter (exemple 2) will be considered here:

The geometric characteristics of the cylinder are:

\[
\frac{L}{a} = 3.5 \quad (4.9)
\]

\[
\frac{a}{h} = 260
\]

In the problem solved by Flügge concerning buckling of a cylinder under bending moments mentioned in IV.1, it was seen that it was sufficient to use only a four by four determinant made up of the coefficients of \(C_4, C_5, C_6\) and \(C_7\) of the system of four equations obtained from an expression similar to \((4.8)\) for which \(n\) was taken to be 4, 5, 6 and 7 respectively.
From the previous analysis of the same cylinder (4.9) under uniform axial load, it was found that the circumferential nodes \( n = 4, 5, 6 \) and 7 are the most significant for practical purposes as far as the vibrations of the cylinder are concerned. This conclusion may be observed in fig. 3.6 as curves \( n = 4, 6 \) and 7 are close to each other and are the nearest to the curve \( n = 5 \) which contains the minimum value of the frequency and the critical load.

With this in mind, equation (4.8) will be evaluated for \( n = 4, 5, 6 \) and 7. The corresponding four equations in matrix form become:

\[
\begin{bmatrix}
0.191 - 91\Omega & 39.8 P_1 & 0 & 0 \\
93 P_1 & 0.348 - 233\Omega & 93 P_1 & 0 \\
0 & 189 P_1 & 0.948 - 474\Omega & 189 P_1 \\
0 & 0 & 347 P_1 & 2.81 - 86\Omega \\
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_5 \\
c_6 \\
c_7 \\
\end{bmatrix} = 0
\] (4.10)

Equating the determinant of the matrix in (4.10) to zero yields an equation of 4th degree in \( \Omega \). The lowest values of \( \Omega \) are plotted in fig. 4.3.

The continuous curve in fig. 4.3 shows the same behaviour found in the cases of uniform axial load and torsion. The frequency decreases as the bending moment increases and the frequency becomes zero when the maximum value of the stress in compression is approximately 1.4 times of that found in the example of uniform axial compression. This result is in agreement with the experimental work carried out by Donnell.
IV.3  Solution of a Simply Supported Cylindrical Shell under Axial Load Combined With Bending Moment by Galerkin's Method

In chapter III Galerkin's method was used as an alternative to study the vibrations of a cylinder under uniform axial load and torsion. In the present case of vibration of a cylinder under axial force and bending moments at its ends, Galerkin's method will be also used as another form of solution.

Batdorf's modified Donnell's equation given by (2.25) is considered:

\[ Q(w) = kV^l\frac{h}{w} + (1-\nu^2)\frac{h}{w} + P_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} = 0 \quad (4.11) \]

Non-uniform initial axial stresses at the end of the cylinder fig. (4.2) is equivalent to a uniform axial force plus a bending moment. The non-uniform axial force \( P_x \) is then given by:

\[ P_x = P_c + P_b \cos\phi \quad (4.12) \]

Substituting (4.12) into (4.11) yields

\[ Q(w) = kV^l\frac{h}{w} + (1-\nu^2)\frac{h}{w} + (P_c + P_b \cos\phi) \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} = 0 \quad (4.13) \]

The assumed radial deflection function \( w \) is the same as that in (4.2) which fulfils the required boundary conditions:

\[ w = \sin \lambda x \cos\omega t \sum_{n=0}^{\infty} C_n \cos n\phi \quad (4.14) \]

According to Galerkin's method outlined in III.5, for the present problem equation (3.48) becomes:
\[ \int_{0}^{2\pi} \int_{0}^{\frac{2\pi}{\lambda}} Q(w) \sin \frac{\pi n a}{\lambda} x \cos \phi \, dx \, d\phi = 0 \]  \hspace{1cm} (4.15)

where \( p = 1, 2, 3, \ldots \) \( q = 1, 2, 3, \ldots \)

Substitution of equation (4.13) with the solution (4.14) into expression (4.15), by integrating over the indicated limits, the following expression is obtained:

\[ C_n \left[ k(\lambda^2 n^2)^2 + (1-v^2) \frac{\lambda^4}{(\lambda^2 n^2)^2} - \lambda^2 P_c n^2 \right] - \frac{\lambda^2 P_d}{2} \left[ C_0 + C_{n-1} + C_{n+1} \right] = 0 \]  \hspace{1cm} (4.16)

It is assumed that in this last expression \( C_{-1} = 0 \) and the two equations for \( n = 0 \) and \( n = 1 \) are usually of little importance.

Expression (4.16) represents an infinite number of equations. Again equating the determinant of this system to zero yields an equation of nth order in \( \Omega \), only the smallest value of \( \Omega \) being of interest.

Expression (4.16) will be evaluated for the cylinder used in the experimental work reported later to investigate the behaviour of the lowest natural frequency when the cylinder is required to withstand axial load and bending moment simultaneously. The characteristics of the cylinder are: Stainless steel \( E = 30 \times 10^6 \text{lb/in}^2 \) Poisson's ratio \( v = 0.3 \), length \( l = 26'' \), mean radius \( a = 3'' \), thickness \( h = 0.012'' \). The results are plotted graphically in fig. 4.4. It is seen also that the critical value of the axial stress in bending when the frequency tends to zero is greater than the critical value of the stress in axial compression. For different values of the axial force and bending moment the solution to the problem is given by a smooth, continuous surface.
One can conclude that a cylindrical shell subjected to axial force and bending moment has a lowest natural frequency which is associated with a complex displacement field, i.e., the circumferential displacement shape is the sum of several harmonic displacement shapes and the most influential value of $n$ is that associated with the lowest natural frequency of free vibrations of the unstressed cylindrical shell.

IV.4 Dynamic Stability of Circular Cylindrical Shells Under Periodic Longitudinal Forces

Previous works on dynamic stability of elastic systems such as columns, rings, shells etc., have had a common characteristic. The problem has been reduced (exactly or approximately) to a second order differential equation with periodic coefficients (Mathieu-equation). Bolotin (1964) has shown that under harmonic excitation, the problem of dynamic stability in the general case reduces to systems of second order differential equations with periodic coefficients.

Mathieu in 1868 introduced his equation in the study of the vibrational modes of an elliptical membrane:

$$\frac{d^2f}{dx^2} + (a - b^2 \cos 2x)f = 0 \quad (4.16)$$

Since this equation has received a lot of attention in the past (see for example McLachlan (1954)), the problem of dynamic stability has been partially solved. One of the most
interesting characteristics of this equation is that for certain relationships between its coefficients \((a, b^2)\) it has solutions which are unbounded (unstable). The regions of stability and instability are shown in fig. 4.5a. In this diagram the shadowed areas between curves are unstable regions, i.e., points \(a, b^2\) in those regions yield unbounded (unstable) solutions. Otherwise the regions outside the shadowed areas are bounded (stable). The curves bounding the regions of stable and unstable solutions are regarded as belonging to the unstable region.

In regard to cylindrical shells, Yao (1963) studied the problem of dynamic stability for several combinations of the static and periodic axial and radial loads. He solved explicitly particular cases presenting the region of instability.

Bolotin (1964) presented the general formulation for thin cylinders by means of a set of three Mathieu type homogenous second order differential equations.

Iwanowski (1966) investigated the response of a cantilevered cylindrical shell with axis vertical, subjected to harmonic excitation at the base. The direction of excitation was vertical, and it was noted that radial vibration occurred at half the excitation frequency. This phenomenon is due to dynamic instability of the shell when subjected to in-plane axial inertia loading. The region of instability predicted analytically was in fair agreement with that observed experimentally.

The present work is concerned with the response of a cylindrical shell subjected to fluctuating bending moment at the ends. This problem has not to the author's knowledge hitherto appeared in the literature. The case of fluctuating axial load which is analytically simpler will be observed first.
IV.4 A simply supported cylindrical shell under constant and fluctuating axial compression

Consider the differential equation (2.24) which describes the behaviour of a thin cylinder under the action of initial axial load $P_x$

$$k v^2 w + v^\frac{d^2 w}{dx^2} + (1-v^2) \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0 \quad (2.24)$$

If $P_x$ is a periodic function of time given by

$$P_x(t) = P_o + P_t \cos \theta t \quad (4.17)$$

where $\theta$ is the circular frequency in radians/s.

Substituting (4.17) into (2.24) the differential equation governing the behaviour of a cylindrical shell under fluctuating axial load is given by

$$k v^2 w + v^\frac{d^2 w}{dx^2} \left[ (P_o + P_t \cos \theta t)^2 \right] + (1-v^2) \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0 \quad (4.18)$$

The solution of (4.18) is required in the form

$$w = f(t) \cos n\phi \sin \lambda x \quad (4.19)$$

which satisfies the end conditions for a simply supported cylindrical shell.

By a direct substitution of (4.19) into (4.18), it can be shown that (4.18) is identically satisfied if the function $f(t)$ is determined from the following differential equation
\[
\frac{d^2 f(t)}{dt^2} + \omega^2 (1 - \frac{P_0}{P^*}) \left[ 1 - \frac{P_t}{(P^* - P_0)} \cos \omega t \right] f(t) = 0 \tag{4.20}
\]

where
\[
\omega = \sqrt{\frac{k(\lambda^2 + n^2)^{4}}{\Delta(\lambda^2 + n^2)^2}} \tag{4.21}
\]
is the frequency of free vibrations of an unloaded cylindrical shell already discussed in section III.3 and

\[
P^* = \frac{k(\lambda^2 + n^2)^{4}}{\Delta(\lambda^2 + n^2)^2} \tag{4.22}
\]
is the critical buckling load of cylindrical shells given in chapter III by equation (3.43).

