FORM AND MOTION OF COMETARY TAILS

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DECLARATION

This work has been carried out under the supervision of Professor A. J. Meadows. I certify that the work has not been accepted in substance for any degree, and is not being concurrently submitted in candidature for any degree.

September 1976
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My thanks are due to the Polytechnic of Wales for leave of absence, and for the use of computing and other facilities.

I am grateful also to Dr R. L. Waterfield for kindly receiving me at Woolston Observatory and for the loan of his plates of Comet Bennett.
This work deals with the geometrical problems inherent in the observation of cometary tails, and with attempts at explanation of their forms.

Following the introduction, a review of the geometrical aspects of tail observation is presented in Chapter 2. New formulae are developed for projection from photographic to orbital plane, and for foreshortening etc. Tail orientation is fully treated and expressions are derived for orientation error in this projection. Equations for an elliptic orbit are developed in addition to those for the usual assumption of a parabolic orbit. Perspective error resulting from bisection of apparent tail images is investigated, and appropriate formulae derived. A set of observations of Comet Bennett 1969i is analysed and the results compared with those of other workers.

The influence of the solar wind on Type I tails is reviewed in Chapter 3 with special reference to dynamical aberration: some results for Comet Bennett are presented. Oscillation of Type I tails is then discussed. A comparison made between the cases of Comet Burnham 1960 II and Comet Halley 1835 III lends some support to the view that the oscillations are due to external influences on the tail. Formulae are developed for the progressive change in Type I tail orientation to be expected as the comet pursues its orbit.

Mechanical theories of Type II tails are reviewed in Chapter 4, special attention being given to tail orientation. The accurate calculation of syndyne and synchrone curves is treated, and the formulae are extended to allow of any value of \( \gamma \). Tail analysis by syndynes and synchrones is considered; results are presented for the time variation in the initial orientation of these curves. Implications for mixed tails and possible coupling between tails are discussed.

Finally, Chapter 5 briefly summarises the conclusions and gives suggestions for further work.
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CHAPTER 1

INTRODUCTION
1.1 Historical

From ancient times the advent of a bright comet has always attracted much attention. To early observers comets seemed to differ from other celestial objects in almost every respect. Stars and planets were seen as well-defined points of light, the Sun and Moon as luminous discs, whereas comets showed no sharp boundaries, being just diffuse areas of light. Attached to the brightest region of the comet might be a long and conspicuous tail, hazy, and diminishing in brightness away from the head. Such an appendage certainly served to put comets into a class of their own among celestial bodies.

In addition to their strange appearance, the arrival of comets and their apparent motion on the celestial sphere were quite unpredictable and displayed none of the regularity associated with the stars and planets. As a result, during the many centuries when astronomy and astrology were inseparable, comets were regarded as omens of impending events - usually calamitous in nature. There is no doubt that the tails of comets always held a special fascination: changes in their brightness, length and shape were closely watched. Fanciful interpretation of the shape then became an important factor in deciding upon the portent of a particular comet. One beneficial result of the attention given to comets by the ancients is that many were carefully recorded, and these early observations are proving to be of value in present day investigations, e.g. Kiang's work on Halley's comet (Kiang, 1973).

As to the nature of comets, the ancients could only indulge in speculation based on ignorance. Aristotle declared that comets were
'exhalations of the Earth', existing in a supposed region of fire above the Earth's surface. The familiar picture of comet tails extending upward from the horizon no doubt helped to form this impression. No progress toward understanding the nature or motion of comets was made until after the close of the middle ages. At this time, Aristotle's dictum still held sway, but in 1577 Tycho Brahe argued that comets were more distant than the Moon, since observation of a comet from widely separated positions showed no shift due to parallax. This demolished the idea of comets as atmospheric phenomena and struck a blow against Aristotelian dogma, according to which no changes could take place in the celestial regions. Apian (16th century) had observed that comet tails always pointed away from the Sun. Galileo and Kepler considered that comets would move in straight lines, while Tycho thought that they moved in circular paths. Though comets came under telescopic observation from early in the 17th century, no clue as to their nature was thereby forthcoming.

Genuine scientific investigation of comets began following Newton's work on gravitation, which made it possible to determine the orbit of a celestial body from several observations of its position. Halley developed convenient methods for computing comet orbits and advanced the epoch-making idea that the comets of 1531, 1607 and 1682 were really the same object, correctly predicting its return in 1758. Comets were shown to be (at least during the time of their observation) members of the solar system, and subject to the same laws that governed the motions of the planets. Thus the time-honoured notion of comets as being lawless and unpredictable was finally eradicated. The science of celestial mechanics developed
considerably during the 18th and 19th centuries, and the study of cometary orbits received significant attention, notably from Olbers. Short-period comets were discovered and their returns accurately predicted. Sufficient data were accumulated to allow the statistical study of orbital characteristics to begin, and as a consequence of the orbital studies, the problem of cometary origin arose. On the observational side, deliberate searching for new comets became an established activity: the names of Messier, Pons and later Barnard, Brooks and others became attached to many cometary discoveries.

Study of the physical nature of comets, as opposed to their motion, was initiated by Bessel in 1836. He put forward a theory of comet tails based on the assumption that they were composed of small particles subject to solar gravity and a repulsive force also obeying an inverse-square law (Sec.4.2). The theory was later extended by Bredikhin and used as a basis for classifying tails. Physical study of comets was of course greatly stimulated when the techniques of photography, spectroscopy and photometry began to be applied in the later part of the 19th century. Halley's comet was intensively studied by these methods at its 1910 apparition. By this time a reasonably clear picture of the nature of comets had emerged. They were thought of as an agglomeration of particles surrounded (when near the Sun) by a gaseous envelope. Spectroscopic examination had provided evidence on the composition of this envelope, and of the tail when present. Progress in cometary research declined during the period 1920 - 1940, when interest was focussed on stellar and galactic astronomy. With the revival of interest in solar system studies over the last 25 years however, cometary research has advanced rapidly.
As far as cometary tails are concerned, it is a little surprising how recently some of the factors controlling their behaviour have come to be appreciated. The dominant influence of the solar wind on gas tails became apparent only in the late 1950's; the adequacy of the radiation pressure theory for dust tails was seriously challenged only ten years ago, to be subsequently reinstated. We return to these matters in Chapters 3 and 4.
1.2 Physical nature and origin of comets

Though this work is concerned with cometary tails, these of course have their origin in the main body, or head, of the comet and are necessarily regarded as a component of the overall comet model.

The older 'classical' model of a comet as consisting of a loose agglomeration of meteoric particles has given way in recent years to a model in which virtually all the mass of the comet is concentrated in a solid nucleus. This was proposed originally by Whipple (1950, 1951) on the basis of the secular acceleration of Comet Encke. The nucleus is considered to be an 'icy conglomerate' containing various parent substances interspersed with meteoric material. On being heated by solar radiation, the ices sublimate and the released gas and meteoric particles go to form the coma and tail. The reaction of the nucleus to ejection of material gives rise to the small 'nongravitational' effects in the motion of the comet (Marsden and Sekanina, 1971,1973,1974). Nuclear radii and masses are difficult to determine. According to Roemer (1966), photographic evidence shows that many have radii of less than 1 km, while the great majority would be under 10 km; corresponding masses, indirectly estimated, are $\sim 10^{16} - 10^{20}$ gm. Structurally, the nucleus is presumed to be inhomogeneous and to have a brittle, layered nature. Then thermal stresses can cause fragmentation of the surface or even large scale splitting, consistent with cometary outbursts and break up into two or more parts. Though this model is now widely accepted, there is still minority support (principally by Lyttleton, 1972,1975) for the older 'sandbank' model: the observed nucleus is dismissed as an optical illusion, or put down to a condensation in
the swarm of particles. The situation is still not completely clear-cut, but perhaps the most convincing recent evidence for the existence of a solid nucleus is the regularity of the nongravitational effects in cometary motion. A most valuable future space probe experiment would be to search for a solid nucleus in a comet (Rh. Lüst, 1969).

At present, it is impossible to identify with certainty the parent molecules in the nucleus which dissociate to give the radicals observed in cometary spectra. H and OH are certainly overwhelmingly abundant in comets, but only very recently has H$_2$O actually been detected (the H$_2$O ion in Comet Kohoutek: Herzberg and Lew, 1974). Studies of the nongravitational effects also confirm that H$_2$O is the principal icy substance in the nucleus. Other molecules probably present are CH$_4$, NH$_3$, CO$_2$, C$_2$N$_2$ etc., but more complicated molecules may also occur and the relationship of the daughter molecules to the parents is not entirely clear. A difficulty of the icy conglomerate model, pointed out by Delsemme and Swings (1952), is that most of the assumed parent substances are so volatile that they would largely evaporate as the comet approached the Sun, and production of daughter molecules would be very small around perihelion, contrary to observation. The parent substances are thus presumed to be present in forms which are much less volatile, viz. in clathrates (solid hydrates) such as CH$_4$·6H$_2$O. The clathrates can take the form of icy grains (Delsemme and Wenger, 1970) which could be stripped from the nucleus by evaporating gases to form a halo around it. Then the source of the radicals would be co-extensive with the halo, and dissociation lifetimes would include the grain lifetimes. This is in accord with
observation, which indicates that the source of the radicals is not confined to the nucleus, but is spread over a region of radius $\sim 10^4$ km.

The cometary atmosphere or coma is composed of neutral molecules and dust particles, being roughly spherical in shape and centred on the nucleus. The light intensity varies approximately as $R^{-1}$ (R being the distance from the nucleus) and can be detected out to distances $\sim 10^5 - 10^6$ km from the nucleus. For a given comet, the size of the coma normally varies, from being quite small at large heliocentric distances to a maximum at $\sim 1.5 - 2$ a.u., and then decreasing again for closer approach to the Sun (this being known as 'contraction of the coma' - Sec.4.2). Cometary heads are not always featureless, but may display patterns known as jets, fans, halos, envelopes, etc. (Bobrovnikov, 1951). Measurement of the expansion velocity of the circular halos is usually cited as evidence for the radial expansion of the coma material at $\sim 0.5$ km per sec. The presence of fans, however, may imply non-isotropic emission of material, as emphasised by Wurm (1974). Spectroscopic investigation has revealed the presence of CH, CN, OH, NH, NH$_2$, C$_2$, C$_3$, identified by their resonance bands, excited by fluorescence interaction with the solar continuum radiation. The spectrum of a comet at large heliocentric distance is continuous; the continuum is redder than sunlight and consistent with scattering by micron-sized particles (Liller, 1960). As the comet approaches the Sun, the bands appear in a fairly definite order, beginning with CN at $r \sim 3$ a.u., but their relative intensity may vary greatly for different comets. Many new features in cometary spectra have recently been detected by infrared and ultraviolet observations. The latter
have confirmed the superabundance of OH as compared to CN (Code, Houck and Lillie, 1972). Radio observations have detected $C_2CN$ and $CH_3CN$ in Comet Kohoutek. There is naturally a wide variation in the total gas density in the coma: near the nucleus this is estimated as $\sim 10^2 - 10^3$ molecules per cm$^3$, compared to $\sim 10^2 - 10^3$ per cm$^3$ in the peripheral regions.

In larger comets, the coma merges into the third main component part, the tail. This we discuss in Sec. 1.3 and remainder of the present work. A fourth component part, possibly present in all comets, is a large surrounding envelope of hydrogen. Cometary observation made for the first time outside the Earth's atmosphere (from the Orbiting Astronomical Observatory) resulted in the discovery of these envelopes around Comet Bennett 1970 II and Comet Tago-Sato-Kosaka 1969 IX (Bertaux and Blamont, 1970). These ultraviolet observations, made on the Lyman-α line, show that hydrogen atoms are released in numbers many times larger than those of 'visible' molecules making up the cometary head as normally seen, and that the gas output rate of a comet must be far larger than that indicated by the molecules normally observed (Biermann, 1971).

General physical conditions in the head of the comet must of course have a profound influence on the tail. Sublimation of the parent substances, followed by dissociation and ionisation provide the gases in the coma and ion tails, while the dust released may form a dust tail. Thus the composition of the (presumed) nucleus is a most important factor; however, the form and behaviour of tails is also subject to external influences (Sec. 1.4). The Finson-Probststein (1968) theory links the behaviour of dust tails with conditions in the head of the comet (Sec. 4.4). From data on the tail, estimates
can be made of the gas and dust emission rates in the head. In other ways also information regarding the head and/or nucleus, difficult to obtain directly, can be got via studies of the tail. Some other implications may be mentioned. The nucleus is generally assumed to rotate, the period being $\sim 1^d$. There is recent supporting evidence for this from studies of the inner coma of Comet Bennett by Larson and Minton (1972). Nuclear rotation is taken by Finson and Probstein to be a factor tending to produce a uniform surface temperature and hence a spherically symmetric gas emission. The rotation might possibly be connected with oscillation of cometary tails (Sec. 3.5) and with changing internal structures in gas tails (Sec. 3.3). If clathrate grains are stripped from the nucleus, these themselves may form a tail (Sekanina, 1973). Finally, the mechanical theory of dust tails (Sec. 4.2) is formulated to take account of the release of particles within a 'sphere of action' which might be identified with the 'classical' comet model.

Much of the present day interest in comets is centred on the problem of their origin and evolution. There have been numerous theories of cometary origin, the difficulty being to devise a theory which does not directly contradict the meagre observational evidence. One of the two main theories postulates an origin within the solar system, and the other from interstellar dust clouds (subsequent to the formation of the solar system). There are variants of each theory (briefly reviewed by Marsden, 1974). The first theory is now usually associated with Oort, according to whom the Sun possesses a surrounding 'comet cloud', $\sim 10^{11}$ in number and at a distance of $\sim 50,000$ a.u. These comets are supposed to have been formed at about the distance of Jupiter or Saturn; stellar perturbations serve
to divert the occasional 'new' comet towards the inner part of the solar system. The original protagonist of the alternative theory was Lyttleton, according to whom interstellar material is accreted by the Sun to form comets on the 'sandbank' model. McCrea (1975) has proposed a theory in which icy conglomerate comet nuclei are formed from the interstellar icy silicate grains.
1.3 Cometary tails and their classification

Comets display tails as a result of the Sun's influence. For much the greater part of their lifetime, comets do not possess a tail; even in the neighbourhood of perihelion, many comets show little or no trace of tail structure, but a minority develop spectacular examples of that most characteristic feature of comets. Heat from the Sun is responsible for releasing from the nucleus the necessary material for tail formation; the behaviour of tails is then largely controlled by factors external to the comet. This relationship of the tail to its environment underlies the work presented in Chapters 3 and 4. Because of their extension on the sky, some geometrical problems are involved in the observation of tails; these are treated in Chapter 2.

In classical times, Pliny listed twelve types of comet tail, the categories being likened to swords, flaming torches, etc. There have been several modern scientific approaches to classification, and these are referred to below. Observation reveals that there are two basic kinds of tail, the difference between them being confirmed by spectroscopy.

Tails of the first kind are long ($\sim 10^7 - 10^9$ km), narrow ($\sim 10^5 - 10^6$ km), have only small curvature if any, and point almost directly away from the Sun. They normally appear only when the comet is within $\sim 2$ a.u. of the Sun, the most celebrated exception being Comet Humason 1961e which showed a detectable tail of this kind at over 5 a.u. These tails often show considerable internal structure: rays, filaments, streamers, etc., and bright knots or kinks which change noticeably on a time scale of $\sim 1^d$. Spectroscopic measurements show that they are
composed of ionised molecules with $\text{CO}^+$ predominant, followed by $\text{N}_2^+$ and others: $\text{CO}_2^+$, $\text{CH}^+$, $\text{OH}^+$. There are probably other species present whose emission bands lie outside the visible range. Typical ion densities are estimated at $10^2 - 10^3$ per cm$^3$. Tails of this kind are variously known as Type I, ion, gas or plasma tails.

The second distinctive kind of tail is generally shorter ($\sim 10^7$ km), broader and more strongly curved than the above kind. These tails rarely show any internal structure, and may point in a direction appreciably away from the extended radius vector from the Sun. Their spectrum is continuous - a reflected solar spectrum showing that they are composed of dust particles. Hence their designation as dust or Type II tails. Photometric studies indicate typical particle sizes of $\sim 1 \mu$m (Liller, 1960), a finding confirmed by investigation of the mechanics of the tails (Chapter 4). The separation between particles at a representative distance into the tail is $\sim 5$ m, corresponding to an average density of $10^{26}$ gm per cm$^3$.

A given comet may exhibit a tail of either type, or both simultaneously. 'New' comets are usually very dusty and have prominent Type II tails (e.g. Comet Bennett 1970 II, which also showed Type I features). There is a probable evolutionary connection between the tail type and the number of perihelion passages made, viz., a diminishing dust component. 'Pure' comets of either type are known, though the number of pure Type II comets is small. When both types have a significant presence, this is now referred to as a case of mixed tails. Correct assignment of tail type by inspection of the image on the photographic plate is not always easy, despite the dissimilarities mentioned above. Actual separation of the two types
on the plate in the mixed case is favoured by a high latitude of the Earth on the comet's orbital plane.

The terms Type I, Type II were introduced originally by Bredikhin in the 19th century along with a third category, Type III. This classification was based on analysis of a large number of tails by the mechanical theory (Sec. 4.2) and the average derived values of the parameter $1 - \mu$, the ratio of the repulsive force to the Sun's gravitational force (Sec. 1.4):

<table>
<thead>
<tr>
<th>Type</th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 - \mu$</td>
<td>18</td>
<td>0.5-2.2</td>
<td>0.0-0.3</td>
</tr>
</tbody>
</table>

Though the method is invalid for Type I tails (Sec. 3.2), the classification is still useful, Type III now being merged with Type II as there are no essential differences involved. S.V. Orlov (see Bobrovnikov, 1951) in 1942 subdivided Types I and II on the basis of concepts from the mechanical theory; this scheme can now be disregarded. The present day use of the two basic classes, qualified by the terms 'pure' or 'mixed' is adequate, as emphasised by Belton (1965).

Anomalous or anti-tails, i.e., tails apparently pointing toward the Sun have occasionally been seen. The best known instances are those of Comets Arend-Roland 1956h and Kohoutek 1973f. With the former, an extremely sharp spike extending toward the Sun was seen at the time of transit of the Earth through the comet's orbital plane. The phenomenon is easily explained as an effect of projection, the tail material concerned actually lying outside the
orbital path of the comet. This material is composed of relatively large particles emitted with very small velocities; the circumstances may be analysed by the methods explained in Chapter 4. Sekanina (1974) has listed criteria which must be satisfied for the appearance of an anti-tail. No new classification is required for these tails: they are accommodated by Type II.

In the following chapters, we are largely concerned with the overall or bulk tail (as opposed to constituent detail features). This large-scale approach necessarily involves orientations. As a preliminary, we consider the external factors which influence tail behaviour.
1.4 External influences on cometary tails

External influences, significant and otherwise, which could affect tails are the following: solar and planetary gravity, the gravitational attraction of the nucleus, solar radiation, an ambient medium (moving or static), magnetic fields.

There is no doubt of the significance of solar gravitation; the attraction of the nucleus itself is extremely small in comparison. It is readily calculated for example that the nuclear attraction on a particle at a representative distance of $10^6$ km from a nucleus of mass $10^{17}$ gm at a distance of 1 a.u. from the sun is about $10^{-12}$ times the solar attraction. On the form of cometary tails, therefore, nuclear gravitation has a negligible effect. It may be relevant, however, for particles emitted at very low speeds in connection with anomalous tails (Gary and O'Dell, 1974).

**Solar radiation**

The fact that comet tails almost invariably point away from the sun implies a repulsive force originating in that body. That light could exert a pressure was not experimentally confirmed until the end of the 19th century: following this, the role of radiation pressure as a major external influence was accepted.

The radiation pressure force is directed radially outward and varies as the inverse square of the solar distance $r$. Since the Sun's gravitational force also varies as $r^{-2}$, the resultant, or effective, force is also central and inverse-square. This effective force could be in the direction of the Sun (reduced
gravity), or in the opposite direction if the radiation pressure is the stronger of the two; in any event, the motion of a particle can be calculated by Keplerian orbit mechanics.

If \( f_g \) is the gravitational force on a particle and \( f_r \) is the radiation pressure force on the particle, then the effective force is

\[
f_{\text{eff}} = f_g - f_r
\]

and the quantity \( \mu \) is defined as

\[
\mu = \frac{f_{\text{eff}}}{f_g}.
\]

It is more usual to refer to the quantity

\[
1 - \mu = \frac{f_r}{f_g}
\]

which for the given particle is a constant, independent of \( r \).

The range of these quantities is thus:

<table>
<thead>
<tr>
<th>( 1 - \mu )</th>
<th>Nett attraction</th>
<th>Nett repulsion</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>No repulsive force</td>
<td>Nett force zero</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>+( \infty )</td>
</tr>
<tr>
<td>( \mu )</td>
<td>1</td>
<td>-( \infty )</td>
</tr>
</tbody>
</table>

Considering a particle of diameter \( d \), density \( \rho \), the radiation force can be expressed as

\[
f_r = \frac{\alpha \rho}{c} \left( \frac{E_\nu}{4\pi r^2} \right) \left( \frac{\pi d^2}{4} \right),
\]

* There is some looseness in definitions found in the literature.
where $E_s$ is the total solar radiation, $c$ the velocity of light and $Q_{pr}$ the scattering efficiency for radiation pressure. The value of $Q_{pr}$ depends on whether the scattering medium is absorbing or a dielectric (Kerker, 1969). In the former case, for relevant values of $d$ and wavelengths, $Q_{pr} \sim 1 - 2$; in the latter, $Q_{pr} \sim 0.5$. The corresponding gravitational force is

$$f_2 = \frac{g m_o (\pi d^3 P)}{\gamma^2}$$

where $g$ is the constant of gravitation and $m_o$ the mass of the Sun. Assuming that the nature of the scattering is independent of the particle size, we have

$$1 - \mu = \frac{f_2}{f_3} = \frac{c}{\rho d} \quad (2)$$

where

$$C = \frac{3 E_s}{8 \pi c g m_o} Q_{pr}$$

$$= (1.19 \cdot 10^4) Q_{pr}$$

on inserting standard numerical values.

The expression for $1 - \mu$ contains the product $\rho d$, so that a knowledge of its value cannot yield $\rho$ or $d$ independently. Also, the relative effect of radiation pressure increases as the particle size diminishes. Only when $d \lesssim 10 \mu$ does the effect become important. ($1 \mu = 10^{-4} \text{ cm}.$, not to be confused with $\mu$ in the above equations). Taking $Q_{pr} = 1$ and two values of $\rho$ corresponding to icy and metallic particles, typical values of $1 - \mu$ and $d$ are:
The dust particles normally found in Type II tails fall into this range of sizes, for which the nature of the scattering is substantially unchanged. With molecules however, the question must be considered afresh (Sec. 3.2).
CHAPTER 2

TAIL OBSERVATION: THE GEOMETRICAL PROBLEM
2.1. General

Comet tails are unusual among astronomical objects in that they often show very appreciable extension across the sky. Apparent tail lengths of a few degrees are common, while lengths of several tens of degrees are observed with bright comets and much greater extensions have been recorded. All tails are of course seen in projection against the 'plane of the sky', which is in effect transferred on to the photographic plate when exposures are made. The term 'photographic plane' is thus used synonymously with 'plane of the sky'.

Thus no comet tail is observed 'as it is'. The image on the photographic plate gives no impression of the relative distances of various parts of the tail, and it is obvious that perspective effects can greatly alter the form of the observed tail as compared with its actual form. The line of sight from the Earth may for instance almost coincide with the length of a tail, which will then appear very much shortened. Further, the curvature of a flat tail will be obscured if the Earth happens to lie in or near the plane of the tail. Despite these radical possible effects of observation, the purely geometrical factors involved are sometimes not fully appreciated, even in serious literature.

Simultaneous observation from widely separated positions would enable some progress to be made toward finding the true form of the tail. The Earth's surface however is too small for the purpose, and space probes have not yet been used to make this type of observation; there are, of course, obvious technical difficulties. Most comet tails are apt to change in form over short periods of time, so that
unfortunately, observation from one point at different times is not
equivalent to observation from different points at the same time.
It is clear therefore, that some fundamental assumption regarding
the actual form of the tail is required before observations can be
interpreted. We return to this in Sec. 2.3.

An associated geometrical problem is that of tail orientation:
it is important to be able to infer the 'true' orientation in the
orbital plane from the (observed) position angle of the tail, and
vice-versa.

