THE MATHEMATICAL MODELLING OF
BALL-JOINTS WITH FRICTION

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Thesis submitted for the degree of Doctor of Philosophy at
the University of Leicester.

June 1987
ACKNOWLEDGEMENTS

The author would like to express his gratitude for the assistance provided by Professor G. D. S. MacLellan during the course of this work. Thanks are due to the Central Electricity Generating Board for their financial assistance which made the completion of the project possible, and to Dr. C. Osgood at the Berkeley Nuclear Laboratories. The author would also like to thank the staff of the drawing office, laboratories and workshop of the Department of Engineering; in particular Mr. D. Pratt, Mrs. K. P Baglin, Mr. C. Morrison and especially Mr. P. Preem.
Nomenclature

Unless otherwise stated at the appropriate point in the text the notation used in this thesis is as follows:

- \(a, b, c, d, e, g, k\) - Dimensions within the spherical joint

- \(c_i = \frac{1-v_i}{E_i}\) - Young's moduli

- \(E_1, E_2\) - Resultant frictional force

- \(f_1 = \frac{F}{N}\) - Applied load

- \(f_2 = \frac{M_S}{rN}\) - Pressure

- \(f_3 = \frac{M_R}{rN}\) - Maximum pressure

- \(I = \int_0^{\infty} 2\sin \alpha \cos \alpha \cos \left(\frac{\alpha \pi}{2}\right) d\alpha\) - Normal reaction

- \(l_x, m_x, n_x\) - Direction cosines of a vector x

- \(M_S\) - Moment of slip friction

- \(M_R\) - Moment of rotational friction

- \(M\) - Total frictional moment

- \(N\) - Normal reaction

- \(P\) - Applied load

- \(p\) - Pressure

- \(P_0\) - Radii of socket and ball

- \(r\) - Radius of joint

- \(x = \frac{f_1}{\sqrt{1 + f_1^2}}\) - Radius of joint
\( \alpha, \beta \) - Angular co-ordinates of a point on the contact area

\( \alpha_0 \) - Angle defining the extent of the contact area

- Angle between the axis of rotation and the line of action of the normal reaction

\( \theta \) - Angle between the axis of rotation and the direction of the applied load

\( \mu \) - Coefficient of friction

\( v \) - Poisson's ratio

\( w \) - Angular velocity
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CHAPTER ONE

Introduction

The Central Electricity Generating Board relies heavily on mechanical systems in the control and operation of its generating plant. In its nuclear plants many of these mechanisms operate in the hostile environment of the reactor core where lubrication in the conventional sense is not possible. As a result, frictional forces can be relatively high and so must be taken into consideration in design assessments.

To aid in the design and assessment of its plant items the CEGB has developed (in collaboration with Mechanical Dynamics Inc. of Ann Arbor, Michigan) two computer programs AMP2D (Advanced Mechanisms Program for 2-Dimensions) and AMP3D-ADAMS (Advanced Mechanisms Program for 3-Dimensions - Automatic Dynamic Analysis of Mechanical Systems). AMP2D was developed to simulate a two-dimensional mechanical system of links, pin-joints, pulleys, etc., with a minimum of input data and AMP3D-ADAMS was developed to extend the range of simulations to three-dimensional mechanisms.

A model of the effects of friction within a two-dimensional mechanism has been incorporated into AMP2D
(Threlfall, 1978). This model is based on the assumption that the frictional force produced is directly proportional to the normal reaction, although, in order to overcome the computationally undesirable effect of a step change in the frictional force on the reversal of motion, the model also assumes that the friction produced is displacement and velocity dependent at velocities close to zero.

Unfortunately, the friction model used in AMP2D cannot be simply adapted for use in AMP3D-ADAMS. The reason for this can be seen when considering the characteristics of the additional joints that have to be modelled in order to simulate three-dimensional mechanisms, particularly those of the spherical joint. In such a joint the resistance to the relative motion due to friction is determined not only by the coefficient of friction and the magnitude of the loading, but also by the nature of the pressure distribution within the joint which results from that loading. Consequently the frictional effects produced can no longer be readily determined by the simple expressions used in the friction model incorporated into AMP2D. There is a further difficulty in adapting this model in that the expressions for the velocity and displacement dependence of the frictional forces produced cannot be simply extended to joints where the sliding velocity is not constant over the surface of contact, as is the case with the spherical joint.
Thus, in order to include frictional forces in the simulations provided by AMP3D-ADAMS or any other three-dimensional mechanism simulation package, it is necessary to develop a mathematical model of the effects of friction in a spherical joint. The basis of this thesis has been the development of such a model and its assessment by comparing the results that it provides with those obtained experimentally.

Chapter two of this thesis gives the background to the project, describing the two programs AMP2D and AMP3D-ADAMS and in some detail the friction model incorporated into AMP2D. An extensive literature survey revealed that little work had been done in the area of the mathematical modelling of friction in spherical joints. The one significant paper that did come to light, as a result of a search done by the CEGB on the DAILTECH database, was that by Smirnov (1981) which gave expressions for the frictional force and moments produced in a spherical joint. These expressions were found to be incorrect but when corrected they became the basis for the required mathematical model. The derivation of these expressions is described in chapter three. Chapter four extends the analysis to take into account the interaction between the frictional forces produced and the pressure distribution in the joint. Other refinements necessary to provide an adequate model are described in chapter five.
The theory indicates that the angle between the direction of the applied load and the axis of rotation of the joint is a major factor in determining the magnitude and direction of the frictional moment produced in a spherical joint. In order to verify this prediction an experimental rig was designed to allow the frictional moment on a socket, produced by a ball rotating in it, to be measured for a range of values of this angle. The apparatus and the experimental procedure, together with a summary of the results obtained, are described in chapter six. The results were obtained for various values of the coefficient of friction in the range from 0.05 to 1.1.

A comparison of these results obtained experimentally with those given by the theoretical analysis confirms the assertion that the angle described above is a significant factor in the determination of the frictional moment produced in a spherical joint. It also indicates that the model developed provides a satisfactory means of determining the frictional effects produced in a spherical joint.
CHAPTER TWO

Background

2.1 AMP2D

AMP2D, the 'Advanced Mechanisms Program for 2-Dimensions', (Osgood, Threlfall, 1983) is the third in a series of computer programs designed to simulate two dimensional mechanical systems. This series began with the program DAMN (Smith, 1971) which was developed at the University of Michigan under the guidance of Professor M. A. Chace. DAMN was then refined and developed to produce the program DRAM, the 'Dynamic Response of Articulated Machinery' (Chace, Angell, 1975). An early version of this program was bought by the CEGB in 1977 and used as the basis for AMP2D. Meanwhile, the team at the University of Michigan formed their own company called Mechanical Dynamics Inc., and continued to develop DRAM along different lines from AMP2D.

AMP2D was developed in order to provide computer models of a wide range of two dimensional mechanical systems so that their response to imposed forces or motions could be simulated and the resulting joint forces and motions of the parts within the system determined. Most two dimensional mechanical systems which can be described in
terms of rigid parts connected by joints and acted upon by forces and motion generators can be modelled by AMP2D, including those with many degrees of freedom and those with none. AMP2D allows the parts to be joined to form closed loops or open chains or a mixture of both.

One part in every system modelled by AMP2D must be assumed to be motionless. This is called the ground part and acts as a reference for the system. The other parts may have mass or inertia as required by the model. The shape of the parts and other points of interest such the centre of mass and the position of the joints are defined using markers. In AMP2D the joints available to connect the parts are either pin joints or slides although other forms of contact such as cams can be simulated.

The relative motion between the parts can either be imposed directly using a motion generator or can result from the forces acting between them. AMP2D provides models for forces such as springs and dampers as well as motion generators which may have a constant rate or may vary harmonically. It can also model impact and more complex forces can be described to the program via the user expressions. Unusual generator functions or particularly complex forces can also be input using Fortran routines. Joint reaction forces and torques are handled automatically by the program and so do not have to be specified in the definition of the mechanism.
The information on the structure of the mechanism and its drives is given to AMP2D in the form of a series of statements. Each statement begins with a keyword that identifies the item such as a part, marker, force, etc., which is defined in that statement.

Further statements specify the information required as output. Each of these statements requests either the relative linear and angular displacement, velocity or acceleration between any two points in the system or the force and torque in a joint. Other statements control the duration of the simulation, the number of measurements to be made during that simulation and the form in which the results are to be presented. It is also necessary to define the system of units to be used, the error levels and the gravitational force.

2.2 The Friction Model for AMP2D

The computer simulation of mechanisms was developed primarily for the automotive industry where it is desirable to keep frictional forces at a low level and as a result friction was not included in these simulations. However, as the CEGB are interested in mechanisms in nuclear systems where frictional forces can be relatively high, a model of the effects of friction was developed which could be incorporated into AMP2D (Threlfall, 1978).
This model was based on the assumption that, when sliding takes place, the frictional force produced, \( F \), is proportional to the normal reaction, \( R \), between the surfaces in contact, i.e.:

\[
F = \mu R \tag{2.1}
\]

where the constant \( \mu \) is known as the coefficient of friction.

The frictional force acts to oppose the motion of the sliding surfaces and so its direction depends upon the relative motion of these surfaces (Fig.2.1). Thus, this simple model has the disadvantage that there is an instantaneous change in the frictional force from +\( F \) to -\( F \) as the relative velocity of the sliding surfaces passes through zero upon a reversal of their motion. This is computationally undesirable as it would cause severe problems for the integration routines used in AMP2D.

This problem was overcome by incorporating into the friction model the displacement dependence of friction during very small motions. This observed characteristic of friction is probably due to the elasticity and plasticity of the surfaces in contact (Bowden and Tabor, 1964) and results in the frictional force behaving for very small motions like a spring with hysteresis (Fig.2.2). Initially, the frictional force is zero for
Figure 2.1 Simple Frictional Force - Velocity Relation
Figure 2.2  Frictional Force - Displacement Relation

Showing Hysteresis Loop
zero displacement. As the displacement increases the frictional force rises rapidly at first and then more slowly until \( F = \mu R \). If the velocity then changes sign, the frictional force rapidly changes to \( F = -\mu R \). However, if the velocity drops to zero the frictional force remains at its current value which means that the model is capable of dealing with static friction and self-jamming.

This displacement dependence of friction was included in AMP2D using a model proposed by Dahl (1968) where:

\[
\frac{dF}{dx} = \gamma (F-|\mu R| s)^2 \tag{2.2}
\]

where:

- \( x \) is the displacement
- \( s \) is the sign of the sliding velocity
- \( \gamma \) is a constant chosen by the user

The CEGB derived an equation from this model (Osgood, 1983) which enabled a factor, \( f \), to be calculated for each integration time step in the program where \( f \) is defined as:

\[
f = \frac{F}{|\mu R|} \tag{2.3}
\]

and is determined by:

\[
f = \frac{x_o f_{L-19} s \Delta x(f_{L-s})}{x_o - 19 \Delta x(f_{L-s})} \tag{2.4}
\]
where:

- $f_i$ is the value of $f$ at the last time step
- $\Delta x$ is the displacement during the current time step
- $x_0$ is the sliding distance (measured from the point at which the velocity was last zero) at which $|f| = 0.95$.

A second observed characteristic of friction which is significant enough to require being incorporated into the friction model used in AMP2D is the decrease in the frictional force from a high "static" value during initial movements to a lower "dynamic" one as the velocity increases. To model this characteristic of friction the program calculates an instantaneous friction coefficient $\mu_I$ using the equation:

$$\mu_I = (\mu_s f - \mu_d) e^{-\frac{v}{v_0}} + \mu_d$$

where:

- $f$ is the factor defined above
- $\mu_s$ is the coefficient of static friction
- $\mu_d$ is the coefficient of dynamic friction
- $v$ is the velocity of the sliding surfaces
- $v_0$ is the velocity at which the changeover from the static to the dynamic coefficient of friction is 95% complete

The two equations 2.4 and 2.5 enable AMP2D to model the displacement and velocity dependence of friction, the user of the program providing the values for the coefficients of static and dynamic friction and for $x_0$ and $v_0$ in order to fit the model to the situation being simulated.
2.3 AMP3D-ADAMS

Mechanical Dynamics Inc., of Ann Arbor, Michigan, who had developed the program DRAM which was the basis for AMP2D, went on to develop a further program called ADAMS, the 'Automatic Dynamic Analysis of Mechanical Systems' (Orlana, 1973) to simulate three dimensional mechanical systems. This program is also used by the CEGB, although because there are slight differences between the version run by the CEGB and that sold by Mechanical Dynamics Inc., the CEGB version is known as AMP3D-ADAMS (Advanced Mechanisms Program in 3-Dimensions - Automatic Dynamic Analysis of Mechanical Systems).

AMP3D-ADAMS (Osgood and Threlfall, 1984) has been developed to do in three dimensions what AMP2D is capable of in two dimensions. Although its method of working is different, the use and applications of the program are virtually identical to those described for AMP2D. The main difference, apart from the obvious fact that AMP2D is limited to the simulation of two dimensional mechanisms, is that AMP3D-ADAMS can model a greater number of different types of joint. As well as the pin-joints and slides modelled in AMP2D, AMP3D-ADAMS can also simulate spherical, universal, cylindrical, screw, rack and pinion, and planar joints.

It is the simulation of these additional joints that
prevents the straight-forward adaption of the friction model used in AMP2D for incorporation into AMP3D-ADAMS. The problem lies in the assumption, used by the friction model in AMP2D, that the resistance to motion due to friction in a joint is directly proportional to the transmitted load in that joint. While this is a reasonable assumption for the simple pin-joints and slides modelled in AMP2D, it does not hold for several of the other joints simulated in AMP3D-ADAMS. This can be clearly seen when considering a spherical joint (Fig.2.3).

The frictional resistance in such a joint is dependent not only on the coefficient of friction and the magnitude of the loading but also on the nature of the pressure distribution over the contact area between the ball and the socket in the joint which is produced by that loading. The variation in the pressure distributed over the contact area results in a similar variation in the frictional forces produced which means that the total frictional effects in the joint cannot be simply determined. In addition, the size of the contact area and its position relative to the axis of rotation of the joint will also affect the frictional resistance produced.

The expressions for the velocity and displacement dependence of friction used in the model incorporated into AMP2D can also not be simply adapted to the case of a joint such as a spherical joint. This is due to the fact
Figure 2.3  Cross-section of a Spherical Joint
that as the joint rotates the velocity and displacement of points on the contact area will vary with their distance from the axis of rotation, shown in Fig. 2.3, which means that the effect on the magnitude of the frictional forces produced will also vary over the contact area.

Thus, the development of a mathematical model of the frictional effects in a spherical joint was necessary if a friction model is to be produced for AMP3D-ADAMS or any other three-dimensional mechanism simulation package.
CHAPTER THREE

Some Frictional Effects in a Spherical Joint

3.1 The Expressions Obtained by Yu. P. Smirnov

The starting point for the analysis of the effects of friction in a spherical joint was the paper written by Smirnov (1981) which gave a series of expressions to be used in the determination of the frictional force and moment produced in a spherical joint. These expressions are given below in the notation used by Smirnov.

In order to derive the expressions Smirnov began by assuming that the contact between the ball and the socket of a spherical joint under load would occur over a circular area with an axisymmetric distribution of pressure over that area about its centre. He defined the size of this contact area by the angle $\alpha_0$ (Fig.3.1) and assumed that the pressure, $n$, was distributed along the radial co-ordinate $\alpha$, according to the cosine law:

$$n = n_0 \cos\left(\frac{\alpha \pi}{2\alpha_0}\right)$$  \hspace{1cm} (3.1)

where $n_0$ is the maximum pressure at $\alpha=0$.

Smirnov then stated that the magnitude of the normal
Figure 3.1  Cross-section of a Spherical Joint in Oblique Projection Showing the Notation Used by Yu. P. Smirnov
reaction, \( N \), resulting from the distributed pressure was given by the expression:

\[
N = 2\pi n \int_0^{\alpha_0} \cos \alpha \cos \left(\frac{\pi \alpha}{2\alpha_0}\right) d\alpha
\]  
(3.2)

Assigning the symbol \( h \) to the integral in equation 3.2, Smirnov gave the following expression for the magnitude of the vector \( \bar{s} \), which he defined as the sum of the frictional forces acting over the contact area:

\[
\frac{\bar{s}}{N} = \frac{f}{m h} \int_0^{\pi} \int_0^{\alpha_0} \frac{2 \cos \left(\frac{\pi \alpha}{2\alpha_0}\right) (\cos \vartheta \sin \varphi - \sin \vartheta \cos \varphi \cos \psi) d\vartheta d\psi}{\sqrt{1 - (\cos \vartheta \cos \psi + \sin \vartheta \sin \psi \cos \varphi)^2}}
\]  
(3.3)

where he also defined:

- \( f \) as the coefficient of slip friction of the materials in contact
- \( \vartheta \) as the azimuthal co-ordinate of a point on the contact area
- \( \varphi \) as the angle between the axis of rotation of the joint (the angular velocity vector \( \omega \)) and the radius vector \( r \), drawn from the centre of the joint to the centre of the contact area, which is coincident with the normal reaction \( N \).

Smirnov derived the frictional moment produced in the joint in the form of two components - the moment of slip friction, \( M_s \), and the moment of rotational friction, \( M_R \).

He gave the following expressions for the magnitudes of these two components:
\[ M = \frac{f}{rN} \int_0^{\pi/2} \int_0^{2\pi} \frac{\alpha^2 \cos(\frac{\alpha}{\pi}) (\sin \gamma - \sin \alpha \sin \gamma \cos \beta - \sin \alpha \cos \cos \cos \cos \beta) d\alpha d\beta}{\sqrt{1 - (\cos \alpha \cos \gamma + \sin \sin \gamma \cos \beta)^2}} \]  

(3.4)

\[ M_R = \frac{f}{rN} \int_0^{\pi/2} \int_0^{2\pi} \frac{\alpha^2 \sin \cos (\frac{\alpha}{\pi}) (\sin \alpha \cos \gamma - \cos \sin \gamma \cos \beta) d\alpha d\beta}{\sqrt{1 - (\cos \alpha \cos \gamma + \sin \sin \gamma \cos \beta)^2}} \]  

(3.5)

Smirnov also gave three expressions which defined the directions of the frictional force and the two components of the frictional moment. These expressions are:

\[ \phi = -f_1 N \frac{\mathbf{w} \times \mathbf{r}}{|\mathbf{w} \times \mathbf{r}|} \]  

(3.6)

where \( f_1 = \frac{\phi}{N} \)

\[ M = \frac{-f_2 N}{|\mathbf{w} \times \mathbf{r}|} \frac{\mathbf{r} \times \mathbf{w} \times \mathbf{r}}{|\mathbf{w} \times \mathbf{r}|} \]  

(3.7)

where \( f_2 = \frac{M}{rN} \)

\[ M_R = -f_3 N r \frac{\text{sgn}(\mathbf{w} \times \mathbf{r})}{rN} \]  

(3.8)

where \( f_3 = \frac{M_R}{rN} \)

Equation 3.6 indicates that the frictional force produced in the joint acts in a direction perpendicular to the
plane in which the axis of rotation and the normal reaction \( N \) both lie. Equations 3.7 and 3.8 indicate that the frictional moment lies within this plane and that the moment of slip friction and the moment of rotational friction are perpendicular components of this moment, the moment of rotational friction being coincident with the normal reaction, \( N \).

Unfortunately, Smirnov gave no details of how he obtained the expressions given above. Thus, it was necessary to determine how they were derived in order to fully understand them and to be able to develop from them an adequate mathematical model of the effects of friction in a spherical joint.

3.2 The Pressure Distribution

In order to derive expressions for the frictional effects produced in a spherical joint it is first necessary to consider the nature of the contact between the ball and the socket in the joint and to determine the form of the pressure distribution between them.

The nature of the contact between two elastic bodies with spherical surfaces is well known (Burr, 1981) and can be seen to apply in the case of a spherical joint. As the radius of the socket cavity is necessarily greater than
that of the ball, contact will occur at a single point when there is no load being transmitted through the joint (Fig.3.2a). When a load is applied to the joint the ball and the socket will elastically deform around the point of contact to produce an area of contact (Fig.3.2b). From the symmetry of the spherical surfaces it can be deduced that this area will be bounded by a circle, whose radius will be defined by the angle $\alpha_o$.

The extent of this contact area depends upon a number of factors including the magnitude of the load, the clearance between the ball and the socket, and the elastic properties of the materials from which the joint is made. As a result there is a wide range of possible sizes of the contact area, over which it is unfortunately not possible to readily determine the pressure distribution. However, there are two specific situations in which it is possible to determine the pressure distribution; one when the dimensions of the contact area are small compared to those of the ball and socket and the other when contact occurs over a full hemisphere.

In the first of these situations the pressure distribution was determined by Heinrich Hertz (Timoshenko and Goodier, 1970). Hertz's analysis shows that in this case the pressure distribution is hemispherical in nature with the maximum pressure at the centre of the contact area. Although this analysis does assume that there is no
**Figure 3.2a**  Contact in an Unloaded Spherical Joint  
(CLEARANCE EXAGGERATED)

**Figure 3.2b**  Contact in a Loaded Spherical Joint  
(CLEARANCE EXAGGERATED)
friction between the surfaces in contact, Goodman (1962) has extended the theory to include the effects of friction and he states that it is acceptable to assume that the normal stress distribution and thus the pressure distribution remain unchanged by the effects of friction. Under the hemispherical pressure distribution the pressure \( p \) at a point on the contact area is given by the expression:

\[
p = p_o \cos\left(\frac{\alpha \pi}{2 \alpha_0}\right) \quad (3.9)
\]

where:

- \( \alpha \) is the angular distance from the centre of the contact area
- \( p_o \) is the maximum pressure at \( \alpha = 0 \).

The second situation for which an approximation to the pressure distribution can be determined is that when contact occurs over a full hemisphere. If the ball and the socket are assumed to have the same elastic properties the situation is approximately equivalent to that when a concentrated force, \( P \), is acting orthogonally to the boundary plane of a semi-infinite body (Fig.3.3). The radial stress distribution about the point of loading in such a halfspace has been determined and is given by Lur'e (1964) as:

\[
\sigma_R = \frac{P}{2 \pi m R} \left[ -\frac{4m-2}{\cos \Theta + (m-2)} \right] \quad (3.10)
\]
Figure 3.3  Force Acting on the Boundary Plane of a Semi-Infinite Body
where:

- \( m \) is Poisson's number, the inverse of Poisson's ratio, \( v \)
- \( R \) is the radial distance from the point of loading
- \( \theta \) is the angle measured from the vertical.

Taking the Poisson's ratio for steel, \( v=0.3 \), this equation gives:

\[
\sigma_R = \frac{P}{\pi R^2} \left(0.2 - 1.7 \cos \theta \right) \tag{3.11}
\]

Thus the pressure distribution over a hemisphere of radius \( r \) fitting perfectly in a hemispherical cavity in the halfspace can be taken as being approximately given by:

\[
p = \frac{P}{\pi r^2} \left(1.7 \cos \theta - 0.2 \right) \tag{3.12}
\]

Determining the pressure distribution relative to the maximum pressure \( p_o \) which occurs at \( \theta = 0 \) and noting that the angle \( \theta \) is equivalent to the angle \( \alpha \) defined previously, equation 3.12 becomes:

\[
P = p_o \left(1.133 \cos \alpha - 0.133 \right) \tag{3.13}
\]

Extending the hemispherical pressure distribution defined
previously to contact over a full hemisphere, i.e. $\alpha_0 = 90^\circ$, equations, 3.9 becomes:

$$p = p_0 \cos \alpha$$

The comparison of the two equations, 3.13 and 3.14, shows that the pressure distribution can be considered to be approximately hemispherical when contact occurs over a full hemisphere. There is a discrepancy between the values that would be given by these two expressions when the angle $\alpha$ is close to $90^\circ$ at which point equation 3.13 gives negative values for the pressure. However, this can be seen to be a consequence of the assumption that the joint could be represented by a continuous body. For an actual joint the discontinuity between the ball and the socket means that the pressure cannot be less than zero, i.e. there can only be compressive forces between the ball and the socket not tensile ones.

Having determined that the pressure distribution can be considered to be hemispherical in nature in the two situations considered, it would seem reasonable to assume that the pressure distribution is also hemispherical in nature when the extent of the contact area lies between these two extremes. This is particularly so as this type of distribution has the necessary characteristics of being axisymmetric, of having the maximum pressure at the centre of the contact area, and of having no pressure at the
edges. The validity of this assumption can be seen in the comparison of the theoretical values produced using it with those obtained experimentally.

3.3 The Normal Reaction

The next stage in the analysis of the effects of friction in a spherical joint is the determination of the normal reaction, $N$, resulting from the pressure acting over the contact area. The distribution of this pressure is assumed to be given by the expression in equation 3.9 where:

$$p = p_0 \cos \left(\frac{\alpha \pi}{2 \alpha_0}\right)$$

As the radii of the ball and the socket in a spherical joint are usually very similar, the local deformation required to produce the contact area will normally not be so great that this area cannot be assumed to be a spherical cap with a radius of curvature $r$, where $r$ is the radius of the joint. Thus it is possible to determine the normal reaction resulting from the pressure acting over the contact area by first considering the reaction due to the pressure on a ring element of this area and then extending this result to the entire contact area.

The area $dA_r$, of a ring element of width $r d\alpha$, as shown in
Fig. 3.4, is given by:

\[ dA^\tau = 2\pi rsin\alpha rd\alpha \]

\[ = 2\pi r^2 sin\alpha d\alpha \]

From the symmetrical nature of the ring element, it can be seen that the resultant reaction \( dN^\tau \) over the whole ring, due to the pressure \( p \), will act along the line from the centre of the contact area \( Z \) to the centre of the joint \( O \) and is given by:

\[ dN_r = 2\pi r^2 sin\alpha d\alpha p_0 \cos\frac{\alpha \pi}{2 \alpha_0} \cos \alpha \]

\[ = 2\pi r^2 p_0 sin\alpha cos\alpha \cos\frac{\alpha \pi}{2 \alpha_0} d\alpha \]

Extending this result to the entire contact area, the total normal reaction is given by:

\[ N = \int \int 2\pi r^2 p_0 sin\alpha cos\alpha \cos\frac{\alpha \pi}{2 \alpha_0} d\alpha \]

\[ = \pi r^2 p_0 \int_{0}^{\alpha_0} 2 sin\alpha cos\alpha \cos\frac{\alpha \pi}{2 \alpha_0} d\alpha \]  

(3.15)

Representing the integral in this expression by the symbol \( I \), the normal reaction is given by:

\[ N = \pi r^2 p_0 I \]  

(3.16)

where:

\[ I = \int_{0}^{\alpha_0} 2 sin\alpha cos\alpha \cos\frac{\alpha \pi}{2 \alpha_0} d\alpha \]  

(3.17)

Comparing the expression obtained for the normal reaction
Figure 3.4: The Contact Area in a Spherical Joint
in equation 3.15 with that given by Smirnov in equation 3.2:

\[ N = 2\pi\rho_o \int_{\alpha_0}^{\alpha} \alpha^2 \cos \alpha \cos \left( -\frac{\alpha \pi}{2\alpha_0} \right) d\alpha \]

there can be seen to be several discrepancies between the two expressions. Similar discrepancies were also found between the expressions derived for the frictional force and moments and those given by Smirnov. Smirnov was contacted in an attempt to discover the reason for these discrepancies and he indicated that the errors were in his calculations.