For convenience, equation (4.20) is represented in the following form:

\[
\frac{d^2 f(t)}{dt^2} + \omega^2 (1 - 2 \mu \cos \omega t) f(t) = 0 \tag{4.23}
\]

where \( \bar{\omega} \) is the frequency of the free vibrations of the cylinder loaded by a constant axial load \( P_0 \),

\[
\bar{\omega} = \omega \sqrt{1 - \frac{P_0}{P^*}} \tag{4.24}
\]

and \( \mu \) is a quantity containing the amplitude of the excitation \( P_t \).

\( \mu \) is called the excitation parameter which is given by:

\[
\mu = \frac{P_t}{2(P^* - P_0)} \tag{4.24}
\]

Equation (4.23) is a Mathieu equation which was outlined above with the coefficients \( a = \bar{\omega}^2 \) and \( b = \bar{\omega} \sqrt{2\mu} \). One way of describing the regions of instability and stability of equation
Equation (4.23) is by seeking a solution in the form of a trigonometric series with a period 2 $T$,

$$f(t) = \sum_{i=1, 3, 5}^{\infty} \left( a_i \sin \frac{i\omega t}{2} + b_i \cos \frac{i\omega t}{2} \right)$$ (4.25)

Investigation of equation (4.23) (see Bolotin (1964) p.20) with the assumed solution for very small values of $\mu$ has proved that the loss of dynamic stability occurs for the parameter values

$$\frac{n}{2\bar{\omega}} = 1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{n}$$ (4.26)

The instability region corresponding to the value of $1$ in equation (4.26) is called the first or fundamental instability region; that corresponding to $\frac{1}{2}$ is called the second, etc. The first region of instability is the most dangerous and has therefore the greatest practical importance

$$\theta = 2\bar{\omega}$$ (4.26)

The interpretation of this region is that, for a cylinder with one end free to move in the longitudinal direction, for each cycle of the transverse vibration, the moving end of the cylinder completes 2 cycles in the longitudinal direction. Thus, if the frequency of the applied load $\theta$ is twice the natural frequency $\bar{\omega}$, then resonance occurs.

Yao (1963) and Iwanowski (1966) mentioned above only discussed the first region of instability in their analysis of cylindrical shells. In the present application also this first region will be considered.

The result (4.26) for the value of $1$ corresponds to taking the first term of the series (4.25), i.e.
\[ f(t) = a \sin \frac{\theta t}{2} + b \cos \frac{\theta t}{2} \quad (4.27) \]

Substituting this expression into the Mathieu equation (4.23), the following expression is obtained.

\[ \theta = 2\bar{\omega} \sqrt{1 + \mu} \quad (4.28) \]

which defines the first region of instability for a simply supported cylindrical shell.

Equation (4.28) is used to plot the principal region of instability (fig. 4.5b) for the cylinder previously considered in section IV.3. A steel cylindrical shell is used of 26" length, mean radius \( a = 3" \) and thickness \( h = 0.012" \). The dotted lines of fig. 4.5b show the instability region in the \((\theta, P_0)\) plane for \( \frac{P_t}{P_0} = 0.3 \). The frequency \( \bar{\omega} \) considered was the minimum natural frequency of a cylinder under initial axial load which, for the present example, corresponds to the nodal configuration \( n=3, m=1 \). The shadowed area indicates the region of dynamic instability.

IV.4 b Simply supported cylindrical shell under constant axial load and fluctuating bending moment

In this case of constant axial load and fluctuating bending moment, expression (4.1) for external load takes the following form:

\[ P_x(t) = P_o + P_t \cos \theta t \cos \phi \quad (4.29) \]

in which \( \phi \) has been already defined in fig. 4.1. By substitution of (4.29) into (2.24) the differential equation governing the
present problem is obtained

\[ Q(w) = k v^2 w + v^4 \left[ P_0 + P_t \cos \theta \cos \phi \right] \frac{\partial^2 w}{\partial x^2} + (1 - \nu^2) \frac{\partial^2 w}{\partial x_t^2} + \Delta v^4 t \frac{\partial^2 w}{\partial t^2} = 0 \]  

(4.30)

It is observed that (4.30) is no longer a differential equation with constant coefficients in \( \phi \). Therefore a different technique for the solution has to be sought. The Galerkin method outlined in section III.5 will be used.

Assuming a more general expression for the radial deflection function \( w \) which satisfies the boundary conditions of a simply supported cylindrical shell,

\[ w = \sin \lambda x \sum_{n=0}^{\infty} f_n(t) \cos n\phi \]  

(4.31)

According to the Galerkin method, the following double integral has to be performed

\[ \int_{0}^{2\pi} \int_{0}^{\ell} Q(w) \cos \phi \sin \frac{q \pi s}{\ell} \, dx \, d\phi = 0 \]  

(4.32)

where \( p = 1, 2, 3 \ldots \) \( q = 1, 2, 3, \ldots \)

Substituting (4.30) into (4.32) with the assumed solution (4.31) and carrying out the indicated integrations, the following expression is obtained:

\[ \frac{1}{\omega_n^2} \frac{d^2 f_n(t)}{dt^2} + (1 - 2 \mu \cos \theta t) f_n(t) - \mu b_n \cos \theta t (f_{n-1}(t) + f_{n+1}(t)) = 0 \]  

(4.33)
in which

\[ b_n = \frac{(\lambda^2+n^2)^2 + 2\lambda^2 + 6n^2 + 1}{(\lambda^2+n^2)^2} \]  

(4.34)

The problem of the dynamic stability of a cylindrical shell under fluctuating bending moment leads therefore to the system (4.33) of ordinary differential equations.

The resulting system (4.33) consists of an infinite number of differential equations. However, the problem is reduced to a finite number of equations, where the number of equations is selected according to the necessary accuracy of the calculations.

Introducing the vector,

\[ F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix} \]

and the matrices

\[ B = \begin{pmatrix} b_{11} & b_{12} & b_{1n} \\ b_{21} & b_{22} & b_{2n} \\ \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & b_{nn} \end{pmatrix} \]

\[ C = \begin{pmatrix} \frac{1}{\omega_1} \\ \frac{1}{\omega_2} \\ \vdots \\ \frac{1}{\omega_n} \end{pmatrix} \]

\[ E = \{1, 1, 1, \ldots, 1\} \]

where \( C \) and \( E \) are diagonal matrices, the system of differential equations (4.33) can be written:
\[
C \frac{d^2 F(t)}{dt^2} + \left| E - \mu \cos \theta t \right| B = 0
\]  

(4.35)

The principal or fundamental region of dynamic instability of the system of Mathieu equations (4.35) is also sought by assuming as a solution the first term of a series

\[
F(t) = \sum_{i=1,3,5}^{=} (a_i \sin \frac{i \omega t}{2} + b_i \cos \frac{i \omega t}{2})
\]

where \(a_i\) and \(b_i\) are vectors which are independent of time.

It has been shown (Bolotin (1964) p.220) that the boundaries of the principal regions of instability for the system (4.35) are given by

\[
F \pm \frac{i}{2} \mu B - \frac{i}{2} e^{2C} = 0
\]  

(4.36)

This determinantal equation (4.36) will be solved for the same cylinder considered in the previous example of vibrations of a cylinder under fluctuating axial load in order to compare the regions of stability.

In section IV.2 it was shown that for the example presented accurate results were obtained when the only values of \(n\) considered were 4, 5, 6, 7. These values of \(n\) are associated with the most dominant modes of vibration under static axial loading, these are shown in fig. 3.6.

A similar technique is therefore employed here to evaluate (4.36). The dominant modes of vibration of the cylindrical shell when subjected to static axial loading can be determined from examination of fig. 4.6. The most dominant modes are seen to be associated with an axial wave number of \(m\) of unity and values of circumferential wave number \(n\) of 2, 3, 4 and 5. The
curve $n = 4$ lies very near to the curve $n = 3$ which is associated with the lowest natural frequency and the critical buckling load of the shell.

With this in mind, the determinant (4.36) will be evaluated for $n = 3$ and $n = 4$.

$$
egin{vmatrix}
1 + \mu - \frac{\theta^2}{4\omega_3^2} & \pm \frac{1}{2} \mu b_3 \\
\pm \mu & 1 + \frac{1}{2} \mu b_4 - \frac{\theta^2}{4\omega_4^2}
\end{vmatrix} = 0 \quad (4.37)
$$

from which

$$
(1 + \mu - \frac{\theta^2}{4\omega_3^2})(1 + \frac{1}{2} \mu b_4 - \frac{\theta^2}{4\omega_4^2}) = \frac{1}{4} \mu^2 b_3 \quad (4.38)
$$

This last expression defines the regions of instability in the same diagram of fluctuating axial load. Fig. 4.5b.

It is interesting to note that the number of branches of the area corresponding to bending moment depends on the number of equations taken from expression (4.33). For the present case two equations were considered, those for $n = 3$ and $n = 4$.

A better approximation is not of very much practical interest since by increasing the number of equations in (4.33) more branches of the area will appear above the one which contains the lowest values of the frequency $\theta$, however the increase in amount of work is considerable. It is concluded therefore that a good approximation is obtained when the values of $n$ taken are those which yield the lowest values of the frequency in free vibrations.
An important remark is that the branch with the minimum values of the frequency $\theta$ contains the whole region of instability for the cylinder under fluctuating axial load.