Methods of determining the projected form and orientation of
tails are somewhat scattered in the literature; in this chapter we
present a systematic review of the subject, paying special attention
to definitions of the tail axis and to errors, and developing some
new formulae. A set of observations of Comet Bennett 1969i is then
analysed.
2.2 The geometrical background to observation

If perturbations are neglected, the orbits of the Earth and comets are conic sections with the Sun in one focus. The Earth's orbit is extremely well determined, and its position at any time is available from the standard almanacs, the disturbing effects of the other planets having been taken into account. For each new comet however, the orbit must be determined. Since the images are usually faint and diffuse, the precision attainable is not equal to that for planets. The long-period comets (usually the ones displaying prominent tails) have nearly-parabolic orbits, while the short-period comets (usually much fainter and having little or no tail) have elliptic orbits of moderate eccentricity. All comets, on discovery, are assumed to have parabolic orbits: this facilitates the computation of the orbit, and the resulting ephemeris is sufficiently accurate to enable the comet to be followed for a reasonable time. The assumption of a strictly parabolic orbit for a long-period comet is well justified in this work: errors in the context of tail geometry are negligible. Similarly, any departure from the osculating orbit due to perturbations and nongravitational effects may be ignored. From the orbital elements, the position of the comet head at a given time may be computed; this and the position of the Earth supply the essential information for a complete description of the Sun-Earth-comet configuration.

In a parabolic orbit, the perihelion distance $q$ determines the size of the orbit.

In an elliptic orbit, the semi-major axis $a$ determines the size of the orbit and the eccentricity $e$ its shape.
For both, the orientation of the orbital plane in space is specified by the angles $i$, the inclination of this plane to the plane of the ecliptic, and $\mathcal{N}$, the longitude of the ascending node of the orbit; the direction of the major axis within the orbital plane is given by the angle $\omega$.

The set of elements is completed in both cases by giving the time of perihelion passage, $T$.

The polar equation of a conic is

$$\frac{p}{r} = 1 + e \cos \nu$$

(3)

where $p$ is the semi-latus rectum $= 2q$ for a parabola ($e = 1$) and $\alpha (1-e^2)$ for an ellipse. $\nu$ is the true anomaly.

![Fig. 1](image)

Orbital coordinates may be set up as shown in Fig. 1. $x$ points to perihelion, $y$ to $\nu = 90^\circ$ and $z$ to the pole of the orbit such that $x,y,z$ form a right-handed system.

Then for a parabolic orbit we have

$$\begin{align*}
\mathbf{r} &= 2q \left/ (1 + \cos \nu) \right. = q \sec^2 \frac{\nu}{2} = q (1 + \beta^2) \\
\beta &= \tan \frac{\nu}{2} \\
\mathbf{r} &= \mathbf{v} \cos \nu \mathbf{e}_x + \mathbf{r} \sin \nu \mathbf{e}_y \\
&= q (1 - \beta^2) \mathbf{e}_x + 2q \beta \mathbf{e}_y
\end{align*}$$

(1a)
where \( \hat{x}, \hat{y} \) are the unit vectors in the \( x, y \) directions respectively.

\( \beta \) is determined by solving Barker's equation for the time of observation \( t \):

\[
\beta + \frac{1}{3} \beta^3 = \frac{k (t - T)}{\sqrt{2} \sigma y^{3/2}} \tag{5}
\]

where \( k \) is the Gaussian gravitational constant, the units being astronomical units and days.

For an elliptic orbit, the eccentric anomaly \( E \) is related to the other quantities by

\[
\begin{align*}
\tau \sin \nu &= b \sin E, \quad \tau \cos \nu = a (\cos E - e)
\end{align*}
\]

where \( b \) is the semi-minor axis. Then

\[
\tau = a (1 - e \cos E) \tag{6}
\]

Also,

\[
\tau = a (\cos E - e) \hat{x} + b \sin E \hat{y} \tag{6a}
\]

and

\[
\tan \frac{\nu}{2} = \beta = \left( \frac{1 + e}{1 - e} \right)^{1/2} \tan \frac{E}{2} \tag{6b}
\]

\( E \) is determined by solving Kepler's equation for the time of observation \( t \):

\[
M = E - e \sin E \tag{7}
\]

where the mean anomaly \( M = \eta (t - T), \quad \eta = k a^{-3/2} \) being the mean motion of the comet, usually expressed in degrees per day.

The orbital coordinates \( x, y, z \) \((z = 0)\) of a comet are thus readily determined. Other orthogonal systems of coordinates are useful for various purposes; these are as follows.
(i) **Ecliptic.** Origin at the Sun, \( x \) directed to the vernal equinox, \( y \) to longitude \( 90^\circ \) and \( z \) to the pole of the ecliptic.

(ii) **Equatorial.** Origin at the Sun, \( x \) directed to the vernal equinox, \( y \) in a plane parallel to that of the Earth's equator such that the angle between \( x \) and \( y \) is \( 90^\circ \) and \( z \) to the north celestial pole.

(iii) **Heliographic.** Origin at the Sun, \( x \) directed to the node of the solar equator, \( y \) to heliographic longitude \( 90^\circ \) and \( z \) to the solar north pole.

(iv) A second heliographic system in which \( x \) is directed to the intersection of the solar equator and a plane of constant longitude passing through the apex of the solar motion. This plane defines an alternative zero of heliographic longitude. \( y \) is directed to the new longitude \( 90^\circ \) and \( z \) to the solar north pole.

(v) **Cometocentric.** Origin at the comet nucleus, \( \xi \) directed along the prolonged radius vector, \( \eta \) in the comet's orbital plane at right angles to \( \xi \) and pointing in the direction from which the comet has come. \( \xi \) completes the right-handed set \( \xi, \eta, \zeta \) and is thus oppositely directed to the orbital coordinate \( z \).

The utility of (i) and (ii) is obvious; (iii) and (iv) are relevant in solar wind studies, the latter on account of possible symmetrical distortion of the interplanetary cavity about the chosen zero of longitude (Brandt, 1963).
The most convenient method of dealing with the rotations involved in changing from one set of coordinates to another is to use rotation matrices. The three fundamental matrices for positive rotations of a system of right-handed rectangular axes about the x, y and z directions are (Danby, 1962)

About x-axis: \[ P(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \]

About y-axis: \[ Q(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \]

About z-axis: \[ R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

(in each case, \( \theta \) is the angle of rotation)

For a combination of two or more rotations, the matrix describing the nett result can be found by matrix multiplication.

To transform from orbital to ecliptic coordinates, three rotations are easily envisaged, described successively by \( R(-\omega) \), \( P(-\iota) \) and \( R(-\lambda) \). Thus the new coordinates are given by

\[ R(-\lambda)P(-\iota)R(-\omega) \begin{bmatrix} x \\ y \\ z \end{bmatrix} \]

To obtain equatorial coordinates, a further rotation specified by \( P(-\varepsilon) \) is required; the final matrix is then

\[ P(-\varepsilon)R(-\lambda)P(-\iota)R(-\omega) \]
being the obliquity of the ecliptic.

Heliographic coordinates may be obtained from ecliptic coordinates by two rotations, giving the matrix

\[ P(\xi)R(\lambda) \quad (10) \]

where \( \lambda \) is the longitude of the node of the solar equator and \( \xi \) the inclination of the latter to the ecliptic. The alternative heliographic system mentioned above requires a further rotation, leading to

\[ R(\kappa)P(\xi)R(\lambda) \quad (11) \]

where the solar apex has heliographic longitude \( \kappa \) in the conventional system.

Conversion of orbital coordinates to cometocentric coordinates requires the two rotations giving the matrix

\[ P(\nu)R(\nu) \]

in which

\[ P(\nu) = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

and a translation corresponding to a shift of origin from the Sun to the comet. Thus

\[ \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = P(\nu)R(\nu) \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} - \begin{bmatrix} \nu \\ 0 \\ 0 \end{bmatrix} \quad (12) \]

(If we write \( x', y', z' \) for \( x, y, z \) respectively, and these latter quantities now refer to the comet, we have \( \sin \nu = y'/\tau, \cos \nu = x'/\tau \) and the equation implies that

\[ \nu \xi = \nu x' + \tau y' - \tau \zeta \]
\[ \nu \eta = \nu y' - x'y' \quad (13) \]

a relationship used in Sec. 4.2.)
The geocentric equatorial coordinates of the Sun are tabulated in the Astronomical Ephemeris. From these coordinates $X, Y, Z$, the heliocentric equatorial coordinates of the Earth are obtained by merely reversing the sign of each quantity. The orbital coordinates of the Earth (i.e., relative to a specified comet orbit) will then follow on multiplication by the matrix

$$ R(\omega)P(\iota)R(\varpi)P(\epsilon) $$

which is the inverse and thus the transpose of the matrix (9)

The further transformation (10) will yield the cometocentric coordinates of the Earth.

In order to compute the apparent position of the comet in the sky, the geocentric equatorial coordinates $\xi^E, \eta^E, \zeta^E$ are obtained from the corresponding heliocentric coordinates $x^h, y^h, z^h$ by a simple translation:

$$
\begin{align*}
\xi^E &= x^h + X = \Delta \cos \delta \cos \alpha \\
\eta^E &= y^h + Y = \Delta \cos \delta \sin \alpha \\
\zeta^E &= z^h + Z = \Delta \sin \delta
\end{align*}
$$

where $\Delta = \sqrt{\xi^E^2 + \eta^E^2 + \zeta^E^2} = \rho$ is the Earth-comet distance and $\alpha, \delta$ the R.A. and declination of the comet.

If we denote the ecliptic coordinates of the comet by $x_{Ec}, y_{Ec}, z_{Ec}$

heliographic $x_h, y_h, z_h$

alternative heliographic $x_{eh}, y_{eh}, z_{eh}$

orbital coordinates of the Earth by $x_{eo}, y_{eo}, z_{eo}$

several useful quantities may be derived as follows.
Ecliptic latitude and longitude:

\[ b_c = \tan^{-1} \frac{Z_{Ec}}{\sqrt{x_{Ec}^2 + y_{Ec}^2}} \quad \ell_c = \tan^{-1} \frac{y_{Ec}}{x_{Ec}} \]  

(16)

Heliographic latitude and longitude:

\[ b_h = \tan^{-1} \frac{Z_h}{\sqrt{x_h^2 + y_h^2}} \quad \ell_h = \tan^{-1} \frac{y_h}{x_h} \]  

(17)

Alternative heliographic latitude and longitude:

\[ b_{\delta n} = \tan^{-1} \frac{Z_{\delta n}}{\sqrt{x_{\delta n}^2 + y_{\delta n}^2}} \quad \ell_{\delta n} = \tan^{-1} \frac{y_{\delta n}}{x_{\delta n}} \]  

(18)

(\( = b_h \))

Cometocentric latitude of the Earth on the comet's orbital plane:

\[ \lambda_1 = \tan^{-1} \frac{Z_{Ec}}{\sqrt{\rho^2 - z_{Ec}^2}} \]  

(19)

'True anomaly' of the projection of the Earth's radius vector \( R \) on the comet's orbital plane:

\[ \nu_E = \tan^{-1} \frac{Z_{\delta n}}{x_{\delta n}} \]  

(20)

Heliocentric latitude of the Earth on the comet's orbital plane:

\[ \lambda_3 = \tan^{-1} \frac{Z_{\delta n}}{\sqrt{R^2 - Z_{\delta n}^2}} \]  

(21)

Phase angle (the angle subtended at the comet by the Earth and Sun):

\[ \alpha_p = \cos^{-1} \frac{\rho^2 + \gamma^2 - R^2}{2\gamma\rho} \]  

(22)

Also,

\[ \alpha = \nu_E - \nu \]  

(23)

Elongation of the comet:

\[ \alpha_E = \cos^{-1} \frac{\rho^2 + R^2 - \gamma^2}{2R\rho} \]  

(24)
Several of the geometrical quantities defined above are shown in Fig. 2. $\gamma$ and $\rho$ are basic in physical studies of the comet while the various angles give a full description of the existing geometrical configuration. The position of the Earth in relation to the comet's orbital plane is particularly important in studies of cometary tails; when the Earth is exactly in this plane, $z_{\infty} = \lambda = \lambda_{1} = 0$. Alternatively, this happens when the ecliptic longitude of the Earth equals the longitude of one of the nodes of the comet's orbit.
2.3 Geometry of the tail

Before any geometrical treatment of the tail can be given, a basic assumption concerning the location of the tail in space is necessary. (Sec. 2.1.) The simplest and most reasonable assumption is that the tail is located in the comet's orbital plane.

This assumption implies that (i) the tail is a plane structure, (ii) the plane of the tail coincides with the comet's orbital plane. It is of course recognised that (i) is not strictly true, but departures from the general plane of the tail are taken to be small compared with the extent of the tail in that plane. Physical considerations (Sec. 4.4) confirm that for Type II tails, emission velocities from the head are very small compared with the typical orbital velocity of the comet, so that the 'thickness' of the tail should be small in comparison with its length. Moreover, since solar radiation acts in the direction of the radius vector, implication (ii) above is to be expected. For Type I tails, the situation is not so clear; the dominating influence on these tails is the solar wind. (Sec. 3.2.) This might not be strictly radial in direction, and is unsteady. In consequence, neither (i) nor (ii) above can be accepted with the same confidence as in the case of Type II tails.

It is possible to test the basic assumption by observation. Twice in each year the Earth must pass through the orbital plane of any comet. At these times, the position angle of the tail should coincide with the position angle of the prolonged radius vector (or differ from it by 180°, this being a perspective effect); the tail, then seen 'side-on', should appear straight and its thickness small.
in comparison with its length. In practice, the visibility of the comet and the necessity of taking a photograph at precisely the right time make the evidence somewhat difficult to obtain. Over a long period of time however, the test has been applied to several comets, and in general the assumption is supported. Mammano and Wurm (1965) have studied Comet Daniel, 1907d (Type I tail) and conclude that the assumption is justified. The work presented in Sec. 2.8 supports it in the case of Comet Bennett 1969i. Belton (1965) has considered examples of Type II tails with a similar result, but queries the evidence from the unusual Comet Humason, 1961e, which displayed an apparent Type I tail at a large heliocentric distance. The validity of the assumption has also been questioned in the case of Comet Burnham 1959k (Malaise 1963). Interesting support for the assumption comes from the behaviour of anomalous tails. Photographs of Comet Arend Roland 1956h, taken when the Earth was in the comet's orbital plane on 1957 April 26 show the anomalous tail as a very thin 'spike' precisely in the sunward direction. This, and the changing aspect of the anomalous tail on either side of this date are consistent with the interpretation of the tail as a very thin sheet of material located in the orbital plane (Larsson-Leander, 1957, 1958).

Apart from a minority of Type I tails therefore, the available evidence supports the basic assumption. It is well to bear in mind however that this evidence is drawn from an extremely limited sample of the unlimited mass of desirable but unattainable observations.
Remembering that the appearance of a comet tail in the sky is completely dependent on the Earth-Sun-comet configuration existing at the time, a procedure is required for the derivation of its 'true' shape, i.e., its form in the orbital plane in consequence of the above assumption. This latter form is fundamental in the theoretical treatment of tails. The problem resolves itself into that of projection of a given point in the photographic plane on to the Comet's orbital plane. Formulae for this purpose were given by Bessel (1836), and alternative methods have since been proposed by several other authors*. The various schemes are listed and criticised by G. & A. Bernasconi (1966). All are dependent on rather involved spherical trigonometry, and successive authors have sought to reduce the labour entailed, revealed by the number of 'elementary operations' having to be carried out. This was an important consideration in the era of hand-computing, but is much less relevant at the present time, when automatic computing is so widely used.

The opposite process, i.e., the projection of a point in the orbital plane on to the photographic plane has recently been called for in connection with the Finson-Probstcin theory of dust tails (Sec. 4.4). The method described by Finson and Probstcin (1968), based on the use of cometocentric coordinates, is convenient for automatic computation; its inverse (derived below) would be equally suitable for projection from the photographic on to the orbital plane.

* It is worth noting that the formulae of Pape (1859), improved by A. Bernasconi (1965), differ from all others in assuming that the tail has the shape of a circular cone with its axis in the orbital plane.
The photographic plane is shown in Fig. 3 as a square normal to the vector \( \mathbf{\rho} \). It is taken that the telescope is directed to the comet nucleus; errors introduced are negligible if the telescope is directed to within a degree or so of the nucleus. The L-axis is directed toward the Earth while the M- and N-axes lie in the photographic plane. The direction of M is taken to be that of the projection of the positive \( \xi \)-axis onto the photographic plane and N completes the right-handed set \( L,M,N \).

If the Earth's cometocentric coordinates are \((\xi_e, \eta_e, \zeta_e)\) and \( \rho = \Delta = \sqrt{\xi_e^2 + \eta_e^2 + \zeta_e^2} \) is the Earth-comet distance, the transformation to the cometocentric system \( L,M,N \) is given by

\[
\begin{align*}
L &= \frac{\xi_e}{\Delta} \xi + \frac{\eta_e}{\Delta} \eta + \frac{\zeta_e}{\Delta} \zeta \\
M &= \frac{1}{\Delta \sqrt{\eta_e^2 + \zeta_e^2}} \xi - \frac{\xi_e \eta_e}{\Delta \sqrt{\eta_e^2 + \zeta_e^2}} \eta - \frac{\xi_e \zeta_e}{\Delta \sqrt{\eta_e^2 + \zeta_e^2}} \zeta \\
N &= \frac{\xi_e}{\Delta \sqrt{\eta_e^2 + \zeta_e^2}} \eta - \frac{\eta_e}{\Delta \sqrt{\eta_e^2 + \zeta_e^2}} \xi \quad (25)
\end{align*}
\]

A specified point in the tail (assumed in the orbital plane) will have coordinates \( \xi_T, \eta_T, 0 \) in the cometocentric system. The corresponding \( L,M,N \) coordinates are

\[
\begin{align*}
L_o &= \frac{\xi_e}{\Delta} \xi_T + \frac{\eta_e}{\Delta} \eta_T \\
M_o &= \frac{1}{\Delta \sqrt{\eta_e^2 + \zeta_e^2}} \xi_T - \frac{\xi_e \eta_e}{\Delta \sqrt{\eta_e^2 + \zeta_e^2}} \eta_T \\
N_o &= \frac{\xi_e}{\Delta \sqrt{\eta_e^2 + \zeta_e^2}} \eta_T \quad (26)
\end{align*}
\]
In the same system, the Earth's coordinates are $L_e = \Delta$, $M_e = N_e = 0$. The required projection of the given tail point on to the photographic plane is the intersection of the straight line joining $L_0$, $M_0$, $N_0$, and $L_e$, $M_e$, $N_e$, with the plane $L = 0$. The $M, N$ coordinates of the projected point are then found to be

\[
M_T = \frac{\Delta}{(\Delta^2 - \xi_e \xi_r - \eta_e \eta_r)} \left\{ \sqrt{\eta_e^2 + \xi_e^2} \xi_T - \frac{\xi_e \eta_e \eta_T}{\sqrt{\eta_e^2 + \xi_e^2}} \right\}
\]

\[
N_T = \frac{\Delta \xi_e \eta_T}{\sqrt{\eta_e^2 + \xi_e^2} (\Delta^2 - \xi_e \xi_r - \eta_e \eta_r)}
\]

When the orbital elements of the comet are known, $\xi_e, \eta_e, \xi_e$, and $\Delta$ may be computed for a given time by the methods described in Sec. 2.2. Formulae (27) then give (to within a suitable scale factor) the position on the photographic plate corresponding to an assumed tail point. (The direction of the $M$-axis on the plate may be fixed by a method described in Sec. 2.4.)

To obtain formulae giving $\xi_T, \eta_T$ when $M_T, N_T$ are known, we may rewrite (27) as a pair of linear equations in $\xi_T, \eta_T$.

We find that

\[
\begin{align*}
A \xi_T + B \eta_T &= P \\
C \xi_T + D \eta_T &= Q
\end{align*}
\]

where

\[
A = \xi_e M_T + \Delta \sqrt{\eta_e^2 + \xi_e^2} \\
B = \eta_e \left( M_T - \frac{\Delta \xi_e}{\sqrt{\eta_e^2 + \xi_e^2}} \right) \\
C = \xi_e \sqrt{\eta_e^2 + \xi_e^2} N_T \\
D = \eta_e \sqrt{\eta_e^2 + \xi_e^2} N_T + \Delta^2 \xi_e
\]
Then
\[
\begin{align*}
\xi_T &= \frac{PD - BQ}{AD - BC} \\
\eta_T &= \frac{AQ - CP}{AD - BC}
\end{align*}
\] (29)

The values of \(M_T, N_T\) corresponding to a particular point in a comet tail may be measured (to a suitable scale) on the photographic plate: formulae (29) then yield the 'true' position of the point in the comet's orbital plane. By applying the procedure for a number of points on the borders of the apparent tail, the 'true' form of the tail in the orbital plane may be delineated.

An apparent tail may be shorter than its actual length.

![Diagram](image)

Assuming that the tail lies in the comet's orbital plane, and is close to the prolonged radius vector, we have in Fig.

\[C'T = CT \sin \alpha_F\]

The foreshortening factor \(F \sim \sin \alpha_F\).

If the phase angle \(\alpha_F < 90^\circ\), the tail is directed away from the Earth; if \(> 90^\circ\), toward the Earth.
If the tail deviates appreciably from the prolonged radius vector, particularly if the true tail length is not small in comparison with the Earth-Comet distance $\rho$, an improved procedure is as follows.
Given that the cometocentric coordinates of the tail point $T$ (Fig. 5) are $\xi_T, \eta_T$, we have $\tan \epsilon = \eta_T / \xi_T$.

Then from the triangle $SPC$,

$$SP = \frac{r \sin \epsilon}{\sin(\lambda_1 + \epsilon)}; \quad CP = \frac{r \sin \lambda_1}{\sin(\lambda_1 + \epsilon)}$$

From the triangle $SEP$,

$$PE^2 = R^2 + SP^2 - 2R \cdot SP \cos \lambda_1$$

From the triangle $EPC$,

$$\cos \alpha'_F = \frac{\rho^2 + CP^2 - PE^2}{2 \rho \cdot CP}$$

and the foreshortening factor $F \sim \sin \alpha'_F$.

Other quantities of interest are $d_0$ and $d_\phi$, the distances of the tail point from the Sun and Earth respectively. For the former we have immediately,

$$d_0^2 = r^2 + CT^2 + 2r \cdot CT \cos \epsilon$$

or

$$d_0^2 = r^2 + \xi_T^2 + \eta_T^2 + 2r \xi_T$$

Also, from the triangle $ECT$,

$$d_\phi^2 = \rho^2 + CT^2 - 2 \rho \cdot CT \cos ECT$$

or

$$d_\phi^2 = \rho^2 + \xi_T^2 + \eta_T^2 + 2 \rho \sqrt{\xi_T^2 + \eta_T^2} \cos \alpha'_F$$

Fig. 5 helps to explain the 'tail lengthening' and aperture broadening' effects referred to in Sec. 2.8. These effects depend on the latitude angle $\lambda_1$ and also on the angle $PCQ$ in the comet's orbital plane between the projected vector $\rho$ and the extension $CP$ of the tail (or the tail itself).
We have
\[ CQ = \sqrt{\rho^2 - z_i^2} \; ; \; \; SA = \sqrt{R^2 - z_i^2} \]
then
\[ \cos SCQ = \frac{r^2 + CQ^2 - SA^2}{2r \cdot CQ} \]
Thus the required angle PCQ = SCQ - \epsilon = b', say. We may define
a tail lengthening factor as \( \sin b' \),
(aperture broadening
2.4. Tail orientation

In the preceding section, the transformation of points in the tail between the orbital and photographic planes has been considered. Other than to make the basic assumption that the tail lies in the orbital plane and to deal with tail points not lying in the radius vector, the idea of tail direction has not yet been treated.