3.4 The Frictional Force

Having determined the normal reaction resulting from the pressure distributed over the contact area between the ball and the socket in a spherical joint, it is then possible to derive an expression for the frictional force produced over this contact area when the ball is rotated in the socket. The spherical joint, with the ball rotating at an angular velocity \( \omega \) about an axis inclined at an angle \( \gamma \) to the line of action of the normal reaction, \( N \), is shown in Fig.3.5.

The frictional force over the whole contact area can be obtained by first determining the frictional force on an
Figure 3.5  **Spherical Joint with the Axis of Rotation at an Angle \( \gamma \) to the Normal Reaction N**
elemental area, whose position at the point A is defined by the angles $\alpha$ and $\beta$, and then extending this result to cover the entire contact area. The area $dA$ of the element can be seen from Fig.3.5 to be given by:

$$dA = r \Delta \alpha \cdot r \sin \alpha \Delta \beta$$

$$= r^2 \sin \alpha \Delta \alpha \Delta \beta$$

The pressure distribution is given by equation 3.9 as:

$$p = p_0 \cos \left( \frac{\alpha \pi}{2 \alpha_0} \right)$$

Thus, the normal reaction on the element, $dN$, due to the pressure is given by:

$$dN = r^2 p_0 \sin \alpha \cos \left( \frac{\alpha \pi}{2 \alpha_0} \right) d\alpha d\beta$$

and so the magnitude of the frictional force on the element, $dF$, which is proportional to the normal reaction, is given by:

$$dF = \mu dN$$

$$= \mu r^2 p_0 \sin \alpha \cos \left( \frac{\alpha \pi}{2 \alpha_0} \right) d\alpha d\beta$$  \hspace{1cm} (3.18)

where $\mu$ is the coefficient of friction.

The frictional force will act to oppose the motion of the element. Thus as the element is on the surface of the ball rotating about the axis represented by the vector $\mathbf{w}$, the
frictional force will act in the plane that passes through the point A and is also perpendicular to the axis of rotation. In this plane, which is shown in Fig.3.6, the frictional force acts orthogonally to the line drawn from the point A to C, the point at which the axis of rotation passes through the plane, in the direction which opposes the rotation.

The point A lies on the arc SUT produced by the intersection of the plane described above and the contact area between the ball and the socket. Because of the symmetrical nature of the contact area and the pressure distribution over it, the point D, which lies opposite A on this arc as shown in Fig.3.6, will have a frictional force acting on it of equal magnitude to that on A.

The frictional forces at these two points can each be divided into two components; one of which is coincident with the line drawn from A to D, the other being perpendicular to this line. From Fig.3.6 it can be seen that the two components perpendicular to the line AD are equal in magnitude but act in opposite directions. Thus it is only the components coincident with the line that need to be taken into consideration in determining the total frictional force produced in the joint. Fig. 3.6 shows that the magnitude of this component at the point A, \( dF \cos \phi \), depends upon the value of the cosine of the angle \( \phi \), which is given by:
The Boundary of the Contact Area

\[ dF \sin \phi \]

\[ dF \cos \phi \]

Figure 3.6 A View of the Spherical Joint Along the Axis of Rotation
\[
\cos \theta = \frac{b}{c} = \frac{b}{\sqrt{a^2 + b^2}} \tag{3.19}
\]

where \(a\), \(b\) and \(c\) denote the lengths of the sides of the triangle \(ABC\).

From Fig.3.5 the distance \(a\) from \(A\) to \(B\) can be seen to be given by:

\[a = rsin\alpha sin\beta \tag{3.20}\]

The distance \(b\) from \(C\) to \(B\) can be determined geometrically from the dimensions of the joint shown in Fig.3.7. From this diagram it can be seen that:

\[b = (d + e) \sin \gamma - g\]

where:

\[d = r\cos \alpha\]
\[e = k\tan \gamma\]
\[g = k/\cos \gamma\]

From Fig.3.5:

\[k = rsin\alpha cos\beta\]

Thus the distance \(b\) is given by:

\[b = (r\cos \alpha + k\tan \gamma) \sin \gamma - k/\cos \gamma\]
\[= r(\cos \alpha \sin \gamma + \sin \alpha \cos \beta \tan \gamma \sin \gamma - \sin \alpha \cos \beta / \cos \gamma)\]
Figure 3.7  The Dimensions within a Spherical Joint
Substituting the expressions obtained for a and b into equation 3.19 the cosine of the angle $\phi$ is given by:

$$
\cos \phi = \frac{\cos \alpha \sin \beta - \sin \alpha \cos \beta \cos \phi}{\sqrt{\sin^2 \alpha \sin^2 \beta + (\cos \alpha \sin \beta - \sin \alpha \cos \beta \cos \phi)^2}}
$$

Appendix 1 shows that the expression:

$$
\sin^2 \alpha \sin^2 \beta + (\cos \alpha \sin \beta - \sin \alpha \cos \beta \cos \phi)^2
$$

is equivalent to:

$$
1 - (\cos \alpha \cos \phi + \sin \alpha \cos \beta \sin \phi)^2
$$

Thus the cosine of the angle $\phi$ can be given as:

$$
\cos \phi = \frac{\cos \alpha \sin \beta - \sin \alpha \cos \beta \cos \phi}{\sqrt{1 - (\cos \alpha \cos \phi + \sin \alpha \cos \beta \sin \phi)^2}} \quad (3.22)
$$

and so the frictional force component $dF \cos \phi$ is given as:

$$
dF \cos \phi = \frac{dF(\cos \alpha \sin \beta - \sin \alpha \cos \beta \cos \phi)}{\sqrt{1 - (\cos \alpha \cos \phi + \sin \alpha \cos \beta \sin \phi)^2}} \quad (3.23)
$$

Substituting the expression obtained in equation 3.18 for $dF$, the component $dF \cos \phi$ is given by:
Extending this expression to cover the entire contact area, the total frictional force, F, is given by:

\[
F = \mu r^2 p \int_0^{\alpha_0} \int_0^{\beta_0} \frac{\sin \alpha \cos \left( \frac{\alpha \pi}{2 \alpha_0} \right) \left( \cos \alpha \sin \beta - \sin \alpha \cos \beta \cos \alpha \right)}{\sqrt{1 - (\cos \alpha \cos \beta + \sin \alpha \cos \beta \sin \beta)^2}} \, d\alpha \, d\beta \quad (3.25)
\]

Taking the expression obtained for the magnitude of the normal reaction, which is given in equation 3.16 as:

\[
N = \pi r^2 p \text{I}
\]

where I is given in equation 3.17 as:

\[
I = \int_0^{\alpha_0} 2 \sin \alpha \cos \alpha \cos \left( \frac{\alpha \pi}{2 \alpha_0} \right) \, d\alpha
\]

the magnitude of the resultant frictional force, F, can be found in terms of its ratio to that of the normal reaction. This removes the term for the maximum pressure, which is unlikely to be known, from the resulting expression given by:

\[
\frac{F}{N} = \frac{\mu}{\pi I} \int_0^{\alpha_0} \int_0^{\beta_0} \frac{\sin \alpha \cos \left( \frac{\alpha \pi}{2 \alpha_0} \right) \left( \cos \alpha \sin \beta - \sin \alpha \cos \beta \cos \alpha \right)}{\sqrt{1 - (\cos \alpha \cos \beta + \sin \alpha \cos \beta \sin \beta)^2}} \, d\alpha \, d\beta \quad (3.26)
\]

The equivalent expression given by Smirnov in equation 3.3
is:

\[
N = \frac{\mu}{\pi h} \int_{0}^{2\alpha_0} \frac{2\alpha \cos(\frac{\alpha \pi}{2\alpha_0}) (\cos \alpha \sin \beta - \sin \alpha \cos \beta \cos \alpha \beta \cos \alpha \beta)}{\sqrt{1 - (\cos \alpha \beta + \sin \alpha \beta \sin \beta)^2}} d\alpha d\beta
\]

where from equation 3.2:

\[
h = \int_{0}^{\alpha_0} 2\alpha \cos \alpha \cos(\frac{\alpha \pi}{2\alpha_0}) d\alpha
\]

As the total frictional force over the contact area is comprised solely of components acting parallel to the line AD, shown in Fig. 3.6, its direction will also be parallel to this line. From Figs. 3.5 and 3.6 it can be seen that the line AD is perpendicular both to the vector \( \mathbf{w} \) representing the axis of rotation and to the direction of the normal reaction, \( \mathbf{N} \). This means that the direction of the frictional force can be obtained from the vector product of \( \mathbf{w} \) and \( \mathbf{N} \). As the direction of the components of the frictional force is from A towards D, the vector \( \mathbf{F} \) representing the resultant frictional force over the contact area, and shown in Fig. 3.8, can be given by:

\[
\mathbf{F} = f_1 \mathbf{N} \frac{\mathbf{w} \times \mathbf{N}}{|\mathbf{w} \times \mathbf{N}|}
\]

(3.27)

where the factor \( f_1 \) represents the ratio \( \frac{\mathbf{F}}{\mathbf{N}} \) given in equation 3.26.
Figure 3.8  The Directions of the Frictional Force and Moments Produced in a Spherical Joint.
3.5 The Frictional Moment

As well as the resultant frictional force described in the previous section, the frictional forces acting over the contact area of the spherical joint also produce a moment acting about the centre of the joint. As the direction of this frictional moment is not known, it is necessary to determine the moment in the form of components acting about mutually perpendicular axes. In order to simplify the method of deriving the expressions required to evaluate these components, the most convenient axes to consider are as follows:

1) the axis coincident with the line of action of the normal reaction \( N \), shown by the line \( OZ \) in Fig. 3.5,

2) the axis shown by the line \( POQ \) in Fig. 3.5 which is perpendicular to the first axis and lies in the plane containing the normal reaction \( N \) and the axis of rotation shown by the vector \( w \),

3) the axis, perpendicular to the plane described above, which passes through the centre of the joint, the point \( O \) in Fig. 3.5.

A consideration of the frictional forces acting on the contact area, as given below, shows that there is no
component of the frictional moment acting about the third axis described above. In the previous section it has been shown that the frictional forces acting on the elemental areas at the points A and D shown in Figs. 3.5 and 3.6 can both be represented by two components $dF\cos\theta$ and $dF\sin\theta$. The $dF\cos\theta$ components have been defined as being coincident with the line AD and Fig. 3.5 shows that this line is parallel to the axis being considered. Thus the $dF\cos\theta$ components of the frictional force produce no moment about this axis.

It has also been shown in the previous section that the $dF\sin\theta$ components of the frictional forces at A and D are equal in magnitude and act in opposite directions. From Fig. 3.5 and 3.6 it can be seen that the lines of action of these $dF\sin\theta$ components are of equal distance from the third axis described above. Thus, when taken together, these components also produce no moment about the third axis. As A and D can be taken to represent any pair of points on opposite sides of the contact area, it can be seen that there is no component of the total frictional moment acting about this third axis.

Thus the total frictional moment acts about an axis lying in the plane described above as containing the line of action of the normal reaction, $N$, and the axis of rotation of the joint. For convenience the two perpendicular components of the frictional moment lying in this plane
are described by the same terms as used by Smirnov. Thus the component acting about an axis coincident with the line of action of the normal reaction is called the moment of rotational friction and the other component is called the moment of slip friction.

3.5.1 The Moment of Slip Friction

The moment of slip friction has been defined as the component of the total frictional moment acting about the axis represented by the line POQ, shown in Fig.3.5. This component can be determined by considering the moment produced about the axis by the frictional force acting on the elemental area at A and then integrating to obtain the total moment produced about this axis by the frictional forces acting over the entire contact area.

It has been shown that the frictional force on the element at A can be divided into two components, \( dF \cos \phi \) and \( dF \sin \phi \), of known directions as shown in Fig.3.6. The expression for the component \( dF \cos \phi \) is given in equation 3.23 as:

\[
dF \cos \phi = \frac{dF \cos \phi (\sin \phi \cos \beta - \sin \alpha \cos \phi \cos \beta)}{\sqrt{1 - (\cos \alpha \cos \phi + \sin \alpha \cos \phi \sin \beta)^2}}
\]

where from equation 3.18:
\[ dF = \mu r^2 p_0 \sin \alpha \cos \left( \frac{\alpha \pi}{\alpha_0} \right) d\alpha d\beta \]

To determine the other component, \( dF \sin \theta \), it is necessary to find an expression for the sine of the angle \( \theta \). From the geometry of the triangle ABC in Fig 3.6 it can be seen that:

\[ \sin \theta = \frac{a}{\sqrt{a^2 + b^2}} \]

Substituting the expressions obtained in equations 3.20 and 3.21 for \( a \) and \( b \), this expression becomes:

\[ \sin \theta = \frac{\sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha \sin^2 \beta + (\cos \alpha \sin \alpha - \sin \alpha \cos \beta \cos \alpha)^2}} \]

Appendix 1 shows that:

\[ \sin^2 \alpha \sin^2 \beta + (\cos \alpha \sin \alpha - \sin \alpha \cos \beta \cos \alpha)^2 \]

is equivalent to:

\[ 1 - (\cos \alpha \cos \beta + \sin \alpha \cos \beta \sin \alpha)^2 \]

Thus the magnitude of the frictional force component, \( dF \sin \theta \) can be given by:

\[ dF \sin \theta = \frac{dF \sin \alpha \sin \beta}{\sqrt{1 - (\cos \alpha \cos \beta + \sin \alpha \cos \beta \sin \alpha)^2}} \quad (3.28) \]

As stated before, the component of the frictional force
\( \text{d}F \cos \theta \) at A acts along the line AD. Fig. 3.7 shows that the perpendicular distance from this line to the axis represented by POQ is given by the distance \( d \) where:

\[ d = r \cos \alpha \]

Thus the moment, \( dM_{SC} \), which acts about the axis represented by POQ and is due to the frictional force component \( \text{d}F \cos \theta \), is given by:

\[
dM_{SC} = \frac{dF (\cos \alpha \sin \gamma - \sin \alpha \cos \beta \cos \gamma) \cdot r \cos \alpha}{\sqrt{1 - (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2}} = \frac{dF \rho (\cos^2 \alpha \sin \beta - \sin \alpha \rho \cos \beta \cos \gamma)}{\sqrt{1 - (\cos \alpha \cos \beta + \sin \alpha \rho \sin \beta)^2}} \quad (3.29)\]

Fig. 3.6 shows that the other component of the frictional force, \( \text{d}F \sin \theta \), acts in a direction parallel to the line CB at a distance \( a \) which from equation 3.20 is given by:

\[ a = r \sin \alpha \sin \beta \]

Fig. 3.7 shows that the line CB lies in the same plane as the axis represented by POQ but inclined by the angle \( \gamma \) to it.

Thus the moment, \( dM_{SS} \), which acts about the axis represented by the line POQ and is due to the component \( \text{d}F \sin \theta \) of the frictional force on the element, is given by:
\[ dM_{SS} = \frac{dF \sin \alpha \sin \beta \sin \gamma \cdot \sin \gamma \cdot \sin \alpha \sin \beta}{\sqrt{1 - (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2}} \]

\[ = \frac{dF \sin^2 \alpha \sin^2 \beta \sin \gamma}{\sqrt{1 - (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2}} \quad (3.30) \]

From Figs. 3.5 and 3.6 it can be seen that both of the moments \( dM_{SC} \) and \( dM_{SS} \) act in the same direction about the axis represented by POQ. Thus the total moment \( dM_S \) about this axis due to the frictional force on the element at A can be found by taking the sum of these moments and is given by:

\[ dM_S = dM_{SC} + dM_{SS} \]

\[ = \frac{dF r (\cos^2 \alpha \sin \gamma - \sin \alpha \cos \beta \cos \gamma + \sin^2 \alpha \sin^2 \beta \sin \gamma)}{\sqrt{1 - (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2}} \]

\[ = \frac{dF r (\sin \alpha - \sin \alpha \cos \beta \cos \gamma - \sin^2 \alpha \sin \gamma)}{\sqrt{1 - (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2}} \]

\[ = \frac{dF r (\sin \gamma - \sin \alpha \cos \beta \cos \gamma - \sin^2 \alpha \cos \beta \sin \gamma)}{\sqrt{1 - (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2}} \]

Substituting the expression given in equation 3.18 for the frictional force \( dF \), this expression for the moment \( dM_S \) becomes:

\[ dM_S = \mu r^3 \rho \sin \alpha \cos \left( \frac{\pi r}{2 \alpha} \right) (\sin \gamma - \sin \alpha \cos \beta \cos \gamma) \frac{- \sin^2 \alpha \cos \beta \sin \gamma}{\sqrt{1 - (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2}} \]
Extending this result to cover the whole of the contact area, the moment of slip friction $M_S$ is given by:

$$M_S = \mu r^3 p_o \int_{\alpha_0}^{2\pi} \int_{\beta_0}^{\beta_\infty} \sin\alpha\cos\left(\frac{\alpha\pi}{2\alpha_0}\right)(\sin\beta - \sin\alpha\cos\beta\cos\psi) \left(-\sin\alpha\cos\beta\sin\psi\right) d\beta d\alpha \frac{1}{\sqrt{1 - (\cos\alpha\cos\beta + \sin\alpha\cos\beta\sin\psi)^2}} \right) \right]$$

(3.32)

Then, taking the expression given for the normal reaction $N$ in equation 3.16 as:

$$N = \pi r^2 p_o I$$

where $I$ given by equation 3.17 as:

$$I = \int_{\alpha_0}^{2\pi} \int_{\beta_0}^{\beta_\infty} 2\sin\alpha\cos\alpha\cos\left(\frac{\alpha\pi}{2\alpha_0}\right) d\alpha$$

the magnitude of the moment of slip friction can be found in terms of its ratio to the magnitude of the normal reaction, as the magnitude of the frictional force was before. The ratio is given by:

$$\frac{M_S}{rN} = \frac{\mu}{\pi I} \int_{\alpha_0}^{2\pi} \int_{\beta_0}^{\beta_\infty} \sin\alpha\cos\left(\frac{\alpha\pi}{2\alpha_0}\right)(\sin\beta - \sin^2\alpha\sin\alpha\cos^2\beta) \left(-\sin\alpha\cos\beta\cos\psi\sin\alpha\cos\beta\cos\psi\right) d\beta d\alpha \frac{1}{\sqrt{1 - (\cos\alpha\cos\beta + \sin\alpha\cos\beta\sin\psi)^2}} \right) \right)$$

(3.33)

The equivalent expression given by Smirnov in equation 3.4 is:

$$\frac{M_S}{rN} = \frac{\mu}{\pi h} \int_{\alpha_0}^{2\pi} \int_{\beta_0}^{\beta_\infty} \frac{2\cos\left(\frac{\alpha\pi}{2\alpha_0}\right)(\sin\beta - \sin^2\alpha\sin\alpha\cos^2\beta - \sin\alpha\cos\beta\cos\psi)}{\sqrt{1 - (\cos\alpha\cos\beta + \sin\alpha\cos\beta\sin\psi)^2}} d\beta d\alpha$$

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where from equation 3.2:

\[ h = \int_{0}^{\alpha_0} 2\alpha^2 \cos \alpha \cos \left( \frac{\alpha \pi}{2 \alpha} \right) d\alpha \]

The moment of slip friction has been defined as the component of the total frictional moment acting about the axis which is perpendicular to the line of action of the normal reaction \( N \) and also lies within the plane containing both that line of action and the axis of rotation of the joint represented by the vector \( \omega \). Thus the direction of the moment of slip friction can be obtained from the vector product of the vector \( N \) and the vector product \( \omega \times N \). As shown in Figs. 3.5 and 3.6 the directions of the components of the frictional forces acting on the contact area indicate that the vector, \( M_s \), representing the moment of slip friction is directed along the line POQ from P towards Q. Thus the vector \( M_s \), shown in Fig. 3.8, is given by:

\[
M_s = \frac{-f_2 \tau}{|\omega \times N|} N \times (\omega \times N) \quad (3.34)
\]

where the factor \( f_2 \) represents the ratio \( \frac{M_s}{rN} \) given in equation 3.33.
3.5.2 The Moment of Rotational Friction

The moment of rotational friction has been defined as the component of the frictional moment acting about the axis coincident with the line of action of the normal reaction, which is shown by the line OZ in Fig.3.5. As with the moment of slip friction, this component of the total frictional moment can be determined by considering the moments produced about the axis by the two components of the frictional force on the elemental area, \( dF \cos \theta \) and \( dF \sin \theta \), and then integrating to extend this result to the entire contact area.

From Figs.3.6 and 3.7, it can be seen that the perpendicular distance from the line of action of the component \( dF \cos \theta \) to the axis represented by the line OZ is given by the distance \( k \), while Fig.3.5 shows that:

\[
k = r \sin \alpha \cos \beta
\]

Combining this expression for \( k \) with that given for the component \( dF \cos \theta \) in equation 3.23, the expression for the moment \( dM_{RC} \), about the axis represented by OZ and due to the component \( dF \cos \theta \) is given by:

\[
dM_{RC} = \frac{dF \sin \alpha \sin \beta - \sin \alpha \cos \beta \cos \gamma \cdot r \sin \alpha \cos \beta}{1 - (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2}
\]
\[
\frac{dF\sin\alpha(\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos^2\beta \cos\gamma)}{\sqrt{1 - (\cos\alpha\cos\delta + \sin\alpha\cos\beta\sin\gamma)^2}} (3.35)
\]

As shown in the previous section the component \(dF\sin\phi\) acts parallel to the line CB in Fig.3.6 at a distance \(a\) given by equation 3.20 as:

\[a = r\sin\alpha \sin\beta\]

and Fig.3.7 shows that the line CB lies in the same plane as the line OZ but inclined at an angle of \((90^\circ - \delta)\) to it.

Thus using this expression for the distance \(a\) and that given for the component \(dF\sin\phi\) in equation 3.28, the moment, \(dM_{RS}\), about the axis represented by OZ and due to the component \(dF\sin\phi\) of the frictional force on the elemental area is given by:

\[
dM_{RS} = \frac{dF\sin\alpha \sin\beta \cos\gamma \cdot r \sin\alpha \sin\beta}{\sqrt{1 - (\cos\alpha\cos\delta + \sin\alpha\cos\beta\sin\gamma)^2}}
\]

\[= \frac{dF\sin^2\alpha \sin^2\beta \cos\gamma}{\sqrt{1 - (\cos\alpha\cos\delta + \sin\alpha\cos\beta\sin\gamma)^2}} (3.36)\]

It can be seen from Figs.3.5 and 3.6 that the two moments \(dM_{RC}\) and \(dM_{RS}\) due to the frictional force components \(dF\cos\phi\) and \(dF\sin\phi\) act in opposite directions about the
axis represented by the line OZ. Thus the resultant moment \( dM_R \), acting about this axis and due to the frictional force on the elemental area at A, is given by the difference between \( dM_{RC} \) and \( dM_{RS} \). The direction of this moment can be defined as being positive when it acts in opposition to the rotation of the joint. As it is the moment \( dM_{RS} \) due to the component \( dF \sin \theta \) which acts in this direction, this means that the resultant moment, \( dM_R \), is given by:

\[
dM_R = dM_{RS} - dM_{RC}
\]

\[
dM_R = dF \sin \alpha (\sin \alpha \sin^2 \beta \cos \gamma - \cos \alpha \cos \beta \sin \gamma + \sin \alpha \cos^2 \beta \cos \gamma) \frac{1}{\sqrt{1 - (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2}}
\]

Substituting the expression given in equation 3.18 for the magnitude of the frictional force \( dF \), this expression for the moment \( dM_R \) becomes:

\[
dM_R = \mu r^3 p_o \sin^2 \alpha \cos \left( \frac{\alpha \pi}{2 \alpha_o} \right) (\sin \alpha \cos \gamma - \cos \alpha \cos \beta \sin \gamma) \frac{d \alpha d \beta}{\sqrt{1 - (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2}}
\]

Integrating to extend this result to the entire contact area, the moment of rotational friction, \( M_R \), is given by:
As previously, if the expression for the normal reaction $N$ is taken as given by equation 3.16:

$$N = \pi r^2 p_o I$$

where the term $I$ is given by equation 3.17 as:

$$I = \int_0^{\alpha_o} 2\sin\alpha\cos\alpha\cos\left(\frac{\alpha\pi}{2\alpha_o}\right) d\alpha$$

the magnitude of the moment of rotational friction can also be given in terms of its ratio to the magnitude of the normal reaction. This ratio is given by:

$$\frac{M_R}{rN} = \frac{\mu}{\pi I} \int_0^{2\pi} \int_0^{\alpha_o} \frac{\sin^2\alpha\cos\left(\frac{\alpha\pi}{2\alpha_o}\right)(\sin\alpha\cos\alpha - \cos\alpha\sin\alpha \sin\alpha)}{\sqrt{1 - (\cos\alpha\cos\alpha + \sin\alpha\cos\alpha\sin\alpha)^2}} d\alpha d\beta$$

(3.39)

This expression can be compared to the equivalent one given by Smirnov in equation 3.6 as:

$$\frac{M_R}{rN} = \frac{\mu}{\pi h} \int_0^{2\pi} \int_0^{\alpha_o} \frac{\sin^2\alpha\cos\left(\frac{\alpha\pi}{2\alpha_o}\right)(\sin\alpha\cos\alpha - \cos\alpha\sin\alpha \sin\alpha)}{\sqrt{1 - (\cos\alpha\cos\alpha + \sin\alpha\cos\alpha\sin\alpha)^2}} d\alpha d\beta$$

where from equation 3.2:

$$h = \int_0^{\alpha_o} 2\alpha^2 \cos\alpha\cos\left(\frac{\alpha\pi}{2\alpha_o}\right) d\alpha$$
The vector \( M_R \) representing the moment of rotational friction has been defined as being coincident with the line of action of the normal reaction, \( N \), shown by the line OZ in Fig.3.5. As the moment has been defined as being positive when it acts in opposition to the rotation of the joint, the vector \( M_R \) is directed from O towards Z provided the angle between the vector \( N \) representing the normal reaction and the vector \( w \) representing the angular rotation of the joint is less than 90°. If this angle is greater than 90° the orientation of the vector \( M_R \) is reversed because the direction of the rotation relative to it is reversed. To show this characteristic of the moment, the expression for the vector \( M_R \), shown in Fig.3.8, is given by:

\[
M_R = -f_3 rN \text{ sgn}(w.N) \tag{3.40}
\]

where the factor \( f_3 \) represents the ratio \( \frac{M_R}{rN} \) given in equation 3.39 and where the term \( \text{ sgn}(w.N) \) is either +1 or -1 depending upon the sign of the scalar product of the vectors \( w \) and \( N \).