Finally it is worth pointing out that calculations were carried out for the cylinder equivalent to the cooling tower considered as example 2 in chapter III. It was found that taking the self-weight of the cylinder as $P_0$ the resulting instability region turned out to be insignificant.
Fig 4.1  CYLINDER SUBJECTED TO EXTERNAL BENDING MOMENT

Fig 4.2  CYLINDER SUBJECTED TO EXTERNAL AXIL FORCE AND BENDING MOMENT
Fig. 4.3 VIBRATIONS OF A CYLINDER UNDER BENDING

SIMPLY SUPPORTED CYLINDRICAL SHELL

\[ \omega = 0 \quad M_x = 0 \quad \text{AT} \quad x = 0, \frac{L}{x} \]

\[ \frac{L}{x} = 3.5 \quad \frac{b}{h} = 260 \quad \nu = 0.3 \]

\[ m = 1 \]

- Bending
- Uniform Axial Compression

AXIAL COMPRESSION

\[ P_x = \frac{1 - \nu^2}{E} \int \]

\[ 2.11 \times 10^{-3} \]

\[ 2.85 \times 10^{-3} \]

FREQUENCY FACTOR \( \nu^2 \)
SIMPLY SUPPORTED CYLINDRICAL SHELL

\[ \omega = 0 \quad M_x = 0 \]

AT \( x = 0, \ \frac{L}{3} \)

\[ \frac{L}{3} = 8.6 \omega \]

\[ \frac{a}{h} = 250 \]

\[ \nu = 0.3 \]

\[ m = 1 \]

**Fig. 4.4** THEORETICAL LOWER NATURAL FREQUENCY OF A CYLINDER UNDER COMPRESSION WITH BENDING MOMENT.
Fig 4.5a INSTABILITY REGIONS FOR MATHIEW EQUATION

Simply Supported Cylindrical Shell

\[ \omega = 0 \quad M_x = 0 \quad \text{at} \quad x = 0, \frac{b}{a} \]

\[ \frac{b}{a} = 8.46 \quad \frac{a}{h} = 250 \quad \frac{P_t}{P_s} = 0.3 \]

\[ f = 30 \times 10^6 \text{ p.s.i.} \quad V = 0.3 \quad m = 1 \]

Fluctuating Bending Moment

Fluctuating Axial Load Only

Fig 4.5b REGION OF INSTABILITY FOR A CYLINDER UNDER FLUCTUATING LOADS
SIMPLY SUPPORTED CYLINDRICAL SHELL

\[ \omega = 0 \quad M_k = 0 \quad \text{at} \quad x = 0, \frac{E}{\alpha} \]

\[ \frac{E}{\alpha} = 8, \omega \quad \frac{a}{h} = 250 \quad I = 30 \times 10^6 \text{ps.i} \]

\[ \psi = 0.3 \quad m = 1 \]

**Fig 4.6 VIBRATIONS OF A CYLINDER UNDER AXIAL COMPRESSION**
CHAPTER V

CRACKS IN VIBRATING CYLINDRICAL SHELLS

V.1 Introduction

Stress concentrations in structural elements are the cause of plastic hinges in certain materials and cracking in others. While plastic analysis has received a lot of attention, the effect of cracks on the behaviour of structures remains unknown in the majority of cases. The reason is that the problem becomes complex because of the lack of information regarding the depth and number of cracks in each case, and from an analytical point of view the element is no longer a continuous system.

It should be borne in mind that a structure or structural element can be seriously damaged by cracking. In general, fractures can render the structure more flexible, consequently reducing its stiffness. In dynamics and stability analysis, a reduction in stiffness would reduce the natural frequency and lower the buckling load.

In concrete structures, subjected to tensile stresses, cracking can always be identified, particularly in thin structures (plates and shells). Regarding the collapse of the cooling towers at Ferrybridge, the Committee of Inquiry (1967) has issued a comprehensive report in which cracking is shown to play an important role. The Committee reported the following: long vertical hairline cracks extending from the ring beam upwards to just below the throat were found on all standing towers; they were distributed at random around the circumference with a maximum crack width of
Cracks covered a very significant proportion of the height of shell. It was also reported that the cause of cracking might be attributed to wind-induced vibration or foundation movements.

Despite the conclusions given in the above-mentioned study as to the main cause for the collapse, the behaviour of a cracked vibrating cooling tower remains unknown. It is the aim of the present work to find out the effect of meridional cracking on vibrating cylindrical shells.

V.2 Vibrations of Rings and Arches

Prior to solving the problem of vibrations for cracked cylindrical shells, Fig. 5.1, a study of ring and arch vibrations is first presented, since the modes of vibrating cylinders in the circumferential direction are much the same as those of vibrating rings or arches.

On the literature concerning vibrations of arches, Lamb (1888) derived the differential equation for vibrations of a part of a circular ring and solved it only for the case where the length of the arch is small in comparison to the radius of curvature. The exact solution for an arch of any curvature becomes extremely complicated. Den Hartog (1928) presented an approximate technique (using the Rayleigh-Ritz method) for the solution of simply supported and clamped arches. In the present problem the Galerkin method in a slightly different manner to that used before will be considered.

For a complete circular ring vibrating in its plane, the natural frequency is reduced if cracks are induced across the
thickness of the ring. For analytical purposes, the bending stiffness of any cracked section may be neglected (i.e. each crack will be considered as a hinge). Therefore the maximum number of cracks that a ring can withstand is three since a system with four cracks (or hinges) becomes a mechanism whose frequency is zero.

V.2 a Equations of motion

The displacements and angles for the arch are shown in fig. 5.2a. Where \( w \) and \( v \) are the radial and circumferential displacements, \( \psi \) is a central angle having any value between 0 and \( 360^\circ \).

From the arch element of fig. 5.2b the equilibrium equations may be written

\[
\frac{1}{a} \frac{\partial Q}{\partial \phi} + \frac{N}{a} = \frac{m^2 v}{a t^2}
\]

\[
\frac{1}{a} \frac{\partial N}{\partial \phi} - \frac{Q}{a} = \frac{m^2 v}{a t^2}
\]

\[
\frac{1}{a} \frac{\partial M}{\partial \phi} + Q = 0
\]

where \( M, Q \) and \( N \) are the bending moment, shear force and normal force. The radius of the arch is denoted by \( a \) and \( m \) is the mass per unit volume.

The non-extensional condition for rings states

\[
\frac{\partial v}{\partial \phi} - v = 0
\]
The expressions of normal force $N$ and bending moment $M$ in terms of displacements for curved bar are

$$N = \frac{AE}{a} \left( \frac{\partial u}{\partial \phi} - w \right)$$

$$M = \frac{EI}{a^2} \left( \frac{\partial u}{\partial \phi} + \frac{\partial^2 w}{\partial \phi^2} \right)$$

from the last equation of (5.1) the expression for the shear force can be obtained.

$$Q = -\frac{1}{a} \frac{\partial M}{\partial \phi}$$

Substituting this last expression into the first two equations of (5.1), using relations (5.3) after some simplifications the two resulting equations are:

$$-\frac{\partial^4 w}{\partial \phi^4} - \frac{\partial^3 u}{\partial \phi^3} - \frac{ma}{EI} \frac{\partial^2 w}{\partial t^2} = 0$$

$$\frac{\partial^3 w}{\partial \phi^3} + \frac{\partial^2 u}{\partial \phi^2} - \frac{ma}{EI} \frac{\partial^2 w}{\partial t^2} = 0$$

which represent the balance of the components of forces along the radial and circumferential directions respectively.

V.2 b The Galerkin method

Galerkin's method was outlined previously when the field equations were reduced to only one differential equation. When the equilibrium of the system is described by a set of differential equations, the Galerkin method can also be used, (see for example Duncan (1937)).
In what follows, the procedure is described very briefly.

In elastic bodies the displacement field may be described by

\[ u = \sum_r q_r U_r(x, y, z) \]
\[ v = \sum_r q_r V_r(x, y, z) \quad (5.6) \]
\[ w = \sum_r q_r W_r(x, y, z) \]

where \( q_r \) is constant and \( U_r, V_r \) and \( W_r \) are chosen to satisfy the boundary conditions.

The equations of equilibrium may be written

\[ \sum_{e} X_x = 0 \]
\[ \sum_{e} Y_y = 0 \]
\[ \sum_{e} Z_z = 0 \quad (5.7) \]

where \( X, Y \) and \( Z \) are the components of force in the \( x, y \) and \( z \) directions. \( X_e, X_i \) and \( X_a \) are the elastic, inertia and external forces respectively in the \( x \) direction per unit volume.

Hence, the Galerkin equation is:

\[ \int \int \int [ U_r(X_e + X_i + X_a) + V_r(Y_e + Y_i + Y_a) + W_r(Z_e + Z_i + Z_a) ] dx dy dz = 0 \]
\[ (5.8) \]

Using expressions (5.5), Galerkin's equation (5.8) for arches takes the form

\[ \frac{\psi}{2} \left\{ \left[ \frac{3}{2} \frac{3}{2} \frac{h}{h} \frac{h}{h} - \frac{3}{2} \frac{3}{2} \frac{h}{h} \frac{h}{h} - \frac{h}{h} \frac{h}{h} \frac{h}{h} \frac{h}{h} \right] \psi + \left[ \frac{3}{2} \frac{3}{2} \frac{h}{h} \frac{h}{h} + \frac{3}{2} \frac{3}{2} \frac{h}{h} \frac{h}{h} - \frac{h}{h} \frac{h}{h} \frac{h}{h} \frac{h}{h} \right] \psi \right\} \left( \frac{3}{2} \frac{3}{2} \frac{h}{h} \frac{h}{h} + \frac{3}{2} \frac{3}{2} \frac{h}{h} \frac{h}{h} - \frac{h}{h} \frac{h}{h} \frac{h}{h} \frac{h}{h} \right) \]
\[ = 0 \]
\[ (5.9) \]
V.2 c Vibrations of a circular ring with two hinges

The vibrations of a ring with two cracks lying on a line of symmetry, fig. 5.3a can be solved by considering the vibrations of an arch which subtends an angle of 180°, simply supported and with free movements of the ends along the line of symmetry (horizontal direction is taken for convenience) fig. 5.3b.