The great majority of comet tails have an apparent width which is much smaller than the apparent length; it is easy therefore to associate with these tails some intuitive idea of direction. Appreciable curvature in a tail makes it necessary to define this more precisely: the most natural definition of apparent tail direction is to take the direction of the tail as it emerges from the head of the comet. This is still, of course, a somewhat imprecise concept and is discussed more fully in Sec.2.7. For present purposes we assume that given a comet photograph, it is possible to mark on it a straight line through the head which defines the tail axis and hence the direction of the tail at the head. The position angle of this line is then a measure of the apparent orientation of the tail. If, in the orbital plane, the tail deviates from the radius vector, the apparent tail axis will not coincide with the projection of the former on the sky. Due to projection effects, the apparent angle between the two may differ widely from its 'true' (orbital plane) value. In recent years, increasing attention has been paid to cometary tail orientations and their significance in theories of tail behaviour. The relevant quantity in these studies is the angle $\epsilon$ in the orbital plane between the prolonged radius vector and the projected tail axis of the comet.
The deduction of this angle from measurements made in the photographic plane is thus a matter of importance.

Projection on to the photographic plane of the prolonged radius vector has been mentioned in Sec. 2.3 as the datum for fixing the L,M axes in this plane. This could be done by using the known $(\alpha, \xi)$ positions of Sun and comet with spherical trigonometry; however, a method suggested by Osterbrock (1958) is more convenient in that the projection of the negative velocity vector of the comet can also be readily obtained.

If $\mathbf{t}$ is a vector identical in direction with the tail axis as defined above, the situation in the orbital plane of the comet is illustrated in Fig. 6 (a) and the corresponding picture on the plane of the sky in Fig. 6 (b).
Here, $\mathbf{V}$ denotes the velocity vector of the comet, tangential to the orbit and making an angle $\gamma$ with the prolonged radius vector. $x$, $\xi$ and $-V$ in (b) are of course the projections of the corresponding quantities in (a).

If $i, j, k$ are unit vectors along the axes in standard geocentric equatorial coordinates and $\mathbf{P}$ the Earth-comet vector,

$$\mathbf{P} = P \cos \alpha \cos \delta i + P \sin \alpha \cos \delta j + P \sin \delta k$$

(cf. equations 15). For a variation in $\mathbf{P}$,

$$d\mathbf{P} = dP_i i + dP_j j + dP_k k$$

where $dP_i, dP_j, dP_k$ are the components as in equation (30). Now $dP_i, dP_j, dP_k$ are readily expressed in terms of $d\rho, d\alpha, d\delta$. Then if the variation $d\rho$ is specified, equation (31) leads to the relations

$$\begin{align*}
\rho \cos \delta \, d\alpha &= -\sin \alpha \, dP_i + \cos \alpha \, dP_j \\
\rho \, d\delta &= -\cos \alpha \sin \delta \, dP_i - \sin \alpha \sin \delta \, dP_j + \cos \delta \, dP_k
\end{align*}$$

giving the corresponding variations in right ascension and declination.
Taking the variation $d\rho$ to be in the direction of the radius vector,

$$d\rho = C_1 \gamma$$

(33)

where the constant $C_1$ may be set equal to unity without affecting the result. For a particular time of observation, Barker's equation (5) is solved for the quantity $\beta$ (assuming a parabolic orbit for the comet). Equation (4a) then gives the components of $\gamma$ in the orbital system $i_x, i_y$, and the transformation (9) the components $d\rho_x, d\rho_y, d\rho_z$ in the system $i, j, k$. Equations (32) then enable the position angle $\phi$ to be determined from

$$\tan \phi = \frac{\cos \delta}{d\alpha}$$

(34)

The velocity vector $V$ is of course tangential to the orbit of the comet. Since the tail is normally found to lag the radius vector (i.e., it is situated on the side of the radius vector from which the comet has come), it is more convenient to work with the vector $-V$; this quantity in fact plays an important part in the sequel. To obtain the position angle $\Psi$ of its projection on the sky, the variation $d\rho$ is taken as a constant multiple of $-d\tau = -\dot{\gamma}$,

$$d\rho = -C_2 \dot{\gamma}$$

(35)

Equation (1a) gives

$$\dot{x} = 2\beta (-q\beta \dot{i}_x + q\dot{i}_y)$$

and, choosing $C_2 = (2\beta)^{-1}$,

$$d\rho = q \beta \dot{i}_x - q \dot{i}_y$$

As before, the transformation (9) then gives the components $d\rho_x, d\rho_y, d\rho_z$ in the system $i, j, k$ and equations (32) the position angle $\Psi$ from

$$\tan \Psi = \frac{\cos \delta}{d\alpha}$$

(36)
The polar equation of the parabolic orbit is

\[ r = q \sec^2 \frac{\nu}{2}, \]

\( \nu \) being the true anomaly. To find the angle \( \gamma \) between the velocity vector \( \mathbf{v} \) and the prolonged radius vector, we have

\[ \tan \gamma = r \frac{d\nu}{dr} \]

\[ = q \sec^2 \frac{\nu}{2} \left/ q \sec^2 \frac{\nu}{2} \tan \frac{\nu}{2} \right. \]

i.e.,

\[ \tan \gamma = \frac{1}{\tan \frac{\nu}{2}} = \frac{1}{\beta} \quad (37) \]

Then the radial and transverse components of the orbital velocity relative to the Sun are

\[ \begin{align*}
V_r &= V \cos \gamma \\
V_\tau &= V \sin \gamma
\end{align*} \quad (38) \]

We now examine in detail the geometrical situation in the orbital plane of the comet, and make the connection between the angle \( \epsilon \) specifying the tail orientation and the position angle \( \theta \).

Fig. 7
In Fig. 7, \( \xi \) is a unit vector in the direction of the prolonged radius vector, \( \eta \) a unit vector in the direction of the negative velocity vector and \( \hat{t} \) a unit vector along the tail axis. The angle between \( \xi \) and \( \eta \) is \( \tan(180^\circ - \gamma) = -1/\beta \).

Then

\[
\hat{t} = c_3 \xi + c_1 k \eta \]

where the ratio of the components along the radial and negative orbital directions is \( c_1 k / c_3 = k \).

Vectors in the directions of \( \xi \), \( \eta \) are \( C_j \hat{r} \) and \( -C_j \hat{z} \)

Thus

\[
\hat{t} = c_3 \xi - c_1 \eta \]

i.e.,

\[
c_3 \xi = c_4 c_2 |r| \hat{r} ; \quad c_3 \eta = c_4 A c_2 |r| \hat{z}
\]

and

\[
k = \frac{A c_1 |x|}{c_2 |z|} = \frac{A c_1}{c_2} \sqrt{\frac{c^2(1 - \beta^2)}{4c^2 \beta^2} + \frac{4\gamma^2 \beta^2}{4c^2 \beta^2}} = \frac{A c_1}{c_2} \frac{1 - \beta^2}{2\beta^2} \quad (39)
\]

With the previous choice of constants, \( C_i = 1 \), \( C_j = (2\hat{\beta})^{-1} \); this gives

\[
k = A \sqrt{1 - \beta^2} = A \sec \frac{\gamma}{2} \quad (40)
\]

The position angles \( \phi \) and \( \psi \) have each been obtained by division of the left-hand sides of equations (32). If the right-hand sides are set equal to \( c, d \) respectively in the case of \( \phi \) and \( a, b \) respectively in the case of \( \psi \), then since these expressions are linear in the components of a vector along the tail axis, the
observed position angle of the latter can be written as

$$\tan \theta = \frac{a + Ac}{b + Ad} \quad (4.1)$$

If $\theta$ is available from observation, then equations (4.0) and (4.1) give

$$h = \left\{ \frac{a - b \tan \theta}{d \tan \theta - c} \right\} \sqrt{1 + \beta^2}. \quad (4.2)$$

![Fig. 8](image)

To deduce the angle of lag $\epsilon$ in the orbital plane, Fig. 8 shows that

$$\tan \epsilon = \frac{TQ}{CP + PQ}$$

where

$$TQ = TP \sin \gamma; \quad PQ = -TP \cos \gamma.$$  

Hence

$$\tan \epsilon = \frac{\sin \gamma}{\frac{(CP)}{TP} - \cos \gamma}$$

or

$$\tan \epsilon = \frac{\sin \gamma}{h - \cos \gamma}, \quad (4.3)$$

where $h$ is obtained from equation (4.2). Rearranging, we have

$$h = \sin (\gamma + \epsilon) \csc \epsilon \quad (4.4)$$
The quantity \( h \) expresses the ratio of the components of the tail axis vector in the directions of the prolonged radius vector and the negative velocity vector, and has become known as Osterbrock's parameter (Belton and Brandt, 1966).

As long as the orbit of the comet is known, a measure of the position angle \( \theta \) can therefore lead to the corresponding value of the tail-lag angle \( \epsilon \).

In the investigation of Type II tails, the inverse process is required (Sec. 4.4). Inspection of the above equations shows that there is no difficulty in computing the position angle \( \theta \) given the orbital elements and a value of \( \epsilon \).

Assuming the tail axis vector to be of unit length, the components in the orbital system \((x, y)\) are evidently

\[
\begin{align*}
t_x &= \cos (\gamma - \epsilon), \\
t_y &= \sin (\gamma - \epsilon),
\end{align*}
\]

from which the components in the equatorial system can be obtained by applying transformation (9).

The above treatment has been almost entirely geometrical in nature. It is convenient to include here a note on the computation of velocities, which are of interest in physical considerations.

Equation (14a) yields

\[
\begin{align*}
\dot{V}_x &= -2\beta V \dot{\beta} \\
\dot{V}_\beta &= 2V \dot{\beta}
\end{align*}
\]
while differentiation of Barker's equation (5) gives

\[ \dot{\beta} = \frac{1}{1 + \beta^2} \frac{k}{\sqrt{2} a_{0}^{\nu}}. \]

Then

\[
\begin{align*}
\dot{\nu}_x &= -\frac{\beta}{\sqrt{\nu}} \frac{k}{1 + \beta^2}, \\
\dot{\nu}_y &= \frac{\beta}{\sqrt{\nu}} \frac{k}{1 + \beta^2}
\end{align*}
\]

\((46)\)

Again, the components of the velocity \(V\) in the equatorial system follow on application of transformation (9). Also,

\[ \nu^2 = \dot{\nu}_x^2 + \dot{\nu}_y^2 \]

\[ = \frac{2}{a_{0}^{\nu}} \frac{k^2}{1 + \beta^2} = \frac{2}{(a_{0}^{\nu})} \frac{k^2}{\alpha^2} \]

\((47)\)

If the direction of the projected radius vector is assumed known, we may derive an alternative formula for the orientation angle \(\alpha\) in terms of cometocentric quantities as follows.

Assuming that the tail lies in the orbital plane, we have (in the notation of Sec.2,3) \(\zeta_{\tau} = 0\), and the projected coordinates in the photographic plane of a tail point \(\xi_{\tau}, \eta_{\tau}\) are (equation 27)

\[ M_{\tau} = \frac{\Delta}{(a_{0}^{\nu})^{2} + \zeta_{\tau}^{2}} \left\{ \sqrt{\eta_{\tau}^2 + \xi_{\tau}^2} \xi_{\tau} - \frac{\xi_{\tau} \eta_{\tau}}{\sqrt{\eta_{\tau}^2 + \xi_{\tau}^2}} \eta_{\tau} \right\} \]

\[ N_{\tau} = \frac{\Delta^2 \xi_{\tau} \eta_{\tau}}{\sqrt{\eta_{\tau}^2 + \xi_{\tau}^2} (a_{0}^{\nu})^{2} \zeta_{\tau} - \xi_{\tau}^2 \eta_{\tau}^2} \]

If on the photographic plane, the tail axis vector makes an angle \(\alpha_{\tau}\) with the \(M\)-direction (i.e., the direction of the projected radius vector), then for points on the tail axis, \(N_{\tau} = M_{\tau} \tan \alpha_{\tau}\).

Division of the above equations then gives
\[
\cot \alpha_T = \frac{\gamma_\xi + \xi_\gamma (\frac{\xi_\xi}{\eta_T}) - \frac{\xi_\xi \eta_T}{\Delta \xi}}{\Delta \xi}.
\]

Now in the orbital plane, \( \eta_T = \xi_T \tan \epsilon \), so that
\[
\cot \epsilon = \frac{\Delta \xi \cot \alpha_T + \xi_\gamma \eta_T}{\eta_T^2 + \xi_\gamma^2} \quad (48)
\]

To obtain the projection of the negative velocity vector, we take
\( \epsilon_{-\nu} = 180^\circ - \gamma \). Then from (48),
\[
\cot \alpha_{-\nu} = \frac{1}{\Delta \xi} \left\{ (\eta_\xi^2 + \xi_\gamma^2) \cot (180^\circ - \gamma) - \xi_\gamma \eta_\xi \right\}
\]
or
\[
\cot \alpha_{-\nu} = -\frac{1}{\Delta \xi} \left\{ (\eta_\xi^2 + \xi_\gamma^2) \beta + \xi_\xi \eta_\xi \right\}, \quad (49)
\]
since \( \cot \gamma = \beta \).
2.5 Elliptic orbits

The treatment of tail orientation, etc. in Sec. 2.4 is based on the assumption of a parabolic comet orbit; the orbits of most comets displaying prominent tails are indeed parabolic, or nearly so. For elliptic orbits, the formulae are modified as follows.

We have
\[ \frac{p}{r} = 1 + e \cos \nu \]
where \( p = a(1-e^2) \).

Then
\[ \frac{d\tau}{d\nu} = \frac{pe \sin \nu}{(1 + e \cos \nu)^2} \]
and
\[ \tan \gamma = \frac{r d\nu}{d\tau} = \frac{1 + e \cos \nu}{e \sin \nu} \]
where
\[ \sin \nu = \frac{-2\beta}{1 + \beta^2}, \quad \cos \nu = \frac{1 - \beta^2}{1 + \beta^2}. \]

and \( \beta \) is given by equation (6b), having solved Kepler's equation for the particular time of observation.

Equation (6a) gives the components of \( r \) in the orbital system \( \hat{i}, \hat{j}, \hat{k} \) and the transformation (9) the components \( d\rho, d\rho, d\rho \) in the equatorial system \( \hat{i}, \hat{j}, \hat{k} \). The right-hand sides of equation (32) may then be evaluated (c, d respectively), and as before, the position angle \( \phi \) of the projected radius vector follows from
\[ \tan \phi = \frac{c}{d}. \]

For the projected negative velocity vector, we have from equation (6a)
\[ \dot{r} = \dot{E} (-a \sin E \hat{i} + b \cos E \hat{j}) \]
and choosing \( C_1 = (\dot{E})^{-1} \),
\[ d\dot{E} = a \sin E \hat{i} - b \cos E \hat{j}. \]
Transforming to determine $d\rho_1, d\rho_2, d\rho_3$ and evaluating the right hand sides of equation (32) $(a, b$ respectively) leads to

$$\tan \psi = \frac{a}{b}.$$ 

As before,

$$
A = \frac{A c_1 |r|}{c_1 |\hat{z}|} 
= \frac{A}{(E)^{1/2}} \frac{a (1 - e \cos E)}{\sqrt{E^2 (a^2 \sin^2 E + b^2 \cos^2 E)}} 
= A \sqrt{\frac{1 - e \cos E}{1 + e \cos E}} \tag{50}
$$

as $b^2 = a^2 (1 - \epsilon^2)$. Alternatively, using $\psi = a (1 - e \cos E)$, this becomes

$$A = A \sqrt{\frac{\psi}{2a - r}}.$$ 

Thus the modified version of equation (42) is

$$A = \left\{ \frac{a - b \tan \theta}{d \tan \theta - c} \right\} \sqrt{\frac{\psi}{2a - r}}, \tag{51}
$$

and the angle $\epsilon$ follows as before.

The velocity components in the orbital system are

$$\dot{x} = -a \sin E \dot{E} \quad ; \quad \dot{y} = b \cos E \dot{E}.$$ 

Differentiation of Kepler's equation $\n(t - T) = E - e \sin E$ yields

$$\dot{E} = \frac{\nu}{1 - e \cos E} = \frac{k a^{-3/2}}{1 - e \cos E}.$$
Then

\[ \begin{align*}
\dot{r}_x &= -\frac{k a^{2/3} \sin E}{1 - e \cos E} \\
\dot{r}_y &= \frac{k a^{2/3} (1 - e^2) \cos E}{1 - e \cos E},
\end{align*} \]

from which the equatorial components of the velocity may be determined.

Also,

\[ \begin{align*}
V^r &= \dot{r}_x^2 + \dot{r}_y^2 \\
&= \dot{E}^2 \left( a^2 \sin^2 E + b^2 \cos^2 E \right) \\
&= k^2 \left( \frac{2}{c} - \frac{1}{a} \right).
\end{align*} \]
2.6 Error in determining tail orientation

The basic data required in order to determine the angle \( \epsilon \) are the orbital elements of the comet, the solar coordinates and a measure of the position angle \( \theta \). The solar coordinates are tabulated in the standard almanacs to a high degree of precision and (unless the comet has only recently been discovered) the orbital elements will also be accurately known. Measurement of the angle \( \theta \) however is subject to various difficulties, which are discussed in Sec. 2.7. In consequence, \( \theta \) is known to a much lesser degree of precision than the other input quantities, and any errors in \( \epsilon \) are due almost entirely to errors in \( \theta \).

Osterbrock (1958) estimated the effect of errors in \( \theta \) by repeating the calculation of \( h \) for various assumed values of \( \theta \). (This author did not deal directly with the orientation angle \( \epsilon \).) It was found that on certain dates, small changes in the value of \( \theta \) caused comparatively large changes in \( h \) for the comets under investigation, making the computed values of \( h \) almost meaningless.

A better method of indicating the sensitivity of the computed \( \epsilon \) to the input value of \( \theta \) is to derive an expression for \( \frac{\partial \epsilon}{\partial \theta} \), and to compute this rate with every determination of \( \epsilon \).

We have (equation 4.2)

\[
h = \left( \frac{a - b \tan \theta}{d \tan \theta - c} \right) \sqrt{1 + \beta^2}
\]

where

\[
\tan \phi = \frac{c}{d}, \quad \tan \psi = \frac{a}{b}.
\]

Thus,

\[
h = \frac{b}{d} \frac{(\tan \psi - \tan \theta)}{(\tan \theta - \tan \phi)} \sqrt{1 + \beta^2},
\]
or
\[ \sin(\gamma + \epsilon) \csc \epsilon = \frac{b}{d} \frac{(\tan \gamma - \tan \theta)}{(\tan \theta - \tan \phi)} \sqrt{1 + \beta^2}. \]  

Differentiation yields
\[ \left\{ -\sin(\gamma + \epsilon) \csc \epsilon \cot \epsilon + \cos(\gamma + \epsilon) \csc \epsilon \right\}_{\theta} \frac{\partial}{\partial \theta} \]
\[ = \frac{b}{d} \sqrt{1 + \beta^2} - \sec^2 \theta \frac{(\tan \theta - \tan \phi)}{(\tan \theta - \tan \phi)^2} \]
and rearranging,
\[ \csc \epsilon \left\{ \cos(\gamma + \epsilon) - \sin(\gamma + \epsilon) \cot \epsilon \right\}_{\theta} \frac{\partial}{\partial \theta} = \frac{b}{d} \sqrt{1 + \beta^2} \frac{\sec^2 \theta (\tan \phi - \tan \psi)}{(\tan \theta - \tan \phi)^2} \]

Using equation (54) to eliminate \(\frac{b}{d} \sqrt{1 + \beta^2}\), this becomes
\[ \frac{\partial e}{\partial \theta} \left\{ \cos(\gamma + \epsilon) - \sin(\gamma + \epsilon) \cot \epsilon \right\} = \frac{\sin(\gamma + \epsilon)}{\sec^2 \theta (\tan \phi - \tan \psi)} \frac{\sec^2 \theta (\tan \phi - \tan \psi)}{(\tan \theta - \tan \phi)(\tan \phi - \tan \theta)} \]
or
\[ \frac{\partial e}{\partial \theta} = \frac{1}{\cot(\gamma + \epsilon) - \cot \epsilon} \frac{\sec^2 \theta (\tan \phi - \tan \psi)}{(\tan \theta - \tan \phi)(\tan \phi - \tan \theta)} \]  

which may be rewritten as
\[ \frac{\partial e}{\partial \theta} = \frac{\sec^2 \theta \sin^2 \epsilon (\cot(\gamma + \epsilon)(\tan \phi - \tan \psi))}{(\tan \theta - \tan \phi)(\tan \phi - \tan \theta)} \]  

Though it does not appear explicitly in this expression, the cometocentric latitude of the Earth \(\lambda\) (Sec. 2.2) clearly plays an important part in deciding the precision with which \(\epsilon\) can be determined. If \(\lambda\) is small, \(\epsilon\) will be very sensitive to small changes in \(\theta\). Moreover, errors (not taken into account in this section) due to extension of the tail outside the orbital plane would then also become significant. If the Earth lies in the comet's orbital plane, no estimate of \(\epsilon\) is possible, and in equation (56), \(\tan \phi = \tan \psi = \tan \theta\).
Expression (56) for \( \frac{2\pi}{2\eta} \) remains unaltered if the orbit is elliptical.

A comprehensive computer program (Appendix 1) has been prepared, based on the methods set out in the present chapter, which gives a detailed analysis of the geometrical and dynamical circumstances for a comet observation (parabolic or elliptic orbit), treating also the tail orientation, errors and dynamical aberration calculations (Sec. 3.4).
2.7 The tail axis

Comet tails vary enormously in appearance on the photographic plate. One tail may be thin, straight and well defined while another is broad, curved and diffuse. Notwithstanding these variations, some method of fixing the tail axis must be decided upon in order that a comet photograph may yield a quantitative measure of the tail orientation. In this section we consider critically various methods occurring in the literature.

The objective in determining tail axes should be to devise some operational method readily applicable to differing types of tail on photographs of varying quality; the procedure should fix the axis with as little uncertainty as possible and not be susceptible to errors of a subjective kind. These desiderata cannot be fully realised in practice. In some early work, e.g., Glancy (1909), the method of fixing the tail axis is not specified and even in some recent work, there is vagueness about what is actually done. Below are listed the methods employed by various authors:

(i) Osterbrock (1958), Maffei (1961) "...line...drawn...parallel to the axis of the tail...error...in judging just where the axis of the tail lies: it is not perfectly symmetric, and different people bisect it in different ways."

(ii) Belton and Brandt (1966) Tail axis "measured at the root of the tail...line bisecting the comet tail at the head....difficulty of judging the direction which bisects the tail because of the width, extent, curvature and amorphous nature of the tail images".
(iii) Wurm and Marmano (1964), Marmano and Wurm (1965). These authors consider the behaviour of individual rays in Type I tails. "The rays from both sides of the radius vector (sic) converge to the same position angle. To define the tail axis by this position angle suggests itself."

(iv) Malaise (1963). Again considering Type I tails, the axis is taken as the direction of the longest ray.

(v) Chincarini (1971). The axis is taken as the average direction of the two brightest rays.

(vi) Bernasconi and Pansecchi (1971). The method of these authors differs from all the above in that the apparent tail axis (on the photographic plate) is not determined. The apparent tail outline is projected on to the orbital plane.

![Diagram](image)

**Fig. 2**

On the orbital plane, a circular arc of arbitrary radius 0.05 a.u. is drawn with centre at the nucleus O to intersect the tail boundaries at A and B. The bisector OT of angle AOB is then taken as the tail axis in the orbital plane.
In addition to these, Finson and Probststein (1968) have referred to a tail axis defined as the locus of the apparent maxima in the light intensity over the tail. All the foregoing are methods of determining a tail axis from observation; we are not concerned here with 'theoretical' definitions which cannot be ascertained in this way, e.g. the locus of particles emitted from the nucleus with zero velocity, useful in the theory of Type II tails.

Throughout this work, tail direction is taken to refer to the direction at the head of the comet, and the tail axis is a certain straight line passing through the head of the comet. Methods (i) - (vi) above are consistent with this. Sometimes however, the term 'axis' is intended to refer to a line, generally curved, passing down the tail, as in the definitions of the preceding paragraph. To avoid confusion we shall use the term 'datum line' for the latter.

The definition (iii) of Wurm and Mammano is essentially a kinematic one: to realise it in practice requires a series of exposures separated by a few hours or less. Thus the method is inapplicable when only a single exposure is available. A real change in direction of the tail axis during the period of observation would confuse this method. The authors claim that in many cases the rays converge 'symmetrically' (i.e., from each side) to the axis. If this is so, the average direction of the rays on a single exposure would give the axis. It is also suggested that for unsymmetrical configurations of rays, the longest should be used to define the axis. This is identical with (iv). Chincarini's definition (v) uses the two brightest rays. These are often also the longest rays, but the resulting axis will not generally coincide with that from the
single longest ray or from the average of a symmetrical configuration. (The rays or streamers used here are normally practically straight, so that difficulties from curvature do not arise). Methods (iii), (iv) and (v) can, of course, be applied only to Type I tails exhibiting ray structure.

