Simultaneous Stabilization of Multivariable Linear Systems

Thesis submitted for the degree of
Doctor of Philosophy
at the University of Leicester

By

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August 1995
To
my parents
my wife
and
my sons & daughters.
The simultaneous stabilization of a collection of systems has received considerable attention over a number of years. The practical motivation for a solution to the simultaneous stabilization problem (SSP) stems from the stability requirements of multimode systems in practical engineering. For example, a real plant may be subjected to several modes due to the failure of sensors and nonlinear systems are often represented by a set of linear models for design purposes. To examine these problems, it is necessary to establish a simultaneous stabilization theory. This dissertation considers the problem of simultaneously stabilizing a set of linear multivariable time-invariant systems. Three methodologies are presented.

The first method is based on finding new approaches to solving the strong stabilization problem (i.e. stabilization by a stable controller) which can then be used in the SSP of two plants. New sufficient conditions and algorithms are derived for the solution to this problem. The second method utilizes robust stability theory applied to a “central” plant obtained from a given set of plants. A generalized two-block $L_\infty$-optimization problem is formulated and solved to find the central plant. The third method utilizes the parametrization of all stabilizing controllers. Sufficient conditions for the existence of a solution are derived and in the case of two plants a formula is derived for finding a simultaneously stabilizing controller.

The work advances the theory of the SSP (and the Strong Stabilization Problem) by introducing and investigating several new approaches, and deriving new sufficient conditions. The work is less successful in deriving practical algorithms for the SSP except in the second method where a reliable algorithm is given for finding a central plant on which existing robust stabilization methods can be applied. This method is illustrated by its application to helicopter control.

Abstract

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Chapter 1

Introduction

1.1 Background and Motivation

A primary task in control engineering is to design a feedback compensator for a given plant so that the controlled plant meets certain design specifications. Feedback is essential because of uncertain signals (disturbances) and uncertain models (unknown parameters and unmodelled dynamics typically at high frequencies). A feedback controller is said to be robust if the design specifications are met in the presence of these uncertainties. A control engineer will therefore often attempt to classify the uncertainties before a controller is designed then this knowledge can be used to direct the design process and also to assess the robustness of the design before implementation.

In this thesis we are primarily interested in model uncertainty as opposed to signal uncertainty. A variety of methods have emerged for capturing model uncertainty in a plant and then for using this information in a controller design. For example, gain and phase margins in classical control and \( H^\infty \) norm-bounded perturbations in more advanced optimal control. In the latter the uncertain plant is assumed to be in a set, an infinite set of models, defined around a nominal plant description. An alternative approach is to assume that the main characterizing features of the plant can be modelled by a finite set of plants, and then to design a controller which will meet the design specifications for every plant in the set. The finite set of plants might have resulted from several linearizations of a nonlinear plant model at different operating points, or from selecting, in a linear
model, different combinations of extreme values of parameters known to lie within certain ranges. In either case we reduce the problem to that of designing a controller for a finite set of plants, and then assume that the controller will also work for the possible plant models not included. This, of course, is often not a valid assumption, and in practice a considerable amount of simulation may be needed to validate (gain confidence in) a design, and in some areas of applications e.g. aerospace, it is sometimes the case that several controllers will be designed for several linear models and the controllers will then be scheduled together or used in some switching strategy. Nevertheless, there has been for many years a strong interest in wanting to know whether a single controller exists which can simultaneously meet performance specifications for several plant models; and further how this single controller should be designed. The practical interest in this problem is clear because of the ease of implementation and savings in design effort that are likely to follow from a single controller, and the theoretical interest is great because of the rich mathematical content of the underlying problems. Research has tended to concentrate on stabilization, rather than performance in general, but as stability is the most important performance requirement this is perhaps not so surprising.

In this thesis, the simultaneous stabilization problem will be the main theme. The objective will be to look for new algorithms and new approaches for solving and investigating this technically difficult problem.

The simultaneous stabilization problem was introduced by Saeks and Murray (1982) [80], for single-input single-output systems and was extended to multi-input multi-output systems by Vidyasagar and Viswanadham (1982) [95]. The problem can be stated as follows: given a set of plants \( G_0(s), G_1(s), \ldots, G_{i-1}(s) \), does there exist a single compensator \( K(s) \) which stabilizes all of them, and if so, find such a \( K(s) \)? Since the simultaneous stabilization problem is closely related to the strong stabilization problem, i.e. stabilization by a stable controller, the latter will also be a major topic in this thesis.

The problem of simultaneously stabilizing a set of linear multivariable time-invariant systems is still an open problem. Several algorithms exist for single-input single-output systems, but for the multi-input multi-output case, when there are more than two plants in the set, the problem is not yet fully solved. The early work on multivariable systems...
shows that the simultaneous stabilization problem of \( l \geq 2 \) plants is equivalent to simultaneously stabilizing \( l - 1 \) plants using a single stable controller. Necessary and sufficient conditions for the existence of such a stable controller are given in [95, 96]. However, a computational procedure to verify these conditions, and furthermore to obtain a controller, is not yet available. Hence, the simultaneous stabilization problem remains an open and difficult one, that warrants further research in order to obtain practical solutions.

1.2 Thesis Contribution

In this thesis, we present some new methods for addressing the problems of strong stabilization and simultaneous stabilization of multivariable linear time-invariant systems. The main aim is to develop useful theoretical results which might lead to practical techniques for solving these two problems or to insights which might help others develop improved algorithms, or to a better understanding of the limitations of using a single fixed gain controller.

The major contributions of the thesis are covered in Chapters 4 to 7 and may be summarized as follows:

- The strong stabilization problem is studied in detail in Chapter 4. For different types of multivariable linear time-invariant systems, different approaches are developed. When the given systems are minimum phase and proper, the problem can be solved using an approach which uses a stable inverse, however, the strictly proper case is more technically involved. When the given systems are non-minimum phase, three different formulations are given for the solution. Firstly, the problem is formulated as a Nevanlinna-Pick interpolation problem in \( \mathbb{H}_\infty \). In the second formulation, the problem is formulated as an \( \mathcal{H}_\infty \)-optimization problem. Finally, the strong stabilization problem is formulated as a stable projection problem. Again the problem is more difficult in the strictly proper case.

- The solution of the simultaneous stabilization problem for (two or more) multivariable systems is discussed in Chapter 5. A new methodology is proposed to solve this problem by breaking it down into two stages. First, a central plant is obtained
from the set of given plants. A generalized two-block $L_\infty$-optimization problem is formulated and reduced to a generalized one-block $L_\infty$-optimization problem. The solution of this generalized $L_\infty$-optimization problem gives the central plant. Necessary mathematical tools to achieve these results are developed. In the second stage, robust stabilization theory is used as a sufficient condition for the solution of the simultaneous stabilization problem. The main results of Chapter 5 have been published in [84].

- A new state-space formulation for the simultaneous stabilization problem is given in Chapter 6 using the Youla parametrization of all stabilizing controllers. Different conditions are obtained for different types of controllers. Also, different algorithms are proposed for the solution of the two-plant case. Some of the ideas discussed in Chapter 6 have been published in [86].

- In Chapter 7, the theory developed in Chapter 5 is applied to a practical design problem. The results discussed in this chapter have been published in [87].

1.3 Organization of the Thesis

This thesis consists of 8 chapters of which this is the first. Summaries of the other chapters are given below.

Chapter 2: Preliminaries

In this chapter, we review relevant results from linear systems theory for use in later chapters. Included are controllability, observability, algebraic equations, coprime factorizations, inner-outer factorizations, norms of systems, stability and performance, best approximation and the singular value decomposition.

Chapter 3: Strong and Simultaneous Stabilization Problems

In this chapter, an overview is given of the strong and simultaneous stabilization problems. Related literature on the simultaneous stabilization problem is reviewed and its current
status is discussed.

Chapter 4: The Strong Stabilization Problem

In this chapter we consider the strong stabilization problem for multivariable linear time-invariant and proper systems. We separately consider minimum and non-minimum phase systems. The solution for minimum phase systems requires a stable inverse of a particular stable matrix transfer function, while the solution for non-minimum phase systems requires an inner-outer factorization, where the outer part is unimodular in $\mathcal{H}_\infty$. When the systems are strictly proper, the two cases are formulated such that the conditions for the existence of a stable inverse or unimodularity of the outer part are satisfied. When the systems are non-minimum phase, the Nevanlinna-Pick interpolation algorithm is used to find a stable controller. Two other approaches are proposed to solve the strong stabilization problem by formulating it as a stable projection in $\mathcal{RH}_\infty$, and as an $\mathcal{H}_\infty$-optimization problem. The chapter ends with an illustrative example to demonstrate the connection between the strong stabilization and the simultaneous stabilization of two plants.

Chapter 5: Simultaneous Stabilization Via Robust Stabilization Theory

In this chapter, the simultaneous stabilization problem for linear multivariable time-invariant systems is reformulated as a robust stabilization problem about a central plant found using a generalized two-block $L_\infty$-optimization problem. The generalized two-block problem is reduced to an ordinary two-block problem with the condition that the solution has a particular number of unstable poles. A generalized one-block $L_\infty$-optimization problem is defined and its solution is given. Finally, robust stabilization theory is used to provide a sufficient condition for solution to the simultaneous stabilization problem.

Chapter 6: A State Space Approach to the Simultaneous Stabilization Problem

In this chapter we use the Youla parametrization of all stabilizing controllers. The simultaneous stabilization problem for $l$ multivariable linear time-invariant systems is reduced
to that of expressing Youla free stable parameters $Q_i(s), i = 1, \ldots, l - 1$, as a function
of a stable parameter $Q_0(s)$. Solutions for $Q_0(s)$ such that the $Q_i(s)$ are stable for each $i$
are discussed for two types of controller: strictly proper and bi-proper (meaning proper
but not strictly proper). Sufficient conditions for the existence of a $Q_0(s)$ are obtained
for each case. A method for finding the $Q_0(s)$ is developed. It involves redefining the
problem as that of stabilizing a real matrix by varying some of its unknown elements.
When a certain condition is satisfied, a closed form solution is given in the two-plant case,
otherwise numerical algorithms are given to solve this problem.

Chapter 7: Helicopter Controller Design: An Example of Simultaneous Stabilization Via Robust Stabilization

In this chapter, we consider an illustrative design problem. A simultaneously stabilizing
controller for a high performance nonlinear helicopter is designed. The theoretical results
of Chapter 5 are used and their effectiveness is demonstrated.

Chapter 8: Conclusions and Future Research

This final chapter contains concluding remarks and suggestions for further research.

1.4 Notation

1.4.1 Symbols

All systems considered in this thesis are linear, multivariable, finite-dimensional and time-
invariant, and are defined by real rational transfer function matrices. All derivations are
carried out in continuous time.

A proper transfer function matrix is represented in state-space form as
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} := C(sI - A)^{-1}B + D,
\]
alternatively written as \((A, B, C, D)\), where \(A, B, C,\) and \(D\) are real matrices and \(I\) is the identity matrix of appropriate dimensions. If \(D = 0\), the system is called a strictly proper system. If \(D\) has full rank, the system is called bi-proper.

A system is said to be stable if it is analytic in the right half plane. A real or complex matrix \(A\) is said to be stable if the real parts of all of its eigenvalues are in the open left half plane.

Standard notation, as indicated below, is used throughout the thesis. Any other notation will be clear from the context.

- \(\mathbb{R}\) The field of real numbers.
- \(\mathbb{C}\) The field of complex numbers.
- \(A \in \mathbb{C}^{m \times r}\) matrix whose elements are in \(\mathbb{C}\).
- \(A^T\) The transpose of \(A\).
- \(\bar{A}\) The complex conjugate of a matrix \(A \in \mathbb{C}\), i.e. \(\bar{A} = [\bar{a}_{ij}]\) where \(a_{ij}\) are the elements of \(A\).
- \(\bar{A}^T\) The complex conjugate transpose of \(\bar{A} \in \mathbb{C}\).
- \(G^*(s)\) Parahermitian conjugate of \(G(s)\), i.e. \(G^*(s) = G^T(-s)\).
- \(A^\dagger\) The pseudo-inverse of \(A\).

- \(C_+, C_-\) The open right and open left complex half planes respectively.
- \(C_{+c}\) The extended open right complex half plane, i.e., \(C_+ \cup \infty\).
- \(\mathcal{RH}^{p,m}_{+\infty}\) The space of proper, rational \(p \times m\) matrix-valued functions of \(s \in \mathbb{C}\) analytic in \(C_+\) (i.e. poles in \(C_-\)).
- \(\mathcal{RH}\) Same as \(\mathcal{RH}^{p,m}_{+\infty}\).
- \(\mathcal{RH}^{p,m}_{+\infty,-}\) Similar to \(\mathcal{RH}^{p,m}_{+\infty}\) but is analytic in \(C_-\).
- \(\mathcal{RL}_2\) Lebesgue space of real rational matrices whose elements are strictly proper and have no poles on the \(jw\) axis.
- \(\mathcal{RH}_2\) The space of strictly proper rational and stable functions.
- \(\mathcal{RH}_2^\perp\) The orthogonal complement of \(\mathcal{RH}_2\) in \(\mathcal{RL}_2\).
- \(\mathcal{RH}^{p,m}_{+\infty} = \mathcal{RH}^{p,m}_{+\infty,+} \cup \mathcal{RH}^{p,m}_{+\infty,-}\).
- \(\mathcal{RH}^{p,m}_{+\infty,-k}\) Has exactly \(k\) poles in \(C_+\).
- \(\mathcal{RH}^{p,m}_{+\infty,-(k\pm)}\) Has no more than \(k\) poles in \(C_-\).
Ch. 1. INTRODUCTION

$||G(s)||_{rms}$  
RMS gain of $G(s)$, equal to its $\mathcal{H}$-norm.

$||x(t)||_2$  
$L_2$-norm of a real vector valued signal $x(t)$.

$||x(t)||_{rms}$  
RMS norm of a real vector valued signal $x(t)$.

$||A||$  
The 2-norm of a complex matrix $A$.

$S(G)$  
The set of all stabilizing controllers of $G(s)$.

$adj(o)$  
The adjoint of $(o)$.

$col(o)$  
The number of columns of $(o)$.

$unit$  
A rational scalar function $g(s) \in \mathcal{RH}_\infty$ is a unit in $\mathcal{RH}_\infty$ if $g^{-1}(s) \in \mathcal{RH}_\infty$.

$unimodular$  
A square rational function $G(s) \in \mathcal{RH}_\infty$ is unimodular in $\mathcal{RH}_\infty$ if $G^{-1}(s) \in \mathcal{RH}_\infty$.

1.4.2 Abbreviations

ARE  
Algebraic Riccati Equation

CLHP  
Closed Left-Half Plane

CRHP  
Closed Right-Half Plane

GTBP  
Generalized Two-Block Problem

GOBP  
Generalized One-Block Problem

LCF  
Left Coprime factorization

LFT  
Linear Fractional Transformation

LSDP  
Loop Shaping Design Procedure

LTI  
Linear Time Invariant

LQG  
Linear Quadratic Gaussian

MIMO  
Multi-Input Multi-Output

OOBP  
Ordinary One-Block Problem

OTBP  
Ordinary Two-Block Problem

RCF  
Right Coprime Factorization

RMS  
Root Mean Square

RST  
Robust Stabilization Theory
**Ch. 1. INTRODUCTION**

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<th>Acronym</th>
<th>Description</th>
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<td>SISO</td>
<td>Single-Input Single-Output</td>
</tr>
<tr>
<td>SS</td>
<td>Simultaneous Stabilization</td>
</tr>
<tr>
<td>SSP</td>
<td>Simultaneous Stabilization Problem</td>
</tr>
<tr>
<td>StSP</td>
<td>Strong Stabilization Problem</td>
</tr>
<tr>
<td>SVD</td>
<td>Singular Value Decomposition</td>
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Chapter 2

Mathematical Preliminaries

2.1 Introduction

In this chapter we introduce some well-known results on continuous time linear time-invariant systems, which will be particularly useful in the following chapters.

Section 2.2, on linear systems theory, includes the state-space form of a transfer function matrix and operations on linear systems. Controllability and observability definitions are introduced in Section 2.3, and algebraic equations such as Lyapunov and Riccati equations are reviewed in Section 2.4. In Section 2.5, definitions and an important Lemma on coprime factorizations are given. Relevant norms of systems are introduced in Section 2.6. Definitions of relevant ideas from stability theory and performance theory, such as internal stability, nominal stability, robust stability and nominal performance are given in Section 2.7. The definition of inner and outer functions with some important theorems are given in Section 2.8. In Section 2.9, we give a brief summary of the 'best' approximation problem, the Nehari problem, and the definitions of related operators such as the Multiplicative (Laurent) operator, Toeplitz operator and Hankel operator. Finally, balanced realizations are reviewed in Section 2.10, together with the singular value decomposition.
2.2 Some Systems Theory

2.2.1 Transfer Functions

Consider a linear state-space model \( G \) described by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = 0 \quad (2.2.1) \\
y(t) &= Cx(t) + Du(t) \quad (2.2.2)
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \) is the state vector of the system, \( u(t) \in \mathbb{R}^m \) is the control vector and \( y(t) \in \mathbb{R}^r \) is a vector of measurements, and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{r \times n}, \) and \( D \in \mathbb{R}^{r \times m} \) are real matrices.

Taking Laplace transform of (2.2.1)-(2.2.2), the resulting transfer function is

\[ Y(s) = G(s)U(s) \]

where

\[ G(s) = C(sI - A)^{-1}B + D \quad (2.2.3) \]

which can be represented in a packed matrix as

\[ G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (2.2.4) \]

2.2.2 Operations on Linear Systems

Under state similarity transformation, \( \dot{x} = Tx \), system \( G \) becomes

\[ G(s) = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \begin{bmatrix} TA^{-1} & TB \\ CT^{-1} & D \end{bmatrix} \quad (2.2.5) \]

where \( T \) is invertible.

Given two systems defined by \( G_1(s) = (A_1, B_1, C_1, D_1) \) and \( G_2(s) = (A_2, B_2, C_2, D_2) \), a state-space representation of the series-connected system is given by

\[ G_1(s)G_2(s) = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \]
There may be cancellations between the poles of one system and the transmission zeros of the other, in which case this realization will not be minimal even if $G_1(s)$ and $G_2(s)$ are minimal.

Given $G_1(s)$ and $G_2(s)$ as above, the state-space representation of the parallel-connection system is given by

\[
G_1(s) \pm G_2(s) = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \pm \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 \pm C_2 & D_1 \pm D_2 \end{bmatrix} \tag{2.2.7}
\]

Again, the realization may not be minimal even if $G_1(s)$ and $G_2(s)$ are minimal.

Given the system

\[
G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{2.2.8}
\]

we can obtain a state-space representation for the inverse system as

\[
G^{-1}(s) = \begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ -D^{-1}C & D^{-1} \end{bmatrix} \tag{2.2.9}
\]

provided that the system is square (i.e., $r = m$) and $D$ is non-singular. If the system is not square but $D$ does have full row (column) rank, then one can define a right (left) inverse $G^\dagger$ by replacing $D^{-1}$ in (2.2.9) with the pseudoinverse $D^\dagger$.

A state-space representation for the dual system of (2.2.8) is given by

\[
G^\dagger(s) = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}, \tag{2.2.10}
\]

### Mathematical Preliminaries

Given $G_1(s)$ and $G_2(s)$ as above, the state-space representation of the parallel-connection system is given by

\[
G_1(s) \pm G_2(s) = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \pm \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 \pm C_2 & D_1 \pm D_2 \end{bmatrix} \tag{2.2.7}
\]
and the parahermitian conjugate system by
\[
G^*(s) = G^T(-s) = \begin{bmatrix} -A^T & -C^T \\ B^T & D^T \end{bmatrix}.
\] (2.2.11)

Given the system
\[
G(s) = \begin{bmatrix} A_1 & Q & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D \end{bmatrix}
\] (2.2.12)
a diagonalization of \(G(s)\) is given by
\[
G(s) = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D \end{bmatrix} = \begin{bmatrix} A_1 & 0 & B_1 - XB_2 \\ 0 & A_2 & B_2 \\ C_1 & C_1X + C_2 & D \end{bmatrix}
\] (2.2.13)

where \(X\) solves
\[
A_1X - XA_2 + Q = 0.
\] (2.2.14)

Another type of operation that appears quite frequently is that of adjoining several systems into one. We shall describe three such operations.

The first operation is the formation of a "fat" system by adjoining \(G_1(s)\) and \(G_2(s)\) as
\[
\begin{bmatrix} G_1 & G_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & C_2 & D_1 & D_2 \end{bmatrix}
\] (2.2.15)

If \(G_1\) and \(G_2\) have identical \(A\) and \(C\) matrices (a frequent occurrence) then a saving in state dimension may be achieved by realizing \(G_1\) and \(G_2\) as
\[
\begin{bmatrix} A & B_1 & B_2 \\ C & D_1 & D_2 \end{bmatrix}.
\] (2.2.16)
If $G_1$ and $G_2$ have the same $A$ and $B$ matrices then in a similar manner we define a "tall" system by stacking $G_1$ and $G_2$ vertically as

$$
\begin{bmatrix}
G_1 \\
G_2
\end{bmatrix} = \begin{bmatrix}
A & B \\
C_1 & D_1 \\
C_2 & D_2
\end{bmatrix}.
$$

(2.2.17)

Finally, we can define an operation that creates a block-diagonal system as

$$
\begin{bmatrix}
G_1 & 0 \\
0 & G_2
\end{bmatrix} = \begin{bmatrix}
A_1 & 0 & B_1 & 0 \\
0 & A_2 & 0 & B_2 \\
C_1 & 0 & D_1 & 0 \\
0 & C_2 & 0 & D_2
\end{bmatrix}.
$$

(2.2.18)

The operations defined by (2.2.16), (2.2.17), and (2.2.18) may be readily extended to the case of $k$ systems.

### 2.3 Controllability and Observability

For the system (2.2.1)-(2.2.2), the pair $(A, B)$ is controllable if, for each $t_1 > 0$ and final state $x_1$, there exists a continuous input $u(t)$ such that the solution of (2.2.1) satisfies $x(t_1) = x_1$.

**Lemma 2.1** The following are equivalent:

1. $(A, B)$ is controllable.
2. The matrix $[B \ AB \ \ldots \ \ A^{n-1}B]$ has independent rows.
3. The matrix $[A - \lambda I \ B]$ has rank $n$ for all eigenvalues of $A$ in $C$.
4. $\lambda_i(A + BF)$ ($i = 1, \ldots, n$) can be freely assigned subject to complex conjugate pairs by suitable choice of $F$.

The pair $(A, B)$ is stabilizable when there exists $F$ such that $A + BF$ is stable.
Lemma 2.2  The following are equivalent:

1. \((A, B)\) is stabilizable.

2. The matrix \[
\begin{bmatrix}
A - \lambda I & B
\end{bmatrix}
\] has rank \(n\) for all eigenvalues of \(A\) in \(\mathbf{CRHP}\).

3. The unstable modes of the system matrix \(A\) are controllable.  \(\Box\)

Lemma 2.3  The following are equivalent:

1. \((A, C)\) is observable.

2. The matrix \[
\begin{bmatrix}
C^T & A^T C^T & \ldots & (A^T)^{n-1} C^T
\end{bmatrix}^T
\] has independent columns.

3. The matrix \[
\begin{bmatrix}
A^T - \lambda I & C^T
\end{bmatrix}^T
\] has rank \(n\) for all eigenvalues of \(A\) in \(\mathbf{C}\).

4. \(\lambda_i(A + HC)\) \((i = 1, \ldots, n)\) can be freely assigned subject to complex conjugate pairs by suitable choice of \(H\).  \(\Box\)

The pair \((A, C)\) is detectable when there exists an \(H\) such that \(A + HC\) is stable.

Lemma 2.4  The following are equivalent:

1. \((A, C)\) is detectable.

2. The matrix \[
\begin{bmatrix}
A^T - \lambda I & C^T
\end{bmatrix}^T
\] has rank \(n\) for all eigenvalues of \(A\) in \(\mathbf{CRHP}\).

3. The unstable modes of the system matrix \(A\) are observable.

4. \((A^T, C^T)\) is stabilizable.  \(\Box\)

The observability (controllability) index is the number of states in \(A\) which can be observed (controlled) via \(C\) \((B)\). A matrix \(A\) will be called stable if all its eigenvalues are in \(\mathbf{C}\).
2.4 Algebraic Equations

There are two algebraic equations, Lyapunov and Riccati equations, which are especially important in state-space methods. In this section, we will review some of their basic properties and related results for use later in the thesis.

2.4.1 Lyapunov Equations

First we state a lemma about the solution of the Sylvester equation.

**Lemma 2.5** [4] For the Sylvester equation,
\begin{equation}
BX + XA = M \tag{2.4.1}
\end{equation}
where $A \in C^{n\times n}$, $B \in C^{m\times n}$, $M \in C^{m\times n}$ are given matrices, there exists a unique solution $X$ if and only if \( \lambda_i(A) + \lambda_j(B) \neq 0 \), \( \forall i = 1, \ldots, n \) and \( j = 1, \ldots, m \).

In particular, if $B = A^T$, (2.4.1) is called a Lyapunov equation for which the following facts are well known.

**Lemma 2.6** [4] Consider the Lyapunov equation
\begin{equation}
A^T X + XA = -C^T C \tag{2.4.2}
\end{equation}

Let $A$ be stable. Then

1. the real solution $X$ is unique and $X \geq 0$;
2. the rank of $X$ is equal to the observability index of $(C, A)$;
3. if $(C, A)$ is observable, then $X > 0$.

Lemma 2.6 states that the solution of the Lyapunov equation is unique and positive (semi) definite in a stable system. Similar results exist for dual pair $(A, B)$ which is related to controllability.
2.4.2 Algebraic Riccati Equation

Consider the following algebraic Riccati equation (ARE)

\[ A^T X + XA - XRX + Q = 0 \]  \hspace{1cm} (2.4.3)

where \( A, B, Q \in \mathbb{R}^{n \times n} \), \( R = R^T \) and \( Q = Q^T \). For (2.4.3), we define a corresponding Hamiltonian matrix as

\[ H = \begin{bmatrix} A & -R \\ -Q & -A^T \end{bmatrix} \]  \hspace{1cm} (2.4.4)

and define a unique stabilizing solution to (2.4.3) by \( X = X^T \) and \( \text{Re}[\lambda_r(A-RX)] < 0 \), \([26]\). We may denote the stabilizing solution via its Hamiltonian matrix as

\[ X := \text{Ric} \begin{bmatrix} A & -R \\ -Q & -A^T \end{bmatrix}. \]  \hspace{1cm} (2.4.5)

When \( Q = 0 \), a special case of the ARE is the following Lyapunov equation

\[ AY + YA^T - R = 0 \]

where \( Y = X^{-1} \).

2.5 Coprime Factorization in \( \mathcal{RH}_\infty \)

A number of well-known results on coprime factorization found in Vidyasagar, \([96]\) will be used in this thesis and are summarized below.

**Definition 2.1** Suppose \( M, N \in \mathcal{RH}_\infty \) have the same number of columns. Then \( M \) and \( N \) are right coprime if and only if there exist \( X, Y \in \mathcal{RH}_\infty \) such that

\[ XM + YN = I. \]  \hspace{1cm} (2.5.1)

The relation (2.5.1) is called (right) Bezout identity. It is possible to represent a possible unstable transfer function in terms of two stable, coprime factors using a right coprime factorization which is defined as follows.
Definition 2.2. The pair \((N, M)\), where \(M, N\) are assumed to be in \(\mathbb{R}H_\infty\), is a Right Coprime Factorization (RCF) of \(G(s)\) if and only if

(a) \(M\) is square and \(\det(M) \neq 0\)

(b) \(G = NM^{-1}\), and

(c) \(N\) and \(M\) are right coprime.

Left coprimeness and a Left Coprime Factorization (LCF) can be defined in an analogous way. Thus if \((\tilde{M}, \tilde{N})\) is a LCF of \(G(s)\), then \(G = \tilde{M}^{-1}\tilde{N}\).

Lemma 2.7. Let \((N, M), (\tilde{M}, \tilde{N})\) be any RCF and LCF of \(G(s)\). Suppose \(\vec{X}, \vec{Y} \in \mathbb{R}H_\infty\) satisfy

\[ \vec{X} M - \vec{Y} N = I. \]  

Then there exist \(X, Y \in \mathbb{R}H_\infty\) such that

\[ \begin{bmatrix} \vec{X} & -\vec{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I. \]  

\(\square\)

The ordered pair of matrices in (2.5.3) is called a doubly coprime factorization of \(G(s)\).

State space constructions of a doubly coprime factors will be described in Chapter 6, Section 2.

For a given \(G(s)\), there are many coprime pairs. A special pair is a normalized coprime factorization which always exist, see for example [62], and satisfies

\[ \tilde{M} \tilde{M}^* + \tilde{N} \tilde{N}^* = I \]  

(for a LCF) \hspace{1cm} (2.5.4)

\[ M^* M + N^* N = I \]  

(for a RCF) \hspace{1cm} (2.5.5)

and will be treated later in Chapter 7, Section 7.3.
2.6 Norms of Systems

In this section we review methods of measuring the size of an LTI system with input \( u \) and transfer function \( G(s) \). Of interest are the \( \mathcal{H}_2 \) norm and the \( \mathcal{H}_\infty \) norm.

The \( \mathcal{H}_2 \) norm of the stable transfer function matrix \( G(s) \) is defined as

\[
\| G(s) \|_2 = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}(G(j\omega)G^*(j\omega))d\omega \right)^{1/2}
\]  

(2.6.1)

and measures, for example, the RMS response of its output when the input is a white noise process, [12]. The LQG theory is concerned with minimizing \( \| T \|_2 \) for a suitably specified \( T(s) \).

The \( \mathcal{H}_\infty \) norm of the stable transfer function matrix \( G(s) \) is defined as

\[
\| G(s) \|_\infty = \sup_{\omega \in \mathbb{R}} \left( \lambda_{\text{max}} [G^*(j\omega)G(j\omega)] \right)^{1/2}
\]

\[
= \sup_{\omega \in \mathbb{R}} \left( \sigma_{\text{max}} [G(j\omega)] \right)
\]

(2.6.2)

and is importantly interpreted as the \( L_2 \) or RMS gain of the system \( G(j\omega) \). This is because the RMS gain of a transfer function is defined as

\[
\| G(s) \|_{\text{rms}} : = \sup_{\| u \|_{\text{rms}} \neq 0} \| Gu \|_{\text{rms}}
\]

(2.6.3)

which coincides with its \( L_2 \) gain

\[
\| G(s) \|_{\text{rms}} = \sup_{\| u \|_{\text{rms}} \neq 0} \frac{\| Gu \|_2}{\| u \|_2}
\]

(2.6.4)

where

\[
\| u(t) \|_2 : = \left( \int_0^T u^2(t)dt \right)^{1/2}
\]

\[
\| u(t) \|_{\text{rms}} : = \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T u^2(t)dt \right)^{1/2}
\]

Remark 2.1 If the transfer function is scalar, then

\[
\| g(s) \|_\infty = \sup_{\omega \in \mathbb{R}} |g(j\omega)|
\]

(2.6.5)
That is, in frequency domain, the $\mathcal{H}_\infty$-norm is the highest value of the Bode magnitude plot. On the Nyquist diagram, the $\mathcal{H}_\infty$-norm is the maximum modulus of the frequency response $G(j\omega)$ over all real frequency $\omega$, i.e., the maximum distance from the origin to the Nyquist diagram of $G(j\omega)$.

In control theory, the $\mathcal{H}_\infty$-norm of a closed-loop transfer function matrix can be interpreted as the worst case energy gain. (Actually, $\|G\|_\infty^2$ is the worst case energy gain, since $\|u\|_2^2$ represents energy). Hence, minimizing the $\mathcal{H}_\infty$-norm of a transfer function matrix is equivalent to minimizing the energy in the output signal due to the energy in the input signal. Finally, the $\mathcal{H}_\infty$-norm of a closed-loop transfer function is a particularly useful measure to minimize because it enables robust stability guarantees to be made.

2.7 Stability Theory

In this section we briefly review the notions of internal stability, nominal stability and robust stability. For more details the reader may see [27, 34, 61, 96].

2.7.1 Internal Stability

Consider the block diagram in Figure 2.1 which represent the two equations

$$
\begin{bmatrix}
  z \\
  y
\end{bmatrix} =
\begin{bmatrix}
  P_{11} & P_{12} \\
  P_{21} & P_{22}
\end{bmatrix}
\begin{bmatrix}
  w \\
  u
\end{bmatrix}, \quad u = Ky,
$$

(2.7.1)

where the plant $P$ and the controller $K$ are assumed to be fixed proper transfer matrices. The feedback system in Figure 2.1 is said to be well-posed if and only if all signals in the system are well defined.
Figure 2.1: Standard Feedback Control Configuration.

Figure 2.2: Internal Stability.
Definition 2.3 \textbf{(Internal Stability)} Consider Figure 2.2 and the transfer function

\[
T(s) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rightarrow \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}:
\]

\[
T(s) = \begin{pmatrix}
(I + K(s)G(s))^{-1} & - (I + K(s)G(s))^{-1} K(s) \\
(I + G(s)K(s))^{-1} G(s) & (I + G(s)K(s))^{-1}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
T_{11}(G,K) & T_{12}(G,K) \\
T_{21}(G,K) & T_{22}(G,K)
\end{pmatrix}
\] (2.7.2)

The feedback system is called internally stable if and only if \(T(s) \in \mathcal{H}_\infty\).

Consider again Figure 2.1. We have the following theorem regarding internal stability of the system.

\textbf{Theorem 2.8} [26] Consider a minimal realization of the system \(P(s)\) in Figure 2.1:

\[
P(s) = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}.
\] (2.7.3)

There exists a feedback controller \(K(s)\) such that the feedback system in Figure 2.1 is internally stable if and only if \((A, B_2)\) is stabilizable and \((C_2, A)\) is detectable.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{general_control_system_with_uncertainty.png}
\caption{General Control System with Uncertainty.}
\end{figure}
2.7.2 Nominal Stability

Consider the block diagram in Figure 2.3. We generally think of \( M(G, K) \) as a nominal, i.e., unperturbed plant, while \( \Delta \) represents the allowable perturbations or uncertainties in the plant. \( \Delta \) is assumed to be stable, i.e., \( \Delta \in \mathbb{R}^{\mathcal{H}_\infty} \), and \( \|\Delta\|_\infty \leq 1 \).

The \( \Delta \) generally has a block diagonal structure with many blocks. If the actual plant uncertainty \( \tilde{\Delta} \) is not less than or equal to 1 in magnitude at all frequencies, we simply construct a frequency-dependent weighting functions \( W(s) \) such that \( \tilde{\Delta} = \Delta W(s) \) with \( \sigma_{\text{max}}(\Delta) \leq 1 \) for all frequencies, and absorb \( W(s) \) into the nominal plant.

If \( \Delta \) has a block diagonal structure, assuming it is a full block matrix, it can lead to conservative results. In order to take advantage of the block diagonal structure, the structured singular value analysis [26] needs to be used.

The input-output behaviour of the perturbed system with a particular perturbation \( \Delta \) is simply \( \mathcal{F}_u(M(G, K), \Delta) \), where \( \mathcal{F}_u(M(G, K), \Delta) \) is the upper linear fractional transformation of \( M(G, K) \) and \( \Delta \) and is computed as the closed loop transfer function of Figure 2.3. The upper linear fractional transformation \( \mathcal{F}_u(M(G, K), \Delta) \) can be computed from Figure 2.3 as follows. Let us write \( M(G, K) \) as

\[
M(G, K) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}
\]

Then from Figure 2.3, we have

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]

Then

\[
y_1 = M_{22}u_2 + M_{21}u_1
\]

But \( u_1 \) can be computed as

\[
u_1 = \Delta y_1
\]

where

\[
y_1 = M_{11}u_1 + M_{12}u_2.
\]
So
\[ u_1 = \Delta(M_{12}u_1 + M_{12}u_2) \]
\[ u_1 = (I - \Delta M_{11})^{-1} \Delta M_{12}u_2 \]
\[ = \Delta(I - M_{11}\Delta)^{-1}M_{12}u_2 \]

Then
\[ y_2 = M_{22}u_2 + M_{21}u_1 \]
\[ = \left[ M_{22} + M_{21} \Delta (I - M_{11}\Delta)^{-1}M_{12} \right] u_2 \]
\[ = \mathcal{F}_u(M(G,K),\Delta)u_2 \]

where
\[ \mathcal{F}_u(M(G,K),\Delta) = M_{22} + M_{21} \Delta (I - M_{11}\Delta)^{-1}M_{12}. \quad (2.7.4) \]

To determine whether the nominal system is stable, we can check the stability of \( \mathcal{F}_u(M(G,K),0) \). But \( \mathcal{F}_u(M(G,K),0) = M_{22} \), therefore the closed-loop system is nominally stable if and only if \( M_{22} \) is stable.

Note that the lower fractional transformation \( \mathcal{F}_l(P,K) \) can be derived in a same way as for \( \mathcal{F}_u(M(G,K),\Delta) \) before. For example in Figure 2.1, the transfer function from \( w \) to \( z \) is given as \( \mathcal{F}_l(P,K) \) where \( \mathcal{F}_l(P,K) \) is defined as
\[ \mathcal{F}_l(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \quad (2.7.5) \]
where \( P_{ij} \) are as defined in equation (2.7.1).

### 2.7.3 Robust Stability

In general, the standard configuration of a feedback control system with a controller \( K \) and a perturbation \( \Delta \) representing uncertainty may be depicted as in Figure 2.4. As far as robust stability concerned, we may consider Figure 2.5 instead of Figure 2.4 where \( M \) is defined as the closed loop transfer function from \( u_1 \) to \( y_1 \) and is given by \( \mathcal{F}_l(P,K) \) (the upper loop in Figure 2.4 is open in this case). Then we may state the robust stability condition as follows, [27]: The closed loop system in Figure 2.5 remains stable for all
\( \Delta \in \mathcal{H}_\infty \) with \( \|\Delta\|_\infty \leq 1 \) if and only if
\[
\|M\|_\infty < 1.
\]

In the synthesis of a robustly stabilizing controller we need to find a controller, \( K(s) \), which nominally stabilizes the system and satisfies the above norm condition.

\[\begin{array}{c}
u_1 \\
\Delta \\
u_2 \\
P(s) \\
u_3 \\
K(s)
\end{array} \]

Figure 2.4: General Frame Work.

\[\begin{array}{c}
u_1 \\
\Delta \\
M
\end{array} \]

Figure 2.5: Robust Stability Test.
2.7.4 Nominal Performance

Good performance for a feedback regulator usually means that the change in the regulated output is small for large changes in the system disturbance input (sensitivity minimization). In other words, performance is good if the closed-loop transfer function is "small" at all frequencies. Many control problems (e.g. tracking) can be stated in this way by appropriately defining the input and the output of the system and by adding appropriate weights reflecting the goals for the design. The selection of weights requires engineering judgement.

In the nominal performance problem, we need to find a controller, $K(s)$, which internally stabilizes the nominal system and satisfies

$$\|\mathcal{F}(P,K)\|_\infty < 1 \quad (2.7.6)$$

where $\mathcal{F}(P,K)$ is the transfer function from $u_2$ to $y_2$ in Figure 2.4 (with the upper loop open, i.e. $\Delta = 0$). The robust performance problem is defined as follows: Find a stabilizing controller, $K(s)$, such that we have satisfactory performance for all possible $\Delta$, [26].

2.8 Inner-Outer Transfer Functions

Inner and outer functions are relevant to many aspects of control theory, especially, in $\mathcal{H}_\infty$ optimization. In this section, the definitions of these functions with their properties will be given.

The function $G \in \mathbb{R}^{r \times m}$ is called all-pass if $G^*G = I$. When $G$ is stable, it is called inner if $G^*G = I$ and co-inner if $GG^* = I$. Note that $G$ need not be square, but $G$ being inner implies $r \geq m$. If $G$ is a nonsquare inner matrix, then a matrix $G^\perp$ can be found such that $\begin{bmatrix} G & G^\perp \end{bmatrix}$ is square and inner; $G^\perp$ is called a complementary inner factor (CIF).

A square function $G(s) \in \mathbb{R}_{\infty}$ is called minimum phase if it is of full rank for all $s$ in the open RHP; it is called outer if it is in $\mathbb{R}\mathcal{H}_\infty$ and is of full rank for all $s$ in the open RHP; and it is called unimodular if it is in $\mathbb{R}_{\infty}$ and keeps full rank for all $s$ in the extended RHP, i.e. $G(\infty)$ is of full rank as well.
The next lemma is useful in characterizing inner functions in terms of a state-space realization [37].

**Lemma 2.9** Let $G = (A, B, C, D)$ be stable and let $L_r$ and $L_o$ be the controllability and observability gramians\(^1\) of $G$, respectively. Then $G$ is inner if and only if
\[
D^T C + B^T L_o = 0 \\
D^T D = I.
\]
Moreover, when $G$ is square and inner, the realization $(A, B, C, D)$ is minimal if and only if
\[
L_r L_o = I.
\]

\[\square\]

It is clear from this lemma that, all the nonzero Hankel singular values\(^1\) of a square inner matrix are equal to one, where the Hankel singular values are defined as the square roots of the eigenvalues of $L_r L_o$ [37].

**Definition 2.4** An *inner-outer factorization* of a function $G \in \mathcal{RH}_\infty$ is a factorization
\[
G(s) = G_i(s) G_o(s)
\]
where $G_i(s)$ is inner and $G_o(s)$ is outer.

The following lemma is given in [34].

**Lemma 2.10** Every transfer function in $\mathcal{RH}_\infty$ has an inner-outer factorization. \[\square\]

The methods of finding $G_i(s)$ and $G_o(s)$ in terms of a state-space representation of $G(s)$ will be described in Chapter 4, Section 4.2.

\(^1\)For the definitions of controllability and observability gramians, and Hankel singular values see Section 2.10.
2.9 Best Approximation Problem

In this section we describe a best approximation problem, called the Nehari problem, and give some definitions about operators on Hilbert space which are used in finding its solution. Note that an $\mathcal{H}_\infty$ optimization problem can be reduced to a Nehari problem.

2.9.1 The Nehari Problem

The Nehari Problem may be defined as follows [37]: given $R \in \mathbb{R}\mathcal{L}_\infty$, find a function $X \in \mathbb{R}\mathcal{H}_\infty$ such that

$$\|R - X\|_\infty$$

is minimized.

Let

$$\gamma_0 := \inf_{X \in \mathbb{R}\mathcal{H}_\infty} \|R - X\|_\infty$$

(2.9.1)

where

$$\|R - X\|_\infty = \sup_{\omega \in \mathbb{R}} |R(j\omega) - X(j\omega)|.$$

Then a function $X$ in $\mathbb{R}\mathcal{H}_\infty$ satisfying

$$\gamma_0 = \|R - X\|_\infty$$

will be called an optimal solution. The problem is thus to best approximate, in $\mathcal{L}_\infty$-norm, a given transfer function by a stable one. To get the best approximation, some operator theory tools are needed. Definitions of related operators are given next. For a subspace $\mathcal{X}$ of $\mathcal{L}_2$, we will use $\Pi_\mathcal{X}$ to denote the orthogonal projection from $\mathcal{L}_2$ to $\mathcal{X}$. For example, $\Pi_{\mathcal{H}_2}$ is the orthogonal project from $\mathcal{L}_2$ to $\mathcal{H}_2$.

2.9.2 Toeplitz Operators

**Definition 2.5** Let $G \in \mathcal{L}_\infty$. The **Multiplicative (Laurent) operator generated by $G$** is denoted by $\mathcal{M}_G$, and defined as

$$\mathcal{M}_G : \mathcal{L}_2 \rightarrow \mathcal{L}_2$$

$$f \rightarrow \mathcal{M}_G f = Gf.$$
Obviously $M_G$ is linear. In fact $\|M_G\| = \|G\|_\infty$, so $M_G$ is bounded. A related operator is the Toeplitz operator, the restriction of $M_G$ to $H_2$, which maps $H_2$ to $H_2$.

Definition 2.6 Let $G \in L_\infty$. The Toeplitz operator generated by $G$ is denoted by $T_G$, and defined as

$$T_G : H_2 \to H_2$$

$$f \mapsto T_G f = (\Pi_{H_2} M_G) f.$$  

The next theorem summarizes some useful properties of the Toeplitz operator.

Theorem 2.11 [27] Let $G \in R L_\infty$. Then

$$\|T_G\| = \|M_G\| = \|G\|_\infty$$

It is well known that a matrix representation of $T_G$ is the (infinite) Toeplitz matrix. More details on Toeplitz operators can be found in [34] and the references therein.

2.9.3 Hankel Operators

The solution of the Nehari optimization problem (2.9.1) is very closely linked with Hankel operator theory. In the following we collect some useful facts about Hankel operators [34]

Definition 2.7 Let $G \in R L_\infty$. The Hankel operator with symbol $G$, denoted by $\Gamma_G$, is defined as

$$\Gamma_G : H_2^+ \to H_2$$

$$f \mapsto \Gamma_G f = (\Pi_{H_2} M_G) f.$$  

Note that Toeplitz and Hankel operators are different in the sense that their domains are different, i.e. the domain of Toeplitz operator is $H_2$ while it is $H_2^+$ for Hankel operator.

Let $(A, B, C, D)$ be a state-space realization of the stable part of $G$. Then it is well known that the norm of a Hankel operator generated by $G$ is given by

$$\|\Gamma_G\| = \lambda_1 (L_2 L_2)^{1/2} = \sigma_1 (G).$$
That is, the norm of a Hankel operator is given by its largest Hankel singular value. This fact plays an essential role in the solution to the best approximation problem using state-space models [37].

An important fact about the Hankel operator generated by a real-rational function is that it has finite rank equal to the number of its nonzero Hankel singular values. But, Toeplitz operators do not have this property.

Lemma 2.12 Let $G = [G]_+ + [G]_-$ decomposed as $G = \{G\}_+ \oplus \{G\}_-$ where $[G]_+$ is the stable part of $G$ and $[G]_-$ is the anti-stable part. Then

$$
\Gamma_G = \Gamma_{[G]_+},
$$

and the rank of $\Gamma_{[G]_+}$ is given by the number of nonzero eigenvalues of the product of its gramians, $L_2L_2$. □

From the above lemma it is clear that $\Gamma_{[G]_-} = 0$.

To link the Nehari problem (2.9.1), we also define below an operator mapping $\mathcal{H}_2$ to $\mathcal{H}_2^*$ which is actually the dual of the Hankel operator.

Definition 2.8 Let $G \in \mathcal{RL}_\infty$. The dual Hankel operator generated by $G$ is denoted by $\Gamma_G^*$, and defined as

$$
\Gamma_G^* : \mathcal{H}_2 \longrightarrow \mathcal{H}_2^*
$$

$$
f \longrightarrow \Gamma_G^* f = (\Pi_{\mathcal{H}_2} M_G) f .
$$

The next lemma gives a lower bound for the Nehari problem (2.9.1).

Lemma 2.13 Let $G \in \mathcal{RL}_\infty$, then for any fixed $X \in \mathcal{RH}_\infty$,

$$
\| R - X \|_\infty \geq \| \Gamma_G^* \| .
$$

□

This lemma shows that $\| \Gamma_G^* \|$ is a lower bound for the distance from $R$ to $\mathcal{RH}_\infty$. 
2.10 Balanced Realizations

A balanced realization of a transfer function matrix serve as a starting point in many concepts in control such as model reduction or controller size reduction techniques. Hence a brief review is given here.