In applying Galerkin's method to the present problem, the assumed solution for the displacements w and v should satisfy the boundary conditions.

At $\phi = \pm \frac{\pi}{2}$ fig. 5.3b, the conditions are prescribed by

\begin{align*}
v &= 0 \\
w &\neq 0 \\
Q &= \frac{dv}{d\phi} + \frac{d^2w}{d\phi^2} = 0 \\
M &= \frac{d^2v}{d\phi^2} + \frac{d^3w}{d\phi^3} = 0
\end{align*}

(5.10)

The last two conditions describe the vanishing of bending moment and shear force given by (5.3) and 5.4). Using the non-extensional condition (5.2) these two conditions become

\begin{align*}
\frac{d^2w}{d\phi^2} - w &= 0 \\
\frac{d^3w}{d\phi^3} + \frac{dw}{d\phi} &= 0
\end{align*}

(5.11)
For harmonic vibrations, a set of displacement functions $w, v$ which satisfy the boundary conditions (5.10) may be

$$w(\phi, t) = W(\phi) \cos \omega t$$
$$v(\phi, t) = V(\phi) \cos \omega t$$  \hspace{1cm} (5.12)

in which

$$W(\phi) = 79.79994 - 33.8207\phi^2 + 1.99711\phi^4 - 85.47495 \cos \phi$$

$$V(\phi) = 79.79994 - 11.2736\phi^3 - 0.39942\phi^5 - 85.47495 \sin \phi$$  \hspace{1cm} (5.13)

The displacement functions $W(\phi), V(\phi)$ may be found in the following manner:

Assume a function $f(\phi)$ for equations (5.11)

$$\frac{d^2 v}{d\phi^2} + w = f(\phi)$$

$$\frac{d^3 v}{d\phi^3} + \frac{dw}{d\phi} = \frac{df(\phi)}{d\phi}$$  \hspace{1cm} (5.111)

This function $f(\phi)$ can be a polynomial or a trigonometric series. Initially $f(\phi)$ and $\frac{df(\phi)}{d\phi}$ should vanish at $\phi = \pm \frac{\pi}{2}$ in order to satisfy the conditions (5.11). Finally by integrating the first equation of (5.111) and the non-extendational condition (5.2), $W(\phi)$ and $V(\phi)$ are found.

The assumed function $f(\phi)$ which yielded (5.13) is

$$f(\phi) = \frac{120}{\pi^2} - \frac{960}{\pi^4} \phi^2 + \frac{120}{\pi^6} \phi^4$$  \hspace{1cm} (5.14)

and the modes for $V(\phi)$ and $V(\phi)$ given by (5.13) are plotted in figure 5.4a.
Upon substitution of (5.12) into the Galerkin expression (5.9), performing the integrations between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$, the circular frequency $\omega$ is found:

$$\omega^2 = 2.3625 \frac{EI}{ma} \quad (5.15)$$

Another expression for $f(\phi)$ can be a trigonometric function

$$f(\phi) = 1 + \cos 2\phi \quad (5.16)$$

which yields

$$W(\phi) = 1 - \frac{1}{3} \cos 2\phi - \frac{\pi}{2} \cos\phi \quad (5.17)$$

$$V(\phi) = \phi - \frac{1}{6} \sin 2\phi - \frac{\pi}{2} \sin\phi$$

and for the circular frequency

$$\omega^2 = 2.36 \frac{EI}{ma} \quad (5.18)$$

The modes for the displacements $W(\phi)$ and $V(\phi)$ (5.17) given as trigonometric functions are also plotted in fig. 5.4b. It is seen that trigonometric expressions for $f(\phi)$ are more useful in this analysis, even through the improvement in the solution is not significant, the simplification in the numerical work is considerable.

For purposes of comparison, the following table gives the expressions for the circular frequency of arches with different boundary conditions.
TABLE 1

CIRCULAR FREQUENCY OF A RING AND ARCHES WITH DIFFERENT BOUNDARY CONDITIONS

<table>
<thead>
<tr>
<th>Condition</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Clamped arch</td>
<td>$\omega^2 = 19.18 \frac{EI}{ma}$</td>
</tr>
<tr>
<td>(2) Complete ring</td>
<td>$\omega^2 = 7.2 \frac{EI}{ma}$</td>
</tr>
<tr>
<td>(3) Simply supported arch</td>
<td>$\omega^2 = 5.15 \frac{EI}{ma}$</td>
</tr>
<tr>
<td>(4) Simply supported arch without restriction in the horizontal direction.</td>
<td>$\omega^2 = 2.34 \frac{EI}{ma}$</td>
</tr>
</tbody>
</table>

For solutions 1, 2 and 3 see Den Hartog (1928) and Timoshenko (1928). The result of the present analysis is given in (4).

From Table 1 it can be observed that the gradual release of the restrictions at the ends reduces the frequency. This behaviour is well known in vibrating beams.
V.3 **Vibration of Cracked Cylinders**

A similar technique used for arches is considered as a first approach to solve the problem of vibrating cylindrical shells with meridional cracks.

By using the set of equilibrium equations (2.22), the Galerkin equation (5.18) for cylindrical shells becomes

\[
\left\{ \begin{array}{l}
\int_0^{2\pi} \left[ \frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2} \frac{\partial^2 u}{\partial \phi^2} + \frac{1+\nu}{2} \frac{\partial^2 v}{\partial x \partial \phi} - \frac{\partial w}{\partial x} \right] u dx d\phi \\
+ \left[ \frac{1+\nu}{2} \frac{\partial^2 u}{\partial \phi^2} + \frac{1-\nu}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial \phi^2} - \frac{\partial w}{\partial \phi} \right] v dx d\phi \\
+ \left[ \frac{p}{2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial \phi} - \nu \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial t^2} \right] w dx d\phi = 0
\end{array} \right.
\]

\[(5.19)\]

V.3 a **Simply supported cylindrical shell with two meridional cracks**

Meridional cracks on a cylindrical shell are also considered as hinges in the circumferential direction which create new boundary conditions. For two cracks defining a symmetry plane in the cylinder (besides the boundary conditions in the longitudinal direction) the following conditions should be satisfied

\[
\begin{align*}
\left. \begin{array}{l}
v(x) = 0 \\
\phi = \pi \\
w(x) \neq 0 \\
\phi = 0 \\
w(x) \neq 0 \\
\phi = \pi
\end{array} \right\} (5.20)
\end{align*}
\]
Again the last two expressions in (5.20) define the bending moment \( M_\phi \) and shear force \( Q_\phi \) in the circumferential direction given by Flügge (1934) or Timoshenko (1940).

For harmonic vibrations

\[
\begin{align*}
v(\phi, x, t) &= V(\phi) \sin \lambda x \cos \omega t \\
w(\phi, x, t) &= W(\phi) \sin \lambda x \cos \omega t
\end{align*}
\]  

(5.21)

in which \( \lambda \) has been defined previously by equation (3.36).

The modes \( V(\phi) \) and \( W(\phi) \) are assumed to be in the form of those used in the arch analysis. From (5.17)

\[
\begin{align*}
V(\phi) &= A\phi + B\phi^3 + E\phi^5 - F \sin 2\phi - G \sin \phi \\
W(\phi) &= A + 3B\phi^2 + 5E\phi^4 - 2F \cos 2\phi - G \cos \phi
\end{align*}
\]  

(5.22)

the relationship between \( V(\phi) \) and \( W(\phi) \) is obtained by using the non-extensional condition (5.2). The reason for five constants \( A, B, E, F \) and \( G \) in (5.22) is to satisfy the five boundary conditions (5.20). Upon substitution of (5.21) into (5.20) a system of five equations results from which the constants \( A, B, E, F \) and \( G \) are
obtained (the displacement $W$ at $\phi = 0$ and $\phi = \frac{\pi}{2}$ for the second and third conditions in (5.20), are those obtained in the arch analysis Fig. 5.4b). For the example 2 of chapter III, it is found that

$$v = (5.98791 - 0.80450\phi^3 + 0.03338\phi^5 - 0.14049 \sin 2\phi - 6.60692\sin\phi)\sin \lambda x \cos \omega t$$

$$w = (5.98791 - 2.41350\phi^2 + 0.166904\phi^4 - 0.28098\cos 2\phi - 6.60692\cos \phi)\sin \lambda x \cos \omega t$$

(5.23)

This set of displacements satisfy the boundary conditions (5.20) around the circumference of the cylinder and the boundary conditions in the longitudinal direction for a simply supported cylindrical shell.

In view of (3.2) and (3.5) of section III.2, the expression for $U(\phi)$ can be similar to that of $W(\phi)$. However $W(\phi)$ is obtained by setting to zero the equilibrium equation corresponding to the $u$ direction, (first term in Galerkin's equation) i.e.

$$\frac{3^2 u}{\partial x^2} + \frac{1-v}{2} \frac{3^2 u}{\partial \phi^2} = -\frac{i+\nu}{2} \frac{3^2 v}{\partial x^2} + \nu \frac{3w}{\partial x}$$

(5.24)

Substituting $v$ and $w$ given by (5.23) and integrating, the expression for $u$ becomes

$$u = (1.81196 - 0.60185\phi^2 + 0.06507\phi^4 - 0.04002 \cos 2\phi - 1.79601\cos \phi)\cos \lambda x \cos \omega t$$

(5.25)

Finally by substituting $u$, $v$ and $w$ into the Galerkin equation (5.19) and integrating over the indicated limits the following relation is obtained

$$\omega = 0.0032714 - 0.8239627 P_x$$

(5.26)
Equation (5.26) is shown as a full line in fig. 5.5 and can be compared with the corresponding results for an uncracked shell.