Methods (i), (ii) and (vi) are independent of internal structure in a tail: they depend essentially on the observed outline of the tail. (i) and (ii), which are conveniently treated together, have been applied in many cases without regard to tail type; (vi) is peculiar to the authors in question. The general procedure in (i) and (ii) may be obvious enough in an intuitive way: we shall try here to analyse it in some detail. Two procedures may be envisaged: these are illustrated in Fig. 10, (a) and (b). In (a), tangents are drawn to the outline at the points where the tail is judged to leave the head. Through the nucleus O, lines OA, OB are drawn parallel to the tangents. The tail axis is then the bisector of angle AOB. In (b), the datum line (shown dotted) bisecting the tail outline is drawn; the tangent to this line at the nucleus O then defines the tail axis. In practice, neither (a) nor (b) is consciously followed, the axis being fixed by eye. This must inevitably introduce a subjective element into the determination. It is doubtful whether any improvement could be obtained by actually carrying out the above constructions since the drawing of the tangents in (a) and the datum line in (b) are themselves liable to subjective error.
The tail boundaries, particularly the 'trailing' one, may be more or less diffuse. Also, the position of the nucleus must be estimated rather than observed in the great majority of cases.

(The outline depicted in Fig. 10, (a) and (b) is fairly typical of a large number of comets; it must be remembered however that a given image need not resemble this at all. A pure Type I tail might have a highly condensed head from which a single straight ray emerges.

At the other extreme would be a comet with a broad, diffuse, irregular
tail for which the axis would be very poorly defined. With images of the kind in Fig. 10, the two boundaries may be sensibly parallel for some distance away from the head, or may even converge to a point, as for some distant Type II tails). It will thus be clear that tail axis fixed by methods (i) and (ii) will nearly always be liable to subjective error, which may be much increased for some types of tail image. It is also true that in the literature, many more tail axes have been fixed by these methods than by any other.

From the position angle of the tail axis, as determined by any of the methods (i) - (v), the orientation angle $\epsilon$ may be deduced (Sec. 2.4). The underlying assumption is that the tail is flat, and lies in the orbital plane. In method (vi) of Bernasconi and Pansecchi, the same assumption is made but the 'true' form of the whole tail is obtained by projection on to the orbital plane before the procedure described above is applied. Evidently, the tail axis resulting from this method cannot be taken to be determined at the head of the comet: any curvature in the tail will increase the value of $\epsilon$, a most undesirable feature. It may be asked whether the 'bisection' method (i), (ii) applied to the 'true' form of the tail would be a superior method of determining the angle $\epsilon$. This would certainly be the case if the 'true' form of the tail could be quickly and accurately found. Unfortunately, on most comet photographs, the head region is overexposed, tending to spread the image. This and the probable diffuse apparent boundaries makes it difficult to select points, the projections of which are to give the 'true' form of the tail near the head. Also, the labour entailed in
measuring, transforming and re-plotting a sufficient number of points would be very considerable. Bisection of the apparent tail is, by contrast, a quick and convenient method, giving a single numerical quantity readily transformed into the angle $\epsilon$. It is, however, subject to a potential defect which we now discuss.

It was pointed out by A. Bernasconi (1969) that bisection of the apparent tail rather than the 'true' tail (assuming this to be flat and in the orbital plane) must introduce an error of perspective into the deduced value of the tail orientation angle $\epsilon$. The only exception would be if the plane defined by the Earth and the true tail axis (the plane EPT in Fig. 5) were normal to the comet's orbital plane, or approximately, for the comet with a small lag angle $\epsilon$, when $\nu = \nu_\epsilon$ or $\nu = \nu_\epsilon + 180^\circ$ in Fig. 2.

The same type of error would affect methods (iii) and (v) above; it is of course avoided in (vi). To eliminate this error, Bernasconi suggested that the two 'tail bounds' should be transformed to the orbital plane and the angle between them then bisected to give the true tail axis. The 'tail bounds' are not defined by Bernasconi.

We shall take the apparent angle between the tail bounds to be the angle AOB in Fig. 10 (a). This angle may also be used to define the apparent aperture of the tail. A rather extreme example is chosen by Bernasconi to demonstrate how large this perspective error may become.

For Comet Arend-Roland 1956h on 1957 April 22.9 (near the transit of the Earth through the comet's orbital plane) there is a large discrepancy between the values of $\epsilon$ given by the two methods. The example is not very useful as the two-dimensional model of the tail cannot in any case be used in these circumstances. It is of more
interest to consider the situation for typical positions of the Earth in relation to the comet as follows.

The computer program described in Appendix 1 is readily modified to compute the values of $\epsilon$ corresponding to two values $\theta_1$, $\theta_2$ of the position angle; if the observed position angle at a certain time of observation is $\theta$, we take

$$\begin{align*}
\theta_1 &= \theta + \delta \theta \Rightarrow \epsilon_1 \\
\theta_2 &= \theta - \delta \theta \Rightarrow \epsilon_2
\end{align*}$$

where $\delta \theta$ may be chosen to represent an assumed apparent semi-aperture of the tail. Then the orientation angle $\epsilon'$ is correctly given by

$$\epsilon' = \frac{1}{2} (\epsilon_1 + \epsilon_2)$$

and the true aperture of the tail is $|\epsilon_1 - \epsilon_2|$. The procedure more usually followed in bisecting an apparent tail is

$$\theta = \frac{1}{2} (\theta_1 + \theta_2) \Rightarrow \epsilon.$$

Using some observations of Comet Arend-Roland tabulated by Belton and Brandt (1966), the difference between $\epsilon'$ and $\epsilon$ was computed for an assumed apparent aperture of $10^\circ$. (Table 1.) $\lambda_1$ is the latitude of the Earth on the comet's orbital plane.

For the observations given (with the exception of the last), the difference $\epsilon' - \epsilon$ is small, and the effect unimportant compared with
the uncertainty in ε due to the difficulty of fixing the position angle θ. The last entry confirms that the effect can become significant for small values of λᵋ, but the model is then untenable as mentioned above.

As an alternative to finding ε' - ε via three transformations θ → ε, we may derive an expression for the difference as follows. For a fixed time of observation, ε may be regarded as a function of θ only:

ε = f(θ)

A change δθ in θ leads to a change δε in ε given by

δε = f'(θ) δθ + \frac{1}{2!} f''(θ) (δθ)^2 + \ldots

A similar trial using the observations of Comet Bennett given in Sec.2.8 gave the same result.
If $\delta \theta$ corresponds to the change from one tail bound to the other, 

$$\theta_2 = \theta_1 + \delta \theta$$

and

$$\epsilon_2 = \epsilon_1 + \delta \epsilon$$

$$= \epsilon_1 + f'(\theta) \delta \theta + \frac{1}{2!} f''(\theta) (\delta \theta)^2 + \ldots$$

Then

$$\epsilon' = \frac{1}{2} (\epsilon_1 + \epsilon_2)$$

$$= \epsilon_1 + \frac{1}{2} f'(\theta) \delta \theta + \frac{1}{4} f''(\theta) (\delta \theta)^2 + \ldots$$

(57)

For bisection of the apparent tail the step is $\frac{1}{2} (\theta_2 - \theta_1)$ or $\frac{1}{2} \delta \theta$

giving

$$\epsilon = \epsilon_1 + f'(\theta) (\frac{1}{2} \delta \theta) + \frac{1}{2!} f''(\theta) (\frac{1}{2} \delta \theta)^2 + \ldots$$

$$= \epsilon_1 + \frac{1}{2} f'(\theta) \delta \theta + \frac{1}{8} f''(\theta) (\delta \theta)^2 + \ldots$$

(58)

and the difference in orientation angle from the two procedures

follows from (57) and (58):

$$\epsilon' - \epsilon \sim \frac{1}{8} f''(\theta) (\delta \theta)^2.$$  

(59)

Equation (55) of Sec. 2.6 gives

$$f'(\theta) = \frac{1}{\cot (\gamma + \epsilon) - \cot \epsilon} \left( \frac{\sec^2 \theta \tan \phi - \tan \theta}{\tan \theta - \tan \psi} \right),$$

and differentiation yields

$$f''(\theta) = \frac{cosec^2 (\gamma + \epsilon) - cosec^2 \epsilon}{\cot (\gamma + \epsilon) - \cot \epsilon} \{ [f'(\theta)]^2 + F(\theta), G(\theta) \}$$

(60)

where

$$F(\theta) = \frac{\sec^2 \theta \tan \phi - \tan \psi}{(\tan \theta - \tan \psi)(\tan \phi - \tan \theta)},$$

$$G(\theta) = \frac{2 \tan \theta \tan \psi (\tan \phi - \tan \theta) - \sec^2 \theta (\tan \phi - 2 \tan \theta + \tan \psi)}{(\tan \theta - \tan \psi)(\tan \phi - \tan \theta)}$$
Somewhat simpler expressions for the derivatives may be obtained from the formula in terms of cometocentric quantities

\[ \cot \varepsilon = \frac{\Delta \xi_c \cot \alpha_r + \xi e \eta e}{\eta e^2 + \xi^2} \]

Here, \( \alpha_r \) is the angle between the tail axis vector and the projected radius vector (on the photographic plane). Evidently,

\[ \frac{\partial \varepsilon}{\partial \eta} = \frac{\partial \varepsilon}{\partial \alpha_r} \quad \text{and} \quad \frac{\partial^2 \varepsilon}{\partial \eta^2} = \frac{\partial^2 \varepsilon}{\partial \alpha_r^2} . \]

We find

\[ \frac{\partial^2 \varepsilon}{\partial \alpha_r^2} = \frac{\Delta \xi_c}{\eta e^2 + \xi^2} \left( \frac{\sin \varepsilon}{\sin \alpha_r} \right)^2 \quad (= f'(\theta)) \]

and

\[ \frac{\partial^2 \varepsilon}{\partial \alpha_r^2} = \frac{2 \Delta \xi_c}{\eta e^2 + \xi^2} \left( \frac{\sin \varepsilon}{\sin \alpha_r} \right) \left\{ \frac{\sin \alpha_r \cos \varepsilon}{\sin \alpha_r} \frac{\partial \xi_c}{\partial \alpha_r} - \frac{\sin \varepsilon \cos \alpha_r}{\sin \alpha_r} \right\} \quad (= f''(\theta)) \]

Computation of \( \varepsilon' - \varepsilon \) from equation (59), with \( f''(\theta) \) from (60) and (61) as a check, is included in the computer program given in Appendix 1. The values obtained agree well with those in Table 1, with exception of the last entry. Expression (59) is of course an approximation which can be invalidated by a rapidly varying \( f''(\theta) \) and insufficiently small \( \delta \theta \).
An analysis of some observations of Comet Bennett (1969i)

Comet Bennett (1969i = 1970II) was discovered as an 8th-magnitude object in Tucana on 1969 December 28 by J. C. Bennett (Pretoria). It reached perihelion on 1970 March 20 and during this month moved rapidly northward in the sky, reaching a maximum brilliancy of about zero total magnitude. One of the great comets of the century, Comet Bennett was most intensively observed and studied. From late March 1970 onward, it was well placed for observation from northern latitudes, remaining above magnitude 4.0 until early May. It was still being followed, at about magnitude 19, at the end of the year.

In common with many 'new' comets, Comet Bennett had a high dust content, spectroscopic observations showing the continuum to be strong in both the head and tail. Prominent Type I features were also visible over most of the period February-May, with considerable activity apparent in these features in April. A remarkably condensed knot was seen on April 5, as much as 8° from the comet's head and several photographs taken about this time show long Type I rays diverging widely from the main body of the tail. Around the time of perihelion passage, the Type II tail reached a length of some 25°, accompanied by a fainter but almost equally long Type I tail. The length of the latter fluctuated somewhat, being reported at about 10° on March 31 and increasing to more than 25° on April 5. The length of the dust tail remained more nearly constant during April, being estimated at 22° on April 30. As expected, it straightened and become narrower near the time of the Earth's transit through the comet's orbital plane on May 5. No anti-tail was detected.
### TABLE 2

**Comet Bennett 1969i. Elements (Epoch 1970 April 4.0 E.T.)**

1970 March 20.0 J.D. = 1970.0 E.T. = J.D. 2,10665.51586

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**May**

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| 12    | 30  | 4.01392     | 0710.51392 | 46 | 26.6 | 61 | 9.8 | 321.2 |
| 13    | 30  | 5.96137     | 0712.46137 | 55 | 20.2 | 61 | 55.9 | 322.5 |
| 3     | 20  | 22.93611    | 0729.43611 | 2 5 | 20.5 | 66 | 22.9 | 329.9 |
| 15    | 30  | 25.95556    | 0732.45556 | 16 | 19.2 | 66 | 54.9 | 328.7 |
| 16    | 30  | 27.96320    | 0734.46320 | 23 | 22.8 | 67 | 14.7 | 329.1 |

**June**

| 17    | 30  | 1.95760     | 0739.45760 | 40 | 8.6 | 67 | 59.9 | 325.9 |
| 18    | 40  | 2.97570     | 0740.47570 | 43 | 25.4 | 68 | 8.5 | 326.4 |
| 19    | 40  | 3.95560     | 0741.45560 | 46 | 32.2 | 68 | 16.7 | 336.9 |
Photographs of Comet Bennett were obtained by R. L. Waterfield and his collaborators at Woolston Observatory during the period 1970 April 1 - June 3. The plates were kindly made available to the author by Dr. Waterfield. They were taken with the 15 cm $f$4.5 Cooke triplet lens at the Observatory, the emulsion being Kodak 0a0.

The nineteen photographs comprised three distinct groups: ten (Group 1) were distributed over the period April 1-14, three (Group 2) between May 3-5 and six (Group 3) between May 22 - June 3. During this time the comet was moving steadily northward through Pegasus, Andromeda and Cassiopeia: the later photographs showed the comet against the rich star fields of the Milky Way. The tail was at its most prominent in the Group 1 photographs; it was still well defined in those of Group 2, but had diminished considerably in Group 3. It was possible to distinguish Type I and Type II features on several plates. Near the head of the comet however, the two types could not be distinguished, effectively implying the same tail orientation for each.

All nineteen plates were contact-printed on heavy grade paper, and subsequent measurements carried out on the prints. In preparation for these measurements, the position of the comet was computed for each time of observation (taken as that of mid-exposure), assuming the orbital elements given by Marsden (1970). The basic data are listed in Table 2.

With the position ($\alpha, \delta$) of the comet head known for each print, comparison could be made with star charts. The atlas used was the
Smithsonian Star Atlas of Reference Stars and Nonstellar Objects, (Cambridge, Mass., 1969) along with the accompanying catalogue. Identification of the field stars in the neighbourhood of the comet's head proved to be quite easy except in the case of print No.1, where they were faint due to the short exposure time. The positions of several such field stars were then taken from the catalogue and recorded for each photograph. Fig. 11 shows the comet on 1970 April 10 (print no.8).
Comet Bennett 1970 April 10.08 (Observation No. 8)

Fig. 11
Comet Bennett 1970 April 10.08 (Observation No.8)

Fig.11
In order to determine the position angle of the tail axis, the position angle of a line between two suitable reference stars must first be obtained. The stars selected for print No.8 are labelled (1) and (2) in the figure, with catalogue positions as follows:

(1) \( \alpha = 22^h 34' 30".251 \quad \delta = +39^\circ 58' 14".38 \)

(2) \( \alpha = 22^h 52' 53".216 \quad \delta = +39^\circ 40' 58".95 \)

The selection of reference stars used for the P.A. determination is influenced by a number of factors. They must be (i) close to the comet's head, (ii) sufficiently far apart to define accurately a straight line drawn between them, but (iii) not so far that the curvature of the coordinate system would appreciably influence the procedure described below; (iv) optimally, the line between them should make an angle of about 90° with the tail axis. Fortunately, these conditions were reasonably well satisfied with nearly all the observations. On some of the prints, the star images became short trails: the reference line, drawn through the mid-point of each trail, is then rather less well defined.

In the plane approximation, the P.A. of the reference line is given by

\[
\tan B = \frac{\cos \delta \, d\alpha}{d\delta}
\]

Fig. 12
Then in Fig. 12, the angle ACD between the reference line and the tail axis is measured. We then have
\[ \hat{\text{NDT}} = \hat{\text{CDB}} = \hat{\text{ACD}} - B \]
and the P.A. of the tail axis is \( \theta = 360^\circ - \hat{\text{NDT}} \). Different configurations are of course possible but the procedure is similar.

All necessary lines were drawn on a thin transparent film held securely in contact with the print. The film protects the print, and all lines etc. drawn are easily erased. The lines were prolonged on the film (larger than the prints), and angles measured directly with a large finely-graduated protractor.

The most difficult part of the whole procedure was the fixing of the apparent tail axis on each print. This problem has been discussed in general terms in Sec. 2.7. Here, the method adopted was that of bisection of the tail image where the tail is judged to leave the head of the comet. Several determinations of the P.A. were made on each print, the value of \( \theta \) in Table 3 being the mean of these. A mean deviation from the mean of about 1°.5 was noted for the Group 1 prints. (There is in fact a subjective element in fixing the tail axis: ideally, determinations by different persons should be made).

As mentioned above, the tail images on the Group 3 prints become much smaller and fainter, adding to the difficulty of fixing the tail axis.

The computer program described in Appendix 1 was used to calculate the various quantities defining the geometry of the situation. These are of course functions of the time of observation.
Input of the measured position angle $\theta$ enables the quantities $\epsilon, \frac{\partial \epsilon}{\partial \theta}$ to be computed also. Some of the geometrical quantities, together with $\epsilon$ and $\frac{\partial \epsilon}{\partial \theta}$ are listed in Table 3.

During the period (about 13 days) covered by the Group 1 observations, the latitude $\lambda$ of the Earth on the comet's orbital plane progressively diminished from 49° to 21°, with a corresponding increase in $\left| \frac{\partial \epsilon}{\partial \theta} \right|$. The greater the value of $|\lambda|$, the easier it becomes to distinguish between (differing) orientations of Type I and Type II tails. In the present case, no difference could be distinguished at the most favourable times, nor indeed at any of the others. The mean value of $\epsilon$ over the Group 1 observations is 9°; with the indicated values of $\frac{\partial \epsilon}{\partial \theta}$ and a mean deviation from the mean of about 1°.5 in the measurements of $\theta$, it is plain that the values of $\epsilon$ cannot be taken to indicate any definite trend. The first three observations of Group 3 show about the same average value of $\epsilon$, but the next two (Nos. 17 and 18) indicate a decrease in this quantity. Despite the improving geometrical circumstances (as implied by $\frac{\partial \epsilon}{\partial \theta}$), the latter observations must be treated with caution on account of the faint tail images. The last observation (No. 19) gives a seemingly discordant value of $\epsilon$, probably due to appreciable error in fixing the tail axis. All nineteen observations were made on the post-perihelion side of the orbit, the comet having passed through perihelion some 12 days before the first of the series. The values of $\epsilon$, and trends in relation to stage in the orbit, are discussed in Sec. 4.6.

Near the time of transit of the Earth through the orbital plane of the comet (May 5.16), the uncertainty in $\epsilon$ is so great that no
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values are entered for the Group 2 observations \( \frac{3e}{\sigma} \sim -25 \) on May 4. At this time also, the two-dimensional model of the tail is unreliable, as any extension of the tail outside the orbital plane will seriously affect the deductions made. The main interest in these observations is the 'side-on' view of the tail obtained near the transit time. Fig. 13 shows the comet on May 4.01, one day before the transit. As in other cases, the tail has straightened, and to within observational error, coincides with the radius vector.
Comet Bennett 1970 May 14,01 (Observation No.12)

Fig. 13
The fact that $\theta \sim \phi$ for the three Group 2 observations, and the thin, straight appearance of the tail supports the evidence quoted in Sec. 2.3. for a two-dimensional model. A tendency for the tail to increase in length is also apparent, being due to the greater depth of tail material (spread in the orbital plane) encountered by the line of sight. This effect is again referred to below.

The abrupt change in $\psi$ between observations 12 and 13 gave some cause for concern when first noted. These observations straddle the time of transit of the Earth through the comet's orbital plane. However, no such abrupt change had been noted in similar circumstances while testing the computer program with Comet Arend-Roland. A diagram of the geometrical configuration (Fig. 14) provides the explanation.

Fig. 14
The comet is at C, with true anomaly $92^0.14$ and the Earth at E with $\nu_e = 5^0.8$. It is seen that the phase angle $\alpha_p$ and the angle $\gamma$ between the comet's direction of motion and the radius vector differ by only $0^0.3$. At the transit therefore, the line of sight to the comet is nearly tangential to the comet's orbit. This causes the projected negative velocity vector $-V$ to swing rapidly through a large angle as the Earth passes through the orbital plane.

The perpendicular motion of the Earth through the plane (inclination $\iota$ of the comet orbit $\sim 90^0$) further assists the rapid change in the angle $\psi$. A computation of the various angles at a short time interval around the time of transit, given in Table confirms the explanation.

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<td>-0.0</td>
<td>43.4</td>
<td>43.8</td>
<td>323.0</td>
<td>326.9</td>
</tr>
<tr>
<td>5.4</td>
<td>5.8</td>
<td>92.6</td>
<td>0.1</td>
<td>43.2</td>
<td>43.7</td>
<td>323.1</td>
<td>341.1</td>
</tr>
<tr>
<td>5.6</td>
<td>5.8</td>
<td>92.8</td>
<td>0.2</td>
<td>43.1</td>
<td>43.6</td>
<td>323.1</td>
<td>350.3</td>
</tr>
</tbody>
</table>
We note that at the precise time of transit, $\psi = \phi = 323^\circ 0$ as expected; $\psi$ in fact changes by over $90^\circ$ in one day. When the Earth passed through the orbital plane of Comet Arend-Roland (1957 April 25) no such rapid change in $\psi$ occurred as the line of sight from the Earth made an angle $\sim 90^\circ$ with the comet's velocity vector.

Similar rapid changes to that discussed here could occur in the angles $\phi$ or $\theta$ if the line of sight from the Earth happens to lie close to the vectors $\tau$ or $\xi$ respectively during a transit of the Earth through the comet's orbital plane. These circumstances must however be rare.

Regarding the computed angle $\psi$, it is worth remarking that the position angle $\psi + 180^\circ$ does not give the position angle of the apparent motion of the comet on the sky. The latter is influenced by the motion of the Earth as well as the comet. We may illustrate this by reference to observation No.1 of Comet Bennett.

**TABLE 5**

<table>
<thead>
<tr>
<th>No.</th>
<th>DATE</th>
<th>R.A.</th>
<th>Dec</th>
<th>$\theta$</th>
<th>$\phi$</th>
<th>$\psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>April 1.1598</td>
<td>22</td>
<td>21</td>
<td>91</td>
<td>19</td>
<td>35.7</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>284.1</td>
</tr>
<tr>
<td>(ii)</td>
<td>1.2098</td>
<td>17.8</td>
<td>4.0</td>
<td></td>
<td></td>
<td>290.1</td>
</tr>
<tr>
<td>(iii)</td>
<td>1.2098</td>
<td>1.8</td>
<td>1.8</td>
<td></td>
<td></td>
<td>290.0</td>
</tr>
</tbody>
</table>

Row (i) of Table 5 contains entries identical with those of Tables 2,3 and Row (ii) gives values computed for a time $0^d.05$ later, while Row (iii) gives data for the same time, but using the solar coordinates for Row (i). Thus Row (ii) refers to the actual situation of a moving Earth and Row (iii) to a 'fixed Earth'. It is easily
calculated from the stated \((\alpha, \delta)\) positions that the actual apparent movement of the comet between (i) and (ii) is in position angle \(13^\circ.9\). With a fixed Earth, the movement between (i) and (iii) would be in position angle \(350^\circ.3\).

In Fig. 15 (a), the direction of motion of the comet is seen to be almost at right angles to the tail axis vector (a fact that is evident from the successive observations used above), but not opposite to the direction of \(-Y\). 'Fixing' the Earth, however, puts the direction of motion opposite to \(-Y\), as shown in (b).