3.6 The Evaluation of the Expressions

To summarize the results obtained in the preceding analysis, the frictional effects produced in a spherical
Joint have been shown to consist of a frictional force $F$ and a frictional moment given in the form of two perpendicular components, the moment of slip friction $M_S$ and the moment of rotational friction $M_R$. Expressions for the force $F$ and the two moments $M_S$ and $M_R$ have been derived and are given, in equations 3.27, 3.34 and 3.40 respectively, as:

\[ F = f_1 N \frac{w \times N}{|w \times N|} \]

\[ M_S = \frac{-f_2 r}{|w \times N|} N \times (w \times N) \]

\[ M_R = -f_3 r N \text{sgn}(w \cdot N) \]

where the expressions for the factors $f_1$, $f_2$, and $f_3$ are given by equations 3.26, 3.33 and 3.39 as:

\[ f_1 = \frac{\mu}{\pi I} \int_{\alpha_0}^{2\pi} \int_{\beta_0}^{2\pi} \sin \alpha \cos \left( \frac{\alpha \pi}{2 \alpha_0} \right) \left( \cos \alpha \sin \gamma - \sin \alpha \cos \beta \cos \gamma \right) d\alpha d\beta \]

\[ \frac{1}{\sqrt{1 - (\cos \alpha \cos \beta + \sin \alpha \cos \beta \sin \gamma)^2}} \]

\[ f_2 = \frac{\mu}{\pi I} \int_{\alpha_0}^{2\pi} \int_{\beta_0}^{2\pi} \sin \alpha \cos \left( \frac{\alpha \pi}{2 \alpha_0} \right) \left( \sin \gamma - \sin^2 \alpha \sin \gamma \cos^2 \beta \right) \left( -
\right. \sin \alpha \cos \beta \cos \gamma \right) d\alpha d\beta \]

\[ \frac{1}{\sqrt{1 - (\cos \alpha \cos \beta + \sin \alpha \cos \beta \sin \gamma)^2}} \]

\[ f_3 = \frac{\mu}{\pi I} \int_{\alpha_0}^{2\pi} \int_{\beta_0}^{2\pi} \sin^2 \alpha \cos \left( \frac{\alpha \pi}{2 \alpha_0} \right) \left( \sin \gamma \cos \alpha \cos \beta \sin \gamma \right) d\alpha d\beta \]

\[ \frac{1}{\sqrt{1 - (\cos \alpha \cos \beta + \sin \alpha \cos \beta \sin \gamma)^2}} \]
and where the term I is given by equation 3.17 as:

\[ I = \int_0^{\phi_0} 2\sin\alpha \cos \alpha \cos \left( -\frac{\alpha \pi}{2 \phi_0} \right) d\alpha \]

These expressions indicate that the magnitude and direction of the frictional force and moment produced in a spherical joint are dependent upon the following factors:

1) the direction of the axis of rotation represented by the vector \( \mathbf{w} \)

2) the magnitude and -

3) the direction of the normal reaction \( \mathbf{N} \)

4) the coefficient of friction \( \mu \)

5) the angle \( \phi_0 \) defining the extent of the contact area

6) the radius, \( r \), of the spherical joint and

7) the angle \( \gamma \), the angle between line of action of the normal reaction \( \mathbf{N} \) and the axis of rotation.

Of these factors, three - the radius of the joint, the coefficient of friction and the direction of the axis of rotation - are independent variables which have to be
supplied in order to determine the frictional force and moment. The angle \( \gamma \) can be determined from the directions of the normal reaction and the axis of rotation. The angle \( \alpha_0 \) is dependent upon a number of additional factors as well as the magnitude of the normal reaction \( N \) which in turn results from the sum of the applied load on the joint and the frictional force produced. The methods of determining both the angle \( \alpha_0 \) and the magnitude and direction of the normal reaction \( N \) are described in the following chapters.

As well as these factors the determination of the frictional force and moment produced is also dependent upon the evaluation of the integrals contained within the expressions given above. The integral given in equation 3.17 as:

\[
I = \int_{0}^{\alpha_0} 2 \sin \alpha \cos \alpha \cos \left( \frac{\alpha \pi}{2 \alpha_0} \right) d\alpha
\]

has been evaluated in Appendix 2 and is given as:

\[
I = \frac{\alpha_0 (8 \alpha_0 - 2 \pi \sin 2 \alpha_0)}{16 \alpha_0^2 - \pi^2} \quad \text{for } \alpha_0 \neq \frac{\pi}{4}
\]

\[
I = \frac{1}{4} \quad \text{for } \alpha_0 = \frac{\pi}{4}
\]

Unfortunately the evaluation of the double integrals in
equations 3.26, 3.33 and 3.39 is not as simple. As these integrals are not comprised of elementary functions, they have to be evaluated numerically, which is done using Simpson's rule for two-dimensions (Salvadori and Baron, 1952).

The three double integrals contained within the expressions are of the form:

\[ \int_{\alpha_0}^{\alpha_0} \int_{\beta_0}^{\beta_0} f(\alpha, \beta) d\alpha d\beta \]

These integrals can be evaluated by first integrating with respect to \( \alpha \) while regarding \( \beta \) as a constant and then integrating the resultant function with respect to \( \beta \). Thus the double integral can be considered as:

\[ \int_{\alpha_0}^{\alpha_0} \left[ \int_{\beta_0}^{\beta_0} f(\alpha, \beta) d\alpha \right] d\beta \quad (3.41) \]

Simpson's rule for a single integral evaluates that integral by the following formula:

\[ \int_{a}^{b} f(x) dx = \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \]

where:

\[ h = \frac{b - a}{n} \]
and where:

- **n** is the number of subintervals over the range of integration and must be even.
- **h** is the width of the subintervals.
- **f(x_i)** is the value of the function being integrated at the point \( x = x_i \) on the range of integration.

This approximation is extended to the double integrals being considered by dividing the range of the 'outer' integral, as shown by the expression in 3.41, into an even number of subdivisions and then, using the given value of \( \beta \), evaluating the 'inner' integral \( \int_0^{\alpha} f(\alpha, \beta) d\alpha \) at each subinterval by Simpson's rule. These values for the 'inner' integral at each subinterval can then be used to calculate the 'outer' integral by the same formula.

A computer program was written to evaluate, by this method of numerical integration, the expressions given for the factors \( f_1, f_2, \) and \( f_3 \), which contain the double integrals as shown in equations 3.26, 3.33 and 3.39. The results obtained are shown in Fig.3.9. As the factors are directly proportional to the coefficient of friction \( \mu \), the results are given in the form of the ratios of the three factors to the coefficient of friction. These ratios are plotted against values of the angle \( \chi \), the angle between the line of action of the normal reaction and the axis of rotation, for various values of the angle \( \alpha_0 \), which defines the extent of the contact area.
Figure 3.9a  The ratios $\frac{f_1}{\mu}$, $\frac{f_2}{\mu}$ and $\frac{f_3}{\mu}$ plotted against the angle $\gamma$ for $\alpha_0 = 5^\circ$. 
Figure 3.9b. The ratios $f_1/\mu$, $f_2/\mu$, and $f_3/\mu$ plotted against the angle $\gamma$ for $\alpha_0 = 30^\circ$. 
Figure 3.9c. The ratios \( f_1/\mu \), \( f_2/\mu \) and \( f_3/\mu \) plotted against the angle \( \gamma \) for \( \alpha = 60^\circ \)
Figure 3.9d  The ratios $f_1/\mu$, $f_2/\mu$ and $f_3/\mu$ plotted against the angle $\gamma$ for $\alpha_0 = 90^\circ$
The results show a number of characteristics which correspond with the expected behaviour of the frictional force and moments and these are described in the following section.

3.7 The Characteristics of the Frictional Force and Moments

3.7.1 The Frictional Force

The first thing that can be seen from the characteristics of the factor $f_1$, as shown in Fig. 3.9, is that there is no resultant frictional force produced when the axis of rotation is aligned with the direction of the normal reaction ($\gamma = 0^\circ$). This is due to the fact that in this position the axis of rotation passes through the centre of the contact area and so the axisymmetric nature of the pressure distribution over the contact area means that the sum of the frictional forces produced is zero.

As the angle increases, the computed results show that the frictional force also increases until the point is reached when the axis of rotation no longer passes through the contact area ($\gamma > \alpha_0$). At this point the magnitude of the frictional force, as shown by the factor $f_1$, is close to its maximum value which is finally reached when the axis of rotation is perpendicular to the
direction of the normal reaction ($\gamma = 90^\circ$).

This behaviour results from the fact that the frictional force produced at a point on the contact area acts tangentially to the perpendicular line from the axis of rotation to that point, in the direction that opposes the rotation. Thus, when the axis of rotation passes through the contact area, a proportion of the frictional forces, produced over that area will act in opposing directions; and, as the point at which the axis of rotation intersects the contact area moves towards the edge of the area, this proportion decreases and so the resultant frictional force increases. Once the axis is beyond the contact area the frictional forces act largely in the same general direction and so produce approximately the maximum resultant force. Further increases in the angle then only produce slight increases in the uniformity of the directions of the frictional forces, resulting in a slight increase in the magnitude of the resultant frictional force.

The computed results presented in Fig.3.9 also show that the ratio of the factor $f_1$ to the coefficient of friction $\mu$ has a maximum value of approximately 1.0 when the angle $\alpha_0$ is small which increases slightly for larger sizes of the contact area. This is due to the fact that when the contact area is small it is also roughly flat. The magnitude of the frictional force produced at a point on
This area is given by the product of the normal reaction at that point and the coefficient of friction. Thus if the contact area is roughly flat and the frictional forces over it act largely in the same direction, as when the axis of rotation does not pass through it, the ratio of the resultant frictional force to the resultant normal reaction, which is the definition of the factor $f_1$, is approximately equivalent to the value of the coefficient of friction $\mu$.

As the size of the contact area increases it can no longer be considered to be approximately flat. Thus, there are components of the frictional forces at the leading and trailing edges of contact area as the joint rotates which act in opposing directions. This should result in a reduction in the magnitude of the resultant frictional force produced. However, the increasing curvature of the contact area also means that there are components of the normal reactions at points diametrically opposite each other on the contact area acting in opposing directions with the result that, for the same resultant normal reaction, the reaction at these points is higher producing a higher frictional force. These two effects counteract each other, and so the maximum value of the factor $f_1$ does not vary greatly until the contact area is very large. It then increases because the second effect is significant over all but the centre of the contact area and not just at the leading and trailing edges.
3.7.2 The Moment of Slip Friction

The moment of slip friction exhibits characteristics very similar to those of the frictional force as can be seen by comparing the two factors $f_1$ and $f_2$ in Fig.3.9. The similarity results from the definition of this moment as the component of the total frictional moment acting about the axis, perpendicular to the line of action of the normal reaction, which lies in the plane containing both this line of action and the axis of rotation, as shown in Fig.3.8. Thus, when the axis of rotation is aligned with the direction of the normal reaction ($\varphi = 0^\circ$) it is perpendicular to the axis the moment is defined as acting about and so the magnitude of the moment is zero. It follows that the moment reaches its maximum value when the axis of rotation is aligned with the axis which the moment is defined as acting about ($\varphi = 90^\circ$).

Fig.3.9 shows that when the angle $\alpha_0$ is small the values of the factors $f_1$ and $f_2$ are practically identical. This is due to the fact that, when the contact area is small, the moment of slip friction can be considered to be produced solely by the resultant frictional force. As described previously, when small the contact area can be considered to be a flat surface. This means that, because of its position relative to the axis about which the moment of slip friction is defined as acting, the sum of the moments produced about this axis by the components of
the frictional forces that are not part of the resultant frictional force is zero. The perpendicular distance from the contact area, when small and roughly flat, to the axis about which the moment of slip friction acts is approximately equal to the radius of the joint \( r \). Thus, in this situation, the magnitude of the moment of slip friction \( M_S \) is approximately equal to the product of the radius \( r \) and the resultant frictional force \( F \). Then considering the definitions of the factors \( f_1 \) and \( f_2 \) which are:

\[
\begin{align*}
f_1 &= \frac{F}{N}, \quad f_2 = \frac{M_S}{rN}
\end{align*}
\]

it can be seen that they will be roughly equal.

When the contact area is larger, the increased curvature of the surface means that there are components of the frictional forces, in addition to those included in the resultant force, which contribute to the moment of slip friction and this results in the values of the factor \( f_2 \) increasing relative to the corresponding values of \( f_1 \) as shown in Fig.3.9. At the same time the increased curvature also means that the perpendicular distance from the axis about which the moment of slip friction is defined as acting to the line of action of the resultant frictional force produced over the contact area is reduced and so the increase in the magnitude of the moment and
3.7.3 The Moment of Rotational Friction

As shown by the factor \( f_3 \) in Fig. 3.9 the moment of rotational friction has completely different characteristics from those described for the moment of slip friction and the frictional force. Its maximum value occurs when the axis of rotation is aligned with the direction of the normal reaction \( \gamma = 0^\circ \), which is as expected because this moment is defined as acting about the axis coincident with the direction of the normal reaction. Thus it follows that the magnitude of this moment is zero when the axis of rotation is perpendicular to the direction of the normal reaction \( \gamma = 90^\circ \).

The computed results also show that the maximum value of the moment of rotational friction is very dependent on the size of the contact area as given by the angle \( \alpha_o \). The moment is very small for a small area of contact and increases rapidly with the size of the contact area. This is due to the fact that the axis about which this moment is defined as acting passes through the centre of the contact area. Thus the greater the size of the contact area, the greater is the moment produced by the frictional forces about that axis.
When contact occurs over a complete hemisphere ($\alpha_0 = 90^\circ$), Fig. 3.9d shows that the maximum value of the factor $f_3$ is equal to the coefficient of friction $\mu$. This is a result of the defined hemispherical pressure distribution acting over a complete hemisphere.

The results plotted in Fig. 3.9 show that there is a change in the way the moment of rotational friction varies with the angle $\phi$ at approximately the point when the axis of rotation no longer passes through the contact area. This is due to the fact that, as described in Section 3.7.1, when the axis of rotation intersects the contact area a proportion of the frictional forces produced over that area act in opposing directions. It is these opposing forces that largely produce the moment about the axis through the centre of the contact area and as the point of intersection of the axis of rotation moves towards the edge of the contact area the proportion of opposing forces, and thus the moment, is reduced. When the axis of rotation no longer passes through the contact area the remaining moment is produced by the curvature in the directions of the frictional forces which are still tangential to the perpendicular lines from the axis of rotation. This remaining moment is steadily reduced as the axis of rotation continues its shift away from the contact area.
CHAPTER FOUR

The Frictional Effects in Relation to the Applied Load

The previous chapter has shown how expressions were derived that enable the frictional force and the components of the frictional moment to be determined relative to the magnitude and direction of the normal reaction within a spherical joint. However, as the normal reaction results from a combination of the applied load acting on the joint and the frictional force produced within it, the means of determining the frictional effects relative to the magnitude and direction of the applied load cannot be obtained by a straight-forward extension of these expressions. It is necessary to consider the effect of the frictional force on the normal reaction if that force and the components of the frictional moment are to be determined relative to the applied load.

4.1 The Frictional Force

Fig. 4.1 shows a spherical joint viewed in the plane containing the applied load $\mathbf{P}$ and the resulting frictional force $\mathbf{F}$. The normal reaction $\mathbf{N}$ is given by the sum of these two forces. Thus:
Figure 4.1  Cross-section of a Spherical Joint in Oblique Protection Through the Plane Containing the Applied Load and the Frictional Force
\[ P + F + N = 0 \]  \hspace{1cm} (4.1)

Fig. 4.1 also shows the axis of rotation which is represented by the angular velocity vector \( \mathbf{w} \) and which does not lie in the plane containing the forces mentioned above. The angle \( \theta \) is the angle between the axis of rotation and the direction of the applied load \( P \) while the angle between the axis of rotation and the line of action of the normal reaction has already been denoted as the angle \( \gamma \).

By selecting a particular co-ordinate system for the joint as shown in Fig. 4.2, the vectors \( \mathbf{w}, \mathbf{P} \) and \( \mathbf{F} \), representing the axis of rotation, the applied load and the resultant frictional force respectively, can be expressed as:

\[ \mathbf{w} = w(-\sin\theta \mathbf{i} + \cos\theta \mathbf{k}) \]  \hspace{1cm} (4.2)

\[ \mathbf{P} = -P \mathbf{k} \]  \hspace{1cm} (4.3)

\[ \mathbf{F} = F(1_p \mathbf{i} + m_F \mathbf{j} + n_F \mathbf{k}) \]  \hspace{1cm} (4.4)

where \( 1_p, m_F \) and \( n_F \) are the direction cosines of the frictional force \( \mathbf{F} \).

Then, by substituting the expressions given for the applied load \( \mathbf{P} \) and the frictional force \( \mathbf{F} \) into equation 4.1, the normal reaction \( N \) is shown to be given by:
Figure 4.2  The Orientation of the Spherical Joint Relative to the Chosen Co-ordinate System
\[ N = -F_{\rho}\hat{\rho} - F_{m}\hat{m} + (P-F_{n})k \]  

(4.5)

These equations show that in order to determine the frictional force \( F \) it is necessary to derive expressions for its magnitude \( F \) and direction cosines \( l_{F}, m_{F} \) and \( n_{F} \) in terms of known quantities. For a friction model which is to be incorporated into a simulation package such as AMP3D-ADAMS, the known quantities are the magnitude and direction of the applied load, the direction of the axis of rotation and from these the angle \( \theta \) between them. All of these quantities would be supplied by the program at each step of the simulation.

The required expressions can be derived from those obtained for the frictional force relative to the normal reaction in section 3.4, where equation 3.27 gives:

\[ F = f_{1}N \frac{w \times N}{|w \times N|} \]

and where the factor \( f_{1} \), representing the ratio \( F/N \), is given in equation 3.26 as:

\[
\frac{F}{N} = \frac{\mu}{\pi I} \int_{\alpha_{0}}^{2\pi} \frac{\sin \alpha \cos (\frac{\alpha}{2}) (\cos \alpha \sin \gamma - \sin \alpha \cos \beta \cos \gamma) \, d\alpha \, d\gamma}{\sqrt{1 - (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2}}
\]

By substituting the ratio \( F/N \) for the factor \( f_{1} \) in equation 3.27, the frictional force \( F \) can be given as:
From this expression it can be seen that the direction of the frictional force $\mathbf{F}$ is perpendicular to that of the normal reaction $\mathbf{N}$. Thus the magnitude of the scalar product $\mathbf{N} \cdot \mathbf{F}$ of these two vectors is zero. Using the expressions for the frictional force $\mathbf{F}$ and the normal reaction $\mathbf{N}$ which are shown in equations 4.4 and 4.5, the scalar product is given by:

$$\mathbf{N} \cdot \mathbf{F} = [-F_i\mathbf{i} - F_m\mathbf{j} + (P - F_n)\mathbf{k}] \cdot [F(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})]$$

$$= -F^2lF^2 - F^2mF^2 - F^2nF^2 + FnFP$$

From the definition of direction cosines:

$$l_F^2 + m_F^2 + n_F^2 = 1$$

Thus, the scalar product $\mathbf{N} \cdot \mathbf{F}$ is given by:

$$\mathbf{N} \cdot \mathbf{F} = FnFP - F^2 = 0$$

The solution to this equation is either that $F = 0$ which is the case when no frictional force is produced or that the direction cosine $n_F$ is given by:

$$n_F = \frac{F}{P} \quad (4.7)$$
The other two direction cosines $l_F$ and $m_F$ can be obtained from the direction of the frictional force which can be determined from the expression in equation 4.6. Using the expressions given in equations 4.2 and 4.5, the vector product $\mathbf{w} \times \mathbf{N}$ is given by:

$$\mathbf{w} \times \mathbf{N} = [w(-\sin \theta_i + \cos \theta_k)] \times [-F_{l_F} \mathbf{i} - F_{m_F} \mathbf{j} + (P - F_{n_F}) \mathbf{k}]$$

$$= w[F_{m_F} \sin \theta_k + (P - F_{n_F}) \sin \theta_j - F_{l_F} \cos \theta_j + F_{m_F} \cos \theta_i]$$

$$= w(F_{m_F} \cos \theta_i + [(P - F_{n_F}) \sin \theta - F_{l_F} \cos \theta] \mathbf{j} + F_{m_F} \sin \theta_k)$$  \hspace{1cm} (4.8)

The magnitude of this vector product $|\mathbf{w} \times \mathbf{N}|$ is given by:

$$|\mathbf{w} \times \mathbf{N}| = w \sqrt{F_{m_F}^2 \cos^2 \theta + [(P - F_{n_F}) \sin \theta - F_{l_F} \cos \theta]^2 + F_{m_F}^2 \sin^2 \theta}$$

$$= w \sqrt{F_{m_F}^2 + [(P - F_{n_F}) \sin \theta - F_{l_F} \cos \theta]^2}$$  \hspace{1cm} (4.9)

Substituting the expressions obtained for the vector product $\mathbf{w} \times \mathbf{N}$ and its magnitude $|\mathbf{w} \times \mathbf{N}|$ into equation 4.6, the frictional force $F$ is given by:

$$F = F \frac{(F_{m_F} \cos \theta_i + [(P - F_{n_F}) \sin \theta - F_{l_F} \cos \theta] \mathbf{j} + F_{m_F} \sin \theta_k)}{\sqrt{F_{m_F}^2 + [(P - F_{n_F}) \sin \theta - F_{l_F} \cos \theta]^2}}$$

Equating the $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components of this expression for the frictional force with those given in equation 4.4, the following expressions are obtained for the three direction cosines $l_F$, $m_F$ and $n_F$:
\[ l_F = \frac{F m_F \cos \theta}{\sqrt{F^2 m_F^2 + [(P - Fm_F) \sin \theta - Fm_F \cos \theta]^2}} \]
\[ m_F = \frac{(P - Fm_F) \sin \theta - Fm_F \cos \theta}{\sqrt{F^2 m_F^2 + [(P - Fm_F) \sin \theta - Fm_F \cos \theta]^2}} \]
\[ n_F = \frac{F m_F \sin \theta}{\sqrt{F^2 m_F^2 + [(P - Fm_F) \sin \theta - Fm_F \cos \theta]^2}} \]

Equating this expression for the direction cosine \( n_F \) with that given in equation 4.7, the following expression is produced:

\[ \frac{F}{P} = \frac{F m_F \sin \theta}{\sqrt{F^2 m_F^2 + [(P - Fm_F) \sin \theta - Fm_F \cos \theta]^2}} \]

which gives:

\[ \sqrt{F^2 m_F^2 + [(P - Fm_F) \sin \theta - Fm_F \cos \theta]^2} = P m_F \sin \theta \quad (4.10) \]

Using the relation given in this equation, the expressions for the direction cosines \( l_F \) and \( m_F \) can be reduced to:

\[ l_F = \frac{F \cos \theta}{P \sin \theta} \quad (4.11) \]
\[ m_F = \frac{(P - Fm_F) \sin \theta - Fm_F \cos \theta}{P m_F \sin \theta} \]
Then substituting the expressions given for the direction cosines $l_F$ and $n_F$ in equations 4.11 and 4.7 into this expression for $m_F$ gives:

$$m_F = \frac{(P-F^2/P)\sin\theta - F^2\cos^2\theta/P\sin\theta}{P\sin\theta}$$

which gives:

$$m_F^2 = 1 - \frac{F^2}{P^2} - \frac{F^2\cos^2\theta}{P^2\sin^2\theta}$$

Thus the direction cosine $m_F$ is given by:

$$m_F = \sqrt{1 - \frac{F^2}{P^2\sin^2\theta}} \quad (4.12)$$

Equations 4.7, 4.11 and 4.12 give expressions for the direction cosines of the frictional force $F$ in terms of its magnitude $F$. This magnitude can be determined using the factor $f_1$, given in equation 3.26, where from the definition of the factor:

$$F = f_1N \quad (4.13)$$

where $N$ is the magnitude of the normal reaction $N$, given in equation 4.5. Substituting the expressions given for the direction cosines in equations 4.7, 4.11 and 4.12 into this equation, the normal reaction $N$ is given by:
\[ N = \frac{-F^2 \cos \theta}{P \sin \theta} - F \sqrt{1 - \frac{F^2}{P^2 \sin^2 \theta}} + \left( \frac{F}{P} \right)^2 \]  

(4.14)

Then the magnitude of the normal reaction will be given by:

\[ N = \frac{F^4 \cos^2 \theta}{P^2 \sin^2 \theta} + F^2 \left( 1 - \frac{F^2}{P^2 \sin^2 \theta} \right) + P^2 - 2F^2 + \frac{F^4}{P^2} \]

\[ = \sqrt{\frac{F^4 (\cos^2 \theta - 1)}{P^2 \sin^2 \theta} + \frac{F^4}{P^2} + P^2 - F^2} \]

\[ = \sqrt{P^2 - F^2} \]  

(4.15)

Substituting this expression for the magnitude of the normal reaction into equation 4.13, the magnitude of the frictional force \( F \) can be expressed in terms of the factor \( f_1 \) and the magnitude of the applied load \( P \) as follows:

\[ F = f_1 \sqrt{P^2 - F^2} \]

\[ F^2 = f_1^2 (P^2 - F^2) \]

\[ F^2 = \frac{f_1^2 P^2}{1 + f_1^2} \]

\[ F = \frac{f_1 P}{\sqrt{1 + f_1^2}} \]  

(4.16)

This expression for the magnitude of the frictional force can then be substituted into equations 4.6, 4.11 and 4.12.
to give the direction cosines $l_F$, $m_F$ and $n_F$ in terms of the factor $f_1$, the magnitude of the applied load $P$ and the angle $\theta$ between the axis of rotation and the direction of the applied load. The expressions for the direction cosines become:

$$l_F = \frac{f_1 \cos \theta}{\sin \theta \sqrt{1 + f_1^2}}$$  \hspace{1cm} (4.17)

$$m_F = \sqrt{1 - \frac{f_1^2}{(1 + f_1^2) \sin^2 \theta}}$$  \hspace{1cm} (4.18)

$$n_F = \frac{f_1}{\sqrt{1 + f_1^2}}$$  \hspace{1cm} (4.19)

4.2 The Moment of Slip Friction

Expressions for the magnitudes and directions of the moment of slip friction and the moment of rotational friction relative to the applied load can be derived in the same way as those for the magnitude and direction of the frictional force were. The moment of slip friction $M_S$ can be expressed in terms of its magnitude and direction cosines as:

$$M_S = M_S (l_M S^i + m_M S^j + n_M S^k)$$  \hspace{1cm} (4.20)

The expressions for the magnitude and direction cosines
can then be derived using those obtained for the moment of slip friction relative to the normal reaction in section 3.6 where equation 3.34 gives:

\[
M_S = \frac{-f_2 r}{|w_wn|} (NwxwN)
\]

and where the factor \( f_2 \) representing the ratio \( M_S/rN \) is given in equation 3.33 as:

\[
M_S = \frac{\mu}{rN \pi l} \int_{\alpha_\theta}^{\pi} \frac{\sin \alpha \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \left( \sin \gamma - \sin^2 \alpha \sin^2 \beta \right) \sin \alpha \cos \beta \cos \theta \sin \theta \, d\alpha \, d\beta}{\sqrt{1 - \left( \cos \alpha \cos \theta + \sin \alpha \cos \beta \sin \theta \right)^2}}
\]