In fig. 5.5 equation (5.26) shows higher values for $\sqrt{n}$ and $P_X$ than those minima corresponding to an uncracked shell. This result may be explained by remembering that equation (5.26) is associated with a circumferential wave number $n=2$ which was obtained in the analysis of arches, whereas the minimum values for an uncracked shell occurs when $n=5$. This is confirmed by observing in fig. 5.5 that the curve $n=2$ for an uncracked shell lies off the graph. Several other nodal patterns may be tried in order to lower the values given by equation (5.26).

The main difficulty in the present method seems to be in predicting the appropriate deflection functions which are associated with the minimum values of the frequency and initial axial load. Several solutions were assumed for a vibrating cylinder with four cracks and the results obtained were always a little higher than the minimum values of an uncracked shell. It was found in the experimental work (reported in the next chapter), that the nodal configuration associated with the lowest value of the frequency for a cylindrical shell changes as cracks along the meridians are induced on the shell. This experimental evidence makes it even more difficult to assume the correct displacement functions.

Even though the proposed method can be useful for arches and cylinders with small numbers of cracks, the numerical work increases considerably as the number of cracks on the cylinder increases. In view of the complexities and the amount of work necessary to solve the problem of a cylinder with a large number of cracks, the solution will be found starting from a different point of view.
When the modes of vibration of structures are uncoupled the lowest natural frequency is associated with a simple displacement shape, as is the case for vibrations of a stretched string or the flexural vibrations of a beam. When the modes of vibration are coupled and the strain energy in the structure has components due to two forms of displacement such as bending and stretching or bending and twisting then it is possible for the lowest natural frequency to be associated with a relatively complex displacement shape. Arnold and Warburton (1948) observed this phenomenon when investigating the vibration of cylindrical shells, and were able to explain this phenomenon from construction of the variation of strain energy due to bending and stretching of the shell with circumferential wave number $n$.

In the present work, the strain energy of the shell will be considered and the influence of shell thickness on the ratio of extensional to inextensional strain energy for a given circumferential wave number will be investigated.

Consider expression (2.10) for the strain energy for an unloaded cylindrical shell

$$S = \frac{Et}{2(1-v^2)} \int_0^{2\pi a} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 + 2v \frac{\partial u}{\partial x} \frac{\partial \psi}{\partial x} \frac{2v}{a} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial s} \frac{1-v}{2} \frac{\partial w}{\partial x} + \frac{3v}{2} \frac{\partial w}{\partial x} \right] dx ds$$

$$+ \frac{Et}{2(1-v^2)} \int_0^{2\pi a} \left[ \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial x} \right)^2 + 2v \frac{\partial v}{\partial x} \frac{\partial \psi}{\partial x} \frac{2v}{a} + \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial s} \frac{1-v}{2} \frac{\partial \psi}{\partial x} + \frac{3v}{2} \frac{\partial \psi}{\partial x} \right] dx ds$$

(5.27)
For a simply supported cylindrical shell, the displacement functions may be expressed

\[
\begin{align*}
    u &= U(t) \cos \lambda x \cos \phi \\
    v &= V(t) \sin \lambda x \sin \phi \\
    w &= W(t) \sin \lambda x \cos \phi
\end{align*}
\]  
\( (5.28) \)

where \( U(t) \), \( V(t) \), and \( W(t) \) are functions of time only.

Upon substitution of \( (5.28) \) into \( (2.28) \) the expression for the strain energy becomes

\[
S = \frac{\pi Eh^2}{4a(1-v^2)} \left\{ U^2 \lambda^2 + (nV-W)^2 + 2n\lambda U(W-nV) + \left(\frac{1-v}{2}\right)(\lambda V-nU)^2 \right. \\
+ \frac{h^2}{12a^2} \left[ \lambda^4 W^2 + n^2 W^2 + 2n^2 \lambda^2 W^2 + 2(1-v)\lambda^2 n^2 W^2 \right] \left\}
\]  
\( (5.29) \)

In harmonic vibrations \( U(t) \), \( V(t) \) and \( W(t) \) may be written

\[
\begin{align*}
    U &= A \cos \omega t \\
    V &= B \cos \omega t \\
    W &= C \cos \omega t
\end{align*}
\]  
\( (5.30) \)

By substituting the values for \( U \), \( V \) and \( W \) given by \( (5.30) \) into \( (5.29) \) the maximum strain energy takes the form

\[
S_{\text{max}} = \frac{\pi Eh^2 a^2}{4a(1-v^2)} \left\{ \lambda^2 \left(\frac{A}{C}\right)^2 + (1-n^2)^2 + 2n\lambda \frac{A}{C}(1-n^2) \lambda + \left(\frac{1-v}{2}\right)(\frac{B}{C} - \frac{A}{C})^2 \right. \\
+ \frac{h^2}{12a^2} \left[ n^2 + 2n\lambda^2 n^2 + 2(1-v)\lambda^2 n^2 \right] \left\}
\]  
\( (5.31) \)
The terms with coefficients \( \frac{h}{a} \) in this last expression correspond to the strain energy due to stretching. Those with coefficients \( \frac{h^3}{a^3} \) correspond to the strain energy due to bending.

Hence

**Stretching energy**

\[
S_s = \frac{E n i c^2}{4(1-v^2)} \eta_s
\]  

(5.32)

**Bending energy**

\[
S_b = \frac{E n i c^2}{4(1-v^2)} \eta_b
\]  

(5.33)

where \( \eta_s \) and \( \eta_b \) are the energy factors for stretching and bending given by

\[
\eta_s = \frac{h}{a} \left[ \frac{\lambda^2 (\frac{A}{c})^2 + (1-n^2 B_c)^2}{1+n^2 B_c} + 2v \lambda (\frac{A}{c}) + \frac{1-v^2}{2} (\frac{B}{c} - n^2) \right]
\]  

(5.34)

\[
\eta_b = \frac{h^3}{12a^3} \left[ \lambda^4 + 4n^2 \lambda^2 + 2(1-v)\lambda^2 n^2 \right]
\]  

(5.35)

the ratios \( \frac{A}{c} \) and \( \frac{B}{c} \) are obtained by considering the Lagrange equation

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{U}} \right) - \frac{\partial T}{\partial U} = - \frac{\partial S}{\partial U}
\]  

(5.36)

in which \( S \) is the strain energy and \( T \) the Kinetic energy given by (2.14)

\[
T = \frac{p h}{2} \int v \left( \dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right) dA
\]  

or by using (5.28)

\[
T = \frac{\pi p h a \lambda}{4} \left[ \dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right]
\]  

(5.37)
From (5.30), (5.31), (5.36) and 5.37)

\[
\left[ \lambda^2 + \left( \frac{1-\nu}{2} \right) n^2 - \Omega \right] A - \left( \frac{1-\nu}{2} \right) \lambda B + v \lambda C = 0 \tag{5.38}
\]

which, with the other two Lagrange equations in \( V \) and \( W \) similar to 5.36, gives

\[
-\left( \frac{1+\nu}{2} \right) \lambda n A + \left[ \left( \frac{1-\nu}{2} \right) \lambda^2 + n^2 - \Omega \right] B - n C = 0 \tag{5.39}
\]

\[\nu \lambda A - n B + \left[ 1 - \Omega + k \left( \lambda^2 + n^2 \right)^2 \right] C = 0\]

Finally from (5.38) and (5.39) the ratios \( \frac{A}{C} \) and \( \frac{B}{C} \) are

\[
\frac{A}{C} = \frac{n \frac{1+\nu}{2} \lambda - v \lambda^3 \frac{1-\nu}{2} - v \lambda n^2 + v \lambda \Omega}{\lambda^4 + \lambda^2 n^2 + \lambda^2 n^2 \left( \frac{1-\nu}{2} \right)^2 + n \left( \frac{1-\nu}{2} \right) \Omega \left( n^2 + 2 \lambda^2 + \Omega \right) - n^2 \Omega \left( \frac{1-\nu}{2} \right) - \left( \frac{1+\nu}{2} \right)^2 \lambda^2 n}
\tag{5.40}
\]

\[
\frac{B}{C} = \frac{n \lambda^2 + \left( \frac{1-\nu}{2} \right) n^3 - n \Omega - \left( \frac{1+\nu}{2} \right) \lambda^2 n}{\lambda^4 + \lambda^2 n^2 + \lambda^2 n^2 \left( \frac{1-\nu}{2} \right)^2 + n \left( \frac{1-\nu}{2} \right) \Omega \left( n^2 + 2 \lambda^2 + \Omega \right) - n^2 \Omega \left( \frac{1-\nu}{2} \right) - \left( \frac{1+\nu}{2} \right)^2 \lambda^2 n}
\]

The strain energy factors \( n_s \) and \( n_b \) given by (5.34) and (5.35) using ratios (5.40) are plotted against the number of circumferential waves \( n \) in fig. 5.6, for the cylinder tested experimentally

\( E = 30 \times 10^6 \text{ lb/in}^2, \ \nu = 0.3, \ l = 26'' , \ a = 3'' \ \text{and} \ h = 0.012'' \).

A decrease in stiffness of a shell can be achieved by considering a thinner cylinder with the other dimensions the same. Plotting the results for strain energy in the same fig. 5.6 for the cylinder whose thickness is \( h/2 \). The following conclusions can be drawn:
(a) The influence of shell thickness \( h \) on the strain energy factor due to bending, \( \eta_b \), is seen to be appreciable but the strain energy factor due to stretching \( \eta_s \) is seen to be almost independent of shell thickness.

(b) The circumferential wave number \( n \), associated with the minimum value of total strain energy factor \( (\eta_b + \eta_s) \), which corresponds to the lowest natural frequency, is seen to increase as the thickness of the shell is reduced.