Let \( G(s) = (A, B, C, D) \) be an asymptotically stable and minimal, \( l \times m \) system having \( n \) states. The associated controllability and observability gramians are defined as

\[
L_c = \int_0^\infty e^{At}BB^Te^{A^Tt}dt
\]

and

\[
L_o = \int_0^\infty e^{A^Tt}C^TCe^{At}dt
\]

respectively. By integrating the corresponding matrix differential equations:

\[
\frac{d}{dt}e^{At}BB^Te^{A^Tt} = Ae^{At}BB^Te^{A^Tt} + e^{At}BB^Te^{A^Tt}A^T
\]

\[
\frac{d}{dt}e^{A^Tt}C^TCe^{At} = A^Te^{A^Tt}C^TCe^{At} + e^{A^Tt}C^TCe^{At}A
\]

from 0 to \( \infty \), respectively, it can be shown that \( L_c \) and \( L_o \) satisfy the following Lyapunov equations

\[
AL_c + L_cA^T + BB^T = 0
\]

\[
A^TL_o + L_oA + C^TC = 0.
\]

The controllability gramian \( L_c \) is symmetric, positive definite and may be factored as

\[
L_c = S^TS
\]

using Cholesky factorization. Similarly, the observability gramian \( L_o \) may be factored as

\[
L_o = R^TR, \quad L_o > 0.
\]

Hankel singular values of the system \( G(s) \) are defined to be the positive square roots of the eigenvalues of \( L_cL_o \) (or equivalently \( L_oL_c \)), i.e.,

\[
\sigma_i^H = [\lambda_i (L_cL_o)]^{1/2} = [\lambda_i (L_oL_c)]^{1/2} \quad i = 1, \cdots, n.
\]

Define \( U \) and \( V \) to be the singular vectors of the singular value decomposition of the product \( SR^T \). Then

\[
SR^T = U\Sigma V^T
\]
where
\[ \Sigma = \text{diag} \left( \sigma_i \left( SR^T \right) \right). \] (2.10.9)

Suppose the state is transformed by a nonsingular matrix \( T_b \) to \( \bar{x} = T_b x \) to yield the realization
\[
G(s) = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} = \begin{bmatrix} T_b AT_b^{-1} & T_b B \\ CT_b^{-1} & D \end{bmatrix}.
\] (2.10.10)

Then the gramians \( L_c \) and \( L_o \) are transformed to
\[
\tilde{L}_c = T_b L_c T_b^T
\] (2.10.11)
and
\[
\tilde{L}_o = T_b^{-T} L_o T_b^{-1}
\] (2.10.12)
and thus are not invariant under coordinate transformations. However, the Hankel singular values are invariant since
\[
\lambda_i(\tilde{L}_c \tilde{L}_o) = \lambda_i(T_b L_c L_o T_b^{-1}) = \lambda_i(L_c L_o).
\]

If a nonsingular matrix \( T_b \) is chosen as
\[
T_b = \Sigma^{1/2} U^T S^{-T}
\] (2.10.13)
then the gramians are equal and diagonal, i.e.,
\[
\tilde{L}_c = \tilde{L}_o = \text{diag} \left( \sigma_1^H, \sigma_2^H, \cdots, \sigma_n^H \right)
\] (2.10.14)
and by convention \( \sigma_1^H \geq \sigma_2^H \geq \cdots \geq \sigma_n^H > 0 \), where \( \sigma_i^H \) are the Hankel singular values of the system \( G(s) \) since
\[
\{ \sigma_i^H \} = \left\{ \lambda_i \left( L_c L_o \right) \right\}^{1/2} = \left\{ \lambda_i \left( SR^T R \right) \right\}^{1/2} = \left\{ \lambda_i \left( R^T SR \right) \right\}^{1/2} = \{ \sigma_i \left( SR^T \right) \}
\]
where \( \{ \sigma_i \left( SR^T \right) \} \) is the set of singular values of \( SR^T \).

The state-space realization (2.10.10) is called a balanced realization, which implies that the controllability and observability gramians are both equal to the diagonal matrix of
the Hankel singular values. The states of such a realization are balanced between controllability and observability. Thus they represent a convenient structure for model reduction since those states having weak controllability can be neglected without causing any imbalance in controllability and observability properties of the remaining states. Hence, the Hankel singular values give good indication of the "minimal" dimension of a system.
Chapter 3

The Strong and Simultaneous Stabilization Problems

3.1 Introduction

The first explicit statement of the simultaneous stabilization problem was introduced in Saeks and Murray (1982) [80] for single-input single-output systems and by Vidyasagar and Viswanadham (1982) [95] for multi-input multi-output systems and stated as follows. Given plants $G_0, ..., G_i$, does there exist a single compensator $K$ that stabilizes all of them and if so, what is $K$? This problem can be viewed as a problem of reliable stabilization, where $G_0$ is the nominal description of a particular plant which changes to $G_1, ..., G_i$ in the case of some failure (e.g. failure of sensors, severance of loops or software breakdown). Another application for this problem arises when one tries to use a compensator for nonlinear systems. As the operating point for the nonlinear system changes, so does its corresponding linearized model. If each of these linearized models can be stabilized using a common compensator, then a fixed (i.e. independent of operating point) compensator can be used for the nonlinear system, with a saving in complexity. Naturally, performance specifications would demand for more than just stability, but this would be the first step in assessing whether it was feasible to use a single controller.

In this chapter a literature survey about the simultaneous stabilization problem of multi-variable linear time-invariant systems, as well as single input single output systems, will be given. The status of the problem in the multivariable case will be stated clearly as a motivation for the work that will be presented later in this thesis. Also the strong
stabilization problem will be stated along with some definitions and related results from the literature.

This chapter is organized as follows. Section 3.2 gives a brief introduction to the simultaneous stabilization problem with a literature survey for single input single output systems in Subsection 3.2.1 and a similar literature for multivariable systems in Subsection 3.2.2. In Section 3.3, the strong stabilization problem is defined with some definitions and related results for multivariable systems. The current status of the simultaneous stabilization problem of multivariable time-invariant systems is presented in Section 3.4. Conclusions are given in Section 3.5. The chapter includes no new work but sets the scene for later chapters.

3.2 Related literature

In their paper [80], Saeks and Murray developed geometric conditions for simultaneous stabilization and state that their solution is "mathematical in nature and not intended for computational implementation." They indicated that computational criteria are only known for the case of two plants. The work of Vidyasagar and Viswanadham [95] is concerned with a multi-input multi-output generalization of the single-input single-output results of Saeks and Murray. To this end, they proved that the problem of simultaneously stabilizing \( k + 1 \) plants is equivalent to the problem of simultaneously stabilizing \( k \) plants with the added requirement that the compensator itself is stable. As far as computational criteria are concerned, the results in Vidyasagar and Viswanadham imply a complete solution for the two plant case only, i.e. upon reducing the two plant problem to that of finding a stable compensator for a single auxiliary plant, one can apply the parity interlacing property, which will be defined later in this chapter, to check for the existence of such a compensator.

In the following two Subsections, literature related to the simultaneous stabilization problem of single input single output systems as well as for multi-input multi-output systems will be given.
3.2.1 Single-Input Single-Output Systems

Following the results of Saeks and Murray [80], Saeks et al. [81], Vidyasagar and Viswanadh [95], the issue of finding a computationally feasible test for simultaneous stabilizability (for three or more plants) has been raised again by many authors, specially for SISO systems.

Emre in (1983) [31], considered SISO plants and the problem of finding a computational test was solved for a special case obtained by imposing a constraint that all l+1 closed-loop systems must end up having the same characteristic polynomial.

In Kwakernaak (1985) [57], it was shown that a family of plants can be simultaneously stabilized if all plants have the same number of transmission zeros which are all located in the strict left half plane and the high frequency behaviour of the plant transfer matrices satisfy some restrictions.

Using a completely different proof, Barmish and Wei (1985) [3] obtained a similar result for a family of SISO systems that are minimum phase systems with one sign high frequency gain. In contrast to Kwakernaak (1985), the number of transmission zeros of each plant is not required to be the same and the compensator was constructed through an iterative algorithm.

The SISO case was considered in detail by Debowski and Kurylowicz (1986),[20] only for the case of 3 plants where they extended the method in Saeks et al. [81]. They gave necessary and sufficient conditions for the existence of a stabilizing compensator and proposed an algorithm for its design.

In (1988), Hollot [45] presented a sufficient condition for the SSP using Kharitonov-like approximations and the Q-parameterization of all stabilizing compensators.

Schmitendorf and Hollot (1989) [89] also considered the simultaneous stabilization of a collection of linear single-input systems. They obtained a sufficient condition for the existence of a stabilizing linear state feedback controller.

Wu et al. (1990) [106] developed an alternative sufficient condition for the stabilizability of a set of single-input systems using an algebraic stability criterion for polynomials. A
linear programming approach was used to check for the condition and hence obtain the controller.

Howitt and Luus (1991) [46], considered the problem of stabilizing a collection of single-input systems with linear state feedback. They developed a necessary and sufficient condition for the existence of a suitable controller and provided a computational algorithm to construct a simultaneously stabilizing controller once its existence is established. Linear programming was used to minimize the maximum eigenvalue.

In (1993), Wei [104] discussed the SSP of discrete systems with a number of assumptions describing the set of allowable plants.

Starting from any stabilizing full state feedback gain for each plant, Broussard and McLean [13] presented an algorithm to solve the SSP using a constant gain decentralized control law.

In (1994), Paskota et al. [71] solved the SSP problem using linear state feedback. Hurwitz's necessary and sufficient conditions were used as the required set of constraints for simultaneous stability. Fonte et al. [32] used a fixed order controller and a linear programming technique to find the parameters of the controller that solve the SSP.

Various authors have considered the simultaneous stabilization problem for different types of systems and different controllers to overcome the intrinsic limitation of LTI controllers. One possible strategy is to allow time varying compensation. Khargonekar et al. (1985) [53] showed that a finite number of plants can be simultaneously stabilized by a digital periodic time-varying dynamic controller. Discrete time simultaneous pole assignment for a finite number of plants was achieved by Kabamba (1987) and Kabamba, Yang (1991) [50] and Ünyeliglu and Özgüler (1992) [94] using a generalized sampled-data hold function which comprises a periodic time-varying function with a sampling scheme.

Petersen (1987) [73] studied the problem of stabilizing, via non-linear state feedback control, a collection of single-input systems represented by state-space models. The main result of this paper was a sufficient condition for the existence of a stabilizing non-linear controller.
Zhang et al. (1993) [116], considered the simultaneous stabilization of \( k \) continuous time SISO LTI plants using a single continuous-time LTI compensator incorporated with a sampler and zeroth-order hold function. Sufficient conditions and a procedure for the design of controller were presented. SS of nonlinear systems was also studied in [77, 108].

Finally, from Blondel's monograph (1994),[10] on the simultaneous stabilization of linear SISO plants we extract: "At the present time there exists no tractable condition for simultaneous stabilization of three or more linear systems."

### 3.2.2 Multi-Input Multi-Output Systems

The first study of the SSP in a multivariable framework was reported by Vidyasagar and Viswanadham (1982).[95]. It was shown that for two \( r \times m \) plants, one can generically stabilize them simultaneously provided either \( r \) or \( m \) is greater than one. This result is further generalized in Ghosh and Byrnes (1983) [35] where it was shown that generic simultaneous stabilizability of \( l \ r \times m \) plants is guaranteed if \( l \leq \max(r, m) \). If a set of \( l+1 \) plants \( P_i, i = 0, \ldots, l \), is given, it is possible to construct a set of \( l \) plants \( \tilde{P}_i, i = 1, \ldots, l \), such that the plants \( \tilde{P}_i \) are simultaneously stabilizable by a stable controller. Necessary and sufficient conditions for the existence of such a stable controller that simultaneously stabilizes \( \tilde{P}_i, i = 1, \ldots, l \), are given but they are intractable, [96].

A particular case of the SS was reported by Alos (1983), [1]. He considered a plant \( G_0 \) in which one or more of its outputs are liable to fail yielding an equivalent plant \( G_1 \). A necessary and sufficient condition was obtained.

In (1985) [76], Sabri studied the SS of a class of plants which are almost disturbance decoupling, minimum phase, and all members of the class have the same order of infinite zeros and have high frequency gain matrices satisfying a certain condition. The restriction on the high frequency gain matrices is equivalent to requiring that the sign of high frequency gain for SISO plants be known.

A state space procedure for proper systems which are, minimum phase and the high frequency gain matrices have the same sign was given in (1986) [63]. Based on the idea that generically a multivariable plant can be stabilized by a rank one compensator, a
sufficient condition for the existence of a solution for $I \times r \times m$ plants where $I \leq \max(r, m)$ was provided. In this method, the coprime factors of the controller were computed. Then the controller was formed from its coprime factors. The controller may not exist due to the inversion required when forming the controller from its coprime factors.

In (1987) [98], a study of the SSP of a collection of plants, where each plant has a different domain of stability associated with it, was given. The objective is to find a fixed common controller $K(s)$ such that the poles of the closed loop transfer function $T(G_i, K)$ are all inside the corresponding given domain of stability $D_i$, for $i = 0, ..., l$. Assuming that one of the domains (say $D_0$) is a subset of the others, an intractable necessary and sufficient conditions for the existence of a common controller are derived.

In (1988), Wei et al [102] gave a generalization of the results in Barmish and Wei (1985) [3] of the SISO case, to a family of MIMO systems. The MIMO plants are represented by their transfer function matrices which are assumed to be proper, rational, minimum phase and have the same high frequency gain sign.

In (1992), Wu et al. [107], extend the results of Wei et al. [102] to a class of diagonally dominant systems.

Recently, in (1995), Chen et al [19], modified the method of (1986) [63] for strictly proper systems, to overcome some of the difficulties. They considered the case when $lr < r + m$, where $l$ is the number of the plants.

After this survey on the simultaneous stabilization problem, we will present next the current status of this problem for MIMO linear time-invariant systems. Before doing so, a definition of a related problem, the strong stabilization problem, is given first in the next section.

### 3.3 Strong Stabilization

In this Section the definition of the strong stabilization problem (StSP) will be given. A summary of some of the well-known results on this problem for MIMO and related properties such as the parity interlacing property (PIP) will be given. First some definitions:
Définition 3.1  
A plant $P$ is strongly stabilizable if there exists a stable controller that stabilizes $P$.

Définition 3.2  
A rational scalar function $g(s) \in \mathbb{RH}_\infty$ is a unit in $\mathbb{RH}_\infty$ if $g(s)^{-1} \in \mathbb{RH}_\infty$, i.e. $g(s)$ is bi-proper and minimum-phase.

Définition 3.3  
A square rational matrix function $G(s) \in \mathbb{RH}_\infty$ is unimodular in $\mathbb{RH}_\infty$ if $G(s)^{-1} \in \mathbb{RH}_\infty$.

A study of strong stabilizability is of interest for its own sake. In addition, several other problems in reliable stabilization are related to strong stabilizability such as simultaneous stabilization which will be studied later in this Chapter.

The SISP was first addressed and solved for SISO systems by Youla et al. in (1974), [111], who gave an easy tractable condition to check whether a given plant is strongly stabilizable or not. A general procedure to construct a stable controller that stabilizes a given plant is not provided. Some attempts to provide such a controller were given in [43, 49, 78, 103] and the references therein for special cases.

In the next subsection we will summarize the strong stabilization condition for MIMO systems and the definition of related terms.

3.3.1 Necessary and Sufficient Condition

There exists a very simple necessary and sufficient condition, based on the numbers and locations of the real poles and zeros of a MIMO plant in the extended right half plane $\mathbb{C}_+$, to check for strong stabilizability of a given plant.

Before we state the main results of this Section it is necessary to define blocking zeros and to introduce the notation $S(G)$ to denote the set of all controllers that stabilizes a MIMO system $G(s) \in \mathbb{R}^{n \times m}(s)$.

Définition 3.4  
An $s_o \in \mathbb{C}_+$ where $G(s_o) = 0$ is called a blocking zero of $P$. 

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Theorem 3.1 [96] Given $G \in \mathbb{R}^{n \times m}$, let $\sigma_1, \ldots, \sigma_i$ denote the real blocking zeros of $G$ in $\mathbb{C}^+$ (including $\infty$ if $G$ is strictly proper), arranged in ascending order. Let $\eta_i$ denote the number of poles in the interval $(\sigma_i, \sigma_{i+1})$, counted according to their McMillan degrees, and let $\eta$ denote the number of odd integers in the sequence $\{\eta_1, \ldots, \eta_n\}$. Then every $K \in S(G)$ has at least $\eta$ poles in $\mathbb{C}^+$. Moreover, this lower bound is exact in that there is a $K \in S(G)$ with exactly $\eta$ poles in $\mathbb{C}^+$.

Corollary 3.2 [96] $G \in \mathbb{R}^{n \times m}(s)$ is strongly stabilizable if and only if the number of real poles of $G$ (counted according to their McMillan degrees) between any pair of real $\mathbb{C}^+$-blocking zeros of $G$ is even.

Remark 3.1 The property described in Corollary 3.2 is referred to as the parity interlacing property (PIP).

Remark 3.2 Both Theorem 3.1 and Corollary 3.2 are natural generalization of the corresponding scalar results.

The following Corollary is an alternative form of Corollary 3.2.

Corollary 3.3 [96] Suppose $G \in \mathbb{R}^{n \times m}(s)$, and let $(N_G, M_G)$ be any RCF of $G$. Then $G$ is strongly stabilizable if and only if $|M_G(\infty)|$ has the same sign at all the $\mathbb{C}^+$-blocking zeros of $G$.

The essential difference between a strongly stabilizable plant and one that is not can be stated as follows [96]. If $G$ is stabilized using a stable compensator, then the resulting stable transfer function matrix has the same number of $\mathbb{C}^+$-zeros as the original plant. On the other hand, stabilization using an unstable compensator always introduces additional $\mathbb{C}^+$-zeros into the closed-loop transfer function matrix beyond those of the original plant. As it is known that the RHP zeros of a plant affect its ability to track reference signals and / or to reject disturbances, it is preferable to use a stable stabilizing compensator whenever possible.
3.4 The Simultaneous Stabilization Problem

The question of simultaneously stabilizing a set of linear systems was formulated some years ago [80, 95]: Let $G_0, ..., G_i$, be scalar (multivariable) linear time invariant systems. Under what condition does there exist a single compensator $K$ that stabilizes all of them and if so, what is $K$?

In this section we will study in detail the general mathematical form of the SSP reported in [95, 96]. First we give the parametrization of all stabilizing controllers.

3.4.1 All Stabilizing Controllers

Consider the feedback configuration of Figure 3.1, where $G(s)$ is a given plant, and $K(s)$ is a controller to be designed for internal stabilization.

Let $G(s)$ have the state-space realization

$$G(s) := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$  \hspace{1cm} (3.4.1)$$

with $(A, B)$ controllable and $(A, C)$ observable, where the dimensions of the real matrices $(A, B, C, D)$ are $n \times n$, $n \times m$, $r \times n$ and $r \times m$, respectively.
Let $G(s)$ have a doubly coprime factorization

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

(3.4.2)

and $X, Y, \tilde{X}$ and $\tilde{Y}$ satisfy the Bezout identity, i.e.,

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M \\ N \\ X \end{bmatrix} = I$$

(3.4.3)

where the matrices $(N, M, \tilde{N}, \tilde{M}, X, Y, \tilde{X}, \tilde{Y})$ belong to (the set of all real rational matrices whose elements are stable and proper). The matrices $(N, M, X, Y, \tilde{N}, \tilde{M}, \tilde{X}, \tilde{Y})$ can each be expressed in state-space form as follows, by choosing a real matrix $F$ such that $A + BF$ is stable and similarly by choosing a real matrix $H$ so that $A + HC$ is stable:

$$\begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} = \begin{bmatrix} A + BF & B - H \\ F & I \\ C + DF & D \end{bmatrix}$$

(3.4.4)

and

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \begin{bmatrix} A + HC & -(B + HD) \\ F & I \\ C & -D \end{bmatrix}$$

(3.4.5)

Then it is well known ([34], [61]) that the set of all stabilizing controllers for the given plant $G(s)$ can be parameterized by

$$K(s) = -(Y - MQ)(X - NQ)^{-1}$$

$$Q(s) \in \mathcal{RH}_\infty$$

(3.4.6)

where $Q(s)$ has the state-space realization

$$Q(s) = \begin{bmatrix} A_q & B_q \\ C_q & D_q \end{bmatrix}$$

(3.4.7)

and the dimensions of the matrices $(A_q, B_q, C_q, D_q)$ are $n_q \times n_q$, $n_q \times r$, $m \times n_q$, and $m \times r$, respectively.
3.4.2 The Status of the Simultaneous Stabilization Problem

First the problem of simultaneously stabilizing two given \( r \times m \) strictly proper plants \( G_0 \) and \( G_1 \) is studied.

Assume that for \( i = 0,1 \), we have available an RCF \( (N_i, M_i) \) and an LCF \( (\tilde{M}_i, \tilde{N}_i) \) of \( G_i \), together with matrices \( X_i, Y_i, \tilde{X}_i, \tilde{Y}_i \in \mathbb{R}^{n \times n} \), such that

\[
\begin{bmatrix}
\tilde{X}_i & -\tilde{Y}_i \\
-N_i & M_i
\end{bmatrix}
\begin{bmatrix}
M_i & Y_i \\
N_i & X_i
\end{bmatrix}
= \begin{bmatrix}
I_m & 0 \\
0 & I_r
\end{bmatrix}.
\]  

(3.4.8)

**Theorem 3.4** [95, 96] Define

\[
A_i = X_q M_i - Y_q N_i; \quad B_i = -\tilde{N}_0 M_i + \tilde{M}_0 N_i.
\]

(3.4.9)

Then \( \det A_i \neq 0 \), so that \( A_i^{-1} \) is well defined, and \( (B_1, A_1) \) are right coprime. Moreover \( \exists \) a \( K(s) \) that stabilizes both \( G_0 \) and \( G_1 \) if and only if \( \exists \) an \( R \in \mathbb{R}^{m \times r} \) that stabilizes the associated plant \( B_1 A_1^{-1} \). 

\( \square \)

**Outline of the Proof:** The sets of compensators that stabilize each \( G_0 \) and \( G_1 \) are given by

\[
S(G_0) = -(\tilde{X}_0 - Q_0 \tilde{N}_0)^{-1}(\tilde{Y}_0 - Q_0 \tilde{M}_0)
\]

\[
S(G_1) = -(\tilde{X}_1 - Q_1 \tilde{N}_1)^{-1}(\tilde{Y}_1 - Q_1 \tilde{M}_1).
\]

(3.4.10)

Hence \( G_0 \) and \( G_1 \) can be simultaneously stabilized if and only if \( \exists \) \( Q_0 \) and \( Q_1 \) such that

\[
(\tilde{X}_0 - Q_0 \tilde{N}_0)^{-1}(\tilde{Y}_0 - Q_0 \tilde{M}_0) = (\tilde{X}_1 - Q_1 \tilde{N}_1)^{-1}(\tilde{Y}_1 - Q_1 \tilde{M}_1).
\]

(3.4.11)

Let \( K \) denote the compensator defined in the above equation. Since both sides of (3.4.11) give LCF’s of \( K \), it is well known that the coprime factorization in \( \mathbb{R}^{m \times r} \) are unique up to multiplication by a unimodular matrix \( U \in \mathbb{R}^{m \times r} \). Therefore, (3.4.11) holds if and only if there exist a unimodular matrix \( U \) such that

\[
(\tilde{X}_0 - Q_0 \tilde{N}_0) = U(\tilde{X}_1 - Q_1 \tilde{N}_1)
\]

\[
(\tilde{Y}_0 + Q_0 \tilde{M}_0) = U(-\tilde{Y}_1 + Q_1 \tilde{M}_1).
\]

(3.4.12)

Thus \( G_0 \) and \( G_1 \) can be simultaneously stabilized if and only if \( \exists \) stable \( Q_0 \) and \( Q_1 \) and a unimodular matrix \( U \) such that (3.4.12) holds. In [96] theorem 5.4.2 states that this will
be the case if and only if there exists a stable $M_0$ such that $A_1 + M_0 B_1$ is unimodular. To prove this statement, rewrite the two equations in (3.4.12) as:

$$\begin{bmatrix} I & Q_0 \end{bmatrix} \begin{bmatrix} \dot{X}_0 & -\dot{Y}_0 \\ -\tilde{N}_0 & \tilde{M}_0 \end{bmatrix} = U \begin{bmatrix} I & Q_1 \end{bmatrix} \begin{bmatrix} \dot{X}_1 & -\dot{Y}_1 \\ -\tilde{N}_1 & \tilde{M}_1 \end{bmatrix}, \quad (3.4.13)$$

and recall from (3.4.8) that

$$\begin{bmatrix} \tilde{X}_1 & -\tilde{Y}_1 \\ -\tilde{N}_1 & \tilde{M}_1 \end{bmatrix}^{-1} = \begin{bmatrix} M_1 & Y_1 \\ N_1 & X_1 \end{bmatrix}. \quad (3.4.14)$$

Multiplying both sides of (3.4.13) by the matrix in (3.4.14) gives

$$\begin{bmatrix} I & Q_0 \end{bmatrix} \begin{bmatrix} \tilde{X}_1 & -\tilde{Y}_0 \\ -\tilde{N}_0 & \tilde{M}_0 \end{bmatrix} \begin{bmatrix} M_1 & Y_1 \\ N_1 & X_1 \end{bmatrix} = U \begin{bmatrix} I & Q_1 \end{bmatrix}. \quad (3.4.15)$$

Or

$$\begin{bmatrix} I & Q_0 \end{bmatrix} \begin{bmatrix} A_1 & S_1 \\ B_1 & T_1 \end{bmatrix} = U \begin{bmatrix} I & Q_1 \end{bmatrix}. \quad (3.4.16)$$

These are the equations to be satisfied, where the definitions of $S_1$ and $T_1$ are self-evident.

Equation (3.4.16) holds if and only if $A_1 + M_0 B_1$ is unimodular for some stable $M_0$. To show this, suppose first that (3.4.16) holds for some suitable $Q_0, Q_1, U$ and let $M_0 = Q_0$. Then from equation (3.4.16) $A_1 + M_0 B_1$ is unimodular, since $A_1 + Q_0 B_1 = U$. Then (3.4.16) holds with

$$U = A_1 + M_0 B_1, \quad Q_0 = M_0, \quad \text{and} \quad Q_1 = U^{-1}(X_1 + Q_0 Y_1). \quad (3.4.17)$$

In the same way the discussion holds for the case where $G_0$ and $G_1$ are not necessarily strictly proper.

It can be seen from the above discussion that the set for all compensators that simultaneously stabilize $G_0$ and $G_1$ is given by

$$-(\tilde{X}_0 - M_0\tilde{N}_0)^{-1}(\tilde{Y}_0 - M_0\tilde{M}_0), \quad M_0 \in \mathcal{M} \quad (3.4.18)$$

where $\mathcal{M}$ is the set of all $M_0 \in \mathcal{RM}_{\infty}^{\text{incr}}$ such that $A_1 + M_0 B_1$ is unimodular, i.e. $\mathcal{M}$ is the set of all stable compensators that stabilizes $B_1 A_1^{-1}$. 
In the multivariable case an explicit expression for the set $\mathcal{M}$ is not available, but $\mathcal{M}$ can be explicitly described in the case of single-input single-output systems, [95].

**Corollary 3.5** [95] Suppose $G_0$ is strictly proper and stable, and $G_1$ is strictly proper. Then $G_0$ and $G_1$ can be simultaneously stabilized if and only if $G_1 - G_0$ can be stabilized by a stable compensator.

In the following, the above stated results on the simultaneous stabilization problem of two plants will be extended to more than two plants.

Given plants $G_0, G_1, \ldots, G_l$ we would like to know whether or not there exists a compensator $K$ that stabilizes all the plants. By proceeding as in the two plants problem, the following result is stated.

**Theorem 3.6** [96] Suppose $G_0, G_1, \ldots, G_l$ are given plants. Define

$$A_i = x_0 M_i - \bar{y}_0 N_i; \quad B_i = - \bar{N}_i M_i + \bar{M}_i N_i.$$  \hspace{1cm} (3.4.19)

Then $G_0, G_1, \ldots, G_l$ can be simultaneously stabilized if and only if there exists a stable $M_0$ such that $A_i + A_0 B_i$ is unimodular for $i = 1, \ldots, l$.

**Outline of the Proof** Similar to the two plant case, there exists a $K$ such that $(G_i, K)$ is stable for $i = 0, \ldots, l$, if and only if there exist matrices $Q_0, \ldots, Q_l$ in $\mathcal{R}M_{\infty}^{mxr}$ such that

$$(\bar{x}_0 - Q_0 \bar{N}_0)^{-1}(\bar{y}_0 - Q_0 \bar{M}_0) = (\bar{x}_i - Q_i \bar{N}_i)^{-1}(\bar{y}_i - Q_i \bar{M}_i) \quad \text{for } i = 1, \ldots, l. \hspace{1cm} (3.4.20)$$

Next (3.4.20) is true if and only if there exist unimodular matrices $U_1, \ldots, U_l$ such that

$$(\bar{x}_0 - Q_0 \bar{N}_0) = U_i(\bar{x}_i - Q_i \bar{N}_i) \quad \text{for } i = 1, \ldots, l.$$

Rewrite (3.4.21) as

$$\begin{bmatrix} I & Q_0 \\ \bar{x}_0 - \bar{y}_0 & -\bar{N}_0 \end{bmatrix} = U_i \begin{bmatrix} I & Q_i \\ \bar{x}_i - \bar{y}_i & -\bar{N}_i \end{bmatrix} \begin{bmatrix} I & Q_i \\ \bar{x}_i - \bar{y}_i & -\bar{N}_i \end{bmatrix}^T, \quad \text{for } i = 1, \ldots, l. \hspace{1cm} (3.4.22)$$
and recall from (3.4.8) that

\[ \begin{bmatrix} \bar{X}_i & -\bar{Y}_i \\ -\bar{N}_i & \bar{M}_i \end{bmatrix}^{-1} = \begin{bmatrix} M_i & Y_i \\ N_i & X_i \end{bmatrix}. \quad (3.4.23) \]

Multiplying both sides of (3.4.22) by the matrix in (3.4.23) gives

\[ X_0 - N_0 M_0 = U_i, \quad Q_i = 1, \ldots, l, \quad (3.4.24) \]

or

\[ \begin{bmatrix} I & Q_0 \\ -\bar{N}_0 & \bar{M}_0 \end{bmatrix} \begin{bmatrix} M_i & Y_i \\ N_i & X_i \end{bmatrix} = U_i \begin{bmatrix} I & Q_i \end{bmatrix} \quad \text{for} \quad i = 1, \ldots, l. \quad (3.4.25) \]

Therefore \( G_0, \ldots, G_i \) can be simultaneously stabilized if and only if there exist \( Q_0, \ldots, Q_i \) and unimodular matrices \( U_1, \ldots, U_l \) such that (3.4.25) holds.

The theorem is proved if it can be established that (3.4.25) holds if and only if \( A_i + M_0 B_i \) is unimodular for some \( M_0 \in \mathbb{R}^{m_{\infty}} \).

Accordingly, suppose first that (3.4.25) holds for suitable \( Q_0, \ldots, Q_i, U_1, \ldots, U_l \) and select \( M_0 = Q_0 \). Then the first equation of (3.4.25) gives \( A_i + Q_0 B_i = U_i, \quad i = 1, \ldots, l \).

Conversely, suppose \( A_i + M_0 B_i \) is unimodular for some \( M_0 \). Then (3.4.25) holds with \( U_i = A_i + M_0 B_i, \quad M_0 = Q_0 \) and from the second equation of (3.4.25), \( Q_i = U_i^{-1}(X_i + Q_0 Y_i) \).

Theorem 3.6 reduces the SSP of \( l+1 \) plants into an equivalent problem of simultaneously stabilizing \( l \) equivalent plants, say \( P_i = B_i A_i^{-1} \) where \( A_i \) and \( B_i \) are as defined in equation (3.4.25) for \( i = 1, 2, \ldots, l \), by a single stable controller. The equivalent problem is more restrictive than the original problem if \( l > 1 \). At present the criterion of the above theorem is not computationally verifiable except when \( l = 1 \), i.e. for two plants only.

### 3.5 Conclusion

In this chapter the definitions of the strong and the simultaneous stabilization problems are given along with a survey of the solutions of the SSP. From this survey we see that, except for special cases, the SSP of \( l+1 \) MIMO systems is reduced into an equivalent problem of simultaneously stabilizing \( l \) equivalent plants. The equivalent problem is
more restrictive than the original problem if \( I > 1 \). At present there is no computational procedure to verify the conditions presented in the previous Section except when \( I = 1 \), i.e. for two plants only where the problem is reduced to a strong stabilization problem. Therefore, the simultaneous stabilization problem of MIMO as well as of SISO systems for \( I > 1 \) remains open and difficult, and warrants further research towards obtaining a satisfactory practical solution.
Chapter 4

Methods for Solving the Strong Stabilization of MIMO Systems

4.1 Introduction

The strong stabilization problem (StSP) is defined as that of finding a stable controller that stabilizes a given plant. Methods which attempt to solve this problem have already been referred to in Chapter 3.

The purpose of this chapter is to present new approaches to the strong stabilization problem. A sufficient condition for the solution of this problem [96] is used as a basis to develop theoretical results leading to computational procedures and new insights into the solution of the StSP. The problem can be categorized into two cases, minimum phase and non-minimum phase systems.

In the case of minimum phase proper systems, the problem is reformulated into that of finding a stable inverse. The parameterization of all stable controllers of a given multi-variable minimum phase system is based on the existence of a stable left/right inverse of a particular stable, rational transfer-function matrix. If the system is strictly proper or does not have full rank at infinity, an inverse is not defined. In which case, we propose a method which first gets a bi-proper transfer function from a strictly proper one. After finding a stable inverse, we recover the controller which will be improper in this case. This practical problem of improperness can be overcome by a special choice of a free
unimodular parameter \( U \) or by multiplying the resulting controller by a lag compensator of a specified order.

When the system is non-minimum phase, we formulate the strong stabilization problem as an interpolation problem in \( \mathcal{RH}_\infty \). The formulation requires an *inner – outer* factorization, where the outer part is *unimodular* in \( \mathcal{RH}_\infty \). Recall that unimodular means that the function and its inverse are in \( \mathcal{RH}_\infty \). Unimodularity is required because the controller depends on its inverse. If the system is non-minimum phase and strictly proper, then the controller may be improper. This problem can be solved by modifying the high gain term of the system, i.e. by putting the \( D \)-term equal to \( \epsilon I \) where \( |\epsilon| \ll 1 \) and \( I \) is the identity matrix. When a certain condition is satisfied, we propose a new method to avoid this modification of the \( D \)-term.

In \([23, 24, 78, 92, 115]\), different stabilization problems are formulated as *Nevanlinna – Pick* interpolation problems. All these methods consider the SISO case only. In \([15]\) MIMO systems are considered and the Youla parameterization of all stabilizing controllers is used to find the system’s closed-loop transfer function matrix in terms of the so-called Youla parameter \( Q \). Then the stabilization problem is formulated as an interpolation problem in \( \mathcal{RH}_\infty \) to find a stable \( Q \). The matrix version of the *Nevanlinna – Pick* problem is modified and used to solve this interpolation problem.

The *inner – outer* factorization of bi-proper systems, i.e. systems that have full rank at \( C_{1+} \), can be found in \([18, 26]\). For strictly proper systems or systems which do not have full rank at infinity, the problem is solved in \([44, 110, 117]\), but in this case the outer part is not a unimodular transfer function in \( \mathcal{RH}_\infty \). We will show in this chapter that a method using a *sub-generalized derivative sequence* can be used to find an *inner – outer* factorization for strictly proper systems or systems which does not have full rank at infinity. This method is computationally simpler than that in \([117]\).

Finally, two new approaches not based on the Nevanlinna-Pick theory to solve the StSP are given. In the first approach, the problem is formulated as an optimization problem in \( \mathcal{RH}_\infty \). In the second, a stable projection is used which makes the unstable part of a stabilizing controller zero, and the formulation of the problem is reduced into that of finding full rank constant gain output feedback.
The Chapter is organized as follows. Section 4.2 summarizes the mathematical tools employed: e.g. stable inverse, inner-outer factorization, modification of a sub-generalized derivative sequence that can be used to simplify the computations in finding IOF of strictly proper systems, and the Matrix Nevanlinna-Pick interpolation problem. The strong stabilization problem is defined in Section 4.3. Its formulation as a stable inverse problem or as an interpolation problem is given in Sections 4.4-4.5. Section 4.6 formulates the problem as an $H_{\infty}$-optimization problem, while Section 4.7 formulates the StSP as a stable projection in $H_{\infty}$. Section 4.8 formulates the StSP via the Youla free parameter $Q$. Because of the strong connection between the strong stabilization problem and the simultaneous stabilization problem, a solution for the simultaneous stabilization problem of two MIMO systems is described in Section 4.9 as an application of the StSP. The Chapter ends with conclusions in Section 4.10.

4.2 Preliminaries

In this section we summarize the necessary mathematical tools for the StSP. We will need, in subsequent sections, to find stable inverses, and to perform inner-outer factorizations, especially for strictly proper systems. Also we are going to make use of the Nevanlinna-Pick interpolation theory. Therefore, in this section we will present these general techniques suitably modified for the problem at hand.

4.2.1 Stable Inverse

One approach to solve the strong stabilization problem requires a stable inverse when the system is minimum phase. In this subsection, a state space approach to find a stable inverse will be given.

In terms of a given transfer-function matrix an inverse is defined as follows, [72]:

**Definition 4.1** Any linear multivariable system with an $r \times m$ transfer-function matrix, $G(s)$, has $G_R^+(s)$ (resp. $G_R^-(s)$) as its right (resp. left) inverse if and only if $G(s)G_R^+(s) = I_r$ (resp. $G_R^-(s)G(s) = I_m$).
The transfer-function matrix $G^r(s)$ (resp. $G^l(s)$) is called a right (resp. left) inverse of $G(s)$.

The existence of a right (resp. left) inverse of $G(s)$ is established by the following lemma.

**Lemma 4.1** [72] *The transfer-function matrix $G(s)$ has a right (resp. left) inverse if and only if*

$$\text{rank } [G(s)] = r \text{ (resp. } m)$$

*over the field of rational functions in } s.

For

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{4.2.1}$$

with $D$ full rank, the state-space representation of the right or left inverse of $G(s)$ as defined in Chapter 2 is given by

$$G^r(s) = \begin{bmatrix} A - BD^+C & BD^+ \\ -D^+C & D^+ \end{bmatrix} \tag{4.2.2}$$

where $D^+$ is the right or the left inverse of $D$ respectively.

It is known that $G(s)$ has a stable right (resp. left) inverse if and only if $G(s)$ has full row (resp. column) rank for all $s \in C_{\infty}$ [72]. Assume that $G(s) \in \mathbb{RH}_\infty^{n \times m}$ has full row rank for all $s \in C_{\infty}$. Then the next lemma gives a state space representation of all stable right inverse of $G(s)$. Before we state the lemma we need a real transformation matrix $T$ to decompose $D$ into $DT = [I_r 0]$. Using the Singular Value Decomposition, let $D = U[\sum_{i} 0]V^T$. Then $T = V \begin{bmatrix} \sum_{i} U^T & 0 \\ 0 & I_{m-r} \end{bmatrix} \in \mathbb{R}^{m \times m}$. Also partition $BT$ as $BT = [B_1 B_2]$.

**Lemma 4.2** *Let $G(s)$ be given as in (4.2.1), then*

1. $G^r(s)$ given by

$$G^r(s) = \begin{bmatrix} A - B_1C - B_2F & -B_1 \\ T \begin{bmatrix} F \\ C \end{bmatrix} & T \begin{bmatrix} I_r \\ 0 \end{bmatrix} \end{bmatrix} \tag{4.2.3}$$
is a right inverse of $G(s)$ for any real matrix $\mathcal{F}$ of appropriate dimensions.

2. $G_R^+(s)$ can be made stable if and only if $(A - B_1C, B_2)$ is stabilizable.

**Proof.** Let $G(s)$ be as defined in (4.2.1), then

$$G(s)G_R^+(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A - B_1C - B_2\mathcal{F} & -B_1 \\ T & T \end{bmatrix} = \begin{bmatrix} A & BT \\ 0 & A - B_1C - B_2\mathcal{F} \end{bmatrix} \begin{bmatrix} C & T \\ DT & T \end{bmatrix}$$

Using the definitions of $DT$ and $BT$, as given before, and the similarity transformation matrix

$$X = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$$

it can be easily shown that

$$G(s)G_R^+(s) = I_r.$$

Since $(A - B_1C, B_2)$ is stabilizable, there exists a real matrix $\mathcal{F}$ such that $(A - B_1C - B_2\mathcal{F})$ is stable, and vice versa.

For completeness, the dual result for a stable left inverse is given in Lemma 4.3. Let $G(s) \in \mathbb{R}_s^{n \times m}$ have full column rank for all $s \in C_{+e}$. Let $T$ decompose $D$ into $TD = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$ and partition $TC$ into $TC = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$, where $T = \begin{bmatrix} V\Sigma^{-1} & 0 \\ 0 & I_{m-m} \end{bmatrix} U^T \in \mathbb{R}^{r \times r}$.

**Lemma 4.3** Let $G(s)$ be given as in as defined above, then

1. $G_L^+(s)$ given by

$$G_L^+(s) = \begin{bmatrix} A - B\mathcal{C}_1 - \mathcal{H}\mathcal{C}_2 \\ -C_1 \\ [I_m \ 0]T \end{bmatrix}$$

is a left inverse of $G(s)$ for any real matrix $\mathcal{H}$ of appropriate dimensions.
2. $G_1^*(s)$ can be made stable if and only if $(A - BC_1, C_2)$ is detectable.

4.3.2 Inner-Outer factorization (IOF)

In this subsection we review the inner–outer factorization problem. First some definitions:

Definition 4.2 A stable transfer function matrix $G(s)$ with no more columns than rows is inner if $G^*(s)G(s) = I$ where $G^*(s) = G^T(-s)$.

Definition 4.3 A stable transfer function matrix $G(s)$ with no more rows than columns is co-inner if $G(s)G^*(s) = I$.

Definition 4.4 A stable transfer function matrix $G(s)$ is outer if all of its finite zeros are stable.

Let

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{RH}_{\infty}^{n \times m}. \quad (4.2.5)$$

When $G(s)$ is tall and left invertible, Chu and Doyle [18, 26], Doyle [26] gave a state space representation for the IOF of $G(s)$. In [110], the IOF of a wide and right invertible $G(s)$ is given. In [117], the IOF is given when $G(s)$ is strictly proper and $CB$ has full column rank or when $G(s)$ is proper but $D$ has less than full rank. In [44], the IOF is given when $G(s)$ is strictly proper with $j\omega$-axis zeros.

In the first two approaches of the IOF [18, 26, 110], the outer part is a unimodular function, while in the other two approaches [44, 117], it is not. Our formulation to the StStP will require the outer part to be unimodular in $\mathbb{RH}_\infty$.

In the following lemma, the inner-outer factorization when $G(s)$ is left invertible is reviewed first.

Lemma 4.4 [18, 26] Assume $G(s)$ in (4.2.5) is left invertible on the extended imaginary axis. Let $D$ be factorized as $D := Q \begin{bmatrix} R \\ 0 \end{bmatrix}$ where $Q$ is orthogonal and $R$ is upper
triangular, i.e. the QR factorization of D. Define \( H := (D^TD)^{-1} = (R^TR)^{-1} \). Then an inner-outer factorization of \( G(s) \) is given by
\[
G_i(s) := \begin{bmatrix} A - BF & BR^{-1} \\ C - DR & DR^{-1} \end{bmatrix}, \quad \text{and} \quad G_o(s) := \begin{bmatrix} A \\ RF \\ R \end{bmatrix},
\]
where \( F := H(D^TC + B^TX) \) and \( X \) is the stabilizing solution to the Riccati equation
\[
(A - BHD^TC)^T X + X(A - BHD^TC) - XBHB^TX + C^T(I - DHD^T)C = 0.
\] (4.2.7)

When \( G(s) \) is right invertible, the above lemma may be applied to \( G^T(s) \) or the method of [110] can be used.

In the formulation of the strong stabilization problem of non-minimum-phase systems, approaches [18, 26, 110] may be used to find the IOF when the plants are proper and have full rank at infinity. The approaches in [44, 117] can be used to find the inner-outer factorization of strictly proper systems or systems which do not have full rank at infinity, but in this case the outer part is not unimodular and this will create a problem in finding a stable inverse of the outer part since it is strictly proper. To overcome this problem, we will use an algorithm based on a sub-generalized derivative sequence. This is summarized in the next subsection. It is a modification of the method in [72] where a sequence of pure derivative elements is used to find the inverse of a strictly proper system. Then it will be shown how this sub-generalized derivative sequence can be used to find inner-outer factorizations of strictly proper systems.

### 4.2.2.1 Sub-Generalized Derivative Sequences

In this subsection we will use the idea studied in [72] with a suitable modification to find the inverse of a strictly proper system. The modification that will be presented is to use a sub-generalized derivative in each iteration instead of using a sub-ordinary derivatives. The idea is to find a multiplicative diagonal sequence consisting of the term \((s + \gamma)I\) or a sub-part of this term for a constant \( \gamma > 0 \). We choose \( \gamma \) to be greater than zero to avoid generating more zeros in the right half plane. When we multiply a given strictly
proper system by this sequence a bi-proper system is obtained. Then the known methods of doing inner-outer factorization of proper plants [18, 26, 110] can be used to find the IOF of the new system. Finally we recover the IOF of the original system.

First the following lemma is stated so that we can work in the state space framework.

**Lemma 4.5** [63] Given a strictly proper system \( N(s) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \), then the system \( N_{\gamma} = (s + \gamma)N \) may be realized as

\[
N_{\gamma} = \begin{bmatrix} A & (A + \gamma I)B \\ C & CB \end{bmatrix}
\]

or

\[
N_{\gamma} = \begin{bmatrix} A \\ C(A + \gamma I) \end{bmatrix} \begin{bmatrix} B \\ CB \end{bmatrix}
\]

**Corollary 4.6** Given \( N(s) \) as before and for constants \( \alpha \) and \( \gamma \), the system \( N_{\alpha \gamma} = (\alpha s + \gamma)N \) may be realized as

\[
N_{\alpha \gamma} = \begin{bmatrix} A \\ C(\alpha A + \gamma I) \end{bmatrix} \begin{bmatrix} B \\ \alpha CB \end{bmatrix}
\]

Let \( N(s) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in RH_{\infty}^{m \times n} \). Assume that \( r > m \). We propose the following algorithm to get a bi-proper system, \( \tilde{N} \), from the strictly proper system \( N(s) \) using the result stated in Lemma 4.5. In the same way Corollary 4.6 can be used.

**Algorithm 4.1**

**Step 0:** Set \( i = 0 \), and choose a suitable \( \gamma > 0 \), then compute

\[
\tilde{N}_0 = (s + \gamma)N(s) = \begin{bmatrix} A & B \\ C(A + \gamma I) & CB \end{bmatrix} = \begin{bmatrix} A & B \\ C_0 & D_0 \end{bmatrix}
\]
Let $q_i = \text{rank}(D_i)$. If $q_i = m$ stop, else there exists an orthogonal $T_i \in \mathbb{R}^{n \times m}$ such that $T_i D_i = \begin{bmatrix} D_{11} \\ 0 \end{bmatrix}$ where $\text{rank}(D_{11}) = q_i$.

**Step II:** Partition $T_i C_i$ accordingly to $T_i C_i = \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix}$. Then

$$\hat{N}_i = \begin{bmatrix} A & B \\ C_{11} & D_{11} \\ C_{12} & 0 \end{bmatrix}$$

(4.2.9)

**Step III:** Compute

$$\hat{N}_{i+1} = \begin{bmatrix} I_r & 0 \\ 0 & (s + \gamma)I_{r-m} \end{bmatrix} \begin{bmatrix} A & B \\ C_{11} & D_{11} \\ C_{12} & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C_{11} & D_{11} \\ C_{12}(A + \gamma I) & C_{12}B \end{bmatrix}$$

(4.2.10)

**Step IV:** Set $i = i + 1$, then go to Step I.

The $T_i$ that is used in each step above is equal to $U_i^T$ of the singular value decomposition of $D_i$, $D_i = U_i \begin{bmatrix} \Sigma_i & 0 \\ 0 & 0 \end{bmatrix} V_i^T$.