Substituting the ratio \( M_S/rN \) for the factor \( f_2 \) in equation 3.34, the moment of slip friction can be given as:

\[
M_S = \frac{-M_S (NwxwN)}{N |wxN|} \quad (4.21)
\]

From the definition of the factor \( f_2 \) the magnitude of the moment of slip friction \( M_S \) is given by:

\[
M_S = rNf_2
\]

Substituting the expression for the magnitude of the normal reaction \( N \), given in equation 4.15, into this equation gives:

\[
M_S = rf_2 \sqrt{P^2 - F^2}
\]

Then, using the expression given for the magnitude of the
frictional force $F$ in equation 4.16, the magnitude of the moment of slip friction $M_S$ can be expressed as:

$$M_S = r f_2 \sqrt{p^2 - \frac{f_1^2 p^2}{(1+f_1^2)}}$$

$$= \frac{r f_2 p}{\sqrt{1+f_1^2}}$$  \hspace{1cm} (4.22)

The expressions for the direction cosines of the moment of slip friction can be derived using the expression for the direction of the moment obtained from equation 4.21. This involves determining an expression for the triple vector product $N \times t \times N$. The determination of this expression can be simplified by replacing the term $f_1/\sqrt{1+f_1^2}$ by the symbol $x$. The expressions for the magnitude and direction cosines of the frictional force $F$ given in equations 4.16 - 4.19 then become:

$$F = P x$$  \hspace{1cm} (4.23)

$$l_p = \frac{x \cos \theta}{\sin \theta}$$  \hspace{1cm} (4.24)

$$m_p = \sqrt{1 - \frac{x^2}{\sin^2 \theta}}$$  \hspace{1cm} (4.25)

$$n_p = x$$  \hspace{1cm} (4.26)

Substituting the expression given in equation 4.23 for the
magnitude of the frictional force \( F \) into equation 4.14, the normal reaction \( N \) can be expressed as:

\[
N = P \left( \frac{x^2 \cos \theta}{\sin \theta} \hat{i} - x \sqrt{1 - \frac{x^2}{\sin^2 \theta}} \hat{i} + (1-x^2) \hat{k} \right) \quad (4.27)
\]

Then, substituting the expressions given for the magnitude and direction cosines of the frictional force \( F \) in equations 4.23-4.26 into equation 4.8, the vector product \( wxN \) can be given as:

\[
w \times N = wP \left( x \cos \theta \sqrt{1 - \frac{x^2}{\sin^2 \theta}} \hat{i} + \left[ (1-x^2) \sin \theta - \frac{x^2 \cos^2 \theta}{\sin \theta} \right] \hat{i} \\
+ x \sin \theta \sqrt{1 - \frac{x^2}{\sin^2 \theta}} \hat{k} \right)
\]

\[
= wP \left( x \cos \theta \sqrt{1 - \frac{x^2}{\sin^2 \theta}} \hat{i} + \frac{\sin^2 \theta - x^2}{\sin \theta} \hat{j} + x \sin \theta \sqrt{1 - \frac{x^2}{\sin^2 \theta}} \hat{k} \right)
\]

The triple vector product \( N \times w \times N \) can then be obtained by taking the vector product of the normal reaction \( N \) given in equation 4.27 and the vector product \( wxN \) given above. This gives:

\[
N \times w \times N = wP^2 \left[ \frac{-x^2 \cos \theta}{\sin \theta} \hat{i} - x \sqrt{1 - \frac{x^2}{\sin^2 \theta}} \hat{i} + (1-x^2) \hat{k} \right]
\]
\[ x \left( x \cos \theta \sqrt{1 - \frac{x^2}{\sin^2 \theta}} \right) + \frac{\sin^2 \theta}{\sin \theta} i + x \sin \theta \sqrt{1 - \frac{x^2}{\sin^2 \theta}} k \]

\[ = wP^2 \left[ -\frac{x^2 \cos \theta (\sin^2 \theta - x^2)}{\sin^2 \theta} k + x^3 \cos \theta \sqrt{1 - \frac{x^2}{\sin^2 \theta}} i \right. \]

\[ + x^2 \cos \theta \left( 1 - \frac{x^2}{\sin^2 \theta} \right) k - x^2 \sin \theta \left( 1 - \frac{x^2}{\sin^2 \theta} \right) i \]

\[ + x^2 (1-x^2) \cos \theta \sqrt{1 - \frac{x^2}{\sin^2 \theta}} i - \frac{(1-x^2)(\sin^2 \theta - x^2)}{\sin \theta} i \]

\[ = wP^2 \left[ \frac{x^2 - \sin^2 \theta}{\sin \theta} i + x \cos \theta \sqrt{1 - \frac{x^2}{\sin^2 \theta}} i \right] \quad (4.28) \]

The other factors in the expression in equation 4.21 giving the direction of the moment of slip friction are the magnitudes of the vector product \( w x N \) and the normal reaction \( N \). By substituting the expression in equation 4.10 into equation 4.9, the magnitude of the vector product is given by:

\[ |w x N| = wP m_F \sin \theta \]

Substituting the expression given in equation 4.25 for the direction cosine \( m_F \) into this equation gives:

\[ |w x N| = wP \sin \theta \sqrt{1 - \frac{x^2}{\sin^2 \theta}} \quad (4.29) \]

The magnitude of the normal reaction \( N \) can then be
obtained by substituting the expression for the magnitude of the frictional force \( F \) in equation 4.23 into equation 4.15 to give:

\[
N = P \sqrt{1 - x^2}
\]  
(4.30)

Then, substituting the expressions, given for the triple vector product \( N_x w x N \) in equation 4.28, the magnitude of the vector product \( w x N \) in equation 4.29 and the magnitude of the normal reaction \( N \) in equation 4.30, into equation 4.21, the moment of slip friction \( M_S \) is given by:

\[
M_S = \frac{-M_S w P^2 \left[ \frac{x^2 - \sin^2 \theta}{\sin \theta} i + x \cos \theta \sqrt{1 - \frac{x^2}{\sin^2 \theta}} j \right]}{[P \sqrt{1 - x^2} \sqrt{w P \sin \theta \sqrt{1 - \frac{x^2}{\sin^2 \theta}}}]}
\]

\[
= M_S \left[ \frac{\sqrt{\sin^2 \theta - x^2}}{\sin \theta \sqrt{1 - x^2}} i - \frac{x \cos \theta \sqrt{1 - x^2}}{\sin \theta \sqrt{1 - x^2}} j \right]
\]

Equating the \( i \), \( j \) and \( k \) components of this expression for the moment of slip friction with those given in equation 4.20, the following expressions are obtained for the direction cosines of the moment of slip friction, \( l_{MS} \), \( m_{MS} \) and \( n_{MS} \):

\[
l_{MS} = \frac{1}{\sin \theta} \sqrt{\frac{\sin^2 \theta - x^2}{1 - x^2}}
\]
Replacing the symbol x by the term $f_1/\sqrt{1+f_1^2}$ the expressions for the direction cosines become:

$$\begin{align*}
L_{MS} &= -\frac{x\cos\theta}{\sin\theta\sqrt{1-x^2}} \\
N_{MS} &= 0
\end{align*}$$

$$m_{MS} = \frac{-x\cos\theta}{\sin\theta\sqrt{1-x^2}}$$

$$N_{MS} = 0$$

(4.31)

$$m_{MS} = \frac{-f_1\cos\theta}{\sin\theta\sqrt{(1+f_1^2)(1-f_1^2)}}$$

$$N_{MS} = 0$$

(4.33)

4.3 The Moment of Rotational Friction

The moment of rotational friction $M_R$ can be expressed in terms of its magnitude and direction cosines as:

$$M_R = M_R (L_{MRi} + m_{MRj} + n_{MRk})$$

(4.34)
The expressions for this magnitude and these direction cosines can then be obtained in the same way as those for the magnitude and direction cosines of the moment of slip friction were. The expressions for the moment of rotational friction $M_R$ relative to the normal reaction $N$ were derived in section 3.7 where equation 3.40 gives:

$$M_R = -f_3 rN \ \text{sgn}(w.N)$$

and where the factor $f_3$ representing the ratio $M_R/rN$ is given in equation 3.39 as:

$$f_3 = \frac{\mu}{\pi} \int_0^{2\pi} \int_0^{\alpha_0} \sin^2 \alpha \left( \frac{\pi}{2 \alpha} \right) (\sin \alpha \cos \alpha - \cos \alpha \sin \alpha) \, d\alpha \, d\beta$$

Substituting the ratio $M_R/rN$ for the factor $f_3$ in equation 3.40, the moment of rotational friction can be given as:

$$M_R = -M_R \frac{N}{N} \ \text{sgn}(w.N) \quad (4.35)$$

The magnitude of the moment of rotational friction $M_R$ is obtained from the definition of the factor $f_3$ as:

$$M_R = rNf_3$$

Using the expression for the magnitude of the normal reaction $N$ which can be deduced from equation 4.22, the
expression for the magnitude of the moment of rotational friction \( M_R \) becomes:

\[
M_R = \frac{rf_3^P}{\sqrt{1+f_1^2}} \quad (4.36)
\]

The expressions for the direction cosines of the moment of rotational friction can be derived using the expression for the direction of the moment which can be obtained from equation 4.35. Using the expression given for the normal reaction \( N \) in equation 4.27 and that given for its magnitude \( N \) in equation 4.30, the term \( N/N \) is given by:

\[
\frac{N}{N} = \frac{-x^2 \cos \theta}{\sin \theta \sqrt{1-x^2}} \mathbf{i} - x \frac{\sin^2 \theta - x^2}{\sin^2 \theta (1-x^2)} \mathbf{j} + \sqrt{1-x^2} \mathbf{k} \quad (4.37)
\]

where \( x \) represents the term \( \frac{f_1}{\sqrt{1+f_1^2}} \).

The scalar product \( \mathbf{w.N} \) can be determined using the expression for the normal reaction \( N \) in equation 4.27 and that given for the vector \( \mathbf{w} \) in equation 4.2. It is given by:

\[
\mathbf{w.N} = wP\left( -\sin \theta i + \cos \theta k \right) \cdot \left( \frac{-x^2 \cos \theta}{\sin \theta} i - x \frac{1-x^2}{\sin^2 \theta} j + (1-x^2)k \right)
\]

\[
= wP [x^2 \cos \theta + (1-x^2) \cos \theta]
\]

\[
= wP \cos \theta
\]
Thus the sign of the scalar product $\mathbf{w} \cdot \mathbf{N}$ is given by the sign of the cosine of the angle $\theta$, i.e.:

$$\text{sgn}(\mathbf{w} \cdot \mathbf{N}) = \text{sgn}(\cos \theta) \quad (4.38)$$

Substituting the expressions obtained in equations 4.37 and 4.38 into equation 4.35, the moment of rotational friction can be expressed as:

$$M_R = M \left[ \frac{x^2 \cos \theta}{\sin \theta \sqrt{1-x^2}} i + x \sqrt{\frac{\sin^2 \theta - x^2}{\sin^2 \theta (1-x^2)}} j - \sqrt{1-x^2} k \right] \text{sgn}(\cos \theta)$$

Equating the $i$, $j$ and $k$ components of this expression for the moment of rotational friction with those given in equation 4.34, the following expressions are obtained for the direction cosines of the moment $l_{MR}$, $m_{MR}$ and $n_{MR}$:

$$l_{MR} = \frac{x^2 \cos \theta}{\sin \theta \sqrt{1-x^2}} \text{sgn}(\cos \theta)$$

$$m_{MR} = x \sqrt{\frac{\sin^2 \theta - x^2}{\sin^2 \theta (1-x^2)}} \text{sgn}(\cos \theta)$$

$$n_{MR} = -\sqrt{1-x^2} \text{sgn}(\cos \theta)$$

Replacing $x$ by the term $f_1 / \sqrt{1+f_1^2}$ the expressions for the direction cosines become:
The results obtained in this chapter can be summarized as follows. The co-ordinate system in a spherical joint was defined so that the vector $\mathbf{w}$ representing the axis of rotation and the vector $\mathbf{P}$ representing the applied load
are given by equations 4.2 and 4.3 as:

\[ w = w(-\sin \theta_i + \cos \theta_k) \]

\[ P = -P_k \]

where \( \theta \) is the angle between the axis of rotation and the line of action of the applied load. This meant that the frictional force \( F \) could be given by equation 4.4 as:

\[ F = F(1_p i + m_p j + n_p k) \]

where its magnitude and direction cosines were given by equations 4.16-4.19 as:

\[ F = \frac{f_1 P}{\sqrt{1 + f_1^2}} \]

\[ l_F = \frac{f_1 \cos \theta}{\sin \theta \sqrt{1 + f_1^2}} \]

\[ m_F = \sqrt{1 - \frac{f_1^2}{(1 + f_1^2) \sin^2 \theta}} \]

\[ n_F = \frac{f_1}{\sqrt{1 + f_1^2}} \]

and where the factor \( f_1 \) representing the ratio \( F/N \)
was given by equation 3.26 as:

\[ f_1 = \frac{\mu}{nI} \int_{0}^{2\pi} \frac{\sin \alpha \cos \left( \frac{\alpha \pi}{2 \alpha_0} \right) (\cos \alpha \sin \gamma - \sin \alpha \cos \beta \cos \gamma) \, d\alpha \rho}{\sqrt{1 - (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2}} \]

The frictional moment was then given in the form of two perpendicular components, the moment of slip friction \( M_S \) and the moment of rotational friction \( M_R \). These moments are expressed in equations 4.20 and 4.34 respectively as:

\[ M_S = M_S (l_{MS} + m_{MS} + n_{MS}) \]

\[ M_R = M_R (l_{MR} + m_{MR} + n_{MR}) \]

where their magnitudes and direction cosines are given by equations 4.22, 4.31-4.33, 4.36 and 4.39-4.41 as:

\[ M_S = \frac{rf_2 P}{\sqrt{1+f_1^2}} \]

\[ l_{MS} = \sqrt{1 - \frac{f_1^2 \cos^2 \theta}{\sin^2 \theta}} \]

\[ m_{MS} = -\frac{f_1 \cos \theta}{\sin \theta} \]

\[ n_{MS} = 0 \]

\[ M_R = \frac{rf_3 P}{\sqrt{1+f_1^2}} \]
\[ l_{MR} = \frac{f_1^2 \cos \theta}{\sin \theta \sqrt{1+f_1^2}} \text{sgn}(\cos \theta) \]

\[ m_{MR} = \frac{f_1}{\sin \theta} \sqrt{\sin^2 \theta - \frac{f_1^2}{1+f_1^2}} \text{sgn}(\cos \theta) \]

\[ n_{MR} = \frac{-1}{\sqrt{1+f_1^2}} \text{sgn}(\cos \theta) \]

and the two factors \( f_2 \) and \( f_3 \), representing the ratios \( M_S/rN \) and \( M_R/rN \) respectively, are given by equations 3.33 and 3.39 as:

\[ f_2 = \frac{\mu}{\pi} \int_0^{2\pi} \int_0^{\pi} \sin \alpha \cos \left( \frac{\alpha \pi}{2} \right) \left( \sin \theta \sin^2 \alpha \sin \cos^2 \beta - \sin \alpha \cos \alpha \cos \beta \cos \gamma \right) \, d\alpha \, d\beta \]

\[ f_3 = \frac{\mu}{\pi} \int_0^{2\pi} \int_0^{\pi} \sin^2 \alpha \cos \left( \frac{\alpha \pi}{2} \right) \left( \sin \alpha \cos \theta - \cos \alpha \cos \beta \sin \gamma \right) \, d\alpha \, d\beta \]

\[ f_3 \left( \frac{\alpha \pi}{2} \right) \left( \sin \alpha \cos \theta - \cos \alpha \cos \beta \sin \gamma \right) \, d\alpha \, d\beta \]

\[ \sqrt{1 - (\cos \alpha \cos \beta + \sin \alpha \cos \beta \sin \gamma)^2} \]

The frictional moment is given in the form of two perpendicular components because it is convenient to derive expressions for these components rather than for the total moment. This total frictional moment \( M \) can then be obtained by taking the sum of the components and is given in the form:

\[ M = M(l_{MR} + m_{MR} + n_{MR}) \]
where its magnitude and direction cosines are given by:

\[
M = \sqrt{M_{S}^2 + M_{R}^2} \quad (4.43)
\]

\[
I_{M} = \frac{M_{S}I_{MS} + M_{R}I_{MR}}{M} \quad (4.44)
\]

\[
m_{M} = \frac{M_{S}m_{MS} + M_{R}m_{MR}}{M} \quad (4.45)
\]

\[
n_{M} = \frac{M_{S}n_{MS} + M_{R}n_{MR}}{M} \quad (4.46)
\]

The results given above show that the determination of the magnitudes and directions of the frictional force and moment produced in a spherical joint depends upon the evaluation of the three factors \( f_{1}, f_{2} \) and \( f_{3} \). In chapter three it was shown that these three factors can be evaluated numerically provided that the values of the angles \( \alpha_{0} \) and \( \gamma \) are known. The method of determining the angle \( \alpha_{0} \) is described in the following chapter. The angle \( \gamma \) has been defined as the angle between the axis of rotation, represented by the vector \( \mathbf{w} \), and the line of action of the normal reaction \( \mathbf{N} \), as shown in Fig.4.1.

Thus, in order to determine the effects of friction relative to the applied load, it is necessary to derive an expression for the angle \( \gamma \) in terms of the angle \( \Theta \), which is the angle between the axis of rotation and the line of action of the applied load.
From the definition of the angle $\gamma$, this expression can be obtained using the equation for the vector product $\vec{wxN}$ which can be re-arranged to give:

$$\sin \gamma = \frac{|\vec{wxN}|}{\vec{wN}}$$

Using the expression for the magnitude of the vector product $\vec{wxN}$ given in equation 4.29 and the expression for the magnitude of the normal reaction $N$ given in equation 4.30, the above expression becomes:

$$\sin \gamma = \frac{\sqrt{1 - \frac{x^2}{\sin^2 \theta}}}{\frac{wp}{\sqrt{1-x^2}}}$$

where $x$ represents the ratio $\frac{f_1}{\sqrt{1+f_1^2}}$. Replacing the term $x$ by this ratio, the expression becomes:

$$\sin \gamma = \frac{\sqrt{(1+f_1^2)\sin^2 \theta - f_1^2}}{\sqrt{1+f_1^2-f_1^2}}$$

i.e. $\sin^2 \gamma = \sin^2 \theta - f_1^2 \cos^2 \theta$ (4.46)

As the factor $f_1$ has been shown to contain a complex double integral involving the angle $\gamma$, which can only be evaluated numerically, it follows that the value of the angle $\gamma$ cannot be determined analytically from equation
4.46. However, this equation can be used to determine the value of the angle for several particular values of the angle $\theta$. Firstly, when the axis of rotation is coincident with the line of action of the applied load, i.e. the angle $\theta$ is either $0^\circ$ or $180^\circ$, there is no resultant frictional force produced because the axis of rotation is passing through the centre of the axi-symmetric pressure distribution over the contact area, and so the factor $f_1$ has a value of zero. Equation 4.46 then becomes:

$$\sin^2 \varphi = 0$$

i.e. $\varphi = 0^\circ$ or $180^\circ$

The defined orientations of the angles $\theta$ and $\varphi$ shown in Fig. 4.1 indicate that when $\theta = 0^\circ$, $\varphi = 0^\circ$ and when $\theta = 180^\circ$, $\varphi = 180^\circ$.

The two other situations in which the angle $\varphi$ can be determined from equation 4.46 occur when the angle $\theta$ is $90^\circ$ and when it is $270^\circ$. In both of these positions the equation becomes:

$$\sin^2 \varphi = 1$$

i.e. $\varphi = 90^\circ$ or $270^\circ$

Again from the orientation of the two angles as shown in
Fig. 4.1, it can be seen that when \( \theta = 90^\circ \), \( \gamma = 90^\circ \) and when \( \theta = 270^\circ \), \( \gamma = 270^\circ \). Equation 4.46 also shows that the angle \( \gamma \) only has the same value as the angle \( \theta \) at these particular values, which means that the angle \( \gamma \) will always be in the same quadrant as the angle \( \theta \). This enables the intermediate values of the angle \( \gamma \) to be determined by iteration. The iterative routine used is shown in Fig. 4.3.

Having determined the appropriate quadrant from the value of the angle \( \theta \), the value of the angle \( \gamma \) is initially set at the mid point of that quadrant. Using this value of the angle \( \gamma \), the factor \( f_1 \) is evaluated and the error obtained when these values of the angle \( \gamma \) and the factor \( f_1 \) are substituted into equation 4.46 is determined. If this error is below a set value, the value of the angle \( \gamma \) is taken as correct. If it is not the value of \( \gamma \) is increased or decreased as appropriate and the procedure repeated until the correct value of the angle \( \gamma \) is obtained.

This iterative routine, together with the expressions derived for the magnitudes and directions of the frictional force and moments, was incorporated into the computer program used to evaluate the factors \( f_1, f_2 \) and \( f_3 \) numerically. The results obtained using this program are shown in Figs. 4.4-4.7. Fig. 4.4 shows the magnitude and direction of the
The Iterative Routine Used to Determine the Value of the Angle $\gamma$

**Figure 4.3**

\[ n = \text{INT} \left( \frac{2\theta}{\pi} \right) \]

I.e. \( n \) is Set at the Value of \( \left( \frac{2\theta}{\pi} \right) \) Truncated to Integer Form

\[ \gamma = (2n + 1) \frac{\pi}{4} \]

\[ x_{T} = \frac{\pi}{4} \]

\[ x_{T} = \frac{x_{T}}{2} \]

Determine $f_{1}$ from $\gamma$

\[ \text{ERROR} = \int_{0}^{f_{1}} \cos^{2} \theta + \sin^{2} \gamma - \sin^{2} \theta \]

Is $|\text{ERROR}| < 0.00001$?

- Yes
  - \[ \gamma = \gamma - x_{T} \]
  - Use Value of $\gamma$ Given

- No
  - Is $\text{ERROR} > 0.0$?
    - Yes
      - $\gamma = \gamma + x_{T}$
    - No
      - $\gamma = \gamma + x_{T}$
frictional force $F$ plotted against values of the angle $\theta$. As the magnitude of the frictional force is directly proportional to that of the applied load $P$ it is shown in the form of the ratio $F/P$. The direction is given in the form of the three direction cosines $l_F$, $m_F$ and $n_F$. Fig.4.5 shows the magnitude and direction cosines of the total frictional moment plotted against the values of the angle $\theta$. The magnitude of the moment is directly proportional to the magnitude of the applied load and to the radius of the joint $r$, so it is given in the form of the ratio $M/rP$. The magnitude and direction cosines of the normal reaction $N$ are shown plotted against values of the angle $\theta$ in Fig.4.6, the magnitude being given in the form of the ratio $N/P$.

The three sets of graphs shown in Figs.4.4-4.6 all show results obtained for various values of the coefficient of friction $\mu$ in the range from 0.2 to 1.0 for a particular value of the angle $\alpha_o$, which defines the size of the contact area, of $30^\circ$. Fig.4.7 shows the magnitude and direction cosines of the frictional moment plotted against values of the angle $\theta$ for various values of the angle $\alpha_o$ in the range from $5^\circ$ to $90^\circ$ and for a particular value of the coefficient of friction of $\mu = 1.0$. Thus Figs.4.4-4.6 show the characteristics of the frictional force and moment produced for a range of values of the coefficient of friction with a contact area of an extent that is likely to occur and Fig.4.7 shows the characteristics of
Figure 4.4a  The Ratio $F/p$ Plotted against the Angle $\theta$ for a Range of Values of the Coefficient of Friction $\mu$
Figure 4.4b  The Direction Cosine $l_F$ Plotted against the Angle $\theta$ for a Range of Values of the Coefficient of Friction $\mu$
Figure 4.4c. The direction cosine $m_f$ plotted against the angle $\theta$ for a range of values of the coefficient of friction $\mu$. 
Figure 4.4d: The Direction Cosine $n_F$ Plotted against the Angle $\theta$ for a Range of Values of the Coefficient of Friction $\mu$
Figure 4.5a. The ratio $\frac{M}{rP}$ plotted against the angle $\theta$ for a range of values of the coefficient of friction $\mu$. 

(Description of the graph)
Figure 4.5b  The Direction Cosine $l_m$ Plotted against the Angle $\theta$ for a Range of Values of the Coefficient of Friction $\mu$
Figure 4.5c  The Direction Cosine $m_m$ plotted against the angle $\theta$ for a range of values of the coefficient of friction $\mu$. 
Figure 4.5d

The Direction Cosine $n_M$ Plotted against the Angle $\theta$ for a Range of Values of the Coefficient of Friction $\mu$
Figure 4.6a  The Ratio $N/p$ Plotted against the Angle $\theta$ for a Range of Values of the Coefficient of Friction $\mu$
Figure 4.6b  The Direction Cosine $l_N$ Plotted against the Angle $\Theta$ for a Range of Values of the Coefficient of Friction $\mu$
Figure 4.6c  The Direction Cosine $m_N$ plotted against the angle $\theta$ for a range of values of the Coefficient of Friction $\mu$. 
The direction cosine $n_n$ plotted against the angle $\theta$ for a range of values of the coefficient of friction $\mu$. 

Figure 4.6d
Figure 4.7a: The ratio $M/\eta P$ plotted against the angle $\theta$ for a range of values of the angle $\alpha_0$. 
The direction cosine $L_m$ plotted against the angle $\theta$ for a range of values of the angle $\phi_0$. 

Figure 4.7b
Figure 4.7c  The Direction Cosine $m_m$ plotted against the angle $\theta$ for a range of values of the angle $\alpha_0$. 
Figure 4.7a

The direction cosine $n_M$ plotted against the angle $\theta$ for a range of values of the angle $\alpha_0$. 
the frictional moment for various sizes of the contact area when the friction is high to show the effect of varying the area clearly. The characteristics shown are described in the following section.

4.5 The Characteristics of the Frictional Force and Moment

4.5.1 The Frictional Force

As a result of the interaction between the frictional force and the normal reaction produced in a spherical joint, which has been analysed in this chapter, the characteristics of the magnitude and direction of the frictional force can only be described with reference to those of the normal reaction and even then they cannot always be described in a straight-forward manner.

The basic characteristics of the magnitude of the frictional force, shown in Fig.4.4a, are largely those described for the factor $f_\perp$ in section 3.9.1 and seen in Fig.3.9b. When the axis of rotation is aligned with the line of action of the applied load i.e. $\theta = 0^\circ$, there is no resultant frictional force produced because in this position the axis of rotation passes through the centre of the contact area over which the pressure producing the frictional forces is axisymmetrically distributed. As the
angle $\theta$ increases the point of intersection of the axis of rotation moves towards the edge of the contact area and so the resultant frictional force steadily increases as described in section 3.9.1. When the axis of rotation no longer intersects the contact area the magnitude of the frictional force is close to its maximum value which is finally reached when the axis of rotation is perpendicular to the line of action of the normal reaction i.e. $\gamma = 90^\circ$. As shown in section 4.4, this occurs when the axis of rotation is perpendicular to the applied load i.e. $\theta = 90^\circ$.