V.5 **Stiffness and Natural Frequency of Cylindrical Shells**

From the previous analysis, it was seen that by reducing the thickness of the shell the total strain energy factor decreases. For the particular case studied when the thickness of the cylinder is reduced by \( \frac{1}{2} \), the minimum value for the total strain energy factor is lower and the number of circumferential waves associated with the minimum value of the strain energy changes from \( n = 3 \) to \( n = 4 \) (see fig. 5.6).

This conclusion lead to an investigation of the behaviour of the natural frequency as the thickness of the shell decreases. Using expression (3.39) the frequency is plotted against the number of circumferential waves \( n \) for the same cylinder previously considered for different thicknesses \( h \) and \( h/2 \) (see fig. 5.7).

It is also observed that by reducing the thickness of the shell, the natural frequency decreases and the nodal configuration in the circumferential direction \( n \) associated with the lowest value of the frequency also changes from \( n = 3 \) to \( n = 4 \).

The same behaviour was observed experimentally. By increasing the number of cracks (reducing stiffness) the curves of frequencies move gradually downwards to the right and become wider.
V.6 Effect of Meridional Cracking on the Natural Frequency of Vibrating Cylindrical Shells

In sections V.4 and V.5 the effect of changes in thickness on the natural frequency of vibrating cylinders was studied. Effect of cracks on the natural frequency can now be studied by finding the effect of cracks on the shell stiffness by using the natural frequency-stiffness relation.

Jones (1959) investigated the effect of small changes in mass and stiffness on the natural frequencies of an n-degree-of-freedom system. He considered the effect of small changes in the coefficients of the kinetic and strain energy functions when expressed in terms of normal co-ordinates. A practical disadvantage of his theory is that it requires a knowledge of the natural frequencies of the unmodified system ("original" system).

A similar technique will be considered in the present work since the original system (uncracked shell) is fully soluble.

Assuming the displacement field in the form

\[ u(x, \phi, t) = U_r(x, \phi) \sin \omega_r t \]
\[ v(x, \phi, t) = V_r(x, \phi) \sin \omega_r t \]
\[ w(x, \phi, t) = W_r(x, \phi) \sin \omega_r t \]

where \( U_r, V_r \) and \( W_r \) specify the rth normal mode of vibration whose associated natural circular frequency is \( \omega_r \).

The maximum kinetic energy (5.37) can be written

\[ T_r = \frac{1}{2} \omega_r^2 \int_0^{2\pi} \int_0^t \rho h [U_r^2 + V_r^2 + W_r^2] \, d\phi \, dx \]
and the maximum strain energy (5.27)

\[ S_r = \frac{E}{2(1-\nu^2)} \int_0^{2\pi} \int_0^l \left( \frac{3V_r}{a} + \frac{1}{a^2} \left( \frac{3V_r}{a} - W_r \right)^2 \right) \, dx \, d\phi + \frac{1-\nu}{2} \left( \frac{1}{a} \frac{3U_r}{a} + \frac{3V_r}{a} \right)^2 \, dx \, d\phi \]

\[ + \frac{E}{24(1-\nu^2)} \int_0^{2\pi} \int_0^l \left( \frac{3^2W_r}{a^2} + \frac{1}{a^4} \left( \frac{3^2W_r}{a^2} \right)^2 \right) \, dx \, d\phi + \frac{2(1-\nu)}{a^2} \left( \frac{3^2W_r}{a^2} \right)^2 \, dx \, d\phi \]

(5.43)

The field displacements (5.41) can be expressed in terms of the normal modes, i.e.

\[ u = \sum_{s=1}^{n} \chi_s U_s \]

\[ v = \sum_{s=1}^{n} \chi_s V_s \]

\[ w = \sum_{s=1}^{n} \chi_s W_s \]

(5.44)

where the quantities \( \chi_s \) are the normal co-ordinates.

It may be shown that expressions for kinetic and strain energy \( T \) and \( S \) can be written in terms of the normal co-ordinates \( \chi_s \)

\[ T = \sum_s \frac{T_s}{\omega_s^2} \left( \frac{d\chi_s}{dt} \right)^2 \]

(5.45)

\[ S = \sum_s T_s \chi_s^2 \]

(5.46)
Denoting the modal shapes and frequency of the modified system (by $\bar{U}_r$, $\bar{V}_r$, $\bar{W}_r$ and $\bar{\omega}_r$),

then

$$\bar{U}_r = U_r + \Delta U_r$$

$$\bar{V}_r = V_r + \Delta V_r \quad (5.47)$$

$$\bar{W}_r = W_r + \Delta W_r$$

and

$$\bar{\omega}_r^2 = \omega_r^2 + \Delta(\omega_r^2) \quad (5.48)$$

where $\Delta U_r$, $\Delta V_r$, $\Delta W_r$ and $\Delta(\omega_r^2)$ are the changes in mode shape and natural frequency due to a change in the stiffness of the shell. The mass of the cracked shell remains constant thus no corrections to $\Delta U_r$, $\Delta V_r$, $\Delta W_r$ and $\Delta(\omega_r^2)$ are necessary.

The modes $\bar{U}_r$, $\bar{V}_r$ and $\bar{W}_r$ may be expressed in terms of the normal modes of the original system.

$$\bar{U}_r = \sum_s \chi_s(r) U_s$$

$$\bar{V}_r = \sum_s \chi_s(r) V_s \quad (5.49)$$

$$\bar{W}_r = \sum_s \chi_s(r) W_s$$

where the normal co-ordinates $\chi_s(r)$ are to be determined.

Choosing the following condition

$$\chi_r(r) = 1 \quad (5.50)$$
then, expressions (5.49) and (5.47) become

\[ \Delta U_r = \sum_{s \neq r} x_s(r) U_s \]

\[ \Delta V_r = \sum_{s \neq r} x_s(r) V_s \]  \hspace{1cm} (5.51)

\[ \Delta W_r = \sum_{s \neq r} x_s(r) W_s \]

It is understood that in (5.51) the summation is for all values of \( s \) except \( s = r \).

For free vibrations the displacement field can be given by

\[ u = \overline{U}_r \sin \overline{\omega}_r t \]

\[ v = \overline{V}_r \sin \overline{\omega}_r t \]  \hspace{1cm} (5.52)

\[ w = \overline{W}_r \sin \overline{\omega}_r t \]

This assumption is valid only if the changes in the original system are small. Upon substitution of (5.44) and (5.49) into (5.52) it yields

\[ x_s = x_s(r) \sin \overline{\omega}_r t \]  \hspace{1cm} (5.53)

The maximum strain energy of the modified system is

\[ S_r = \sum_{s} T_s x_s(r) + \delta S_r \]  \hspace{1cm} (5.54)

where \( \delta S_r \) is the change of strain energy due to the change in stiffness of the modified system, from (5.43) \( \delta S_r \) is given by
This expression gives the increase or decrease in strain energy for the modified system. For the problem concerning cracks, $\delta h$ in (5.55) is the decrease in thickness and $\alpha$ is the subtended angle for the decreased thickness, fig. 5.8. The integration with respect to $x$ has to be performed from zero to $\alpha$ if cracks are induced in the whole length of the cylinder.

The maximum kinetic energy remains unchanged since for a cracked section the mass remains constant.

Hence from (5.45) and (5.53)

$$T_r' = \sum \frac{T_s}{\omega_s^2} \chi_s^2(r)$$

Equating (5.54) and (5.56)

$$\left( \frac{\omega_r^2 - \omega_r^2}{\omega_r^2} \right) \frac{T_r}{\omega_r^2} + \sum \chi_s^2(r) \left( \frac{\omega_r^2 - \omega_s^2}{\omega_s^2} \right) \frac{T_s}{\omega_s^2} = \delta r = 0$$
Applying Rayleigh's principle which states

\[ \frac{\partial \omega_r}{\partial \chi_s(r)} = 0 \quad s \neq r \quad (5.58) \]

from which \((n-1)\) equations can be obtained.

Differentiating (5.57) with respect to \(\chi_s(r)\), it is obtained for \(s \neq r\)

\[ 2\chi_s(r)(\omega_r^2 - \omega_s^2) \frac{T_s}{\omega_s^2} - \frac{3}{3} \frac{\partial \delta S_r}{\partial \chi_s(r)} = 0 \quad (5.59) \]

With the \((n-1)\) equations given by (5.59) and the equation (5.57) it is possible to obtain the \(n\) quantities \(\chi_s(r) \ (s \neq r)\) and \(\omega_r\).