If a comet is photographed against a region of the sky thinly populated by stars, it may be difficult to find convenient reference stars to establish a datum line from which to determine the position angle \(\theta\) of the tail (Fig. 12). It may also happen that stars in the neighbourhood of the comet's head are too faint to have been catalogued. Photographs of comets often show star trails: in some cases, these could be useful in the determination of \(\theta\). The trails are of course produced by the motion of the
comet against the sky background during the exposure time, the telescope having been guided to follow the comet. Under optimum conditions of sufficiently rapid motion of the comet, accurate guiding and the absence of peculiar effects due to refraction, the trails may be straight and well-defined. The orbital parameters of the comet being known, the position angle of its motion may be computed as above; an ideal trail near the comet head can then be assigned this position angle. (The correct direction along the trail is easily decided, knowing the approximate north direction.) The star trails in Fig. 16 are quite well-defined, and could be used to determine the position angle of this tail.

Comet Borely 1903c, 1903 Aug 1, Exposure 45 min. (30-inch Thompson refractor, Greenwich Observatory)
Bernasconi and Pansecchi (1971) have made a study of the tail of Comet Bennett based on six photographs obtained between 1970 March 28 and April 30. Their results are summarised in Table 6.

**TABLE 6**

<table>
<thead>
<tr>
<th>No.</th>
<th>DATE</th>
<th>$\lambda$</th>
<th>$\epsilon$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>March 28.155</td>
<td>60.7</td>
<td>25.5</td>
<td>19.0</td>
</tr>
<tr>
<td>2</td>
<td>30.166</td>
<td>51.9</td>
<td>27.5</td>
<td>21.5</td>
</tr>
<tr>
<td>3</td>
<td>April 1.121</td>
<td>41.5</td>
<td>24.0</td>
<td>23.0</td>
</tr>
<tr>
<td>4</td>
<td>11.142</td>
<td>26.4</td>
<td>24.5</td>
<td>32.0</td>
</tr>
<tr>
<td>5</td>
<td>11.120</td>
<td>21.4</td>
<td>22.5</td>
<td>36.5</td>
</tr>
<tr>
<td>6</td>
<td>30.099</td>
<td>3.6</td>
<td>32.5</td>
<td>134.0</td>
</tr>
</tbody>
</table>

These observations will be referred to as BP1, BP2, etc. BP1 and BP2 pre-date the observations used in the foregoing analysis, but BP3-5 fall within the period of the previous Group 1 observations. Further, there are three near-coincidences in time: Nos. BP3, BP4, BP5 with Nos. 3, 9, 10 respectively. Observation BP6, made 5 days before the transit of the Earth through the Comet's orbital plane, may be compared with the previous Group 2 observations.

The basic assumption made by Bernasconi and Pansecchi is the same as that made above, viz., that the tail is flat, and lies in the orbital plane. These authors' result for the orientation angle $\epsilon$ from observations BP1-5 is, however, some three times larger than the value of 9° deduced above. A comparison of the respective methods used is therefore necessary.
No measures of the position angle of the apparent tail were made by Bernasconi and Pansecchi. They selected a number of points (~10) on the apparent tail borders, and projected these on to the orbital plane. Joining the projected points then produces a picture of the 'true' form of the tail. The mode of definition of ε by these authors has been discussed in Sec. 2.7: curvature of the 'true' tail increases the value of ε on this definition. It is clear from the marked curvature of the tails in the orbital plane diagrams given that ε is substantially increased in consequence. An interesting comparison of results is possible on account of the close proximity in time of observations 3, BP3 and 9, BP4. (Observations 10 and BP5 cannot be compared as no diagrams are given by Bernasconi and Pansecchi for the latter).

**TABLE 7**

<table>
<thead>
<tr>
<th>No.</th>
<th>DATE</th>
<th>ε(1)</th>
<th>ε(2)</th>
<th>ε(3)</th>
<th>ε(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1970</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>April 4.160</td>
<td>9.8</td>
<td>11.1</td>
<td>8.0</td>
<td>24.0</td>
</tr>
<tr>
<td>BP3</td>
<td>4.121</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>11.134</td>
<td>10.8</td>
<td>18.3</td>
<td>17.0</td>
<td>24.5</td>
</tr>
<tr>
<td>BP4</td>
<td>11.142</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

ε(1) is the value of ε given in Table 3.

ε(2) is computed from an estimate of the position angle θ of the tail on Bernasconi and Pansecchi's diagram of the photographic plate.

ε(3) is the value of ε obtained by the bisection at the head of the 'true' form of the tail as projected by Bernasconi and Pansecchi.
$\epsilon(4)$ is the result of Bernasconi and Pansecchi's definition, as in Table 6.

The diagrams given by Bernasconi and Pansecchi must be treated with some caution; the reproduction of the photographic plates may not be accurate - the comet head and tail is represented by a linear outline, which can only approximate to the appearance on the plate. As mentioned previously, the projected outline of the comet in the head region must always be subject to considerable doubt; moreover, in the present case, very few points selected for projection lie near the head. Nevertheless, the increase in $\epsilon$ entailed by Bernasconi and Pansecchi's definition is evident in the difference between $\epsilon(3)$ and $\epsilon(4)$. Using the same reference stars as in the reduction of observations 3 and 9 (which could be identified on the plate diagrams given), the position angle of the tail for observations BP3 and BP4 could be estimated and the resulting $\epsilon(2)$ computed. Bearing in mind the always-present difficulty of fixing tail axes, and the effect of using a diagram instead of the plate (or print), the agreement between $\epsilon(1)$ and $\epsilon(2)$ is quite good for observations 3, BP3. The agreement is not so good for observations 9, BP4. Here, it was noticed that the north line, as drawn on the plate diagram of BP4, seemed to be $2^o$ in error. This could have an effect on the form of the projected tail, as the position angles of the projection points (presumably measured from the north line) are used in the method of reduction chosen by Bernasconi and Pansecchi. It thus appears that the values of $\epsilon$ given as results by Bernasconi and Pansecchi should be reduced appreciably for comparison with the results obtained here. On the basis of the
limited data available, it may be concluded that Bernasconi and Pansecchi's results imply a value of $\epsilon$ (measured at the head) a few degrees higher than that obtained here.

We may add a note regarding Bernasconi and Pansecchi's reference to the very large increase in the deduced true aperture of the tail in their last observation (BP6 in Table 6). This can hardly be real; it is ascribed to what the authors call 'optical foreshortening'. The cause is of course the failure of the two-dimensional model. As the latitude $\lambda_t$ decreases, the line of sight to the tail becomes more nearly parallel to the orbital plane, and will traverse greater depths of cometary material spread outside the plane. This affects the apparent aperture, and with a small value of $\lambda_t$ has a profound effect of the inferred 'true' aperture. For a given $\lambda_t$, the effect will be at its greatest when the line of sight falls at right angles to the tail, i.e. the factor $\sin b' = 1$. A slight lengthening of the tail is also observed, due to the line of sight encountering greater depths of material spread out in the orbital plane (or nearly so).
CHAPTER 3

TYPE I TAILS AND THE SOLAR WIND
3.1 General

The characteristic differences between Type I and Type II tails have been discussed in Sec.1.3. Spectroscopic evidence is conclusive in deciding tail type, but this can only be obtained from the brightest comets. It can be difficult to distinguish between the different kinds of tail from photographs alone (i.e., single photographs. Separate photographs on different emulsions can be most useful for distinguishing tails). On account of this, and a lack of awareness of the physical processes at work in ion tails, until comparatively recently the two types were seldom treated as essentially distinct.

Within the last twenty years, the existence and nature of the solar wind have been established, due in no small measure to the study of Type I comet tails. The details of the interaction of the solar wind with a comet are complex, and as yet only partly understood. The production of tail ions, the development of 'rays' and the acceleration of internal features are aspects requiring better explanation; from the 'gross tail' viewpoint however, the dynamical aberration hypothesis is well established.

Tail kinematics and the solar wind are first reviewed in this chapter; dynamical aberration is treated and some results given for Comet Bennett. We then consider the oscillation of cometary tails with reference to Comet Halley and Comet Burnham, and finally investigate the time trend in the aberration angle of a comet.
3.2 Repulsive forces and tail kinematics

Before there was any clear distinction between types of tail, Bessel (1836) formulated the mechanical theory. Extended by Bredikhin and others, this method of analysis was applied to several comets irrespective of differences in the nature of the tails. By the syndyne comparison method (see Sec.4.2), values of the parameter \( \lambda \) ranging from less than 1 to about 20 were found, this forming the basis of Bredikhin's system of tail classification. With the very slight curvature of Type I tails, the precision of the results was poor, but the high values of the repulsive force for these tails stood in marked contrast to values between 0 and 1 for what are now styled Type II tails. Values obtained for the emission velocities of the 'particles', \( \sim 10^7 \) \( \text{km per sec.} \) or more, were also remarkably high.

The fine structure displayed by Type I tails makes possible alternative methods of determining the repulsive forces at work. 'Clouds', 'condensations', 'knots', 'edges' or other features may be distinguished and followed over a period of time. The lifetime of these structures varies very widely, for the same comet and from one comet to another. For very active tails there can be noticeable motion of a feature in a few hours, and the whole appearance of the tail can be entirely different on consecutive nights. (e.g. Comet 1943I); features in less active tails have been followed over periods of several days. The instrumental problems involved in the case of active tails are considerable. Fast Schmidt cameras are required to reduce exposure times to a minimum so as to reduce the blurring of features which are not very sharp at best.

An obvious procedure, used by Bredikhin and other earlier workers, is to measure the actual positions \( (\alpha, \delta) \) of a tail structure at three
or more separated times and to use conventional methods of orbit
determination to find the orbital elements appropriate to the
structure. The problem is simplified by assuming that the structure
moves in the comet's orbital plane, thus fixing the quantities i and
Ω. The equation of motion of the structure is taken as

$$\ddot{x} + \frac{\mu x}{r^3} = 0,$$

with similar equations in

$$\ddot{y}, \ddot{z},$$

involving the quantity $\mu$ (Sec. 4.2). Since the basic observations
must be reasonably well separated, the method is practicable only when
a feature can be followed over a period of days; even then the
inevitable diffuseness and consequent lack of precision in the
measured $(\alpha, \delta)$ render the results uncertain. Numerical values of the
remaining orbital elements and $1-\mu$ are obtained. Values of the
eccentricity $e$ slightly exceeding 1 show that the features have
hyperbolic orbits; the parameter $1-\mu$ is poorly determined but the
values obtained (summarised by Kopff, 1929) ranged up to an order of
magnitude greater than the largest found by the syndyne -comparison
method. A few typical values are:

<table>
<thead>
<tr>
<th>Comet</th>
<th>$1-\mu$</th>
<th>Investigator</th>
</tr>
</thead>
<tbody>
<tr>
<td>1910 II (Halley)</td>
<td>194</td>
<td>A. Orlov</td>
</tr>
<tr>
<td></td>
<td>70</td>
<td>S. V. Orlov</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>Gondolatsch</td>
</tr>
</tbody>
</table>

The lack of precision in $1-\mu$ is emphasised by the fact that all three
investigators made their measurements on the same tail structure.

Improved instrumentation has in recent years yielded much better
photographs of the fine detail in many Type I tails. The movement of
small-scale features over short time intervals is often possible, and
led to intensive study of the kinematics of the tails. Hoffmeister (1943) studied Comet 1943 I (Whipple-Fedtke-Tevzadze) and was able to trace the progress of as many as 17 different tail features on plates taken during one night. An individual feature was often observed on four or five plates exposed at intervals during the night. Instead of deriving orbital elements, the basic method is to compute the position of the structure (in the comet's orbital plane) relative to the comet nucleus. A series of positions and corresponding times then being known, mean velocities between the points are readily calculated. On the assumption that it is constant over the range involved, the acceleration between neighbouring pairs of points will follow. The general motion is practically in the continuation of the radius vector, as Type I tails normally make an angle of at most a few degrees with this. Hoffmeister was however the first to show with certainty that some features had moved across the general tail direction, a development inexplicable on the radiation pressure hypothesis.

The connection between the measured accelerations and the parameter \( 1 - \mu \) requires some care: inaccurate statements on this occur quite frequently in the literature. If \( f_r \) is the repulsive force (in the radius vector) on a molecule of mass \( m \), the acceleration is

\[
\alpha = \frac{f_r}{m}.
\]

But \( 1 - \mu = f_r/f_y \), where \( f_y \) is the Sun's gravitational attraction on the molecule, \( Gm_o m/r^2 \), \( m_o \) being the Sun's mass. Hence

\[
\alpha = (1 - \mu) \frac{Gm_o}{r^2} = (1 - \mu) \frac{Gm_o}{(\gamma_{au} \times 1.5 \times 10^{12})^2} \text{ cm per sec}^2
\]

(61)
Now the acceleration due to solar gravity at a distance of 1 a.u. is

\[ a_0 = \frac{G m_o}{(1.5 \times 10^{13})^2} \]

so that

\[ 1 - \mu = \left( \frac{a}{a_0} \right) r_{au}^2 \]  

(62)

Inserting standard numerical values, \( a_0 = 0.594 \text{ cm per sec}^2 \), and

\[ 1 - \mu \sim (1.7 r_{au}^2) a \]  

(63)

As Type I tails are normally observed around \( r_{au} \sim 1 \), \( 1 - \mu \) is of the same order as the acceleration in cm per sec\(^2\) (but not equal numerically to \( a \) except at a special value of \( r \)). In the case of radiation pressure, \( f_r \) varies as \( r^{-2} \) and \( 1 - \mu \) is constant; the acceleration would then also vary as \( r^{-2} \). The results, however, do not confirm any such variation.

Velocities derived by Hoffmeister and others are typically in the range 50-150 km per sec, and occasionally as high as 200 km per sec. The derived values of \( 1 - \mu \) show considerable variation, but confirm and extend upward the values obtained by the orbital method. For Comet 1943I, Hoffmeister found a mean acceleration of 55 cm per sec\(^2\) corresponding to \( 1 - \mu \sim 194 \) (mean radius vector 1.44 a.u.).

In addition to the generally high values of \( 1 - \mu \), the variations revealed by the kinematic studies are of the greatest interest and significance. With several different structures in a tail under observation simultaneously, the respective accelerations can vary widely. Also, the same structure can possess different accelerations at different times. It is found that the accelerations and velocities are markedly greater in the outer parts of the tail than in its central region. If tail rays are present, these display the greatest
accelerations, reaching $10^3$ cm per sec$^2$. The outer streamers generally show the greatest values of all. Stumpff (1959) found for Comet 1908 III (Morehouse) $l-\mu$-values of 20-50 in the central region of the tail, but values of $10^3$ and higher in streamers on both sides of the tail. Similar results were obtained for Comet 1957d (Mrkos) by Rh. Lüst (1962), $l-\mu$-30-70 in the central region and $10^2-10^3$ in the side-streamers. A peculiar 'edge' structure was visible in the latter tail during August 23-25 1957: for this, Lüst found $l-\mu$-12. As well as moving down the tail, this structure was remarkable in that it appeared to rotate about the main tail. A spiral motion of this kind would hardly be produced by radiation pressure force.

On account of the more or less diffuse nature of the structures observed in the kinematic method, the numerical results cannot have a high degree of accuracy. Projection errors will also contribute to loss of accuracy: these errors are minimised when the Earth has a high latitude relative to the comet's orbital plane. Emission velocities for the Type I tails are higher than for the dust particles in Type II tails; it is therefore likely that deviations from the orbital plane for Type I tails will be greater, leading to more serious projection error. In active Type I tails also, the turbulent motions observed would hardly be confined to the orbital plane. Caution is thus needed in interpreting the observations when the geometrical circumstances are unfavourable. It is also possible that one part of the tail may be seen in projection against another, e.g. side-streamers against the main body of the tail, and an observed motion attributed to the wrong component of the tail. This however is unlikely by reason of the relative brightnesses involved.
A potentially more serious criticism of the kinematic method is that the observed motions may not be 'bulk motions', but some kind of wave motion through the tail. Bulk motion must of course exist in order to disperse the gas/dust output of the comet head, but the passage of a wave might be superimposed on this.
Inadequacy of radiation pressure for Type I tails

For the case of the molecules constituting these tails, the expression for $1-\mu$ derived for dust particles in Sec. 1.4 must be re-formulated. It is now inappropriate to use integrated solar radiation: discrete resonance transitions of the molecules are considered.

The radiation pressure is the product of the photon momentum per unit area per second and the effective area of the molecule:

$$f_r = \frac{\pi F_r}{c} \frac{\pi e^2}{m c} f$$

where $e$ is the electronic charge, $m$ the molecular mass and $f$ the oscillator strength ($f$-value) of the appropriate resonance transition. The flux at the comet is $\pi B_\nu (R_\odot / \tau )^2 = F_\nu$, where $R_\odot$ is the solar radius and $B_\nu$ is given by Planck's law, written with the usual notation as

$$B_\nu = \frac{2 \hbar c}{\lambda^3} \frac{1}{e^{\hbar c/\lambda kT} - 1}$$

Also,

$$f_\nu = \frac{g m o m}{\nu^2}$$

so that

$$1-\mu = \frac{f_r}{f_\nu} = \frac{2 \pi^2 \hbar c^2 R_\odot^2}{c^3 g m o m m_e \lambda^3} \frac{f}{e^{\hbar c/\lambda kT} - 1}$$

Inserting numerical values, Wurm (1943) found for the CO$^+$ molecule in the photographic range,

$$1-\mu \sim 47f.$$ 

At the time, the $f$-value was unknown, but thought to be of the order of unity, in which case $1-\mu$ would be of approximately the required order, though somewhat on the small side. More recent determinations, however, put the value of $f$ at about 0.002 (Dalby, 1962). The
resulting $1 - \mu \sim 0.1$ is then too small by about three orders of magnitude. (There are resonance transitions in the ultraviolet which have $f$-values $\sim 1$, but for these wavelengths the spectral intensity of the radiation diminishes appreciably so that from equation (63a), $1 - \mu$ is again small).

It is therefore clear that the radiation pressure force is far too small to explain the accelerations quoted in this section. Confirmation that the observed motions are not due to radiation pressure is provided by two other considerations: (i) motions across the tail axis, and occurrence of turbulent motions, (ii) the fact that observed accelerations are considerably greater in the side-streamers than in the main body of the tail, each being at practically the same distance from the Sun. The tails are optically thin, and if radiation pressure were responsible, the accelerations should be equal at the periphery and in the interior of the tail. The observed inequality indicates that the agency responsible is more effective in the outer regions than in the interior of the tail.
3.3 The solar wind and its interaction with comets

It is now accepted that a continuous stream of gas leaves the Sun and moves outward through the solar system. The study of the interaction of this gas, the solar wind, with the various members of the solar system has received much attention in recent years. Now thought to have a mean proton and electron density of about $5 \text{ per cm}^3$ (with a much smaller component of heavier nuclei) and to move with a mean velocity of about $450 \text{ km per sec}$, the influence of the solar wind on an insubstantial object such as a comet tail might be expected to be important. Indeed, the study of Type I comet tails played a decisive part in establishing the existence of the solar wind as now understood.

A number of lines of evidence led to the idea that the Sun might be emitting streams of charged particles. The connection between solar activity and the occurrence of magnetic storms and aurorae had been noted in the nineteenth century, and the velocity of the supposed particles estimated from the delay involved. The connection was not widely accepted at the time, but as a result of continuing study of auroral and geomagnetic phenomena during the present century, the idea of 'solar streams' gained currency.

Following the work of Störmer on aurorae based on the paths of electrons in a dipole field, Chapman and Ferraro sought to explain auroral and magnetic storms on the basis of neutral streams of particles. The zodiacal light had long been taken to imply the existence of interplanetary matter, and in the 1950's was temporarily attributed to electron scattering*. Studies of time variations in the intensity of galactic cosmic rays also indicated that clouds of

* This explanation was abandoned in favour of scattering by dust particles.
charged particles carrying a magnetic field could be reaching the
neighbourhood of the Earth: the observed sudden decreases in
intensity ('Forbush' decreases) were put down to the shielding
effect of this magnetised plasma. Significant too was the observed
variation of intensity with the sunspot cycle.

Newton and Encke considered that the shapes of comet tails might
be explained by the presence of a resisting medium; Bessel (1836)
however, on the basis of his mechanical theory (Sec.4.2), concluded
that the shapes could be explained without such a medium. Bessel, of
course, was not aware of the essential difference between Type I and
Type II tails. A resisting medium was again invoked in an attempt to
explain the orientation of Type II tails of distant comets, (Belton,
1965 and Sec.4.3), but abandoned as being unable to produce the
observed orientations. In the case of Type I tails, however, it is
now widely held that the solar wind, as a continuously expanding
resistive medium, has a dominant influence on their behaviour.

The work of Hoffmeister (1943) on the motion of features dis-
cernible in Type I comet tails has been referred to in Sec.3.
and subsequently, the inadequacy of radiation pressure for the
production of these motions. Hoffmeister was also the first to study
the orientations of Type I tails in a systematic way. He plotted the
tangent of the orientation angle $\epsilon$ against the comet's velocity
perpendicular to the radius vector, obtaining an approximate straight
line. Despite correctly attributing the effect to electromagnetic
forces due to corpuscular streams, Hoffmeister did not actually
analyse the situation and deduce a value for the solar wind velocity.

Up to the 1950's, the available evidence supported the view that
at various times, streams of charged particles were released by the Sun to move outward through interplanetary space. Biermann (1951) put forward the epoch-making suggestion that the solar emission is continuous in nature. This was based on his interpretation of the behaviour of ionic comet tails, in particular that recorded by Hoffmeister. The two aspects of tail behaviour considered by Biermann were (i) orientations, (ii) kinematics.

(i) In general, Type I comet tails were found to lag the radius vector by a few degrees. Biermann proposed an explanation analogous to that of stellar aberration by Bradley: the tail orientation being determined by the vector velocities of the comet in its orbit and the radial stream of gas from the Sun. This explanation has become known as dynamical aberration, and is treated in Sec. 3.4. Confirmation that the hypothesis of aberration is a true one is provided by the findings on dispersion in the angle $\epsilon$ (Sec. 3.4) and by the fact that a minimum velocity of the solar wind (Brandt, 1967; Brandt and Heise, 1968) indicated by a minimum value of the quantity $\xi$ (Sec. 3.4) agrees with a theoretical prediction by Axford, Dessler and Gottlieb (1963). The systematic difference in $\epsilon$ between direct and retrograde comets also corroborates the aberration interpretation if the solar wind is assumed to possess an azimuthal component.

(ii) Studies of the motion of identifiable features in Type I tails by Hoffmeister and others have been described in Sec. 3.2. Biermann proposed that the observed accelerations were due to the transfer of momentum to the tail features from the corpuscular streams originating in the Sun. Biermann's model for the process is however, now regarded as being only qualitatively successful.
Studies of Type I comet tails thus played a vital part in pointing to the existence of a continuous corpuscular emission from the Sun. Theoretical models for the solar corona were constructed by Chapman and later by Parker. Parker (1958) put forward a hydrodynamic model involving the flow of gas away from the Sun, contrasting with the static model of Chapman. The flow occurs in consequence of the coronal temperature of $2.10^6$ K; this continuous 'evaporation' of the corona was termed the solar wind by Parker. His analysis led to a transonic solution giving velocities in agreement with the value of 400 km per sec deduced by Biermann from Hoffmeister's work. Parker also predicted that as a result of solar rotation, the magnetic field lines originating in the Sun would take the form of an Archimedean spiral. Conclusive proof of the existence of a solar wind was provided by direct measurements from space probes. A probe has to be sent to a sufficient distance from the Earth to avoid the influence of the Earth's magnetosphere; early results were not inconsistent with the expected data, but readings were obtained only for short time intervals and in one case the probe was near the boundary of the magnetosphere. In 1962, however, the Mariner 2 probe obtained readings over a continuous period of three months, confirming the picture of the solar wind as now accepted: a detectable solar wind flux was always present; the velocity varied between about 300 and 850 km per sec, with an average of some 500 km per sec; the average proton/electron density was about 5 per cm$^3$. Correlations of velocity with geomagnetic activity were found and streams of particularly high velocity (associated with the solar M-regions) recurred at 27-day intervals. Heavier particles were also found to be present.
So well established is the existence of the solar wind that the converse standpoint to that of earlier times is now admissible: instead of inferring the existence of the wind from comet tail behaviour, the effect on comet tails (and other objects in the solar system) of the solar wind may be investigated. On account of the explanation of tail lag angle by aberration, Type I tails have been likened to natural 'windsocks' in the solar wind: from their orientations, the corresponding solar wind velocities can be deduced and a picture of the velocity field built up. This has been done, notably by Brandt and his co-workers. Comets are, of course, by no means confined to the plane of the ecliptic, so that they are particularly useful in gathering data relating to regions difficult of access to artificial space probes. As far as dynamical aberration is concerned, the actual mechanism of the interaction with the solar wind is immaterial; the only tacit assumption is that the tail near the comet head be symmetrical with respect to the local wind direction, as for the geomagnetic tail (Ness and Donn, 1966). It should be emphasised that the comet tail in question here is the 'gross' tail. The orientation and general behaviour of constituent structures such as rays or streamers requires separate treatment. We return to this below.