In the next we will show how to use the sequence of operators in the above algorithm to obtain a bi-proper system from a strictly proper as might be required in finding an inverse or doing inner-outer factorization.

Let us denote the sequence of generalized derivatives as

$$S_0 = (s + \gamma)I_r$$
\[ S_i = \begin{bmatrix} I_r & 0 \\ 0 & (s + \gamma)I_r \end{bmatrix} \text{ for } i = 1, 2, \ldots \]

For example assume we have the following equation, which will be described in the next section,
\[ (U - \bar{M}) = \bar{N}C_r \quad (4.2.11) \]
where \(U, \bar{M},\) and \(\bar{N}\) are some rational transfer functions with \(\bar{N}\) is strictly proper and we want to solve for \(C_r\). If we want to find \(C_r\) by finding the right inverse of \(\bar{N}\), we need to perform the algorithm described before using successive multiplications by the operator \(S_i\) and the transformation matrix \(T_i\) as follows.

\[
\begin{align*}
S_0(U - M) &= \bar{N}C_r \\
S_0(T_0S_0(U - M)) &= S_0\bar{N}C_r \\
T_0S_0(U - M) &= T_0S_0\bar{N}C_r \\
S_1T_0S_0(U - M) &= S_1T_0S_0\bar{N}C_r \\
T_1S_1T_0S_0(U - M) &= T_1S_1T_0S_0\bar{N}C_r \\
&\vdots \\
S_i \ldots T_1S_1T_0S_0(U - \bar{M}) &= S_i \ldots T_1S_1T_0S_0\bar{N}C_r
\end{align*}
\]

The process stops when \([S_i \ldots T_1S_1T_0S_0\bar{N}]\) has full rank at infinity. Denote this sequence of operations by \(T_k\), where
\[ T_k = \prod_{i=k}^0 S_iT_i. \quad (4.2.12) \]

Then equation (4.2.11) becomes
\[ T_k(U - \bar{M}) = T_k\bar{N}C_r \]
and \(C_r\) can be obtained as
\[ C_r = RT_k(U - \bar{M}) \]
where \(R\) is the right inverse of \(T_k\).

It is worth mentioning that, recently, an algorithm similar to the above mentioned one has appeared in [19]. More iterations are involved in the algorithm of [19] since it operates on \(i\) rows \((i = 1, 2, \ldots)\) of the initial matrix \[
\begin{bmatrix} C_0 & D_0 \end{bmatrix}.
\]
In the next subsection, the above algorithm will be used to find an IOF of a strictly proper system.

4.2.2.2 IOF of Strictly Proper System

In this subsection we will illustrate how the sub-generalized derivative sequence discussed above can be used to find IOF of strictly proper systems.

Let \( N(s) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in \mathbb{RH}_{\infty}^{m} \). We may then obtain from \( N \) a new bi-proper matrix, say \( \tilde{N} \), by multiplying \( N \) by the sequence \( T_\delta \) given in (4.2.12). We get

\[ \tilde{N} = NT_\delta . \]  

Then we may use the results of Lemma 4.4 to do inner-outer factorization on \( \tilde{N} \) as \( \tilde{N} = \tilde{N}_i\tilde{N}_o \) where \( \tilde{N}_o \) is unimodular. Then the IOF of \( N \) is given by

\[ N = \tilde{N}_i(\tilde{N}_oT_\delta^{-1}) = \tilde{N}_i\tilde{N}_o \]  

\( \tilde{N}_o \) in this form is no longer unimodular. The case when \( N(s) \) is proper but does not have full rank at infinity is solved in [117] using a procedure similar to the above one. When \( CB \) has full rank, formulas for IOF of \( N(s) \) are also reported in [117], but the procedure involves the computation of the observability Gramian and the solution of a spectral factorization problem and a generalized spectral factorization.

4.2.3 The Matrix Nevanlinna-Pick Interpolation Theory

There has been considerable interest in the application of the classical Nevanlinna-Pick interpolation theory to the design of robust control systems in the scalar case. See, for example, [115] for applications to minimum sensitivity design, and [55] for applications to robust stabilization. A multivariable extension of the scalar case of the Nevanlinna-Pick theory was first used by Chang et al [15]. He reduced the optimal disturbance reduction problem to the matrix Nevanlinna-Pick theory developed in [21] via inner-outer factorization. This approach was also adopted in [70, 96].
This subsection is devoted to the matrix version of the Nevanlinna-Pick problem. As we shall see, the strong stabilization problem of multivariable non-minimum phase systems can be formulated as an interpolation problem in \( \mathcal{H}_\infty \).

The matrix version of the Nevanlinna-Pick interpolation problem can be stated as follows:

Given complex numbers \( \lambda_1, \ldots, \lambda_n \) with \( |\lambda_i| < 1 \) \( \forall \ i \), together with complex matrices \( F_1, \ldots, F_n \) with \( \|F_i\| \leq 1 \) \( \forall \ i \), find all matrix-valued functions \( \Phi \) (if any) such that \( \|\Phi\|_\infty \leq 1 \) and \( \Phi(\lambda_i) = F_i, \ i = 1, 2, \ldots, n \), where \( \| \cdot \| \) denotes the norm of a complex matrix and \( \| \cdot \|_\infty \) denotes the infinitive norm of a matrix-valued function.

Before proceeding to the interpolation algorithm, it is established in [96] that one may assume that \( \|F_i\| < 1 \) \( \forall \ i \), without any loss of generality. This is explained in the following lemma.

**Lemma 4.7 [96]** Suppose \( \Phi \in \mathcal{H}^{\infty, \infty} \) and \( \|\Phi\|_\infty \leq 1 \). Suppose \( \|\Phi(z_0)\| = 1 \) for some \( z_0 \) in the open unit disk, \( \mathcal{D} \), and let \( r \) denote the multiplicity of 1 as a singular value of \( \Phi(z_0) \). Then \( \|\Phi(z)\|_\infty = 1 \) \( \forall \ z \in \mathcal{D} \). Moreover, there exist unitary matrices \( U \in \mathbb{C}^{k \times k}, V \in \mathbb{C}^{k \times k} \) such that

\[
\tilde{U}^T \Phi V = \begin{bmatrix}
I_r & 0_{k-r, k} \\
0_{k-r, r} & \Phi(z)
\end{bmatrix},
\]

where \( \|\Phi(z)\|_\infty < 1 \) \( \forall \ z \in \mathcal{D} \). \( \square \)

This means that if \( \|F_i\| = 1 \) for some values of \( i \), among these select one for which the multiplicity of 1 as a singular value is the largest, and renumber it as \( F_1 \). Then applying SVD on \( F_1 \) to get unitary matrices \( U, V \) such that

\[
\tilde{U}^T F_1 V = \begin{bmatrix}
I_r & 0 \\
0 & G_1
\end{bmatrix},
\]

where \( r \) is the multiplicity of 1 as a singular value of \( F_1 \) and \( \|G_1\| < 1 \). Then generate \( G_i \) such that

\[
\tilde{U}^T F_i V = \begin{bmatrix}
I_r & 0 \\
0 & G_i
\end{bmatrix}, \ i = 2, \ldots, n.
\]
and \(|G_i| < 1\). Now, by Lemma 4.7 any \(\Phi\) such that \(|\Phi|_{\infty} \leq 1\) and \(\Phi(\lambda_i) = F_i\) must satisfy

\[
U^T \Phi(z)V = \begin{bmatrix}
I_r & 0 \\
0 & \Psi(z)
\end{bmatrix} \quad \forall z \in D. \tag{4.2.17}
\]

Now replace the original interpolation problem by the following modified problem: Find a \(\Psi\) such that \(|\Psi|_{\infty} \leq 1\) and \(\Psi(\lambda_i) = G_i \forall i\). There is a one-to-one correspondence between solutions \(\Phi\) of the original problem and \(\Psi\) of the modified problem via (4.2.17).

Now the main result on the matrix Nevanlinna-Pick problem is stated in the following theorem.

**Theorem 4.8** [96] Suppose \(\lambda_1, \ldots, \lambda_n\) are distinct complex numbers and \(F_1, \ldots, F_n\) are complex matrices with \(|\lambda_i| < 1\) and \(|F_i| < 1 \forall i\). Define a partitioned matrix

\[
P = \begin{bmatrix}
P_{11} & \cdots & P_{1n} \\
\vdots & \ddots & \vdots \\
P_{n1} & \cdots & P_{nn}
\end{bmatrix}, \quad \text{where} \quad P_{ij} = \frac{1}{1 - \lambda_i \lambda_j}(I - F_i^T F_j). \tag{4.2.18}
\]

Then there exists a \(\Phi \in \mathcal{H}_{\infty}^{k \times l}\) such that \(|\Phi|_{\infty} < 1\) and \(\Phi(\lambda_i) = F_i \forall i\), if and only if the matrix \(P\) is positive definite.

The next subsection summarizes the steps involved in the Nevanlinna-Pick Algorithm.

### 4.2.3.1 Nevanlinna-Pick Algorithm

Given \(\tilde{M}(s) \in \mathcal{RH}_{\infty}\) with \(\mu_0 = \sup_s \sigma(\tilde{M}(s))\) and a finite set of numbers, \(s_i > 0\), an optimal \(\tilde{\Phi} \in \mathcal{RH}_{\infty}\) with \(|\tilde{\Phi}|_{\infty} = \mu_0\) will be constructed such that \(\tilde{\Phi}(s_i) = \tilde{M}(s_i) \forall i\).

The following transformation \(z = \frac{s - s_i}{1 + s_i}\) is used to transform the closed right half of the \(s\)-plane into a unit disk, \(D\), centered at the origin of the \(z\)-plane. Define

\[
\Psi(z) := \frac{1}{\mu_0} \tilde{\Phi}(s)|_{s=(1-z)/(1+z)} \tag{4.2.19}
\]

and

\[
W_i := \frac{1}{\mu_0} \tilde{M}(s_i)|_{s=(1-z)/(1+z)}, \quad i = 1, 2, \ldots, n. \tag{4.2.20}
\]
Now the problem becomes as follows: Find a $\Psi(z)$ which is analytic in the unit disk such that $||\Psi||_{\infty} \leq 1$ and

$$\Psi(\xi_i) = W_i, \quad i = 1, \ldots, n. \quad (4.2.21)$$

where $\xi_i = \frac{1 - s_i}{1 + s_i}$ are the interpolation points. Following the notations in [15, 21], define

$$L(E) := \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} (I - EE^T)^{-1/2} & -(I - EE^T)^{-1/2}E \\ -(I - EE^T)^{-1/2}E^T & (I - EE^T)^{-1/2} \end{bmatrix}, \quad (4.2.22)$$

where $E$ is a constant matrix with $\sigma_{\text{max}}(E) < 1$ and $E^{1/2}$ is the Hermitian square root of $E$, i.e. if $E = RR^T$, we denote $R$ by $E^{1/2}$. Define the linear fractional transformations

$$\Gamma_{L(E)} : X \mapsto (AX + B)(CX + D)^{-1} \quad (4.2.23)$$

$$y(\xi, z) := \begin{cases} \frac{\xi - z}{1 - \xi z} & \text{if } \xi \neq 0 \\ -z & \text{if } \xi = 0 \end{cases} \quad (4.2.24)$$

and

$$\Gamma_{L(E), y(\xi, z)} : X \mapsto Y = y(\xi, z)^{-1}(AX + B)(CX + D)^{-1} \quad (4.2.25)$$

then

$$\Gamma_{L(E), y(\xi, z)}^{-1} : Y \mapsto X = [y(\xi, z)YC - A]^{-1}[B - y(\xi, z)YD] \quad (4.2.26)$$

where $A, B, C$ and $D$ are as defined in equation (4.2.22). The following lemma establishes the existence of $\Gamma_{L(E)}$.

**Lemma 4.9** [96] $\Gamma_{L(E)}$ defined by (4.2.23) is well defined and

$$||\Gamma_{L(E)}|| \leq 1 \quad (\text{resp. } < 1) \iff ||X|| \leq 1 \quad (\text{resp. } < 1). \quad (4.2.27)$$

The steps described above are summarized in the following algorithm.

**Algorithm 4.2 (Nevanlinna-Pick Algorithm):**

**Step 0 (Initialization):** Let $W_i^\lambda = W_i, \quad i = 1, 2, \ldots, n$ as defined in equation (4.2.20) and $\xi_i$ be the interpolation points as defined before.
Step 1 (Forward Step): Compute $\begin{bmatrix} A^{k-1} & B^{k-1} \\ C^{k-1} & D^{k-1} \end{bmatrix} = L(W_{k-1}^{k-1})$ using Equation (4.2.22) and $y(\xi_{k-1}, \xi_i)$ using equation (4.2.24). Then set

$$W_i^k = \frac{1}{y(\xi_{k-1}, \xi_i)} \left( A^{k-1} W_i^{k-1} + B^{k-1} \right) \left( C^{k-1} W_i^{k-1} + D^{k-1} \right)^{-1}, \quad i = k, \ldots, n,$$

$$k = 2, \ldots, n. \quad (4.2.28)$$

Step 2 (Backward Step): Let $\Psi_n(z) = W_n^k$. Then compute

$$\Psi_{n-i}(z) = \left[ y(\xi_{n-i}, z) \Psi_{n-i+1}(z) C^{n-i} - A^{n-i} \right]^{-1} \left[ B^{n-i} - y(\xi_{n-i}, z) \Psi_{n-i+1}(z) D^{n-i} \right],$$

$$i = 1, \ldots, n - 1, \quad (4.2.29)$$

where $z$ is variable. Then set

$$\Psi(z) = \Psi_1(z)$$

and

$$\Phi(s) = \mu_0 \Psi(z) |_{z = (1-s)/(1+s)}.$$

Remark 4.1 In the above algorithm, for simplicity, we assume that the interpolation points $\xi_i$, $i = 1, \ldots, n$ are distinct. If this is not the case, then the modification in [109] can be used.

Remark 4.2 The second assumption is that we assume $||W_i|| < 1$, $i = 1, \ldots, n - 1$. If $||W_{k-1}^{k-1}|| = 1$ then we may add the following modifications to the $k^{th}$ iteration of the above algorithm, [15]:

Step 1 (Forward Step): Do the singular value decomposition of $W_{k-1}^{k-1}$ as

$$U_k^T W_{k-1}^{k-1} V_k^{-1} = \begin{bmatrix} I & 0 \\ 0 & W_{k-1}^{k-1} \end{bmatrix}$$

and

$$U_i^T W_i^{k-1} V_i^{-1} = \begin{bmatrix} I & 0 \\ 0 & W_i^{k-1} \end{bmatrix}, \quad i = k, \ldots, n.$$
Then compute
\[
\begin{bmatrix}
\hat{A}^{k-1} & \hat{B}^{k-1} \\
\hat{C}^{k-1} & \hat{D}^{k-1}
\end{bmatrix} = L(\hat{W}_{k-1}^{k-1}) \text{ using Equation (4.2.22)}.
\]
Then set
\[
W_i^k = \frac{1}{y(\xi_{k-1}, \xi_i)} \left( \hat{A}^{k-1} \hat{W}_{i}^{k-1} + \hat{B}^{k-1} \right) \left( \hat{C}^{k-1} \hat{W}_{i}^{k-1} + \hat{D}^{k-1} \right)^{-1}, \quad i = k, \ldots, n.
\]

**Step 2 (Backward Step):** Compute
\[
\hat{\Psi}_{h-1}(z) = \frac{1}{y(\xi_{h-1}, z)} \varphi(z) \left[ \hat{\Psi}_h(z) \right]^{-1} \left[ \hat{C}^{h-1} - y(\xi_{h-1}, z) \hat{\Psi}_h(z) \right]^{h-1}.
\]
Then set
\[
\Psi_{k-1}(z) = U_{k-1} \begin{bmatrix} I & 0 \\ 0 & \hat{\Psi}_{k-1}(z) \end{bmatrix} \hat{V}_{k-1}^T.
\]

**Remark 4.3** We may overcome the occurrence of the case in Remark 4.2 by increasing \( \mu_0 \) to guarantee that \( ||W_i|| < 1 \) in (4.2.20).

**Remark 4.4** A list of a computer program, using MATHEMATICA software, to implement the symbolic computations involve in Algorithm 4.8 is given in Appendix B.

In this subsection we have described briefly the matrix version of the Nevanlinna-Pick interpolation theory along with a recursive algorithm to implement it. This algorithm will be referred to in the next sections when we reduce the strong stabilization problem of multivariable non-minimum phase systems into an interpolation problem in \( \mathcal{H}_{\infty} \).

In this section we have also summarized the necessary mathematical tools which will be used in the rest of this chapter to study the strong stabilization problem. Although mainly standard some results have required modification e.g. the stable inverse, and inner-outer factorization, especially for strictly proper systems.

In the next section, the strong stabilization condition will be stated. The main objective of this chapter is to look for mathematical methods which can be used to satisfy this condition and which might lead to implementable solutions.
4.3 Strong Stabilization Problem

The strong stabilization problem is formulated as follows: Given an unstable plant $G$, when does there exist a stable compensator $C$ that stabilizes $G$ and if so, what is $C$?

We first state mathematically the condition for strong stability of a given MIMO plant. Then in the next sections we will give suitable mathematical formulations towards the solution of the StSP.

In Chapter 3, it is stated in Corollary 3.2 that $G$ is strongly stabilizable if and only if the number of real poles of $G$ (counted according to their McMillan degrees) between any pair of real right half plane blocking zeros (including $\infty$ if $G$ is strictly proper) is even. An alternative necessary condition to the parity interlacing property is the positive definiteness of the Nevanlinna-Pick Matrix defined in (4.2.18).

The following theorem will state the general stability condition. The corollary which follows will give an alternative condition for strong stabilizability.

**Theorem 4.10** [96] Suppose a plant model $G$ and a compensator $C$ in $\mathbb{R}^{r \times m}(s)$. Let $(N_p, M_p), (\tilde{N}_p, \tilde{M}_p)$ be any RCF and LCF of $G$, and let $(N_c, M_c), (\tilde{N}_c, \tilde{M}_c)$ be any RCF and LCF of $C$. Under these conditions, the following are equivalent:

(i) The pair $(G, C)$ is stable.

(ii) The matrix $\tilde{N}_c N_p + \tilde{M}_c M_p$ is unimodular.

(iii) The matrix $\tilde{N}_p N_c + \tilde{M}_p M_c$ is unimodular.

**Corollary 4.11** [96] Let $(N_p, M_p), (\tilde{N}_p, \tilde{M}_p)$ be any RCF and LCF of $G$. If $C$ is stable and $G$ is strongly stabilizable, then the following are equivalent:

1. The pair $(G, C)$ is stable.

2. The matrix $CN_p + M_p$ is unimodular.

3. The matrix $\tilde{N}_p C + \tilde{M}_p$ is unimodular.
In the next sections we will develop some theoretical methods that will satisfy the strong stabilization condition stated in Corollary 4.11 and may lead to practical implementations.

4.4 Strong Stabilization Problem of Minimum Phase Systems

In this section a formulation of the strong stabilization problem for minimum phase systems as a stable inverse problem is given. The systems under consideration are strictly proper, i.e. \( D = 0, \) \( (D = G(\infty)). \) This procedure can be easily extended to the case when \( D \) has rank deficiency or \( D \) has full rank.

Given \( G(s) \in \mathbb{R}^{r \times s} \) and \((\tilde{M}, \tilde{N}) \in \mathbb{R} \mathcal{H}_\infty \) a left coprime factorization of \( G(s) \) and \((N, M)\) a right coprime factorization. Then, as stated in Section 4.3, the condition for strong stabilization is

\[
\tilde{M} + \tilde{N}C_r = U \tag{4.4.1}
\]

or

\[
M + C_lN = U \tag{4.4.2}
\]

for some stable \( C_r \) or \( C_l \) and any unimodular matrix \( U \) in \( \mathbb{R} \mathcal{H}_\infty \).

Rewrite (4.4.1) and (4.4.2) as

\[
(U - \tilde{M}) = \tilde{N}C_r, \tag{4.4.3}
\]

and

\[
(U - M) = C_lN. \tag{4.4.4}
\]

If \( G(s) \) is strictly proper, then \( \tilde{N}(s) \) and \( N(s) \) are also strictly proper. Therefore we need to multiply (4.4.3) and (4.4.4) by the sequence \( T_b \), defined in (4.2.12), such that we can perform stable inversion.

\[
T_b(U - \tilde{M}) = T_b\tilde{N}C_r = \tilde{N}C_r, \tag{4.4.5}
\]

and

\[
(U - M)T_b = C_lNT_b = C_l\tilde{N}. \tag{4.4.6}
\]

Now \( \tilde{N} \) and \( \tilde{N} \) have full rank for all \( s \in C_{++} \).

Now, the solution for \( C_r \) and \( C_l \) can be found as follows. Assume there exists a stable right inverse of \( \tilde{N} \) denoted by \( R_\sigma \) and a stable left inverse of \( \tilde{N} \) denoted by \( R_l \) such that
\( \tilde{N} R_r = I_r \) and \( R_i \tilde{N} = I_m \). \( R_r \) or \( R_i \) can be computed as described in Section 4.2.1. Then we may state the following result.

**Lemma 4.12** If \( \tilde{N}(\tilde{N}) \) defined as in (4.4.5)/(4.4.6) has a stable inverse \( R_r(R_i) \), then strong stabilization controllers, \( C_r(C_i) \) of \( G(s) \) may be obtained by

\[
C_r = R_r T_b (U - \tilde{M}) \quad \text{if} \quad m \geq r, \tag{4.4.7}
\]

or

\[
C_i = (U - M) T_b R_i \quad \text{if} \quad m \leq r, \tag{4.4.8}
\]

where \( U \) is some unimodular transfer function matrix in \( \mathbb{R}^{m \times m} \).

**Proof.** If \( G(s) \) is wide, then \( N R_r = I_r \). So multiplying (4.4.7) by \( \tilde{N} \) from the left we obtain (4.4.5). But if \( G(s) \) is tall, then \( R_i \tilde{N} = I_m \). Then (4.4.6) is obtained by multiplying (4.4.8) from the right by \( \tilde{N} \).

If the system under study has full rank at infinity, then \( T_b \) in equations (4.4.7-4.4.8) equals to \( I \). If the system has rank deficiency at infinity, then the sequence \( T_b \) needs to be adjusted accordingly.

The following algorithm summarizes the procedure described above to design a stable controller for a strictly proper minimum phase system \( G(s) \) where \( m > r \).

**Algorithm 4.3:**

**Step 0 (Initialization):** \( G(s) \) is a given strictly proper and minimum phase plant.

**Step 1:** Find LCF for \( G(s) \) so that \( G = \tilde{M}^{-1} \tilde{N} \).

**Step 2:** Use Algorithm 4.1 to find a sequence \( T_b \) such that \( T_b \tilde{N} \) is bi-proper.

**Step 3:** Use the result of Lemma 4.2 to find a right stable inverse for \( T_b \tilde{N} \), i.e. find \( R_r \) such that \( T_b \tilde{N} R_r = I \).

**Step 4:** Assume any unimodular matrix \( U \in \mathbb{R}^{m \times m} \).

**Step 5:** Use equation (4.4.7) to find the controller \( C_r \) and stop.
If \( m < r \), we can follow the same procedure as above with appropriate changes, i.e.
compute RCF instead of LCF in step 1 and so on (as described previously in this section).

**Remark 4.5** In step 4 of algorithm 4.3, if \( U = I \), the controller obtained will be proper.
If \( U \) is dynamic, the controller will be improper. In this case we multiply the controller by
the term \( \left( \frac{1}{\sigma + n} \right)^n \) for some positive integer \( n \) such that the resulting controller is proper.

The following two examples demonstrate the above results.

**Example 4.1** This example demonstrates the strong stabilization of minimum phase
systems. The plant that we will use in this example is an unstable and tall, i.e. the
number of inputs are less than the number of outputs. For later work, the plant that we
are going to stabilize is an auxiliary plant which will be formed from the following two
plants.

\[
P_0 = \begin{bmatrix}
    0 & 1 & 1 \\
    -1 & 0 & 0 \\
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    1 & 0 & 0
\end{bmatrix}
\]

\[
P_1 = \begin{bmatrix}
    0 & 1 & 1 \\
    -9 & 0 & 0 \\
    1 & -1 & 0 \\
    -1 & 1 & 0 \\
    1 & -1 & 0
\end{bmatrix}
\]

These two plants are reported in [19] except that we interchange the transpose of the
input matrix with the transpose of the output matrix to get a tall system. The auxiliary
plant is defined in [96] as \( P = B_0 A_0^{-1} \) where \( A_0 \) and \( B_0 \) are defined as follows
\[
\begin{bmatrix}
    A_0 \\
    B_0
\end{bmatrix} = \begin{bmatrix}
    \tilde{X}_0 & -\tilde{Y}_0 \\
    -\tilde{N}_0 & \tilde{M}_0
\end{bmatrix} \begin{bmatrix}
    M_1 \\
    N_1
\end{bmatrix}
\]
where $\tilde{N}_0, \tilde{M}_0, M_1, N_1$ are LCF and RCF of $P_0$ and $P_1$ respectively, i.e. $P_0 = \tilde{M}_0^{-1}\tilde{N}_0$, and $P_1 = N_1M_1^{-1}$. $\tilde{X}_0, \tilde{Y}_0$ satisfy the Bezout identity $\tilde{X}_0\tilde{M}_0 - \tilde{Y}_0\tilde{N}_0 = I$. Then $A_0$ and $B_0$ are

$$
A_0 = \begin{bmatrix}
-2.121 & -0.9863 & 1.294 & -0.6883 & 0.08665 \\
-1.322 & -4.252 & -5.359 & 9.974 & 1.107 \\
0 & 0.2078 & -0.4181 & -0.9472 & 1.142 \\
0 & 0 & 1.811 & -3.209 & -0.6805 \\
0 & 0 & 0 & 5.878 & 1
\end{bmatrix}
$$

and

$$
B_0 = \begin{bmatrix}
-3.001 & 2.113 & 0.04866 & -1.343 & 1.373 \\
6.154 & -5.444 & 1.908 & 6.715 & -0.6702 \\
-2.308 & 1.014 & -2.51 & -3.48 & 0.7887 \\
2.364 & -2.724 & 0.3689 & 0.9543 & 0.2113 \\
0 & 0 & -0.366 & 1.366 & 0 \\
0 & 0 & -1 & -1 & 0 \\
0 & 0 & -0.366 & 1.366 & 0
\end{bmatrix}
$$

Then the plant $P$ that we are going to stabilize in this example is given as

$$
P = \begin{bmatrix}
-3.422 & -5.635 & -2.028 & -3.194 & 1.048 \\
2.443 & -1.911 & -0.07678 & 5.469 & 1.889 \\
1.471 & -1.859 & -2.643 & 0.3333 & 1.115 \\
3.541 & 2.323 & 0.3333 & 1.976 & 0.2989 \\
0 & 0 & -0.2588 & 0.9659 & 0 \\
0 & 0 & -0.7071 & -0.7071 & 0 \\
0 & 0 & -0.2588 & 0.9659 & 0
\end{bmatrix}
$$

The system $P$ is unstable with the following poles $[0.0799 \pm 3.7170i, -1.5555, -4.6043]$. The StSP as defined in (4.4.4) is to find, for some unimodular transfer function $U \in \mathbb{H}_{\infty}$, a stable controller $C_1$ such that

$$A_0 + C_1B_0 = U.$$

The plant $P$ is strictly proper and minimum phase and so is $B_0$. Using the technique which is developed in this section $C_1$ can be found as follows. From equation (4.4.6) we
have

\[ U - A_0 = C_i B_0 \]
\[ = C_i T_1^{-1} T_0 B_0 \]

where \( T_0 = (s + \gamma)I \). Then \( T_0 B_0 \) is computed for any \( \gamma > 0 \), say \( \gamma = 2 \), and is found to be

\[
T_0 B_0 = \begin{bmatrix}
-3.001 & 2.113 & 0.04866 & -1.343 & 1.373 \\
6.154 & -5.444 & 1.908 & 6.715 & -0.6702 \\
-2.308 & 1.014 & -2.51 & -3.48 & 0.7887 \\
2.364 & -2.724 & 0.3689 & 0.9543 & 0.2113 \\
4.073 & -4.092 & 0.6906 & 5.309 & 0 \\
-0.0574 & 1.71 & 0.1409 & 0.5258 & -1 \\
4.073 & -4.092 & 0.6906 & 5.309 & 0
\end{bmatrix}
\]

which is proper and has the following stable left inverse

\[
R_0 = \begin{bmatrix}
-3.413 & 32.38 & 12.78 & 3.676 & -1.225 & 0 \\
0.9352 & -15.25 & -5.851 & -1.742 & 1.09 & 0 \\
0 & 25.57 & 6.665 & 4.775 & 0.5571 & 0 \\
0 & 0 & 1.795 & 0 & -1 & 0
\end{bmatrix}
\]

Then \( C_i T_1^{-1} \) is given by \( C_i T_1^{-1} = (U - A_0)R_0 \), for \( U = 1 \), (see Remark 4.5) as

\[
C_i T_1^{-1} = \begin{bmatrix}
-2.121 & -0.9863 & 1.294 & -0.6883 & 0 & 0 & 0.1555 & 0 & -0.69665 & 0 \\
-1.322 & -4.292 & -5.369 & 9.974 & 0 & 0 & 1.986 & 0 & -1.107 & 0 \\
0 & 0.2078 & -0.4181 & -0.5472 & 0 & 0 & 2.061 & 0 & -1.142 & 0 \\
0 & 0 & 1.811 & -3.309 & 0 & 0 & -1.221 & 0 & 0.6805 & 0 \\
0 & 0 & 0 & 0 & -3.413 & 32.38 & 12.78 & 3.676 & -1.225 & 0 \\
0 & 0 & 0 & 0 & 0.9352 & -15.25 & -5.851 & -1.742 & 1.09 & 0 \\
0 & 0 & 0 & 0 & 0 & 25.57 & 6.665 & 4.775 & 0.5571 & 0 \\
0 & 0 & 0 & 0 & -5.878 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The stable controller \( C_i \) is recovered and is given as
The minimal realization of $C_i$ is given as
\[
\begin{bmatrix}
-4.783 & -5.432 & 0.4707 & 3.353 & -3.347 & 9.112 & -0.3962 & -1.011 & 0 \\
0.04308 & -1.507 & 0.1284 & -0.4798 & 7.091 & 2.513 & 0.9524 & -0.05665 & 0 \\
0 & 0.09515 & -0.5999 & 7.698 & -39.33 & -7.675 & -4.844 & 1.332 & 0 \\
0 & 0 & 1.16 & 0.07912 & -11.77 & -2.283 & -2.835 & -0.08885 & 0 \\
0 & 0 & 2.703 & -12.49 & -2.367 & -1.15 & 1.071 & 0 \\
0 & 0 & 0 & 12.63 & -0.698 & 2.336 & 1.431 & 0 \\
0 & 0 & 0 & 0 & 0 & 14.68 & 0 & -4 & 0
\end{bmatrix}
\] and has the following eigenvalues $[-1.0000, -3.0000, -3.0000, -4.0000, -4.0000, -5.0000].$

The eigenvalues of the closed-loop system $T(P, C_i)$ are given as $[-1.0000, -1.0000, -2.0000, -2.9996, -3.0002, -3.0002, -3.9999 \pm 0.0002i, -4.0002, -5.0000]$ and a minimal realization gives the following closed-loop eigenvalues $[-1.0000, -2.0000, -3.0000, -4.0000].$

Example 4.2 This example demonstrates the strong stabilization of minimum phase system when the given system is fat, i.e. the number of inputs are more than the number of outputs. The plant that we will use in this example is an auxiliary plant formed, as in example 4.1, from the following two fat plants which are reported in [19].

\[
R_0 \triangleq \begin{bmatrix}
0 & 1 & 1 & 0 & 1 \\
-1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
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The auxiliary plant is computed as before from its RCF \((A_0, B_0)\) and is found to be

\[
P = \begin{bmatrix}
-1.665 & -4.775 & -1.963 & 3.397 & 0.2244 & -0.1452 & 0.2244 \\
0.1862 & -0.9541 & -1.057 & -3.281 & -1.325 & 0.7574 & -1.325 \\
0 & 4.377 & -0.5646 & -4.743 & -1.716 & 1.873 & -1.716 \\
0 & 0 & 0.9374 & -2.816 & 0 & -0.8041 & 0 \\
0 & 0 & 0 & 1.244 & 0 & 0 & 0
\end{bmatrix}
\]

Since the system is fat we will use equation (4.4.1) to solve the StSP. We therefore need to find first the LCF of \(P, (\tilde{A}_0, \tilde{B}_0)\) and they are given as

\[
\tilde{A}_0 = \begin{bmatrix}
-1.665 & 4.775 & 1.968 & -37.3 & -38.55 \\
-0.1862 & -0.9541 & -1.057 & 11.97 & 17.34 \\
0 & 4.377 & -0.5646 & -23.45 & -21.27 \\
0 & 0 & 0.9374 & -10.82 & -9.097 \\
0 & 0 & 0 & 0.8794 & 1
\end{bmatrix}
\]

and

\[
\tilde{B}_0 = \begin{bmatrix}
-1.665 & 4.775 & 1.968 & -37.3 & -0.3173 & 0.2053 & -0.3173 \\
-0.1862 & -0.9541 & -1.057 & 11.97 & -1.874 & 1.071 & -1.874 \\
0 & 4.377 & -0.5646 & -23.45 & -2.426 & 2.649 & -2.426 \\
0 & 0 & 0.9374 & -10.82 & 0 & -1.137 & 0 \\
0 & 0 & 0 & 0.8794 & 0 & 0 & 0
\end{bmatrix}
\]

Then the StSP as defined in (4.4.1) is to find, for some unimodular transfer function \(U \in RH_{\infty}\) a stable controller \(C_r\) such that

\[\tilde{A}_0 + \tilde{B}_0 C_r = U.\]

The plant \(P\) is strictly proper and minimum phase and so is \(\tilde{B}_0.\) Using the technique which is developed in this section, \(C_r\) can be found as follows. From equation (4.4.5) we
have

\[ U - \tilde{A}_0 = \tilde{B}_0 C_r \]
\[ = \tilde{B}_0 T_1 T_1^{-1} C_r \]

where \( T_1 = (s + \gamma)I \). Then \( \tilde{B}_0 T_1 \) is computed for \( \gamma = 2 \) and is found to be

\[
\begin{bmatrix}
-1.559 & 2.293 & 5.051 \\ 0.476 & -2.106 & -17.36 \\ 0 & 0.476 & -8.335 \\ 0 & 0 & 7.797 \\
-2.454 & 2.414 & -2.454 \\ -1.258 & 0.4541 & -1.258 \\ -0.2565 & 1.411 & -0.2565 \\ -1.258 & -0.2565 & 0 \\
0 & 0 & 0 \\ 0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

which is proper and has the following stable right inverse

\[
\begin{bmatrix}
-3.156 & -0.1268 & -3.733 & 1.543 \\ 0.4043 & -0.9611 & 69.67 & -1.848 \\ -0.3265 & 0.182 & -7.883 & 1.492 \\ 0 & 0 & 0 & 0 \\
0 & 0 & -0.3379 & -7.799 & -1 \\
0 & 0.08961 & -29.37 & 0
\end{bmatrix}
\]

Then \( T_1^{-1} C_r \) is given by \( T_1^{-1} C_r = R_r \left(U - \tilde{A}_0\right) \), for \( U = 1 \), as

\[
\begin{bmatrix}
-3.156 & -0.1268 & -3.733 & 0 & 0 & 0 & 0 & -1.357 \\ 0.4043 & -0.9611 & 69.67 & 0 & 0 & 0 & 0 & 1.625 \\ -0.3265 & 0.182 & -7.883 & 0 & 0 & 0 & 0 & -1.312 \\ 0 & 0 & 0 & 0 & -1.665 & 4.775 & 1.968 & -37.3 & -38.55 \\ 0 & 0 & 0 & -0.1862 & -0.9541 & -1.057 & 11.97 & 17.34 \\ 0 & 0 & 0 & 4.377 & -0.5646 & -23.45 & -21.27 \\ 0 & 0 & 0 & 0 & 0.9374 & -10.82 & -9.979 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.3379 & -7.789 & 0 & 0 & 0 & 0.8794 \\
0 & 0.08961 & -29.37 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
The stable controller \( C_r \) is recovered and its minimal realization is given as

\[
C_r = \begin{bmatrix}
-2.428 & 0.8172 & 6.519 & 13.4 & 23.53 & 57.16 & 17.59 \\
0.3251 & -1.94 & 0.6976 & -2.65 & -4.64 & 1.416 & -2.639 \\
-0.01215 & 0.4224 & -0.9697 & -0.3716 & 3.404 & -14.7 & 10.77 \\
-0.01976 & -0.2597 & -4.54 & -10.01 & -16.59 & -4.615 & -19.45 \\
0 & 0 & -0.4185 & 1.397 & -2.198 & 6.441 & -2.419 \\
0 & 0 & 0.6305 & 0.9275 & 2.002 & -6.449 & 1.913 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3.176 & -22.31 & -8 \\
0 & 0 & 0 & 0 & 0 & 0.3862 & -183.5 & 0 \\
\end{bmatrix}
\]

\( C_r \) has the following eigenvalues \([-1.0000, -3.0000, -4.0000, -5.0000, -5.0000, -6.0000] \).

The eigenvalues of the closed-loop system \( T(P, C_r) \) are given as \([-1.0000, -1.0000, -2.0000, -3.0000, -4.0000, -4.9999 \pm 0.0003i, -5.0003, -6.0000, -6.0000]\) and a minimal realization gives the following closed-loop eigenvalues \([-1.0000, -2.0000, -5.0000, -6.0000]\).

In this section, a solution to the strong stabilization problem of multivariable linear and time-invariant minimum phase systems was given. The method was demonstrated by two examples.

In the next section we will discuss the strong stabilization problem of non-minimum phase systems.

### 4.5 Strong Stabilization Via Unimodular \( H_\infty \) Interpolation

In this section we will formulate the strong stabilization problem of non-minimum phase systems as an interpolation problem. We will discuss two cases: the given system is strictly proper in Subsection 4.5.1, and the given system is bi-proper (i.e. has full rank at infinity) in Subsection 4.5.2.

In order to formulate the StSP problem as an interpolation problem in \( H_\infty \), consider the following lemma which appeared as a statement in [96].

**Lemma 4.13** If \( R \in \Re H_\infty \) and \( ||R||_\infty < 1 \), then \( R + I \) is unimodular in \( \Re H_\infty \).
Proof. For each \( s \in \mathbb{C}^+ \), the spectrum of the matrix \((R(s) + I)\) consists of \(\{1 + \lambda_i(s), i = 1, \ldots, n\}\), where \(\lambda_1(s), \ldots, \lambda_n(s)\) are the eigenvalues of \(R(s)\). Since \(\|R\|_\infty < 1\), it follows that the \(\infty\)–norm of the matrix \(R(s)\) is less than one for all \( s \in \mathbb{C}^+ \). Hence \(|\lambda_i(s)| \leq \|R\|_\infty < 1 \; \forall \; i, \; \forall \; s \in \mathbb{C}^+\). As a result \(1 + \lambda_i(s) \neq 0 \; \forall \; i, \; \forall \; s \in \mathbb{C}^+\), so \(|R(s) + I| = \Pi(1 + \lambda_i(s)) \neq 0 \; \forall \; s \in \mathbb{C}^+\). This shows that \(|R + I|\) is a unit of \(\mathcal{H}_\infty\), so that \(R + I\) is unimodular in \(\mathcal{B}_\infty\).

4.5.1 Strictly Proper Non-minimum Phase Systems

In this subsection we will formulate the StSP for strictly proper systems as an interpolation problem as follows.

Given \(G(s) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in \mathbb{R}^{n \times m}\) such that \(CB\) has full rank. The LCF of \(G(s) = \tilde{M}^{-1}\tilde{N}\), is given in state space representation as

\[
\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} = \begin{bmatrix} A + HC & B \\ C & 0 \end{bmatrix} \in \mathbb{R}^{n \times (r+m)}. \tag{4.5.1}
\]

The condition for strong stabilization as stated in Section 4.3 is, for some unimodular \(U\) and stable \(C_r\),

\[
U - \tilde{M} = \tilde{N}C_r. \tag{4.5.2}
\]

We now assume \(m = r\). The solution for the case \(m \neq r\) can be solved by adding zero rows or columns. Let \(T_k\) be the sequence defined in (4.2.12) with \(k = 1\), then (4.5.2) can be written as

\[
T_k(U - \tilde{M}) = T_k\tilde{N}C_r = \tilde{N}C_r. \tag{4.5.3}
\]

\(\tilde{N}\) has the following state-space representation

\[
\tilde{N} = \begin{bmatrix} A + HC & B \\ C(A + \gamma I) & CB \end{bmatrix}
\]

where \(\gamma\) is a free parameter greater than zero. \(\tilde{N}\) has full rank at infinity since \(CB\) has full rank by assumption. Let \(\tilde{N}\) be factorized into an inner-outer factorization as \(\tilde{N} = N_lN_o\),
where \( N \) is unimodular. Then (4.5.3) becomes
\[
T_{\delta}(U - \hat{M}) = N_{\delta}C
\]  
(4.5.4)

Now define \( \phi(s) \) to be the determinant of \( N \), i.e.
\[
\phi(s) = \det(N).
\]  
(4.5.5)

Then \( \phi(s) \) is an inner scalar function whose zeros are the right half-plane zeros of \( N \) and \( \phi N_{\delta}^* \) is also an inner function \([15]\). \( \phi(s)N_{\delta}^* \) can be written as \( \phi(s)N_{\delta}^* = \text{adj}(N_{\delta}) \). Multiply both sides of (4.5.4) by \( \text{adj}(N_{\delta}) \)
\[
\text{adj}(N_{\delta})T_{\delta}(U - \hat{M}) = \phi(s)N_{\delta}C
\]

(4.5.6)
\[
= \phi(s)\hat{C}
\]  
(4.5.7)

where \( \hat{C} = N_{\delta}C \). Since \( \hat{M} \) is bi-proper, by definition, \( \text{adj}(N_{\delta})T_{\delta}\hat{M} \) is improper. To overcome this problem we let
\[
\hat{C} = \hat{N}_{\delta}C \quad \text{and should satisfy the condition } \|R\|_{\infty} < 1 \text{ according to Lemma 4.13.}
\]
Then equation (4.5.7) becomes
\[
\text{adj}(N_{\delta})T_{\delta}R - \text{adj}(N_{\delta})T_{\delta}(\hat{M} - I) = \phi(s)\hat{C}
\]  
(4.5.8)

where \( R \in \mathcal{RH}_{\infty}^{m \times r} \) and should satisfy the condition \( \|R\|_{\infty} < 1 \) according to Lemma 4.13. Then equation (4.5.7) becomes
\[
\text{adj}(N_{\delta})T_{\delta}R - \text{adj}(N_{\delta})T_{\delta}(\hat{M} - I) = \phi(s)\hat{C}
\]

(4.5.9)

From (4.5.1), \( (\hat{M} - I) \) is strictly proper, so \( \text{adj}(N_{\delta})T_{\delta}(\hat{M} - I) \) is proper for \( k = 1 \). Define \( \Phi(s) := \text{adj}(N_{\delta})T_{\delta}R = \phi(s)N_{\delta}^*T_{\delta}R \) and \( \hat{H}(s) := \text{adj}(N_{\delta})T_{\delta}(\hat{M} - I) = \phi(s)N_{\delta}^*T_{\delta}(\hat{M} - I) \).

Then rewrite (4.5.9) as
\[
\Phi - \hat{H} = \phi\hat{C}.
\]  
(4.5.10)

So we want a stable \( \Phi(s) \) and \( \hat{C}(s) \) such that equation (4.5.10) is satisfied. This problem can be considered as an interpolation problem in \( \mathcal{RH}_{\infty} \). The next lemmas shows that \( \hat{C} \) is stable, hence \( C \), if and only if \( \Phi(\xi_k) = \hat{H}(\xi_k) \) where \( \xi_k \) are the RHP zeros of \( \phi \). In the case \( \hat{H} \) is not square, adding some zero row/column vectors to make it square will not affect the problem, \([15]\).

**Lemma 4.14** \([15]\) **Suppose that** \( \hat{H} \in \mathcal{RH}_{\infty}^{m \times m} \) **and all the zeros** \( \{\xi_k, k = 1, \ldots, n\} \) **of** \( \phi(s) \) **are distinct. Then** \( \hat{C}(s) \in \mathcal{RH}_{\infty}^{m \times m} \) **if and only if** \( \Phi(s) \in \mathcal{RH}_{\infty}^{m \times m} \) **and**
\[
\Phi(\xi_k) = \hat{H}(\xi_k) \quad k = 1, \ldots, n.
\]  
(4.5.11)
Proof. If $\hat{C} \in \mathcal{RH}_\infty$, then by (4.5.10), $\Phi \in \mathcal{RH}_\infty$, and from the definition of $\phi(s)$ in (4.5.5), we have (4.5.11). Conversely, if $\Phi \in \mathcal{RH}_\infty$, then by (4.5.10) $\Phi \hat{C} \in \mathcal{RH}_\infty$. The only possibility that $\hat{C} \notin \mathcal{RH}_\infty$ is that $\hat{C}$ has some RHP poles cancelled by the zeros of $\phi$. In particular, suppose $\hat{C}$ has a pole at $z_k$, then $\phi \hat{C}_{|s=z_k} \neq 0$ which contradicts (4.5.11). Therefore $\hat{C} \in \mathcal{RH}_\infty$.

Corollary 4.15 Suppose a scalar function $\phi(s)$ and a matrix transfer function $T(s)$ are both of distinct poles. Let $z_o$ be a zero of $\phi(s)$. If

$$\phi(s)T(s)_{|s=z_o} = 0,$$

then $z_o$ is not a pole of $T(s)$.

Proof. If $z_o$ is a pole of $T(s)$, then at least one entry of $T$ will not be zero, since $T(s)$ has only distinct poles. Hence $z_o$ must not be a pole of $T(s)$.

In the case of $\phi(s)$ has zeros with multiplicities, then we may use the result of the following lemma.

Lemma 4.16 [17] Suppose $\hat{H} \in \mathcal{RH}_\infty^{m \times m}$ and $\phi$ is an inner function having zeros $\xi_k$ with multiplicities $m_k, k = 1, 2, \ldots , n$. Then $\hat{C}(s)$ in (4.5.10) is stable if and only if $\Phi(s) \in \mathcal{RH}_\infty^{m \times m}$ and

$$\Phi^{(l_k-1)}(\xi_k) = \hat{H}^{(l_k-1)}(\xi_k), \quad k = 1, 2, \ldots , n \quad \text{and} \quad \hat{H}^{(l_k-1)}(\xi_k) = (\hat{H}_{st})^{(l_k-1)}(s)|_{s=\xi_k} = l_k = 1, 2, \ldots , m_k.$$

So the Nevanlinna-Pick algorithm described in Section 4.2.3.1 can be used to compute $\Phi(s)$ and $\hat{C}(s)$ in equation (4.5.10).

Before we summarize this subsection in an algorithm and demonstrate it with an example, we have to make the following remarks on the norm of $\hat{H}$ and the norm and stability of $R$. 
Remark 4.6 In the interpolation problem we ask that $\|\Phi\|_\infty < 1$ which is needed to guarantee the unimodularity of $U$ assumed by (4.5.8). One necessary condition for $\|\Phi\|_\infty < 1$ is that $\sigma_{\max}(H) < 1$ at the interpolation points. It is obvious that if $\|\hat{H}\|_\infty < 1$, then this necessary condition is satisfied. In this particular case, the unimodularity of $U$ assumed by (4.5.8) will be guaranteed. But

$$\|\hat{H}\|_\infty = \|\phi(s)N_i^*\hat{T}_k(\hat{M} - I)\|_\infty = \|\hat{T}_k(\hat{M} - I)\|_\infty = \|\hat{M}\|_\infty$$

where $\hat{M}$ has the following state-space representation

$$\hat{T}_k(\hat{M} - I) \triangleq \hat{M} = \begin{bmatrix} A + HC & H \\ C(A + \gamma I) & CH \end{bmatrix}$$

(4.5.13)

So the problem is reduced to choosing suitable $\gamma > 0$ and $H$ such that $\hat{M}$ satisfies the $H_\infty$—norm restriction. If the required $H_\infty$—norm is not satisfied then we may use $\hat{T}_k = (\alpha s + \gamma I)$ and the result of Corollary 4.6 to achieve the required norm.