Multiplying (5.59) by \(\frac{1}{2} \chi_s(r)\)

\[ \sum_{s \neq r} \chi_s(r) \left( \omega_r^2 - \omega_s^2 \right) \frac{T_s}{\omega_s^2} - \frac{3}{3} \frac{\partial \delta S_r}{\partial \chi_s(r)} \chi_s(r) = 0 \quad (5.60) \]

Subtracting (5.60) from (5.57)

\[ \left( \omega_r^2 - \omega_r^2 \right) \frac{T_r}{\omega_r^2} - \delta S_r + \frac{3}{3} \frac{\partial \delta S_r}{\partial \chi_s(r)} \chi_s(r) = 0 \quad (5.61) \]

Differentiating (5.55) with respect to \(\chi_s(r)\), making use of (5.49), (5.51) and assuming

\[ \overline{U}_r = U_r \]
\[ \overline{V}_r = V_r \]
\[ \overline{W}_r = W_r \]

equation (5.61) can be written
\[(\omega_r^2 - \omega^2) \int_0^{2\pi} \int_0^l \rho \left[h \left( u_r^2 + v_r^2 + w_r^2 \right) + \frac{1}{a^2} \left( \frac{\partial v_r}{\partial x} - \frac{\partial u_r}{\partial \phi} \right)^2 + \frac{2\nu}{a} \frac{\partial v_r}{\partial x} \left( \frac{\partial v_r}{\partial \phi} - \frac{\partial w_r}{\partial \phi} \right) \right] dx \, d\phi = 0 \]  

(5.62)

where

\[
\delta s_r = \frac{E}{2(1-\nu^2)} \int_0^l \int_0^\alpha \delta h \left\{ \left( \frac{\partial u_r}{\partial x} \right)^2 + \frac{1}{a^2} \left( \frac{\partial v_r}{\partial \phi} - \frac{\partial u_r}{\partial \phi} \right)^2 + \frac{2\nu}{a} \frac{\partial v_r}{\partial x} \left( \frac{\partial v_r}{\partial \phi} - \frac{\partial w_r}{\partial \phi} \right) \right\} dx \, d\phi + \frac{E}{2b(1-\nu^2)} \int_0^l \int_0^\alpha \delta h \left\{ \left( \frac{\partial^2 w_r}{\partial x^2} \right)^2 + \frac{1}{a^2} \left( \frac{\partial^2 w_r}{\partial \phi^2} \right)^2 \right\} dx \, d\phi
\]

(5.63)

Equation (5.62) gives the relationship between the frequencies of the unmodified and modified systems in terms of the shell stiffness. Calculations were carried out for different numbers of cracks with \( a = 10^6 \) and \( \delta h = \frac{h}{2} \) (see fig. 5.8). For the solution, the following displacement field was assumed

\[
U_r = \cos \phi \cos \frac{m\pi x}{l}
\]

\[
V_r = \sin \phi \sin \frac{m\pi x}{l}
\]

\[
W_r = \cos \phi \sin \frac{m\pi x}{l}
\]

The result is plotted in fig. 5.9 for \( m = 1 \) and different values of \( n \). The solution is bounded by two curves (case 1; uncracked shell with thickness \( h \), case 5; uncracked shell with thickness \( \frac{h}{2} \) (see fig. 5.8)). Intermediate curves were found by
using expression (5.62) for different numbers of cracks given as a percentage of the wholly cracked shell (the shell is supposed to be wholly cracked when the whole thickness is considered as $\frac{h}{2}$). It is observed that the error in the solution increases as the percentage of cracking increases (curve for 45% becomes closer to that of 22%), this is not surprising since the theory was developed for small changes in stiffness. However, when the percentage of cracking is small (say less than 40%) the results are in very good agreement with the experimental results described in the next chapter.
Fig 5.1 MERIDIONAL CRACKING OF A CYLINDRICAL SHELL

Fig 5.2a ARCH DISPLACEMENTS

Fig 5.2b ARCH ELEMENT
Fig 5.3a Ring with two hinges

Fig 5.3b Simply supported arch with free horizontal movements at the ends (idealization of a ring with two hinges)

(a) Radial (ω) and tangential displacements given by (5.13)

(b) Radial (ω) and tangential displacements given by (5.17)

Fig 5.4 FUNDAMENTAL MODES OF VIBRATION FOR THE ARCH SHOWN IN Fig 5.3b
Fig. 5.5. SOLUTION FOR A VIBRATING CYLINDER WITH A CIRCUMFERENTIAL WAVE NUMBER \( n = 2 \)
Fig.5.6. COMPARISON OF STRAIN ENERGY FOR A CYLINDER WITH DIFFERENT THICKNESS $h$ AND $h/2$
Fig. 5.7 COMPARISON OF FREQUENCIES FOR A CYLINDER WITH DIFFERENT THICKNESS $h$ AND $h/2$.
Case 1 uncracked cylinder

Case 2 11% of cracking

Case 3 22% of cracking

Case 4 45% of cracking

Case 5 uncracked cylinder with thickness $h/2$

Fig 5.8 CRACKING IDEALIZATION
SIMPLY SUPPORTED CYLINDRICAL SHELL

\[ \omega = 0 \quad M_x = 0 \quad \text{At } x = 0, \frac{L}{2} \]

\[ L = 26'' \quad a = 3'' \quad h = 0.012'' \]

- Case 1: uncracked cylinder
- Case 2: 11% of cracking
- Case 3: 22% of cracking
- Case 4: 45% of cracking
- Case 5: uncracked cylinder with thickness \( h/2 \)

Fig. 5.9. DECREASE IN FREQUENCY OF A CYLINDER USING PRESENT THEORY (Expression 5.62)
CHAPTER VI

EXPERIMENTAL WORK

VI.1 Introduction

The aim of the experimental work was to confirm the theoretical predictions of the dynamic behaviour of cylindrical shells when subjected to externally applied loads and to investigate the influence of meridional cracks on the modes of vibration.

The first part consisted of a careful search for the natural frequencies of a simply supported cylinder which were taken as a basis for comparison for the subsequent work. The cylinder was then loaded under axail load, bending moment and axial load combined with bending moment carrying out the frequency analysis in each case. Two stainless steel cylinders of the same geometry were tested, one of them to the point of collapse under axial load. Finally, experiments were carried out on an unloaded cylinder with two, four and eight cracks in the meridional direction.

VI.2 Test Specimens

Two test specimens of stainless steel were used with a length to radius ratio \( \frac{l}{a} \) of 8.66, a radius to thickness ratio \( \frac{a}{h} \) of 250 and a thickness of 0.012 inches. Such dimensions were selected because of ease in finding the nodal configuration of the cylinder and to reduce the effects of the boundary conditions, as in practice the boundary conditions assumed in the analysis are difficult to achieve. Specimens were each fabricated from
a steel sheet rolled and welded. The weld appeared to have no appreciable effect on the results obtained.

For computational purposes, the properties of steel used were Young's modulus $30 \times 10^6$ lb/sq.in., Poisson's ratio of 0.3 and $\rho = 7.33 \times 10^{-4}$ lb-sec$^2$/in$^4$.

VI.3 Test Apparatus

The test apparatus was designed in order to apply axial load and bending moment separately or simultaneously on a cylindrical shell with simply supported edges.

The cylinder was supported at its ends in a rigid rig as shown in Figs. 6.1 and 6.3. The end pieces designed to provide the simply supported condition consisted of a thick circular plate, a ring and a spindle as illustrated in Fig. 6.4. The thick circular plate was accurately machined to fit the bore of the cylinder and shaped to provide as near as possible line contact with the inner surface. The ring was adjusted to the outside diameter of the cylinder by means of four screws in order to avoid any possibility of slackness in the fitting. The plate and the ring were then supported by the spindle which was located in appropriate slots of the rigid rig. In this manner the cylinder was pinned to the rig allowing movements of the cylinder in the axial direction since the spindle could slide without friction into the slot in which it was located.

The cylinder was loaded with two hydraulic jacks and pressures were measured with calibrated transducers placed on the noses of the jacks. In order to avoid eccentricities, each
pressure was transmitted through a ball bearing. For the axial force test, the ball bearing was located in the centre of the thick circular plate as can be seen in Fig. 6.4. For the bending moment test, a second jack was placed on the rig at a distance of ten inches from the longitudinal-axis of the cylinder. This was used to load an arm attached to the top ring in order to apply a bending moment to the cylinder, as shown in Figs. 6.1 and 6.3.

The shell was excited by an electro-mechanical vibrator driven by a decade oscillator through an audio amplifier. The vibrator was located in the middle of the length of the cylinder. Local excitation was discovered in the region near the vibrator, consequently the measurements were made at a distance from the vibrator where these localised effects were negligible. A double-beam oscilloscope, a filter and piezo-electric crystals were used to measure frequencies and to find nodal configurations. This apparatus was connected as shown in Fig. 6.5.

The test procedure consisted firstly of applying the desired axial load and/or bending moment to the cylindrical shell. Secondly the frequency of excitation was varied by means of the oscillator until a resonant frequency was obtained. The presence of such a frequency was indicated by a sharp rise in the noise emitted by the vibrating cylinder. With fine adjustments of the oscillator the maximum intensity of the note emitted was obtained. In order to ensure that the cylinder was vibrating with the same frequency as that of the excitation, the cylinder response was detected with a piezo-electric crystal cemented in the middle of the cylinder whose output was fed into the double-beam oscilloscope via the filter. In addition, use was made of
a piezo-electric crystal cemented to the tip of a probe which could be placed at any point on the cylinder. By connecting this to the double-beam oscilloscope via the filter, the nodal configurations were traced for each natural frequency.

VI.4 **Comparison of Test Results With Theory**

VI.4 a **Free vibrations of a simply supported cylindrical shell**

Before the application of loads, an investigation of the free vibrations of the unstressed shell was carried out. Figs. (6.6), (6.7) and (6.8) show the experimental and theoretical curves for the cylinder tested. For the theoretical analysis expression (3.39) was used since it was demonstrated previously that for the ratio $\frac{H}{a}$ of the specimen tested ($\frac{H}{a} = 8.66 < 20$) accurate results are obtained. The discrepancy between the theoretical and experimental curves may be explained by the following

(a) Omission of the in-plane inertia terms $\frac{3^2u}{\partial t^2}$, $\frac{3^2v}{\partial t^2}$ in the theoretical analysis.

When the in-plane inertia terms are taken into account, the values of the frequency are lower than those obtained when only radial inertia is considered. For the present case, this difference in values of the frequency is very small.

(b) Effect of imperfections

During the experiment two natural frequencies were found for each nodal configuration. This phenomenon was first discussed by Rayleigh in 1894 as a result of imperfections. Tobias (1951)
carried out some experiments to study the effect of imperfections and found that the difference in these two frequencies is a measure of the degree of imperfection. In the present experiment the difference of frequencies for one nodal configuration was of the order of \( \frac{1}{4} \) per cent.