Fig. 4.4a shows that as expected the magnitude of the frictional force increases as the coefficient of friction rises. However, it also shows that the magnitude of this increase is steadily reduced with the rise in the coefficient of friction. This reduction is due to the fact that the direction of the frictional force is perpendicular to that of the normal reaction while the normal reaction $N$ is produced by the sum of the applied load $P$ and the frictional force $F$. Thus, as shown in Fig. 4.1, the direction of the frictional force is always such that it acts against the applied load to reduce the magnitude of the normal reaction. This is confirmed by comparing the magnitudes of the frictional force and the normal reaction in Figs. 4.4a and 4.6a. The result is that as the coefficient of friction rises the magnitude of the normal reaction is reduced and so the magnitude of the
frictional force, which is dependent on that of the normal reaction, is not as great as would be expected.

Fig. 4.4a also shows that as the coefficient of friction rises the magnitude of the frictional force reaches the point at which it is close to its maximum value at higher values of the angle \( \theta \). This characteristic results from the effect of the frictional force on the direction of the normal reaction as described below. When the coefficient of friction is low, the frictional force is small and so the normal reaction has roughly the opposite direction to the applied load which acts vertically. This is shown in Fig. 4.6d by the direction cosine \( n_N \) approaching unity as the coefficient of friction is reduced. The direction of the frictional force has been shown in section 3.4 to be perpendicular to both the axis of rotation and the direction of the normal reaction. Thus, when the coefficient of friction is low the frictional force acts largely in the \( j \) direction as given in Fig. 4.2, which is shown in Fig. 4.4c by the direction cosine \( m_F \) approaching unity as the coefficient of friction drops.

As the coefficient of friction rises, the increased frictional force results in a shift of the direction of the normal reaction from the vertical as shown by the direction cosine \( n_N \) in Fig. 4.6d. There is a corresponding shift in the direction of the frictional force towards the vertical, due to the fact that it is perpendicular to the
direction of the normal reaction, which is shown by the increase in the direction cosine \( n_F \) in Fig.4.4d. When the angle \( \theta \) is small the axis of rotation lies close to the vertical in the \( i-k \) place as shown in Fig.4.8. Thus when the direction of the frictional force shifts towards the vertical it also shifts towards the \( i \) direction to remain perpendicular to the axis of rotation. This is shown in Fig.4.4b by the increase in the direction cosine \( l_F \) as the coefficient of friction rises.

As the normal reaction is produced by the sum of the frictional force and the applied load which acts in the vertical or \( k \) direction, the shift in the frictional force towards the \( i \) direction produces a corresponding shift in the direction of the normal reaction. This is shown in Fig.4.6b by the increase in the direction cosine \( l_N \). Fig.4.6b also shows that as the angle \( \theta \) increases the shift in the normal reaction towards this direction also at first increases because of the increase in the magnitude of the frictional force producing it as shown in Fig.4.4a. The shift brings the direction of the normal reaction closer to the axis of rotation with the result that the axis of rotation continues to intersect the area of contact at higher values of the angle \( \theta \) and it is this which causes the magnitude of the frictional force to reach the point close to its maximum at a higher value of the angle \( \theta \). The higher the coefficient of friction is, the greater the frictional force with the consequent
Figure 4.8a  

The frictional force and moment produced when the coefficient of friction is low ($\mu = 0.2$) and the angle $\theta$ is small.
Figure 4.8b

The Frictional Force and Moment

produced when the coefficient of friction is high ($\mu=1.0$) and the angle $\theta$ is small.
Figure 4.8c

The frictional force and moment produced when the coefficient of friction is low ($\mu = 0.2$) and the angle $\theta$ is large.
The frictional force and moment produced when the coefficient of friction is high ($\mu = 1.0$) and the angle $\theta$ is large.
increase in the shift in the direction of the normal reaction towards the axis of rotation, resulting in the magnitude of the frictional force reaching the value close to its maximum at a higher value of the angle \( \theta \).

When the value of the angle \( \theta \) reaches this point at which the magnitude of the frictional force no longer increases, the shift in the direction of the normal reaction away from the vertical and the resulting shift in the direction of the frictional force towards the vertical both cease as shown by the direction cosines \( n_N \) and \( n_F \) in Figs. 4.6d and 4.4d respectively. As the angle \( \theta \) increases, the axis of rotation rotates towards the \( \pm \) direction and so to remain perpendicular to this axis the direction of the frictional force shifts away from the \( i \) direction as shown by the decrease in the direction cosine \( l_F \) in Fig. 4.4b. The combination of these two conditions means that the direction of the frictional force suddenly shifts towards the \( j \) direction as shown by the characteristics of the direction cosine \( m_F \) in Fig. 4.4c.

4.5.2 The Frictional Moment

The basic characteristics of the magnitude of the frictional moment as shown in Fig. 4.5a are largely a combination of those described for the moment of slip friction and the moment of rotational friction in sections.
3.7.2 and 3.7.3 which are shown by the factors \( f_2 \) and \( f_3 \) respectively in Fig. 3.9. At small values of the angle \( \theta \), when the angle \( \psi \) is also small, the frictional moment is comprised mainly of the moment of rotational friction. Then at larger values of the angle \( \theta \) the moment of slip friction predominates. As with the frictional force the magnitude of the moment increases with a rise in the coefficient of friction and for the higher values of the angle \( \theta \) this increase is reduced as the coefficient of friction rises due to the action of the frictional force reducing the magnitude of the normal reaction. However, when the angle \( \theta \) is zero, the magnitude of the frictional moment is directly proportional to the coefficient of friction because there is no resultant frictional force to effect the magnitude of the normal reaction.

Fig. 4.5a shows that the magnitude of the frictional moment reaches its maximum value at higher values of the angle \( \theta \) as the coefficient of friction rises in the same way that the magnitude of the frictional force did. This is due to the effect of the frictional force on the direction of the normal reaction as described in the previous section. Fig. 4.7a shows the variation of the magnitude of the moment with the value of angle \( \alpha \) which results from the fact that the moment of rotational friction, being the component of the moment produced about the axis passing through the centre of the contact area, is very dependent on the size of the area, while the moment of slip
friction, being the component produced about an axis perpendicular to the one passing through the centre of the contact area, is not.

The characteristics of the direction of the frictional moment, as shown in Fig. 4.8 and by the direction cosines \( l_M \), \( m_M \) and \( n_M \) in Fig. 4.5, are also largely a combination of those described for the moment of slip friction and the moment of rotational friction. As stated above, at low values of the angle \( \theta \) the moment of rotational friction is predominant and this moment acts about an axis coincident with the direction of the normal reaction which, as shown by Fig. 4.6, is very close to the vertical in this situation. Thus for small values of the angle \( \theta \) the frictional moment largely acts about the vertical direction as shown by the direction cosine \( n_M \) in Fig. 4.5d.

As the coefficient of friction rises, the increased frictional force produces a shift in the direction of the normal reaction towards the axis of rotation as described before. This means that the angle \( \gamma \) remains low and so the moment of rotational friction continues to predominate at higher values of the angle \( \theta \). As the direction of the normal reaction is still mainly vertical, this results in the direction cosine \( n_M \) shown in Fig. 4.5d remaining at a high value for larger values of the angle \( \theta \).

When the contact area is larger the moment of rotational
friction remains significant for higher values of the angle $\theta$ and so the total moment continues to be directed towards the vertical at these higher values of the angle $\theta$. This is shown in Fig. 4.7d by the direction cosine $n_M$ remaining at a high value, for greater values of the angle $\theta$ as the angle $\alpha$ rises.

For higher values of the angle $\theta$ the moment of slip friction becomes the major component of the friction moment and so the direction of the total moment tends towards that of the moment of slip friction. The direction of the moment of slip friction has been defined as being perpendicular to the line of action of the normal reaction and lying in the plane containing this line of action and the axis of rotation. Thus, when the coefficient of friction is low with the result that the direction of the normal reaction is close to the vertical, the direction of the moment tends towards the $i$ direction as shown by the direction cosine $l_M$ in Fig. 4.5b.

As the coefficient of friction rises the normal reaction is deflected towards the $i$ direction, bringing it closer to the axis of rotation, for the lower values of the angle $\theta$ as described before. Thus, at those values of the angle $\theta$ which are high enough that the moment of slip friction predominates and low enough that the shift in the direction of the normal reaction described above occurs, the direction of the moment is deflected towards the $j$
direction as shown by the direction cosine $m_M$ in Fig.4.5c.

Then, as also described previously, at higher values of the angle $\theta$ the direction of the normal reaction shifts away from the $i$ direction and towards the $j$ direction and this is reflected in the frictional moment being deflected from the $j$ direction towards the $i$ direction. When the coefficient of friction is higher the shift occurs more rapidly at a higher value of the angle $\theta$ as is shown in Figs.4.5b and 4.5c by the direction cosines $l_M$ and $m_M$.

Fig.4.7 shows that when the contact area is small this shift in direction is more prominent which is a result of frictional force reaching its maximum value at much lower values of the angle $\theta$ as shown in Fig.3.9. Then for larger contact areas the shift is less prominent as the moment of rotational friction becomes more significant over a larger range of the angle $\theta$. 
CHAPTER FIVE

The Mathematical Model

This chapter describes the areas in which the theoretical analysis was extended in order to develop, from the expressions derived for the frictional force and moment in the preceding chapters, a mathematical model that will adequately simulate the effects of friction in a spherical joint.

5.1 The Angle $\alpha_o$, Defining the Area of Contact

In section 4.4 of the previous chapter it was shown that the expressions for the frictional force and moment could be evaluated provided that the value of the angle $\alpha_o$, which defines the size of the contact area, was known. However, the extent of the contact area depends upon several factors as described in section 3.2 and so it is necessary to derive an expression for the angle $\alpha_o$ which, if the expressions for the frictional force and moment are to be used in a mathematical model, must be given in terms of quantities whose values are likely to be known by a user of the model.

The expression derived for the angle $\alpha_o$ is based on that
provided by the theory developed by Hertz for the radius of the contact area between two spheres (Timoshenko and Goodier, 1970), which is given in terms of the transmitted load, the relative radii of the spheres and their elastic properties. Unfortunately, Hertz's theory is based on the assumption that the radius of the contact area is very small in comparison with the radii of the two spheres, which is unlikely to be the case in a spherical joint under a normal load.

However, Goodman and Keer (1965) have extended Hertz's analysis in the case of an elastic sphere indenting a spherical cavity in an elastic solid by avoiding the assumption of a small contact area. Their results show that the expression given by Hertz's theory for the radius of the contact area can be extended to relatively large areas of contact provided that the sphere and cavity are of approximately equal radius.

Hertz's theory gives the following expression for the radial arc of the contact area, \( a \), shown in Fig. 5.1:

\[
a = \sqrt{\frac{3PR_1R_2(c_1 + c_2)}{4(R_1 - R_2)}}
\]

where 
- \( P \) is the load
- \( R_1 \) is the radius of the socket
- \( R_2 \) is the radius of the ball
Figure 5.1  Dimensions of the Contact Area
and the factors $c_1$ and $c_2$ are given by:

$$c_1 = \frac{1 - v_1^2}{E_1}$$

$$c_2 = \frac{1 - v_2^2}{E_2}$$

where $v_1$ and $v_2$ are the Poisson's ratios and $E_1$ and $E_2$ are the moduli of elasticity for the socket and ball respectively.

The radii of the ball and socket in a spherical joint are approximately equal. For example, the Rose Bearings catalogue (1983) gives the radial clearance, $R_1 - R_2$, of a normally fitting spherical bearing with a diameter of 25.4mm as being between 0.0075 and 0.015mm, which means that the ratio of the radius of the socket to that of the ball lies in the range from 1.0006 to 1.0012.

Thus the angle $\alpha_o$ defining the area of contact can be determined using the radius $R$ which is the mean radius between those of the ball and the socket. It can be seen from Fig.5.1 that the angle $\alpha_o$ is given, in radians, by the expression:

$$\alpha_o = \frac{a}{R}$$
Substituting the expression given for \( a \) in equation 5.1 into this expression gives:

\[
\alpha_0 = \sqrt{\frac{3P (c_1 + c_2)}{4R^3 (R_1 - R_2)}}
\]

As the radius \( R \) is the mean of the two radii \( R_1 \) and \( R_2 \) which are themselves approximately equal, this expression is practically equivalent to:

\[
\alpha_0 = \sqrt{\frac{3P (c_1 + c_2)}{4R (R_1 - R_2)}} \quad (5.4)
\]

where the two factors \( c_1 \) and \( c_2 \) are given by equations 5.2 and 5.3 respectively.

This expression gives the angle \( \alpha_0 \) in terms of quantities whose values are likely to be known. The radius, radial clearance and the elastic properties of the materials within the joint will have to be supplied by the user of the model. The load will be given by the normal reaction determined for the joint. As the magnitude of the normal reaction depends upon the magnitude and direction of the frictional force produced which in turn depends upon the extent of the contact area, the actual value of the angle \( \alpha_0 \) for a particular spherical joint can only be determined by iteration. The iterative routine used to determine the value of the angle \( \alpha_0 \) in the computer program used to
predict the effects of friction in a spherical joint is shown in Subroutine Alpha of the flow diagram of the program contained in Appendix 3.

The results produced by Goodman and Keer indicate that an accurate value for the angle $\alpha_o$ will be obtained from the expression given in equation 5.4 while the angle is $20^\circ$ or less. Considering these results and the assumptions originally used in determining the expression, it would appear safe to assume that reasonably accurate values will continue to be given as the angle $\alpha_o$ rises up to at least $30^\circ$, provided that the radii of the ball and the socket are as similar as has shown to be the case with a normally fitting spherical joint. As the value of the angle $\alpha_o$ increases further it is likely that the expression will become increasingly inaccurate, giving a value for the angle higher than it actually is.

However, a consideration of the effects of the size of the contact area on the frictional moment produced, which can be seen in Fig.4.7, shows that when the value of the angle $\Theta$ is high, large variations in the value of the angle $\alpha_o$ have little effect on the frictional moments. The angle $\Theta$ is the angle between the direction of the applied load and the axis of rotation in a spherical joint, which tends to have a high value in many applications of spherical joints, particularly when straddle-type spherical joints are used. Thus the accuracy of the value determined for
the angle \( \alpha \) will not be significant particularly if it is less than 70°.

Fig. 5.2 shows values for the angle \( \alpha \) obtained using the expression in equation 5.4 plotted against the load for several values of the radial clearance obtained in a normally fitting joint with a radius of 12.7mm. The Rose Bearings catalogue indicates that the maximum dynamic load on a straddle-type spherical joint of this size will be approximately 25kN and so it can be seen from Fig. 5.2 that under normal loading the value of the angle \( \alpha \) is likely to be less than 70°.

When the value of the angle \( \theta \) is smaller Fig. 4.7 shows that it will be necessary to determine the value of the angle \( \alpha \) more accurately. However, in this configuration the load transmitted by the joint is likely to be much less. For example the maximum axial load that can be safely applied to a straddle-type joint is about 20% of its maximum radial load, depending on the design of the joint. This means that when the value of the angle \( \theta \) is low the size of the contact area is likely to be in the range where it can be more accurately determined by the expression in equation 5.4.

Thus, although the expression derived to determine the value of the angle \( \alpha \) is not likely to be very accurate when the angle is large, it can be taken as being
Joint Radius: 12.7 mm
Poissons Ratios $\nu_1, \nu_2 = 0.3$
Youngs Moduli $E_1, E_2 = 2.05 \times 10^5$ N/m

\[ \alpha \]

Figure 5.2: The value of the angle $\alpha_0$ plotted against that of the applied load for various values of the radial clearance.
sufficient for the purposes of the mathematical model.

5.2 The Reaction to the Effects of Friction

It is necessary to consider the effects the frictional force and moment will have on the mechanism which contains the joint and how the reactions to these effects might alter the conditions at the joint.

Considering first the effect of the frictional force, the previous chapters have shown how the frictional effects over the surface of the ball in the joint can be reduced to a single resultant force acting through, and a single resultant moment acting about, the centre of the joint. The frictional effects produced on the surface of the socket are equal in magnitude and opposite in direction to those on the surface of the ball and so they can be reduced to a single force and moment at the centre of the joint which are equal and opposite to those acting on the ball. Thus the two frictional forces will balance each other and, although they affect the magnitude and direction of the normal reactions within the joint as shown in chapter four, beyond the joint they have no direct effects.

Hence it is only the frictional moment that has an effect on the mechanism containing the joint. As can be seen
from Fig. 5.3, the frictional moment is balanced by a couple produced by the reaction at the next joint or support of the mechanism and an equal and opposite reaction at the centre of the joint. This reaction produced at the centre of the joint affects the resultant load produced on the joint which in turn affects the frictional force and moment produced. However, as the direction and magnitude of this reaction depends upon the structure and configuration of the mechanism, its effect cannot be incorporated into a model which represents the joint independently of the mechanism containing it. When incorporated into a mechanism simulation package such as AMP3D-ADAMS the main program will calculate the reaction produced and so determine the resultant load on the joint at each stage of the simulation.

5.3 The Direction of the Frictional Moment Relative to a General Co-ordinate System

In order to obtain the expressions for the frictional moment produced in a spherical joint as described in chapter four it was necessary to select a particular coordinate system for the joint so that the applied load $P$ and the vector $w$, representing the axis of rotation, were given by equations 4.2 and 4.3 as:
Frictional Forces on Ball and Socket:
\[ F_B + F_S = 0 \]

Frictional Moments on Ball and Socket:
\[ M_B = -M_S \]

Figure 5.3 Reaction to the Effects of Friction
\[ w = w(\sin \theta_i + \cos \theta_k) \]

\[ P = -Pk \]

The frictional moment \( M \) was then given in terms of this co-ordinate system by equation 4.42 as:

\[ M = M(l_M i' + m_M i' + n_M k') \]

where \( M \) and \( l_M \), \( m_M \) and \( n_M \) are the magnitude and direction cosines of the frictional moment given by equations 4.43 to 4.46.

For the model of the frictional effects in the spherical joint to be of value in the simulation of a larger mechanism the expression for the frictional moment has to be given in terms of a general co-ordinate system which can be represented by the unit vectors \( i', j', k' \). In this general co-ordinate system the applied load \( P \) and the vector \( w \) can be expressed as:

\[ w = w(l_w i' + m_w j' + n_w k') \quad (5.5) \]

\[ P = P(l_p i' + m_p j' + n_p k') \quad (5.6) \]

while the frictional moment \( M \) can be given as:

\[ M = M(l_M i' + m_M j' + n_M k') \quad (5.7) \]
where \( l_w', m_w', n_w', l_p', m_p', n_p', l_M', m_M', n_M' \) are the direction cosines of the three vectors relative to the general co-ordinate system.

In the simulation of a mechanism the direction of the applied load and the axis of rotation at a joint will be known by the program and so the relationship between the two co-ordinate systems can be obtained by comparing the equations in which they are given in relation to each co-ordinate system. From equations 4.3 and 5.6 it can be seen that:

\[
k = -(l_p'i' + m_p'i' + n_p'k')
\]

(5.8)

and from equations 4.2 and 5.5:

\[
-sin\theta i' + cos\theta k = l_w'i' + m_w'i' + n_w'k'
\]

(5.9)

By combining equations 5.8 and 5.9 the unit vector \( i' \) can be given as:

\[
i' = \frac{-1}{sin\theta} [(l_w' + l_p' cos\theta)i' + (m_w' + m_p' cos\theta)j' + (n_w' + n_p' cos\theta)k']
\]

(5.10)

An expression for the third unit vector \( j' \) can then be obtained from the vector product \( k x i \) which gives:

\[
j' = -(l_p'i' + m_p'i' + n_p'k') \times \frac{-1}{sin\theta} [(l_w' + l_p' cos\theta)i' + (m_w' + m_p' cos\theta)j' + (n_w' + n_p' cos\theta)k']
\]

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\[
= \frac{1}{\sin \theta} \left[ l_p ' (m_w ' + m_p ' \cos \theta) k' - l_p ' (n_w ' + n_p ' \cos \theta) i' \right.
\]
\[
- m_p ' (l_w ' + l_p ' \cos \theta) k' + m_p ' (n_w ' + n_p ' \cos \theta) i'
\]
\[
+ n_p ' (l_w ' + l_p ' \cos \theta) j' - n_p ' (m_w ' + m_p ' \cos \theta) i' \right]
\]
\[
= \frac{1}{\sin \theta} \left[ (m_p ' - n_p ' m_w ') i' + (n_p ' l_w ' - l_p ' n_w ') j'
\]
\[
+ (l_p ' m_w ' - m_p ' l_w ') k' \right] \quad (5.11)
\]

Expressions for the sine and cosine of the angle \( \theta \) in terms of the unit vectors of the general co-ordinate system can be obtained using the scalar product of the vectors \( w \) and \( p \) which from Fig. 4.2 can be seen to be given by:

\[ w \cdot p = w p \cos (\pi - \theta) \]
\[ = -w p \cos \theta \quad (5.12) \]

In terms of the direction cosines the scalar product is given by:

\[ w \cdot p = w p (l_w ' l_p ' + m_w ' m_p ' + n_w ' n_p ') \quad (5.13) \]

From equations 5.12 and 5.13 the cosine of the angle \( \theta \) can be seen to be given by:

\[ \cos \theta = -(l_w ' l_p ' + m_w ' m_p ' + n_w ' n_p ') \quad (5.14) \]

and so the sine of the angle \( \theta \) is given by:
\[ \sin \theta = \sqrt{1 - (l_w' l_p' + m_w' m_p' + n_w' n_p')^2} \]  \hspace{1cm} (5.15)

The direction cosines of the frictional moment relative to the general co-ordinate system, \( l_M', m_M' \) and \( n_M' \), can then be found by taking the sum of the components of the direction cosines \( l_M, m_M \) and \( n_M \) that act in the relevant direction given by the unit vector \( \hat{i}', \hat{j}' \) or \( \hat{k}' \). Using equations 5.8, 5.10 and 5.11 the direction cosines are given by:

\[ l_M' = l_M \left( \frac{l_w' + l_p' \cos \theta}{\sin \theta} \right) + m_M \left( \frac{m_w' n_p' - n_w' m_p'}{\sin \theta} \right) - n_M l_p' \]  \hspace{1cm} (5.16)

\[ m_M' = m_M \left( \frac{m_w' + m_p' \cos \theta}{\sin \theta} \right) + n_M \left( \frac{n_p' l_w' - l_p' n_w'}{\sin \theta} \right) - n_M m_p' \]  \hspace{1cm} (5.17)

\[ n_M' = n_M \left( \frac{n_w' + n_p' \cos \theta}{\sin \theta} \right) + m_M \left( \frac{l_p' m_w' - m_p' l_w'}{\sin \theta} \right) - n_M n_p' \]  \hspace{1cm} (5.18)

where \( \cos \theta \) and \( \sin \theta \) are given by equations 5.14 and 5.15.

These expressions will give the direction of the frictional moment relative to the general co-ordinate system. The direction of the frictional force or normal reaction can be obtained relative to the general co-ordinate system, if required, by using the same equations after substituting the relevant direction cosines for \( l_M' \),...
5.4 The Displacement and Velocity Dependence of Friction in a Spherical Joint

The displacement and velocity dependence of friction was originally incorporated into the friction model developed for AMP2D to avoid the computationally undesirable effect of an instantaneous change in the frictional force from a positive to a negative value on the reversal of the direction of motion. However, it also allowed the change from a higher 'static' value of the coefficient of friction during initial motion to the lower 'dynamic' value at higher velocities to be modelled as is described in section 2.2.

When the direction of rotation is reversed in a spherical joint there is an instantaneous change in the magnitude and the direction of the frictional moment produced and so it can be seen that the displacement and velocity dependence of friction needs to be incorporated into any model of friction in a spherical joint. However, as described in section 2.3, the model used in AMP2D cannot be simply adapted because the relative velocity of the surfaces in contact and thus the coefficient of friction varies over the area of contact within the joint.
This problem can be overcome using the model developed for friction in a spherical joint in chapters three and four by considering an instantaneous coefficient of friction $\mu_I$ at each elemental area on the contact area. The value of this instantaneous coefficient of friction varies with its position on the contact area and so is contained within the integrals in the expressions for the frictional force and moment. The expressions for the factors $f_1$, $f_2$ and $f_3$ given by equations 3.26, 3.33 and 3.39 then become:

\begin{align}
  f_1 &= \frac{1}{\pi I} \int_0^{2\pi} \int_0^{\pi} \mu_I \sin \alpha \cos \left(\frac{\alpha \pi}{2 \alpha_0}\right) \left(\cos \alpha \sin \gamma - \sin \alpha \cos \beta \cos \alpha \right) \, d\alpha \, d\beta \\
  \frac{1}{\sqrt{1 - (\cos \alpha \cos \beta + \sin \alpha \cos \beta \sin \gamma)^2}} &+ (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2 \\
  f_2 &= \frac{1}{\pi I} \int_0^{2\pi} \int_0^{\pi} \mu_I \sin \alpha \cos \left(\frac{\alpha \pi}{2 \alpha_0}\right) \left(\sin \gamma - \sin^2 \alpha \sin \gamma \cos \beta - \sin \alpha \cos \alpha \cos \beta \cos \alpha \right) \, d\alpha \, d\beta \\
  \frac{1}{\sqrt{1 - (\cos \alpha \cos \beta + \sin \alpha \cos \beta \sin \gamma)^2}} &+ (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2 \\
  f_3 &= \frac{1}{\pi I} \int_0^{2\pi} \int_0^{\pi} \mu_I \sin^2 \alpha \cos \left(\frac{\alpha \pi}{2 \alpha_0}\right) \left(\sin \cos \gamma - \cos \alpha \sin \gamma \cos \beta \right) \, d\alpha \, d\beta \\
  \frac{1}{\sqrt{1 - (\cos \alpha \cos \beta + \sin \alpha \cos \beta \sin \gamma)^2}} &+ (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2
\end{align}

where $I$ is given by equation 3.17 as:

\[ I = \int_0^{2\pi} 2 \sin \alpha \cos \alpha \cos \left(\frac{\alpha \pi}{2 \alpha_0}\right) \, d\alpha \]

The instantaneous coefficient of friction $\mu_I$ can be determined using the expressions that were derived for the friction model incorporated into AMP2D as described in section 2.2. Equation 2.5 gives:
\[ \mu_I = (\mu_s f - \mu_d) e^{-\frac{3v}{V}} + \mu_d \]

where \( \mu_s \) is the coefficient of static friction

\( \mu_d \) is the coefficient of dynamic friction

\( v \) is the sliding velocity of the surfaces in contact

\( s \) is the sign of that velocity

\( V \) is the sliding velocity at which the changeover from 'static' to 'dynamic' friction is 95% complete

\( f \) is a factor calculated for each integration time step of the program and given by equation 2.4 as:

\[ f = \frac{x_0 f_L - 19s\Delta x(f_L-s)}{x_0 - 19\Delta x(f_L-s)} \]

where \( f_L \) is the value of \( f \) at the last time step

\( \Delta x \) is the displacement during the current time step

\( x_0 \) is the sliding distance (measured from the point at which the velocity was last zero) at which \( |f| = 0.95 \)

Within the expressions the only terms that are dependent on the position of the elemental area are the sliding velocity \( v \) and the displacement \( \Delta x \). These two terms can be given by the following expressions:

\[ v = r'w \]
\[ \Delta x = r'\Delta \phi \]
where \( w \) is the speed of rotation of the joint

\[ \Delta \phi \] is the angular displacement of the joint during the current time step of the simulation

\( r' \) is the perpendicular distance from the element to the axis of rotation.