In the following the assumption that $R(s)$ is stable and has $H_\infty$—norm less than one will be discussed. The norm of $R$ will be less than one if the condition in Remark 4.6 is satisfied. That is because $\Phi(s) = \phi(s)N_i^*\hat{T}_kR$, and $\|\Phi(s)\|_\infty < 1$ by computation, and

$$\|\hat{T}_k^{-1}(s)\|_\infty = \frac{1}{\gamma}, \text{ hence for } \gamma > \|R(s)\|_\infty, \|R(s)\|_\infty = \|\hat{T}_k^{-1}(\phi(s)N_i^*) \Phi(s)\|_\infty < 1.$$

The stability of $R$ comes from the following discussion. First some remarks to clarify the situation.

Remark 4.7 $\phi N_i^*$ may not be minimum phase. For example let $N_i$ be given as

$$N_i = \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s-2}{s+2} \end{bmatrix}.$$  

(4.5.14)

Then $\phi N_i^*$ is given as

$$\phi N_i^* = \begin{bmatrix} \frac{s-2}{s+2} & 0 \\ 0 & \frac{s-1}{s+1} \end{bmatrix}.$$
Remark 4.8 \( \phi N_t^* \) and \( N_t \) (also \( \phi \)) share the same zeros.

\[
N_t^* N_t = I
\]
\[
\phi N_t^* N_t = \phi I.
\]

For \( s_o \) being a zero of \( \phi(s) \), (also of \( N_t(s_o) \) by definition), assume that \( N_t(s_o) \) has rank \( k \), \( 0 < k < \text{col}(N_t) \). Let \( \xi \notin \text{Null}(N_t(s_o)) \), i.e. \( N_t(s_o)\xi \neq 0 \). However

\[
(\phi N_t^*)(s_o) \cdot N_t(s_o) \cdot \xi = \phi(s_o)\xi = 0
\]

thus \( \phi N_t^* \) loses rank at \( s_o \). Therefore, \( s_o \) is a zero of \( \phi N_t^* \). Now let \( s_o \) be a zero of \( \phi N_t^* \). If \( s_o \) is not a zero of \( N_t \), then \( N_t(s_o) \) is full rank and square (by assumption \( m = r \)). (It is obvious that \( s_o \) is not a zero of \( \phi(s) \) either). In the equation

\[
(\phi N_t^*)(s_o) \cdot N_t(s_o) \cdot \xi = \phi(s_o)\xi
\]

for a nonzero vector \( \xi \), the RHS is nonzero and \( N_t(s_o)\xi \) can be any nonzero vector, hence it is a contradiction to the assumption \( s_o \) being a zero of \( \phi N_t^* \). Thus \( s_o \) must be a zero of \( N_t(s) \). If \( k = 0 \), i.e. \( \text{rank}(N_t(s_o)) = 0 \), then \( N_t(s_o) = 0 \) and this case will not occur unless each non-zero entry of \( N_t \) has \( s_o \) as a zero. In such case \( \phi(s) \) will have \( s_o \) with multiplicity greater than one considering the dimension of \( N_t > 1 \). Then \( \phi N_t^* \) has a zero-pole cancellation at \( s_o \). But due to the multiplicity of \( s_o \) in \( \phi(s) \), \( \phi N_t^* = 0 \).

However, in terms of the multiplicity of zeros, they are not necessarily the same in \( N_t \) and \( \phi N_t^* \). For instance let \( N_t(s) = \text{diag}\{\frac{s-1}{s+1}, \frac{s-1}{s+1}, \frac{s-1}{s+1}\} \). Then \( \phi(s) = \left(\frac{s-1}{s+1}\right)^3 \) and \( \phi N_t^*(s) = \text{diag}\{\left(\frac{s-1}{s+1}\right)^2, \left(\frac{s-1}{s+1}\right)^2, \left(\frac{s-1}{s+1}\right)^2\} \).

Remark 4.9 From equation (4.5.10) we have

\[
\Phi(s) = \phi N_t^* T_b(\tilde{M} - I) + \phi \hat{C}.
\]

where \( \Phi(s) = \phi N_t^* T_b R \). Multiply both sides with \( \phi^{-1} N_t \) we get

\[
\phi^{-1} N_t \Phi(s) = T_b(\tilde{M} - I) + N_t \hat{C}.
\]

Since the RHS is stable, so is the LHS. But \( \Phi(s) = \phi N_t^* T_b R \). Then

\[
\phi^{-1} N_t \Phi(s) = \phi^{-1} N_t \phi N_t^* T_b R
\]

\[
= T_b R
\]
so \( T_0R \) is stable.

We conclude that \( R \) is stable, considering the structure of \( T_0 \) and the suggestion made in Remark 4.6 on choosing the constants in this sequence.

**Remark 4.10** The assumption that \( CB \) should have full rank can be relaxed only if \( CH \) in (4.5.13) is zero or does not have full rank. If this is the case, then we can take \( k = 2 \) in the sequence \( T_0 \), or even larger.

The following algorithm summarizes the procedure described in this subsection to design a stable controller for a strictly proper and non-minimum phase MIMO system via unimodular interpolation in \( \mathcal{RH}_\infty \).

**Algorithm 4.4:**

**Step 0 (Initialization):** \( G(s) \) is a given strictly proper and non-minimum phase plant.

**Step 1:** Use the parity interlacing property\(^1\) of Theorem 3.1 to check if the given plant is strongly stabilizable. If so continue, otherwise stop.

**Step 2:** Find LCF for \( G(s) \) so that \( G = \tilde{M}^{-1}\tilde{N} \).

**Step 3:** Use Algorithm 4.1 to find a sequence \( T_0 \) such that \( \tilde{N} = T_0\tilde{N} \) is bi-proper.

**Step 4:** Find an inner-outer factorization of \( \tilde{N} \), i.e. \( \tilde{N} = N_0N_e \).

**Step 5:** Define a scalar inner function \( \phi(s) \) as

\[
\phi(s) = \frac{(s - s_1)(s - s_2)\cdots(s - s_k)}{(s + s_1)(s + s_2)\cdots(s + s_k)}
\]

where \( s_i, i = 1,\ldots, k \) are the RHP zeros of \( N_0 \).

**Step 6:** Compute \( \hat{H} = \text{adj}(N_0)T_0(\tilde{M} - I) \).

**Step 7:** Compute \( \|\hat{H}\|_\infty \). If \( \|\hat{H}\|_\infty < 1 \) continue, else go to Step 3 and change the constants \( \alpha \) and \( \gamma \) of the sequence\(^2\) \( T_0 \).

\(^1\)This step is not implemented in this work.

\(^2\)See Remark 4.6.
Step 8: Solve the interpolation\(^3\) problem in equation (4.5.11) using Algorithm 4.2.

Step 9: Compute \(\hat{C}\) from equation (4.5.10), hence the stable controller is given as \(C = N_o^{-1}\hat{C}\).

The following example illustrates the ideas described in this subsection.

Example 4.3 Given a strictly proper system

\[
G = \begin{bmatrix}
\frac{1}{\sqrt{2}(s+9)(s+3)} & \frac{1}{s+1} & 0 \\
\frac{1}{\sqrt{2}s+1} & 0 & \frac{s-2}{(s+2)^2}
\end{bmatrix}
\]

\[
= \hat{M}^{-1}\hat{N}
\]

where

\[
\hat{M} = \begin{bmatrix}
\frac{s-0.1}{s+1} & \frac{s-3}{(s+9)(s+3)} & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \hat{N} = \begin{bmatrix}
\frac{1}{\sqrt{2}(s+1)]^2} & 0 \\
\frac{1}{\sqrt{2}s+1} & 0 \\
0 & \frac{s-2}{(s+2)^2}
\end{bmatrix}
\]

Since \(\hat{N}\) is strictly proper, we choose \(T_i = (0.1s + 0.9)I\), and find the inner-outer factorization of \(T\hat{N}\) as

\[
T\hat{N} = N_iN_o
\]

where \(N_i\) and \(N_o\) are given as

\[
N_i = \begin{bmatrix}
\frac{1}{\sqrt{2}(s+1)} & 0 \\
\frac{1}{\sqrt{2}(s+1)} & 0 \\
0 & \frac{(s-2)}{(s+2)}
\end{bmatrix}, \quad N_o = \begin{bmatrix}
\frac{s+9}{10(s+1)} & 0 \\
0 & \frac{s+9}{10(s+2)}
\end{bmatrix}
\]

Define a scalar inner function \(\phi(s)\) as

\[
\phi(s) = \frac{(s-1)(s-2)}{(s+1)(s+2)}
\]

\(^3\)A MATHEMATICA program for this step is given in Appendix B.
\( \phi(s) \) has the same RHP zeros as \( N_t \). Then \( \phi(s)N^* \) is given as

\[
\phi(s)N^* = \begin{bmatrix}
\frac{1}{\sqrt{2}}(s-2) & \frac{1}{\sqrt{2}}(s-1)(s-2) \\
0 & (s-1)
\end{bmatrix}
\]

which is co-inner. The interpolation problem as derived before is to find \( \Phi \in \mathcal{H}_\infty \) with \( \|\Phi\|_\infty < 1 \) such that equation (4.5.11) is satisfied and \( C \) in equation (4.5.10) is stable.

The \( \hat{H} \) is computed as

\[
\hat{H} = \phi(s)N^*T(M-I)
\]

which has an \( \infty \)-norm of 0.7036 < 1. We want to find a stable \( \Phi(s) \) which will satisfy the interpolation condition. That is find a stable \( \Phi(s) \) such that

\[
\Phi(s) = \hat{H}(s)_{z=z_1}
\]

where \( z_1 \) are the zeros of \( \phi(s) \) given as \( z = 1, 2 \). Then

\[
\hat{H}(1) = \begin{bmatrix}
\frac{1}{6\sqrt{2}} & \frac{1}{60\sqrt{2}} & 0 \\
0 & 0 & 0
\end{bmatrix}
\text{ and } \hat{H}(2) = 0_{3\times2}
\]

A candidate \( \Phi(s) \) which satisfies the interpolation condition in (4.5.19) and makes \( C \) stable in equation (4.5.10) may be given as

\[
\Phi(s) = \begin{bmatrix}
\frac{-1.1(s-2)}{2\sqrt{2}(s+2)} & \frac{-1}{20\sqrt{2}(s+2)} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

which has an \( \infty \)-norm of 0.7786 < 1. Since \( \Phi(s) = \phi(s)N^*T R \), the StSP is solvable if we can recover a stable \( R \) from \( \Phi(s) \) with an \( \infty \)-norm less than one as stated in
Lemma 4.13. But this condition is satisfied in this example since \( \| \hat{H} \|_\infty < 1 \) (see Remark 4.6). To justify whether these requirements are satisfied or not, let us recover \( R(s) \) from \( \Phi(s) = \phi(s)N^cT_I R \). One candidate such \( R \) which satisfies the stability and norm requirements is

\[
R(s) = \begin{bmatrix}
-11 & 1 & 0 \\
2(s + 9) & 2(s + 9) & 0 \\
0 & 0 & 0 \\
s + 9y_{11} & 10 & 10 \\
s + 9y_{12} & s + 9y_{13} & s + 9y_{13}
\end{bmatrix}
\]

where \( y_{11}, y_{12} \) and \( y_{13} \) are any proper stable scalar functions that make \( R(s) \) stable and has \( \infty - norm \) less than one. One choice is \( y_{11} = y_{12} = y_{13} = 0 \). Then

\[
R(s) + I = \begin{bmatrix}
2s + 7 & 1 & 0 \\
2s + 18 & 2(s + 9) & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
(R(s) + I)^{-1} = \begin{bmatrix}
2s + 18 & -1 & 0 \\
2s + 7 & 2s + 7 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

which is stable. So \( (R + I) \) is unimodular. Now the required controller can be computed as

\[
\hat{C} = \phi^{-1}(\Phi - \hat{H})
\]

\[
\hat{C} = \begin{bmatrix}
0.44 \\
\frac{s + 1}{\sqrt{2}} \\
0
\end{bmatrix}
\]

But \( \hat{C} = N^cC \) which gives \( C \) as

\[
C = N^{-1}_c \hat{C}
\]

\[
C = \begin{bmatrix}
\frac{10(s + 1)}{s + 9} & 0 & \frac{0.44}{\sqrt{2}} & \frac{3}{20\sqrt{2}} & s + 3 \\
0 & \frac{10(s + 2)}{s + 9} & 0 & 0 & 0 \\
\frac{4.4(s + 1)}{\sqrt{2}} & \frac{(s + 1)^2}{s + 9} & -\frac{3}{2\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Now we may check the result by computing the value of \( \tilde{M} + \tilde{N}C \) and see if it is unimodular.

\[
U = \tilde{M} + \tilde{N}C
\]
4.4 (a + 1) \quad \frac{3}{2} (s + 3)(s + 9)
\begin{align*}
&= \frac{s^2 + 6.7s + 1.3}{(s + 1)(s + 9)} \quad \frac{s - 9}{s + 9} \\
&= \frac{2.2}{s^2 + 11.2s + 26.25} \quad \frac{0.55(s - 9)(s + 3)}{(s + 9)(s + 3)}
\end{align*}

The determinant of $U$ is
\[
\text{det}(U) = \frac{s^2 + 6.7s + 1.3}{(s + 1)(s + 9)} \quad \frac{s - 9}{s + 9} \\
\frac{(s + 1)(s + 9)}{s + 9} \quad \frac{4(s + 9)(s + 3)}{(s + 9)(s + 3)}
\begin{align*}
&= \frac{s^2 + 6.7s + 1.3}{(s + 1)(s + 9)} \quad \frac{s - 9}{s + 9} \\
&= \frac{2.2}{s^2 + 11.2s + 26.25} \quad \frac{0.55(s - 9)(s + 3)}{(s + 9)(s + 3)}
\end{align*}
\]

which is a unit in $\mathbb{RH}_\infty$, so $U$ is unimodular in $\mathbb{RH}_\infty$ which means that the strong stabilization condition is satisfied.

In the next subsection we are going to formulate the StSP of proper non-minimum phase systems which have full rank at infinity as an interpolation in $\mathbb{RH}_\infty$. The same procedure discussed above will be followed with some modifications. A sufficient condition for the solution of the problem will be stated.

4.5.2 Proper Non-minimum Phase Systems

In this subsection we are going to formulate the StSP of proper non-minimum phase systems which have full rank at infinity as an interpolating problem in $\mathbb{RH}_\infty$. The same procedure discussed in the previous subsection will be followed with some modifications. A sufficient condition for the solution of the StSP problem will be stated.
Given $G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{m \times m}$ such that $D$ has full rank and $G(s)$ has the LCF $(\tilde{M}, \tilde{N})$ as defined in (4.5.1). Define the unimodular matrix $U$ as $U = \frac{1}{2}(R + T)$, where $T(s)$ is a unimodular matrix in $\mathcal{RH}_\infty$, and the IOF of $\tilde{N}$ as $\tilde{N} = N_t N_e$. Then the condition for the StSP from equation (4.5.2), $U - \tilde{M} = \tilde{N} C_r$, becomes

$$\frac{1}{2} R - \left( \frac{1}{2} T \right) = N_t N_e C_r. \quad (4.5.20)$$

Define an inner scalar function $\phi(s)$ whose zeros are the RHP zeros of $N_t$. Denote by $Z_\phi$ the set of zeros of $\phi(s)$. Multiply both sides of (4.5.20) by the inner matrix function $\phi(N_t^*)$ to get

$$\frac{1}{2} \phi N_t^* R - \phi N_t^* \left( \frac{1}{2} T \right) = \phi N_e C_r$$

$$\Phi(s) - H(s) = \phi \tilde{C} \quad (4.5.21)$$

where $\Phi(s) = \frac{1}{2} \phi N_t^* R$ and $H(s) = \phi N_t^* \left( \frac{1}{2} T \right)$. Define $\mu_\phi$ as

$$\mu_\phi = \|H\|_\infty = \left\| \phi N_t^* \left( \frac{1}{2} T \right) \right\|_\infty = \left\| \frac{1}{2} T \right\|_\infty. \quad (4.5.22)$$

Then define for some real constant $\varepsilon_0 > 0$,

$$\Psi(s) = \frac{1}{\mu_\phi + \varepsilon_0} \phi(s) \quad (4.5.23)$$

and

$$\bar{H}(s) = \frac{1}{\mu_\phi + \varepsilon_0} H(s). \quad (4.5.24)$$

Then the $\mathcal{H}_\infty$-interpolation problem is to find $\Psi(s) \in \mathcal{RH}_\infty$ such that

$$\Psi(s_i) = \bar{H}(s_i) \quad \forall \ s_i \in Z_\phi \quad (4.5.25)$$

with

$$\|\Psi\|_\infty \leq 1. \quad (4.5.26)$$

Now let us find the condition on $T$ such that $U$ is unimodular. From (4.5.23)

$$\Phi(s) = (\mu_\phi + \varepsilon_0) \Psi(s)$$

$$\frac{1}{2} \phi N_t^* R = (\mu_\phi + \varepsilon_0) \Psi(s) \quad (4.5.27)$$
So \[ \frac{\lambda}{2} \| R \|_\infty \leq (\mu_0 + \epsilon_0) \| \Psi \|_\infty \leq (\mu_0 + \epsilon_0). \] But the condition for \( U \) to be unimodular is that \( \| RT^{-1} \|_\infty < 1. \) But \[ \| RT^{-1} \|_\infty \leq \| R \|_\infty \| T^{-1} \|_\infty \leq \frac{2}{\lambda} \left( \| \tilde{M} - \frac{\lambda}{2} T \|_\infty + \epsilon_0 \right) \| T^{-1} \|_\infty. \] (4.5.28)

So for the condition \( \| RT^{-1} \|_\infty < 1 \) to hold, we see from (4.5.28) that the following condition should be satisfied.

\[ \left( \| \tilde{M} - \frac{\lambda}{2} T \|_\infty + \epsilon_0 \right) \| T^{-1} \|_\infty \leq \frac{\lambda}{2}. \] (4.5.29)

The above results are summarized in the following theorem.

**Theorem 4.17** If there exist a unimodular \( T \in \mathcal{RH}_\infty \) and \( \lambda \geq 2 \) satisfying the following \( \infty \)-Norm bound

\[ \left( \| \tilde{M} - \frac{\lambda}{2} T \|_\infty + \epsilon_0 \right) \| T^{-1} \|_\infty \leq \frac{\lambda}{2}. \] (4.5.30)

then \( U = \frac{\lambda}{2} (R + T) \) is unimodular in \( \mathcal{RH}_\infty \). \( \square \)

**Remark 4.11** We add \( \epsilon_0 > 0 \) in (4.5.23) to satisfy the requirement in the Nevanlinna-Pick Algorithm that \( \| \tilde{H}(s) \| < 1 \). This requirement will simplify the computations, especially in the first iteration of the algorithm.

**Remark 4.12** When \( \lambda = 2 \) and \( \epsilon_0 = 0 \) inequality (4.5.30) becomes \( \| \tilde{M} - T \|_\infty \| T^{-1} \|_\infty < 1. \) If there were such a unimodular \( T \) which satisfied the above inequality, then \( \tilde{M} \) is also unimodular \([96]\), which is not the case here since the plant is unstable. Therefore, \( \lambda \) should be greater than 2.

**Remark 4.13** The problem which is left as an open problem is the existence of a unimodular transfer function \( T(s) \) that will satisfy the condition in Theorem 4.17. The lemmas and corollaries in the next section will clarify this point further.

The following algorithm summarizes the procedure described above to design a stable controller for a bi-proper and non-minimum phase MIMO system via unimodular interpolation in \( \mathcal{RH}_\infty \).
Algorithm 4.5:

**Step 0 (Initialization):** $G(s)$ is a given bi-proper, non-minimum phase and strongly stabilizable system.

**Step 1:** Find LCP for $G(s)$ so that $G = \tilde{M}^{-1} \tilde{N}$.

**Step 2:** Choose a unimodular matrix $T \in \mathbb{RH}_{\infty}$ and a constant $\lambda > 2$ such that the condition in Theorem 4.17 is satisfied.

**Step 3:** Compute $\mu_{\epsilon}$ according to equation (4.5.22).

**Step 4:** Choose a real constant $\epsilon_{\epsilon} > 0$, and compute $\tilde{H}(s)$ from equation (4.5.24).

**Step 5:** Solve the interpolation problem in equations (4.5.25-4.5.26) following the procedure outlined in Algorithm 4.4 and Algorithm 4.2.

**Step 6:** Compute $\tilde{C}$ from equation (4.5.21), hence the stable controller is given as $C = N_o^{-1} \tilde{C}$.

In the previous two subsections, the strong stabilization problem of multivariable linear time-invariant and non-minimum phase systems is formulated as an interpolation problem in $\mathbb{RH}_{\infty}$. The Nevanlinna-Pick algorithm was recommended to solve this interpolation problem. This algorithm involves symbolic computations which makes the problem difficult to solve especially for systems which have many zeros in the right half plane. To avoid the computational problem, new simple methods are needed to solve the StSP problem.

In the next two sections the StSP will be solved in a way such that the interpolation problem is avoided.

### 4.6 Strong Stabilization Via $H_\infty$ Optimization

In this section we are going to formulate the strong stabilization problem as an $H_\infty$-optimization problem. The solution of this $H_\infty$-optimization problem may lead to a
solution to the StSP. First we give the following corollaries and lemma.

**Corollary 4.18** If $g(s)$ is stable and $t(s)$ is a unit in $\mathcal{RH}_\infty$ such that $\|t^{-1}(s)g(s)\|_\infty \leq 1$, then $g(s) + t(s)$ is a unit in $\mathcal{RH}_\infty$.

**Proof.** From Lemma 4.13 it is shown that $x(s) + 1$ is a unit in $\mathcal{RH}_\infty$ for a stable function $x(s)$ with $\mathcal{H}_\infty$-norm less than 1. Therefore $t^{-1}(s)g(s) + 1$ is a unit in $\mathcal{RH}_\infty$. Since $t(s)$ itself is a unit, hence $g(s) + t(s) = t(s)[t^{-1}(s)g(s) + 1]$ is a unit in $\mathcal{RH}_\infty$. □

**Example 4.4** Let $g(s) = \frac{s - 10}{s + 1}$ and $t(s) = \frac{cs + d}{as + b}$, where $a, b, c, d$ are constants to be chosen such that $t(s)$ is a unit in $\mathcal{RH}_\infty$, and $\|t^{-1}(s)g(s)\|_\infty \leq 1$. Then

$$t^{-1}(s)g(s) = \frac{as + b}{cs + d} \cdot \frac{s - 10}{s + 1} = \frac{as^2 + (b - 10)s - 10b}{cs^2 + (c + d)s + d}$$

Now we want values for $a, b, c$ and $d$ such that $t(s)$ is a unit in $\mathcal{RH}_\infty$, and $\|t^{-1}(s)g(s)\|_\infty \leq 1$. For example, one choice is $a = 1, b = 11, c = 2$ and $d = 111$. Then $t(s)$ is a unit in $\mathcal{RH}_\infty$ with $\|t(s)\|_\infty \simeq 10.09$ and $\|t^{-1}(s)g(s)\|_\infty \simeq 0.991$, and $\|g(s)\|_\infty \simeq 10$. Therefore

$$g(s) + t(s) = \frac{s - 10}{s + 1} + \frac{2s + 111}{s + 11} = \frac{(s - 10)(s + 11) + (2s + 111)(s + 1)}{(s + 1)(s + 11)} = \frac{3s^2 + 114s + 1}{(s + 1)(s + 11)}$$

which is a unit in $\mathcal{RH}_\infty$.

**Corollary 4.19** If $g(s)$ is proper, stable and non-minimum phase function, then there exists a unit $t(s)$ in $\mathcal{RH}_\infty$ such that $g(s) + t(s)$ is also a unit in $\mathcal{RH}_\infty$.

**Proof.** The proof is based on finding $t(s)$ which is a unit and satisfy the norm condition $\|t^{-1}g\|_\infty < 1$. □
Example 4.5 Let \( g(s) = \frac{s - 2}{s + 5} \) and \( t(s) = \frac{s + a}{s + b} \), \( a, b > 0 \). Then

\[
g(s) + t(s) = \frac{s - 2 + s + a}{s + 5 + s + b} = \frac{(s - 2)(s + b) + (s + 5)(s + a)}{(s + 5)(s + b)} \]

\[
= \frac{s^2 + (b - 2)s - 2b + s^2 + (5 + a)s + 5a}{(s + 5)(s + b)}
\]

\[
= \frac{2s^2 + (b + 3 + a)s + 5a - 2b}{(s + 5)(s + b)}
\]

which will be a unit if \( a \) and \( b \) are chosen such that

\[5a - 2b > 0\]

An extension of the above two lemmas to the multivariable case will be stated next.

**Lemma 4.20** If \( V(s) \) is stable, then \( V(s) + P(s) \) is unimodular in \( \mathcal{RH}_\infty \) if \( P(s) \) is unimodular in \( \mathcal{RH}_\infty \) and \( \|P^{-1}V\|_\infty < 1 \).

**Proof.** Let \( U = V + P \), then

\[P^{-1}U = P^{-1}V + I.\]

If \( \|P^{-1}V\|_\infty < 1 \), then \( (P^{-1}V + I) \) is unimodular \( \mathcal{RH}_\infty \) by Lemma 4.13. So \( P^{-1}U \) is unimodular \( \mathcal{RH}_\infty \). Since \( P \) is unimodular, then \( U \) is unimodular. Hence \( V + P \) is unimodular in \( \mathcal{RH}_\infty \).

**Lemma 4.21** If \( V(s) \) is strictly proper and stable with \( \|V\|_\infty < 1 \), then there always exists a unimodular function \( P(s) \in \mathcal{RH}_\infty \) such that \( U(s) = V(s) + P(s) \) is unimodular in \( \mathcal{RH}_\infty \).

**Proof.** Let \( V(s) \) and \( P(s) \) be given in state space form as

\[
V(s) = \begin{bmatrix} A_v & B_v \\ C_v & 0 \end{bmatrix}, \quad P(s) = \begin{bmatrix} A_p & B_p \\ C_p & I \end{bmatrix}.
\]
Then
\[ U(s) = V(s) + P(s) = \begin{bmatrix} A_v & B_v \\ 0 & A_p \\ C_v & B_p \end{bmatrix}. \]

The condition for \( U(s) \) to be unimodular is that \( U^{-1}(s) \) is stable, i.e. the matrix
\[
\tilde{A} = \begin{bmatrix} A_v - B_v C_v & -B_v C_p \\ -B_p C_v & A_p - B_p C_p \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\]
is stable. Since \( \|V\|_{\infty} < 1 \), then \( (V(s) + I) \) is unimodular, hence \( (A_v - B_v C_v) \) is stable.

Define \( R = A_{21} Y + A_{22}^T \), where \( Y \) is the positive definite solution of the following Lyapunov equation
\[ A_{11} Y + Y A_{11}^T = -Q_1 \]
for some positive definite matrix \( Q_1 \). Define the positive definite matrix \( X \) as
\[ X = \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \]
then in the Lyapunov equation
\[ \tilde{A} X + X \tilde{A}^T = -C \]
\( \tilde{A} \) is stable if \( X > 0 \) and \( C > 0 \) \cite{5}. A candidate \( C > 0 \) is the following
\[ C = \begin{bmatrix} Q_1 & -R^T \\ -R & Q_2 + R Q_1^{-1} R^T \end{bmatrix} \]
where \( Q_2 > 0 \). This can be seen by multiplying \( C \) by \( T \) and \( T^T \) where \( T \) is given as
\[ T = \begin{bmatrix} I & 0 \\ R Q_1^{-1} & I \end{bmatrix}. \]
We then get
\[ T C T^T = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \]
which is positive definite since \( Q_1 \) and \( Q_2 \) are positive definite. Hence, for \( \tilde{A} \) being stable, we may request
\[ A_{22} + A_{22}^T = -(Q_2 + R Q_1^{-1} R^T). \]
That is, in terms of \( (A_p, B_p, C_p) \),

\[
(A_p - B_p C_p) + (A_p - B_p C_p)^T = -(Q_2 + R Q_1^{-1} R^T)
\]

or

\[
A_p + A_p^T = -(Q_2 + R Q_1^{-1} R^T - B_p C_p - C_p^T B_p^T).
\]

Hence, \( A_p \) will be stable if the RHS in the above equation is negative definite. For some choice of \( B_p, C_p, \) and \( Q_2 > 0 \), \( A_p \) can be made stable. For example, we may choose \( Q_2 \) as

\[
Q_2 = B_p C_p + C_p^T B_p^T + \alpha I
\]

for some \( \alpha > 0 \) such that \( Q_2 > 0 \).

**Corollary 4.22** For any strictly proper and stable transfer function \( V(s) \), there always exists a unimodular transfer function \( P(s) \) in \( \mathcal{W} \text{-} \mathcal{L}_\infty \) such that \( V(s) + P(s) \) is unimodular in \( \mathcal{RH}_\infty \).

**Proof.** Let \( \gamma > \|V\|_\infty \) and

\[
U(s) = V(s) + P(s).
\]

Multiply both sides by \( \frac{1}{\gamma} \),

\[
\frac{1}{\gamma}U(s) = \frac{1}{\gamma}V(s) + \frac{1}{\gamma}P(s).
\]

Since \( \frac{1}{\gamma}V \) is unimodular in \( \mathcal{RH}_\infty \), there always exists a unimodular transfer function \( Q(s) \) in \( \mathcal{RH}_\infty \) such that \( \frac{1}{\gamma}V(s) + Q(s) \) is also unimodular in \( \mathcal{RH}_\infty \). Therefore, \( \frac{1}{\gamma}U(s) \) and \( U(s) \) are unimodular in \( \mathcal{RH}_\infty \). Hence, for \( P(s) = \gamma Q, \) \( V(s) + P(s) \) is unimodular in \( \mathcal{RH}_\infty \).

Now the formulation of the StSP may be represented as an \( \mathcal{H}_\infty \)-optimization problem as follows. Given the RCF \( M \) and \( N \) of a plant \( G \in \mathcal{R} \mathcal{L}_\infty \) such that \( G = N M^{-1} \), then for some unimodular matrix \( U \in \mathcal{RH}_\infty \), we want to find a stable controller \( C(s) \) such that

\[
M + CN = U
\]
Assume \( r \leq m \) and define \( U \) in (4.6.1) as \( U = R + \gamma_0 T \), where \( \gamma_0 \) is a constant and \( T \) is a unimodular transfer function in \( \mathcal{RH}_\infty \). \( U \) will be unimodular in \( \mathcal{RH}_\infty \) if \( R \in \mathcal{RH}_\infty \) and \( \| T^{-1} R \|_\infty < \gamma_0 \). Then (4.6.1) becomes

\[
M - \gamma_0 T + CN = R. \tag{4.6.2}
\]

So the problem becomes

\[
\min_{C \in \mathcal{RH}_\infty} \| (M - \gamma_0 T) + CN \|_\infty \tag{4.6.3}
\]

Let \( G_1 = M - \gamma_0 T \). Then (4.6.3) becomes

\[
\min_{C \in \mathcal{RH}_\infty} \| G_1 + CN \|_\infty. \tag{4.6.4}
\]

Let \( N \) be written in an inner-outer factorization as \( N = N_o N_i \), then equation (4.6.4) becomes

\[
\min_{C \in \mathcal{RH}_\infty} \| G_1 + CN \|_\infty = \min_{C \in \mathcal{RH}_\infty} \| G_1 + C N_o N_i \|_\infty
= \min_{C \in \mathcal{RH}_\infty} \| G_1 + \begin{bmatrix} N_i & 0 \\ N_i^{-1} \end{bmatrix} \|_\infty \tag{4.6.5}
\]

where \( N_i^{-1} \) is a complementary inner factorization (CIF) chosen such that \( \begin{bmatrix} N_i \\ N_i^{-1} \end{bmatrix} \) is square and inner. From the inner invariance property of \( \| \cdot \|_\infty \), (4.6.5) becomes

\[
\min_{C \in \mathcal{RH}_\infty} \left\| G_1 + \begin{bmatrix} C N_o & 0 \\ N_i & N_i^{-1} \end{bmatrix} \right\|_\infty
= \min_{C \in \mathcal{RH}_\infty} \left\| G_1 \begin{bmatrix} N_i & (N_i^{-1})^* \end{bmatrix} + \begin{bmatrix} C N_o & 0 \end{bmatrix} \right\|_\infty
= \min_{C \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} G_1 N_i & G_1(N_i^{-1})^* \end{bmatrix} + \begin{bmatrix} C N_o & 0 \end{bmatrix} \right\|_\infty
= \min_{C \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} G_1 N_i + C N_o & G_1(N_i^{-1})^* \end{bmatrix} \right\|_\infty
= \min_{Q \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} R_1 + Q & R_2 \end{bmatrix} \right\|_\infty \tag{4.6.6}
\]

where

\[
Q = C N_o \quad R_1 = G_1 N_i^* \quad R_2 = G_1(N_i^{-1})^*. \]
Equation (4.6.6) is in the form of a generalized 2-block problem (GTBP) since $R_1$ and $R_2$ are in $\mathcal{R}_{\infty}$. To reduce it into the ordinary 2-block problem (OTBP), write $R_1$ in terms of stable and anti-stable parts as

$$R_i(s) = R_{is}(s) + R_{ia}(s)$$  \hspace{1cm} (4.6.7)

and factorize $R_2$ as

$$R_2 = R_{2u}R_{2s}$$  \hspace{1cm} (4.6.8)

where $R_{2s}$ is stable and inner. Then (4.6.6) becomes

$$\min_{Q \in \mathcal{H}_{\infty}} \left\| \begin{bmatrix} R_1 + Q & R_2 \\ \end{bmatrix} \right\|_{\infty}$$

$$= \min_{Q \in \mathcal{H}_{\infty}} \left\| \begin{bmatrix} R_{1u} + (Q + R_{1a}) & R_{2u}R_{2s} \\ \end{bmatrix} \right\|_{\infty}$$

$$= \min_{Q \in \mathcal{H}_{\infty}} \left\| \begin{bmatrix} R_{1u} + (Q + R_{1a}) & R_{2u}R_{2s} \\ I & 0 \\ 0 & R_{2s} \\ \end{bmatrix} \right\|_{\infty}$$

$$= \min_{\hat{Q} \in \mathcal{H}_{\infty}} \left\| \begin{bmatrix} R_{1u} + \hat{Q} & R_{2u} \\ \end{bmatrix} \right\|_{\infty}$$  \hspace{1cm} (4.6.9)

where $\hat{Q} = Q + R_{1a}$. Now (4.6.10) is in the form of OTBP whose solution can be obtained numerically using the well-known technique called "$\gamma$-iteration".

**Remark 4.14** If the dimension of $Q(s)$ is such that $r \geq m$, then the two-block formulation will be reduced to the following ordinary 1-block problem (OBBP)

$$\min_{\hat{Q} \in \mathcal{H}_{\infty}} \left\| R_{1u} + \hat{Q} \right\|_{\infty}.$$  \hspace{1cm} (4.6.11)

**Remark 4.15** If we define $\gamma_0$ in equation (4.6.2) as $\gamma_0 = \|T\|_\infty \|T^{-1}\|_\infty$, then $\|T^{-1}R\|_\infty < \gamma_0$, the condition for $U$ to be unimodular in $\mathcal{R}_{\infty}$, will be satisfied if $\|R\|_\infty < \|T\|_\infty$.

**Remark 4.16** A condition for the solution of the StSP via the $\mathcal{H}_{\infty}$ optimization approach described in this section is to choose a unimodular function $T(s)$ in equation (4.6.2) such that $\sigma_{\max}\left( \begin{bmatrix} R_{1u} & R_{2u} \end{bmatrix} \right) < \gamma_0$ for $r < m$ or $\sigma_{\max}(R_{1u}) < \gamma_0$ for $r \geq m$. A criteria to choose such $T(s)$ is not available yet.

**Remark 4.17** A state space representation for the multiplicative/inner factorization given in (4.6.8) will be given in Chapter 5, Section 2.3.
The following algorithm summarizes the procedure described above to design a stable controller for a bi-proper and non-minimum phase system via $\mathcal{H}_\infty$ Optimization. This algorithm can be used also for strictly proper and non-minimum phase system by letting $G(\infty) = \epsilon I$, for some $\epsilon \ll 1$.

**Algorithm 4.6:**

**Step 0 (Initialization):** $G(s) \in \mathbb{R}^{c \times m}$ is a given bi-proper, non-minimum phase and strongly stabilizable plant. (If $G(s)$ is strictly proper, let $G(\infty) = \epsilon I$, for some $\epsilon \ll 1$).

**Step 1:** Find RCF for $G(s)$ so that $G = NM^{-1}$.

**Step 2:** Choose a unimodular matrix $T(s) \in \mathbb{H}_\infty$ and define $\gamma_0 = ||T||_\infty ||T^{-1}||_\infty$.

**Step 3:** Let $G_1 = M - \gamma_0 T$.

**Step 4:** Find an inner-outer factorization for $N$ as $N = \begin{bmatrix} N_o & 0 \\ N_i & N_i^* \end{bmatrix}$.

**Step 5:** Set $Q = CN_o$, $R_1 = G_1 N_i^*$, $R_2 = G_1(N_i^*)^*$ and find the factorizations in equations (4.6.7) and (4.6.8).

**Step 6:** If $\sigma_{\text{max}} \left( \begin{bmatrix} R_{1u} & R_{2u} \end{bmatrix} \right) < \gamma_0$ for $r < m$ or if $\sigma_{\text{max}}(R_{1u}) < \gamma_0$ for $r \geq m$ continue, else go to Step 2 for another choice of $T(s)$.

**Step 7:** Solve the 2-block problem in equation (4.6.10) or the 1-block problem in equation (4.6.11). Then compute the controller $C$.

The above algorithm presents a trial-and-error approach to solve the StSP. That is by choosing a simple unimodular transfer function $T(s)$, we may define $\gamma_0 = ||T||_\infty ||T^{-1}||_\infty$ and calculate the minimum cost function value in (4.6.10) or (4.6.11); if it is less than $||T||_\infty$, then appropriate $C$ and $R$ can be derived to satisfy (4.6.2) and to form a solution for the StSP.

The following example demonstrates the theory presented in this section.

---

6See Remarks 4.15 and 4.16.

7See Remark 4.17.
Example 4.6 Given the plant

\[ G(s) = \frac{(s - 0.5)(s - 2)}{(s - 1)^2} \cdot N \]

where

\[ N = \frac{(s - 0.5)(s - 2)}{(s + 0.5)(s + 2)} \quad \text{and} \quad M = \frac{(s - 1)^2}{(s + 0.5)(s + 2)}. \]

This system is unstable and non-minimum phase. We choose a unit function \( T(s) \in \mathbb{R}H_\infty \) as

\[ T = \frac{0.3s + 8.1}{s + 8}. \]

There is a restriction on the \( H_\infty \)-norm of \( T \) and \( T^{-1} \). See Remarks 4.15 and 4.16. For \( \gamma_0 = 1 \), \( M - \gamma_0 T \) becomes

\[ (M - \gamma_0 T) = \begin{bmatrix} 0.7s^3 + 2.82s^2 - 21.575s + 5.57 \\ \frac{s^2 + 10.5s + 21s + 8} \end{bmatrix} \]

and

\[ (M - \gamma_0 T)N^* = \begin{bmatrix} 0.7s^3 + 4.57s^4 - 13.625s^3 - 45.047s^2 - 7.45s + 5.57 \\ \frac{s^2 + 8s + 4.25s - 34s^2 + s + 8} \end{bmatrix} \]

\[ = [(M - \gamma_0 T)N^*]_+ + [(M - \gamma_0 T)N^*]_-. \]

where

\[ [(M - \gamma_0 T)N^*]_+ = -\frac{0.0159s^2 + 0.0397s + 0.0159}{s^3 + 10.5s + 21s + 8} \]

\[ [(M - \gamma_0 T)N^*]_- = \frac{0.7s^2 - 2.764s + 0.6982}{s^3 - 2.5s + 1}. \]

Now, the equation

\[ M + CN = R + \gamma_0 T \]

\[ M - \gamma_0 T + CN = R \]

is reduced to

\[ [(M - \gamma_0 T)N^*]_+ + [(M - \gamma_0 T)N^*]_- + C = R \]

or

\[ Z + Q = R \]
where $Z = [(M - \gamma_tT^*)N]_+$ which is unstable, and $Q = [(M - \gamma_tT^*)N]_+ + C$ which we aim to make stable. Now the problem as in (4.6.11) is to solve the following 1-Block problem

$$\min_{Q \in \mathcal{RH}_\infty} \|Z + Q\|_\infty < 1.$$  \hspace{1cm} (4.6.12)

Since the maximum Hankel singular value $\sigma^H_{\text{max}}(Z) \approx 0.2037$, a solution of (4.6.12) is possible. One such solution\(^8\) is

$$Q = \frac{0.7s^3 + 10.4428s + 0.7036}{s^4 + 12.1284s + 1.0002}$$

with $\gamma = 0.25$. The stable controller is given as

$$C = -(Q + [(M - \gamma_tT^*)N]_+)$$

$$= -\frac{0.7s^3 + 16.0264s^2 + 84.0532s + 5.6129}{s^3 + 20.1284s^2 + 98.0278s + 8.0017}$$

$$= -\frac{(s + 0.0830368)(s + 8.0001)(s + 12.0453)}{(s + 0.536383)(s + 1.87214)(s + 12.0453)}.$$

The StSP is solvable by the stable controller $C$ if

$$\|T^{-1}R\|_\infty < 1$$

according to Corollary 4.18. Now $\|T^{-1}R\|_\infty \approx 0.8055 < 1$, so $M + CN$ is a unit in $\mathcal{RH}_\infty$ and its minimal realization is given by

$$U = M + CN$$

$$= \frac{0.3s^3 + 3.1015s^2 + 6.031s + 2.3889}{s^3 + 20.1284s^2 + 98.0278s + 8.0017}$$

$$= \frac{3(s + 0.536383)(s + 1.87214)(s + 7.92981)}{(s + 0.0830368)(s + 8.0001)(s + 12.0453)}$$

which is stable and a unit in $\mathcal{RH}_\infty$.

The solution of the 1-Block problem in (4.6.12) depends on choosing $\gamma > \sigma^H_{\text{max}}(Z)$. If we choose $\gamma > 0.35$ when solving the 1-Block problem in (4.6.12), $U$ will be stable but not a unit in $\mathcal{RH}_\infty$. Varying $\gamma$ between $\sigma^H_{\text{max}}(Z)$ and $\gamma_{\text{max}} (= 0.35 in this case) will locate the poles and zeros of $U$ at different regions in the $s$-plane. So this parameter gives some freedom to locate the poles of the closed-loop system.

---

\(^8\)The solution of 1-Block problem will be discussed in Chapter 5.
4.7 Strong Stabilization Via State Space Projection

In the previous section a procedure was given to find a stable controller for a given multivariable linear and time-invariant system. The procedure discussed is based on selecting a unimodular transfer function in $\mathcal{RH}_\infty$ such that a certain condition is satisfied. A clear criterion is yet to be found to select such a function. In this section we provide another technique to find a stable controller on the assumption that the given plant is strongly stabilizable.

Assume a given plant $G(s)$ which may be factorized as $G(s) = N(s)M^{-1}(s)$, where $N(s)$ and $M(s) \in \mathbb{R} \mathcal{H}_\infty$ are RCF of $G(s)$ with appropriate dimensions. Let $G(s)$ be define in state space representation as

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

If $DD^T > 0$, then $M(s)$ and $N(s)$ with $N(s)$ is inner are given by

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} A + BF & BR^{-1} \\ F & R^{-\frac{1}{2}} \\ C + DF & DR^{-\frac{1}{2}} \end{bmatrix} \in \mathbb{R} \mathcal{H}_\infty$$

where

$$R = DD^T$$

$$F = -R^{-1}(B^TY + D^TC)$$

$$Y = Ric \begin{bmatrix} A - BR^{-1}D^TC & -BR^{-1}B^T \\ -C^TDY + D^TC & -(A - BR^{-1}D^TC)^T \end{bmatrix}.$$  

A necessary and sufficient condition for $G(s)$ to be strongly stabilizable is that there exists a unimodular matrix $U \in \mathbb{R} \mathcal{H}_\infty$ such that

$$U - M = CN.$$  

In the following we will state the conditions on the parameters of $U$ that will satisfy (4.7.3). Let $U$ be given in state-space representation as

$$U = \begin{bmatrix} A_u & B_u \\ C_u & D_u \end{bmatrix}.$$
Then
\[ U - M = \begin{bmatrix} A_u & 0 & B_u \\ 0 & A + BF & BR^{-\frac{1}{2}} \\ C_u & -F & D_u - R^{-\frac{1}{2}} \end{bmatrix}. \] (4.7.5)

From (4.7.3), since \( N \) is inner, \[
C = (U - M)N^* = [(U - M)N^*]_+ + [(U - M)N^*]_- = C_s + C_{us}.
\] (4.7.6)

So a sufficient condition for \( C_s \) to be stable is that \( C_s = 0 \).

Let us find \( C_{us} \) in state-space form. Algebraic manipulations shows that \((U - M)N^*\) can be written as
\[
(U - M)N^* = \begin{bmatrix} A_u & 0 & B_u(DR^{-\frac{1}{2}})^T + X_1(C + DF)^T \\ 0 & A + BF & BR^{-1}D^T + X_2(C + DF)^T \\ C_u & -F & (D_u - R^{-\frac{1}{2}})(DR^{-\frac{1}{2}})^T \end{bmatrix}
+ \begin{bmatrix} (A + BF)X_1 - FX_2 + (D_u - R^{-\frac{1}{2}})(BR^{-\frac{1}{2}})^T \\ (C + DF)^T \end{bmatrix} = C_s + C_{us}.
\]

Equation (4.7.7) can be written in the form of the following two Lyapunov equations
\[
A_uX_1 + X_1(A + BF)^T + B_u(BR^{-\frac{1}{2}})^T = 0 \quad (4.7.8)
\]
\[
(A + BF)X_2 + X_2(A + BF)^T + BR^{-1}B^T = 0.
\]

For \( C_{us} \) to be zero, the following sufficient condition should be satisfied:
\[
C_sX_1 - FX_2 + (D_u - R^{-\frac{1}{2}})(BR^{-\frac{1}{2}})^T = 0 \quad (4.7.9)
\]
Equation (4.7.9) has two unknowns, \( C_u \) and \( D_u \), which are parameters of the required unimodular matrix \( U \in \mathbb{R}^{n \times n} \). If \( X_1 \) is invertible, then solving (4.7.9) for \( C_u \) in terms of \( D_u \) giving

\[
C_u = \left\{ FX_2 - (D_u - R^{-\frac{1}{2}})(BR^{-\frac{1}{2}})^T \right\} X_1^{-1} \quad (4.7.10)
\]

But for \( U \) to be unimodular, we impose the following condition on \( C_u \) and \( D_u \). For some stable \( A_u \) and stabilizable \( (A_u, B_u) \),

\[
A_u - B_u D_u^{-1} C_u \quad (4.7.11)
\]

should be stable. Substitute for \( C_u \) from (4.7.10) in (4.7.11) and define \( A_1, B_1 \) and \( C_1 \) as

\[
A_1 = A_u + B_u (BR^{-\frac{1}{2}})^T X_1^{-1} \\
B_1 = B \\
C_1 = FX_2 X_1^{-1} + R^{-1} B^T X_1^{-1} \quad (4.7.12)
\]

Then equation (4.7.11) is reduced to

\[
A_1 - B_1 D_u^{-1} C_1. \quad (4.7.13)
\]

So, the StSP is reduced to finding a full rank constant matrix \( K \) such that \((A_1 - B_1 KC_1)\) is stable with \( D_u = K^{-1} \). The result of the above analysis is summarized in the following theorem.