Biermann (1953) attempted a detailed analysis of the accelerations of tail features to be expected as a result of momentum transfer from the solar wind. Production of CO$^+$ ions from the neutral molecule could result from charge exchange with the protons in the solar wind. Biermann studied theoretically the interaction between a proton-electron stream (the solar wind) and an ion-electron gas (cometary
plasma), deducing the acceleration produced on the ions as

$$\frac{dv_i}{dt} \sim \frac{e^+ m_e N_e v_e}{\sigma m_i} = 10^{-4.2} \frac{m_e N_e v_e}{m_i}$$

where $m_e$ is the electron mass, $m_i$ the ion mass, $N_e$ the electron density in the solar wind stream, $v_e$ the stream velocity of the electrons, $e$ the electronic change and $\sigma$ the electrical conductivity. Assuming that $v_e = 1000$ km per sec, the equation shows that to produce an acceleration of $10^2$ cm per sec$^2$, $N_e$ must be $\sim 600$ cm$^{-3}$. This was in accordance with other evidence available at the time, but is now known to be too high by two orders of magnitude. For an acceptable average solar wind velocity of 500 km per sec the position is worse, as $N_e$ must then be doubled to $1200$ cm$^{-3}$. In this model however, magnetic fields are not taken into account.

The presence of long straight filaments in Type I tails has been generally accepted as evidence for magnetic fields in these tails. The occasional presence of helical structures supports this view. Typical radii for the filaments are $3000 - 4000$ km., the radius being interpreted as the Larmor radius for the CO$^+$ ions.

The manner in which a comet may 'sweep up' magnetic lines of force and the important role which may be played by magnetic fields has been stressed by many authors, notably Alfvén (1957). Magnetic fields would certainly enhance the coupling between the solar wind and the tail ions and possibly make feasible the production of the observed accelerations by solar wind action. Other difficulties however remain, including the question of how the solar wind can have sufficient access to knots in the interior of the tail, where
the observed accelerations may still be considerable (Sec. 3.2).
The fine structure of Type I tails has received much attention in
recent years, many contributions being made by Wurm and his
associates (e.g. Wurm, 1968, Wurm and Mammano, 1972). Interest is
focused on the head region, and the development of the individual
rays. Photographs taken at intervals of a few hours or less show
that rays first appear at large angles (60° or more) to the tail
axis; they lengthen and turn toward the axis, the rate of the turning
motion decreasing as they approach the axis. The time scale of the
process is some 15-25 hours. There is apparent symmetry of the ray
system about the axis, the picture having been likened to a folding
umbrella. The origin of the tail streamers appears to be near the
nucleus on the sunward side, perhaps within a distance of 1000 km.

A fully satisfactory detailed theory of the interaction of the
solar wind with a comet has yet to be attained. Such a theory should
account for the observed fine structure behaviour, including the
ionisation mechanism, as well as the motion of tail features on a
larger scale. The ionisation mechanism is not well understood since
the development of individual rays on the scale observed implies
time scales of $10^3 - 10^4$ sec for the production of $\mathrm{CO}^+$, whereas
photoionisation and charge exchange require time scales of $\sim 10^6$ sec.
Biermann, Brosowski and Schmidt (1967) have proposed a hydrodynamic
model for the interaction in which a flow of neutral molecules
issues radically from the nucleus. When ionised (by photoionisation),
they are quickly accelerated to the mean flow velocity of the
surrounding plasma. The model applies only to the sunward side of
the nucleus, and it is found that a detached shock occurs at a stand-
of distance of $\sim 10^6$ km. There is also a contact discontinuity $\sim 10^5$ km from the nucleus, separating the mixed solar wind and cometary plasma from the pure cometary plasma. While this fluid-type treatment is a useful quantitative method of approach, it has been emphasised, particularly by Wurm, that it takes no account of the internal structures observed in the Type I tails.
3.4 Dynamical Aberration

Accepting that the behaviour of Type I comet tails is governed by the solar wind in the manner described in Sec. 3, the analysis proceeds as follows. This is due originally to Biermann (1951) and developed by Belton and Brandt (1966), Pflug (1966) and others.

If \( \mathbf{w} \) is the solar wind velocity, \( \mathbf{v} \) the orbital velocity of the comet and \( \mathbf{T} \) the tail axis vector, the fundamental equation for Type I tails is

\[
\mathbf{T} = \mathbf{w} - \mathbf{v}
\]

an equation which still holds good whether or not \( \mathbf{w} \) and \( \mathbf{T} \) lie in the comet's orbital plane (Fig. 17), and which implies that the tail vector points in the direction of the solar wind as seen by an observer on the comet head.

The assumption of a radial plasma flow for the solar wind enables an expression to be derived for \( \epsilon \), the tail orientation angle as defined in Sec. 2.4. The vectors \( \mathbf{w} \) and \( \mathbf{\hat{e}} (\equiv \mathbf{T}) \) then lie in the orbital plane. In Fig. 18,
\[ t = \omega - \nu \]

and

\[ BD = V \cos (180^\circ - y) = -V \cos y \]
\[ AB = V \sin (180^\circ - y) = V \sin y \]

\( y \) being the angle between the comet's velocity vector and the prolonged radius vector. Then

\[ \tan \epsilon = \frac{AB}{CD + BD} \]

\[ \tan \epsilon = \frac{V \sin y}{\omega - V \cos y} \] (65)

This formula, derived from dynamical considerations, may be compared with formula (13), Sec. 2.4, where the treatment is purely geometrical in nature. The two formulae are completely equivalent if

\[ \omega = \pm V. \] (66)

The quantity \( \omega \) (Osterbrock's parameter) expresses the ratio of the components of the tail axis vector in the directions of \( \nu \) and \( -\nu \)
and in the present context is the ratio of velocities $w/V$. Again,
\[ h = \sin (\gamma + e) \csc \epsilon. \]  
(67)

From the nature of the derivation here, the angle $\epsilon$ is often referred to as the aberration angle.

In equation (65), $V \cos \gamma$ is normally small in comparison with $w$, so that
\[ \tan \epsilon \sim \frac{h}{w} V \sin \gamma. \]

A graph of $\tan \epsilon$ against the velocity of the comet perpendicular to the radius vector will thus be approximately a straight line, as found by Hoffmeister (1943).

Observations of Type I comet tails may be analysed as described in Chapter 2 to determine the aberration angle $\epsilon$ and hence the solar wind velocity from (66). This method of investigating the velocity field of the solar wind has been used notably by Belton and Brandt (1966). Errors in $\epsilon$ will of course give rise to errors in $w$. The effect on $\epsilon$ of uncertainty in the measurement of the position angle $\epsilon$ was discussed in Sec. 2.6; we may apply a similar treatment to $w$.

In Sec. 2.6, we derived the equation
\[ \frac{\Delta \epsilon}{\Delta \theta} = \frac{1}{\cot (\gamma + e) - \cot \epsilon} \frac{\sec \beta (\tan \phi - \tan \gamma)}{(\tan \beta - \tan \gamma)(\tan \phi - \tan \beta)} \]  
(68)

From equations (66) and (67) we have
\[ w = V \sin (\gamma + e) \csc \epsilon, \]
whence
\[ \frac{\Delta w}{\Delta \epsilon} = V \left\{ \cos \epsilon \csc \epsilon \cos (\gamma + e) - \sin (\gamma + e) \csc \epsilon \cot \epsilon \right\}, \]
\[ = w \left\{ \cot (\gamma + e) - \cot \epsilon \right\} \]  
(69)
on eliminating $V$. 

Now
\[
\frac{\partial \omega}{\partial \theta} = \frac{2\omega}{\partial \epsilon} \frac{\partial \epsilon}{\partial \theta}
\]

so that from (68) and (69),
\[
\frac{\partial \omega}{\partial \theta} = \omega \frac{\sec \theta (\tan \phi - \tan \psi)}{(\tan \theta - \tan \gamma)(\tan \phi - \tan \theta)}
\]  

(70)

an expression quoted (but not derived) by Belton and Brandt (1966). The coefficient of \( \omega \) on the right-hand side (\( F(\theta) \) in Sec.2.7.) occurs also in \( \frac{\partial \epsilon}{\partial \theta} \), facilitating computation. In determinations of \( \omega \) by the aberration method, it is convenient to tabulate \( \frac{\partial \omega}{\partial \theta} \) so that uncertainties in \( \omega \) due to those in \( \theta \) may be evident.

Dropping the assumption of a strictly radial solar wind, expression (65) for the aberration angle may be generalised.

Resolving the solar wind velocity into a radial component \( \omega_r \) and an azimuthal component \( \omega_\phi \) (measured positive in the sense of the solar rotation), the geometrical situation in the orbital plane is as shown in Fig. 19.

![Diagram](image-url)
Here, \( CA = \omega_r; \quad GA = \omega_\phi \cos i' \),

where \( i' \) is the inclination of the comet's orbit to the plane of the solar equator. Then \( \hat{BF} = 180^\circ - \gamma \), and

\[
EF = V \sin \gamma \quad ; \quad GF = -V \cos \gamma .
\]

Thus

\[
tan \, \varepsilon = \frac{DE}{EF-GA} = \frac{EF-GA}{CA+GF} \quad \text{or} \quad tan \, \varepsilon = \frac{V \sin \gamma - \omega_\phi \cos i'}{\omega_r - V \cos \gamma} \quad \text{(71)}
\]

This equation was used as the basis of a method for measurement of the azimuthal component \( \omega_\phi \) (Brandt and Heise, 1968).

The technique is a statistical one: the mean aberration angles of direct and retrograde comets should show a systematic difference since \( \cos i' \) changes sign for the latter.

Taking \( \omega_r \) and \( \omega_\phi \) to be constant, and neglecting \( V \cos \gamma \ll \omega_r \)

\[
\omega_r tan \, \varepsilon = V \sin \gamma - \omega_\phi \cos i'
\]

and averaging for direct comets,

\[
\left\langle \frac{1}{\omega_r} \right\rangle \left\langle \tan \varepsilon \right\rangle_D = \left\langle V \sin \gamma \right\rangle_D - \left\langle \omega_\phi \right\rangle \left\langle |\cos i'| \right\rangle_D
\]

For retrograde comets,

\[
\left\langle \frac{1}{\omega_r} \right\rangle \left\langle \tan \varepsilon \right\rangle_R = \left\langle V \sin \gamma \right\rangle_R + \left\langle \omega_\phi \right\rangle \left\langle |\cos i'| \right\rangle_R
\]

(\( 1/\omega_r \) is averaged since it is continuous for zero values of \( \varepsilon \)).

Adding and subtracting these equations gives

\[
\left\langle \frac{1}{\omega_r} \right\rangle^{-1} = \frac{\left\langle V \sin \gamma \right\rangle_R + \left\langle V \sin \gamma \right\rangle_D}{\left\langle \tan \varepsilon \right\rangle_R + \left\langle \tan \varepsilon \right\rangle_D} - \left\langle \omega_\phi \right\rangle \frac{\left\langle |\cos i'| \right\rangle_D - \left\langle |\cos i'| \right\rangle_R}{\left\langle \tan \varepsilon \right\rangle_R + \left\langle \tan \varepsilon \right\rangle_D} \quad \text{(72)}
\]

\[
\left\langle \omega_\phi \right\rangle = \frac{\left\langle \tan \varepsilon \right\rangle_R - \left\langle \tan \varepsilon \right\rangle_D}{\left\langle |\cos i'| \right\rangle_R + \left\langle |\cos i'| \right\rangle_D} \left\langle \frac{1}{\omega_r} \right\rangle^{-1} = \frac{\left\langle V \sin \gamma \right\rangle_R - \left\langle V \sin \gamma \right\rangle_D}{\left\langle |\cos i'| \right\rangle_R + \left\langle |\cos i'| \right\rangle_D} \quad \text{(73)}
\]
The coefficients of $\langle \omega_\phi \rangle$ in (72) and $\langle \psi_{\phi r} \rangle$ in (73), also the last term in (73) depend on differences in $\langle \cos \phi \rangle$, $\langle \tan \epsilon \rangle$, $\langle \psi_{\phi r} \rangle$ respectively between direct and retrograde comets in the sample of observations used. These differences were found to exist in the sample of about 600 Type I tail observations used by Brandt and Heise:

<table>
<thead>
<tr>
<th></th>
<th>Direct</th>
<th>Retrograde</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle \tan \epsilon \rangle$</td>
<td>0.062</td>
<td>0.096</td>
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<tr>
<td>$\langle \cos \phi \rangle$</td>
<td>0.90</td>
<td>0.72</td>
</tr>
<tr>
<td>$\langle \psi_{\phi r} \rangle$</td>
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<td>31.7</td>
</tr>
</tbody>
</table>

These values then gave $\langle \psi_{\phi r} \rangle \sim 450$ km per sec and $\langle \omega_\phi \rangle \sim 8.4$ km per sec.
This was the first measurement of the azimuthal component $\omega_\phi$.

The above data imply $\langle \epsilon \rangle \sim 3^\circ.7$ for direct comets and $\langle \epsilon \rangle \sim 5^\circ.5$ for retrograde comets, the latter being, as expected, the larger.

Efforts have been made to use the cometary data to link variations in the solar wind velocity with geomagnetic activity (using the $K_P$ index). Both $\omega_\phi$ and $\omega_\phi$ show an increase with $K_P$ agreeing with space probe results; however, the fact that the Earth and comet are usually far apart is an obvious drawback for the method.

After removing effects due to errors and variations of the quantities in equation (71), an inherent r.m.s. dispersion of about $3^\circ.5$ is found in the sample of values of $\epsilon$, indicating a corresponding dispersion of the solar wind about its mean direction. This agrees well with space probe results.
Brandt, Roosen and Harrington (1972, 1973) have further
generalised the analysis of tail orientations to allow for a
meridional flow of the solar wind, \( \omega_h \). The technique is again a
statistical one. Position angles on the plane of the sky are
computed from an assumed solar wind velocity vector \( (\omega_r, \omega_\phi, \omega_\theta) \);
statistical programs are then used to minimise the sum of squares
of the differences between the computed and observed position angles.
The best solution indicates a meridional flow away from the solar
equator of 2.6 km per sec at 1 a.u. for solar latitudes of \( \pm 45^\circ \).

**Comet Bennett**

As mentioned in Sec. 2.8, Comet Bennett, though having a
predominant dust tail, also displayed strong Type I features,
particularly in April 1970. Near the head, both types of tail
appeared to show the same orientation. On the basis of dynamical
aberration, the implied (radial) solar wind velocity, and associated
quantities, were computed by means of the program described in
Appendix 1, and are given in Table 8.
### TABLE 8
Comet Bennett 1969

<table>
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<th>$\theta$</th>
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<th>$V$</th>
<th>$k$</th>
<th>$w$</th>
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<td>1</td>
<td>April</td>
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<td>2.29</td>
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</tbody>
</table>

$V, w$ in km per sec.
Oscillation of cometary tails

An intriguing phenomenon detected in a very few Type I tails is that of tail oscillation, i.e., a periodic or quasi-periodic time variation of the aberration angle $\varepsilon$. Malaise (1963) drew attention to the case of Comet Burnham 1960II: this comet displayed a 'pure' Type I tail, and passed close to the Earth in April-May 1960. During this period Malaise obtained 26 photographs, and measurements made on these indicated that the tail appeared to oscillate with an amplitude of some $15^\circ$ and a period of $3^\circ\cdot 9$. The mean direction about which the oscillations took place was initially that of $\varepsilon \sim 8^\circ$, but this progressively changed: after slightly more than three complete periods between April 21 and May 3, the mean direction had become coincident with the radius vector ($\varepsilon \sim 0^\circ$). (The decreasing value of $\varepsilon$, on which the oscillations seem to be superimposed, is the natural trend in the post-perihelion period; this is treated in Sec. 3.6).

This report attracted considerable attention and some doubts were expressed regarding the reality of the oscillations. Ness and Donn (1966) proposed an alternative explanation in which the apparent periodicity is due to the turning of individual tail rays toward the axis with a spacing of a few days. Malaise's measurements, they claimed, had not all been made on the same ray in the tail structure. As discussed in Sec. 3.3, individual rays first appear at large angles to the axis, and then close in to it at a decreasing rate; measurements on several different rays could then approximately reproduce parts of the wave-like plot of $\varepsilon$ against time obtained by Malaise. However, Brandt (1968) pointed out that additional observations of Comet
Burnham not available to Malaise provided extra points in agreement with Malaise’s curve of $\epsilon$ against time. They also extended the curve. Also, most of the photographs given by Malaise show the tail as a single long straight ray emerging from a condensed circular coma. Though short secondary rays are occasionally present, it seems doubtful that confusion could arise in determining the tail direction as a 'gross' feature.

It was thought at the time that this 'tail-wagging' phenomenon was unique to Comet Burnham; A. J. Meadows, however, drew the author’s attention to the fact that Bessel (1836) had recorded a similar effect with the tail of Comet Halley at its 1835 apparition. Between 1835 October 2 and October 25, he observed an oscillation of amplitude 60° and average period $\frac{1}{4}$ day. In particular, on 1835 October 12, Bessel made several measurements over a period of 6 hours, during which time the angle between the tail axis and the extended radius vector increased by some 35°. Bessel’s measurements were, of course, made visually, and his values for angles must be subject to greater error than that to be expected with modern photographic measurements. However, the strikingly large amplitude of 60° and the short-period increase of 35° are much too great to be accounted for by observational error, and must be taken to represent genuine variations in $\epsilon$. Moreover, the increase referred to here is not in accord with the interpretation of Ness and Donn.

The reality of the time-variation of $\epsilon$ for Comet Halley suggests that the oscillations observed with Comet Burnham were also genuine. It was considered that a comparison of the positions of the two

*Rahe and Donn (1969) mentioned that Winnecke had also observed oscillations with the tail of Comet 1862 III.
comets during their 'tail-wagging' periods would be of interest. The positions at 5-day intervals were computed (essentially by the methods described in Sec. 2.2.), using the elements listed by Porter (1961).

### TABLE 9

<table>
<thead>
<tr>
<th>Comet</th>
<th>Date</th>
<th>Heliocentric distance (a.u.)</th>
<th>Ecliptic latitude(°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Halley 1835 III</td>
<td>1835 Oct. 5</td>
<td>1.07</td>
<td>+ 7.8</td>
</tr>
<tr>
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<td></td>
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</tr>
<tr>
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<td>15</td>
<td>0.91</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20</td>
<td>0.84</td>
</tr>
<tr>
<td></td>
<td></td>
<td>25</td>
<td>0.77</td>
</tr>
<tr>
<td>Burnham 1960 II</td>
<td>1960 Apr. 20</td>
<td>0.85</td>
<td>+ 8.8</td>
</tr>
<tr>
<td></td>
<td></td>
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<td>0.94</td>
</tr>
<tr>
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</tr>
<tr>
<td></td>
<td>May 5</td>
<td>1.11</td>
<td></td>
</tr>
</tbody>
</table>

Table 9 shows that the heliocentric positions of the two comets during the relevant periods of time were very similar. (They were in fact also at similar distances from the Earth - 0.20 a.u. for Comet Halley and 0.23 a.u. for Comet Burnham at the middle of the respective time interval). The similarity in position, combined with the strong correlation in period of oscillation, lend some support to the view that the tail-wagging effect is at least partly controlled by factors external to the comet, i.e., by conditions in
the interplanetary medium. Likening the tail to a plasma sheath carrying a cluster of magnetic lines of force, the oscillations could be due to a hydromagnetic perturbation of this system. Taking the length of the tail of Comet Burnham as $2.10^6$ km and the Alfvén velocity in the region concerned as 7 km per sec, the characteristic time for an oscillation of the tail is $\approx 3.10^5$ sec. This compares favourably with the observed oscillation period of $3.10^5$ sec.

Malaise considered possible explanations based on a varying initial velocity of the tail ions due to rotation of the nucleus, and a jet of ions rotating with the nucleus, but thought these mechanisms unlikely to produce the observed effects.

Subsequently, Ershkovich and Chernikov (1972) have studied the natural oscillations of Type I tails, based as above on the magnetic tail model. The problem is analogous to the vibration of a string of length $\ell$ excited by a blow from a flat hammer of length $\rightarrow \ell$ with velocity $v_\perp$. If the $z$-axis is taken parallel to the undisturbed field $B$ and the $y$-axis perpendicular to this, the wave equation

$$\frac{\partial^2 y}{\partial t^2} - \nu^2 \frac{\partial^2 y}{\partial z^2}$$

is solved under the conditions (i) $y(o,t) = 0; y(\ell,t) = 0$; one end of the tail fixed to the comet head, the other 'frozen' into the undisturbed solar wind, (ii) $y(o,t) = 0; (\partial^2 y/\partial z^2)_t = 0$ : one end fixed as before, the other free. $\nu = B/\sqrt{4\pi\rho}$ is the Alfvén velocity, $\rho$ being the plasma density. The solutions are readily obtained by the Fourier method. The natural (fundamental) oscillation periods are for case (i) $\tau_0 = 2\ell/\nu_\perp$, and for case (ii) $\tau_0 = 4\ell/\nu_\perp$. For acceptable values of $\ell$ and $\nu_\perp$, these lead to periods of $10^5 - 10^6$ sec,
in agreement with the observations and the characteristic time suggested above. Ershkovich and Chernikov also find that abrupt changes in the velocity of the solar wind of the order of a few km per sec can account for the amplitude of the oscillations observed with Comet Bennett.

This latter interpretation seems to be the most plausible at present. However, Wurm prefers to attribute the effect to 'slight variations in the emission conditions for ions at their source'. (e.g. Wurm and Mammano, 1972). He has consistently put forward the view that features of Type I tail behaviour are attributable to internal rather than external causes.
3.6 Progressive change in the aberration angle

Though, as indicated in the preceding sections, considerable attention has been given to dynamical aberration and its consequences, the progressive change to be expected in the aberration angle of a given comet has not been examined explicitly. Assuming a radial solar wind, the aberration angle depends on the wind velocity \( w \), the orbital velocity \( V \) of the comet and the angle between the tangent to the orbit and the radius vector. The last two quantities may be computed if the orbital elements are known; though \( w \) is not, of course, a constant in practice, it is still of interest to consider it so, and investigate the changes in the aberration angle \( \epsilon \) occasioned purely by the motion of the comet in its orbit.

We have

\[
\tan \epsilon = \frac{V \sin \gamma}{w - V \cos \gamma}
\]

Also, (Sec. 2.2 and 2.4) for a parabolic orbit,

\[
V^2 = \frac{2k^2}{q(1+\beta^2)} ; \quad \sin \gamma = \frac{1}{\sqrt{1+\beta^2}} , \quad \cos \gamma = \frac{\beta}{\sqrt{1+\beta^2}} .
\]

Thus

\[
V \sin \gamma = \sqrt{\frac{2}{q}} \frac{k}{1+\beta^2}
\]

\[
V \cos \gamma = \sqrt{\frac{2}{q}} \frac{\beta}{1+\beta^2}
\]

and substituting in the aberration equation,

\[
\tan \epsilon = \frac{1}{\sqrt{\frac{2}{q} \frac{k}{(1+\beta^2) - \beta}}}
\]

It is more convenient to relate the aberration angle to time rather than the quantity \( \beta \). We have
\[ \beta + \frac{1}{3} \beta^3 = \frac{k(t - \tau)}{\sqrt[3]{2} q^{3/2}} \]  

Then \( \beta \) may be found for a given time \( t - \tau \) (relative to perihelion) and \( \epsilon \) computed from (74).

It is clear from the aberration formula that \( \epsilon \) will not be symmetrical about perihelion, \( \psi \) and \( \sin \gamma \) are symmetrical but \( \cos \gamma \) changes sign. Also, \( \epsilon \) will be a maximum for a post-perihelion position of the comet.