From Fig. 3.6 it can be seen that the perpendicular distance \( r' \) is the distance from \( A \) to \( C \) which is denoted by the symbol \( c \) where:

\[ c = \sqrt{a^2 + b^2} \]

Expressions for the distances \( a \) and \( b \) have been determined in section 3.4 and are given by equations 3.20 and 3.21 as:

\[ a = rsin\alpha sin\beta \]
\[ b = r(cos\alpha sin\beta - sin\alpha cos\beta cos\gamma) \]

Thus the perpendicular distance \( r' \) is given by:

\[ r' = r \sqrt{\sin^2\alpha \sin^2\beta + (\cos\alpha \sin\beta - \sin\alpha \cos\beta \cos\gamma)^2} \]

which from Appendix 1 can be seen to reduce to:

\[ r' = r \sqrt{1 - (\cos\alpha \cos\gamma + \sin\alpha \cos\beta \sin\gamma)^2} \]

Substituting this expression into those given for the sliding velocity \( v \) and the displacement \( \Delta x \) produces the
following equations:

\[ v = rw\sqrt{1 - (\cos \alpha \cos \beta + \sin \alpha \cos \beta \sin \delta)^2} \quad (5.22) \]

\[ x = r\Delta \phi \sqrt{1 - (\cos \alpha \cos \beta + \sin \alpha \cos \beta \sin \delta)^2} \quad (5.23) \]

Substituting these expressions into those for the instantaneous coefficient of friction \( \mu_I \), the factors \( f_1 \), \( f_2 \) and \( f_3 \) can be evaluated by numerical integration as before and so the frictional force and moment can be determined using this adaption of the friction model already developed, provided the user of the model supplies the values of the coefficients of static and dynamic friction, \( \mu_s \) and \( \mu_d \), and the values for \( v_o \) and \( x_o \).

Fig.5.4 shows the effect of the velocity dependence of friction on the frictional moment determined using the model described above. The graph shows the variation in the frictional moment, as the coefficient of friction changes from a high 'static' value to a lower 'dynamic' value with the increasing angular velocity, for various values of the angle \( \theta \), the angle between the axis of rotation and the direction of the applied load. It can be seen that when the angle \( \theta \) is small with the axis of rotation passing through or close to the contact area the variation in the frictional moment is more gradual than when the angle \( \theta \) is large. This is due to the fact that when points on the contact area are close to the axis of
Figure 5.4: The ratio $M/rP$ plotted against the angular velocity $\omega$ for various values of the angle $\theta$. 
rotation, their sliding velocities remain small and so within the range where the friction is velocity dependent for much higher values of the angular velocity.

It will be shown in the following section that it may be useful to obtain some relatively simple and approximate expressions for the values of the frictional force and moment. This is due to the fact that the numerical integration and the iterations used in the model require a considerable amount of computer time and also that absolute accuracy is not required by a computer simulation which aims to be a guide to what happens rather than a precise method of prediction.

This means that if numerical integration is to be avoided the velocity and displacement dependence has to be determined by a more approximate method than described above. From the equations given above, it can be seen that the only factor in the expressions determining the instantaneous coefficient of friction $\mu_I$ which is dependent upon the position of the element is its perpendicular distance $r'$ from the axis of rotation. Thus if an approximately average value can be taken for the perpendicular distances from the axis of rotation of all the elements on the contact area, the coefficient of friction does not need to be included in the integration.

This can be demonstrated using a simple approximation for
the average perpendicular distance $r'$. When the contact area is small and the angle $\theta$ is large the mean value for the distance $r'$ is given by the perpendicular distance from the centre of the contact area to the axis of rotation. From Fig.3.5 it can be seen that this distance is given by:

$$r' = r \sin \gamma$$

As the contact area increases in size the mean position of the elements on it will move from its centre towards the centre of the joint. Taking this effect into account, as well as the pressure distribution which results in greater frictional effects at the centre of the contact area, a suitable approximation for the distance $r'$ can be given by:

$$r' = r \sin \gamma \cos(\frac{\alpha}{2})$$

However, this approximation is based on the assumption that the distance $r'$ is equivalent to the perpendicular distance from the mean position of the elements on the contact area to the axis of rotation, which is not valid when the axis of rotation passes though the contact area. Thus, the approximation has to be adapted to cover this situation. A possible approximation that could be used is:
\[ r' = r \sin \theta \cos \left( -\frac{\alpha_0}{3} \right) \quad \alpha_0 < \theta < 90^\circ \]

\[ r' = r \sin \theta \cos \left( -\frac{\alpha_0}{3} \right) + \sin \left( -\frac{\alpha_0 - \theta}{2} \right) \quad 0^\circ < \theta < \alpha_0 \]

Fig. 5.5 shows a comparison of the values of the frictional moment obtained using this approximation with those obtained previously. It can be seen that the approximation gives results closely comparable with those obtained previously when the angle \( \theta \) is large. This is due to the fact that when the angle \( \theta \) is large there is little variation in the sliding velocity over the contact area. When the angle \( \theta \) is smaller there is a greater difference between the values obtained using the approximation and those obtained previously. This is due to the greater variation in the sliding velocity over the contact area, although the difference could probably be reduced by determining a more accurate approximation.

### 5.5 Approximate Expressions

As stated briefly in the previous section, the numerical integration and the iterations necessary to determine the frictional effects in a spherical joint require considerable computer time. Thus if the friction model based on these routines is to be incorporated into a computer simulation program such as AMP3D-ADAMS, it will greatly increase the time and therefore the cost of the
Figure 5.5  As Figure 5.4 but showing the effect of the approximate method of determining the velocity dependence of friction
simulation. However, the purpose of such a program is not to provide precise results but to give an approximate simulation of events. Thus reasonably accurate approximations to the expressions used to determine the frictional effects in a spherical joint would provide adequate results and save on time and expense.

As the frictional force has been shown to have no direct effect beyond the spherical joint, it is only the expressions for the frictional moment $M$ for which approximations have to be obtained. In equation 4.42 this moment is given by:

$$ M = M(1_{M1} + m_{M1} + n_{Mk}) $$

while its magnitude and direction cosines are given by equations 4.43 to 4.46 as:

$$ M = \sqrt{M_s^2 + M_r^2} $$

$$ 1_M = \frac{M_s1_{MS} + M_r1_{MR}}{M} $$

$$ m_M = \frac{M_s m_{MS} + M_r m_{MR}}{M} $$

$$ n_M = \frac{M_s n_{MS} + M_r n_{MR}}{M} $$
The magnitudes and directions of the two perpendicular components of this moment, the moment of slip friction $M_S$ and the moment of rotational friction $M_R$, are given by equations 4.22, 4.31-4.33 and 4.36 and 4.39-4.41 as:

$$M_S = \frac{rf_2^p}{\sqrt{1+f_1^2}}$$

$$l_{MS} = \sqrt{1 - \frac{f_1^2 \cos^2 \theta}{\sin^2 \theta}}$$

$$m_{MS} = \frac{-f_1 \cos \theta}{\sin \theta}$$

$$n_{MS} = 0$$

$$M_R = \frac{rf_3^p}{\sqrt{1+f_1^2}}$$

$$l_{MR} = \frac{f_1 \cos^2 \theta}{\sin^2 \theta \sqrt{1+f_1^2}} \text{sgn}(\cos \theta)$$

$$m_{MR} = \frac{f_1}{\sin \theta} \sqrt{\frac{\sin^2 \theta}{1+f_1^2} - \frac{f_1^2}{1+f_1^2}} \text{sgn}(\cos \theta)$$

$$n_{MR} = \frac{-1}{\sqrt{1+f_1^2}} \text{sgn}(\cos \theta)$$
Considering these expressions it is clear that they could easily be evaluated if simple approximate expressions were determined for the factors $f_1$, $f_2$ and $f_3$. Fig. 3.9 shows how the ratio of each of these three factors to the value of the coefficient of friction varies with the values of the angle $\gamma$, which is the angle between the line of action of the normal reaction and the axis of rotation, and the angle $\alpha_0$ which defines the extent of the contact area. It can be seen from these graphs that, when the value of the angle $\gamma$ is high and that of the angle $\alpha_0$ is low, the values of the factors $f_1$ and $f_2$ tend towards the value of the coefficient of friction and the value of the factor $f_3$ tends towards zero.

It can also be seen from equation 4.46 that the value of the angle $\gamma$ is high when the value of the angle $\theta$, the angle between the axis of rotation and the direction of the applied load, is also high.

Thus it can be seen from Fig. 3.9a that for a wide range of the higher values of the angle $\theta$ when the angle $\alpha_0$ is small the factors $f_1$, $f_2$ and $f_3$ can be taken as being given as:

\[ f_1 = f_2 = \mu \]

\[ f_3 = 0 \]
Substituting these values for the three factors into the equations given previously, the frictional moment \( M \) can be given approximately, when the angle \( \theta \) is large and the angle \( \alpha_o \) is small, by:

\[
M = M(l_M^1 + m_M^1 + n_M^k)
\]

while the magnitude and direction cosines are given by:

\[
M = M_S = \frac{\mu r P}{\sqrt{1 + \mu^2}}
\]

\[
l_M = l_{MS} = \sqrt{1 - \frac{\mu^2 \cos^2 \theta}{\sin^2 \theta}}
\]

\[
m_M = m_{MS} = \frac{-\mu \cos \theta}{\sin \theta}
\]

\[
n_M = n_{MS} = 0
\]

For slightly higher values of the angle \( \alpha_o \) the magnitude of the factor \( f_3 \) has to be taken into consideration as can be seen from Fig.3.9b. However, a good approximation to the frictional moment can be obtained in this case by assuming a uniform variation of the factor \( f_3 \) over the higher values of the angle \( \theta \). Thus the three factors could be given by:
\[ f_1 = f_2 = \mu \]
\[ f_3 = \mu \frac{2\alpha_o^2}{3} \left( \frac{\pi}{2} - \theta \right) \]

which when substituted into the expressions for the frictional moment give a good approximation over a range of the smaller values of the angle \( \alpha_o \).

This approximation may be of some value in the case being considered in the next section - that of a straddle-type joint (where the angle \( \theta \) is always high) under low loads, so that the contact area is small and thus the value of the angle \( \alpha_o \) is low. However, in general, where the angle \( \theta \) is likely to be small or the contact area large a more wide-ranging approximation to the three factors will be required.

As can be seen from the expressions in equations 3.26, 3.33 and 3.39, the magnitudes of the factors \( f_1, f_2 \) and \( f_3 \) depend upon the values of the angles \( \alpha_o \) and \( \gamma \) and the coefficient of friction \( \mu \). The angle \( \gamma \) is shown by equation 4.46 to be dependent on the angle \( \theta \). Then, as the factors are directly proportional to the coefficient of friction \( \mu \), good approximations for them can be found by fitting surfaces to the variations of the ratios of these factors to the coefficient of friction against the angles \( \theta \) and \( \alpha_o \). There are computer packages available to
do this but it has not yet been attempted and is one of the outstanding requirements if the model developed is to be incorporated into a computer simulation program such as AMP3D-ADAMS.

5.6 Straddle-type Spherical Joints

Spherical joints are commonly used in the form of the straddle-type joint, also known as a spherical bearing, which is shown in Fig. 5.6. The friction model developed in the previous chapters can be applied directly to this type of joint when the load on the joint is of low magnitude and is largely radial in direction i.e. it acts roughly perpendicular to the axis of the straddle as shown in Fig. 5.6.

The direct application of the model is limited to these conditions because it is necessary for the contact between the ball and the straddle support to occur over a circular area, which is one of the assumptions the model is based upon. Thus the extent of the circular contact area predicted by the formula in section 5.1 must remain within the limits given by the edges of the straddle support.

The range of loads applied to a straddle-type joint for which the model is valid can be determined by taking the dimensions of a typical joint such as supplied by the Rose
**Figure 5.6a**  
**Straddle-type Spherical Joint**  
$(\theta = 90^\circ)$

**Figure 5.6b**  
**Straddle-type Spherical Joint**  
$(\theta < 90^\circ)$
Bearings catalogue (1983). For example, a joint with a radius R of 12.7mm and a radial clearance \((R_2-R_1)\) in the range 0.0075-0.015mm has a contact area defined by the angle \(\alpha_o\) which is given by equation 5.4 as:

\[
\alpha_o = \frac{3N(c_1+c_2)}{4R(R_2-R_1)} \text{ radians}
\]

The factors \(c_1\) and \(c_2\) are given by equations 5.2 and 5.3 as:

\[
c_1 = \frac{1-v_1^2}{E_1} \quad c_2 = \frac{1-v_2^2}{E_2}
\]

The Poisson's ratios \(v_1\), \(v_2\) and the Young's Moduli \(E_1\) and \(E_2\) can be taken for a stainless steel joint as:

\[
v_1 = v_2 = 0.3 \quad E_1 = E_2 = 2.05\times10^9 \text{ N/m}^2
\]

while N is the resultant load produced by the sum of the applied load and the frictional force in the joint.

Then, assuming an average radial clearance of \((R_2-R_1)\) = 0.01mm, the expression in equation 5.4 can be evaluated to give:

\[
\alpha_o = 0.0374 \sqrt[3]{N} \text{ radians}
\]

The extent of the straddle, which is shown by the angle Y
in Fig. 5.6, is given for this joint as $32^\circ$. The friction model applies provided the predicted contact area is contained with the confines of the straddle support. Thus the sum of the angle $\alpha_0$ and the angle $\psi$, which is the angular movement of the centre of the contact area towards the edge of the straddle, must be less than the angle $Y$. For the particular joint being considered, the maximum angle of misalignment, shown by the angle $X$ in Fig. 5.6, is $11^\circ$, which gives a minimum value of the angle $\theta$ for the joint of $79^\circ$. Taking this value of the angle $\theta$ and a high value for the coefficient of friction of $\mu = 1.0$, the value of the angle $\psi$ is predicted by the friction model, using a computer program, as being roughly $12^\circ$ over a wide range of values of the angle $\alpha_0$. Thus the maximum possible value of the angle $\alpha_0$ is given by:

$$\alpha_0 = Y - \psi$$
$$= 32^\circ - 12^\circ = 20^\circ$$

The magnitude of the resultant load $N$ can then be determined, from the expression given for the angle $\alpha_0$ above, as:

$$N = \left(\frac{\alpha_0}{0.0374}\right)^3$$
$$= 813 \text{ Newtons}$$

Using the computer program, the friction model shows that
in this situation the ratio of the resultant load $N$ to the load applied to the joint $P$ is given by:

$$N/P = 0.778$$

Thus in the joint described above the applied load $P$ can have a magnitude of up to 1,000 Newtons while the friction model can still be applied. If the coefficient of friction is lower the possible magnitude of the applied load $P$ is higher. For example, when the coefficient of friction has a value of $\mu = 0.2$ the applied load can be as high as 3,000 Newtons and the friction model can still be applied.

However, when the applied load is higher than that described above or there is a significant degree of axial load on the joint the contact area reaches the edge of the straddle-support and so is no longer circular. In these cases the friction model will have to be revised and adapted if it is to be used to predict the effects of friction within a straddle-type spherical joint.
CHAPTER SIX

Experimental Apparatus, Procedure and Results

6.1 The Apparatus

Having developed a theoretical model of the effects of friction in a spherical joint, experimental results were then required to determine whether the model provides a reasonable accurate estimation of the frictional moment produced in an actual joint. Thus it was necessary to be able to measure the frictional moment produced in a joint while varying the conditions at that joint so that the model could be confirmed to be reasonably accurate over a range of the factors predicted to affect the frictional moment.

The previous chapters have indicated that there are three main factors which affect the frictional moment produced in a spherical joint. The first is the angle $\theta$, which is the angle between the direction of the applied load and the axis of rotation in the joint. The second is the coefficient of friction and the third main factor is the angle $\alpha_0$ which defines the extent of the area of contact and which depends upon a number of other factors including the magnitude of the load and the radial clearance in the
An investigation found no evidence of any previous experimental work done to measure the frictional moment produced in a spherical joint while these factors are varied and so the apparatus had to be specially designed for this task. There were two particular problems to be overcome in the design of this apparatus. The first was that the direction of the frictional moment varied with the changing conditions which meant that the moment had to be measured in the form of three perpendicular components. The second problem was that the friction produced in the apparatus had to be negligible when compared to that produced in the joint if reasonable results were to be obtained.

A number of possibilities were considered before the design of the apparatus shown in Figs.6.1 to 6.3 was decided upon. This design involved the measuring of the frictional moment produced on the socket of a joint which was loaded vertically from below. The ball was rotated in this socket by means of an attached shaft whose angle to the vertical could be altered to vary the angle $\theta$ between the direction of the applied load and the axis of rotation. The coefficient of friction and the extent of the contact area could then be varied by using a series of different balls and sockets.
Figure 6.1 The Gimbal Mechanism
Figure 6.2 The Supporting Bracket for the Shafts Rotating the Ball
Fig. 6.1 shows the global mechanism which allowed the frictional impact on the socket to be measured. This mechanism allowed the socket to rotate freely in any of three perpendicular directions about the ball. These rotations were then prevented by a series of strain-gauged rollers, attached to the top of the cylinder containing the ball. These rollers were used in the base of the apparatus.

The inner sleeve of the global mechanism was connected to its outer ring by a pair of joints that allowed the socket and inner sleeve to rotate freely about a horizontal axis passing through the centre of the joint. This rotation is shown in Figure 6.3, which is a general view of the experimental apparatus.

**Figure 6.3**

**A General View of the Experimental Apparatus**
Fig. 6.1 shows the gimbal mechanism which allowed the frictional moment on the socket to be measured. This mechanism allowed the socket to rotate freely in any of three perpendicular directions about the ball. These rotations were then prevented by a series of strain-gauged levers, some of which can be seen in Figs. 6.1 and 6.2. The strain-gauged levers were connected via amplifiers to the chart recorder shown in Fig. 6.3. The strain-gauges were calibrated so that from the plots produced on the chart recorder the components of the frictional moment in each of the three perpendicular directions could be calculated.

The design of the gimbal mechanism can be considered more closely by referring to Fig. 6.4. Between the cylinder containing the socket and the inner sleeve of the gimbal mechanism there was a cylindrical roller bearing which allowed the socket to rotate freely about the vertical axis. Then, as shown in Fig. 6.1, the rotation in either direction about the vertical axis was prevented by a strain-gauged lever, attached to the top of the cylinder containing the socket, which acted against a post fixed to the base of the apparatus.

The inner sleeve of the gimbal mechanism was connected to its outer ring by a pair of joints that allowed the socket and inner sleeve to rotate freely about a horizontal axis passing through the centre of the joint. This rotation
Figure 6.4  THE GIMBAL MECHANISM
was then resisted by a pair of strain-gauged levers, attached to the outer ring of the gimbal, which acted against the inner sleeve. The outer ring was connected to the base of the apparatus using a second pair of joints that allowed the mechanism to rotate freely about another horizontal axis passing through the centre of the joint, perpendicular to the first. The rotation about this axis was resisted by a further pair of strain-gauged levers which were attached to the base of the apparatus as shown in Fig.6.2.

As stated before, one of the main problems in obtaining useful results was to reduce the friction produced in the gimbal mechanism so that it would be negligible when compared to that produced in the joint. In order to achieve this, needle bearings were used in the joints of the gimbal mechanism and, as can be seen in Fig.6.1, flat roller bearings were placed between the strain-gauged levers and the inner gimbal sleeve to reduce the frictional resistance to rotation at these points of contact.

However, the main source of frictional resistance to the rotation of the socket came from the point at which the socket was loaded from below using a 5:1 loading arm attached beneath the base-plate of the apparatus. This problem was eventually overcome by designing an air-bearing which was incorporated between the end of the
loading arm and the base of the cylinder containing the socket. This air-bearing considerably reduced the amount of frictional resistance to the rotation of the socket and made it possible to obtain some reasonable results from the experiments.

The ball was rotated in the socket by means of an electric motor, attached underneath the base of the apparatus, which can be seen in Fig.6.3. This motor was connected to a control box which allowed the speed of the motor to be varied as well as enabling it to be run in reverse. As the motor had a maximum speed of 7,000 r.p.m. and it was necessary to rotate the ball at speeds of less than 1 r.p.m. - if hydrodynamic lubrication was to be avoided when there were lubricants between the ball and the sockets, which would greatly reduce the friction produced - a considerable reduction in the speed of rotation was required. This was achieved using a 10:1 gearbox attached beneath the base-plate and two sets of pulleys and timing belts, one of which connected the motor to the gearbox and the other of which can be seen connecting the drive-shafts shown in Fig.6.2. This resulted in a reduction in the motor speed by a hundredfold.

Fig.6.2 shows the supporting bracket for the drive-shafts. The bevel-gear assembly shown allowed the entire bracket to be rotated about a horizontal axis passing through the centre of the joint. This enabled the angle \( \theta \) between the
direction of the applied load and the axis of rotation to be varied through approximately $40^\circ$ either side of the vertical. While an experiment was performed the bracket was held in place against a stand, connected to the base-plate, by a pair of bolts and washers.

To enable the ball and socket to be easily removed for either replacement or examination the housing around each of the two bearings on the drive-shaft attached to the ball could be unscrewed and removed as can be seen from Fig. 6.2. It was necessary to have a thrust bearing between a shoulder on the shaft and the lower bearing housing to take the reaction due to the load applied to the ball through the socket. A split washer held in place by an O-ring was then placed between the thrust bearing and the shoulder on the shaft to enable the shaft to be removed easily. This arrangement is clearly shown in Fig. 6.1. The ball was held in the shaft by a grub screw and so was easily detached. Once the shaft had been removed the socket could be taken from the gimbal mechanism after the uppermost strain-gauged lever had been unscrewed.

The sockets were produced in the Engineering Department's workshop using a vertical milling machine. The cylinders in which the sockets were cut were held at an angle of $45^\circ$ to the milling tool and slowly rotated as the tool generated the spherical socket. The sockets were then ground and polished to provide a smooth surface finish.
The balls for the apparatus were specially made by the Speedright Gauge and Tool Co. of Coventry to ensure the necessary sphericity and surface finish, and to ensure that the axis of rotation of the shaft into which the balls were fitted would pass through the centre of each ball.

6.2 The Procedure

The aim of the experimental work was to obtain several sets of measurements of the frictional moment produced in a spherical joint over a range of values of the angle θ, which is the angle between the direction of the applied load and the axis of rotation, for different values of the coefficient of friction and the size of the contact area. These measurements could then be compared with the corresponding values given by the model to show that the model can predict the frictional moment produced in a spherical joint with reasonable accuracy.

As shown in the previous section, the configuration of the apparatus could be altered to give a range of values of the angle θ up to roughly 40° either side of the vertical. The coefficient of friction was initially to be varied by using balls and sockets of different materials such as tool steel, stainless steel, mild steel and brass, and the size of the contact area by using balls and sockets with
different radial clearances. Thus the experimental work was to consist of obtaining a set of measurements of the frictional moment over a range of the angle $\theta$ for a particular ball and socket and then changing the ball and socket to give different values of the coefficient of friction and the size of the contact area for which a set of measurements could be obtained.

Before each set of measurements were taken the ball and socket were cleaned by ultrasonic vibration in a paraffin bath, and then wiped with acetone to remove any grease before being assembled in the apparatus. Careful cleaning was necessary as any particle of dirt between the ball and the socket would cause surface damage when the ball was rotated, which would distort the measurements being taken and, particularly when softer materials were being used, would make the ball or socket useless for further experimental work.

Initially, it was attempted to rotate the ball in the socket without any form of lubrication in order to produce high values of the coefficient of friction. However, under these conditions it was found that the ball, which was generally of a harder material, tended to pick up particles of the softer material of the socket and then scour grooves in the socket as it rotated.

To prevent this damage occurring it was decided to 'run
in' each pair of the balls and sockets before taking the required measurements. This was done using a thin oil, Tellus R10, as a lubricant between the ball and socket. As the damage was likely to occur when the angle $\Theta$ was small, with the axis of rotation then intersecting the contact area close to the point of maximum pressure at its centre, the 'running-in' was done with the angle $\Theta$ at the maximum value allowed by the constraints of the apparatus design. The 'running-in' consisted of first rotating the ball with no load applied to the joint, and then gradually increasing the load on the loading arm in 1/2kg steps until a load of 4kg was reached, the maximum that could be supported by the air-bearing because of its limited area. Taking into consideration the frictional losses in the pivot of the loading arm this represented a load of approximately 190N on the joint.

After the 'running-in' the first sets of measurements were taken using the thinnest available oil, Tellus R5, as a lubricant between the ball and socket to reduce the likelihood of damage occurring while the measurements were being taken. The first measurement of each set was taken with the angle $\Theta$, the angle between the axis of rotation and the direction of the applied load, being $35^\circ$ which was the largest angle that could be obtained with this apparatus. The ball was initially rotated in the socket with no load applied. The load was then gradually increased to the maximum of about 190N while the three
components of the frictional moment were recorded on the chart plotter. The load was then removed, the direction of rotation of the ball was reversed and the magnitudes of the three components of the frictional moment were recorded for this direction of rotation.

This was done so that the difference between the magnitudes of each component of the frictional moment during forward and reverse rotation could be determined and then compared to these differences as calculated by the theoretical model. It was necessary to compare the theoretical and experimental results in this way because of the difficulty in determining datum values from which measurements in a single direction of rotation could be taken. This resulted from the fact that when no load was applied the cylinder containing the socket rested on a shoulder on the inner sleeve of the gimbal, as shown in Fig.6.4, and when the load was applied this raised the socket by about a millimetre to clear the shoulder and bring it into contact with the ball. This movement caused a change in the values recorded by the strain-gauges, which meant that the values recorded when no load was applied were not true representations of the zero values. There was also the fact that the remaining frictional resistances in the gimbal mechanism meant that the values recorded by the strain-gauges did not return to exactly the same values after each measurement as had been shown before.
After the measurements during forward and reverse rotations were completed, the angle between the axis of rotation and the vertical direction of the applied load was reduced by 5°, and the measurements of the components of the frictional moment during forward and reverse rotations were repeated. This was continued, reducing the angle between the axis of rotation and the vertical by 5° each time, until a set of measurements were obtained for values of the angle between 35° and 10°. It was not attempted to obtain results for smaller angles than this because of the strong likelihood of damage occurring between the ball and socket.