**Theorem 4.23** A sufficient condition for \( G(s) \) to be strongly stabilizable is that there exists a full rank constant gain output feedback \( K \) such that \((A_1 - B_1 KC_1)\) is stable. □

**Lemma 4.24** If the condition in Theorem (4.23) is satisfied, then a stable controller that stabilizes \( G(s) \) is given by

\[
\begin{bmatrix}
A_u & 0 & B_u (DR^{-\frac{1}{2}})^T + X_1 (C + DF)^T \\
0 & A + BF & BR^{-1} D^T + X_2 (C + DF)^T \\
C_u & -F & (D_u - R^{-\frac{1}{2}})(DR^{-\frac{1}{2}})^T
\end{bmatrix}
\]

□
Therefore, in order to solve the StSP, one needs to find some unimodular matrix

\[ U = \begin{bmatrix} A_u & B_u \\ C_u & D_u \end{bmatrix}, \]

and then compute the stable controller that satisfies the condition \( M + CN = U \). As a summary, to find such \( U \), choose, almost an arbitrary, stable \( A_u \) and \( B_u \) with appropriate dimensions. The only condition on \( A_u \) and \( B_u \) is that the solution \( X_1 \) of the Lyapunov equation in (4.7.8) is invertible which can be easily satisfied. Then solve equation (4.7.8) for \( X_1 \) and \( X_2 \) such that \( X_1 \) is invertible. Compute the triple \( (A_1, B_1, C_1) \) from equation (4.7.12) and find a full rank matrix \( K \) such that \( (A_1 - B_1 KC_1) \) is stable. Then set \( D_u = K^{-1} \) and compute \( C_u \) using equation (4.7.10).

**Remark 4.18** If \( N(s) \) is not inner, then we can use inner-outer factorization technique to reduce (4.7.3) to

\[ U - M = CN_u N_i \]  

(4.7.14)

and proceed as above.

**Remark 4.19** An algorithm to find a full rank constant output feedback gain is listed in Chapter 6.

**Remark 4.20** If \( G(s) \) is strictly proper, i.e. \( D = 0 \), then an inner \( N \) in (4.7.3) does not exist. Also \( N_u \) in (4.7.14) will be strictly proper. To overcome this problem we may let the \( D \) term of the given system be equal to \( \epsilon I \). Another way to overcome the above problem is to solve for the stable part of \( CN_u \) as above. Then it may be possible to recover \( C \) from \( C_u \) if \( C_u \) is strictly proper.

The following algorithm summarizes the procedure described above to design a stable controller for a bi-proper and non-minimum phase system via state space projection. This algorithm can also be used for strictly proper and non-minimum phase system by letting \( \tilde{G}(\infty) = \epsilon I \), for some \( \epsilon \ll 1 \).

**Algorithm 4.7:**
Step 0 (Initialization): \( G(s) \in \mathbb{R}^{n \times m} \) is a given bi-proper, non-minimum phase and strongly stabilizable plant. (If \( G(s) \) is strictly proper, let \( G(\infty) = \varepsilon I \), for some \( \varepsilon < 1 \).)

Step 1: Find a RCF for \( G(s) \) as in equation (4.7.2).

Step 2: Choose a stable matrix \( A_u \) and \( B_u \) with appropriate dimensions.

Step 3: Solve the two Lyapunov equations in equation (4.7.8) for \( X_1 \) and \( X_2 \).

Step 4: If \( X_1 \) is invertible continue, otherwise go to Step 2 and do some changes in \( A_u \) and \( B_u \). This process is trial and error.

Step 5: Compute the triple \((A_1, B_1, C_1)\) according to equation (4.7.12).

Step 6: Find a full rank constant output feedback \( K \) such that \((A_i - B_1 K C_i)\) is stable. This is a static output feedback problem; see comments on page 152.

Step 7: Set \( D_u = K^{-1} \) and compute \( C_u \) from equation (4.7.10).

Step 8: Compute the stable stabilizing controller given in Lemma 4.24.

Example 4.7 The following illustrative example demonstrates the above theory. Let a plant be given as

\[
G = \begin{bmatrix}
2 & 6 & 5 & 1 & 0 \\
0 & 9 & 8 & 1 & 1 \\
6 & 3 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1
\end{bmatrix}
\]

whose poles are \([0.695, -3.197, 13.5]\). Choose two parameters \( A_u \) and \( B_u \) of a uni-modular matrix \( U = (A_u, B_u, C_u, D_u) \). \( C_u \) and \( D_u \) will be determined later. \([A_u, B_u]\) are chosen arbitrarily as

\[
\begin{bmatrix}
A_u \\
B_u
\end{bmatrix} = \begin{bmatrix}
-98 & -71 & -39 & 1 & 1 \\
-71 & -118 & -75 & 0 & 1 \\
-39 & -75 & -73 & 1 & 0
\end{bmatrix}
\]
where $A_u$ is stable. The eigenvalues of $A_u$ are $[-49.83, -14.35, -224.8]$. The only condition on $A_u$ and $B_u$ is that the solution $X_1$ of the Lyapunov equation in (4.7.8) is invertible which can be easily satisfied. Then solve equation (4.7.8) for $X_1$ and $X_2$ we get

$$X_1 = \begin{bmatrix} 0.01841 & 0.01755 & 0.006776 \\ -0.03123 & -0.001392 & 0.009677 \\ 0.03372 & 0.002138 & -0.01081 \end{bmatrix}$$

and

$$X_2 = \begin{bmatrix} 0.1015 & -0.01136 & 0.0431 \\ -0.01136 & 0.2679 & -0.145 \\ 0.0431 & -0.145 & 0.2277 \end{bmatrix}$$

Then $A_1, B_1$ and $C_1$ of equation (4.7.12) are

$$A_1 = \begin{bmatrix} 12.1 & 155.1 & 140 \\ 0.361 & -405.4 & -380.2 \\ -0.2629 & 438.5 & 411.1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -20.39 & 755.7 & 715.4 \\ 16.46 & -1050 & -992.3 \end{bmatrix}$$

The full rank constant output feedback $K$ that stabilizes the triple $(A_1, B_1, C_1)$ is given by

$$K = \begin{bmatrix} 26.47 & 39.45 \\ 56.25 & 15.01 \end{bmatrix}$$

The eigenvalues of $(A_1 - B_1 K C_1)$ are $[-5526, -103.6 \pm 50.67i]$. Therefore, the rest of the parameters of the required unimodular $U$ are computed as $D_u = K^{-1}$ and $C_u$ from equation (4.7.10) and they are given as

$$\begin{bmatrix} C_u \\ D_u \end{bmatrix} = \begin{bmatrix} -21.62 & 766.1 & 726 & -0.008243 & 0.02166 \\ 16.3 & -1070 & -1012 & 0.03088 & -0.01454 \end{bmatrix}$$
The eigenvalues of $U^{-1}$ are $[-5526, -103.6 \pm 50.67\%]$. The stable controller $C_s$ is given as

$$
\begin{bmatrix}
-98 & -71 & -39 & 0 & 0 & 0 & 0.7616 & 0.7734 \\
-71 & -118 & -75 & 0 & 0 & 0 & 0.1405 & 1.014 \\
-39 & -75 & -73 & 0 & 0 & 0 & 0.8467 & -0.0137 \\
0 & 0 & 0 & -4.878 & -0.4036 & -0.2239 & 0.2398 & -0.3289 \\
0 & 0 & 0 & -8.688 & -6.221 & -3.917 & 0.3764 & -0.5163 \\
0 & 0 & 0 & 4.189 & -5.817 & -6.693 & -0.6593 & 0.9041 \\
-21.62 & 766.1 & 726 & 6.878 & 6.404 & 5.224 & -1.008 & 0.02166 \\
16.3 & -1070 & -1012 & 1.811 & 8.817 & 6.693 & 0.03088 & -1.015
\end{bmatrix}
$$

The closed-loop eigenvalues are $[-5526, -103.6 \pm 50.67\% - 4.896 - 4.896 - 11.49 - 11.49 - 1.404]$. The controller in this example is not minimal. If a minimal realization of the controller is used, the new closed-loop eigenvalues are $[-5526, -103.6 \pm 50.67\% - 11.49 - 4.896 - 4.896 - 1.404]$.

4.8 StSP Via the Youla Parameter $Q$

The main results presented in this chapter, for solving the StSP, are based on Corollary 4.11. That is, find a stable controller $C$ such that $\tilde{M} + \tilde{N}C$ is unimodular in $\mathcal{H}_\infty$. By Corollary 4.11, this guarantees that $(G, C)$ is internally stable. Hence $C$ provides a solution to the strong stabilization problem.

Another way to find a stable, stabilizing controller is to look at the Youla parametrization of all stabilizing controllers directly. Let

$$
C = -(Y - MQ)(X - NQ)^{-1}. \tag{4.8.1}
$$

It is obvious that $(X - NQ)$ being unimodular in $\mathcal{H}_\infty$ gives a stable $C$. On the other hand this is a necessary condition as well because

$$
\tilde{M} + \tilde{N}C = \tilde{M} - \tilde{N}(Y - MQ)(X - NQ)^{-1} \\
= \left\{ \tilde{M}X - \tilde{M}NQ - \tilde{N}Y + \tilde{N}MQ \right\}(X - NQ)^{-1}
$$
\[
\begin{align*}
\{ (\hat{M}X - \hat{N}Y) - (\hat{M}N - \hat{N}M)Q \} (X - NQ)^{-1} \\
= (X - NQ)^{-1}
\end{align*}
\]

Thus, from Corollary 4.11, \(C\) being stable and stabilizing \(G\) implies \(\hat{M} + \hat{N}C\) is unimodular in \(\mathcal{H}_{\infty}\). That is then implies that \(X - NQ\) is unimodular in \(\mathcal{H}_{\infty}\). Hence, the strong stabilization problem is reduced to find a \(Q \in \mathcal{H}_{\infty}\) such that \(X - NQ\) is unimodular in \(\mathcal{H}_{\infty}\). The result of this section can be stated in the following lemma.

**Lemma 4.25** A necessary and sufficient condition for \(C\) to be a stable and stabilizing controller is that \(X - NQ\) is unimodular in \(\mathcal{H}_{\infty}\) for some stable \(Q\).

The methods of Sections 4.5-4.7 can be used to find a stable \(Q\) such that \(X - NQ\) is unimodular in \(\mathcal{H}_{\infty}\).

### 4.9 Simultaneous Stabilization: The Two Plants Case

It is known that the problem of simultaneous stabilization of two plants can be reduced to an interpolation problem with units in \(\mathcal{H}_{\infty}\). Most of the work done in this direction is limited to single-input single-output plants characterized by real rational transfer functions. See for example [24, 25].

The mathematical formulation of the simultaneous stabilization of two MIMO plants as a strong stabilization problem of a related auxiliary plant is given in [96]. The solution of this formulation for MIMO case has not been reported yet. But for the SISO case it is reported in [20].

In the following subsection the formulation is reviewed and the solution can be sought using one of the methods described in the previous Sections.
4.9.1 Forming The Auxiliary Plant

For two plants \( P_0 \) and \( P_1 \), define an auxiliary plant as follows: Assume the coprime factorization of \( P_0 \) and \( P_1 \) are given as

\[
P_0 = N_0 M_0^{-1} = \tilde{M}_0^{-1} \tilde{N}_0 \tag{4.9.1}
\]
\[
P_1 = N_1 M_1^{-1} \tag{4.9.2}
\]

and there exist \( \tilde{X}_0 \) and \( \tilde{Y}_0 \) that satisfy Bezout identity \( \tilde{X}_0 M_0 - \tilde{Y}_0 N_0 = I \). Define \( A_0 \) and \( B_0 \) as

\[
A_0 = \tilde{X}_0 M_1 - \tilde{Y}_0 N_1 \tag{4.9.3}
\]
\[
B_0 = -\tilde{N}_0 M_1 + \tilde{M}_0 N_1 \tag{4.9.4}
\]

or, write them as

\[
\begin{bmatrix}
A_0 \\
B_0
\end{bmatrix} =
\begin{bmatrix}
\tilde{X}_0 & -\tilde{Y}_0 \\
-\tilde{N}_0 & \tilde{M}_0
\end{bmatrix}
\begin{bmatrix}
M_1 \\
N_1
\end{bmatrix}.
\tag{4.9.4}
\]

Then the auxiliary plant \( G_0 \) is defined as

\[
G_0 = B_0 A_0^{-1}. \tag{4.9.5}
\]

\( A_0 \) and \( B_0 \) are the right coprime factorization of \( G_0 \).

**Theorem 4.26** [96] Given \( A_0 \) and \( B_0 \) as defined above. There exists a controller \( C(s) \) that stabilizes the two plants in (4.9.1,4.9.2) if and only if there exist \( M_0 \) (proper, rational, and stable matrix) that stabilizes the auxiliary plant \( G_0 \) or \( A_0 + M_0 B_0 \) is unimodular in \( \mathcal{RH}_\infty \).

If there exists such \( M_0 \), then the controller that simultaneously stabilizes \( P_0 \) and \( P_1 \) is given by

\[
C = - \left( \tilde{X}_0 - M_0 \tilde{N}_0 \right)^{-1} \left( \tilde{Y}_0 - M_0 \tilde{M}_0 \right). \tag{4.9.6}
\]

**Remark 4.21** If the two plants, \( P_0 \) and \( P_1 \), are strictly proper, then \( G_0 \) is strictly proper. But if one of the plants is proper, then \( G_0 \) is proper.
Example 4.8. This example demonstrates how the SSP of two plants can be solved as a StSP. The following two plants which are reported in [19] will be simultaneously stabilized by a single controller.

\[
P_0 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
P_1 = \begin{bmatrix} 0 & 1 & 1 & -1 & 1 \\ -9 & 0 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

According to the theory of this section, an auxiliary plant is first formed from \( P_0 \) and \( P_1 \) as \( P = B_0A_0^{-1} \) where \( A_0 \) and \( B_0 \) are defined by equation (4.9.4). The auxiliary plant \( P \) is computed in Example 4.2 and is given as

\[
P = \begin{bmatrix} -1.665 & -4.775 & -1.968 & 3.397 & 0.2244 & -0.1452 & 0.2244 \\ 0.1862 & -0.9541 & -1.057 & -3.281 & -1.325 & 0.7574 & -1.325 \\ 0 & 4.377 & -0.5646 & -4.748 & -1.716 & 1.873 & -1.716 \\ 0 & 0 & 0.9374 & -2.816 & 0 & -0.8041 & 0 \\ 0 & 0 & 0 & 1.244 & 0 & 0 & 0 \end{bmatrix}
\]

A stable controller which will stabilize this plant is also given in Example 4.2 as

\[
C_r = \begin{bmatrix} -2.428 & 0.8172 & 6.519 & 13.4 & 23.53 & 57.16 & 17.59 \\ 0.3251 & -1.94 & 0.6976 & -2.65 & -4.64 & 1.416 & -2.639 \\ -0.01215 & 0.4224 & -0.9697 & -0.3716 & 3.404 & -14.7 & 10.77 \\ -0.01976 & -0.2597 & -4.54 & -10.01 & -16.59 & -4.615 & -19.45 \\ 0 & 0 & -0.4185 & 1.397 & -2.198 & 6.441 & -2.419 \\ 0 & 0 & 0.6305 & 0.9275 & 2.002 & -6.449 & 1.913 \end{bmatrix}
\]

Then simultaneous stabilization controller for the two plants \( P_0 \) and \( P_1 \) can be computed according to equation (4.9.6) with \( M_0 = C_r \). A minimal realization for this controller is given as
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CloselOop T.F.

Table 4.1: The Closed-loop Eigenvalues, $T(P_i, K_0)$ for $i = 1, 2, 3$.

<table>
<thead>
<tr>
<th>$T(P_0, K_0)$</th>
<th>$T(P_1, K_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1.0000 - 2.0000 - 3.0000 - 4.0000 - 5.0000 - 5.0000 - 6.0000$</td>
<td>$-3.0000 - 4.0000 - 4.9996 - 5.0002 \pm 0.0003, -6.0000, -6.0000$</td>
</tr>
</tbody>
</table>

The closed-loop eigenvalues for the two systems are given in Table 4.1. This controller locates the eigenvalues of the closed-loops at the locations shown in Table 4.1.

4.10 Conclusion

In this chapter we considered the strong stabilization problem of multivariable systems. The problem is categorized into minimum and non-minimum phase systems. When the given systems are minimum phase, the solution requires a stable inverse of a particular stable matrix-transfer function, while for non-minimum phase systems, the solution requires an inner-outer factorization whose outer part is unimodular in $\mathcal{RH}_\infty$. We modified the formulation of the two cases so that the conditions for the existence of stable inverse or unimodularity of the outer part are satisfied. When the systems are strictly proper the problem is more technically involved.
When the systems are non-minimum phase, the StSP is formulated as an interpolation problem in $\mathcal{RH}_\infty$ and the matrix version of the Nevanlinna-Pick Algorithm can be used to find the controller. The symbolic computations involved in the algorithm can be done using the MATHEMATICA or MAPLE software packages. Again the problem is more difficult in the strictly proper case. A program written in MATHEMATICA has been developed to implement the Nevanlinna-Pick Algorithm and a listing of this program is given in Appendix B.

Finally, two new approaches not involving interpolation to solve the StSP for nonminimum phase systems were proposed. In the first approach, the problem is formulated as an optimization problem in $\mathcal{RH}_\infty$, where a 2-Block problem, or a 1-Block problem, is obtained. The other approach is based on a stable projection. The formulation in this approach is reduced to that of finding a full rank constant gain output feedback.
The methods and algorithms developed in this chapter for different types of MIMO linear time-invariant systems are summarized diagrammatically in Figure 4.1. A summary for the free parameters that the user should select as inputs to these algorithms is given in Table 4.2.

There is a strong connection between the StSP and the SSP of two plants, and the solution of the first can be applied to solve the second. This was illustrated by a numerical example.
Chapter 5

Simultaneous Stabilization Via Robust Stabilization Theory

5.1 Introduction

The formulation of the Simultaneous Stabilization Problem (SSP) of $l$ MIMO plants as proposed in Saeks and Murray [80] and Vidyasagar and Viswanadham [95], came up with necessary and sufficient conditions for the existence of a solution for an equivalent problem of simultaneously stabilizing $l - 1$ plants by a single stable controller. However, a computational procedure to verify these conditions was not provided. In this sense, the SSP remains an open and difficult problem. In this Chapter, we give an alternative formulation which is relatively easy to solve but which in effect provides only a sufficient condition for the SSP.

We consider the SSP for MIMO systems in two stages. We first formulate and solve a generalized $\mathcal{L}_\infty$ 2-Block optimization problem (GTBP) to obtain a central plant from the set of plants to be stabilized. We then use results from robust stabilization theory to obtain a controller and a sufficient condition for its solution to the SSP. The novelty of the procedure is in the determination of the central plant. Note also that our procedure introduces some conservatism since we aim to robustly stabilize over a wider set of plants than that represented in the problem statement.

This Chapter is organized as follows. In Section 5.2, a generalized $\mathcal{L}_\infty$ 2-Block problem
will be presented as a mechanism for finding the central plant, along with the necessary mathematical tools. A state space representation for the GTBP will be presented in Section 5.3. A discussion of the so-called general distance problem in $\mathcal{H}_\infty$ and a definition of a generalized 1-Block problem with its solution are given in Section 5.4 as a preamble to the solution of the GTBP in Section 5.5. Robust stabilization theory is summarized in Section 5.6 and an algorithm is provided to design a controller that may robustly stabilize the central plant and solve the SSP. Conclusions are given in Section 5.7.

5.2 Formulation of the Central Plant Problem

Assume that we are given a set of $n$ unstable plants, say $\{P_1(s), \ldots, P_n(s)\} \in \mathbb{R}^{m \times m}_{\infty}$, with the assumption that all of them have the same number of unstable poles. We are aiming to find a single controller that simultaneously stabilizes all the plants of the given set. One method is to pick one of the given plants and then to use $\mathcal{H}_\infty$-methods to robustly stabilize this plant. The resulting controller can then be tested to see whether it simultaneously stabilizes all the given plants. Which plant model should we select? At present there is no criterion to help in this selection. In this section and the subsequent sections of this chapter, we propose a tractable method to find a suitable plant, that we refer to as the central plant. When stabilizing this central plant, stability and performance requirements on the original plants of the given set may be taken into consideration. A sufficient condition for the controller obtained to simultaneously stabilize the given set of plants is provided from standard robust stabilization theory.

The problem considered here is to find a "central" plant $\tilde{P}(s)$ of the set $\{P_1(s), \ldots, P_n(s)\}$ in the following sense. Each of the given plants can be written as

$$P_i(s) = \tilde{P}(s) + \Delta P_i(s) \quad \forall \ i = 1, \ldots, n. \quad (5.2.1)$$

where $\tilde{P}(s) \in \mathbb{R}^{m \times m}_{\infty}$ with the same number of unstable poles as the given plants and $\tilde{P}(s)$ lies at the "center" of the set $\{P_1(s), \ldots, P_n(s)\}$. By the central plant $\tilde{P}(s)$, we mean
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that \( \bar{P}(s) \) which is the solution of the following \( L_\infty \)-optimization problem

\[
\begin{align*}
\bar{P}(s) & \in \mathbb{R}^{n \times n} \\
\min \left\{ \begin{array}{c}
P_1 - \bar{P} & 0 & \cdots & 0 \\
0 & P_2 - \bar{P} & 0 & \cdots \\
\vdots & 0 & \ddots & \cdots \\
\vdots & \vdots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & P_n - \bar{P}
\end{array} \right\} \\
\end{align*}
\]

The next lemma follows from the definition of the \( L_\infty \)-norm and from the diagonal structure of (5.2.2).

**Lemma 5.1** For (5.2.2) one has the following equality

\[
\begin{align*}
\hat{P}(s) & \in \mathbb{R}^{n \times n} \\
\min \left\{ \begin{array}{c}
P_1 - \hat{P} & 0 & \cdots & 0 \\
0 & P_2 - \hat{P} & 0 & \cdots \\
\vdots & 0 & \ddots & \cdots \\
\vdots & \vdots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & P_n - \hat{P}
\end{array} \right\} \\
\end{align*}
\]

\[
\begin{align*}
\hat{P}(s) & \in \mathbb{R}^{n \times n} \\
\min \left\{ \begin{array}{c}
P_1 - \hat{P} & 0 & \cdots & 0 \\
0 & P_2 - \hat{P} & 0 & \cdots \\
\vdots & 0 & \ddots & \cdots \\
\vdots & \vdots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & P_n - \hat{P}
\end{array} \right\} \\
\end{align*}
\]

The optimization problem given by (5.2.2) is new and nonstandard. A direct way to solve this problem is not yet known, so we will give an approximate approach based on the following lemma.

**Lemma 5.2** Given the set of plants defined above, then the following inequality holds

\[
\begin{align*}
\hat{P}(s) & \in \mathbb{R}^{n \times n} \\
\min \left\{ \begin{array}{c}
P_1 - \hat{P} & 0 & \cdots & 0 \\
0 & P_2 - \hat{P} & 0 & \cdots \\
\vdots & 0 & \ddots & \cdots \\
\vdots & \vdots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & P_n - \hat{P}
\end{array} \right\} \\
\end{align*}
\]
The proof of the above lemma is based on the fact that a compression cannot increase the norm of a matrix. Note that the solution of (5.2.5) will give an upper bound for the problem in (5.2.4). Hence a suboptimal solution for the central plant will be obtained in this work.

Based on the above lemma we may state the following new problem which is an approximating of (5.2.2).

Given \( n \) unstable plants, say \( P_1(s), \ldots, P_n(s) \in \mathbb{R}^{r \times m} \), find a \( \hat{P}(s) \in \mathbb{R}^{r \times m} \) with a specified number of stable and unstable poles such that the following new \( L_\infty \)-optimization problem

\[
\min_{\hat{P}(s) \in \mathbb{R}^{r \times m}} \max_{i = 1, \ldots, n} \| P_i - \hat{P} \|_{L_\infty} \quad (5.2.4)
\]

is satisfied.

The formulation, we propose here, will give an approximate solution to (5.2.2). Equation (5.2.6) can be transformed into the following form

\[
\min_{\hat{P}(s) \in \mathbb{R}^{r \times m}} \| P_1(s) \hat{P}(s) \ldots P_n(s) \|_{L_\infty} \quad (5.2.6)
\]
where \( U_I \) is a unitary matrix, defined in the following theorem. First the definition of unitary matrices.

**Definition 5.1** A real square matrix \( U \) is unitary if and only if \( UU^T = I \).

**Theorem 5.3** The following is an \( n \times n \) unitary matrix,

\[
U_1 = \frac{1}{\sqrt{n}} \begin{bmatrix}
1 & -1 & -1 & \ldots & -1 \\
1 & \beta & -\alpha & \ldots & -\alpha \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & -\alpha & \beta & -\alpha & \ldots & -\alpha \\
1 & -\alpha & \ldots & -\alpha & \beta
\end{bmatrix},
\]

if and only if

\[
\alpha = \frac{1}{1 + \sqrt{n}} \text{ and } \beta = -\alpha + \sqrt{n},
\]

or

\[
\alpha = \frac{1}{1 - \sqrt{n}} \text{ and } \beta = -\alpha - \sqrt{n}.
\]

**Proof.** The proof is based on the definition of unitary matrices and the special structure of \( U_1 \).

**Corollary 5.4** Let \( I \) be an \( n \times n \) identity matrix. Then

\[
U_I = \frac{1}{\sqrt{n}} \begin{bmatrix}
I & -I & -I & \ldots & -I \\
I & \beta I & -\alpha I & \ldots & -\alpha I \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
I & -\alpha I & \beta I & -\alpha I & \ldots & -\alpha I \\
I & -\alpha I & \ldots & -\alpha I & \beta I
\end{bmatrix},
\]

is an \( n^2 \times n^2 \) unitary matrix.
By substituting for \( U \) from equation (5.2.9) into equation (5.2.7) we will get equation (5.2.6).

Since multiplication by a unitary matrix does not change the \( \mathcal{L}_\infty \)-norm, equation (5.2.7) can be re-written as follows:

\[
\begin{bmatrix}
    P_1(s) \\
    P_2(s) \\
    \vdots \\
    P_n(s)
\end{bmatrix}
\begin{bmatrix}
    \sqrt{n}\tilde{P}(s) \\
    0 \\
    \vdots \\
    0
\end{bmatrix}
\leq \begin{bmatrix}
    I_n \\
    0 \\
    \vdots \\
    0
\end{bmatrix}
\]

(5.2.10)

After substituting for \( U \) from equation (5.2.9) into equation (5.2.10), we may get the following equation:

\[
Q \in \mathcal{L}_\infty^{m \times m} \quad \min_{R \in \mathcal{L}_\infty^{m \times m}} \| R_1 - Q \|_{L_\infty},
\]

(5.2.11)

where \( R_1, R_2 \) and \( Q \) are given by

\[
R_1 = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} P_i
\]

(5.2.12)

\[
Q = \sqrt{n}\tilde{P}(s)
\]

(5.2.13)

\[
R_2 = \frac{1}{\sqrt{n}}
\begin{bmatrix}
    P_1 + \beta P_2 - \alpha \sum_{i=3}^{n} P_i \\
    P_2 + \beta P_3 - \alpha \sum_{i=4}^{n} P_i \\
    \vdots \\
    P_n + \beta P_1 - \alpha \sum_{i=2}^{n} P_i
\end{bmatrix}
\]

(5.2.14)

The above results are summarized in the following theorem.
Theorem 5.5

\[
\begin{bmatrix}
P_1(s) & \hat{P}(s) \\
P_2(s) & \hat{P}(s)
\end{bmatrix}
\begin{bmatrix}
P_1(s) \\
P_2(s)
\end{bmatrix}
\begin{bmatrix}
\hat{P}(s) \\
\hat{P}(s)
\end{bmatrix}
\begin{bmatrix}
P_1(s) \\
P_2(s)
\end{bmatrix}
= Q \in \mathbb{R}^{2 \times 2}
\begin{bmatrix}
R_1 - Q \\
R_2
\end{bmatrix}_{L_\infty}
\]

where \( R_1, R_2 \) and \( Q \) are as defined above and \( \hat{P}(s) = \sqrt{n} Q(s) \).

Since the minimization in (5.2.11) is carried out over \( L_\infty \), we may call equation (5.2.11) a Generalized 2-Block Problem (GTBP), which needs to be solved for \( Q(s) \) and hence \( \hat{P}(s) = \sqrt{n} Q(s) \).

In equation (5.2.9), if we interchange the first and the last rows and then the first and the last columns we will obtain a new unitary matrix \( \hat{U}_1 \) given as

\[
\hat{U}_1 = \begin{bmatrix}
\beta I & -\alpha I & \cdots & -\alpha I & I \\
-\alpha I & -\alpha I & \cdots & I \\
\vdots & \ddots & \ddots & \vdots \\
-\alpha I & \cdots & \beta I & I \\
-I & -I & \cdots & I & I
\end{bmatrix}
\]

\( \hat{U}_1 \) will transform (5.2.6) into

\[
Q \in \mathbb{R}^{n \times n}
\begin{bmatrix}
\hat{R}_1 \\
\hat{R}_2 - Q
\end{bmatrix}_{L_\infty},
\]

where

\[
\hat{R}_1 = \frac{1}{\sqrt{n}}
\begin{bmatrix}
\beta P_1 - \alpha \sum_{i=2}^{n} P_i - P_n \\
\beta P_2 - \alpha \sum_{i=2}^{n} P_i - P_n \\
\vdots \\
\beta P_n - \alpha \sum_{i=1}^{n} P_i - P_n \\
\beta P_{n-1} - \alpha \sum_{i=1}^{n-2} P_i - P_n
\end{bmatrix}
\]

\[
\hat{R}_2 = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} P_i
\]

\[
Q = \sqrt{n} \hat{P}(s)
\]
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The need for the alternative formulation given by equation (5.2.15) will be clear in Section 5.4 where a solution will be given to this problem.

Remark 5.1  It is clear that if each of the plants $P_j(s)$ has $k$ modes, then $R_j(s)$ can have $n.k$ modes. This will result, generally, in an optimal plant $\hat{P}(s)$ with $n.k$ modes.

Remark 5.2  If each $P_i(s)$ has $k_i$ unstable poles, $R_i(s)$ will have up to $n.k_i$ unstable poles.

Since $R_j$ in (5.2.11) and $\hat{R}_j$ in (5.2.15), $j = 1, 2,$ are in $\mathcal{RH}_\infty$, a reformulation of the GTP will be given in the next subsection such that $R$ and $\hat{R}$ are in $\mathcal{RH}_\infty$.

5.2.1 The Generalized Two-Block Problem

The GTP formulated above is different from the standard $\mathcal{H}_\infty$ distance problem because $R_1$ and $R_2$ in (5.2.11) are in the space $\mathcal{RL}_\infty$, and, secondly, because the $Q$ sought is an $\mathcal{RC}_\infty$ transfer function matrix with the same number of unstable poles as the $P_i$’s, (in other problem the number of unstable poles of $Q$ may be any pre-specified number).

Therefore, in order to solve the GTP using the available algorithms for solving the standard $\mathcal{H}_\infty$ distance problem, we first need to do some manipulations as follows. Let $R_1(s)$ and $R_2(s)$ be decomposed into

$$R_1(s) = R_{1u}(s) + R_{1s}(s)$$
$$R_2(s) = R_{2u}(s) + R_{2s}(s)$$

where $R_{1u}(s), R_{2u}(s) \in \mathcal{RH}_\infty$ and $R_{1s}(s), R_{2s}(s) \in \mathcal{RH}_\infty$. From the $L_\infty$ norm property [27],

$$\|A + B\|_{L_\infty} \leq \|A\|_{L_\infty} + \|B\|_{L_\infty}$$

we may write equation (5.2.11) as

$$Q \in \min_{\mathbb{R}^{\infty \times m}} \left\| \begin{array}{c} R_1 - Q \\ R_2 \end{array} \right\|_{L_\infty} \leq \min_{\mathbb{R}^{\infty \times m}} \left\| \begin{array}{c} R_{1s} + R_{1u} - (Q_{1s} + Q_{1u}) \\ R_{2s} + R_{2u} \end{array} \right\|_{L_\infty}$$
The first 2-Block problem on the right hand side of (5.2.19) is the ordinary 2-Block problem defined in the literature, see for example [27]. And the second one can be converted into the standard one as we shall see in the next remark. So the GTBP can be reduced into two ordinary 2-Block problems (OTBPs) whose solutions give an upper bound for the optimal solution of (5.2.11), i.e. \( Q = Q_s + Q_u \).

**Remark 5.3** The problem

\[
\min_{Q_u \in \mathcal{H}_{\infty}^{m \times n}} \begin{bmatrix} R_{1u} - Q_u \\ R_{2u} \end{bmatrix}_{\mathcal{H}_{\infty}} + \min_{Q_u \in \mathcal{H}_{\infty}^{m \times n}} \begin{bmatrix} R_{1u} - Q_u \\ R_{2u} \end{bmatrix}_{\mathcal{L}_{\infty}}
\]

(5.2.19)

The above procedure can be used to solve the GTBP, but it is conservative. In the following subsection we propose a new procedure to reduce the GTBP into an OTBP with the same constraint on the solution \( Q(s) \) as before.

### 5.2.2 Reducing the GTBP to an OTBP

Before we reduce the GTBP into an OTBP, we first need to do some manipulations. Let \( R_1(s) \) be decomposed as before into

\[
R_1(s) = R_{1u}(s) + R_{1v}(s)
\]

(5.2.20)

where \( R_{1u}(s) \in \mathcal{H}_{\infty}^{+} \) and \( R_{1v}(s) \in \mathcal{H}_{\infty}^{-} \). Furthermore, let \( R_2(s) \) have an inner/anti-stable factorization of

\[
R_2(s) = R_{2u}(s)R_{2w}(s)
\]

(5.2.21)
where $R_{2n}(s) \in \mathbb{RH}_{\infty,+}$ is inner and $R_{2n}(s) \in \mathbb{RH}_{\infty,-}$. The formula for this inner/anti-stable factorization will be given in the next subsection. Then using the invariance property of inner functions, (5.2.11) can be written as

$$Q \in \mathbb{RL}_{\infty} \begin{bmatrix} I & 0 \\ 0 & R_{2n} \\ \end{bmatrix} \begin{bmatrix} R_{2n} + R_{1u} - Q \\ R_{2n}R_{2u} \end{bmatrix} \in \mathbb{L}_{\infty},$$

(5.2.22)

where $R_{2n}(s) = R_{2n}(-s)$. The above equation can be reduced to

$$Q \in \mathbb{RL}_{\infty} \begin{bmatrix} R_{1u} - \bar{Q} \\ R_{2u} \end{bmatrix} \in \mathbb{L}_{\infty},$$

(5.2.23)

where $\bar{Q} = Q - R_{1u}$, and should be in $\mathbb{RH}_{\infty,-k}$, where $k$ is assumed to be the number of unstable poles of $P_1$ in this problem. In other problems the value of $k$ may be specified as required from the statement of the problems.

Similarly, we can transform (5.2.15) to

$$Q \in \mathbb{RL}_{\infty} \begin{bmatrix} \bar{R}_{1u} \\ \bar{R}_{2u} - \bar{Q} \end{bmatrix} \in \mathbb{L}_{\infty},$$

(5.2.24)

where $\bar{Q} = Q - \bar{R}_{2n}$.

Problems (5.2.23) and (5.2.24) are now ready for solution using the standard procedures of the $H_{\infty}$ model matching problem with a constraint.

### 5.2.3 Multiplicative Inner/Anti-Stable Factorization

In this subsection we derive a state-space representation for the multiplicative inner/anti-stable factorization that we need in equation (5.2.21).

**Lemma 5.6** Let $G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{RL}_{\infty}^{nm}$ with no poles on the $jw$-axis. Then $G$ can be factorized into $G_i$ (inner) and $G_a$ (anti-stable) such that

$$G = G_iG_a$$

where $G_i$ and $G_a$ have the following state-space representations

$$G_i = \begin{bmatrix} -(A + F^TC)^T & -C^T \\ F & I \end{bmatrix}$$
and

\[ G_u = \begin{bmatrix} (A + F^TC) & -(B + F^TD) \\ -C & D \end{bmatrix} \]

where

\[ F = CX \]

and

\[ X = \text{Ric} \begin{bmatrix} -A^T & -C^T C \\ 0 & A \end{bmatrix} \]

A proof of the above Lemma follows by direct algebraic manipulations.

### 5.3 State Space Representation of The Central Plant Problem

In this section we will give a state space representation of \[ \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \] in equation (5.2.11).

Let

\[ P_i(s) = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \quad \forall \ i = 1, 2, ..., n. \]  

(5.3.1)

Then

\[
R_1 = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} P_i = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} 0 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & A_n & \vdots & \vdots \\ \frac{1}{\sqrt{n}} C_1 & \frac{1}{\sqrt{n}} C_2 & \cdots & \frac{1}{\sqrt{n}} C_n & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} D_i \end{bmatrix} \]

(5.3.2)
R_3 may be written as

\[
R_3 = \frac{1}{\sqrt{n}} \begin{bmatrix}
-P_1 + \beta P_2 - \alpha \sum_{i=2}^{n} P_i \\
-P_1 + \beta P_3 - \alpha \sum_{i=2}^{n} P_i \\
\vdots \\
-P_1 + \beta P_n - \alpha \sum_{i=2}^{n} P_i \\
\end{bmatrix} \begin{bmatrix}
R_{21} \\
R_{22} \\
\vdots \\
R_{2,n-1} \\
\end{bmatrix}
\]

(5.3.3)

where

\[
R_{2k} = \frac{1}{\sqrt{n}} (-P_1 + \beta P_k - \alpha \sum_{i=2}^{n} P_i), \quad k = 2, 3, ..., n
\]

\[
= \frac{1}{\sqrt{n}} \begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_n \\
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_n \\
\end{bmatrix}
\begin{bmatrix}
0 \\
A_k \\
\vdots \\
A_n \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
\end{bmatrix}
\begin{bmatrix}
-\alpha C_1 - \alpha C_2 \ldots + \beta C_k \ldots - \alpha C_n - D_1 + \beta D_k - \alpha \sum_{i=2}^{n} D_i \\
\end{bmatrix}
\]

(5.3.4)

where

\[
A = \text{diag} \{A_1, A_2, \ldots, A_n\},
\]

\[
B = [B_1, B_2, \ldots, B_n],
\]

\[
\tilde{C}_k = [-C_1 - \alpha C_2 \ldots + \beta C_k \ldots - \alpha C_n],
\]

(5.3.5)

and

\[
\tilde{D}_k = [-D_1 - \alpha D_2 \ldots + \beta D_k - \alpha D_{k+1} \ldots - \alpha D_n].
\]

(5.3.6)

We may write \(\tilde{C}_k\) as

\[
\tilde{C}_k = \chi_k \odot \tilde{C}
\]

(5.3.7)
where

\[ \tilde{\mathbf{C}} = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix}, \]

(5.3.8)

\[ \chi_k = \begin{bmatrix} -1 & -\alpha & -\alpha & \cdots & +\beta & -\alpha & \cdots & -\alpha \end{bmatrix}, \]

(5.3.9)

the definition of \( \Theta \) is clear from (5.3.5), and \( +\beta \) is the \( k^{th} \) element of \( \chi_k \).

Similarly, we may write \( \tilde{\mathbf{D}}_k \) as

\[ \tilde{\mathbf{D}}_k = \chi_k \odot \tilde{\mathbf{D}}, \]

(5.3.10)

where

\[ \tilde{\mathbf{D}} = \begin{bmatrix} D_1^T & D_2^T & \cdots & D_n^T \end{bmatrix}^T. \]

(5.3.11)

Therefore \( \mathbf{R}_2 \) is given by

\[
\mathbf{R}_2 = \frac{1}{\sqrt{n}} \begin{bmatrix} A & B \\ \tilde{C}_1 & \tilde{D}_1 \\ \tilde{C}_2 & \tilde{D}_2 \\ \vdots & \vdots \\ \tilde{C}_k & \tilde{D}_k \\ \vdots & \vdots \\ \tilde{C}_{n-1} & \tilde{D}_{n-1} \end{bmatrix} := \begin{bmatrix} A & B \\ \tilde{C}_2 & \tilde{D}_2 \end{bmatrix} \]

(5.3.12)

and we can write \( \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \) as

\[
\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} A & B \\ \tilde{C}_1 & \tilde{D}_1 \\ \tilde{C}_2 & \tilde{D}_2 \end{bmatrix} \]

(5.3.13)

Similarly, a state space representation of \( \begin{bmatrix} \tilde{R}_1 \\ \tilde{R}_2 \end{bmatrix} \) in (5.2.15) can be derived.

Before we solve the GTBP we first give in the next section a brief review of the general distance problem.
5.4 The General Distance Problem

First we will give a brief summary to the definitions and solutions of the ordinary 1-Block and 2-Block problems,[26, 27, 51, 37]. Then we will use these methods to get a solution to the GTBP, proposed in (5.2.11) or (5.2.15).

5.4.1 Definition

Given $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \in \mathcal{H}_{\infty}$, the General Distance Problem (GDP) is to find $Q \in \mathcal{H}_{\infty}$ such that

$$\left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_{\infty},$$

(5.4.1)

is minimized. The minimum value, denoted by $\gamma_0$, is the distance from $R$ to the set of (matrix) functions of the form $\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$, $Q \in \mathcal{H}_{\infty}$. This formulation is called the 4-Block Problem. When $\begin{bmatrix} R_{21} & R_{22} \end{bmatrix}$, or $\begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}$, is identically zero, (5.4.1) is reduced to a 2-Block Problem and is referred to as the Ordinary 2-Block Problem (OTBP) to distinguish it from the GTBP that we defined earlier in this Chapter. If both $\begin{bmatrix} R_{11} & R_{22} \end{bmatrix}$ and $\begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}$ are zero, this formulation is known as the 1-Block Problem or the optimal (Hankel) approximation problem [27], which we will refer to as the Ordinary 1-Block Problem (OOBP) to distinguish it from the Generalized 1-Block Problem that we will define later in this Section.

5.4.2 The One-Block Problem

Consider the antistable rational $m \times m$ system $R(s)$ with minimal realization

$$R(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$  
(5.4.2)
The sub-optimal ordinary 1-Block problem is defined as follows:

Given $\gamma > 0$, find $Q \in \mathcal{RH}_\infty$ such that

$$\|R - Q\|_\infty \leq \gamma. \quad (5.4.3)$$

From Nehari's theorem $\gamma \geq \|\Gamma_R\|$, where $\|\Gamma_R\|$ is the Hankel norm of $R$ denoted by $\sigma_1$.

5.4.2.1 Sub-Optimal Solution

In this subsection we review only one particular solution of the above OOBP. This solution is suitable to solve the generalized 1-Block problem that we will define in the next subsection.

**Lemma 5.7** [51] For any $D_1$ with $\sigma_{\text{max}}(D_1) \leq \gamma$, a solution to (5.4.3) is given by

$$Q(s) = \begin{bmatrix} -A^T + N(\gamma)(C^TD_1 - L_0B)B^T & N(\gamma)(C^TD_1 - L_0B) \\ CL_0 - D_1B^T & D - D_1 \end{bmatrix} (5.4.4)$$

$$:= \begin{bmatrix} A_Q & B_Q \\ C_Q & D_Q \end{bmatrix}, \quad (5.4.5)$$

where $L_0$ and $L_c$ are the observability and controllability gramians of $R$, and $N(\gamma) = (\gamma^2I - L_0L_c)^{-1}$. \[\Box\]

5.4.3 The Generalized One-Block Problem

The Generalized 1-Block Problem (GOBP) is defined as follows. Given $R(s) \in \mathcal{RH}_\infty^{\mathbb{R}}$, find $Q \in \mathcal{RH}_\infty^{\mathbb{R}}$, i.e. $Q$ has exactly $k$ unstable poles, such that equation (5.4.3) is satisfied.

In the next theorem we give a solution to this problem.

**Theorem 5.8** $Q(s)$ in (5.4.3) will have $k$ unstable poles if and only if $\sigma_k > \gamma > \sigma_{k+1}$, where the $\sigma_i$s are the Hankel singular values of $R$. \[\Box\]
To prove Theorem 5.8 we begin with a definition on the inertia of matrices, [6].

**Definition 5.2**  The inertia of a matrix $A$, denoted $\text{In}(A)$, is defined as $\text{In}(A) = (\pi, \nu, \delta)$, where $\pi = \pi(A), \nu = \nu(A), \delta = \delta(A)$ are respectively the numbers of eigenvalues of $A$ having positive, negative, and zero real parts.

The following lemmas are also needed in the proof of Theorem 5.8.

**Lemma 5.9**  [39]  Given complex $n \times n$ and $n \times m$ matrices, $A$ and $B$, and a Hermitian matrix $P = P^T$, satisfying

$$AP + PA^T + BB^T = 0$$  (5.4.6)

then

1. There is a unique solution to (5.4.6) if and only if $\lambda_i(A) + \lambda_j(A) \neq 0 \forall i, j$.

2. If $\delta(A) = 0$, then $\pi(A) \leq \nu(P), \nu(A) \leq \pi(P)$.

3. If $\delta(P) = 0$, then $\pi(P) \leq \nu(A), \nu(P) \leq \pi(A)$.

4. If $(A, B)$ is completely controllable then $\pi(A) = \nu(P), \nu(A) = \pi(P)$, $\delta(A) = \delta(P) = 0$. □

**Lemma 5.10**  Let $N_{Lo}$ be as defined in Lemma 5.7. If $\sigma_h > \gamma > \sigma_{h+1}$, then $\nu(N_{Lo}) = k$.

**Proof.**  Assume the system $(A, B, C, D)$ is a balanced realization, then

$$L_o = L_c = \Sigma = \text{diag}(\sigma_1, \sigma_2, ..., \sigma_h, \sigma_{h+1}, ..., \sigma_n).$$  (5.4.7)

But by definition

$$N^{-1} = \gamma^2I - L_oL_c = \text{diag}(\gamma^2 - \sigma_1^2, ..., \gamma^2 - \sigma_h^2, \gamma^2 - \sigma_{h+1}^2, ..., \gamma^2 - \sigma_n^2).$$  (5.4.8)
Since \( \sigma_k > \gamma > \sigma_{k+1} \), then the first \( k \) terms of \( N^{-1} \), and similarly those of \( N \), are negative, while the rest of the terms are positive. Then \( \nu(NL_0) = k \) and \( \pi(NL_0) = n - k \) since \( NL_0 \) is a Hermitian matrix and has full rank for all \( \gamma \neq \sigma_i \).

The following facts are also needed in the proof, (See the proof of Lemma 4.1 in [51]).

- \( A_0 \) does not have any poles on the \( jw \)-axis, (i.e. \( \delta(A_0) = 0 \)).
- \( A_0 \) satisfies the following Lyapunov equation
  \[
  A_0(NL_0) + (NL_0)A_0^T + B_0B_0^T + NC^T(\gamma I - D_1D_1^T)CN^T = 0. \tag{5.4.9}
  \]

**Proof of Theorem 5.8** The proof is immediate from Lemmas 5.9 and 5.10.

When \( G(s) \in \mathbb{RH}_{\infty}^{m \times n} \), a solution for this case is reported in [39]. For completeness we give this solution here, following the next lemma.

**Lemma 5.11** [39] Let \( B \in \mathbb{C}^{n \times m} \), and \( C \in \mathbb{C}^{p \times n} \) have rank \( r \) and satisfy \( C^TC = BB^T \). Then there exists a unitary \( D \in \mathbb{C}^{(m+p-r) \times (m+p-r)} \) where \( D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} \), with \( D_{11} = -(C^T)^\dagger B = -C(B^\dagger)^\dagger \) such that
  \[
  D_{11} = -B \begin{bmatrix} C^T & 0 \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} = 0. \tag{5.4.9}
  \]

The construction of \( D \) is completed by choosing \( D_{12} \) and \( D_{21} \) such that the matrices \( \begin{bmatrix} U_1 & D_{12} \end{bmatrix} \) and \( \begin{bmatrix} W_1 & D_{21} \end{bmatrix} \) are unitary, where \( U_1 \) is obtained from the singular value decomposition of \( C \) as \( C = U_1\Sigma_1V_1 \) and \( W_1 = B^TV_1\Sigma_1^{-1} \). Then the following theorem gives the solution of the complement of the GOBP defined before.