If we set the denominator of the expression for \( \tan \epsilon \) equal to \( P \),

\[ P = \sqrt[3]{2} \frac{w}{k} (1 + \beta^2) - \beta \]

then for a stationary value of \( P \),

\[ \frac{dP}{d\beta} = \sqrt[3]{2} \frac{w}{k} 2\beta - 1 = 0 \]

whence

\[ \beta_m = \frac{k}{\sqrt[3]{2} q \psi} \]  

and this stationary value is a minimum. Substitution in equation (74) gives the corresponding maximum value of \( \tan \epsilon \):

\[ (\tan \epsilon)_{\text{max}} = \frac{1}{\sqrt[3]{2} \frac{w}{k} (1 + \frac{k^2}{2q \omega^2}) - \frac{k}{\sqrt[3]{2} q \psi}} \]

\[ = \frac{2 \sqrt[3]{2} q \psi k \omega}{2 q \omega^2 - k^2} \]  

(77)

This occurs at a time \( \epsilon_m \) (relative to perihelion) given by

\[ \beta_m + \frac{1}{3} \beta_m^3 = \frac{k \epsilon_m}{\sqrt[3]{2} q^{3/2}} \]
or \[
\frac{k}{\sqrt{2} \gamma w} + \frac{k^3}{3(2q)^{3/2} \omega^3} = \frac{k t_m}{\sqrt{2} \gamma \omega^2}
\]

whence \[
t_m = \frac{\gamma}{\omega} + \frac{k^3}{6 \omega^3} \tag{78}
\]

Alternately, remembering that \( \gamma = q(i+\beta) \), the foregoing results may be expressed in terms of the comet's heliocentric distance \( \gamma \). We find

\[
\tan \epsilon = \frac{k \sqrt{2} \gamma}{\omega \gamma - k \sqrt{2} (\gamma - \gamma)} \tag{79}
\]

\[
\gamma_m = \frac{\gamma + \frac{k^2}{2 \omega^2}} \tag{80}
\]

The general shape of the graph of \( \epsilon \) against time may be seen from Fig. 20. The computation is conveniently done by means of a short computer program (Appendix 2) based on equations (74) and (75).

As mentioned above, the curve is not symmetrical about perihelion, and in fact is not symmetrical about the ordinate corresponding to \( \epsilon_{\text{max}} \). It is drawn here for Comet Mrkos 1957d, \( q \sim 0.35 \text{ a.u.} \). As perihelion is approached, \( \epsilon \) increases steadily, the maximum value being reached some 2 d after perihelion; the subsequent decrease in \( \epsilon \) is slightly less rapid than the preceding increase.

From equation (76), the true anomaly at which \( \epsilon_{\text{max}} \) occurs decreases for increasing values of \( q \) and \( \omega \). The corresponding time \( t_m \) decreases as \( \omega \) increases, but increases with \( q \).
Comet Mrkos 1957d
q \sim 0.35 \text{ a.u.}
\nu \sim 325 \text{ km per sec.}
CHAPTER 4

TYPE II TAILS AND 'MIXED' TAILS
General

As mentioned in Sec. 1.1, the idea that the behaviour of comet tails is governed by solar influences is a very old one. The early ideas of Kepler and Newton regarding the pressure exerted by sunlight however, did not receive theoretical and experimental confirmation until the latter part of the nineteenth century, with the work of Maxwell, Lebedev and others. Prior to this, Bessel (1836) as part of his detailed study of Halley's comet, developed a theory of cometary tails based on a repulsive force subsequently identified with solar radiation pressure.

The theory, extended by Bredikhin (see Jaegermann, 1903) became known as the 'mechanical theory of cometary forms', and was applied over a period of many years to various cometary tails without regard to tail type. Though now recognised as being inappropriate for the case of gas tails, the principles of the mechanical theory were taken over by Finson and Probstein (1968) as the foundation of a more sophisticated and successful theory of dust tails.

A full discussion of the 'mechanical theory' and the Finson-Probstein theory is given, and we extend the calculation of particular loci required in the latter to include values of equal to, or greater than 1. We then present some orientation-time properties of the special loci, and consider the relevance of these to time changes in tail orientation and the question of mixed tails.
4.2 Mechanical Theory

Bessel (1836) was the first to establish an analytical theory of cometary forms, based on mechanics. The cometary atmosphere and tail are assumed to be composed of particles moving in the field of a central force originating in the Sun. No assumption is made regarding the nature of the particles, or of the repulsive force other than it be central and vary as the inverse square of the solar distance. Each particle is thus subject to the attractive force of solar gravity and the repulsive force, both varying as the inverse square of the solar distance. For a given particle, the ratio of the two is then constant and independent of the solar distance.

The most important parameter in the theory is the quantity $1-\mu$, defined as in Sec. 1, as

$$1-\mu = \frac{f_r}{f_g}$$

where $f_r$ and $f_g$ are the repulsive and gravitational forces on a particle. A net central force will of course still produce a plane orbit: the theory as developed by Bessel assumes that particles are ejected from the comet head only in the comet's orbital plane.

Bessel's original (1836) presentation of the theory is very terse, while Jaegermann's (1903) account is unobtainable in this country. The standard English sources usually quoted are those of Bobrovnikov (1951) and Wurm (1963); these treatments are however only outline in nature, and contain some errors in the formulae. The derivations were therefore re-worked as follows.
If \( m \) is the mass of the particle, \( m_o \) the mass of the Sun, and \( \mathcal{G} \) the constant of gravitation, then referred to unaccelerated axes \( OA, OB, OC \), we have

\[
m \ddot{X} = \left( -\frac{\mathcal{G} m_o m}{r^2} + \frac{K}{r^4} \right) \cos \Theta; \quad m \ddot{X}_o = \frac{\mathcal{G} m_o m}{r^2} \cos \Theta \quad (81)
\]

where the repulsive force on the particle is \( K/r^2 \) and \( \cos \Theta \) the appropriate direction cosine.

With \( \mathcal{G} = k^2 \) and \( x = X-X_o \), these give

\[
\ddot{X} + \left\{ k^2 (m_o + m) - \frac{K}{r^4} \right\} \frac{\cos \Theta}{r^2} = 0
\]

Now \( m \ll m_o \), so that

\[
\ddot{X} + k^2 m_o \left\{ 1 - \frac{K}{k^2 m_o m} \right\} \frac{\cos \Theta}{r^2} = 0
\]

But for the particle, we have

\[
\frac{\dot{f_r}}{f_r^2} = \frac{K}{r} \cdot \frac{k^2 m_o m}{r^2} = \frac{K}{k^2 m_o m} = 1 - \mu.
\]

Thus introducing a dash to denote quantities pertaining to the particle,
\[ \ddot{x} + \frac{\mu x}{r^3} = 0, \text{ with similar equations in } y, z. \quad (82) \]

For the comet nucleus itself, the repulsive force is negligible, so that \( K = 0 \) and the same procedure gives

\[ \ddot{x} + \frac{\kappa m x}{r^3} = 0, \text{ with similar equations in } y \text{ and } z. \quad (83) \]

The effect of repulsion on the particle is thus to replace the solar mass by \( \mu m_o \).

With the unit of length taken as the astronomical unit, the unit of mass as the solar mass, it is convenient to choose a unit of time so that in these units, \( \gamma = \kappa = 1 \). The unit of time is then \( 1/\kappa = 58^d.13 \), and the equations reduce to

Nucleus: \[ \ddot{x} + \frac{x}{r^3} = 0, \text{ with similar equations in } y \text{ and } z. \quad (84) \]

Particle: \[ \ddot{x} + \frac{\kappa x}{r^3} = 0, \text{ with similar equations in } y \text{ and } z. \quad (85) \]

Following standard practice, the x-axis is chosen to point to the perihelion point of the comet orbit, the y-axis (also in the orbital plane) being on the side of positive true anomaly.

The nucleus is not assumed to be a point mass; particles are assumed to be released from a 'sphere of influence' or 'wirkungssphäre' surrounding the centre of mass.
At the time of observation $t$, the tail is composed of particles released at various earlier times. Fundamentally, the problem is to calculate the position at time $t$ of a particle released at time $t_o$.

The following notation is used:

\[
\begin{array}{ll}
\text{Comet (centre of mass)} & \text{Particle} \\
\text{At time } t_o = t - \tau & x_o, y_o, \tau_o, \nu_o \\
\alpha & x, y, \tau, \nu \\
\end{array}
\]

The circumstances of ejection of the particle are shown in Fig. 22(b); $g$ is the emission velocity relative to the centre of mass, the angles $F$ and $G$ being measured from the radius vector toward the direction from which the comet has come.

The relative coordinates of the particle at time $t_o$ are

\[x_o' - x_o = \alpha, \quad y_o' - y_o = b. \tag{86}\]

The relative velocities of the particle and C.M. at time $t_o$ are

\[\frac{d}{{dt}} x_o' - \frac{d}{{dt}} x_o = \alpha, \quad \frac{d}{{dt}} y_o' - \frac{d}{{dt}} y_o = \beta. \tag{87}\]
4.2

Thus we have

\[
\begin{align*}
\alpha &= -f \cos (\nu_0 + \psi) \\
\beta &= -g \sin (\nu_0 + \psi)
\end{align*}
\]

\[
\begin{align*}
\omega &= -f \sin (\nu_0 + \psi) \\
\beta &= -g \cos (\nu_0 + \psi)
\end{align*}
\]

The relation between orbital and cometocentric coordinates is

(Sec. 2.2.)

\[
\begin{align*}
\tau^2 &= xx' + yy' - \tau^2 \\
\tau \eta &= yx' - xy'.
\end{align*}
\]

Bessel's technique is to express \(x', y'\) in Taylor series:

\[
\begin{align*}
x' &= x'_0 + \frac{dx'_0}{dt} \tau + \frac{d^2x'_0}{dt^2} \frac{\tau^2}{2} + \frac{d^3x'_0}{dt^3} \frac{\tau^3}{6} + \ldots \\
y' &= y'_0 + \frac{dy'_0}{dt} \tau + \frac{d^2y'_0}{dt^2} \frac{\tau^2}{2} + \frac{d^3y'_0}{dt^3} \frac{\tau^3}{6} + \ldots
\end{align*}
\]

Substitution of (90) into (89) would lead to series for \(x, \eta\).

However, it is convenient to express the solution in terms of parameters relating to the comet at the time of observation, \(t\).

Dashed quantities (particle) are replaced by undashed (comet); all quantities suffixed \(o\) are replaced by unsuffixed quantities, the latter being accomplished by the 'retrospective' use of Taylor series,

\[
\mathcal{F}(t_o) = \mathcal{F}(t) - \mathcal{F}'(t) \tau + \mathcal{F}''(t) \frac{\tau^2}{2} - \ldots
\]

From the equations of motion of the particle, (85), we have

\[
\frac{d^2x'}{dt^2} = -\frac{\mu x'}{r^3}; \quad \frac{d^3x'}{dt^3} = -\frac{\mu}{r^3} \frac{dx'}{dt} + \frac{3\mu x'}{r^5} \frac{dy'}{dt}
\]

and (90a) becomes

\[
\begin{align*}
x' &= x'_0 + \frac{dx'_0}{dt} \tau - \frac{1}{2} \frac{\mu x'_0}{r^5} \tau^2 - \frac{1}{6} \left( \frac{\mu}{r^3} \frac{dx'_0}{dt} + \frac{3\mu x'_0}{r^5} \frac{dy'_0}{dt} \right) \tau^3 + \ldots
\end{align*}
\]
with a similar equation for $y'$. The dashed and suffixed quantities are now replaced on the right-hand side. We have

\[
x'_0 = a' + x = a + x - \frac{dx}{dt} \tau - \frac{d^2x}{dt^2} \frac{\tau^2}{2} - \frac{d^3x}{dt^3} \frac{\tau^3}{6} + \ldots \tag{93}
\]

\[
\frac{dx'_0}{dt} = a' + \frac{dx}{dt} = a + \frac{dx}{dt} - \frac{d^1x}{dt^1} \tau - \frac{d^2x}{dt^2} \frac{\tau^2}{2} - \ldots
\]

From the equations of motion of the comet, we have

\[
\frac{d^3x}{dt^3} = -\frac{x}{\tau^3} \quad ; \quad \frac{d^3x}{dt^3} = -\frac{1}{\tau} \frac{dx}{dt} + \frac{3x}{\tau^2} \frac{dr}{dt}
\]

and equations (93) become

\[
x'_0 = a + x - \frac{dx}{dt} \tau - \frac{1}{2} \frac{x}{\tau^2} \tau^2 + \frac{1}{6} \left( \frac{1}{\tau^3} \frac{dx}{dt} - \frac{3x}{\tau^4} \frac{dr}{dt} \right) \tau^3 \ldots \tag{94}
\]

\[
\frac{dx'_0}{dt} = a + \frac{dx}{dt} + \frac{x}{\tau^3} \tau - \frac{1}{2} \left( \frac{1}{\tau^3} \frac{dx}{dt} - \frac{3x}{\tau^4} \frac{dr}{dt} \right) \tau^2 \ldots
\]

with similar equations for $y'_0$, $d y'_0$. The only other dashed quantities remaining in (92) are $\frac{1}{\tau} \frac{dx}{dt}$ and $\frac{1}{\tau} \frac{dy}{dt}$. By relating $\tau'_0$ to $\tau_c$ then using Taylor series to relate $\tau_c$ to $\tau$, we have

\[
\frac{1}{\tau^3} = \frac{1}{\tau^3} - \frac{3}{\tau^5} (ax + by) + \frac{3}{\tau^4} \frac{dx}{dt} \tau
\]

\[
\frac{1}{\tau^4} \frac{d\tau}{dt} = \frac{1}{\tau^4} \frac{dy}{dt} + \frac{ax + by}{\tau^5}
\]

Substituting (94) and (95) into (92) and neglecting all terms of the 2nd degree in $a, b$ in the $\tau^2$, and all terms of the 1st degree in $a, b$ in the $\tau^3$ term, we have

\[
x' = a + x + ax' + \frac{1}{2} \left( \frac{(1-\mu)}{\tau^3} - \frac{ma}{\tau^3} + \frac{3x}{\tau^5} (ax + by) \right) \tau^2
\]

\[- \frac{1}{6} \left( \frac{2(1-\mu)}{\tau^3} \left( \frac{dx}{dt} - \frac{3x}{\tau} \frac{dr}{dt} \right) - \frac{ma}{\tau^3} - \frac{3x}{\tau^5} (ax + by) \right) \tau^3 \ldots \tag{96}
\]
Similarly,
\[ y' = b + y + \beta z + \frac{1}{2} \left\{ (l-m) \frac{\dd x}{\dd t} - \frac{2}{r^3} \frac{\dd z}{\dd t} + 3 \omega \frac{\dd \theta}{\dd t} (\alpha x + \beta y) \right\} z^2 + \frac{1}{6} \left\{ \frac{2}{r^3} \left( \frac{\dd x}{\dd t} - \frac{2}{r^3} \frac{\dd z}{\dd t} \right) - \omega \frac{\dd \theta}{\dd t} - 3 \omega \frac{\dd \theta}{\dd t} (\alpha x + \beta y) \right\} z^3 \ldots \] (96)

Remembering that \( x^2 + y^2 = r^2 \) and \( \frac{dx}{dt} + \frac{dy}{dt} = \frac{d}{dt} \), substitution of (96) in (90) gives
\[ \tau^2 = a x + b y + (\omega x + \beta y) z - \frac{1}{2} \left\{ \frac{1}{r^3} \left( \frac{\dd x}{\dd t} + 2 \omega \frac{\dd \theta}{\dd t} (\alpha x + \beta y) \right) \right\} z^2 + \frac{1}{6} \left\{ \frac{2}{r^3} \left( \frac{\dd x}{\dd t} + \frac{\dd \theta}{\dd t} \right) + \frac{2}{r^3} (\omega x + \beta y) \right\} z^3 \ldots \] (97)

where, in the second equation, \( \frac{dx}{dt} + \frac{dy}{dt} = \frac{e \sin \nu}{\sqrt{p}} \), from the orbit equations of the comet. From orbital theory, we also have \( \frac{dx}{dt} = \frac{e \sin \nu}{\sqrt{p}} \), where \( e \) is the eccentricity and \( p \) the 'parameter' of the orbit.

Using this in the first equation of (97) and substituting the polar forms of \( \alpha, \beta, \omega, \beta \) we obtain
\[ \frac{t}{\rho} = - f \cos (\nu_0 + \dot{\nu}) \cos \nu - f \sin (\nu_0 + \dot{\nu}) \sin \nu \]
\[ + \left\{ \frac{1}{r^3} \left( \frac{\dd x}{\dd t} + 2 \omega \frac{\dd \theta}{\dd t} (\alpha x + \beta y) \right) \right\} \frac{z^2}{2} + \left\{ \frac{1}{r^3} \frac{4 \sqrt{p} \sin \nu}{\sqrt{p}} - \frac{2 \omega \frac{\dd \theta}{\dd t}}{r^3} \left( \cos (\nu_0 + \dot{\nu}) \cos \nu + \sin (\nu_0 + \dot{\nu}) \sin \nu \right) \right\} \frac{z^3}{6} \ldots \] (98)
For uniformity with the foregoing procedure, \( \nu_0 \) in (98) must now be expressed in terms of \( \nu \). Using Taylor series, and the orbital equations to obtain \( \frac{d\nu}{dt}, \frac{d\tau}{dt} \) etc., we find

\[
\begin{align*}
\cos \nu_0 &= \cos \nu + \sin \nu \frac{\sqrt{P}}{\gamma^2} \tau - \left\{ \cos \nu \frac{P}{\gamma^2} - \frac{2e\sin^2 \nu}{\gamma^2} \right\} \frac{\tau^2}{2} \\
\sin \nu_0 &= \sin \nu - \cos \nu \frac{\sqrt{P}}{\gamma^2} \tau - \left\{ \sin \nu \frac{P}{\gamma^2} + \frac{2e\sin \nu \cos \nu}{\gamma^2} \right\} \frac{\tau^2}{2}
\end{align*}
\]

(99)

Finally, expanding all terms \( \frac{\cos (\nu_0 + \frac{P}{\gamma^2})}{\sin \nu_0} \), inserting expressions (99) for \( \frac{\cos \nu_0}{\sin \nu_0} \) and reducing,

\[
\begin{align*}
\xi &= -f \cos F - \left\{ g \cos G + f \sin F \frac{\sqrt{P}}{\gamma^2} \right\} \tau \\
&\quad + \left\{ \frac{1 - \mu}{\gamma^2} - g \sin G \frac{2\sqrt{P}}{\gamma^2} - f \cos F \left( \frac{2e}{\gamma^2} - \frac{P}{\gamma^2} \right) - f \sin F \frac{2e \sin \nu}{\gamma^2} \right\} \frac{\tau^2}{2} \\
&\quad + \left\{ \frac{1 - \mu}{\gamma^2} \frac{4e \sin \nu}{\gamma^2} - g \cos G \left( \frac{4e}{\gamma^2} - \frac{3P}{\gamma^2} \right) - g \sin G \frac{6e \sin \nu}{\gamma^2} \right\} \frac{\tau^3}{3}
\end{align*}
\]

\[
\begin{align*}
\eta &= f \sin F + \left\{ g \sin G - f \cos F \frac{\sqrt{P}}{\gamma^2} \right\} \tau \\
&\quad - \left\{ g \cos G \frac{2\sqrt{P}}{\gamma^2} + f \sin F \left( \frac{4e}{\gamma^2} + \frac{P}{\gamma^2} \right) + f \cos F \frac{2e \sin \nu}{\gamma^2} \right\} \frac{\tau^2}{2} \\
&\quad + \left\{ \frac{1 - \mu}{\gamma^2} \frac{2\sqrt{P}}{\gamma^2} - g \sin G \left( \frac{4e}{\gamma^2} + \frac{3P}{\gamma^2} \right) - g \cos G \frac{6e \sin \nu}{\gamma^2} \right\} \frac{\tau^3}{3}
\end{align*}
\]

(100)

Here, terms in \( \tau^3 \) have been omitted; \( \gamma \) and \( \nu \) refer to the comet at time \( t \).

These equations involve rather many disposable parameters. The quantities \( f, F, g, G \) specifying the circumstances of ejection of the particle are unknown a priori, as is \( 1 - \mu \) which depends on the nature of the particle. If the orbital elements of the comet are known, the values of \( p \) and \( e \) are available while \( \gamma \) and \( \nu \) may be computed for a given time \( t \). Most important is the fact that the truncated expansions are valid only for small values of \( \tau \), i.e. for particles ejected shortly before the time of observation. The formulae can thus be confidently applied only for the head and near-tail regions of the comet.
As discussed in Sec. 1.2, current ideas regarding the nature of the comet head favour a small solid nucleus rather than a comparatively large agglomeration of particles. A non-zero value of the distance $f$ (with the associated angle $F$) might have some relevance in connection with the 'sandbank' model, but considerable simplification of the expressions, consistent with the adoption of the solid nucleus model, is obtained by setting $f = 0$. We then find

$$
\begin{align*}
\xi &= -g \cos \theta \tau + \left( \frac{1 - \mu}{\tau^2} - 2g \sin \theta \frac{\nu}{\tau^2} \right) \frac{\tau^2}{2} \\
&\quad + \left( \frac{1 - \mu}{\tau^2} \sqrt{\frac{\nu}{\tau^2}} - g \cos \theta \left( \frac{\mu}{\tau^2} - \frac{3b}{\tau^2} \right) - g \sin \theta \frac{\nu \sin \nu}{\tau^3} \right) \frac{\tau^3}{3} \\
\eta &= g \sin \theta \tau - \left[ 2g \cos \theta \sqrt{\frac{\nu}{\tau^2}} \right] \frac{\tau^2}{2} \\
&\quad + \left( \frac{1 - \mu}{\tau^2} \sqrt{\frac{\nu}{\tau^2}} - g \sin \theta \left( \frac{\mu}{\tau^2} + \frac{3b}{\tau^2} \right) - g \cos \theta \frac{\nu \sin \nu}{\tau^3} \right) \frac{\tau^3}{3}
\end{align*}
$$

(101)

If $\tau$ is restricted to very small values, the terms in $\tau^3$ and those in $\tau^2$ containing $g$ as a factor may be neglected. From the simplified equations, some deductions relating to the head region of the comet may be made. Assuming that ejection of particles is isotropic at a constant velocity $g$, elimination of $G$ from the equations

$$
\begin{align*}
\xi &= -g \cos \theta \tau + \frac{1}{2} \frac{1 - \mu}{\tau^2} \tau^2 \\
\eta &= g \sin \theta \tau
\end{align*}
$$

(102)

gives

$$
\left( \xi - \frac{1}{2} \frac{1 - \mu}{\tau^2} \tau^2 \right)^2 + \eta^2 = g^2 \tau^2
$$

(103)

The locus at time $t$ of particles which left the nucleus at time $t - \tau$ is therefore a circle of radius $g\tau$ with centre at $(\xi_o, \eta_o)$, where

$$
\xi_o = \frac{1}{2} \frac{1 - \mu}{\tau^2} \tau^2, \quad \eta_o = 0.
$$
The circle moves along the extension of the radius vector with accelerated motion while its radius expands. A second-order approximation (in which \( \eta \neq 0 \)) can also be readily found. Extrapolated to three dimensions, this picture corresponds to a halo being observed in the comet head. Isotropic ejection of particles, and their subsequent location on a moving, expanding spherical 'shell' is also a device used by Finson and Probstein (1960) in their development of the mechanical theory (Sec. 4.4).

Turning to the actual paths of ejected particles, the type of orbit followed depends on the parameter \( 1 - \mu \):

\[
1 - \mu < 1 \quad \text{(diminished gravitation): hyperbola concave to the Sun}
\]
\[
= 1 \quad \text{(forces balanced): straight line}
\]
\[
> 1 \quad \text{(nettreplusion): hyperbola convex to the Sun}
\]

Here, the orbits are referred to the \((x, y)\) coordinates, and it is assumed that \( g \sim 0 \). For the head region, \( \tau \) may be eliminated from equations (102) to give as the particle path

\[
\xi = - \cot G, \eta + \frac{1 - \mu}{2 + \eta^3} \sin^2 G - \eta^2
\]

which represents a system of parabolas passing through the origin of the \((\xi, \eta)\) coordinates.

![Fig. 23](image)

The envelope of these parabolas is itself a parabola,
4.2

We have here the 'fountain' model, the parabolic outline of the head (equation 104) being more or less confirmed by observations of dust heads.