If the set of measurements were obtained without causing any great damage to either the ball or the socket, the procedure was repeated using a smaller amount of oil as the lubricant between the ball and socket, which tended to give a set of measurements for a slightly higher value of the coefficient of friction. Using small amounts of the oil Tellus R5 as the lubricant, several sets of measurements were obtained with the value of the coefficient of friction in the range from 0.05 to 0.1. When no lubricant was used the friction produced rose rapidly, but unfortunately in most cases this was accompanied by damage occurring to the surfaces of either the ball or the socket, which distorted the results produced. However, it was possible, in a few cases, to obtain some useful results for higher values of the
coefficient of friction. Once the ball and socket were damaged it was attempted to re-use them by first polishing them to remove the damage. However, in most cases the damage quickly re-occurred.

For a number of reasons the range of balls and sockets used in the experiments was not as great as originally intended. In the case of the balls, the brass and the mild steel proved to be too soft for the required sphericity to be produced. Thus the balls were either made of tool steel casehardened to Rockwell 58/65C or from stainless steel. In the case of the sockets, it was not possible to obtain any results using brass because the sockets were damaged very easily. Also, it was not possible to produce adequate sockets from the tool steel because of difficulties in machining a spherical cavity with a good surface finish in that material. Thus the sockets used were made from stainless steel and mild steel.

Because of difficulties in machining the sockets to the required sphericity only a few pairs of balls and sockets actually produced useful results. The problem was that if the socket was significantly out of shape the contact between the ball and the socket occurred over a different area to that predicted by the friction model, and so the frictional moment measured would not have been comparable to that given by the model.
6.3 Interpretation of Results

Having obtained a set of measurements of the frictional moment produced in a spherical joint over a range of the angle $\theta$, the angle between the axis of rotation and the direction of the applied load, the next step was to compare these values with those predicted by the friction model described in the previous chapters. The friction model had been incorporated into the computer program, shown in Appendix 3, which can be used to calculate the frictional moment produced in a spherical joint. However, to produce theoretical values corresponding to those obtained experimentally, it was necessary to adapt this program in a number of ways.

First, the effect of the reaction against the frictional moment produced in the socket, at the points where the rotation of the socket was resisted, had to be taken into account. As described in section 5.2 the frictional moment was balanced by couples produced by the reactions at these points and equal and opposite reactions at the centre of the joint. These reactions altered the magnitude and direction of the normal reaction in the joint and so affected the frictional moment produced.

However, as the reaction forces produced were relatively small compared to the applied load the computer program could be adapted using a simple iterative routine, shown
in Fig. 6.5. In this routine the frictional moment and reaction forces were calculated for the load applied. They were then re-calculated for the sum of the applied load and the reaction forces produced. This process was repeated until there was no significant difference between the new reaction forces calculated and those obtained in the previous iteration.

The computer program was also adapted so that it calculated the difference between the frictional moments produced when the ball was rotating first in one direction and then in the reverse direction, as this was the form in which the experimental results were obtained.

Having adapted the computer program in these ways, the main difficulty in comparing the theoretical values it produced with those obtained experimentally was that during the experiments the values of the coefficient of friction and of the radial clearance in the joint could only be very roughly estimated. Thus, to compare the experimental and theoretical results obtained, it was necessary to vary the values of the coefficient of friction, and of the radial clearance, used to calculate the theoretical results until the best possible fit between the two sets of results was found.

This was made easier by the fact that, as can be seen from Fig. 4.7a, there is only a small variation in the magnitude
**Figure 6.5** Iterative Routine Used to Determine the Effect of the Reaction Against the Frictional Moment

*Take the Applied Load as the Initial Value of the Resultant Load on the Joint*

*Determine the Frictional Moment*

*Determine the Reaction Resulting from the Moment*

*Is the Magnitude of the Reaction, \( R_{mag} \), Zero?*

*Yes*  
*Use the Values Determined*

*No*  
*Has the Reaction Been Calculated Before?*

*Yes*  
*\[ P_{er} = \frac{(R_{mag} - R_2)}{R_{mag}} \]*  
*\[ |P_{er}| < 0.01 ? \]*  
*Yes*  
*Use the Values Determined*

*No*  
*R_2 = R_{mag}*

*Take the Resultant Load as the Sum of the Applied Load and the Reaction Calculated*
of the frictional moment over a wide range of the lower values of the angle \( \alpha_0 \), defining the size of the contact area, when the angle \( \theta \) is \( 35^\circ \). Thus the magnitude of the frictional moment, measured when the angle \( \theta \) was \( 35^\circ \), could be used to determine the value of the coefficient of friction as the value of the radial clearance only needed to be known very approximately. Then knowing the value of the coefficient of friction, the value of the radial clearance could be varied until the best fit between the experimental and theoretical values of the frictional moment was obtained over the range of the angle \( \theta \). The results obtained in this way are described in the following sections.

6.4 The Results

The results obtained are shown in Figs. 6.6 to 6.12. They are given in the form of the experimental measurements of the differences between the frictional moments produced on a socket during forward and reverse rotations of the ball plotted, with the corresponding theoretical values given by the friction model, against values of the angle \( \theta \). For each set of results there are four graphs showing the plots of first the total magnitudes of these differences and then the components in the three perpendicular directions which were actually measured by the apparatus. The directions of the components were given using an
i,j,k co-ordinate system corresponding to that used in determining the theoretical results. Thus the k direction is the vertical direction in which the applied load acted and the axis of rotation always remained in the i,k plane.

The first three sets of graphs, Figs. 6.6 to 6.8, show the results obtained with a tool steel ball rotating in a mild steel cup and lubricated with a drop of the Tellus R5 oil. For each set of measurements the coefficient of friction increased giving results which corresponded to theoretical values calculated for coefficients of friction of 0.065, 0.075 and 0.095. The radial clearance was determined to be approximately 1.0x10^{-6}m.

When the direction of the rotation of the ball was reversed the components of the frictional moment in the i and k directions acted in the opposite directions to what they were, which was as expected, while the component in the j direction continued to act in the same direction. Thus, as the components of the frictional moment are approximately equal in magnitude for either direction of rotation when the coefficient of friction is low, the difference between the components in the j direction was close to zero, as shown by the results. It also follows that the variations with the angle 0 of the differences in the components of the frictional moment in the i and k directions corresponded with the variations of the components of the actual frictional moment as described in
Figure 6.6a  The Total Magnitude of the Difference Between the Frictional Moments Produced During Forward and Reverse Rotations Plotted Against the Angle $\theta$ ($\mu = 0.065$)
Figure 6.6b

The component of the difference acting in the direction of the unit vector \( i \) plotted against the angle \( \theta \) (\( \mu = 0.065 \))
Figure 6.6c  The Component of the Difference Acting in the Direction of the Unit Vector $\hat{j}$ Plotted against the Angle $\theta$ ($\mu = 0.065$)
Figure 6.7a

The total magnitude of the difference between the frictional moments produced during forward and reverse rotations plotted against the angle \( \theta \) (\( \mu = 0.075 \))
Figure 6.7b The Component of the Difference Acting in the Direction of the Unit Vector \( \mathbf{i} \) Plotted against the Angle \( \theta \) (\( \mu = 0.075 \))
Figure 6.7c. The component of the difference acting in the direction of the unit vector j plotted against the angle \( \theta \) (\( \mu = 0.075 \))
Figure 6.7d. The component of the difference acting in the direction of the unit vector $\mathbf{b}$ plotted against the angle $\theta$ ($\mu = 0.075$)
Figure 6.8a  The total magnitude of the difference between the frictional moments produced during forward and reverse rotations plotted against the angle $\theta$ ($\mu = 0.095$)
Figure 6.8b  
The Component of the Difference Acting in the Direction of the Unit Vector \( \mathbf{i} \) Plotted against the Angle \( \theta \) (\( \mu = 0.095 \))
Figure 6.8c: The component of the difference acting in the direction of the unit vector $\mathbf{j}$ plotted against the angle $\theta$ ($\mu = 0.095$)
Figure 6.8d  The Component of the Difference Acting in the Direction of the Unit Vector $b$ Plotted Against the Angle $\theta$ ($\mu = 0.095$)
section 4.5. This is shown in the results by the component in the \( k \) direction increasing and that in the \( i \) direction decreasing as the angle \( \theta \) approaches zero.

The graphs of the results show that there was a tendency for the values of the frictional moment measured during the experiments to be greater than the values predicted by the theory when the value of the angle \( \theta \) was small. This tendency varied between each set of results and corresponded to the extent of surface damage detected on the ball and socket. As stated before, particles of the softer material of the socket tended to be 'picked up' by the ball as it rotated and would then scour grooves in the socket resulting in a higher coefficient of friction being observed, particularly at lower values of the angle \( \theta \) where this damage was more likely to occur. In several of the experimental runs the damage distorted the results so much that they were useless but where the damage was less severe the effect on the results was that shown in the graphs.

Figs. 6.9 and 6.10 show some results obtained with the same ball and socket used to produce the previous results after the socket had been repolished to remove some of the damage done to its surface. The radial clearance was thus still approximately \( 1.0 \times 10^{-6} \) m. Small differences in the radial clearance would not have an observable effect on the results because of the small change in the frictional
Figure 6.9a  The total magnitude of the difference between the frictional moments produced during forward and reverse rotations plotted against the angle $\theta$ ($\mu = 0.0775$)
Figure 6.9b The component of the difference acting in the direction of the unit vector $\mathbf{i}$ plotted against the angle $\theta$ ($\mu = 0.0775$)

Experimental Results

Calculated Values
Figure 6.9c: The component of the difference acting in the direction of the unit vector $\hat{j}$ plotted against the angle $\theta$ ($\mu = 0.0775$).
Figure 6.9d  The Component of the Difference Acting in the Direction of the Unit Vector $\mathbf{b}$ Plotted Against the Angle $\theta$ ($\mu = 0.0775$)
Figure 6.10a

The total magnitude of the difference between the frictional moments produced during forward and reverse rotations plotted against the angle $\theta$ ($\mu = 0.088$)
Figure 6.10b  
**The Component of the Difference Acting in the Direction of the Unit Vector \( \vec{i} \) Plotted against the Angle \( \theta \) (\( \mu = 0.088 \))**
Figure 6.10c  The Component of the Difference Acting in the Direction of the Unit Vector $j$ Plotted against the Angle $\theta$ ($\mu = 0.088$)
Figure 6.10a: The component of the difference acting in the direction of the unit vector $b$ plotted against the angle $\theta$ ($\mu = 0.088$).

- Experimental results
- Calculated values
moment produced. A small drop of Tellus R5 oil was used as lubricant again and results were obtained with values corresponding to those calculated for coefficients of friction of 0.0775 and 0.088. The main difference between these results and those obtained previously was that the shaft rotating the ball had been positioned on the other side of the vertical, which resulted in the directions of the components in the $i$ direction being reversed, with the corresponding effect on the direction of the differences between these components, as shown by the graphs.

The final two sets of graphs, given in Figs.6.11 and 6.12, show the results obtained using a different tool steel ball and mild steel socket. In these cases only the slightest smear of the oil was used as a lubricant and as a result the surfaces of both the ball and socket were damaged. However, the damage was evenly spread over the surfaces in contact and not confined to a few grooves, and this meant that some useful measurements of the frictional moment were obtained at higher coefficients of friction. The two sets of experimental results corresponded to calculated values obtained with coefficients of friction of 0.8 and 1.08 and with a radial clearance of $1.0 \times 10^{-6}$ m.

The graphs show that when the coefficient of friction was high the difference in the components in the $j$ direction was not so close to zero as it was before. They also show that the experimental results obtained were more erratic,
Figure 6.11a The total magnitude of the difference between the frictional moments produced during forward and reverse rotations plotted against the angle $\theta$ ($\mu = 0.8$)
Figure 6.11b  The Component of the Difference Acting in the Direction of the Unit Vector $\mathbf{i}$ Plotted against the Angle $\theta$ ($\mu = 0.8$)
Figure 6.11c  

The Component of the Difference Acting in the Direction of the Unit Vector \( \hat{j} \) Plotted against the Angle \( \theta \) (\( \mu = 0.8 \))
Figure 6.11d  The Component of the Difference Acting in the Direction of the Unit Vector $b$ Plotted against the Angle $\theta$ ($\mu = 0.08$)
Figure 6.12a  The total magnitude of the difference between the frictional moments produced during forward and reverse rotations plotted against the angle $\Theta$ ($\mu = 1.08$)
Figure 6.12b The Component of the Difference Acting in the Direction of the Unit Vector i Plotted Against the Angle θ (μ = 1.08)
Figure 6.12c: The component of the difference acting in the direction of the unit vector $j$ plotted against the angle $\theta$ ($\mu = 1.08$)

Experimental results

Calculated values
Figure 6.12d  The component of the difference acting in the direction of the unit vector $\mathbf{b}$ plotted against the angle $\theta$ (µ = 1.08)

Experimental results

Calculated values
for the higher values of the coefficient of friction, in their variation with the angle $\theta$. This was a result of the fact that as the angle $\theta$ changed the portion of the ball in contact with the socket varied and as the surface damage would not be completely even over the ball, the coefficient of friction would vary from position to position.
CHAPTER SEVEN

Conclusions

The aim of this project has been to develop a mathematical model of the effects of friction in a spherical joint, which could then be incorporated into a three-dimensional mechanism simulation package such as the AMP3D-ADAMS computer program.

This aim has been largely achieved, in that a mathematical model of the effects of friction in a spherical joint has been developed and experimental results have been obtained which uphold the validity of this model. However, the model does have a number of limitations, of which the main one is its dependence on the approximate integration of double integrals and the use of iteration to determine the required results. Both of these factors result in a large amount of computing time and thus expense being necessary to obtain the desired results, which make the model, in its present form, unsuitable for incorporation into a mechanism simulation program.

This problem can be overcome by developing some simple approximations to the complex expressions contained within the model, using the method of surface fitting, which would drastically reduce the computing time required.
However, except for certain limited situations described in the thesis, this has not yet been achieved so further work in this area is needed if the model is to be incorporated into a program such as AMP3D-ADAMS.

A second limitation on the model is that it has only been developed for the general case of a ball and cup type of spherical joint, and that except under certain specific conditions described in the thesis, it has not been extended to cover the case of the straddle-type of spherical joint which is a form in which spherical joints are often found in practice.

However, despite these limitations which can be overcome by further research, a model has been developed which can calculate the frictional moment produced in a spherical joint and it can be seen how this moment varies with the conditions at the joint; particularly with the angle between the axis of rotation and the direction of the applied load, the coefficient of friction and the extent of the contact area between the ball and the socket. Also, despite considerable difficulties, experimental results have been obtained which support the validity of the model at high and low values of the coefficient of friction over a range of values of the angle between the axis of rotation and the direction of the applied load.
APPENDIX 1

To show that the expression:

\[ \sin^2 \alpha \sin^2 \beta + (\cos \alpha \sin \gamma - \sin \alpha \cos \beta \cos \gamma)^2 \]

is equivalent to:

\[ 1 - (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2 \]

This is done by expanding the first expression as follows:

\[
\begin{align*}
\sin^2 \alpha \sin^2 \beta + (\cos \alpha \sin \gamma - \sin \alpha \cos \beta \cos \gamma)^2 &= \\
= \sin^2 \alpha \sin^2 \beta + \cos^2 \alpha \sin^2 \gamma - \sin^2 \alpha \cos^2 \beta \sin \gamma - 2 \sin \alpha \cos \alpha \cos \beta \sin \gamma \\
= \sin^2 \alpha \sin^2 \beta + \cos^2 \alpha - \cos^2 \alpha \cos^2 \gamma + \sin^2 \alpha \cos^2 \beta \\
-\sin^2 \alpha \cos^2 \beta \sin^2 \gamma - 2 \sin \alpha \cos \alpha \cos \beta \sin \gamma \\
= \sin^2 \alpha \sin^2 \beta + \cos^2 \alpha - \cos^2 \alpha \cos^2 \gamma + \sin^2 \alpha - \sin^2 \alpha \sin^2 \beta \\
-\sin^2 \alpha \cos^2 \beta \sin^2 \gamma - 2 \sin \alpha \cos \alpha \cos \beta \sin \gamma \\
= 1 - (\cos \alpha \cos \gamma + \sin \alpha \cos \beta \sin \gamma)^2
\end{align*}
\]
The integral given in equation 3.17
\[
I = \int_{0}^{\alpha_0} 2\sin \alpha \cos \alpha \cos \left(\frac{\alpha \pi}{2 \alpha_0}\right) d\alpha
\]
can be evaluated as follows:

\[
I = \int_{0}^{\alpha_0} 2\sin \alpha \cos \alpha \cos \left(\frac{\alpha \pi}{2 \alpha_0}\right) d\alpha
\]
\[
= \int_{0}^{\alpha_0} \sin 2\alpha \cos \left(\frac{\alpha \pi}{2 \alpha_0}\right) d\alpha
\]
\[
= \frac{1}{2} \int_{0}^{\alpha_0} \left( \sin \left[ \frac{4\alpha_0 \pi - \pi}{2 \alpha_0} \right] + \sin \left[ \frac{4\alpha_0 \pi + \pi}{2 \alpha_0} \right] \right) d\alpha
\]
\[
= \frac{\alpha_0}{4\alpha_0 - \pi} \cos \left( \frac{4\alpha_0 \pi - \pi}{2 \alpha_0} \right) + \frac{\alpha_0}{4\alpha_0 + \pi} \cos \left( \frac{4\alpha_0 \pi + \pi}{2 \alpha_0} \right)
\]
\[
= \frac{\alpha_0}{4\alpha_0 - \pi} (1 - \cos[2\alpha_0 - \frac{\pi}{2}]) + \frac{\alpha_0}{4\alpha_0 + \pi} (1 - \cos[2\alpha_0 + \frac{\pi}{2}])
\]
\[
= \frac{\alpha_0}{4\alpha_0 - \pi} (1 - \sin 2\alpha_0) + \frac{\alpha_0}{4\alpha_0 + \pi} (1 + \sin 2\alpha_0)
\]
\[
= \frac{\alpha_0 \left[ (1 - \sin 2\alpha_0)(4\alpha_0 + \pi) + (1 + \sin 2\alpha_0)(4\alpha_0 - \pi) \right]}{16\alpha_0^2 - \pi^2}
\]
\[
\alpha_0 \left( \frac{8\alpha_0 - 2\pi \sin 2\alpha_0}{16\alpha_0^2 - \pi^2} \right)
\]

However, it can be seen that when \( \alpha_0 = \frac{\pi}{4} \) this expression cannot be evaluated because both the denominator and the numerator are then zero and so the expression is indeterminate. This problem can be overcome by taking the value of \( \alpha_0 \) as:

\[
\alpha_0 = \frac{\pi}{4} - \varepsilon
\]

where \( \varepsilon \) is a very small number.
The expression for $I$ then becomes:

$$I = \frac{8(\frac{\pi}{4} - \epsilon)^2 - 2\pi(\frac{\pi}{4} - \epsilon)\sin(\frac{\pi}{2} - 2\epsilon)}{16(\frac{\pi}{4} - \epsilon)^2 - \pi^2}$$

$$= \frac{\frac{\pi^2}{4} - 4\pi \epsilon + 8\epsilon^2 - (\frac{\pi^2}{2} - 2\pi \epsilon)\sin\frac{\pi}{2} \cos 2\epsilon}{-8\pi \epsilon + 16\epsilon^2}$$

For very small numbers $\cos 2\epsilon = 1$. Therefore:

$$I = \frac{-2\pi \epsilon + 8\epsilon^2}{-8\pi \epsilon + 16\epsilon^2}$$

$$= \frac{-\pi + 4\epsilon}{-4\pi + 8\epsilon}$$

Thus as $\epsilon \to 0$, $I \to 1/4$. 
APPENDIX 3

A Computer Program to Evaluate the Effects of Friction in a Spherical Joint

This appendix contains a listing, flow diagram and variable list for the computer program written to evaluate the effects of friction in a spherical joint using the friction model described in this thesis. In the form shown the program calculates the magnitude and direction cosines of the frictional force and moments produced over a range of values of the angle θ, the angle between the axis of rotation and the direction of the applied load, and of the coefficient of friction. The program is described in the form of brief summaries of the main program and its subroutines as follows:

The Main Program THETA - sets the values of the angle θ and the coefficient of friction for which the effects of friction are to be evaluated and calls the subroutines that perform this evaluation. It also contains the iterative routine that determines the value of the angle γ, described in section 4.4, and performs the final stages in the calculation of the magnitudes of the frictional moments as well as determining the magnitude and direction cosines of the normal reaction.
The Subroutine IN gives the data required by the friction model which includes the elastic properties of the joint materials, the dimensions of the joint and the load on the joint. It also prints a summary of this data for the user.

The Subroutine ALPHA contains the iterative routine used to determine the size of the angle $\alpha$, which is described in section 5.1. It also determines the factor $f_1$ from which the magnitude of the frictional force can be obtained.

The Subroutine AYE evaluates the integral $I$ for a particular value of the angle $\alpha$, using the expressions given in Appendix 2.

The Subroutine LOOPS evaluates the double integrals contained within the expressions for the factors $f_1$, $f_2$ and $f_3$ by the method of approximate integration described in section 3.6.

The Subroutines FORCE, MOSLIP, and MOROT evaluate the expressions within the double integrals at each step of the approximate integration.

The Subroutine DIRCOS determines the direction cosines of the frictional forces and moments using the expressions shown in section 4.4.
The Subroutine COORD - determines the direction cosines of the total frictional moment relative to the general coordinate system as described in section 5.3.

The Subroutine REACT - determines the reaction at the joint resulting from the action of the frictional moment on the mechanism containing the joint as described in section 5.2.

The Subroutine OUT - prints the calculated results. It gives the magnitudes and direction cosines of the applied load, the frictional force, the moment of slip friction, the moment of rotational friction, the total frictional moment, the reaction resulting from the moment, and the normal reaction.
LIST OF VARIABLES

A, ALPI, B, B1, B2 : Intermediate variables used in determining the expressions in the subroutines FORCE, MOSLIP, MOROT and AYE

AL, ALO : The angles $\alpha$ and $\alpha_0$

BE : The angle $\beta$

BEO : The maximum value of the angle $\beta$

C1, C2, EX1 : Intermediate variables used in determining the value of the angle $\alpha_0$

C, D(J) : The values of the expressions within the subroutines FORCE, MOSLIP and MOROT

DALO(K1), DGA(K1), DTH : The values of the angles $\alpha_0$, $\gamma$ and $\theta$ measured in degrees

DI, PHI2 : Variables used in the iteration to determine the value of the angle $\alpha_0$

DLL, DLM, DLN : Intermediate variables used in determining the direction cosines of the total frictional moment relative to the general co-ordinate system

DML, DMM, DMN : Intermediate variables used in determining the direction cosines of the total frictional moment relative to the general co-ordinate system

DN(0,K1), PL : Direction cosines of the applied load - $l_p, m_p, n_p$

DN(0,K1), PM : Direction cosines of the frictional force - $l_F, m_F, n_F$

DN(0,K1), PN : Direction cosines of the frictional force - $l_F, m_F, n_F$

DN(2,K1) : Direction cosines of the moment of slip friction - $l_M, m_M, n_M$

DN(3,K1) : Direction cosines of the moment of rotational friction - $l_M, m_M, n_M$

DN(4,K1) : Direction cosines of the total frictional moment - $l_M, m_M, n_M$

DN(5,K1) : Direction cosines of the total frictional moment relative to the general co-ordinate system
DL(6,K1); RL : Direction cosines of the reaction resulting from the frictional moment
DM(6,K1); RM
DN(6,K1); RN

DL(7,K1) ; Direction cosines of the normal reaction -
DM(7,K1)
DN(7,K1)

DS : Intermediate variable used in determining the direction cosines of the frictional moments

E1,E2 : Young's moduli E₁, E₂ for the socket and ball

E(I) : The value of the 'inner integral' of the double integral

EYE : The value of the integral I

FR : The coefficient of friction

FS1,FS2,FS3 : The factors f₁, f₂, f₃

FX : The value of the double integral determined by the subroutine LOOPS

GA : The angle ϑ

G1,G2,XT : Variables used in the determination of the angle ϑ by iteration

Q1,Q2

HL(1),HL(2) : The distances to the points at which the rotation of the socket is resisted

I,J,K,L : Counters used in the approximate integration in subroutine LOOPS

K1,K2,K3,K4
K5,K6,K7 : Counters used in the program

N,M : The number of steps used in the approximate integration

P,Q : The magnitude of the step intervals used in the approximate integration

PI : The constant π

P,PL,PM,PN : See XM(0,K1), DL(0,K1), DM(0,K1), DN(0,K1)

PHIF, SMRF : The ratios of the factors f₁, f₂, f₃ to the coefficient of friction

RMRF

R : The radius of the joint
RC : The radial clearance in the joint
RESN : The resultant load on the joint
RI,RJ,RK : The perpendicular components of the reaction resulting from the action of the frictional moment
RMAG,RL,RM,RN : See XM(6,K1), DL(6,K1), DM(6,K1), DN(6,K1)
SGN : The sign of cos θ
STR(K6) : A character string
TH : The angle θ
U,V,W,H : Variables used in determining the 'outer integral' of the double integral evaluated numerically
V1,V2 : The Poisson's ratios v1, v2 of the socket and ball
WL,WM,WN : Direction cosines of the angular velocity - l_w, m_w, n_w
X,Y,Z,G : Variables used in determining the 'inner integral' of the double integral evaluated numerically
XM(0,K1); P : Magnitude of the applied load on the joint
XM(1,K1); PHN : The ratio of the magnitude of the frictional force to that of the applied load (F/P)
XM(2,K1) : The ratio of the magnitude of the moment of slip friction to the product of the magnitude of the applied load and the radius of the joint (M_s/rP)
XM(3,K1) : The ratio of the magnitude of the moment of rotational friction ...(M_r/rP)
XM(4,K1) : The ratio of the magnitude of the total frictional moment ...(M/rP)
XM(5,K1) : The magnitude of the total frictional moment
XM(6,K1);RMAG : The magnitude of the reaction resulting from the frictional moment
XM(7,K1) : The ratio of the magnitude of the normal reaction to that of the applied load (N/P)
PROGRAM THETA
C DETERMINES THE FRICTIONAL FORCE AND MOMENT IN A SPHERICAL JOINT
C
COMMON/ONE/PI,N,M,ALG,EYE,K3,GA,FX
COMMON/TWO/9,EX1,FR,P,PL,PM,PN
COMMON/THREE/RMAG,RL,RM,RN
COMMON/FOUR/XM0:7,50),TH,PHN,FS1,0GA(50),DALD(50)
COMMON/FIVE/K1,WL,WM,WN,DL(0:7,50),DM(0:7,50),DN(0:7,50)
C CALL IN
DO 6 K2=1,5
PRINT *
PRINT 50,FR
50 FORMAT(1H ,5X,",MU = ",F4.2)
C DTH= 0.0
C RESULTS OBTAINED OVER A RANGE OF THE ANGLE THETA
C DO 3 K1=1,46
TH=DTH*PI/180.0
C AXIS OF ROTATION AND INITIAL APPLIED LOAD DEFINED
WL=-SIN(TH)
WM=0.0
WN=COS(TH)
XM(0,K1)=P
DL(0,K1)=PL
DM(0,K1)=PM
DN(0,K1)=PN
C ALD=(XM(0,K1)*EX1)^2 (1.0/3.0)
IF(ALD.GT.PI/2.0)ALD=PI/2.0
C CALL AYE
C ANGLE GAMMA BETWEEN AXIS OF ROTATION AND NORMAL REACTION DETERMINED
K3=1
IF(SIN(TH).EQ.0.0.OR.COS(TH).EQ.0.0)THEN
GA=TH
CALL ALPHA
ELSE
Q1=2.0*TH/PI
GA=INT(Q1)*PI/2.0+PI/4.0
Q2=1.0
IF(INT(Q1).EQ.1.0R.INT(Q1).EQ.3)Q2=-1.0
XT=PI/4.0
C DO 9 K4=1,20
XT=XT/2.0
CALL ALPHA
G1=(FS1*COS(TH))**2+2*G1**2
G2=G1-SIN(TH)**2
IF(ABS(G2).LT.0.00001)THEN
GOTO 13
ELSE IF(G2.GT.0.0)THEN
GA=GA-Q2*XT
C
ELSE
    GA = GA + Q2*XT
ENDIF
CONTINUE