**Theorem 5.12** [39] Let \( G \in \mathbb{RH}_{\infty}^{m \times n} \), satisfying \( \sigma_k(G) > \sigma_{k+1}(G) \). Then there exists \( Q \in \mathbb{RH}_{\infty}^{(m-k) \times (n-k)} \) such that \( ||G + Q||_{\infty} \leq \sigma \) if and only if \( \sigma \geq \sigma_{k+1}(G) \). Furthermore all solutions to
  \[
  ||G + Q||_{\infty} \leq \sigma = \sigma_{k+1}(G)
  \]
  are given by
  \[
  Q = \mathcal{F}(J, \Phi), \quad \Phi \in \mathbb{RH}_{\infty}^{(m-k) \times (n-k)}, \quad ||\Phi||_{\infty} \leq \sigma
  \]
where \( \mathcal{F}(J, \Phi) \) is the lower fractional transformation of \( J \) and \( \Phi \) and \( J \) is constructed as follows. Let \( G = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & 0 \end{bmatrix} \) be a realization of \( G \) with controllability and observability gramians given by \( \begin{bmatrix} \sigma I_r & 0 \\ 0 & X_2 \end{bmatrix} \) and \( \begin{bmatrix} \sigma I_r & 0 \\ 0 & Y_2 \end{bmatrix} \), respectively. Let the multiplicity of \( \sigma_{k+1} \) and \( Z_2 = X_2Y_2 - \sigma^2 I \) invertible. Define \( D_k = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} \in C^{n \times m-\delta \times (\delta + m - \delta)} \) according to Lemma 5.11 where \( l = \text{rank}(C_1) = \text{rank}(B_1) \), and

\[
\begin{bmatrix} C_1^T & 0 \\ D_k & B_1 & 0 \end{bmatrix} = 0.
\]

Then \( J \) is given by

\[
J = \begin{bmatrix} -A_{12} - B_1B_2^T & B_1 (Z_2^T)^{-1} C_1^T D_{12} \\ -C_2X_2 - \sigma D_{11}B_2^T & \sigma D_{11} & D_{12} \\ -D_{21}B_2^T & D_{21} & 0 \end{bmatrix}, \tag{5.4.10}
\]

where

\[
B_1 = (Z_2^T)^{-1}(Y_2B_2 + \sigma C_1^T D_{11}).
\]

It is clear from this theorem that the computations involved in solving the GOBP is more than that proposed by Theorem 5.8.

### 5.4.4 The Two-Block Problem

An iterative scheme, known as \( \gamma \)-iteration, to solve the general distance problem is considered in [27]. The idea is based on an initial guess for the minimal norm, \( \gamma_0 \), after which the distance problem is simplified to an equivalent best approximation problem which can be solved by existing techniques. The following theorems state the solution of the OTBP.
Theorem 5.13 [27] Consider the following 2-Block problem

\[ \gamma_0 = \min_{Q \in \mathcal{RH}_\infty^{\infty \times m}} \| R - \begin{bmatrix} Q \\ 0 \end{bmatrix} \|_\infty \]  

(5.4.11)

where \( R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \in \mathcal{RH}_\infty^{\infty \times m} \). Then \( \gamma_0 = \| \Gamma_R \| \).

Theorem 5.14 [27] Assume \( Q \in \mathcal{RH}_\infty \) and \( \gamma > \| R \|_\infty \). Then

\[ \| \begin{bmatrix} R_1 - Q \\ R_2 \end{bmatrix} \|_\infty \leq \gamma, \]

(5.4.12)

if and only if

\[ \| (R_1 - Q)M^{-1} \|_\infty \leq 1, \]

(5.4.13)

where \( M \) is the spectral factor of the para-Hermitian matrix \( (\gamma^2 - R_2 R_2^*) \).

Solution to equation (5.4.11) can be obtained by considering the following best approximation (1-Block) problem

\[ Q \in \mathcal{RH}_\infty^{\infty \times m} \| R_1 M^{-1} - Q \|_\infty = \| \Gamma_R M^{-1} \|, \]

(5.4.14)

where \( Q = QM \).

5.5 Solution of The GTBP

For the problem we have proposed in equation (5.2.11), we only need the sub-optimal solution since we are looking mainly for \( Q(s) \in \mathcal{RH}_\infty^{\infty \times k} \), i.e. \( Q(s) \) is analytic in \( C_+ \) except for \( k \) poles in \( C_+ \). Therefore any of the methods described in Section 5.4 can be used to solve the GTBP proposed in Section 5.2.

In the last step of the solution of the ordinary 2-Block problem, we will get a stable \( Q \) if we choose \( \gamma > \sigma_1 \) as required in the solution of the OOBP. But we will choose \( \gamma \) such that \( \sigma_k > \gamma > \sigma_{k+1} \), where the \( \sigma_k \) are the Hankel singular values of \( R_1 M^{-1} \) in (5.4.14). In this case the solution \( Q(s) \) will have \( k \) unstable poles as stated by Theorem 5.8. Therefore we choose a suitable \( \gamma \) such that \( \sigma_k > \gamma > \sigma_{k+1} \).
When using the second form of the 1-Block problem reported in [39], we need to replace $s$ with $-s$ in $\hat{A}$ before we seek the solution.

In the following subsection we will outline the main steps involve in finding a central plant from a given set of plants. We will assume here that all the given plants have the same number of unstable poles. We make this assumption since it is a major one in robust stabilization theory, which will be the second step toward solving the simultaneous stabilization problem.

5.5.1 Algorithm 5.1

The following algorithm will outline the main steps involved in finding a central plant from a given set of plants according to the theory presented in this chapter.

**Step 0 (Initialization):** Given a set of plants $P_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \in \mathbb{R}^{n \times m}$, $i = 1, 2, \cdots, l$ with the same number of unstable poles.

**Step 1:** Form $R_1$ and $R_2$ using equations (5.3.2) and (5.3.12).

**Step 2:** Find the factorization in equation\(^1\) (5.2.20) and equation\(^2\) (5.2.21).

**Step 3:** Reduce the 2-Block problem given in equation (5.2.23) to a generalized 1-Block problem using the result of Theorem 5.14.

**Step 4:** Solve the generalized 1-Block problem obtained in Step 3 using the results of Theorem 5.8 and Lemma 5.7.

**Step 5:** Recover from the result of Step 4 the central plant $\hat{P}(s)$.

In the following section we will describe briefly how robust stabilization theory can be used to robustly stabilize the central plant $\hat{P}(s)$, obtained from Algorithm 5.1. The main idea here is in choosing the weights when forming the augmented plant.

\(^1\)You may use the MATLAB file stabproj.m.

\(^2\)Use the procedure described in Lemma 5.6.
5.6 The Solution of The SSP

In this section, we propose an algorithm for solving the SSP using the formulation we proposed in this Chapter, equation (5.2.11) or (5.2.15), and Robust Stabilization Theory (RST). First, we discuss the relevant Robust Stabilization Theory.

5.6.1 Robust Stabilization Theory

Define a family \( P \) of neighbouring plants to consist of all real-rational matrices \( P + \Delta P \) having the same number (in terms of McMillan degree) of poles in \( \mathbb{C}^+ \) as \( P \). Suppose also that the feedback system is internally stable for \( \Delta P = 0 \). Let \( W \) be the radius function belonging to \( \mathcal{JR}_{\infty} \) and bounding the perturbation \( \Delta P \) in the sense that

\[
||\Delta P(jw)|| < |W(jw)| \quad \forall \ 0 \leq w < \infty. \tag{5.6.1}
\]

Lemma 5.15 [34] A real-rational proper \( K \) stabilizes all plants in \( P \) if and only if \( K \) stabilizes the nominal plant \( P \) and

\[
||WK(I - PK)^{-1}||_{\infty} \leq 1. \tag{5.6.2}
\]

To apply the above result to the new formulation of the SSP, we take \( W \) to be the bound of \( \Delta P(jw) \) for all \( i = 1, 2, \ldots, n \) and \( w \in [0, \infty) \), i.e. we define \( \Delta_a(s) \) as follows

\[
\Delta_a(jw) = \max_{w \in [0, \infty)} (\Delta P_i(jw)), \quad \forall \ i = 1, 2, \ldots, n.
\]

Then we may state the following lemma.

Lemma 5.16 Let \( \tilde{P} \) be as defined in (5.2.11) or (5.2.15). Then one can find a simultaneous stabilizing controller \( K \) if

\[
P(s) = \tilde{P}(s) + \Delta_a(s), \quad \delta[\Delta_a(jw)] \leq W \tag{5.6.3}
\]

is robustly stabilizable.

The algorithm in the following subsection summarizes the steps involved in solving the SSP in conjunction with RST.
5.6.2 Algorithm 5.2

The following algorithm is proposed to solve the SSP.

**Step 0:** Solve (5.2.11) or (5.2.15) for best \( \hat{P}(s) \) using Algorithm 5.1.

**Step 1:** Calculate
\[
\Delta_{a}(jw) = \max_i \max_w \sigma(P_i(jw) - \hat{P}(jw)), \quad \forall w \in [0, \infty).
\]

**Step 2:** Fit\(^3\) the data of step 1 to get a stable, minimum phase weight, \( W(s) \).

**Step 3:** Augment \( W(s) \) and \( \hat{P}(s) \) as in Figure 5.1.

**Step 4:** Run \( H_\infty - \) subroutine to find \( K \).

**Step 5:** If \( \gamma < 1 \), then \( \hat{P}(s) \) is robustly stabilizable and the controller obtained will simultaneously stabilize all the plants given in (5.2.1).

**Step 6:** If \( \gamma > 1 \), check first if the controller stabilizes all the given plants. Otherwise you may go to Steps 1 and 2 for another \( W(s) \).

Since robust stabilization theory gives a sufficient conditions, it is not easy to conclude that if \( \gamma > 1 \) then the given systems are not simultaneously stabilizable.

Another alternative for the above algorithm is to use the Loop-Shaping Design Procedure

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\(^3\) A subroutine "fitmag" in the MATLAB \( \mu \)-toolbox can be used.
(LSDP) of McFarlane and Glover [62] to stabilize the central plant \( \hat{P}(s) \) and then to check if the resulting controller stabilizes the given discrete set of plants. It might be necessary in the design phase to refine this process a few times before one can reach to a simultaneously stabilizing controller. A description of the LSDP will be given in Chapter 7 with application to a real design problem.

5.7 Conclusion

The simultaneous stabilization problem for linear multivariable time-invariant systems has been considered in two stages. First a generalized 2-Block \( L_\infty \) optimization problem was formulated to obtain a "central" plant from the given set of plants to be stabilized. The generalized 2-Block problem was reduced to an ordinary 2-Block problem with the condition that its solution would have a required number of unstable poles. A generalized 1-Block problem was defined and its solution was given as a preamble to the solution of the GTBP. Theoretical results which were needed to perform the above work were developed.

Finally robust stabilization theory was incorporated to provide a sufficient condition for finding a solution to the simultaneous stabilization problem. Simulation results to test the theory presented in this chapter will be reported in Chapter 7.

The novelty of the procedure is in the determination of the central plant. Note also that this procedure introduces some conservatism since we aim to robustly stabilize over a wider set of plants than that represented in the problem statement. Also the central plant obtained is in some sense suboptimal; this is because the original formulation of the problem resulted in a nonstandard optimization problem which is not tackled in this work. Further it should stressed that the criterion used here for defining the "central" plant may not always be the best in practice. For example, it may turn out that the "central" plant may actually be an atypical description of the particular plant even though it is the closest to all the candidate plant models. In the absence of any other knowledge, however, it does seem a practical approach to follow.
Chapter 6

A State Space Approach To The Simultaneous Stabilization Problem

6.1 Introduction

The simultaneous stabilization problem of more than two MIMO plants is not yet solved as already stated in Chapter 3. Also a computational procedure to verify the necessary and sufficient conditions given in [80, 95, 96], for reducing the SSP of \( l (> 2) \) plants into an equivalent SSP of \( l - 1 \) plants by a single stable controller is not available. In this sense therefore the SSP remains an open and difficult problem.

In this chapter, we are going to present a state-space approach for the formulation of the simultaneous stabilization problem of multivariable systems. It is shown that, for two stabilizing controllers of two plants to be equal, i.e. \( K_0(Q_0(s)) = K_1(Q_1(s)) \), a new condition on the Youla parameters \( Q_0(s) \) and \( Q_1(s) \) is to be satisfied. Under certain conditions, closed form solutions for \( Q_0(s) \) and \( Q_1(s) \) may be given.

This method will then be extended to the general case of two or more systems. In general, the problem is reduced to that of searching for a stable real matrix which is an element of a larger matrix that is to be stabilized. When there are only two plants and a certain condition is satisfied, a closed form solution for this stable real matrix will be given, otherwise numerical algorithms will be proposed to solve the problem. Although the problem of more than two plants will be simplified, it is still difficult to solve and further
work is required on the algorithms.

This chapter is organized as follows. In Section 6.2, the parametrization of all stabilizing controllers is reviewed. In Section 6.3 a new formulation for simultaneously stabilizing two MIMO plants is presented. Section 6.4 gives conditions on the existence of a closed form solution as a special case to the solution of the SSP of two plants. When these conditions are satisfied, a closed form controller will be given. In Section 6.5 we extend the formulation to the general case of two systems or more. Section 6.6 discusses different possibilities for solving the general SSP with two or more systems. In Section 6.7 we discuss various numerical algorithms which might be used to satisfy the conditions we obtain. Illustrative numerical examples are given in Section 6.8. Conclusions are given in Section 6.9.

6.2 All Stabilizing Controllers

Consider the feedback configuration of Figure 6.1, where $G(s)$ is a given plant, and $K(s)$ is a controller to be designed for internal stabilization.

Let $G(s)$ have the state-space realization

$$G(s) := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$
with $(A, B)$ controllable and $(A, C)$ observable, where the dimensions of the real matrices $(A, B, C, D)$ are $n \times n$, $n \times m$, $r \times n$ and $r \times m$, respectively.

Let $G(s)$ have a doubly coprime factorization

$$G(s) = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

and $X, Y, \tilde{X}$ and $\tilde{Y}$ satisfy the Bezout identity, i.e.

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I \quad (6.2.1)$$

where the matrices $(N, M, \tilde{N}, \tilde{M}, X, Y, \tilde{X}, \tilde{Y})$ all belong to $\mathcal{RH}_\infty$. The matrices $(N, M, X, Y, \tilde{N}, \tilde{M}, \tilde{X}, \tilde{Y})$ can each be expressed in state-space form as follows, by choosing a real matrix $F$ such that $A + BF$ is stable and similarly by choosing a real matrix $H$ so that $A + HC$ is stable:

$$\begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} := \begin{bmatrix} A + BF & B - H \\ F & I \\ C + DF & D - I \end{bmatrix} \quad (6.2.2)$$

and

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} := \begin{bmatrix} A + HC & -(B + HD) \\ F & I \\ C & -D - I \end{bmatrix} \quad (6.2.3)$$

Then it is well known [34, 61] that the set of all stabilizing controllers for the given plant $G(s)$ can be parameterized by

$$K(s) = -(Y - MQ)(X - NQ)^{-1} \quad (6.2.4)$$

where $Q(s) \in \mathcal{RH}_\infty$ and has the state-space realization

$$Q(s) := \begin{bmatrix} A_q & B_q \\ C_q & D_q \end{bmatrix}$$

where the dimensions of the matrices $(A_q, B_q, C_q, D_q)$ are $n_q \times n_q$, $n_q \times r$, $m \times n_q$ and $m \times r$, respectively.
6.3 New Formulation of the SSP

In this section we will give a new formulation to the problem of simultaneously stabilizing two MIMO systems.

Given two \( r \times m \) plants \( G_0 \) and \( G_1 \in \mathbb{R}L_\infty \). Assume that for \( i = 0,1 \), there exist an RCF \((N_i, M_i)\), an LCF \((\tilde{N}_i, \tilde{M}_i)\) and matrices \( X_i, Y_i, \tilde{X}_i, \tilde{Y}_i \) such that

\[
\begin{bmatrix}
\dot{X}_i & -\tilde{Y}_i \\
-\tilde{N}_i & \tilde{M}_i
\end{bmatrix}
\begin{bmatrix}
M_i & Y_i \\
N_i & X_i
\end{bmatrix}
= \begin{bmatrix}
I_m & 0 \\
0 & I_r
\end{bmatrix}.
\]  

(6.3.1)

All the eight matrices in the above equation are in \( \mathbb{R}H_\infty \).

The condition for simultaneously stabilizing two plants can be written as

\[
K_0(Q_0) = K_1(Q_1)
\]  

(6.3.2)

where \( K_0 \) and \( K_1 \) are the stabilizing controllers of \( G_0 \) and \( G_1 \), respectively, parameterized in terms of \( Q_0 \) and \( Q_1, Q_i \in \mathbb{R}H_\infty \). Then using Youla’s parameterization of all stabilizing controllers, the RCF of \( K_0 \) and the LCF of \( K_1 \), equation (6.3.2) can be written as follows

\[
(Y_0 - M_0 Q_0)(X_0 - N_0 Q_0)^{-1} = (\tilde{X}_1 - Q_1 \tilde{N}_1)^{-1}(\tilde{Y}_1 - Q_1 \tilde{M}_1)
\]

The above equation can be rewritten as

\[
(\tilde{X}_1 - Q_1 \tilde{N}_1)(Y_0 - M_0 Q_0) = (\tilde{Y}_1 - Q_1 \tilde{M}_1)(X_0 - N_0 Q_0).
\]

By expanding the left and right hand sides of the above equation we get

\[
\tilde{X}_1 Y_0 - \tilde{X}_1 M_0 Q_0 - Q_1 \tilde{N}_1 Y_0 + Q_1 \tilde{N}_1 M_0 Q_0
= \tilde{Y}_1 X_0 - \tilde{Y}_1 N_0 Q_0 - Q_1 \tilde{M}_1 X_0 + Q_1 \tilde{M}_1 N_0 Q_0.
\]

Grouping together the terms that contain \( Q_0 \) and any variable with 0 subscript in one factor and doing the same for \( Q_1 \) with some simplifications, we get the following simplified equation

\[
\begin{bmatrix}
Q_1 \\
-\tilde{N}_1 & \tilde{M}_1
\end{bmatrix} + \begin{bmatrix}
\tilde{X}_1 & -\tilde{Y}_1
\end{bmatrix} \begin{bmatrix}
M_0 & Y_0 \\
N_0 & X_0
\end{bmatrix} Q_0 = 0.
\]
For notational simplicity let $A(s, Q_1(s))$ and $B(s, Q_0(s))$ be defined as

$$A(s, Q_1(s)) = \{ Q_1 [-\tilde{N}_1 \tilde{M}_1] + [\tilde{X}_1 -\tilde{Y}_1] \}$$

and

$$B(s, Q_0(s)) = \left\{ \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q_0 + \begin{bmatrix} Y_0 \\ X_0 \end{bmatrix} \right\}$$

Then we have the following equation in $Q_0(s)$ and $Q_1(s)$

$$A(s, Q_1(s))B(s, Q_0(s)) = 0.$$ 

Note that for simplicity, $A(s, Q_1(s))$ and $B(s, Q_0(s))$ will also be later written as $A(Q_1)$ and $B(Q_0)$, respectively.

Before we leave this section, we give a property of $A(s, Q_1(s))$ and $B(s, Q_0(s))$ as follows.

**Lemma 6.1** Let $A(s, Q_1(s))$ and $B(s, Q_0(s))$ be as defined in (6.3.3) and (6.3.4). Then $A(s, Q_1(s)) \neq 0$ and $B(s, Q_0(s)) \neq 0$.

**Proof.** It is easy to see that

$$A(s, Q_1(s)) = \begin{bmatrix} I & Q_1 \end{bmatrix} \begin{bmatrix} \tilde{X}_1 & -\tilde{Y}_1 \\ -\tilde{N}_1 & \tilde{M}_1 \end{bmatrix}.$$ 

If $A(s, Q_1(s)) = 0$, then due to invertibility of $\begin{bmatrix} \tilde{X}_1 & -\tilde{Y}_1 \\ -\tilde{N}_1 & \tilde{M}_1 \end{bmatrix}$, we have

$$\begin{bmatrix} I & Q_1 \end{bmatrix} = 0.$$ 

That is a contradiction. Thus, $A(s, Q_1(s)) \neq 0$. Similarly, it can be shown that $B(s, Q_0(s)) \neq 0$. 

So the problem of finding a single controller that simultaneously stabilizes two MIMO systems is reduced to finding two stable parameters, $Q_0(s)$ and $Q_1(s)$ that satisfy equation (6.3.5).

The method of this section will be extended to the general case of more than two plants. Before doing that, we first give in the next section a solution for a special case of two plants.
6.4 Solution of the SSP of Two Plants: Special Case

In the previous section the simultaneous stabilization problem of two plants was reduced to finding stable functions \( Q_0(s) \) and \( Q_1(s) \) such that the condition \( A(Q_1)B(Q_0) = 0 \) is satisfied, where \( A(Q_1) \) and \( B(Q_0) \) are as defined in equation (6.3.5). In this section we show how to find stable \( Q_0 \) and \( Q_1 \) to satisfy this condition, for a special case (see Theorem 6.2 below).

Let \( A(Q_1) \) and \( B(Q_0) \) of equations (6.3.3-6.3.4) be written as

\[
A(Q_1) = \begin{bmatrix}
\dot{X}_1 & -\dot{Y}_1 \\
I & Q_1
\end{bmatrix} + Q_1 \begin{bmatrix}
-N_1 & M_1
\end{bmatrix}
\]

\[
B(Q_0) = \begin{bmatrix}
Y_0 \\
X_0
\end{bmatrix} - \begin{bmatrix}
M_0 \\
N_0
\end{bmatrix} Q_0
\]

Then using equations (6.2.2-6.2.3) and after some manipulations, the state space representations of \( A(Q_1) \) and \( B(Q_0) \) are

\[
A(Q_1) = \begin{bmatrix}
A_q & B_q C_1 & -B_q D_1 & B_q^1 \\
0 & A_1 + H_1 C_1 & -(B_1 + H_1 D_1) & H_1 \\
C_q & D_q C_1 + F_q & (-D_q D_1 + I) & D_q^1
\end{bmatrix}
(6.4.1)
\]

\[
B(Q_0) = \begin{bmatrix}
A_q + B_q F_0 & B_q C_q & -(B_q D_q^2 + H_0) \\
0 & A_q & -B_q^1 \\
F_0 & C_q & -D_q^1 \\
C_0 + D_0 F_0 & D_0 C_q & -(D_0 D_q^2 + I)
\end{bmatrix}
(6.4.2)
\]

\[
Q_i(s) = \begin{bmatrix}
A_q^i & B_q^i \\
C_q^i & D_q^i
\end{bmatrix} \text{ for } i = 0, 1.
(6.4.3)
\]

**Theorem 6.2** Let \( A(Q_1), B(Q_0) \) and \( Q_i(s) \) be defined as in (6.4.1-6.4.3). For strictly proper systems, if \( H_0, H_1, F_0 \) and \( F_1 \) in (6.2.2-6.2.3) are chosen such that

\[
B_1 F_0 = H_1 C_0
(6.4.4)
\]
\[ B_0F_1 = H_0C_1 \]  \hspace{1cm} (6.4.5)

and

\[ (A_i + B_iF_i + H_iC_i) \text{ for } i = 0, 1 \]  \hspace{1cm} (6.4.6)

are stable, then \( Q_0 \) and \( Q_1 \) that make \( A(Q_1)B(Q_0) = 0 \) may be given by

\[
Q_0 = \begin{bmatrix}
A_1 + B_1F_1 + H_1C_1 & H_1 \\
F_1 & 0
\end{bmatrix} \hspace{1cm} (6.4.7)
\]

\[
Q_1 = \begin{bmatrix}
A_0 + B_0F_0 + H_0C_0 & H_0 \\
F_0 & 0
\end{bmatrix} \hspace{1cm} (6.4.8)
\]

**Proof.** Since \( G_0 \) and \( G_1 \) are strictly proper, let \( D_0 = D_1 = 0 \) in (6.4.1-6.4.2). Then substitute for \( Q_0 \) and \( Q_1 \) from (6.4.7-6.4.8) in (6.4.1-6.4.2). After some elementary matrix operations, it is easy to show that

\[ A(Q_1)B(Q_0) = 0. \]

\[ \square \]

The following proposition is a direct result of Theorem 6.2.

**Proposition 6.3** Given two strictly proper and strongly stabilizable systems, \( G_0(s) \) and \( G_1(s) \). If the conditions given in Theorem 6.2 are satisfied, then a simultaneously stabilizing controller for the two plants, \( G_0(s) \) and \( G_1(s) \), is given by

\[
K(s) = \begin{bmatrix}
A_0 + B_0F_0 + H_0C_0 & B_0F_1 & H_0 \\
H_1C_0 & A_1 + B_1F_1 + H_1C_1 & H_1 \\
F_0 & F_1 & 0
\end{bmatrix} \hspace{1cm} (6.4.9)
\]

\[ \square \]

**Remark 6.1** In Conditions (6.4.4) and (6.4.5), if the two systems have the same state dimension, the number of unknowns are \( n \times (m + r) \) and the number of equations
are $n^2$. To state a condition on the existence of a non-zero solution, let us write (6.4.4) as

$$
\begin{bmatrix}
B_1 \otimes I_n & -I_n \otimes C_0
\end{bmatrix}
\begin{bmatrix}
\text{vec}(F_0) \\
\text{vec}(H_1)
\end{bmatrix} = 0.
$$

(6.4.10)

where $\otimes$ is the Kronecker product and vec($\cdot$) is a column vector consisting of the rows of the matrix ($\cdot$).

**Remark 6.3** The parametrization of all state-feedback matrices $F_i$ such that $(A_i + B_i F_i)$ is stable is given in [68].

The following lemma states a necessary and sufficient condition on the existence of a non-trivial solution to (6.4.10).

**Lemma 6.4** [4] The homogeneous system $Mx = 0$ has a non-trivial solution if and only if the rank of $M$ is less than the number of columns of $M$.

**Remark 6.3** A clear method to solve for the real matrices $F_0, F_1, H_0$ and $H_1$, such that the assumptions on these matrices plus conditions (6.4.4-6.4.6) are satisfied, is not yet available and looks to be a difficult problem in general. It may be formulated as a constrained optimization problem.

**Example 6.1** The following illustrative example demonstrates the above theory. Assume that the following two plants are given.

$$
G_0 = \begin{bmatrix}
0 & 1 & 1 \\
3 & -2 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \text{ and } G_1 = \begin{bmatrix}
0 & 1 & 1 \\
4 & -3 & 1 \\
1 & 0 & 0 \\
0 & 2 & 0
\end{bmatrix}.
$$

The eigenvalues of $G_0(s)$ are $\approx [1 -3]$, and the eigenvalues of $G_1(s)$ are $\approx [1 -4]$.

Candidate $F_0, F_1, H_0$ and $H_1$ which satisfy conditions (6.4.4-6.4.6) are given by:

$$
F_0 = \begin{bmatrix}
-1.8375 & -0.7322 \\
-0.4264 & -0.3058
\end{bmatrix}, H_0 = \begin{bmatrix}
-1.4036 & -1.1729 \\
-1.1729 & -1.2652
\end{bmatrix},
$$
The eigenvalues of \((A_0 + B_0F_0 + H_0C_0)\) are \([-2.9492, -4.5951]\), and the eigenvalues of \((A_1 + B_1F_1 + H_1C_1)\) are \([-3.1609, -7.5258]\).

The controller is

\[
K = \begin{bmatrix}
-3.2411 & -0.9051 & -1.8051 & -0.5430 & -1.4036 & -1.1729 \\
-0.4368 & -4.3032 & -2.4855 & -0.9500 & -1.1729 & -1.2652 \\
-1.8375 & -0.7322 & -3.6426 & -1.0078 & -1.3875 & -0.7322 \\
-2.6903 & -1.3438 & -1.8561 & -7.0441 & -2.6903 & -1.3438 \\
-1.8375 & -0.7322 & -1.8051 & -0.5430 & 0 & 0 \\
-0.4264 & -0.3058 & -0.6803 & -0.4066 & 0 & 0
\end{bmatrix}.
\]

The eigenvalues of \(K\) are \([-8.9478, -1.8224, -3.0318, -4.4291]\). The eigenvalues of the closed loop \(T_0(G_0, K)\) are \([-7.5258, -1.5925, -3.0764, -3.1609, -3.1843, -1.6911]\), and those of \(T_1(G_1, K)\) are \([-4.5037 \pm 2.18881, -5.5225, -2.6970, -2.0026, -2.0016]\).

### 6.5 Formulation of the SSP for \(l\) Plants

In Section 6.3, the SSP for two plants was formulated. A new condition was stated to solve this problem. In this section we will extend the formulation of Section 6.3 for more than two strictly proper plants.

The condition for simultaneously stabilizing two plants, namely

\[
K_0(Q_0) = K_1(Q_1)
\]

is reduced in Section 6.3 to that of finding a stable \(Q_0(s)\) and a stable \(Q_1(s)\) such that

\[
A(Q_1)B(Q_0) = 0.
\]

For \(l \geq 2\) plants, condition (6.5.1) becomes

\[
K_0(Q_0) = K_1(Q_1) = \ldots = K_{l-1}(Q_{l-1})
\]
This condition may be broken down into the following set of equations

\[ K_0(Q_0) = K_1(Q_1) \]
\[ K_0(Q_0) = K_2(Q_2) \]
\[ \vdots \]
\[ K_0(Q_0) = K_{i-1}(Q_{i-1}), \]  

(6.5.4)

which can be reduced, following the arguments in Section 6.3, to

\[ A_i(Q_i)Q_0 = 0 \]
\[ A_2(Q_2)Q_0 = 0 \]
\[ \vdots \]
\[ A_{i-1}(Q_{i-1})Q_0 = 0. \]  

(6.5.5)

Each of the above equations can be written as

\[
\begin{bmatrix}
I & Q_i \\
-\tilde{N}_i & \tilde{M}_i
\end{bmatrix}
\begin{bmatrix}
X_i \\
Y_i
\end{bmatrix}
= \begin{bmatrix}
M_0 & Y_0 \\
N_0 & X_0
\end{bmatrix}
\begin{bmatrix}
Q_0 \\
-\mathbf{I}
\end{bmatrix} = 0 \quad \text{for } i = 1, 2, \ldots, l - 1. \tag{6.5.6}
\]

Let (6.5.6) be written as

\[ \tilde{Q}_i S_i \tilde{Q}_0 = 0 \quad \text{for } i = 1, 2, \ldots, l - 1, \tag{6.5.7}\]

where

\[ \tilde{Q}_0 = \begin{bmatrix}
Q_0(s) \\
-\mathbf{I}
\end{bmatrix} = \begin{bmatrix}
A_0^s & B_0^s \\
C_0^s & D_0^s \\
0 & -\mathbf{I}
\end{bmatrix}, \]

\[ \tilde{Q}_i = \begin{bmatrix}
I & Q_i(s)
\end{bmatrix} = \begin{bmatrix}
A_i^s & 0 \\
C_i^s & I
\end{bmatrix} \begin{bmatrix}
0 & B_i^s \\
I & D_i^s
\end{bmatrix} \quad \text{for } i = 1, 2, \ldots, l - 1 \]

\[ S_i = \begin{bmatrix}
A_i + H_i C_i & -B_i & H_i \\
F_i & I & 0 \\
C_i & 0 & I
\end{bmatrix} \begin{bmatrix}
A_0 + B_0 F_0 & B_0 & -H_0 \\
F_0 & I & 0 \\
C_0 & 0 & I
\end{bmatrix} = \begin{bmatrix}
A_i + H_i C_i & -B_i F_0 + H_i C_0 & -B_i & H_i \\
0 & A_0 + B_0 F_0 & B_0 & -H_0 \\
F_i & F_0 & I & 0 \\
C_i & C_0 & 0 & I
\end{bmatrix} \]
Combine $S_i$ and $\bar{Q}_0$ as

$$\tilde{S}_i = S_i \bar{Q}_0, \quad i = 1, 2, \ldots, l - 1$$

Then equation (6.5.7), with the help of equation (6.5.8), can be written as

$$\begin{bmatrix}
\tilde{Q}_1 & 0 & \cdots & 0 \\
0 & \tilde{Q}_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \tilde{Q}_{l-1}
\end{bmatrix}
\begin{bmatrix}
\tilde{S}_1 \\
\tilde{S}_2 \\
\vdots \\
\tilde{S}_{l-1}
\end{bmatrix} = 0. \quad (6.5.9)$$

In the next section we will discuss the solution of (6.5.9) for the following cases:

- The controller is strictly proper.
- The controller is proper.

### 6.6 Simplification of the SSP for $l$ Plants

In the previous section, the SSP of $l \geq 2$ strictly proper plants was reduced to a set of $l - 1$ equations, (6.5.9), in $l$ unknowns. The unknowns in these equations are the Youla free parameters, $Q_i$. In this section we will give an expression for each $Q_i$, $i = 1, \cdots, l - 1$,.
in state space form, as a function of $Q_0$. Two cases will be considered. The case when the required controller is strictly proper and the case when the required controller is not zero at infinity, i.e. proper but not strictly proper.

### 6.6.1 Strictly Proper Controllers

Let $D_q^i = 0$ for $i = 0, 1, \ldots, l - 1$. Then $S_i$ in equation (6.5.8) becomes

\[
\bar{S}_i = \begin{bmatrix}
A_i + H_i C_i & -B_i F_0 + H_i C_0 & -B_q C_q^0 & -H_i \\
0 & A_0 + B_0 F_0 & B_0 C_0^0 & H_0 \\
F_i & F_0 & C_0^0 & 0 \\
C_i & C_0 & 0 & -I
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A_{M_1} & B_{M_1} \\
C_{M_1} & 0 \\
C_{M_1} & -I
\end{bmatrix}
\]  

Equation (6.6.1) will be satisfied if either one of the following conditions holds:

1. $B_{M_1} = 0$.

2. $C_{M_1} = 0$.

In state space form, the $i^{th}$ equation of (6.5.9) is

\[
\begin{bmatrix}
A_q^i & 0 & B_q^i \\
C_q^i & I & 0
\end{bmatrix}
\begin{bmatrix}
A_{M_1} & B_{M_1} \\
C_{M_1} & 0 \\
C_{M_1} & -I
\end{bmatrix} = 0
\]  

\[
\begin{bmatrix}
A_q^i & B_q^i C_{M_2} & -B_q^i \\
0 & A_{M_1} & B_{M_1} \\
C_q^i & C_{M_1} & 0
\end{bmatrix} = 0
\]

Equation (6.6.3) will be satisfied if either one of the following conditions holds:

1. $B_{M_1} = 0$.

2. $C_{M_1} = 0$.  

Since we are looking for dynamic $Q_i(s)$ for $i = 0, 1, \ldots, l - 1$, we ignore these conditions because $B'_q = 0$ or $C'_q = 0$ implies $Q_i = 0$ due to the assumption that $D'_q = 0$. An alternative way to satisfy equation (6.6.3) is by performing similarity transformation and then solving for $A'_q, B'_q, C'_q$ and the similarity transformation matrix. This method ends up with different alternatives (each is a set of nonlinear equations) to solve for the unknown matrices. Some initial work has been done in this direction and is reported in Appendix A. A special case of this method is given in the following lemma, where the similarity transformation matrix is assumed to be $X = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$.

**Lemma 6.5** Equation (6.6.3) will be satisfied if and only if $A'_q, B'_q$ and $C'_q$ are given by

$$A'_q = A_{Mq} + B_{Mq}C_{Mq} \quad (6.6.4)$$

$$B'_q = B_{Mq} \quad (6.6.5)$$

$$C'_q = C_{Mq} \quad (6.6.6)$$

**Proof.** Assume a similarity transformation matrix $X$ as

$$X = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}.$$  

Using this similarity transformation matrix on the left hand side of equation (6.6.3), we get

$$\begin{bmatrix} A'_q & A_{Mq} + B_{Mq}C_{Mq} & B_{Mq} - B'_q \\ 0 & A_{Mq} & B_{Mq} \\ C'_q & -C'_q + C_{Mq} & 0 \end{bmatrix} = 0. \quad (6.6.7)$$

Equation (6.6.7) will be satisfied if

$$A'_q = A_{Mq} + B_{Mq}C_{Mq}$$

$$B'_q = B_{Mq}$$

$$C'_q = C_{Mq}$$
In this case, $Q_i(s)$ for $i = 1, 2, \ldots, l - 1$ are given in state space form as

$$
Q_i(s) = 
\begin{bmatrix}
A_i & -B_iF_0 & -B_iC_q^0 & -H_i \\
H_0C_i & A_0 + B_0F_0 + H_0C_0 & B_0C_q^0 & H_0 \\
B_q^0C_i & B_q^0C_0 & A_q^0 & B_q^0 \\
F_i & F_0 & C_q^0 & 0
\end{bmatrix}
$$

where the dimensions of the matrices $(A_q^0, B_q^0, C_q^0)$ are $(n_l + n_o + n_q^0) \times (n_l + n_o + n_q^0)$, $((n_l + n_o + n_q^0) \times r)$ and $m \times (n_l + n_o + n_q^0)$, respectively.

So the simultaneous stabilization problem of $l$ strictly proper plants can be solved by choosing $Q_0(s) \in \mathcal{RH}_\infty$ such that $Q_i(s) \in \mathcal{RH}_\infty$, $i = 1, 2, \ldots, l - 1$. These $Q_i$ will satisfy the SS conditions derived in Section 6.5. Hence, a controller $K_{i}(Q_i)$, mentioned in the previous section, will simultaneously stabilize $l$ plants. A discussion on how to obtain $Q_0(s)$ that makes $Q_i(s)$, $i = 1, \ldots, l - 1$, stable will be given in the next section.

In the next subsection an expression for $Q_i(s)$ in terms of $Q_0(s)$ will be derived for the case when the required controller is not zero at infinity.

### 6.6.2 Proper Controllers

Let us write $\tilde{S}_i$ in (6.5.8) as

$$
\tilde{S}_i =
\begin{bmatrix}
\tilde{S}_{i1} \\
\tilde{S}_{i2}
\end{bmatrix},
$$

where the partition is as shown in (6.5.8).

Then the $i^{th}$ equation of (6.5.9) is

$$
\begin{bmatrix}
I & Q_i
\end{bmatrix}
\begin{bmatrix}
\tilde{S}_{i1} \\
\tilde{S}_{i2}
\end{bmatrix} = 0 \text{ for } i = 1, 2, \ldots, l - 1.
$$

For a proper controller, $Q_i(s)$ must be proper. Let us assume that $Q_0(s)$ is proper with $D_q^0$ is not zero. Then from (6.6.9), $Q_i$ is given by

$$
Q_i = -\tilde{S}_{i1}(\tilde{S}_{i2})^{-1}.
$$

From (6.5.8) $\tilde{S}_{i1}$ and $\tilde{S}_{i2}$ are stable for a stable $Q_0(s)$. A sufficient condition for $Q_i$ to be stable is the stability of $(\tilde{S}_{i2})^{-1}$. Now let us study the dependency of $(\tilde{S}_{i2})^{-1}$ on $Q_0(s)$. 

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First of all from (6.5.8), \( \tilde{S}_2 \) is invertible and is given as

\[
\tilde{S}_2 = \begin{bmatrix} A_M & B^D_{M2} \\ C_{M2} & -I \end{bmatrix}
\]

where the superscript \( D_q \) means that \( D_q \neq 0 \) in the expression of \( B^D_{M2} \). Then the inverse of \( \tilde{S}_2 \) is given as

\[
(\tilde{S}_2)^{-1} = \begin{bmatrix} A_M + B^D_{M2}C_{M2} & -B^D_{M2} \\ C_{M2} & -I \end{bmatrix}.
\]

Then \( Q_i(s) \) becomes

\[
Q_i = -\tilde{S}_{11}(\tilde{S}_2)^{-1} = \begin{bmatrix} A_M & B^D_{M2} \\ C_{M2} & D^0_q \end{bmatrix} \begin{bmatrix} A_M + B^D_{M2}C_{M2} & B^D_{M2} \\ C_{M2} & D^0_q \end{bmatrix}^{-1} = \begin{bmatrix} A_M + B^D_{M2}C_{M2} & B^D_{M2} \\ C_{M2} & D^0_q \end{bmatrix} = \begin{bmatrix} A_{Qi} & B_{Qi} \\ C_{Qi} & D^0_q \end{bmatrix}
\]

where

\[
A_{Qi} = A_M + B^D_{M2}C_{M2}
\]

For further work in the next section, \( A_{Qi} \) can be written as

\[
A_{Qi} = \begin{bmatrix} A_{Qi} & -B_{Qi}F_0 \\ H_{Qi}C_i & A_0 + B_0F_0 + H_0C_0 \end{bmatrix} + \begin{bmatrix} -B_{Qi} \\ B_0 \end{bmatrix} D^0_q \begin{bmatrix} C_i & C_0 \\ B^0_qC_i & B^0_qC_0 \end{bmatrix} = \begin{bmatrix} A^0_q \\ B^0_qC_i \\ B^0_qC_0 \end{bmatrix}
\]
So the problem of simultaneously stabilizing \( l \) strictly proper plants with a proper controller is reduced to choosing \( Q_0(s) \in \mathcal{RH}_\infty \) such that \( Q_i(s) \) of equation (6.6.13) are in \( \mathcal{RH}_\infty \) for \( i = 1, 2, \ldots, l - 1 \).

As a summary for this section, the solution of the SSP, with a strictly proper controller or with a controller which is not zero at infinity, is reduced to finding a stable \( Q_0(s) \) such that \( Q_i(s) \) in (6.6.8) and (6.6.13) are stable for all \( i = 1, 2, \ldots, l - 1 \).

It is worth mentioning that the results obtained in this section are similar to the results in [96] in the sense that the SSP of \( l \) plants is reduced to simultaneously stabilizing a new set of \( l - 1 \) auxiliary plants by a stable controller. Simultaneously stabilizing a new set of \( l - 1 \) auxiliary plants by a stable controller is more restrictive than the original problem. Therefore, the method which is presented in this section gives some scope to develop different numerical approaches to obtain a solution. This freedom of choice for employing a suitable numerical technique to get a solution is not found in [96]. Plus, the new \( l - 1 \) auxiliary plants, in our method, are Youla's free parameters.

In the next section we will give some algorithms that can be used to calculate the state space parameters of \( Q_0(s) \) which will stabilize \( Q_i(s) \) as stated before. The problem is redefined as a new problem, i.e. stabilizing a real matrix by varying some of its unknown elements.

### 6.7 Stabilizing a Real Matrix \( A \) by Varying Some of Its Blocks

As we noted from the previous section the \( A \)-matrix of \( Q_i(s) \) is a function of sub-matrices \( A_{q_i}^0, B_{q_i}^0 \) and \( C_{q_i}^0 \). In this section we will discuss several methods that may be used to find appropriate \((A_{q_i}^0, B_{q_i}^0, C_{q_i}^0)\) such that \( Q_i(s) \) is stable with the constraint of \( A_{q_i}^0 \) being stable.
6.7.1 Formulation of the Problem and Solution

For notational simplicity define

\[ A_{i1} = \begin{bmatrix} A_i & -B_iF_0 \\ H_0C_i & A_0 + B_0F_0 + H_0C_0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \quad A_{i2} := \begin{bmatrix} -B_i \\ B_0 \end{bmatrix} \begin{bmatrix} C_i & C_0 \end{bmatrix} \in \mathbb{R}^{2n \times m}, \quad A_0^0 \in \mathbb{R}^{n \times n}, \quad B_0^0 \in \mathbb{R}^{n \times r} \text{ and } C_0^0 \in \mathbb{R}^{m \times n}, \quad i = 1, 2, \ldots, l - 1. \]

Then from (6.6.8), \( Q_i(s) \) becomes

\[
Q_i(s) = \begin{bmatrix} A_{i1} & B_0 \\ C_i & C_0 \end{bmatrix} \begin{bmatrix} -H_1 \\ H_0 \end{bmatrix} - B_i \begin{bmatrix} C_i & C_0 \end{bmatrix} \begin{bmatrix} -H_1 \\ H_0 \end{bmatrix} \quad (6.7.1)
\]

The problem then is to find a stable \( [A_0^0 \ B_0^0] \) such that \( Q_i(s) \) given by (6.7.1) is stable for \( i = 1, 2, \ldots, l - 1. \) We consider this problem in two cases:

- \( A_{i1} \) is stable.

and

- \( A_{i1} \) is not stable.

Also, we consider the simple case of 2 plants first, i.e. \( i = 1. \) The general situation of \( i > 1 \) will be addressed later.

Case I: \( A_{i1} \) is stable

Define \( \bar{A} \) as the A-matrix of \( Q_i(s) \) in (6.7.1), i.e. \( \bar{A} = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{21} & A_0^0 \end{bmatrix} \), if \( A_{i1} \) is stable, then we give the following theorem.

**Theorem 6.6** Define \( R = A_{i1}P + A_{i2}^T \), where \( P \) is the positive definite solution of the following Lyapunov equation

\[
A_{i1}P + PA_{i1}^T = -Q \quad (6.7.2)
\]
for some positive definite \( Q \). Then a stable \( A_q^0 \) that makes \( \bar{A} \) stable is given by

\[
A_q^0 = -\frac{1}{2} \left( I + R Q^{-1} R^T \right).
\] (6.7.3)

Before we give a proof for this theorem, we need the following lemma.

**Lemma 6.7** [5] Suppose \( X \) is the solution of the following Lyapunov equation, where \( C \geq 0 \),

\[
\bar{A} X + X \bar{A}^T = -C.
\] (6.7.4)

Then

(i) \( \text{Re}[\lambda_i(\bar{A})] \leq 0 \) if \( X > 0 \) and \( C \geq 0 \);

(ii) \( \bar{A} \) is stable if \( X > 0 \) and \( C > 0 \);

(iii) \( \bar{A} \) is stable if \( X \geq 0, C \geq 0 \) and \( (C, \bar{A}) \) is detectable; where \( \text{Re}[\lambda_i(\bar{A})] \) is the real part of an eigenvalue \( \lambda_i \) of \( \bar{A} \).

**Proof of Theorem 6.6** For some positive definite \( Q \), a positive definite matrix \( C \) may be given by

\[
C = \begin{bmatrix} Q & -R^T \\ -R & I + R Q^{-1} R^T \end{bmatrix}
\] (6.7.5)

If we want \( \bar{A} \) to be stable, then there should exist an \( X > 0 \) such that

\[
\bar{A} X + X \bar{A}^T = -C.
\] (6.7.6)

It is easy to verify that

\[
X := \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} > 0
\] (6.7.7)

is a solution to (6.7.6). Consequently, from Lemma 6.7 \( \bar{A} \) is stable. The stability of \( A_q^0 \) is obvious.
Hence, in Case I, to make $Q_i(s)$ in (6.7.1) stable, we have total freedom in choosing $B_i^0$ and $C_i^0$, but $A_i^0$ will be given as in (6.7.3).

Case II: $A_{11}$ is not stable

If $A_{11}$ is not stable, then we propose the following approach for the problem.

Applying a state similarity transformation on $Q_i(s)$, (6.7.1) can be written as

$$
Q_i(s) = \begin{bmatrix}
- B_i & 0 & \cdots & 0 \\
B_0 & - B_i & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & - B_i & C_i^0 X \\
\end{bmatrix} + \begin{bmatrix}
C_i^0 X \\
\vdots \\
C_i^0 X \\
F_i + C_i^0 X \\
\end{bmatrix} + \begin{bmatrix}
-H_i \\
\vdots \\
-H_i \\
-H_i \\
\end{bmatrix} \\
A_i^0 - X + G_i^0 B_i^0 - X \\
B_i \\
H_0 \\
0 \\
\end{bmatrix} 
$$

(6.7.8)

where $X$ solves

$$
A_i^0 X - X A_{11} - X \begin{bmatrix}
- B_i \\
B_0 \\
\end{bmatrix} C_i^0 X + B_i^0 \begin{bmatrix}
C_i \\
C_0 \\
\end{bmatrix} = 0. 
$$

(6.7.9)

Equation (6.7.9) is in the form of a non-standard algebraic Riccati equation. Now the problem is to find $X, C_i^0, B_i^0$, and a stable $A_i^0$ that make $Q_i(s)$ stable for $i = 1$.