(c) Effect of boundary conditions.

The end pieces of the cylinder satisfy reasonably the simply supported condition, but due to the fact that such end pieces were designed to withstand external loads, the shell itself was clamped into the end pieces. However, the effect of the boundary conditions is not very influential because of the dimensions of the cylinder.

It can be concluded that from an engineering point of view, the experimental results are in general in good agreement with those predicted by theory.

VI.4 b Vibrations of an initially loaded simply supported cylindrical shell

This phase of the experiment consisted firstly of applying the initial load to the cylinder and then tracing nodal configurations for each frequency detected between 100 and 1250 cycles per second.

Fig. 6.9 shows the theoretical and experimental analysis for the cylinder tested under direct axial load. The theoretical solution was obtained using expression (3.39) which predicts a buckling load of 7.5 tons (point A). The agreement of decrease in frequency as the axial load increases is quite good in the region where the axial load is approximately \( \frac{1}{3} \) of the critical load, but beyond this limit the experimental curve deviates
from the theoretical curve and collapse occurred at 2.97 tons 
(point B in Fig. 6.9). The difference between the experimental 
and theoretical buckling load indicates that effect of imperfections 
is far more important in stability than in dynamic analysis. In 
addition, it is also observed in fig. 6.9 that for the curves 
studied n = 2, 3, 4 and 5 were found to converge to a common point 
(the experimental critical load in buckling point B) as the load 
increases. This is contrary to theoretical predictions which 
associate each nodal configuration with different buckling loads. 

The experimental results obtained suggest that more work 
should be done using large deflection theory as results given 
by linear analysis are in poor agreement with those obtained by 
experiment.

With regard to vibrations of a cylinder under bending 
moment, the results are quoted in fig. 6.10. The bending moment 
was increased gradually. For each applied bending moment, the 
lowest natural frequency was found to be lower than that predicted 
by theory in all cases (the analytical discussion is given in 
Section IV of this thesis). It was also found that the lowest 
natural frequency was associated with the circumferential wave 
number n = 4. As the load was increased the circumferential 
configuration remained the same (n = 4), but the magnitude of 
the displacements in the tension zone increased. This observation 
agrees quite well with the theoretical analysis which predicts 
that the most influential nodal configuration in bending corresponds 
to that for vibration of a cylinder under static axial load.

For the case of axial load and bending moment applied 
simultaneously to the cylinder, the test was carried out in the 
same manner. The lowest natural frequencies were measured for
a constant axial load and an increasing bending moment. The procedure was repeated for several values of axial load. Results are shown in fig. 6.11 for an axial load of 1 ton. The same behaviour of a decrease in natural frequency as the external loads increase was observed.

VI.4c Vibrations of cracked cylindrical shells

Cracks on the shell were produced by scratching the surface of the cylinder. Guides were placed on the cylinder along the meridians. By drawing a sharp chisel along the guides cracks were induced. Due to the small thickness of the shell it was not easy to regulate the cracks' depth. However, precautions were taken to avoid cutting completely through the thickness of the shell.

With two cracks, the cylinder was made to vibrate and its frequency behaviour was then investigated. The procedure was repeated for the same cylinder with four and eight cracks. The corresponding results are shown in figs. 6.12, 6.13 and 6.14. It is observed that as the number of cracks increases the natural frequency for any nodal configuration decreases, and the number of circumferential waves, n, associated with the minimum value of the frequency differs from that for an uncracked shell. This phenomenon was clearly defined for different values of the axial wave number m which confirms the theoretical analysis given in the previous chapter where the cracks' depth was considered to be $h/2$. Excellent agreement is observed as it was shown theoretically that the number of circumferential waves n changes from 3 to 4 when large numbers of cracks are induced in the cylinder.
Fig 6.1 PHOTOGRAPH OF THE TESTED CYLINDER

Fig 6.2 ELECTRONIC EQUIPMENT
Fig. 6.5
VIBRATING EQUIPMENT USED FOR HARMONIC EXCITATION
Fig 6.6  COMPARISON OF FREQUENCIES FOR AN UNLOADED SIMPLY SUPPORTED CYLINDRICAL SHELL FOR m=1 AND DIFFERENT VALUES OF n
Fig 6.7 COMPARISON OF FREQUENCIES FOR AN UNLOADED SIMPLY SUPPORTED CYLINDRICAL SHELL FOR $m=2$ AND DIFFERENT VALUES OF $n$
Fig 6.8 COMPARISON OF FREQUENCIES FOR AN UNLOADED SIMPLY SUPPORTED CYLINDRICAL SHELL FOR m=3 AND DIFFERENT VALUES OF n
SIMPLY SUPPORTED CYLINDRICAL SHELL

$\omega = 0 \quad M_x = 0 \quad \text{At} \ x = 0, \frac{p}{a}$

$a = 8.06 \quad \alpha = 250 \quad \phi = 0.3$

$m = 1$

$\circ$ — Theoretical

$\triangle$ — Experimental

Fig 6.9  COMPARISON OF RESULTS OF FREQUENCY FOR A CYLINDER UNDER AXIL LOAD
Fig. 6.10 COMPARISON OF RESULTS OF FREQUENCY
FOR A CYLINDER UNDER BENDING MOMENT
Fig. 6.11. COMPARISON OF RESULTS OF FREQUENCY
FOR A CYLINDER UNDER BENDING MOMENT
AND AN AXIAL FORCE OF 1 TON
**Fig. 6.12.** EXPERIMENTAL RESULTS FOR A CRACKED CYLINDER FOR $m = 1$
SIMPLY SUPPORTED CYLINDRICAL SHELL
\[ \omega = 0 \quad M_k = 0 \quad \text{At} \quad x = 0, \quad \ell_2/2 \]
\[ \ell_2 \approx 8.66 \quad a/h = 250 \quad \nu = 0.3 \]
ANALYTICAL SOLUTION
- 0 uncracked shell
- EXPERIMENTAL RESULTS
- \( \triangle \) uncracked shell
- • two cracks
- † four cracks

Fig. 6.13. EXPERIMENTAL RESULTS FOR A CRACKED CYLINDER FOR \( m = 2 \)
SIMPLY SUPPORTED CYLINDRICAL SHELL

\[ \omega = 0 \quad M_x = 0 \quad \text{At} \quad x = 0 \quad \ell / 2 \]

\[ \ell / 2 = 8.66 \quad \alpha / h = 250 \quad \nu = 0.3 \]

ANALYTICAL SOLUTION

- uncracked shell

EXPERIMENTAL RESULTS

- uncracked shell
- two cracks
- four cracks
- eight cracks

Fig. 6.14 EXPERIMENTAL RESULTS FOR A CRACKED CYLINDER FOR \( m = 3 \)
CHAPTER VII

CONCLUSIONS

It has been demonstrated that simple expressions can be used which yield results of sufficient accuracy in predicting the vibration characteristics of cylindrical shells. The Donnell type equation has limited applicability being confined to shells within the range $0 < \frac{\lambda}{a} < 20$. This restriction has been removed by considering a Morley type equation which was shown to be valid for all values of $\frac{\lambda}{a}$ and retains the same simplicity as that of Donnell. Furthermore, the analysis becomes simpler for long shells ($\frac{\lambda}{a} > 10$) as the same assumption given by Donnell can be used ($|\frac{\lambda}{n}|^2 < 1$) without any significant effect on the exact solution.

The behaviour of vibrating cylindrical shells under initial external loads was investigated. It was shown theoretically for the type of loads studied (axial, torsion, bending), that the frequency decreases as the external load increases. In the limit the frequency tends to zero and the external load becomes the critical load of buckling. This behaviour was confirmed with the experimental work for the following initial external loads,

(i) axial force
(ii) bending moment
(iii) axial force combined with bending moment.

The comparison of theoretical and experimental results showed good agreement when the load is increased to a point near to collapse, which for the cylinder tested was a third of the theoretical critical load of buckling. The discrepancy of results
beyond this point is probably due to the effect of initial imperfections and makes it worthwhile to reconsider the problem including non-linear geometric terms in the analysis.

Additionally it was shown that by producing the same decrease in frequency the maximum stress $P_1$ due to the external bending moment is higher than the stress $P_0$ of axial compression. It was found that the value of $P_1$ was about 1.4 times the value of $P_0$ for the cylinder considered. Since the analytical solution for axial load is simpler, the above result suggests that this solution may be used in the design of vibrating cylinders subjected to an external bending moment.

The effect of axial load combined with torsion on the dynamical behaviour of cylinders was also studied, and it was found that torsion has a more predominant effect on the natural frequency of the cylinder, the decrease in frequency being more sensitive for a torque than for an axial load. Further, when only torsion is applied, the nodal configuration associated with the lowest natural frequency changes as the torque is increased. The change in nodal configuration depends on the geometry of the cylinder. In the case of axial load the nodal configuration of any natural frequency remains the same during the process of increasing the axial load.

The effect of constant axial load combined with fluctuating axial load and fluctuating bending moment on vibrating cylindrical shells was also investigated. It was shown that the region of instability for static and fluctuating axial load becomes significant only for large values of the static load. For the cylinder considered which was selected to represent one of the Ferrybridge
cooling towers, the regions of instability turned out to be negligible if the self-weight of the cooling tower is taken as the external axial load. Again for design purposes of vibrating cylinders under fluctuating bending moment, the solution for fluctuating axial load can be used which is analytically simpler and is on the safe side.

The behaviour of cylinders with meridional cracking was finally investigated. It was found that the natural frequency of the cylinder decreases as the number of cracks increases. The circumferential wave number n associated with the lowest natural frequency increases as the number of cracks increases. This theoretical prediction was shown to be in good agreement with the experimental results.
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