ENDIF
GGA(K1) = GA = 180.0/PI
DALO(K1) = ALO = 180.0/PI

C
C RATIO OF MAGNITUDES OF FRICTIONAL FORCE AND APPLIED LOAD
C
XM(1, K1) = PHN

C
C RATIO OF MAGNITUDES OF MOMENT OF SLIP FRICTION AND APPLIED LOAD
C
K3 = 2
CALL LOOPS
SMRF = FX/(PI*EYE)
FS2 = ABS(SMRF)*FR
XM(2, K1) = FS2/SQRT(1.0 + FS1**2)

C
C RATIO OF MAGNITUDES OF MOMENT OF ROTATIONAL FRICTION AND APPLIED LOAD
C
K3 = 3
CALL LOOPS
RMRF = FX/(PI*EYE)
FS3 = ABS(RMRF)*FR
XM(3, K1) = FS3/SQRT(1.0 + FS1**2)

C
C RATIO OF MAGNITUDES OF TOTAL FRICTIONAL MOMENT AND APPLIED LOAD
C
XM(4, K1) = SQRT(XM(2, K1)**2 + XM(3, K1)**2)

C
C CALL DIRCOS

C
C MAGNITUDE OF TOTAL FRICTIONAL MOMENT
C
XM(5, K1) = R = XM(0, K1) * XM(4, K1)

C
C CALL COORD
C CALL REACT

XM(6, K1) = RMAG
DL(6, K1) = RL
DM(6, K1) = RM
DN(6, K1) = RN

C
C RATIO OF MAGNITUDES OF NORMAL REACTION AND APPLIED LOAD
C
XM(7, K1) = SQRT(1.0 - XM(1, K1)**2)

C
C DIRECTION COSINES OF NORMAL REACTION
C
DL(7, K1) = -(DL(0, K1) + XM(1, K1)**2/IXM(7, K1)) / XM(7, K1)
DM(7, K1) = -(DM(0, K1) + XM(1, K1)**2/IXM(7, K1)) / XM(7, K1)
DN(7, K1) = -(DN(0, K1) + XM(1, K1)**2/IXM(7, K1)) / XM(7, K1)
DTH = DTH + 2.0

3 CONTINUE
C
C CALL CUT
C      FR = FR + 0.2
6      CONTINUE
C      STOP
END
PROGRAM THETA

CALL IN

K2 = 1.5

PRINT VALUE OF Fr

DTh = 0.0

K1 = 1.46

TH = DTh × Pi/180.0

DETERMINE DIRECTION COSINES OF AXIS OF ROTATION AND THE MAGNITUDE AND DIRECTION OF THE LOAD ON THE JOINT

Alo = √Xm(0,K1) × Ex1

Yes

No

CALL AYE

K3 = 1

X Alo > Pi/2.0

Yes

No

Alo = Pi/2.0

CALL AYE

Q1 = 2.0 × TH/Pi

Q2 = 1.0

GA = Aint(Q1) × Pi/2.0 + Pi/4.0

CALL ALPHA

1 2 3 4
INT(Q1) = 1 OR 3?

X_T = PI/4.0

K4 = 1.2.0

X_T = X_T/2.0

CALL ALPHA

G1 = (FSL x COs(TH))^2 + SIn(GA)^2
G2 = G1 - SIn^2(TH)

|G2| < 0.00001?

G2 > 0.0?

G_A = G_A - Q2 x X_T

G_A = G_A + Q2 x X_T

CONTINUE

D_GA(K1) = G_A x 180/PI
D_ALO(K1) = ALO x 180/PI

X_M(1, K1) = PHN
K3 = 2

CALL LOOPS

SMRF = Fx/(Pi x Eye)
Fs2 = Abs(SMRF) x Fr
Xm(2, K1) = Fs2/1.0 + Fs2^2

K3 = 3

CALL LOOPS

RMRF = Fx/(Pi x Eye)
Fs3 = Abs(RMRF) x Fr
Xm(3, K1) = Fs3/1.0 + Fs3^2

Xm(4, K1) = \sqrt{Xm(2, K1)^2 + Xm(3, K1)^2}

CALL DIRCOS

Xm(5, K1) = Rx Xxm(0, K1) x Xm(4, K1)

CALL COORD

CALL REACT

Xm(6, K1) = RMAG
DL (6, K1) = RL
DM (6, K1) = RM
DN (6, K1) = RN

Xm(7, K1) = \sqrt{1.0 - Xm(1, K1)^2}

Determine the direction cosines of the normal reaction

Dth = Dth + 2.0
8 -> 9
9 -> 10
10

- Continue
- Call Out
- \( Fr = Fr + 2.0 \)
- Continue
- Stop
- END
SUBROUTINE IN

COMMON/ONE/PI,N,M,A00,EXE,K3,GAL,FX
COMMON/THREE/R,E1,F1,P1,P2,P3,P4
COMMON/THREE/R,MAG,RL,AM,RN

TO INPUT REQUIRED DATA

PI=3.1415927
N=16
M=16

ELASTIC PROPERTIES OF JOINT MATERIALS

V1=0.3
V2=0.3
E1=2.05E11
E2=2.05E11
C1=(1.0-V1**2)/E1
C2=(1.0-V2**2)/E2

DIMENSIONS OF AND CONDITIONS AT JOINT

R=0.0127
RC=1.0E-6
EX=3.0*(C1+C2)/(4.0*R*RC)

FR=0.2
P=190.0
PL=0.0
PM=0.0
PN=-1.0

HL(1)=0.0427
HL(2)=0.062

PRINT REQUIRED DATA

PRINT *
PRINT 350,P,PL,PM,PN
PRINT *
PRINT 340,V1,V2
PRINT *
PRINT 330,E1,E2
PRINT *
PRINT 320,R,RC
PRINT *
PRINT 300,HL(1),HL(2)

350 FORMAT(1H,7X,'P = '+'F7.2,C'',F7.4,I +''=7.4, J +''
+ F7.4,K'))
340 FORMAT(1H,5X,'V1 = '+'F4.2,10X,'V2 = '+'F4.2)
330 FORMAT(1H,5X,'E1 = '+'E9.3E2,5X,'E2 = '+'E9.3E2)
320 FORMAT(1H,5X,'R = '+'F5.4,5X,'R2-R1 = '+'E7.2E1)
300 FORMAT(1H,5X,'L1 = '+'F6.4,8X,'L2 = '+'F6.4)

RETURN
END
SUBROUTINE IN

SET VALUES OF
Pi, N, M
V1, V2, E1, E2

C1 = (1.0 - V1^2)/E1
C2 = (1.0 - V2^2)/E2

SET VALUES OF
R, Rc

Ex1 = 3.0 x (C1 + C2)/(4.0 x RxRc)

SET VALUES OF
Fr, P, Pl, Pm, Pn
HL(1), HL(2)

PRINT VALUES OF
P, Pl, Pm, Pn
V1, V2, E1, E2
HL(1), HL(2)

RETURN
SUBROUTINE AYE

TO EVALUATE THE INTEGRAL I (EYE)

COMMON /ONE/PI,N,M,ALO,EYE,K3,GX

IF (ABS(ALO-PI/4.0).LT.0.0001) THEN
  EYE=0.25
ELSE
  A=8.0*ALO**2-2.0*PI*ALO*SIN(2.0*ALO)
  B=16.0*ALO**2-PI**2
  EYE=A/B
ENDIF

RETURN
END
SUBROUTINE AYE

Yes

ALO = \frac{\pi}{4}?

No

A = 8.0 \times ALO^2 - 2.0 \times \pi \times ALO \times \sin (2.0 \times ALO)

B = 16.0 \times ALO^2 - \pi^2

EYE = A/B

EYE = 0.25

RETURN
SUBROUTINE ALPHA

DETERMINES ANGLE ALPHA-0 DEFINING SIZE OF CONTACT AREA

COMMON/ONE/PI,N,M,ALO,EYE,K3,G4,FX
COMMON/TWO/R,EX1,PR,P,PL,PM,PN
COMMON/FOUR/XM(0:7,50),TH,PHN,FS1,DA(50),DALO(50)
COMMON/FIVE/K1,WL,WN,DL(0:7,50),DM(0:7,50),DN(0:7,50)

DO 30 K5=1,20

CALL LOOPS

PHIF=FX/(PI%EYE)
FS1=ABS(PHIF)#FR
PHN=FS1/SQRT(1.0+FS1**2)
RESN=X*K1**SQRT(1.0-PHN**2)
ALO=(RESN*EX1)**(1.0/3.0)
IF(ALO.GT.PI/2.0)ALO=PI/2.0

CALL AYE

IF(KS.GT.1.0)THEN
DI=PHIF-PHIF
IF(ABS(DI).LT.0.001)GOTO 19
ENCIF
PHIF2=PHIF
19 CONTINUE
30 CONTINUE

RETURN

END
**Subroutine Alpha**

1. $K5 = 1, 20$
2. **Call Loops**
3. $\phi f = \frac{F}{(\pi \times \text{EYE})}$
4. $F_s1 = \text{abs}(\phi f) \times F_R$
5. $P_{HN} = \frac{F_{s1}}{\sqrt{1.0 + F_{s1}^2}}$
6. $\text{Resn} = X_{M} (0, K1) \times \sqrt{1.0 - P_{HN}^2}$
7. $A_{LO} = \frac{3}{\text{Resn}} \times \text{Ex1}$
8. **If** $A_{LO} > \frac{\pi}{20}$ **Then**
   - $A_{LO} = \frac{\pi}{20}$
9. **Else**
10. **Call Aye**
11. $K5 > 1$
12. **If** $K5 > 1$ **Then**
13. **Else**
14. $D_i = \phi f - \phi_{i2}$
15. **If** $|D_i| < 0.0001$ **Then**
16. **Else**
17. $\phi_{i2} = \phi f$
18. **Continue**
19. **Return**
SUBROUTINE LOOPS

TO EVALUATE A DOUBLE INTEGRAL BY SIMPSON'S RULE

COMMON/ONE/PI,N,M,ALO,EYE,K3,GA,FX
COMMON/SIX/AL,BE,C
DIMENSION D(0:20),E(0:20)

LIMITS AND INTERVALS SET FOR NUMERICAL INTEGRATION

BE0=2.0*PI
P=BE0/N
Q=ALO/M
BE=0.0

BETA LOOP - TO EVALUATE OUTER INTEGRAL

DO 60 I=0,N
AL=0.0

ALPHA LOOP - TO EVALUATE INNER INTEGRAL

DO 63 J=0,M
IF(K3.EQ.1)THEN
  CALL FORCE
ELSEIF(K3.EQ.2)THEN
  CALL MOSLIP
ELSE
  CALL MOROT
ENDIF
D(J)=C
AL=AL+Q
63 CONTINUE

X=D(0)+D(M)
Y=0.0
Z=0.0

TO OBTAIN THE SUM OF THE VALUES FOR EVEN AND ODD INTERVALS

DO 66 K=1,M-1
G=K/2.0-INT(K/2.0)
IF(G.GT.0.1)THEN
  Y=Y+D(K)
ELSE
  Z=Z+D(K)
ENDIF
66 CONTINUE

THE VALUE OF THE INNER INTEGRAL BY SIMPSON'S RULE

E(I)=(G/3.0)*(X+4.0*Y+2.0*Z)

BE=BE+P
60 CONTINUE

U=E(0)+E(N)
V=0.0
W=0.0
THE SUM OF THE VALUES AT THE EVEN AND ODD INTERVALS

DO 69 L=1,N-1
H=L/2.0-INT(L/2.0)
IF(H.GT.0.1) THEN
  V=V+E(L)
ELSE
  W=W+E(L)
ENDIF
69 CONTINUE

THE VALUE OF THE DOUBLE INTEGRAL BY SIMPSON'S RULE

FX=(P/3.0)*(U+4.0*V+2.0*W)

RETURN
END
SUBROUTINE LOOPS

BOE = 2.0 * Pi
P = BOE / N
Q = AL0 / M
BE = 0.0

I = 0, N

AL0 = 0.0

J = 0, M

K3 = 1 ?

CALL FORCE

D(J) = C
AL = AL + Q

CONTINUE

X = D(0) + D(M)
Y = 0.0
Z = 0.0

K = 1, M - 1

G = K / 2.0 - INT(K / 2.0)

G > 0.1 ?

Y = Y + D(K)
Z = Z + D(K)

CALL MOSLIP

CALL MOROT

CALL FORCE
\begin{align*}
\text{E(I)} & = \left( \frac{Q}{3.0} \right) \times (X + 4.0 \times Y + 2.0 \times Z) \\
\text{BE} & = \text{BE} + P \\
\text{CONTINUE} \\
\text{CONTINUE} \\
U & = E(O) + E(N) \\
V & = 0.0 \\
W & = 0.0 \\
L & = 1, N-1 \\
H & = \frac{L}{2.0} - \text{INT}(\frac{L}{2.0}) \\
\text{H} & > 0.1 \quad \text{YES} \\
\text{H} & > 0.1 \quad \text{NO} \\
V & = V + E(L) \\
W & = W + E(L) \\
\text{CONTINUE} \\
F_X & = \left( \frac{P}{3.0} \right) \times (U + 4.0 \times V + 2.0 \times W) \\
\text{RETURN}
\end{align*}
SUBROUTINE FORCE

EVALUATES THE EXPRESSION IN THE FRICTION FORCE INTEGRAL

COMMON/ONE/PI,N,M,ALG,EYE,K3,GA,FX
COMMON/SIX/AL,EE,C

ALPI=AL*PI/(2.0*ALG)
A=COS(AL)*COS(GA)+SIN(AL)*SIN(GA)*COS(BE)
B1=SIN(AL)*COS(AL)*COS(GA)
B2=COS(AL)*SIN(GA)*COS(BE)
B=SIN(GA)-B2-B1
IF(ABS(A).GE.1.0)THEN
C=0.0
ELSE
C=SIN(AL)*COS(ALPI)*6/SQRT(1.0-A**2)
ENDIF

RETURN
END

SUBROUTINE MOSLIP

EVALUATES THE EXPRESSION IN THE MOMENT OF SLIP FRICTION INTEGRAL

COMMON/ONE/PI,N,M,ALG,EYE,K3,GA,FX
COMMON/SIX/AL,EE,C

ALPI=AL*PI/(2.0*ALG)
A=COS(AL)*COS(GA)+SIN(AL)*SIN(GA)*COS(BE)
B1=SIN(AL)*COS(AL)*COS(GA)
B2=COS(AL)*SIN(GA)*COS(BE)
B=SIN(GA)-B2-B1
IF(ABS(A).GE.1.0)THEN
C=0.0
ELSE
C=SIN(AL)*COS(ALPI)*6/SQRT(1.0-A**2)
ENDIF

RETURN
END

SUBROUTINE MOROT

EVALUATES THE EXPRESSION IN THE MOMENT OF ROTATIONAL FRICTION INTEGRAL

COMMON/ONE/PI,N,M,ALG,EYE,K3,GA,FX
COMMON/SIX/AL,EE,C

ALPI=AL*PI/(2.0*ALG)
A=COS(AL)*COS(GA)+SIN(AL)*SIN(GA)*COS(BE)
B1=SIN(AL)*COS(AL)*COS(GA)
B2=COS(AL)*SIN(GA)*COS(BE)
B=SIN(GA)-B2-B1
IF(ABS(A).GE.1.0)THEN
C=0.0
ELSE
C=SIN(AL)*COS(ALPI)*6/SQRT(1.0-A**2)
ENDIF

RETURN
END
SUBROUTINE FORCE/MOSLIP/MOROT

ALPI = AL x Pi / (2.0 x ALo)

DETERMINE VALUES OF A AND B

|AL| \geq 1.0 ?

YES

C = 0.0

NO

C = \sin(AL) x \cos(ALPI) x B / \sqrt{1.0 - A^2}

RETURN
SUBROUTINE DIRCQS
C
C DETERMINES THE DIRECTION COSINES OF THE FRICTION FORCE AND
C MOMENTS
C
COMMON/FOUR/XM(0:7,50),TH,PHN,FS1,OGA(50),DALD(50)
COMMON/FIVE/K1,WL,WM,WN,DL(0:7,50),DM(0:7,50),DN(0:7,50)
C
FRICITION FORCE DIRECTION COSINES
C
IF(SIN(TH) .EQ. 0.0) THEN
   DL(1,K1)=0.0
   DM(1,K1)=0.0
ELSE
   DL(1,K1)=PHN*COS(TH)/SIN(TH)
   IF(PHN .GT. ABS(SIN(TH))) THEN
      DM(1,K1)=0.0
   ELSE
      DM(1,K1)=SQRT(1.0-(PHN/SIN(TH))**2)
   ENDIF
ENDIF
DN(1,K1)=PHN
C
DIRECTION COSINES OF MOMENT OF SLIP FRICTION
C
DS=SQRT(1.0-PHN**2)
DL(2,K1)=DM(1,K1)/DS
DM(2,K1)=-DL(1,K1)/DS
DN(2,K1)=0.0
C
DIRECTION COSINES OF MOMENT OF ROTATIONAL FRICTION
C
IF(COS(TH) .GT. 0.0) THEN
   SGN=1.0
ELSEIF(COS(TH) .LT. 0.0) THEN
   SGN=-1.0
ELSE
   SGN=0.0
ENDIF
DL(3,K1)=-PHN*DM(2,K1)*SGN
DM(3,K1)=PHN*DL(2,K1)*SGN
DN(3,K1)=-DS*SGN
C
DIRECTION COSINES OF THE TOTAL MOMENT
C
DL(4,K1)=(XM(2,K1)*DL(2,K1)+XM(3,K1)*DL(3,K1))/XM(4,K1)
DM(4,K1)=(XM(2,K1)*DM(2,K1)+XM(3,K1)*DM(3,K1))/XM(4,K1)
DN(4,K1)=(XM(2,K1)*DN(2,K1)+XM(3,K1)*DN(3,K1))/XM(4,K1)
C
RETURN
END
SUBROUTINE DIRCOS

\[ \sin(\theta) = 0.0 ? \]

**Yes**

\[ D_{L}(1, K1) = \frac{p_{HN}}{\sin(\theta)} \]

\[ D_{M}(1, K1) = 0.0 \]

**No**

\[ P_{HN} > |\sin(\theta)| ? \]

**Yes**

\[ D_{M}(1, K1) = 0.0 \]

**No**

\[ D_{M}(1, K1) = \sqrt{1.0 - \left(\frac{p_{HN}}{\sin(\theta)}\right)^2} \]

\[ D_{N}(1, K1) = p_{HN} \]

\[ D_{S} = \sqrt{1.0 - p_{HN}^2} \]

\[ D_{L}(2, K1) = \frac{D_{M}(1, K1)}{D_{S}} \]

\[ D_{M}(2, K1) = -\frac{D_{L}(1, K1)}{D_{S}} \]

\[ D_{N}(2, K1) = 0.0 \]

**Determine Sign of \( \cos(\theta) \)**

\[ D_{L}(3, K1) = -p_{HN} \times D_{M}(2, K1) \times \text{SGN} \]

\[ D_{M}(3, K1) = p_{HN} \times D_{L}(2, K1) \times \text{SGN} \]

\[ D_{N}(3, K1) = -D_{S} \times \text{SGN} \]

**Determine Values of**

\[ D_{L}(4, K1), D_{M}(4, K1), D_{N}(4, K1) \]

RETURN
SUBROUTINE COORD
C DETERMINES THE DIRECTION COSINES OF THE TOTAL FRICTIONAL
C MOMENT RELATIVE TO THE JOINT CO-ORDINATES
C
COMMON/FOUR/XM(0:7,50),TH,PHN,FS1,CGA(50),DALO(50)
COMMON/FIVE/K1,WL,WM,WN,DL(0:7,50),OM(0:7,50),DN(0:7,50)
C
IF(SIN(TH).EQ.0.0)THEN
DL(5,K1)=-ON(4,K1)*DL(0,K1)
DM(5,K1)=-DN(4,K1)*OM(0,K1)
DN(5,K1)=-ON(4,K1)*DN(0,K1)
ELSE
DLL=DL(4,K1)*CO5(TH)+WL)/SIN(TH)
DLM=DM(4,K1)*WN-ON(0,K1)*WM)/SIN(TH)
DLN=ON(4,K1)*DL(0,K1)
DL(5,K1)=DLM-DLL-DLN

DML=DL(4,K1)*OM(0,K1)*COS(TH)+WM)/SIN(TH)
DMM=DM(4,K1)*WN-ON(0,K1)*WM)/SIN(TH)
DNN=DN(4,K1)*OM(0,K1)
DM(5,K1)=DMM-DML-DNN

DNM=DN(4,K1)*ON(0,K1)*COS(TH)+WN)/SIN(TH)
DNN=DN(4,K1)*ON(0,K1)
DN(5,K1)=DNM-DNL-DNN
ENDIF

RETURN
END
Subroutine Coord

\[ \sin(\theta) = 0.0 \? \]

Yes

\[ DL(5,K1) = -Dn(4,K1) \times DL(0,K1) \]

No

Determine Values of DLL, DLM, DLN

\[ DL(5,K1) = DLM - DLL - DLN \]

Determine Values of DML, DMM, DMN

\[ DM(5,K1) = DMM - DML - DMN \]

Determine Values of DNL, DNM, DNN

\[ DN(5,K1) = DNM - DNL - DNN \]

Return
SUBROUTINE REACT

C DETERMINES THE REACTION PRODUCED BY THE FRICTIONAL MOMENT

C

COMMON/THREE/ML(2),RMAG,RL,RM,RN
COMMON/FOUR/XM(0:7,50),TH,PHN,F51,CGA(50),DALD(50)
COMMON/FIVE/K1,WM,WN,DL(0:7,50),DM(0:7,50),DN(0:7,50)

RI=XM(5,K1)*(DM(5,K1)*ML(2)-OM(5,K1)/HL(1))
RJ=XM(5,K1)*DL(5,K1)/HL(1)
RK=0.0
RMAG=SQRT(RI**2+RJ**2+RK**2)

IF(RMAG.EQ.0.0)THEN
  RL=0.0
  RM=0.0
  RN=0.0
ELSE
  RL=RI/RMAG
  RM=RJ/RMAG
  RN=RK/RMAG
ENDIF

RETURN
END
Subroutine REACT

Determine Values of $R_i, R_j, R_k$

$$R_{mag} = \sqrt{R_i^2 + R_j^2 + R_k^2}$$

$R_{mag} = 0.0$ ?

Yes

No

$$R_L = \frac{R_i}{R_{mag}}$$
$$R_M = \frac{R_j}{R_{mag}}$$
$$R_N = \frac{R_k}{R_{mag}}$$

RETURN

$R_L = 0.0$
$R_M = 0.0$
$R_N = 0.0$
SUBROUTINE GUT

PRINTS RESULTS

COMMON/FOUR/XM(0:7,50),TH,PHN,FS1,GA(50),DALC(50)
COMMON/FIVE/K1,ML,WM,WN,DL(0:7,50),DM(0:7,50),DN(0:7,50)
CHARACTER STR(0:7)%5

STR(0)=' P '
STR(1)=' F/P '
STR(2)=' MS/RP '
STR(3)=' MR/RP '
STR(4)=' M/RP '
STR(5)=' M '
STR(6)=' R '
STR(7)=' N/P '

DO 90,K6=0,7
PRINT 
PRINT 100,STR(K6)
PRINT 

100 FORMAT(1H ,2X,' TH ETA',6X,' GAMMA',5X,' ALPHA-0',
+ 6X,AS,10X,' L ',12X,' H ',12X,' N ')

DTH=0.0

DO 93 K7=1,46
IF(K6.EQ.0.0 OR K6.EQ.6) THEN
PRINT 250,DTH,GA(K7),DALC(K7),XM(K6,K7),
+ DL(K6,K7),DM(K6,K7),DN(K6,K7)
ELSE
PRINT 200,DTH,GA(K7),DALC(K7),XM(K6,K7),
+ DL(K6,K7),DM(K6,K7),DN(K6,K7)
ENDIF

200 FORMAT(1H ,X,3(F5.2,5X),F7.5,3(F6X,F7.4))
250 FORMAT(1H ,X,3(F6.2,5X),F7.2,3(F6X,F7.4))

DTH=DTH+2.0

93 CONTINUE
90 CONTINUE

RETURN
END
SUBROUTINE OUT

K6 = 0.7

PRINT HEADINGS FOR RESULTS

DTH = 0.0

K7 = 1.46

PRINT RESULTS FOR VALUE OF DTH

DTH = DTH + 2.0

CONTINUE

RETURN
REFERENCES


Rose Bearings Aerospace Catalogue (1983), Rose Bearings, Saxilby, Lincoln.


At present the effects of friction are not included in three-dimensional mechanism simulation packages because of the difficulty of determining a friction model for joints such as the spherical joint where the frictional resistance to motion depends not only upon the coefficient of friction and the magnitude of the loading on the joint but also on the pressure distribution within the joint resulting from that loading. Thus the basis of this thesis has been the development of a mathematical model of the effects of friction in a spherical joint which could then be incorporated into a mechanisms simulation program.

The model developed has shown that the main factors determining the magnitudes and directions of the frictional effects produced in a spherical joint, apart from the coefficient of friction and the magnitude of the loading, are the extent of the contact area between the ball and the socket and the magnitude of the angle between the axis of rotation of the joint and the direction of the applied load. Experimental results were obtained using apparatus that enabled the frictional moment produced on the socket of a joint to be measured while allowing the angle between the axis of rotation of the ball and the direction of the applied load to be varied between measurements. These results, obtained for a range of values of the coefficient of friction, confirm that this angle is a significant factor in the model and that the model usefully determines the frictional effects produced in a spherical joint.