The first method is given in the following algorithm to solve for $X, C_i^0, B_i^0$, and a stable $A_i^0$.

**Algorithm 6.1:**

1. Find $K$ such that $A_{11} + \begin{bmatrix}
- B_i \\
B_0 \\
\end{bmatrix} K \begin{bmatrix}
C_i \\
C_0 \\
\end{bmatrix}$ is stable; see comments on next page.

2. Factorize $K$ using $QR$ factorization as $K = C_i^0 R_i \implies X = R \begin{bmatrix}
C_i \\
C_0 \\
\end{bmatrix}$.

3. Choose a stable $B$ such that $A_i^0 = X \begin{bmatrix}
- B_i \\
B_0 \\
\end{bmatrix} C_i^0 + B$ is also stable.
4. Find $F$ such that 
\[
\begin{bmatrix}
C_1 & C_0
\end{bmatrix}A_{11} = F
\begin{bmatrix}
C_1 & C_0
\end{bmatrix}.
\]

5. From (6.7.9), solve for $B^o$ as $B^o = RF - BR$.

In step 4 of Algorithm 6.1, $F$ can be found as follows. Let $Z = \begin{bmatrix} C_1 & C_0 \end{bmatrix}$. If $A_{11}$ and $Z$ satisfy the matrix equation
\[
ZA_{11} = ZA_{11}Z^T \left( ZZ^T \right)^{-1} Z
\]
then $F$ is given by
\[
F = ZA_{11}Z^T \left( ZZ^T \right)^{-1}.
\]

If equation (6.7.10) is not satisfied, then there is a possibility to vary some of the elements of $A_{11}$.

In Algorithm 6.1, we need to solve a static output feedback problem. There are some algorithms in the literature that can be used to solve this problem, see for example [58] and the references therein. In the next subsection we will summarize an algorithm reported in [58]. We pick this algorithm from the literature because we notice that it can be modified to solve our original problem stated in Subsection 6.6.1, i.e. to find a stable $Q_o(s) = \begin{bmatrix} A^o & B^o \\ C^o & 0 \end{bmatrix}$ such that $Q_l(s)$ in (6.6.8) is stable. As we shall see, this problem is reformulated as a sub-diagonal static gain output feedback problem.

In the above subsection we discussed how to find a strictly proper $Q_o(s)$ to get a strictly proper controller. If the $Q_o(s)$ that we seek is to be proper as required in Subsection 6.6.2, we can use the same methods as discussed earlier in this subsection. But $A_{11}$ in this case is a little bit different and we have some freedom to make $A_{11}$ stable. This is shown as follows. From equation (6.6.15) $A_{11}$ is defined as
\[
A_{11} = \begin{bmatrix}
A_l & -B_lF_0 \\
H_0C_0 & A_0 + B_lF_0 + H_0C_0
\end{bmatrix} + \begin{bmatrix}
-B_l & 0 \\
B_l & B_0
\end{bmatrix} D^o \begin{bmatrix} C_l & C_0 \end{bmatrix}.
\]

So $A_{11}$ will be stable if $D^o$ is chosen as a stabilizing output feedback. If this is the case, then a closed-form for $A^o$ can be obtained by Theorem 6.6 and hence $Q_o(s)$ is determined with free $B^o$ and $C^o$.

The problem of finding static output feedback will be discussed in the next subsection.
6.7.2 Stabilization with Static Output Feedback

In this subsection we will include an algorithm for assigning the closed-loop system's poles to a set of prescribed positions via static output feedback. We will then discuss how to use this kind of algorithm to solve for $A_0^f, B_0$ and $C_0^f$ in (6.7.1).

Consider the following $n$th-order, $m$-input $r$-output, linear system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

(6.7.12)

to which a static output feedback

$$u = Ky$$

(6.7.13)

is applied. The problem under consideration is to choose $K$ such that the closed-loop poles have the specified values $\{\lambda_i\} =: \Lambda$. The closed-loop characteristic polynomial in this case is

$$p(s) = \det[sI - A - BK(j)]$$

(6.7.14)

In [58], an iterative algorithm for finding the feedback gain matrix $K$ such that the closed-loop poles are essentially assigned at the desired values is proposed on the assumption that $m \times r \geq n$. The following is the idea behind this.

To achieve the above objective, first an $m \times r$ matrix $K$ can be expressed in terms of its column vectors $k_j$ as

$$K = K_{rxr} = \sum_{i=1}^{r} k_i e_i$$
$$= \sum_{j=1}^{r} k_i e_i + k_j e_j$$
$$= K_j + k_j e_j$$

(6.7.15)

where $e_i$ is the $i$th row of $(r \times r)$ identity matrix, and $K_j$ is equal to $K$ except for the $j$th column which has all zero elements. Then the closed-loop characteristic polynomial (6.7.14) may be rewritten as

$$p(s) = \det[sI - A - BK(j)C - Bk_j e_j C]$$
$$= \det[sI - A - BK(j)C], \det[I - (sI - A - BK(j)C)^{-1}Bk_j e_j C]$$
$$= \det[sI - A - BK(j)C] - e_j C \text{adj}(sI - A - BK(j)C)Bk_j$$
$$:= P_j(s) - N_j(s)k_j.$$

(6.7.16)
In the above, for a given $K_j$, (6.7.16) is linear in $k_j$ so that if all other columns are known (or specified), then the $j$th column can be appropriately computed. Thus (6.7.16) provides a basis for the construction of a suitable iterative algorithm for pole placement by output feedback. To ensure that (6.7.16) can be solved for each $j$, consider a partition of $\Lambda$ into $r$ subsets as

$$
\Lambda = \{\lambda_i\} := \{\lambda_1\} \cup \{\lambda_2\} \cup \ldots \cup \{\lambda_r\} := \Lambda_1 \cup \Lambda_2 \cup \ldots \cup \Lambda_r
$$

such that $\Lambda_j$ will be used for solutions of $k_j$. Based on the discussion above, the following iterative algorithm for pole assignment by static output feedback is proposed [58].

Algorithm 6.2:

1. Set $K = K^0$;
2. For $j=1,2,\ldots,r$, solve the following system of linear equations:

$$
\begin{bmatrix}
N_j(\lambda_{ij}) \\
N_j(\lambda_{12}) \\
\vdots \\
N_j(\lambda_{ij})
\end{bmatrix} k_j =
\begin{bmatrix}
\bar{p}_j(\lambda_{ij}) - p_c(\lambda_{ij}) \\
\bar{p}_j(\lambda_{12}) - p_c(\lambda_{12}) \\
\vdots \\
\bar{p}_j(\lambda_{ij}) - p_c(\lambda_{ij})
\end{bmatrix}
$$

(6.7.18)

where $\lambda_{ij} \in \Lambda_j$. For each $k_j$ computed, update the previous values with the current computed values;

3. Compute the eigenvalues $\tilde{\lambda}_i$ of $(A + BK_j)$, and if

$$
\sum_{i=1}^{n}(\lambda_i - \tilde{\lambda}_i)^2 < \epsilon
$$

(6.7.19)

where $\epsilon > 0$ is some specified tolerance, then stop. Otherwise, go to Step 2.

In the following we will re-write the problem of finding the parameters of $Q_0(s)$ in a form suitable to use a modified version of Algorithm 6.2.
Let the A-term of $Q(s)$ in (6.7.1) be written as

$$A_{Q_1} = \begin{bmatrix} A_{11} & -B_i \\ B_i^0 & C_i \\ 0 & A_i^0 \end{bmatrix} \begin{bmatrix} C^q_0 \\ 0 \\ 0 \end{bmatrix}$$

which is in the form of a constant gain output feedback problem.

Now the problem is reduced into that of finding a constant gain output feedback with the restriction that this gain $K$ is block-diagonal with appropriate dimensions, i.e.

$$K = \begin{bmatrix} B_i^0 & 0_{(m \times r)} \\ 0 & C_i^0 \end{bmatrix} \in \mathbb{R}^{(m+r) \times (r+n_q)}.$$

In this way we have complete freedom in choosing a stable $A_i^0$. From the structure of $K$ we see the following:

- The number of elements of $K = (n_q + m) \times (r + n_q) = n_q^2 + n_q(m + r) + mr$.
- The number of pre-assigned elements (i.e. zero elements in $A_T$) is equal to $(n_q^2 + mr)$.
- The number of unknown elements in $K$ is $n_q(m + r)$.
- The number of eigenvalues to be re-assigned is $(2n + n_q)$.

To use the algorithm of pole placement described in Algorithm 6.2, the following condition should be satisfied to solve for the unknowns

$$2n + n_q \leq n_q(m + r).$$

This means that the number of equations is less than or equal to the number of unknowns. This condition enables us to assign some elements of $k_j$ to be zeros and solve for the rest, if $A$ is properly divided into $(r + n_q)$ subsets.
The next algorithm is a modified version of Algorithm 6.2.

Algorithm 6.3:

1. Choose a minimum integer value for \( n_q \) such that inequality (6.7.22) is satisfied.
2. Choose a stable \( A_0^q \in \mathbb{R}^{n_q \times n_q} \).
3. Set \( K = K^0 \).
4. For \( j = 1, 2, \ldots, r + n_q \), solve the following system of linear equations:

\[
\begin{bmatrix}
N_j(\lambda_{1j}) \\
N_j(\lambda_{2j}) \\
\vdots \\
N_j(\lambda_{ij})
\end{bmatrix}
\begin{bmatrix}
k_{j1} \\
k_{j2} \\
\vdots \\
k_{jn_q}
\end{bmatrix}
= 
\begin{bmatrix}
\bar{p}_j(\lambda_{1j}) - p_c(\lambda_{1j}) \\
\bar{p}_j(\lambda_{2j}) - p_c(\lambda_{2j}) \\
\vdots \\
\bar{p}_j(\lambda_{ij}) - p_c(\lambda_{ij})
\end{bmatrix}
\]

(6.7.23)

where \( \lambda_{ij} \in \Lambda_j \) and \( k_j \in \mathbb{R}^{(n_q + m) \times 1} \) is of the form

\[
\begin{bmatrix}
k_{j1} \\
k_{j2} \\
\vdots \\
k_{jn_q} \\
0_1 \\
\vdots \\
0_{n_q}
\end{bmatrix}
\]

for \( j = 1, \ldots, r \) and,

\[
\begin{bmatrix}
0_1 \\
\vdots \\
k_{j,n_q+1} \\
\vdots \\
k_{j,n_q+m}
\end{bmatrix}
\]

for \( j = r + 1, \ldots, r + n_q \).

5. For each \( k_j \) computed, update the previous values with the current computed values;
6. Compute the eigenvalues \( \tilde{\lambda}_i \) of \( (A + BK^jC) \), and if

\[
\max_i |\text{Re}(\tilde{\lambda}_i)| < 0,
\]

stop. Otherwise, go to Step 4.
If there is no convergence after a certain number of iterations, the execution of the algorithm goes to Step 2 to try another $A_q^0$ or to Step 1 to choose a larger $n_q$.

6.8 Numerical Examples

In this section we give illustrative examples to demonstrate the theoretical results developed in this chapter. The plants that we will use in these examples are reported in [19]. The first two plants are given as

\[
P_0 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (6.8.1)
\]

\[
P_1 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -4 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (6.8.2)
\]

**Example 6.2** In the first example we are looking for a strictly proper controller that simultaneously stabilizes the given two plants using the method described in Subsection 6.6.1. Since the given two plants are strictly proper, to get a strictly proper controller, $D_q^0$ must be zero. A candidate $Q_0(s)$ (found by trial and error) that makes $Q_1(s)$ in equation (6.6.8) stable is

\[
Q_0 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]
From equation (6.6.8), \( Q_1(s) \) is given as

\[
\begin{bmatrix}
0 & 1 & 1 & 1 & -2 & -1 & 7 \\
-4 & 0 & -2 & 1 & 1 & 0 & 8 \\
-3 & 0 & -4 & 0 & 2 & 1 & -3 \\
-1 & 0 & -1 & -2 & 0 & 1 & -1 \\
1 & 0 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -4 & 0 \\
-3 & -1 & -1 & 1 & 1 & 0 \\
1 & -5 & 1 & -2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

with the following minimal\(^1\) realization

\[
\begin{bmatrix}
-2.269 & -0.7258 & 0.3535 & 1.711 & -0.8473 & 6.392 \\
0.2715 & -2.825 & 1.157 & 4.811 & 0.8015 & 2.144 \\
0.1631 & 0.4934 & -0.9993 & -1.008 & 0.216 & -2.863 \\
-0.3184 & 0.3929 & 1.195 & -1.401 & -1.326 & 5.09 \\
0.5246 & -0.5893 & -0.3694 & 3.954 & 0.4945 & -6.666 \\
0 & 0 & 0.08088 & -3.49 & 0.9003 & 0 \\
0 & 0 & -0.008065 & 0.569 & 5.539 & 0 \\
0 & 0 & -0.9532 & -0.301 & 0.02953 & 0
\end{bmatrix}
\]

\( Q_1(s) \) is stable with the following eigenvalues: \([-0.4205, -0.9675 \pm 1.9732i, -2.3223 \pm 0.1542i]\). The simultaneous stabilization controller is given, according to the theory of Section 6.3, by

\[
K_0 = -(Y_0 - M_0Q_0)(X_0 - N_0Q_0)^{-1}
\]

(6.8.3)

\[
\begin{bmatrix}
A_0 + B_0F_0 + H_0C_0 + B_0D_0^2C_0 & B_0C_0^2 \\
D_0^2C_0 & A_0^2 & B_0^2 \\
F_0 + D_0^2C_0 & C_0^2 & D_0^2
\end{bmatrix}
\]

(6.8.4)

\(^1\) A system is in a minimal realization if it is controllable and observable and can be computed using the MATLAB file minreal.m
where \((A_q, B_q, C_q)\) and \((A^0_q, B^0_q, C^0_q)\) are the plant \(P_0\) and the Youla free parameter \(Q_0\) in state space form respectively. \(F_0\) and \(H_0\) are state feedback and observer gains.

A minimal realization of \(K_0\) is given as

\[
K_0 = \begin{bmatrix}
-1.333 & 1.106 & 0 & 0 \\
0.5025 & -1.03 & 0.2227 & 0 \\
0.3693 & -1.002 & -4.635 & 3.317 \\
0.8165 & 0.2462 & 1.508 & 0 \\
2.041 & -0.8616 & -0.3015 & 0 \\
0.4082 & 0.8616 & 0.3015 & 0 \\
\end{bmatrix}
\]

The eigenvalues of the closed-loop transfer function \(T(P_0, K_0)\) are found to be \([-1.0000, -1.0000, -1.0000, -2.0000, -2.0000]\) and the eigenvalues of the closed-loop transfer function \(T(P_1, K_0)\) are found to be \([-0.4205, -0.9675 \pm 1.9732i, -2.3223 \pm 0.1542i]\).

Example 6.3 In the second example we are looking for a proper controller. In this case \(D_q^0 \neq 0\). Let us choose \(Q_0(s)\) as before with \(D_q^0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\) instead of \(D_q^0 = 0_{m+1}\). So \(Q_0(s)\) which makes \(Q_1(s)\) stable is

\[
Q_0 = \begin{bmatrix}
-1 & 0 & 1 \\
0 & -4 & 0 \\
1 & 1 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

\(^2\)\(F_0\) and \(H_0\) can be computed using the MATLAB file place.m
and \( Q_1(s) \) is found (using equation (6.6.13)) to be

\[
Q_1 = \begin{bmatrix}
-1 & 1 & 0 & 1 & -2 & -1 \\
-3 & 0 & -1 & 1 & 1 & 0 \\
-2 & 0 & -3 & 0 & 2 & 1 \\
-1 & 0 & -1 & -2 & 0 & 1 \\
1 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -4 \\
-2 & -1 & 0 & -1 & 1 & 1 \\
1 & -5 & 1 & -2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\( Q_1(s) \) is stable with the following eigenvalues \([-0.2266, -1.0848 \pm 1.7156\%, -2.3019 \pm 0.2409\%, -4.0000\%]\). A minimal realization of the simultaneous stabilization controller in this case is given by

\[
K_0 = \begin{bmatrix}
-1.8 & 0.9798 & 0 \\
0.1633 & -0.3667 & 0.3727 \\
0.7303 & -0.5217 & -3.833 \\
0.8944 & 0.7303 & 0.8165 \\
2.236 & 0 & 0 \\
0 & 0.9129 & 0.4082
\end{bmatrix}
\]

The eigenvalues of the closed-loop transfer function \( T(P_0, K_0) \) are found to be \([-1.0000, -1.0000, -1.0000, -2.0000, -2.0000] \) and the eigenvalues of the closed-loop transfer function \( T(P_1, K_0) \) are found to be \([-0.2266, -1.0848 \pm 1.7156\%, -2.3019 \pm 0.2409\%]\).

**Remark 6.4** \( Q_0(s) \) given in the above two examples is chosen arbitrarily. If it is not easy to find such an appropriate \( Q_0(s) \), then the procedures explained in Section 6.7 can be used to find \( Q_0(s) \) which makes \( Q_1(s) \) stable.

**Example 6.4** In this example three plants will be simultaneously stabilized. The three plants are the two plants seen in Examples 1 and 2 before and the third plant is [19]
The first step toward finding a simultaneous stabilization controller is to find a stable $Q_0(s)$ which will make $Q_i(s)$, $i = 1, 2$, of equation (6.6.8) or equation (6.6.13), stable. The $Q_0(s)$ that we choose, by trial and error, is given as

$$Q_0(s) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(6.8.6)

with the following eigenvalues $[-0.6972, -4.3028]$. Then $Q_1(s)$ and $Q_2(s)$ are given as

$$Q_1(s) = \begin{bmatrix} 0 & 1 & 1 & 1 & -2 & -1 & 7 \\ -4 & 0 & -2 & 1 & 1 & 0 & 8 \\ -3 & 0 & -4 & 0 & 2 & 1 & -3 \\ -1 & 0 & -1 & -2 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ -3 & -1 & -1 & -1 & 1 & 1 & 0 \\ 1 & -5 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

which is minimal and
Ch. 6. A STATE SPACE APPROACH TO THE SSP

with the following minimal realization

\[
Q_2 = \begin{bmatrix}
0 & 1 & 2 & -1 & -2 & 0 & 7 \\
-9 & 0 & -2 & 1 & 2 & 0 & 3 \\
-3 & 0 & -4 & 0 & 2 & 1 & -3 \\
-1 & 0 & -1 & -2 & 0 & 1 & -1 \\
1 & 0 & 1 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & -4 & 0 \\
-6 & 1 & -1 & -1 & 1 & 1 & 0 \\
0 & 0 & 1 & -2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

which are stable. A minimal realization of the controller that will simultaneously stabilize the three plants \( P_0, P_1 \) and \( P_2 \) is given according to equations (6.2.4) and (6.5.3) by

\[
K_0 = \begin{bmatrix}
-2.929 & -0.546 & -1.537 & -0.2199 & -1.336 \\
-1.455 & -1.716 & 1.798 & -1.348 & 1.221 \\
-0.3619 & 0.7435 & -3.906 & 0.04337 & -2.685 \\
-0.9801 & -1.632 & 0.192 & -2.449 & 0.7167 \\
0 & 0.277 & -1.349 & 1.45 & 0 \\
0 & -0.103 & 0.9298 & 2.264 & 0 \\
0 & -0.7302 & -0.643 & 0.3309 & 0
\end{bmatrix}
\]

The eigenvalues of the three closed-loop systems are shown in Table 6.1.
Comparing the results obtained here with those obtained in [19] for the same three plants, we see that the controller we obtained is of order 4 while the one that is obtained in [19] is of order 6. Besides that there is a restriction in [19] about the number of plants that can be simultaneously stabilized. This restriction is that the number of plants times the number of outputs should be less than the sum of the number of inputs and the number of outputs. For the same dimension of the plants in hand, only three plants can be simultaneously stabilized by the approach proposed recently in [19]. When the number of outputs is more than one, more restrictions are involved and these are mentioned in detail in Chapter 3.

The restrictions mentioned above are not important in our approach as long as we can find a stable \( Q_0(s) \) that can make \( Q_i(s), \ldots, Q_{l-1}(s) \) each stable. Besides the low order \( K_0(s) \) that is obtained in this work, we also get another two controllers. These are \( K_1(s) \) and \( K_2(s) \) which are computed according to the following equation

\[
K_i = -(\hat{X}_i - Q_i\hat{N}_i)^{-1}(\hat{Y}_i - Q_i\hat{M}_i)
\]

\[
\begin{bmatrix}
A_i & B_i^T C_i \\
B_i C_i^T & A_i + B_i F_i + H_i C_i + B_i D_i C_i + H_i + B_i D_i
\end{bmatrix}
\]

A minimal realization is computed for \( K_i(s) \), \( i = 1, 2 \) and the results are as follows

\[
K_1 = \begin{bmatrix}
-2.929 & 0.546 & 1.537 & 0.2199 & 1.336 \\
1.455 & -1.716 & 1.798 & -1.348 & 1.221 \\
0.3619 & 0.7435 & -3.906 & 0.04337 & -2.685 \\
0.9801 & -1.632 & 0.192 & -2.449 & 0.7167 \\
0 & 0.277 & -1.349 & 1.45 & 0 \\
0 & -0.103 & 0.9298 & 2.264 & 0 \\
0 & -0.7302 & -0.643 & 0.2309 & 0
\end{bmatrix}
\]
and
\[
K_2 = \begin{bmatrix}
-3.979 & 0.7872 & 5.035 & 0.2304 & 8.677 \\
0.2465 & -1.201 & 1.976 & -1.175 & 1.135 \\
0.06605 & 0.6251 & -3.396 & 0.3385 & -2.668 \\
0.1382 & -1.259 & 0.659 & -1.963 & 0.9093 \\
0 & 0.2474 & -1.256 & 1.506 & 0 \\
0 & -0.05053 & 1.041 & 2.018 & 0 \\
0 & -0.6842 & -0.531 & 0.3957 & 0
\end{bmatrix}
\]

These two controllers locate the eigenvalues of the closed-loop systems \(T(P_i, K_j)\) for \(i = 1, 2, 3\) and \(j = 1, 2\) at the locations shown in Table 6.2 and Table 6.3. It is clear from Tables 6.1, 6.2 and 6.3 that, although the state-space representations of the three controllers are different, all of these controllers locate the closed-loop poles at the same locations.

6.9 Conclusion

Using the Youla parametrization of all stabilizing controllers, the simultaneous stabilization problem of \(I\) multivariable \(LTI\) and strictly proper systems is reduced to that of expressing \(I - 1\) of the Youla free parameters \(Q_i(s)\) as a function of a selected one, say,
\(Q_0(s)\). It is required to find a stable \(Q_0(s)\) such that \(Q_i(s)\) in (6.6.8) and (6.6.13) are stable for all \(i = 1, 2, \ldots, t - 1\). Two cases are discussed to get a clear understanding of the problem. For strictly proper controllers a special transformation matrix is used, while for controllers which are bi-proper or have rank deficiency at infinity, a stable inverse is required.

The problem is solved here for two plants, where the problem can be reduced to a new problem of stabilizing a real matrix by varying some of its unknown elements. When a certain condition is satisfied in this new problem, a closed form solution is obtained, otherwise numerical algorithms are provided. Hence a solution of the SSP of two plants can be given in terms of the solution of the new problem. In the case of more than two plants, there are no systematic means for computing a \(Q_0\) that makes \(Q_i, i > 1\), stable. More work is needed to develop the existing theory and to provide numerical algorithms to calculate the parameters of \(Q_0(s)\).
Chapter 7

Helicopter Controller Design: An Example of Simultaneous Stabilization Via Robust Stabilization

7.1 Introduction

In this thesis we have principally looked to develop theoretical methods which may lead eventually to practical solutions of the strong stabilization problem, Chapter 4, and the simultaneous stabilization problem, Chapters 5 and 6. The results obtained in Chapter 5, can be tested on any number of LTI systems, whereas the results in Chapters 4 and 6 are limited to two plants. Therefore, in this chapter we are going to demonstrate the results of Chapter 5 (where the software is fully developed) by a real example.

The chapter presents an application of simultaneous stabilization theory to the design of a control system for a high performance helicopter. The work illustrates how practical design problems may be formulated and solved using linear design techniques, for which a wealth of literature exists.

One approach to control system design for a helicopter is to linearize the system at different operating points, and then to use linear techniques to design a controller for each of the linear models obtained. The linear controllers resulting from the design can then be linked together using gain scheduling techniques to produce a full range nonlinear controller.

The simultaneous stabilization approach is based on designing only one controller for all
the linear models. Our approach is to find a single controller which stabilizes all the models based on the theory presented in Chapter 5, i.e. on first finding a central plant from the set of plants obtained from the linearization process. A robust controller is then designed to stabilize the central plant, taking into consideration the stability of the original models.

It should be emphasized that the helicopter problem is being used only to illustrate the results in Chapter 5. That is, the resulting controller should not be seen as the "best" one might design for helicopter purposes which in general will include a wide range of performance requirements. However, by using a robust design methodology on the "central" plant the final controller does exhibit satisfactory behaviour in the limited tests carried out.

In the next two sections a brief description of the helicopter control problem is given. For more details refer to [33, 93, 100, 112, 113] and the references therein.

7.2 The Helicopter Model Description

The helicopter is unique for flight and control in its ability to ascend and descend vertically, move in any direction horizontally, or hover over a specific spot on the ground, and while doing so turn onto a desired heading. In order to carry out flight manoeuvres, helicopters must have a flight control system which will produce changes in pressure distribution and lift forces of the main aerofoil surfaces. These effects are obtained by varying the pitch of the main and tail rotor blades. The pilot has pitch-control over the following variables:-

- Collective Pitch Lever - The pitch angle of the main rotor blades is collectively changed to cause vertical movement of the helicopter.
- Cyclic Pitch Lever - The main rotor disc is tilted by varying the pitch of the main rotor blades individually to cause horizontal movement.
- Foot Pedals - The pitch angle of the tail rotor blades is changed collectively to cause directional control.
The main purpose of the tail rotor, apart from heading control, is to counteract the main rotor torque, in the absence of which the helicopter would spin around in circles.

The main difference between cyclic and collective pitch control is that in cyclic pitch control the pitch angle of the blade is changed individually during the cycle of rotation so that at any chosen time the angle of one blade is increasing, whilst the angle of the other blade is decreasing, whereas in collective pitch control, the angles of the two blades are changed simultaneously.

The collective pitch lever is normally to the left of the pilot’s seat, and the cyclic stick is normally situated in the front. The open loop responses to these actuations and also for the tail rotor angle are highly coupled, and unstable.

The major inter-axis couplings that occur in the helicopter are those between longitudinal and lateral cyclic; this is due to the second moment of inertia about pitch being about four or five times higher than about roll.

For control purposes the coordinate system illustrated in Figure 7.1 is used. \((u, v, w)\) are components of the total velocity and \((p, q, r)\) the roll, pitch and yaw angular rates about each of the \((x, y, z)\) axis respectively.

![Figure 7.1: Helicopter Body-Fixed Axis](image)

The greater the understanding of the dynamic behaviour of the helicopter, the more effectively the associated control problems can be tackled. The rotor dynamics are extremely important, but not extensively modelled. The aerodynamical effects of the rotor give rise
to the so called rotor flapping and coning modes.

The coning mode effects can be visualised as due to the shape that the main rotor blade tips sweep out, forming a cone in the air. This coning angle increases, that is the blade tips increase in height relative to the helicopter, if a rotor collective input is applied to move the aircraft upwards.

The rotor flapping is a primary task in helicopter stability. It is due to the asymmetrical velocity distribution of the retreating blades relative to the advancing ones. At a typical forward cruising speed of 120 knots, the blades over the nose and the tail have the same velocity distributions, but the retreating tip is revolving 30% slower relative to the advancing tip. This lack of symmetry is a dominant factor in forward flight, and accounts for most of the differences between airplanes and helicopters.

Suppose that the helicopter starts off accelerating with each blade having the same pitch setting (and therefore angle of attack). The difference in velocity produces more lift on the advancing side than the retreating side, thus with a non-rigid rotor the advancing blade lifts up and follows a higher trajectory. This climbing blade, then has a decreased angle of attack compared to the retreating blade, and vice versa on the retreating, which ends up with an equilibrium position when the lift distribution is balanced.

The longitudinal flapping mode described above, and the lateral flapping of the rotor blades due to another load asymmetry over the nose and tail, are characteristics which are responsible for the helicopter behaving better in gusty air than an airplane. This comes from the rotor blades flapping individually in response to gusts, compared to the airplanes' rigid wings transmitting the unsteady gust load directly into the fuselage.

As a summary, the main helicopter characteristics are summarized as follows:

- A high level of inter axis coupling.
- Unstable plant.
- Non-linear and asymmetric.
- Dynamics vary considerably with flight conditions.
Complex aero-elastic rotor models.

Pilots can fly them (despite above) but very tiring over long periods.

7.3 The Helicopter Mathematical Model

The nonlinear helicopter model used for simulation purposes was developed at the Defence Research Agency (DRA), Bedford and is known as the Rationalized Helicopter Model (RHM14).

The RHM covers a wide portion of the working flight envelope of a Lynx-like high performance military helicopter with a four-bladed semi-rigid main rotor. It models the separate aerodynamic force and moment contributions of the main rotor, tail rotor, fuselage, fin and horizontal stabilizer with the main rotor model consisting of rigid constant chord blades hinged with stiffness in flap at the center of rotation. Rotor speed is an additional degree of freedom. Constant left slope and uniform induced flow are assumed and unsteady aerodynamics are ignored.

In terms of physical states, the nonlinear model contains eight rigid body states, three engine states, four simple actuator states and three second order rotor dynamic states, amongst other inherent model states. The rotor states include two flapping modes and one coning mode. Due to some of the complex rotor modes left unmodelled, a good case for requiring a robust stabilization technique is apparent since these rotor modes can have significant effects.

The nonlinear helicopter model can be linearized about a particular operating point in the flight envelope and represented by the usual state-space notation. Controller design will be based on eight-state linearizations at chosen trim positions over the flight envelope from hover up to approximately the maximum allowed of the model (110 knots).

The basic linearized models of the helicopter that will be used in this design study have 8 rigid body states, 4 inputs and 6 outputs, and are unstable. The rotor dynamics will be treated as being uncertain and will be left out of the nominal plant description.

The state space description of the linearized rigid body equations of motion are expressed
in the standard form as:

\[
\dot{x} = Ax + Bu \quad \text{(7.3.1)} \\
y = Cx + Du
\]

where the state \( x \) and input vector \( u \) are:

\[
x = \begin{bmatrix}
\theta \\
\phi \\
p \\
q \\
r \\
u \\
v \\
w
\end{bmatrix} \quad u = \begin{bmatrix}
\theta_o \\
\theta_i \\
\theta_c \\
\theta_w
\end{bmatrix} \quad \text{(7.3.2)}
\]

The state variables and inputs are described in Table 7.1 and Table 7.2 and the measured outputs are shown in Table 7.3. The plant has 4 inputs, therefore, it is possible to independently control only 4 outputs. These are shown in Table 7.4.

The scaling process is always an important issue in multivariable design, since it allows the relative importance of output variations to be properly accounted for. The scalings used in this design are given in Table 7.5.
### Input Description Units

<table>
<thead>
<tr>
<th>Input</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_s (r_1)$</td>
<td>Main Rotor Collective</td>
<td>Deg.</td>
</tr>
<tr>
<td>$\theta_{ls} (r_2)$</td>
<td>Longitudinal Cyclic</td>
<td>Deg.</td>
</tr>
<tr>
<td>$\theta_{lc} (r_2)$</td>
<td>Lateral Cyclic</td>
<td>Deg.</td>
</tr>
<tr>
<td>$\theta_{ts} (r_4)$</td>
<td>Tail Rotor Collective</td>
<td>Deg.</td>
</tr>
</tbody>
</table>

Table 7.2: The Input Variables in the Helicopter Model.

### Output Description

<table>
<thead>
<tr>
<th>Output</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{h} (y_1)$</td>
<td>Heave Velocity</td>
</tr>
<tr>
<td>$\theta(y_2)$</td>
<td>Pitch Attitude</td>
</tr>
<tr>
<td>$\phi(y_3)$</td>
<td>Roll Attitude</td>
</tr>
<tr>
<td>$\dot{\psi} (y_4)$</td>
<td>Heading Rate</td>
</tr>
<tr>
<td>$p(y_5)$</td>
<td>Roll rate</td>
</tr>
<tr>
<td>$q(y_6)$</td>
<td>Pitch rate</td>
</tr>
</tbody>
</table>

Table 7.3: The Output Variables in the Helicopter Model.

### Controlled Output Description Pilot Input Units

<table>
<thead>
<tr>
<th>Controlled Output</th>
<th>Description</th>
<th>Pilot Input</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h (y_1)$</td>
<td>Heave Velocity</td>
<td>Coll.</td>
<td>Ft/sec.</td>
</tr>
<tr>
<td>$\theta(y_2)$</td>
<td>Pitch Attitude</td>
<td>Long.</td>
<td>Rad</td>
</tr>
<tr>
<td>$\phi(y_3)$</td>
<td>Roll Attitude</td>
<td>Latt.</td>
<td>Rad</td>
</tr>
<tr>
<td>$\dot{\psi} (y_4)$</td>
<td>Heading Rate</td>
<td>Coll.(Pedal)</td>
<td>Rad/sec.</td>
</tr>
</tbody>
</table>

Table 7.4: The Controlled Variables in the Helicopter Model.

### Controlled Outputs Scaling

<table>
<thead>
<tr>
<th>Controlled Outputs</th>
<th>Scaling</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h (y_1)$</td>
<td>0.1 Sec/ft.</td>
</tr>
<tr>
<td>$\theta(y_2)$</td>
<td>5 Rad$^{-1}$.</td>
</tr>
<tr>
<td>$\phi(y_3)$</td>
<td>5 Rad$^{-1}$.</td>
</tr>
<tr>
<td>$\dot{\psi} (y_4)$</td>
<td>5 Sec/rad.</td>
</tr>
</tbody>
</table>

Table 7.5: Controller Inputs and Scaling factors.
7.4 A Normalized Coprime Factor Design Procedure

The aim of this section is to summarize the main results on normalized coprime factorization and robust design given in [62], since this will later form the basis of the design procedure used on the "central" plant.

7.4.1 Left Coprime Factorization

Matrices \((\tilde{N}, \tilde{M}) \in \mathcal{H}_\infty\) denote the space of functions with no poles in the closed right half plane, constitute a left coprime factorization (LCF) of a plant model \(G\) if and only if

(a) \(\tilde{M}\) is square, and \(\det(\tilde{M}) \neq 0\)

(b) The plant model is given by

\[ G = \tilde{M}^{-1}\tilde{N} \]  

(7.4.1)

(c) There exist \(\tilde{X}, \tilde{Y} \in \mathcal{H}_\infty\) such that

\[ \tilde{M}\tilde{X} + \tilde{N}\tilde{Y} = I. \]  

(7.4.2)

A left coprime factorization of a plant model \(G\) as defined in (7.4.2) is normalized if and only if

\[ \tilde{N}^* + \tilde{M}^* = I \quad \text{for all } s \in j\mathbb{R} \]  

(7.4.3)

where \(\tilde{N}^* = \tilde{N}^T(-s)\), etc.

Let \(G(s) = C(sI - A)^{-1}B + D\) with \((A, B, C, D)\) minimal. A state-space construction for the normalized right coprime factorization can be obtained in terms of the solution to the generalized control algebraic Riccati equation

\[ (A - BS^{-1}D^TC)^T X + X (A - BS^{-1}D^TC) - XBS^{-1}B^TX + C^T (I - DS^{-1}D^T) C = 0 \]  

(7.4.4)

where \(X \geq 0\) is the unique stabilizing solution and \(S := I + D^TD\). A state-space representation for the normalized left coprime factorization can be similarly obtained. In
particular, if

$$H = -(ZC^T + BD^T)R^{-1}$$

(7.4.5)

where $Z \geq 0$ is the unique stabilizing solution of the generalized filter algebraic Riccati equation

$$(A - BD^TR^{-1}C)^T Z + Z (A - BD^TR^{-1}C) - ZC^TR^{-1}CZ + B(I - D^TR^{-1}D)B^T = 0$$

(7.4.6)

where $R := I + DD^T$, then

$$\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} = \begin{bmatrix} A + HC & B + HD & H \\ R^{-1/2}C & R^{-1/2} \tilde{D} & R^{-1/2} \end{bmatrix}$$

(7.4.7)

is the state-space representation of the normalized left coprime factorization of $G$ such that $G = \tilde{M}^{-1}\tilde{N}$.

A perturbed plant transfer function can be written as

$$G_\Delta = \left(\tilde{M} + \Delta_M\right)^{-1} \left(\tilde{N} + \Delta_N\right)$$

(7.4.8)

where $(\Delta_M, \Delta_N)$ are stable unknown transfer functions which represent uncertainty in the nominal plant model. The robust stabilization objective is to stabilize, using a feedback controller $K$ (Figure 7.2), not only the nominal model $G$, but the family of perturbed plants defined by

$$G_\varepsilon = \left\{\left(\tilde{M} + \Delta_M\right)^{-1} \left(\tilde{N} + \Delta_N\right) : \|\Delta_M, \Delta_N\|_\infty < \varepsilon\right\}$$

(7.4.9)

where $\varepsilon > 0$ is the stability margin. The feedback system $(\tilde{M}, \tilde{N}, K, \varepsilon)$ is said to be robust if and only if $(G, K)$ is internally stable and

$$\left\|\begin{bmatrix} K(I - GK)^{-1}\tilde{M}^{-1} \\ (I - GK)^{-1}\tilde{M}^{-1} \end{bmatrix}\right\|_\infty \leq \varepsilon^{-1}.$$  

(7.4.10)

Therefore, to maximize robust stability of the closed-loop system given in Figure 7.2, one minimizes

$$\gamma := \left\|\begin{bmatrix} K \\ I \end{bmatrix} \left((I - GK)^{-1}\tilde{M}^{-1}\right)\right\|_\infty.$$  

(7.4.11)
The lowest achievable value of $\gamma$ is given by

$$\gamma_0 = (1 - \lambda_{\max}(XZ))^{1/2} \quad \text{(7.4.12)}$$

where $X$ and $Z$ are as defined by equations (7.4.4) and (7.4.6). The optimal controller which achieves this bound is given by, [62]

$$K = \frac{A + BF + \gamma_0^2(L^T)^{-1}ZCT(C + DF)}{B^TX} \begin{bmatrix} \gamma_0^2(L^T)^{-1}ZCT \\ -D^T \end{bmatrix} \quad \text{(7.4.13)}$$

where

$$F = -S^{-1}(D^TC + B^TX),$$

$$L = (1 - \gamma_0^2)I + XZ.$$
7.5 Loop Shaping Design Procedure

The design study presented in this chapter makes use of the loop shaping design procedure (LSDP) [62] to obtain performance/robustness trade-offs, whilst using the robust stabilization method as a means of guaranteeing closed-loop stability.

Due to the conflicting requirements of performance (tracking and disturbance rejection) requiring high gain, and robustness (sensor noise attenuation) requiring low gain, there must be a frequency separation of these objectives. An acceptable compromise comes from performance being (typically) most important at low frequencies, and robust stability at high frequencies. Therefore, to take into account both robust stabilization and performance requirements, shaping of the plant frequency response is needed in order to meet the closed-loop performance requirements.

In practical design applications, the performance specifications are first translated into the frequency domain, and the open-loop plant’s singular value frequency response is given the desired shape. This is achieved by augmentation of the nominal plant model $G$ by pre- and/or post compensators (or weighting functions) $W_1$ and $W_2$ respectively. These weighting functions are chosen to be stable, minimum phase transfer functions. The shaped plant is then robustly stabilized against coprime factor uncertainty, and the controller thus obtained is cascaded with the weights to obtain the final controller.

As a summary, the loop-shaping design procedure consists of three main stages [62]:

1. Loop Shaping. Shape the singular values of the central plant, using filters $W_1$ and $W_2$, to give a desired open-loop frequency response shape. Combine the central plant, and the shaping weights (Figure 7.3) to form the shaped plant $G_s = W_2GW_1$.

2. Robust Stabilization.
   
   (a) Calculate $\epsilon_{\text{max}} = (\gamma_\epsilon)^{-1}$. If $\epsilon_{\text{max}} << 1$, return to step 1 and adjust the shaping weights.
   
   (b) Choose $\epsilon \leq \epsilon_{\text{max}}$, and synthesize a feedback controller, $K_\infty$ (Figure 7.4), which robustly stabilizes the normalized left coprime factorization of the plant $G_s$. 

3. Construct the final feedback controller, $K$, (Figure 7.5) by combining the $H_\infty$ controller, $K_\infty$, with the weighting functions $W_1$ and $W_2$ to give

$$K = W_2 K_\infty W_1.$$  \hspace{1cm} (7.5.1)

For reference tracking, the reference signal is generally fed between $K_\infty$ and $W_1$, so that the closed loop transfer function between the reference $r$ and the plant output $y$ becomes

$$y(s) = (I + G(s)K(s))^{-1} G(s)W_1(s)K_\infty(0)W_2(0)r(s).$$  \hspace{1cm} (7.5.2)

Note that the reference $r$ is connected through a gain $K_\infty(0)W_2(0)$ where

$$K_\infty(0)W_2(0) = \lim_{s \to 0} K_\infty(s)W_2(s),$$  \hspace{1cm} (7.5.3)

to ensure a steady state gain of unity between $r$ and $y$, Figure 7.6.

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![Figure 7.3: The Shaped Plant.](image1)

![Figure 7.4: $H_\infty$-Compensation.](image2)
Figure 7.5: Final Controller.

Figure 7.6: Closed Loop System for Loop Shaping Controller
7.6 Simulation Results

The main objective of this chapter is to robustly stabilize a set of four linear models of the helicopter using a single controller. Apart from stabilizing the set of models, and due to the steps involved in the LSDP, we may also consider the performance such that realistic requirements, [114] are met as far as possible. It should be noted that our effort to improve the performance lies only in choosing the suitable weighting functions $W_1$ and $W_2$ and the alignment frequency.

7.6.1 Central Plant

A central plant (nominal plant) is obtained from four linear models of the helicopter described in this chapter. We use the theoretical results of Chapter 5 to get such a central plant. Since each of the helicopter linear models has 8 states, the central plant obtained from the theory has 32 states. A model reduction technique [62] is used to obtain a reduced order central plant of 18 states.

A comparison between the singular values of this central plant and the singular values of the other plants are shown in Figures 7.7-7.10. At low frequencies, the first and the second singular values of the central plant (solid) are, nearly, in the centre of the corresponding singular values of the other plants. The third and the fourth singular values do not show this property. While at high frequencies, all the singular values of the central plant are in the middle of those of the other plants.

7.6.2 Controller Design

Using the design procedure outlined in Section 7.5, a controller was designed and implemented for the central model as shown in Figure 7.6. The design weights for the central model were

$$W_1 = \text{diag} \left[ \frac{s+5}{s}, \frac{s+5}{s}, \frac{s+5}{s}, \frac{s+5}{s} \right]$$

$$W_2 = \text{diag} \left[ 1, 1, 1, 1 \right].$$

(7.6.1)

The singular values of the augmented plant were aligned at 5 rad/sec.
The controller was then formulated from the robust stabilization of the normalized left coprime factor shaped plant as described earlier in Section 7.4. The order of the controller is 26. This is because that the order of the central plant is 18 and the order of the shaped-central plant is 22. The controller obtained for the central plant simultaneously stabilizes the four linear models of the helicopter.

7.6.3 Frequency and Time Response Results

After obtaining the controller which stabilizes the central plant, step inputs were commanded on all the four outputs at hover (0 knots), 40 knots, 60 knots, and 80 knots. The controller simultaneously stabilized the four linear models obtained at the above mentioned operating points. Relevant frequency responses and time responses showing the control signals and step disturbance rejection properties are shown in Figures 7.11-7.41, for the central plant and the other four plants. From all the plots it can be seen that the design controller provided an acceptable level of decoupling of the controlled outputs and performance.

7.7 Conclusion

In this chapter, the simultaneous stabilization theory presented in Chapter 5 was demonstrated by a realistic example. A central plant was obtained from a set of four linear models of a high performance nonlinear helicopter at different operating points. A stabilizing controller was designed for this central model using a one degree of freedom loop-shaping design procedure. The controller obtained for the central plant simultaneously stabilized the four linear models. The drawback of this technique is the relatively high order of the controller relative to the order of the given plants. However, this can be considered as the price to be paid for controlling four plants simultaneously using one controller only. As already stated the helicopter example was used to illustrate the results of Chapter 5 and not to produce a fully tested controller for the helicopter. The final controller did, however, display satisfactory characteristics as one would perhaps expect given that the tried and tested $\mathcal{H}_\infty$ loop shaping design procedure was applied to the central plant.
Figure 7.7: Singular Values of the Central Plant (solid) & Other 4 Models.

Figure 7.8: Singular Values of the Central Plant (solid) & Other 4 Models.
Figure 7.9: Singular Values of the Central Plant (solid) & Other 4 Models.

Figure 7.10: Singular Values of the Central Plant (solid) & Other 4 Models.
Achieved Loop Shape

Figure 7.11: Achieved Loop Shape (Central Plant).

Sensitivity Function

Figure 7.12: Sensitivity Function (Central Plant).
Figure 7.13: Complementary Sensitivity Function (Central Plant).

Figure 7.14: Heave Axis Step Demand (Central Plant).
Figure 7.15: Pitch Axis Step Demand (Central Plant).

Figure 7.16: Roll Axis Step Demand (Central Plant).
Figure 7.17: Yaw Axis Step Demand (Central Plant).
Figure 7.18: Sensitivity Function (0 knots).

Figure 7.19: Complementary Sensitivity Function (0 knots).
Figure 7.20: Heave Axis Step Demand (0 Knots).

Figure 7.21: Pitch Axis Step Demand (0 Knots).
Figure 7.22: Roll Axis Step Demand (0 Knots).

Figure 7.23: Yaw Axis Step Demand (0 Knots).
Figure 7.24: Sensitivity Function (40 knots).

Figure 7.25: Complementary Sensitivity Function (40 knots).
Figure 7.26: Heave Axis Step Demand (40 knots).

Figure 7.27: Pitch Axis Step Demand (40 knots).
Figure 7.28: Roll Axis Step Demand (40 knots).

Figure 7.29: Yaw Axis Step Demand (40 knots).
Figure 7.30: Sensitivity Function (60 knots)

Figure 7.31: Complementary Sensitivity Function (60 knots).
Figure 7.32: Heave Axis Step Demand (60 knots).

Figure 7.33: Pitch Axis Step Demand (60 knots).
Figure 7.34: Roll Axis Step Demand (60 knots).

Figure 7.35: Yaw Axis Step Demand (60 knots).
Figure 7.36: Sensitivity Function (80 knots).

Figure 7.37: Complementary Sensitivity Function (80 knots).
Figure 7.38: Heave Axis Step Demand (80 knots).

Figure 7.39: Pitch Axis Step Demand (80 knots).
Figure 7.40: Roll Axis Step Demand (80 knots).

Figure 7.41: Yaw Axis Step Demand (80 knots).
Chapter 8

Conclusions and Future Research

8.1 Conclusions

This thesis has addressed the strong and simultaneous stabilization problems with the aim of extending existing theory and developing practical techniques for solving these two problems. Earlier work focuses on existence of solutions and/or is of theoretical interest. This research aimed at providing several approaches/algorithms which can be used in certain situations to find the controller. They therefore represent a practical contribution. We began by studying the Strong Stabilization Problem (StSP) which has very close links to the Simultaneous Stabilization Problem (SSP).

The strong stabilization problem of multivariable systems was categorized into minimum and non-minimum phase systems. The solution for minimum phase systems was formulated such that a stable inverse of a particular stable matrix-transfer function was required. When the systems are strictly proper, the inverse that we seek does not exist. In this case the formulation was modified such that the conditions for the existence of the required stable inverse were satisfied.

For non-minimum phase systems we used three approaches for the StSP. In the first, the Nevanlinna-Pick algorithm was used to find a stable controller. The formulation required an inner-outer factorization in which the outer part had to be unimodular in $\mathcal{H}_\infty$. But for strictly proper systems, the outer part is not unimodular. We therefore introduced a suitable modification to force the required outer part to be unimodular. The Nevanlinna-
Pick algorithm was implemented in MATHEMATICA and a program listing provided in Appendix B.

A second approach to solve the StSP was also given. In this approach the problem was reformulated as an optimization problem in $\mathcal{RH}_\infty$. A sufficient condition for the solution of this optimization problem such that the StSP can be solved was stated.

A third approach to solve the StSP was given where the problem was reformulated as a stable projection in $\mathcal{RH}_\infty$. The parameters of an unknown unimodular matrix were chosen such that the unstable part of the controller was zero. The problem was reduced to that of finding a full rank constant output feedback matrix.

Following the StSP the simultaneous stabilization problem of linear multivariable time-invariant systems was studied. Two approaches were proposed. In the first a single "central" plant was derived from the set of plants to be stabilized which could then be robustly stabilized. To find a central plant a generalized two-block $L_\infty$-optimization problem (GTBP) was proposed. The GTBP was then reduced to an ordinary two-block problem with the condition that the solution would have a required number of unstable poles. A generalized one-block problem was also defined and its solution was given as a preamble to the solution of the GTBP. In this way a central plant was obtained from the solution of the GTBP. Finally robust stabilization theory was used on the central plant to provide a sufficient condition for finding a solution to the simultaneous stabilization problem. Simulation results showed that good results can be achieved. The key novel contribution in this approach is the identification of the central plant.

In the second approach, the simultaneous stabilization problem of $l$ multivariable LTI systems was reduced using Youla’s parameterization of all stabilizing controllers to a problem of finding an initial Youla free stable parameter $Q_0(s)$ such that all the other Youla free parameters, $Q_i(s)$'s, are stable. Different cases were discussed to get a clear understanding of the problem resulting in two methods of solution. For strictly proper controllers, the problem formulation resulted in a square $A$ matrix with some unknown block entries which needed to be solved so that the $A$ matrix was stable. For bi-proper controllers or controllers which do not have full rank at infinity, the problem was reduced to a constant output feedback problem. The problems discussed in this approach were solved
for the case of two plants. When a certain condition is satisfied a closed form solution was given, otherwise numerical algorithms were provided to solve these problems. In the case of more than two plants the problem is not yet solved and more effort is needed to develop the existing numerical algorithms to calculate the parameters of $Q_0(s)$.

Finally, we emphasize that there are no tractable necessary and sufficient conditions to solve the simultaneous stabilization problem of three or more multivariable linear time-invariant systems. In this thesis we have derived sufficient conditions to solve this simultaneous stabilization problem and the related strong stabilization problem. The work advances the theory of the SSP and the StSP by introducing and investigating several new approaches to these problems, and deriving new sufficient conditions. The work, although new results were presented, was less successful in deriving practical algorithms for the SSP of more than two plants except for the approach in Chapter 5 which was based on finding a "central" plant on which existing robust stabilization methods could be applied.

A helicopter design problem was used to illustrate the "central" plant method to simultaneous stabilization.

8.2 Recommendations for Future Research

The methodology used in this thesis to tackle the strong stabilization and the simultaneous stabilization problems raises a number of questions which require further research. In this section, alternative methods, problem formulations and algorithms are suggested:

1. Implementing a full version of the Nevanlinna-Pick algorithm using MATHEMATICA and MATLAB software packages since nontrivial symbolic manipulations are required. Interfacing these two packages is a future task which may be very useful for control designers and researchers.

2. Solving the $H_{\infty}$ minmax problem defined in Chapter 5, Section 2 to get a better central plant.
3. The idea of finding a central plant can be used in other applications, even when the systems are stable. One of the current methods to improve the performance of nonlinear systems, whose linearized models at different operating points are stable, is to pick one of the linearized models and then to apply linear theory to design a controller which meets the performance requirements. Instead, one can find a central plant from the set of linear plants using the theory developed in Chapter 5 and then use the design procedure suggested in Chapter 7, or any other technique, to find a controller that achieves the performance requirements imposed on this central plant. However, for certain plants and certain performance requirements it may be that the "central" plant is not a good choice. This needs further investigation.

4. Solving the approach of the SSP proposed in Chapter 6, Section 6, using a general form of the similarity transformation matrix instead of the special one which we used. Some initial work has been done in this direction and is reported in Appendix A.

5. The algorithms presented in Section 6.7 are restricted to solve the SSP of two plants and require further developments to handle the case of more than two plants.

6. A new algorithm, to satisfy the conditions in Chapter 6 for simultaneously stabilizing more than two plants may be developed incorporating the method of inequalities, (MOI) reported in [105] and the references therein.

7. The design procedure reported in Chapter 7, which is an application of the theory developed in Chapter 5, needs more tuning to achieve better results. There are some parameters that could be varied to study their effects on the results. One of these parameters is the value of $\gamma$ which is used to solve the Generalized 1-Block Problem (GOBP) in Chapter 5. It is assumed that $\gamma$ should satisfy the condition $\sigma_k > \gamma > \sigma_{k-1}$. The effect of varying $\gamma$ in the specified interval is not yet tested. Another idea is to see the effect of the order of the resulting central plant obtained in this process when using model reduction.

8. In some cases, it may be difficult to find one controller that simultaneously stabilizes all the given linear models of a specific nonlinear system. In this situation, one may
try to divide these linear models into two or more groups such that the members of each group can be simultaneously stabilized. Then a gain scheduling technique may be used to schedule the controllers obtained in this process.

9. In other cases, robust stabilization of a set of plants may be achieved simultaneously, but the performance specifications may not be satisfied. So, a procedure to incorporate the 2DOF design technique and gain scheduling technique to achieve good performance is suggested as follows. Let the controller be given as

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} y \\ r \end{bmatrix}$$

(8.2.1)

where $K_1$ is to robustly stabilize all the given plants and $K_2$ is a prefilter to improve the performance. If $K_2$ alone does not satisfy the performance requirements for all or most of the operating points, then a set of controllers $\{K_{2i}\}$, can be used for this purpose instead of $K_2$. Then gain scheduling may be used to implement these controllers.

Besides the above points for future research we also propose the following new direction for solving the simultaneous stabilization problem.

8.2.1 New Formulation for the SSP

Given $K_0$ that stabilizes the plant $G_0$, solve the following optimization problem

$$\min_{Q_i \in \mathbb{R}^{n_{in}}} \|K_1(Q_1) - K_0\|_\alpha$$

(8.2.2)

where $\alpha = 1, 2, \text{or } \infty$ and $K_1(Q_1)$ is a stabilizing controller for $G_1$. $K_1$ is given in the linear fractional transformation form as

$$K_1 = \mathcal{F}_i(J, Q_1)$$

(8.2.3)

and

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

(8.2.4)
$J$ is defined in terms of the state-space realization of $G_1 = (A, B, C, D)$, the state feedback matrix $F$ and the output gain matrix $H$ as

$$J = \begin{bmatrix} A + BF + HC + HDP & -H & B + HD \\ F & 0 & I \\ -(C + DF) & I & -D \end{bmatrix}.$$  (8.2.5)

Equation (8.2.2) can be interpreted as in Figures 8.1-8.2.

![Figure 8.1: Optimizing the Linear Fractional Transformation, $||e_{01}||_{\infty}$.

$M$ in Figure 8.2 is computed as follows. From Figure 8.1 we have

$$\begin{pmatrix} e \\
\eta_1 \end{pmatrix} = \begin{bmatrix} J_{11} - K_0 & J_{12} \\
J_{21} & J_{22} \end{bmatrix} \begin{pmatrix} u \\
\zeta_1 \end{pmatrix} = M \begin{pmatrix} u \\
\zeta_1 \end{pmatrix}$$

where

$$J_1 = \begin{bmatrix} J_{11} & J_{12} \\
J_{21} & J_{22} \end{bmatrix}.$$  

This method can be extended to more than two plants recursively.
The philosophy behind the above idea is that if two plants can be simultaneously stabilized, then the intersection of their sets of controllers should not be empty nor rarely should they intersect at one point. So the proposed optimization method will force the resulting controller(s) to lie in the intersecting region of the two sets.
Appendix A

A General State Space Approach for the SSP of Two Plants

The condition for simultaneously stabilizing two plants was written in Chapter 6 as

\[ K_0(Q_0) = K_1(Q_1) \]  \hspace{1cm} (A.0.1)

where \( K_0 \) and \( K_1 \) are the stabilizing controllers of \( G_0 \) and \( G_1 \) respectively parametrized in term of \( Q_0 \) and \( Q_1 \), \( Q_i \in RH_{\infty} \). The formulation that we gave in Chapter 6 reduced the above condition into the following new condition:

\[
\left\{ Q_1 \left[ \begin{array}{cc} -\tilde{N}_1 & \tilde{M}_1 \\ \tilde{X}_1 & -\tilde{Y}_1 \end{array} \right] + \left[ \begin{array}{c} \tilde{X}_1 \\ -\tilde{Y}_1 \end{array} \right] \right\} \left\{ \begin{array}{c} M_0 \\ N_0 \end{array} \right\} Q_0 - \left( \begin{array}{c} Y_0 \\ X_0 \end{array} \right) \right\} = 0 \]  \hspace{1cm} (A.0.2)

For abbreviation, we let \( A(s,Q_1(s)) \) and \( B(s,Q_0(s)) \) be defined as

\[ A(s,Q_1(s)) = \left\{ Q_1 \left[ \begin{array}{cc} -\tilde{N}_1 & \tilde{M}_1 \\ \tilde{X}_1 & -\tilde{Y}_1 \end{array} \right] + \left[ \begin{array}{c} \tilde{X}_1 \\ -\tilde{Y}_1 \end{array} \right] \right\} \]  \hspace{1cm} (A.0.3)

and

\[ B(s,Q_0(s)) = \left\{ \begin{array}{c} M_0 \\ N_0 \end{array} \right\} Q_0 - \left( \begin{array}{c} Y_0 \\ X_0 \end{array} \right) \]  \hspace{1cm} (A.0.4)

Therefore we have the following equation in \( Q_0(s) \) and \( Q_1(s) \)

\[ A(s,Q_1(s))B(s,Q_0(s)) = 0. \]  \hspace{1cm} (A.0.5)

Equation (A.0.5) implies that \( A(s,Q_1(s)) \neq 0 \) and \( B(s,Q_0(s)) \neq 0 \), but \( AB = 0 \) which was proved in Chapter 6.
Now the problem of simultaneously stabilizing two MIMO systems is reduced to finding
two stable parameters, \( Q_0 \) and \( Q_1 \) that satisfy equation (A.0.5). This method may be
extended for the general case of more than two plants as shown in Chapter 6.

In the following section we will give a state-space representation of \( A(s, Q_1(s)) \) and
\( B(s, Q_0(s)) \), while doing so we will try to diagonalize the intermediate results to reach
to a general approach to solve for \( Q_0 \) and \( Q_1 \) different than that we gave in Chapter 6.

A.1 State Space Representation of \( A(s, Q_1(s))B(s, Q_0(s)) = 0 \)

As in Chapter 6, let \( G_0(s) \) and \( G_1(s) \) be given as
\[
G_0(s) = N_0(s)M_0^{-1}(s) \quad \text{(A.1.1)}
\]
and
\[
G_1(s) = \tilde{M}_1^{-1}(s)\tilde{N}_1(s). \quad \text{(A.1.2)}
\]

Assume that the following eight stable transfer function matrices are exist and given in
state space form as follows.

\[
\begin{bmatrix}
M_0 & Y_0 \\
N_0 & X_0
\end{bmatrix} :=
\begin{bmatrix}
A_0 + B_0F_0 & B_0 & -H_0 \\
F_0 & I & 0 \\
C_0 + D_0F_0 & D_0 & I
\end{bmatrix} \quad \text{(A.1.3)}
\]

and

\[
\begin{bmatrix}
\tilde{X}_1 & -\tilde{Y}_1 \\
-\tilde{N}_1 & \tilde{N}_1
\end{bmatrix} :=
\begin{bmatrix}
A_1 + H_1G_1 & -(B_1 + H_1D_1) & H_1 \\
F_1 & I & 0 \\
C_1 & -D_1 & I
\end{bmatrix} \quad \text{(A.1.4)}
\]

Define \( Q_i \) as
\[
Q_i = \begin{bmatrix} A_i^t & B_i^t \\ C_i & D_i \end{bmatrix} \quad \text{(A.1.5)}
\]

Then \( A(Q_1) \) and \( B(Q_0) \) are given, after some matrix manipulations and similarity trans­
formations, as
\[
A(Q_1, s) = \left\{ Q_1 \begin{bmatrix} -\tilde{N}_1 & \tilde{M}_1 \end{bmatrix} + \begin{bmatrix} \tilde{X}_1 & -\tilde{Y}_1 \end{bmatrix} \right\}
\]
\[ A(x) \times B(x) = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 + H_1 C_1 \end{bmatrix} \begin{bmatrix} -B_1 D_1 + X_2 (B_1 + H_1 D_1) & B_1^2 - X_2 H_1 \\ -(B_1 + H_1 D_1) & +H_1 \end{bmatrix} \]

\[ = \begin{bmatrix} A_0 & B_a \\ C_a & D_a \end{bmatrix} \]

(A.1.6)

where \( X_2 \) solves

\[ A_0 X_2 - X_2 (A_1 + H_1 C_1) + B_1^2 C_1 = 0 \]  

(A.1.7)

and

\[ B(Q_0, s) = \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} Q_0 - \begin{bmatrix} Y_0 \\ X_0 \end{bmatrix} \]

\[ = \begin{bmatrix} A_0 + B_0 F_0 \\ 0 \end{bmatrix} \begin{bmatrix} A_0^0 \\ B_0^0 \end{bmatrix} + X_1 \begin{bmatrix} D_0 D_0^0 - I \\ (C_0 + D_0 F_0) X_1 + D_0 C_0^0 \end{bmatrix} \]

(A.1.8)

where \( X_1 \) solves

\[ (A_0 + B_0 F_0) X_1 - X_1 A_0^0 + B_0 C_0^0 = 0. \]

(A.1.9)

Define \( A(Q_1, s) B(Q_0, s) \) as

\[ A(Q_1, s) B(Q_0, s) = \begin{bmatrix} A_0 & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_0 & B_b \\ C_b & D_b \end{bmatrix} \]

\[ = \begin{bmatrix} A_0 & B_a C_b & B_a D_b \\ C_a & D_a C_b & D_a D_b \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ 0 & A_b \end{bmatrix} \]

\[ = \begin{bmatrix} A_0 & 0 \\ 0 & A_b \end{bmatrix} \begin{bmatrix} B_a D_a - X_3 B_b \\ D_a D_b \end{bmatrix} \]

(A.1.10)
where $X_3$ solves

$$A_a X_3 - X_3 A_a + B_a C_b = 0. \quad (A.1.12)$$

and

$$\bar{A} = \begin{bmatrix} A_a & 0 & 0 & 0 \\ 0 & A_1 + H_1 C_1 & 0 & 0 \\ 0 & 0 & A_0 + B_0 F_0 & 0 \\ 0 & 0 & 0 & A_b \end{bmatrix}. \quad (A.1.13)$$

Then

$$\bar{B} = \begin{bmatrix} B_1^a \begin{bmatrix} -D_1 & I \\ -(B_1 + H_1 D_1) & -H_1 \end{bmatrix} + X_3 \begin{bmatrix} (B_1 + H_1 D_1) & -H_1 \\ -H_1 & B_0 D_0^b - X_1 B_1^b + H_0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} I \\ D_0 \\ 0 \\ I \end{bmatrix}$$

$$- \begin{bmatrix} B_0 D_0^b - X_1 B_1^b + H_0 \\ B_1^b \end{bmatrix}.$$

Since $A_a$ and $A_b$ are diagonal we may assume $X_3$ to be diagonal as follows.

$$X_3 = \begin{bmatrix} X_{311} & 0 \\ 0 & X_{322} \end{bmatrix}. \quad (A.1.15)$$

Then

$$\bar{B} = \begin{bmatrix} \{ -B_1^a D_1 + X_3 (B_1 + H_1 D_1) \} D_0^b - X_{311} \{ B_0 D_0^b - X_1 B_1^b + H_0 \} \\ +(B_1^a - X_3 H_1) \{ D_0 D_0^b - I \} \\ \{ -B_1^a D_1 + X_3 (B_1 + H_1 D_1) \} D_0^b - X_{322} B_1^b + H_1 \{ D_0 D_0^b - I \} \\ B_0 D_0^b - X_1 B_1^b + H_0 \\ B_1^b \end{bmatrix},$$

and

$$\bar{C} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 & \bar{C}_3 & \bar{C}_4 \end{bmatrix}. \quad (A.1.16)$$
where

\[ \begin{align*}
\vec{C}_1 &= C_4^2 \\
\vec{C}_2 &= \vec{C}_2X_2 + D_4^2C_1 + F_4 \\
\vec{C}_3 &= C_4^2X_311 + (-D_4^2D_4 + I)F_0 + D_4^2(C_0 + D_0F_0) \\
\vec{C}_4 &= (C_4^2X_3 + D_4^2C_1 + F_4)X_{322} + (-D_4^2D_4 + I)(F_0X_3 + C_4^2) \\
&\quad + D_4^2[(C_0 + D_0F_0)X_3 + D_0C_4^2] 
\end{align*} \]  

(A.1.17)

and

\[ \begin{align*}
\vec{D} &= \begin{bmatrix} I \\ -D_0 \end{bmatrix} - \begin{bmatrix} I \\ 0 \end{bmatrix} \\
&= D_4^2(D_0 - D_4)D_0^2 + D_0^2 - D_4^2 
\end{align*} \]  

(A.1.18)

Since the matrix, \( \vec{A} \), above is diagonal, we may split the above system into the sum of small subsystems as below.

\[ A(Q_1, s)B(Q_0, s) = \begin{bmatrix}
A_0^2 \\
C_4^2 \\
\end{bmatrix} \begin{bmatrix}
\{-B_4^2D_4 + X_0(B_1 + H_1D_4)\} D_0^2 - X_{311} \{B_4^2D_4^2 - X_1B_0^2 + H_0\} + (B_4^2 - X_2H_1) \{D_4^2D_4^2 - I\} \\
\{C_4^2X_3 + D_4^2C_1 + F_4\} \\
\end{bmatrix} \begin{bmatrix}
\{C_4^2X_311 + (-D_4^2D_4 + I)F_0 + D_4^2(C_0 + D_0F_0)\} \\
\{C_4^2X_3 + D_4^2C_1 + F_4\}X_{322} + (-D_4^2D_4 + I)(F_0X_3 + C_4^2) + D_4^2[(C_0 + D_0F_0)X_3 + D_0C_4^2] \\
\end{bmatrix} \begin{bmatrix}
D_0^2 \\
B_0^2 \\
\end{bmatrix} \\
+ \begin{bmatrix}
\{A_0^2 \} \\
\{A_0^2 \} \\
\end{bmatrix} \begin{bmatrix}
\{B_0D_0^2 - X_1B_0^2 + H_0\} \\
\{B_0D_0^2 - X_1B_0^2 + H_0\} \\
\end{bmatrix} \\
+ \begin{bmatrix}
\{A_0^2 \} \\
\{A_0^2 \} \\
\end{bmatrix} \begin{bmatrix}
\{B_4^2 \} \\
\{B_4^2 \} \\
\end{bmatrix} \\
+ \begin{bmatrix}
\{A_0^2 \} \\
\{A_0^2 \} \\
\end{bmatrix} \begin{bmatrix}
\{B_4^2 \} \\
\{B_4^2 \} \\
\end{bmatrix} \\
+ \begin{bmatrix}
\{A_0^2 \} \\
\{A_0^2 \} \\
\end{bmatrix} \begin{bmatrix}
\{B_4^2 \} \\
\{B_4^2 \} \\
\end{bmatrix} \\
\end{align*} \]  

(A.1.19)

The last term in the above equation is the \( D \)-term of the system.

Now, we will list the possible conditions that will make \( A(Q_1)B(Q_0) = 0 \). From these conditions, the free parameters \( Q_0(s) \) and \( Q_1(s) \) are to be determined.
A.2 Simplification of The Problem

In the following subsections, we will give different approaches to find the parameters of $Q_0(s)$ and $Q_1(s)$. These approaches are based on the new formulation that we got and simplified by equations (A.1.10) and (A.1.19).

A.2.1 Approach I

This approach is based on equation (A.1.10). The condition $A(Q_1)B(Q_0) = 0$ can be satisfied if the following equations are satisfied.

\begin{align*}
B_4D_5 - X_3B_6 &= 0 \quad (A.2.1) \\
C_4X_3 + D_4C_6 &= 0 \quad (A.2.2) \\
D_4D_5 &= 0 \quad (A.2.3) \\
A_4X_3 - X_3A_6 + B_6C_5 &= 0 \quad (A.2.4) \\
(A_6 + B_6F_0)X_3 - X_1A_6^0 + B_6C_6^0 &= 0 \quad (A.2.5) \\
A_1^3X_2 - X_2A_1 + H_1C_1 + B_1C_1 &= 0 \quad (A.2.6) \\
A_4X_3 - X_3A_6 + B_6C_6 &= 0. \quad (A.2.7)
\end{align*}

Since $A_4$ and $A_6$ are diagonal, we can assume that $X_3$ is also diagonal and given as

\begin{equation}
X_3 = \begin{bmatrix} X_{311} & 0 \\ 0 & X_{322} \end{bmatrix}. \quad (A.2.8)
\end{equation}

An expansion of the above equations will result in the following set of equations

\begin{align*}
\{-B_4^1D_4 + X_2(B_4 + H_1D_4)\}D_4^0 - X_{311}\{B_6D_6^0 - X_1B_6^0 + H_6\} + \{B_4^1 - X_2H_1\}\{D_6D_6^0 - I\} &= 0, (A.2.9) \\
\{-B_4^1D_4 + X_2(B_4 + H_1D_4)\}D_4^0 - X_{322}B_6^0 + H_1\{D_6D_6^0 - I\} &= 0, \quad (A.2.10) \\
\{C_4^2X_{311} + (-D_4^1D_4 + I)F_0 + D_4^1(C_0 + D_6F_0)\} &= 0, \quad (A.2.11) \\
(C_4^2X_3 + D_4^1C_1 + F_1)X_{322} + (-D_4^1D_4 + I)(F_0X_3 + C_0^2) + D_4^1\{[C_0 + D_6F_0]X_3 + D_6C_6^0\} &= 0, \quad (A.2.12)
\end{align*}
\((-D_1^2D_1 + I)D_0^2 + D_1^2(D_0D_0^2 - I) = 0.\) (A.2.13)

\[A_1^2X_{311} - X_{311}(A_0 + B_0F_0) + [-B_1^2D_1 + X_2(B_1 + H_1D_1)]F_0 + (B_1^2 - X_2H_1)(C_0 + D_0F_0) = 0\] (A.2.14)

\[[-B_1^2D_1 + X_2(B_1 + H_1D_1)](F_0X_1 + C_0^2) + [-B_1^2 - X_2H_1](C_0 + D_0F_0)X_1 + D_0C_0^2 = 0\] (A.2.15)

\[(A_1 + H_1C_1)X_{322} - X_{322}A_0^2 + [-B_1 + H_1D_1)(F_0X_1 + C_0^2)] + H_1[(C_0 + D_0F_0)X_1 + D_0C_0^2] = 0\] (A.2.16)

and

\[-(B_1 + H_1D_1)F_0 + H_1(C_0 + D_0F_0) = 0,\] (A.2.17)

Equation (A.2.17) has no unknowns, therefore it will be the condition for simultaneously stabilizing two plants.

A.2.1.1 Simplification of The Equations

Let \(\Delta_0 = [(D_0 - D_1)D_0^2 - I]\) and \(\Delta_1 = [D_1^2(D_0 - D_1) + I].\) In the case of strictly proper systems or when \(D_0 = D_1, \Delta_0 = -I\) and \(\Delta_1 = I.\)

After several simplifications for the above set of equations we reach to the following simplified set of equations.

\[B_1^2\Delta_0 = X_2X_{322}B_0^2 + X_{311}\left\{B_0D_0^2 - X_1B_0^2 + H_0\right\},\] (A.2.18)

\[\Delta_1C_0^2 = C_0^2X_{311}X_1 - \left\{C_0^2X_2 + D_0^2C_1 + F_1\right\}X_{322}\] (A.2.19)

\[A_1^2X_{311} - X_{311}(A_0 + B_0F_0) + B_1^2[C_0 + (D_0 - D_1)F_0] = 0,\] (A.2.20)

\[(A_1 + H_1C_1)X_{322} - X_{322}A_0^2 + [-B_1 + H_1(D_0 - D_1)]C_0^2 = 0,\] (A.2.21)

\[B_1^2[C_0 + (D_0 - D_1)F_0]X_1 - X_2[-B_1 + H_1(D_0 - D_1)]C_0^2 + B_1^2(D_0 - D_1)C_0^2 = 0\] (A.2.22)
\[(A_0 + B_0 F_0) X_1 - X_1 A_0^0 + B_0 C_0^0 = 0, \]  \quad (A.2.23)

\[A_1^0 X_2 - X_2 (A_1 + H_1 C_1) + B_1^0 C_1 = 0, \]  \quad (A.2.24)

\[\Delta_1 D_\theta^0 = D_\theta^1 \quad \text{or} \quad D_1^1 \Delta_0 = -D_\theta^0, \]  \quad (A.2.25)

and the following condition

\[(B_1 + H_1 D_1) F_0 = H_1 (C_0 + D_0 F_0). \]  \quad (A.2.26)

Now we have 8 nonlinear equations and one condition in 12 unknowns.

### A.2.1.2 A Suggested Procedure for the Solution

**Step 0**: Choose \(F_0, H_1 \in R^{n \times m}\) such that the condition in (A.2.26) is satisfied and \((A_0 + B_0 F_0)\) and \((A_1 + H_1 C_1)\) are stables.

**Step 1**: Assume a stable \(A_0^0\) and any \(C_0^0\).

**Step 2**: Solve (A.2.21) for \(X_{22}\) and (A.2.23) for \(X_1\).

**Step 3**: Solve (A.2.22) for \(B_1^1\) as a function of \(X_2\), i.e.

\[B_1^1 \left( \left[ C_0 + (D_0 - D_1) F_0 \right] X_1 + (D_0 - D_1) C_0^0 \right) = X_2 [H_1 (D_0 - D_1) - B_1] C_0^0. \]

Let \(Z_0\) and \(Z_1\) be defined as \(Z_0 = \left[ C_0 + (D_0 - D_1) F_0 \right] X_1 + (D_0 - D_1) C_0^0\) and \(Z_1 = [H_1 (D_0 - D_1) - B_1] C_0^0\). Then \(B_1^1\) is given by

\[B_1^1 = X_2 Z_1 Z_0^T. \]  \quad (A.2.27)

**Step 4**: Substitute for \(B_1^1\) in (A.2.24) and simplify it to become

\[A_1^0 X_2 - X_2 (A_1 + H_1 C_1 + Z_1 Z_0^T C_1) = 0. \]  \quad (A.2.28)

**Step 5**: Assume stable \(A_1^0\), solve for \(X_2\) from Step 4, then solve for \(B_1^1\) from Step 3.

**Step 6**: Solve (A.2.20) for \(X_{311}\).
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Step 7: Assume any $D^o_q$ and solve (A.2.25) for $D^i_q$.

Step 8: Solve (A.2.18) for $B^o_q$ and (A.2.19) for $C^o_q$ as follows.

\[
B^o_q = (X_2X_{322} - X_{311}X_1)^\dagger \left\{ B_1^1 \Delta_0 - X_{311}(B_0D^o_q + H_0) \right\} \quad (A.2.29)
\]

\[
C^i_q = - \left\{ \Delta_1C^o_q + (D_1^iC_1 + F_1)X_{322} \right\} (X_2X_{322} - X_{311}X_1)^\dagger \quad (A.2.30)
\]

where $(.)^\dagger$ is the pseudo-inverse notation. It is clear that the existence of $B^o_q$ and $C^i_q$ depend on the existence of the inverse of $(X_2X_{322} - X_{311}X_1)$ since both left and right pseudo-inverse are required. This is the bottle-neck of this approach.

Solution of $Ax=b$ Given the system $Ax=b$ where $A \in \mathbb{R}^{n \times m}$, $x$ is the column $m$-vector with components $x_i$, $i = 1, 2, ..., m$, and $b$ is the column $n$-vector with components $b_i$, $i = 1, 2, ..., n$. Define the augmented matrix $B = (A, b)$. Then the following theorem is stated.

Theorem A.1 The system $Ax=b$ possesses

1. an infinite number of solutions if and only if $\text{rank}(A) = \text{rank}(B) < m$,
2. a unique solution if and only if $\text{rank}(A) = \text{rank}(B) = m$,
3. no solution if and only if $\text{rank}(A) < \text{rank}(B)$.

The generalization of the above theorem is needed in solving some of the equations of this approach.

A.2.2 Approach II

This approach is based on equation (A.1.19). It is clear from this equation that the four subsystems can be made zeros and another set of equations may be solved to get $A(Q_1)B(Q_0) = 0$. This set of equations is stated bellow and its solution is discussed.

\[
\{-B_1^2D_1 + X_2(B_1 + H_1D_1)\}D_2^0 - X_{311} \left\{ B_0D_0^o - X_1B_1^0 + H_0 \right\} + (B_1^1 - X_2H_1) \{ D_0D_0^o - I \} = 0 \quad (A.2.31)
\]
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\[ C_{2}X_{2} + D_{1}^{1}C_{1} + F_{1} = 0, \quad (A.2.32) \]
\[ B_{0}D_{0}^{0} - X_{2}D_{0}^{0} + H_{0} = 0, \quad (A.2.33) \]

\[ (C_{2}X_{2} + D_{1}^{1}C_{1} + F_{1}) X_{2} + (-D_{1}^{1}D_{1}^{1} + I)(F_{0}X_{1} + C_{2}^{0}) + D_{1}^{1} [(C_{0} + D_{0}F_{0})X_{1} + D_{0}C_{2}^{0}] = 0 \quad (A.2.34) \]

\[ (-D_{1}^{1}D_{1}^{1} + I)D_{0}^{0} + D_{1}^{1}(D_{0}D_{0}^{0} - I) = 0. \quad (A.2.35) \]
\[ (A_{0} + B_{0}F_{0})X_{1} - X_{1}A_{0}^{0} + B_{0}C_{0}^{0} = 0, \quad (A.2.36) \]
\[ A_{1}^{1}X_{3} - X_{3}(A_{1} + H_{1}C_{1}) + B_{1}^{1}C_{1} = 0, \quad (A.2.37) \]
\[ A_{0}X_{3} - X_{3}A_{0} + B_{0}C_{0} = 0. \quad (A.2.38) \]

Solving equation (A.2.31) for \( B_{1}^{1} \) and (A.2.34) for \( C_{2}^{0} \) we get

\[ B_{1}^{1} = X_{3}(H_{1} + B_{1}D_{1}^{1}) \quad (A.2.39) \]
\[ C_{2}^{0} = -(D_{0}^{0}C_{0} + F_{0})X_{1}. \quad (A.2.40) \]

Substituting for \( C_{2}^{0} \) and \( B_{1}^{1} \) in (A.2.36) and (A.2.37) we get

\[ (A_{0} - B_{0}D_{0}^{0}C_{0})X_{1} - X_{1}A_{0}^{0} = 0, \quad (A.2.41) \]
\[ A_{1}^{1}X_{3} - X_{3}(A_{1} - B_{1}D_{1}^{1}C_{1}) = 0. \quad (A.2.42) \]

A.2.2.1 Solution of Approach II

In this choice if we put aside equation (A.2.38), since \( X_{3} \) does not appear in the chosen set of equations, then 7 equations are left with 10 unknowns. Therefore we may assume \( A_{0}^{0}, A_{1}^{1}, \) and \( D_{0}^{0} \) to be free parameters and then solve for the rest of the unknowns. The idea behind this assumption is to guarantee that \( A_{0}^{0} \) and \( A_{1}^{1} \) are stable.

Solution I Assume that we let \( X_{3} = X_{3} = I \). Then the following algorithm is proposed.

1. Choose \( D_{0}^{0} \) such that \( (A_{0} - B_{0}D_{0}^{0}C_{0}) \) is stable.
2. Solve (A.2.35) for $D_i^1$.

3. Check if $(A_1 - B_1 D_0^1 C_1)$ is stable.

4. If stability is obtained in step (3) continue, otherwise go to step 1 to choose another $D_0^1$. (If stability is not obtained, see solution II).

5. Solve for the parameters of $Q_0(s)$ and $Q_1(s)$.

Solution II Assume that we could not find $D_0^1$ such that $A_0 - B_0 D_0^1 C_0$ and $A_1 - B_1 D_0^1 C_1$ are stable as required in Solution I. Then we may use the following theorem in solving (A.2.41) and (A.2.42) to get $X_1$ and $X_2$.

**Theorem A.2** The equation

$$AX = XB$$  \hspace{1cm} (A.2.43)

has a non-zero solution if and only if $A$ and $B$ have at least one common characteristic root.

So we may choose $A_0^1$ and $A_1^1$ such that a non-zero solution for $X_1$ and $X_2$ are exist. But we have to impose the following condition on $X_1$ and $X_2$. $X_1$ and $X_2$ should make equations (A.2.32) and (A.2.33) consistent to get $C_1^1$ and $B_0^1$.

A.2.3 Approach III

In this approach we may use the following idea to determine the parameters of $Q_0(s)$ and $Q_1(s)$. For two nonzero systems, $G_1(s) + G_2(s) = 0$ if the following holds:

$$A_1 = A_2$$  \hspace{1cm} (A.2.44)

$$B_1 = B_2$$  \hspace{1cm} (A.2.45)

$$C_1 = -C_2$$  \hspace{1cm} (A.2.46)

$$D_1 = -D_2.$$  \hspace{1cm} (A.2.47)
Using this fact and reasonable combinations of the plants give by equation (A.1.19), after several simplifications and combinations, we get the following new set of equations.

\[
\begin{align*}
A_q^1 & = A_1 + H_1 C_1 \\
A_q^0 & = A_0 + B_0 F_0 \\
B_q^0 & = \{B_0 D_q^0 - X_1 B_q^0 + H_0\} \\
& = (I + X_1)^{-1}(B_0 D_q^0 + H_0) \\
C_q^1 & = -(C_q^1 X_2 + D_q^1 C_1 + F_1) \\
& = -(D_q^1 C_1 + F_1)(I + X_2)^{-1} \\
B_q^1 & = (I + X_2)(H_1 + B_1 D_q^0) - (X_{311} - X_{322}) B_q^0 D_q^0 D_q^1 \\
C_q^0 & = -(F_0 + D_q^0 C_0)(I + X_1) - D_q^0 D_q^1 C_q^1 (X_{311} - X_{322}) \\
& = (A_0 + B_0 F_0) X_1 - X_1 A_q^0 + B_0 C_q^0 = 0 \\
& = A_q^1 X_2 - X_2 (A_1 + H_1 C_1) + B_q^1 C_1 = 0 \\
\left\{D_q^1 (D_0 - D_1) + I\right\} D_q^0 & = +D_q^1. \\
A_q^1 X_{311} - X_{311} (A_0 + B_0 F_0) + B_q^1 C_0 + (D_0 - D_1) F_0 & = 0, \\
(A_1 + H_1 C_1) X_{322} - X_{322} A_q^0 + [-B_1 + H_1 (D_0 - D_1)] C_q^0 & = 0, \\
B_q^1 C_0 + (D_0 - D_1) F_0 X_1 - X_3 [-B_1 + H_1 (D_0 - D_1)] C_q^0 + B_q^1 (D_0 - D_1) C_q^0 & = 0, \text{ and one condition}
\end{align*}
\]

The following condition

\[-(B_1 + H_1 D_1) F_0 + H_1 (C_0 + D_0 F_0) = 0, \]  
So we have 10 unknowns: \(B_q^0, B_q^1, C_q^0, C_q^1, D_q^0, D_q^1, X_1, X_2, X_{311}\) and \(X_{322}\) in 10 equations and one condition.

**A.2.3.1 Strictly Proper Systems**

For strictly proper systems, \(D_0 = D_1 = 0\), (the result also valid for \(D_0 = D_1\)) and from equation (A.2.58) we get

\[D_q^1 = D_q^0\]
The equations (A.2.48-A.2.62) of the previous subsection will be reduced into the following new set of equations

\begin{align*}
A_l^1 &= A_1 + H_1 C_1, \\
A_0^0 &= A_0 + B_0 F_0, \quad (A.2.64) \\
B_{q}^0 &= \left\{ B_0 D_q^0 - X_1 B_q^0 + H_0 \right\}, \\
&= (I + X_1)^{-1}(B_0 D_q^0 + H_0), \quad (A.2.65) \\
C^1_{q} &= -\{ C^1_{q} X_2 + D^1_{q} C_1 + F_1 \}, \\
&= -(D^1_{q} C_1 + F_1)(I + X_2)^{-1}, \quad (A.2.66) \\
B^1_{q} &= (I + X_2)(H_1 + B_1 D^1_{q}) - (X_{311} - X_{322})B^0_{q}, \quad (A.2.67) \\
C^0_{q} &= -(F_0 + D^0_{q} C_0)(I + X_1) - C^0_{q}(X_{311} - X_{322}), \quad (A.2.68)
\end{align*}

\begin{align*}
(A_0 + B_0 F_0)X_1 - X_1 A_0^0 + B_0 C^0_{q} &= 0, \quad (A.2.69) \\
A^1_2 X_2 - X_2 (A_1 + H_1 C_1) + B^1_1 C_1 &= 0, \quad (A.2.70) \\
A^1_2 X_{311} - X_{311} (A_0 + B_0 F_0) + B^1_1 C_0 &= 0, \quad (A.2.71) \\
(A_1 + H_1 C_1)X_{322} - X_{322} A^0_q - B_1 C^0_q &= 0, \quad (A.2.72) \\
B^1_2 C_0 X_1 + X_2 B_1 C^0_q &= 0. \quad (A.2.73)
\end{align*}

and the following condition

\begin{align*}
B_1 F_0 &= H_1 C_0. \quad (A.2.74)
\end{align*}

So we have 10 unknowns: $B^0_q, B^1_q, C^0_q, C^1_q, D^0_q, D^1_q, X_1, X_2, X_{311}$ and $X_{322}$ in 10 simplified equations and one condition.

**A.2.4 Approach IV: Special Case**

In this approach let us assume that we are seeking for constant $Q_0$ and $Q_1$ that make $K_0(Q_0) = K_1(Q_1)$. Then we may let

\begin{align*}
C^0_q &= 0, \\
B^1_q &= 0. \quad (A.2.75)
\end{align*}
to get $Q_0 = D_q^0$ and $Q_1 = D_q^1$. The condition we got in this case can be derived from any of the previous approaches.

From (A.2.72) and (A.2.73) we get $X_{311} = X_{322}$, and equations (A.2.68-A.2.69) are reduced to

$$H_1 + B_1 D_q^1 = 0, \quad (A.2.77)$$

$$F_0 + D_q^0 C_0 = 0, \quad (A.2.78)$$

since $X_1$ and $X_2$ can be zero or identity from (A.2.70-A.2.71). With the help of conditions (A.2.77) and (A.2.78) it is easy to prove that $K_0 = D_q^0$ and $K_1 = D_q^1$. But (A.2.63) implies that $K_0 = K_1$. So the condition for existing $D_q^0$ and $D_q^1$ is that there exist $F_0$ and $H_1$ such that $(A_1 + H_1 C_1)$ and $(A_0 + B_0 F_0)$ are stable and equations (A.2.77-A.2.78) are consistent. Therefore we may state the following lemma.

**Lemma A.3** A sufficient condition for two strictly proper plants to be simultaneously stabilized by one controller is

$$H_1 C_0 = B_1 F_0, \quad (A.2.79)$$

where the controller is a constant matrix equal to $D_q^0$ or $D_q^1$.

**Example 1** Given two unstable plants

$$P_0(s) = \begin{bmatrix} 0.0 & 1.0 & 1.0 & 0.0 \\ 3.0 & -2.0 & 1.0 & 1.0 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \end{bmatrix} \quad (A.2.80)$$

$$P_1(s) = \begin{bmatrix} 0.0 & 3.0 & 1.0 & 0.0 \\ 4.0 & -4.0 & 1.0 & 2.0 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 2.0 & 0.0 & 0.0 \end{bmatrix} \quad (A.2.81)$$

The eigenvalues of $P_0(s)$ are $[1, -3]$ and that of $P_1(s)$ are $[2, -6]$. The controller $K$ is

$$K = D_q^0 = D_q^1 = \begin{bmatrix} 1.8375 & 0.7322 \\ 0.4264 & 0.3058 \end{bmatrix} \quad (A.2.82)$$
The closed loop poles when applying $K$ as a controller are $[-3.1843, -1.6911]$ and $[-1.4533, -7.0718]$.

### A.2.5 Extension of Approach IV

In this section we may state the following claim.

**Claim 1.1:** Given $n$ unstable plants. Then

1. These plants are simultaneously stabilizable if $\exists$ $F_0$ and $H_i$, $i = 1, 2, ..., n - 1$ such that

   $H_i C_0 = B_i F_0$,  \hspace{1cm} (A.2.83)

   and $(A_0 + B_0 F_0)$, $(A_i + H_i C_i)$ are stable for all $i = 1, 2, ..., n - 1$.

2. The controller that stabilizes these plants is constant and is given as a solution to the following equation

   $K C_0 = -F_0$,  \hspace{1cm} (A.2.84)

   or

   $B_i K = -H_i$  \hspace{1cm} (A.2.85)

   for $0 < i \leq n - 1$.

   □

Similar result can be stated for proper systems.

### A.2.6 Problems

The complete solution of the new approach described in this report is not complete until the following problem(s) is/are solved.

**Problem 1:** Given $B_1$ and $C_0$, find $H_1$ and $F_0$ such that $(A_0 + B_0 F_0)$ and $(A_1 + H_1 C_1)$ are stables and

$B_1 F_0 = H_1 C_0$
is satisfied.

This problem appeared in Approach I (A.2.26), Approach III (A.2.62) and (A.2.75), Approach IV (A.2.79) and in the extension of Approach IV (A.2.83).

**Problem 2:** In Solution II of Approach II, $F_1, H_0, D^1_\alpha$ and $D^0_\alpha$ are to be chosen such that equations (A.2.32) and (A.2.33) are consistent when solving for $C^1_\alpha$ and $B^0_\alpha$. 
Appendix B

A MATHEMATICA Program for the Nevanlenna-Pick Algorithm

This a list of an m (mathematica) file to implement the Nevanlinna-Pick interpolation algorithm, written in MATHEMATICA and imported from the MATHEMATICA Editor:

n=2; (* n number of interpolation points *)
Table[w[i,j],i,n,j,i,n];
Table[zeta[i],i,n];
Table[A1[i],i,n-1];
Table[B1[i],i,n-1];
Table[C1[i],i,n-1];
Table[D1[i],i,n-1];
Table[y[i],i,n-1];

(*Defining the function y *)
y[zeta_,z_]:=Abs[zeta]/zeta (zeta - z)/(1 - Conjugate[zeta] . z) /;(zeta != 0)
y[zeta_,z_]:=z /;(zeta == 0)

(* Function to find square root of a matrix *)
sqrtm[a_]:=Block[{k,ai},x[0]=IdentityMatrix[Length[a]]; Do[x[k+1]=1/2 (x[k]+a . Inverse[x[k]]),k,0,12]; ai=x[13]]

(*Defining the matrix L={A,B;C,D} *)
A[l_,E_,type_] := N[Sqrt[1/(1 - E . Conjugate[E])]]; type = "R";
A[l_,E_,type_] := N[Inverse[IdentityMatrix[Length[E]]].
Outer[Times,E,Conjugate[E]]]; type = "C";
A[l_,E_,type_] := N[Inverse[IdentityMatrix[Length[E]]].
E . Transpose[Conjugate[E]]]; type = "M";
B[l_,E_,type_] := A[l,E,type] E /; type = "R";
D[l_,E_,type_] := N[Inverse[IdentityMatrix[Length[E]]].
Outer[Times,Conjugate[E],E]] /; type = "R";
D[l_,E_,type_] := N[Sqrt[1/(1 - Conjugate[E] . E)]] /; type = "C";
D[l_,E_,type_] := N[Inverse[IdentityMatrix[Length[E]]].
Transpose[Conjugate[E]] . E]] /; type = "M";
C[l_,E_,type_] := - D[l,E,type] Conjugate[E] /; type = "R" || type = "M";
C[l_,E_,type_] := D[l,E,type] Conjugate[E] /; type = "C";

(° Defining Gamma °)
gama[E_,X_,k_,i_,type_] := 1/y[zeta[k-1],zeta[i]] (A[k-1,E,type] X + B1[k-1,E,type]) . (Inverse[Outer[Times,Cl[k-1,E,type],X] + D1[k-1,E,type]]) /;
type = "R";
gama[E_,X_,k_,i_,type_] := 1/y[zeta[k-1],zeta[i]] (A[k-1,E,type] . X + B1[k-1,E,type])
(C1[k-1,E,type] . X + D1[k-1,E,type]) /; type = "C"

(° End of the Algorithm°)

Illustrative Example

<<initials.m

(° Defining The Function H(s), The first function°)

(° Zeta : The zeros of interpolations °)
zeta[1] = 0; zeta[2] = 0.5;
w[1,1] = {0.2, 0.4}; w[1,2] = {0.3, 0.7};
type = "C"; k = 2; i = 2;
w[2, 2] = gama[w[1, 1], w[1, 2], k, i, type]

Output

\{0.315827, 0.902693\}

invgama[E1, X1, k, i, type_] := Inverse[y[zeta[1], x] Outer[Times, X1, C1[k - 1, E1, type]] A1[k - 1, E1, type]] . (B1[k - 1, E1, type] - y[zeta[1], x] D1[k - 1, E1, type] X1) ;
type == "C";

phi = invgama[w[1, 1], w[2, 2], 2, 2, "C"]

Output

\{-1.09443 - 0.403696, -0.223607 - 0.353105\} ^ {1.11803 + 0.474317}

\{-0.447214 - 1.00924, -0.223607 - 0.353105\} ^ {1.11803 + 0.474317}

\{0.0472136 + 0.141242\} ^ {1.11803 + 0.474317}

\{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} ^ {1.11803 + 0.474317}

\{0.0472136 + 0.141242\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} ^ {1.11803 + 0.474317}

\{0.023607 + 0.365902\} ^ {1.11803 + 0.474317}

\{0.023607 + 0.365902\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} ^ {1.11803 + 0.474317}

\{0.023607 + 0.365902\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} ^ {1.11803 + 0.474317}

\{0.023607 + 0.365902\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} ^ {1.11803 + 0.474317}

\{0.023607 + 0.365902\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} ^ {1.11803 + 0.474317}

g[z] = Chop[Simplify[%]]

Output

\{0.223607 + 0.385902\} ^ {1.11803 + 0.474317}

\{0.223607 + 0.385902\} \{-0.223607 + 0.353105\} ^ {1.11803 + 0.474317}

\{0.223607 + 0.385902\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} ^ {1.11803 + 0.474317}

\{0.223607 + 0.385902\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} ^ {1.11803 + 0.474317}

\{0.223607 + 0.385902\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} \{-1.09443 + 0.403696\} \{-0.223607 + 0.353105\} ^ {1.11803 + 0.474317}
References


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