Since our primary interest is in the cometary tail, deductions from the mechanical theory for larger values of \( \tau \) will be important. The path followed by a tail particle will of course depend on the initial conditions, \( g, G, f, F \) and the parameter \( 1 - \mu \). A valuable simplification is obtained by assuming that \( g \sim 0 \) (in addition to \( f = 0 \)), the particle being just released from the nucleus. Then we have, to order \( \tau^3 \),

\[
\begin{align*}
\xi & = \frac{1 - \mu}{2r^2} \tau^2 + \frac{2}{3} \frac{1 - \mu}{r^3} \frac{\epsilon \sin \nu}{\sqrt{p}} \tau^3 \\
\eta & = \frac{1 - \mu}{3r^3} \sqrt{p} \tau^3
\end{align*}
\]

(105)

If it is imagined that uniform particles (having a fixed value of \( 1 - \mu \)) are released continuously from some previous instant \( t_e \) to the instant of observation \( t \), the subsequent location of these particles may be found by eliminating \( \tau \) between equations (105). This gives

\[
\xi = \left( \frac{g \tau^2 (1 - \mu)}{8p} \right) \eta^{2/3} + \frac{2 \tau \epsilon \sin \nu}{p} \eta
\]

(106)

This instantaneous locus of particles with a fixed value of \( 1 - \mu \), emitted over a range of time prior to observation was termed by Bredikhin a syndyname (or syndyne). Because expansions have been used, the time span \( t - t_e \) cannot be large, and the expression will be valid only for the near-tail region.
The situation is depicted in Fig. 21(a). As would be expected on dynamical grounds, and as shown by equation (106), the synodyne is tangent to the $\xi$-axis at the nucleus.

Bredikhin also introduced another type of locus, the synchrone. This depends on the idea that the particles which constitute the tail may not be uniform, and may thus have a range of values of $1-\mu$. A synchrone is the instantaneous locus of all particles emitted at a single previous instant, and is depicted in Fig. 21(b). Particles with $1-\mu \sim 0$ would be found at the nucleus end of the synchrone, while those with a maximum value of $1-\mu$ are located at the far end.

Elimination of $1-\mu$ between equations (105) gives, neglecting terms in $\eta$ etc.,

$$\eta = \frac{2p\tau}{\nu(2\nu\sqrt{p} + 4\varepsilon \sin \nu \tau)} \xi,$$  \hspace{1cm} (107)

so that (for small values of $\tau$) the synchrone approximates to a straight line extending from the nucleus. If $\varepsilon$ is the angle it makes with the $\xi$-axis,
\[ \tan \epsilon = \frac{2p \tau}{\tau(3\tau p + 4c \sin \nu \tau)} \]

This is a function of \( \tau \), and 'older' synchrones will be inclined at larger angles to the \( \frac{x}{2} \)-axis.

The synchrone concept was introduced originally as an attempt to explain the rectilinear streaks sometimes observed near the ends of Type II tails ("synchronous bands" or "terminal synchrones"). It was also surmised that certain tails might be formed as a result of a single large outburst of material over a very short time span, the formation then being known as a complete synchrone.

It is of course clear that the syndyne and synchrone concepts are complementary. If a dust tail is formed by the emission of non-uniform particles over a range of time, as must be the case with the majority, systems of syndynes and synchrones may be constructed which have a relevance for the given tail.

![Diagram](image)

- \( Sd' \): Syndyne for greater value of \( 1-\mu \)
- \( Sd^2 \): " " lesser " " \( 1-\mu \)
- \( Sc' \): Synchrone for emission at \( t_1 \)
- \( Sc^2 \): " " " " \( t_2 \)

**Fig. 25**
The mechanical theory, in the development based on expansions, was applied by Bessel, Bredikhin and others to particular comets. From the inception of the theory, the standard method of analysis was via syndynes, the sole use of synchrones being confined to the somewhat rare effects mentioned above. This tendency has indeed persisted up to very recent times: only within the last ten years has adequate attention been given to tail analysis via synchrones. (Sec. 4.6).

Equations (106) and (107), for syndynes and synchrones respectively, have been derived on the assumption that the emission velocity $g \sim 0$. If $g$ has a non-zero value, the emission angle $G$ must be taken into account and a syndyne or synchrone for fixed values of $g$, $G$ would deviate from the curve corresponding to $g \sim 0$. A range of values of $G$ gives rise to further small variations in the locus in question. An infinite variety of syndynes could therefore be constructed for comparison with an actual tail.

In early work, the particles were assumed uniform and the emission of particles isotropic with a common value of $g$. The borders of the tail are then syndynes for $G = \pm 90^\circ$ (at least for observations made not too far from perihelion, so that the angle $\gamma$ of Sec. 2.4. is not too different from $90^\circ$). Elimination of $\gamma$ between equations (101) leads to the syndyne equation

$$\eta = \frac{2 \sqrt{2} p}{3 \sqrt{1 - \mu}} \xi^{3/2} + g \sin G \left[ \frac{\gamma \sqrt{2} \xi}{\sqrt{1 - \mu}} - \frac{4 \pi - \sin \gamma}{3 \sqrt{2} p (1 - \mu)} \right]$$

Using this, Pape (1859) in a typical application of the method, obtained pairs of values ($\xi, \eta$) on the borders of the strongly curved tail of Comet 1858 VI (Donati) and derived.
\[ 1 - \mu = 0.4 \]
\[ g \sin \theta \sim \pm 0.14 \]

whence \[ g \sim 0.14 = 4.2 \text{ km per sec.} \]

These results implied a reduced gravitational force on the particles, well in accord with later work, but an emission velocity now considered to be one order of magnitude too large (Sec. 4.4). The assumptions made are of course too sweeping. It is unlikely in the average case that the emitted particles would be uniform and that their emission velocity would be constant. There is thus a multiplicity of syndynes, and the tail borders cannot be identified with any specific sydnyne, nor indeed need they be syndynes at all.*

The assumption of isotropic emission is probably the least harmful. Sunward emission of particles \((G = 0)\) may be thought to produce an 'axis' sydnyne, which is 'broadened' by the isotropic emission, the amount of broadening depending on the velocity \(g\). An alternative 'axis', not identical with, but close to the first, is got by simply taking \(\theta = 0\). This device is used in the later work of Finson and Froebstein (1968)

There is considerable scope for adjusting the parameters \(g, G\) in attempting to explain peculiar features found in some tails. For example, if the emission is predominantly on one side of the radius vector, the corresponding side of the tail will be more developed than the other; if emission occurs in an 'oscillating jet', represented by periodic variation in \(G\), the tail will have a more or less wavy structure. This type of exercise was much pursued by the

* They could be envelopes of many sydnyne curves.
earlier workers, but in many cases the effort was directed at trying to explain the ray structure or turbulent form of Type I tails for which the mechanical theory is inapplicable. With dust tails, the explanation would have at least qualitative validity.

The mechanical theory, also known as the Bessel-Bredikhin theory in the form set out above, was the first soundly-based theory of cometary tails. It achieved a measure of success in explaining cometary forms, though its quantitative results are unreliable. Inevitably, its mathematical formulation followed the practice of celestial mechanics in using expansion methods: the limitations of the formulae were never fully realised until recently. Even Wurm (1963) could state that "... the real difficulties of the theory of cometary forms are not in the mathematical formulation but in the true explanation and physical understanding of the observations." Yet it is clear that the Bessel-Bredikhin theory cannot be applied in the case of large well-developed dust tails because the values of the variable $\tau$ involved in the formation of the visible tail are so great as to preclude the use of the expansion formulae. This deficiency does not invalidate the basic principles on which the theory is based, viz.,

(i) motion of tail particles under a central inverse-square law of force, the centre being the Sun, and the nett-force attractive, zero or repulsive dependent on the strength of the inherent repulsive force,
(ii) the idea of 'particle independence', i.e., the absence of any interaction between the particles themselves. (The gravitational attractive force between the particles and the nucleus itself is neglected, as it is relatively very small.)

Particle independence also implies the absence of magnetic or electric forces on the particles. Difficulties encountered during the 1960's in understanding the dust tails of the 'distant' comets (e.g. Belton, 1965) led to the view that these latter forces might indeed be operating, but later work has tended to show that they need not be invoked. We return to this in Sec. 4.3.
4.3 The dust tail orientation problem

It became increasingly clear during the 1950s that the governing physical processes for Type I and Type II tails were essentially different. All the applications of mechanical theory to various tails up to that time do not seem to have encountered the orientation problem which then arose in connection with certain Type II tails. Osterbrock (1958) studied the tail orientations of Comet Baade 1954h and Comet Haro-Chavira 1954k and found that the tails lagged the extended radius vector by about 45°. The comets were observed when between 4 and 5 a.u. from the Sun; Beyer (1955) had obtained similar results for distant comets.

These values of $\epsilon \sim 45^\circ$ were highly unusual, and were considered to be a 'problem of the distant comets' - implying that Type II tails observed at lesser radial distances would show values of $\epsilon$ of a few degrees as noted in previous orientation studies. Osterbrock sought an explanation in terms of a drag force on the tail particles, caused by a static ambient medium with an assumed electron density of 20 per cm$^3$, and claimed that the orientations could be accounted for if the tails were composed of hydride molecules, CH, NH, OH. Subsequently, all these assumptions were found to be unjustified.

Rejecting Osterbrock's ideas, Donn (1962) returned to the mechanical theory and attempted an explanation by means of a syndyne fit.
For a single date, the tail (in the orbital plane) is represented by the thick line in Fig. 26; syndynes for various values of \(1-\mu\) are drawn, using the formula

\[
\gamma = \frac{2\sqrt{2} \rho}{3\pi \sqrt{1-\mu}} \frac{1}{t^{3/2}}
\]

(i.e., retaining only the first term of equation 109). The 'fit' is inevitably very approximate. All syndynes are tangent to the \(\xi\)-axis at the origin and are appreciably curved, whereas the observed tail had \(\epsilon \sim 45^\circ\), and was nearly straight. Donn found that a reasonable fit was obtained if \(1-\mu \sim 0.01\), concluding that the mechanical theory gave an adequate explanation and that any assumption of drag forces was unnecessary. Belton, Brandt and Hodge (1963) performed another syndyne fit, finding an optimum value of \(1-\mu \sim 1.7 \times 10^{-3}\). Donn (1962) had omitted to point out that his value of \(1-\mu \sim 0.01\) implied (Sec.1.4) a particle diameter \(d \sim 0.1\) mm \((\rho = 1)\). The new value of \(1-\mu\) implied even larger particles, \(d \sim 0.6\) mm. Supposing that the distant comet tails were composed of similar particles to the nearer and well-observed tails, for which \(d \sim 1\) \((\rho = 1)\), Belton et al reasoned that the mechanical theory failed to explain the orientations, and that a tangential resisting force was in operation.

It may be noted that the mechanical theory was applied via a syndyne fit to the observed orientation, and that the possible presence of large particles was dismissed. Both these points are relevant in the sequel.

The dust tail orientation problem became much more pressing following the work of Belton (1965). He reconsidered the classification of tails (Sec. 1.3), and was able to demonstrate that pure Type II tails had \(\epsilon \sim 45^\circ\) irrespective of heliocentric distance (though the
sample of these tails for which orientation data were available was small). Belton criticised the mechanical theory on several grounds: the implied ejection velocities (Sec. 4.2 and 4.4) were too high by an order of magnitude; a syndyne fit implied large particles, considered inadmissible as above. He also rejected the idea of modelling the tail by a complete synchrone, maintaining that the steady increase of the synchrone lag angle with time as required by not theory (Sec 4.2) was reflected by the observations (of the distant comets, and Comet Arend-Roland). Since the existing mechanical theory appeared to be inadequate, Belton considered a modified version in which the particles are subject to a drag force directed along the relative velocity vector between the interplanetary medium and the comet (in addition to the usual forces). Taking the solar wind to move radially with velocity \( \mathbf{w} \), the equation of motion for the particle is

\[
\ddot{x}' = -\frac{\mu}{r^3} + f(w - \dot{r}),
\]

where \( f \) is the drag force per unit mass per unit relative velocity. As before, the equation of motion of the nucleus is

\[
\ddot{\mathbf{r}} = -\frac{\mu}{r^3}.
\]

Assuming that the particle is emitted from the nucleus with zero initial velocity, the initial conditions are

\[
x_0' = x_0; \quad \dot{x}_0' = \dot{x}_0,
\]

and the \((\xi, \eta)\) coordinates of the particle may be obtained by a similar procedure to that in Sec. 4.2 (on reworking the derivations, a slight difference was found in the \( \tau^3 \) term for \( \gamma \) as compared with that given by Belton).
4.3

\[
\hat{x} = \frac{c^2}{2} \left[ \frac{(1-\mu)}{r^2} + \frac{f}{r^2} \left( \omega - e \sin \gamma \right) \right] + \frac{c^2}{6} \left[ \frac{(3-\mu)F_{\text{w}}}{r^3} - \frac{(3-\mu)}{r^2} \left( f^2 \hat{z} + 2 \hat{z} \right) \right] (\omega - e \sin \gamma \hat{r}) \]

\[
\eta = \frac{c^2}{2} \left[ \frac{\xi}{r^2} \right] + \frac{c^2}{6} \left[ \frac{2(1-\mu)\xi}{r^2} - \frac{(f^2 + 2\hat{z})}{r^2} \right]
\]

If \( \varepsilon' \) is the angle between \( \omega - \hat{z} \) and \( \hat{r} \), we have

\[
|\omega - \hat{z}| \cos \varepsilon' = \omega - V \cos \gamma
\]

\[
|\omega - \hat{z}| \sin \varepsilon' = V \sin \gamma
\]

where \( V = \hat{r} \) and \( \gamma \) is defined as before. Since \( V \cos \gamma = \frac{dy}{dt} \), \( V \sin \gamma = \frac{d\phi}{dt} \),

\[
\hat{x} = \frac{c^2}{2} \left[ \frac{(1-\mu)}{r^2} + \frac{f}{r^2} |\omega - \hat{z}| \cos \varepsilon' \right] + \frac{c^2}{6} \left[ \frac{4(1-\mu)\xi \sin \gamma}{r^3} - \frac{(3-\mu)F_{\text{w}}}{r^3} - \frac{(f^2 + 2\hat{z})}{r^3} |\omega - \hat{z}| \cos \varepsilon' \right]
\]

\[
\eta = \frac{c^2}{2} \left[ f |\omega - \hat{z}| \sin \varepsilon' \right] + \frac{c^2}{6} \left[ \frac{2(1-\mu)\xi}{r^2} - \frac{(f^2 + 2\hat{z})}{r^2} |\omega - \hat{z}| \sin \varepsilon' + \frac{2f\omega \xi}{r^2} \right]
\]

and

\[
\tan \varepsilon = \frac{f |\omega - \hat{z}| \sin \varepsilon'}{1 - \mu + f |\omega - \hat{z}| \cos \varepsilon'} = \frac{q \sin \varepsilon'}{1 + q \cos \varepsilon'}
\]

where

\[
q = \frac{f |\omega - \hat{z}|}{(1-\mu)} = \frac{\text{drag force per unit mass}}{\text{radiation pressure force per unit mass}}
\]

It is shown that \( \varepsilon \) cannot be greater than \( \varepsilon' \); then if the drag force is very strong, \( q \rightarrow \infty \) and \( \varepsilon \rightarrow \varepsilon' \). But as before, \( \omega = kV \) and the observations indicate \( k \sim 1 \). Thus to account for the observed orientations requires that \( \omega \sim V \). This conflicts with all measurements of the solar wind velocity, which show that \( \omega \) is an order of magnitude greater than the orbital velocity \( V \). Further, an estimate of \( q \) puts its value at a small fraction of the radiation pressure force, so that in Belton's view, the modified theory completely failed to explain the observed orientations.

In place of the mechanical theory, Belton invoked electromagnetic interaction between the dust particles and the interplanetary medium with its magnetic field. The dust particles may become 'ionised',
but have a very low charge-to-mass ratio. The essential parameters are poorly known, and this development remained little better than a qualitative attempt at explanation. The dust tail orientation problem thus remained unresolved until the advent of the Finson-Probststein theory (Sec. 4.4). However, the various objections to the mechanical theory mentioned above have largely turned out to be groundless (Sec. 4.6).
4.4 The Finson - Probstein theory

This theory (Finson and Probstein, 1968) represents a very significant advance in the study of dust tails. Apart from providing an explanation for the observed dust tail orientations, the theory gives quantitative results for the dust and head gas emission rates as functions of time, the distribution of dust particle sizes and the particle emission velocity as a function of particle size and time. The theory is an extension of the original ideas of the Bessel - Bredikhin theory, based on the assumption of an essentially continuous emission of particles of varying size from the nucleus. The particles are taken to be accelerated into the tail by a spherically - symmetric gas flow from the nucleus, their subsequent motion being controlled only by solar gravity and radiation pressure. No complications due to drag forces or electromagnetic interactions are introduced.

The flow of a dusty gas in the head region of a comet had previously been investigated by Probstein (1967). The gas mean free paths near the nucleus are $\sim 10 - 10^3$ cm, so that the number of collisions is large and a continuum - type flow results. As the particle sizes are much less than the gas mean free paths, the drag forces on the particles are computed using the free molecular drag coefficient for spheres, dust-dust collisions being neglected as unimportant. It is found that the terminal velocity of the dust particles is attained within about 20 radii of the nucleus. As this distance is so small, the dust can be taken to issue from a point source as far as the tail is concerned. The initial velocity $v_i$ is obtained as a function of particle size, the number rate of emission of the particles and the mass flow rate of the neutral gas. For
reasonable values of the parameters, velocities $v_\xi \sim 0.3 \text{ km per sec}$ are indicated.

Complementary to this 'inner solution' is the treatment of the tail itself on a large scale - the 'outer solution'. As a consequence of the assumed continuous emission of particles, themselves of varying sizes, from the nucleus, the tail at a given time of observation is a composite structure and may be thought of as a superposition of an infinity of elementary syndyne tails or synchrone tails. The former are 'constant particle size' (or constant $1-\mu$) tails, and the latter are 'constant time interval' tails. This recognition of the observed tail as being composite is a key refinement of the older methods, which likened a tail to one syndyne having a fixed $1-\mu$ or a complete synchrone corresponding to emission at one instant. The axis of each elementary tail corresponds to zero emission velocity (Sec. 2.7), the tail being broadened by a finite value of this quantity. The two viewpoints are in fact complementary, the required superposition being equivalent to a double integration over $1-\mu$ and time. The 'syndyne approach' corresponds to $\int_{1-\mu}^{1}$ and the 'synchrone approach' to $\int \int$.

Three fundamental parameters are involved in the calculation: $g(\rho \Delta)$ the size distribution of the particles (assumed constant in time), $N_\Delta(t)$ their number rate of emission as a function of time and $v_\xi(t-\mu, t)$ the variation of initial particle velocity with size and time. If $t_e$ is the time of observation ($t = 0$ at perihelion), for an elementary syndyne tail corresponding to a particular value of $1-\mu$, the particles are emitted at times $t = t_e - \tau$.

The number of particles emitted in the interval $\tau$ to $\tau + \Delta \tau$ with
sizes in the range $\rho d$ to $(\rho d)+d(\rho d)$ is

$$\dot{N}_d(t) d\tau \cdot g(\rho d) d(\rho d)$$

To replace particle densities by observable light intensities, this formula is weighted by its light scattering ability, proportional to $(\rho d)^2$ for a given $\rho$. Now $\rho d \propto (1-\mu)^{-1}$, and it is more convenient to use $1-\mu$ instead of $\rho d$ as a variable. Then the number of particles emitted in time $\tau, \tau+d\tau$ in the range $(-\mu), (-\mu)+d(-\mu)$, weighted by the light scattering ability of the particles is

$$\dot{N}_d(t) d\tau \cdot f(1-\mu) d(1-\mu)$$

where the new function $f$ may be related to $g$. At the time of observation $t_e$, the particles in this fraction are distributed over a spherical surface of radius $\gamma \tau$, centred at the point $(\xi_{cm}, \eta_{cm})$ on the syndyne locus for the particular value of $\tau$. The syndyne tail is composed of an infinity of such surfaces for values of $\tau \rightarrow \infty$. Now $\xi_{cm} = \xi_{cm}(1-\mu, \tau_e)$ and may be computed ab initio from orbit mechanics. (It is recognised that the series expansion methods of the original Bessel-Bredikhin theory are invalid for appreciable values of $\tau$; we return to this point below). To an observer on the Earth, the points $(\xi_{cm}, \eta_{cm})$ are seen in projection on the photographic plane (Sec. 2.3): leading to $M_{cm} = M_{cm} (1-\mu, \tau, t_e)$.

The modified (i.e., proportional to light intensity) surface density is obtained by a line-of-sight integration of the density on the spherical surface. Integrating over all spherical surfaces of different $\tau$ values for a given line of sight leads to the expression

$$\frac{f(1-\mu) d(1-\mu)}{2\pi} \int_{\tau}^{\tau_e} \dot{N}_d d\tau \frac{\dot{N}_d d\tau}{(\nu_0, \tau)^2 \sqrt{1-z_0^2/(\nu_0, \tau)}}$$

where $z_0$ is the distance of the line of sight from the centre of the spherical surface and $\tau, \tau_e$ are the $\tau$-values for which $z_0 = \nu_0, \tau$. 
Introducing a 'hypersonic' approximation based on the fact that relative to the nucleus, the speed $\frac{dx}{dt}$ of the particles along the tail axis is $\gg v_i$, the various spheres can be considered to have the same $\tau$-value. Then $N_\lambda(t)$, $v_i \tau$ and $\frac{dx}{dt}$ can be taken as constants over the range of integration, and the modified surface density for an elementary syndyne tail becomes

$$\frac{N_\lambda f(t-\mu \tau) d(t-\mu \tau)}{2 (v_i \tau) \frac{dx}{dt}}.$$  

The total modified surface density at a point $(M_\lambda, N_\lambda)$ in the comet tail is then

$$D = \int_{(t-\mu \tau)_l}^{(t-\mu \tau)_u} \frac{N_\lambda f(t-\mu \tau) d(t-\mu \tau)}{2 (v_i \tau) \frac{dx}{dt}},$$  

(110)

where $(t-\mu \tau)_l$ are the limiting values of $t-\mu \tau$ for which the given point lies within the individual syndyne tails. $\tau$ in the integrand is related to $t-\mu \tau$ by the requirement that $(M_\lambda, N_\lambda)$ lies on the normal to the syndyne axis for each $t-\mu \tau$, i.e.,

$$\frac{\partial M_\lambda}{\partial \tau} = \frac{M_\lambda - M_{\lambda m}}{N_\lambda - N_{\lambda m}}.$$  

(111)

$D$ from (110) must be evaluated numerically in conjunction with (111). Analogous expressions to these are given by the 'synchrone approach'.

The three basic functional parameters $\dot{N}_\lambda(t)$, $f(t-\mu \tau)$, $v_i(t-\mu \tau, \tau, t)$ are involved in equation (110) for $D$. In applications of the method, the first stage is to determine these functions ($\dot{N}_\lambda$ in a relative sense only) by matching the calculated and observed light intensities. Details of the application to Comet Arend-Roland are presented by Finson and Probststein (1968, 2nd paper). Determination of the functions is an 'inverse' problem, and the matching entails the computation of many cases, judiciously varying the functions to obtain a good fit.

The numerical work is heavy, and the method is feasible only with the use of automatic computing. In the first stage, no assumptions need be
made regarding the nature of the particles, the light scattering or
the state of the nucleus. Assuming in the second stage of the
procedure that the quantity \( Q_p \sim 1 \) (Sec. 1.4), the distribution \( q(\rho,\theta) \)
can be found from \( f(\tau,\theta) \), and hence the size distribution on
assigning a value to \( \rho \). Absolute calibration of the light intensity
and assumptions regarding the albedo and phase angle function of the
particles lead to values for \( \dot{N}_d(t) \) (absolute) and \( \dot{M}_d(\tau) \), the mass
production rate of dust. The third stage is to employ the 'inner
solution' to determine \( \dot{M}_g \), the mass production rate of gas. \( \varsigma \) is
now known, and further assumptions regarding the radius and
temperature of the nucleus are required. There are many complications
and refinements in the detailed application of the method, which is
a considerable undertaking.

In the primary application of the method to Comet Arend-Roland
1956h, Finson and Probststein were able to match the observed isophote
plot remarkably well (though the computation of some 180 models was
necessary). Since a correct prediction of the angle \( \varepsilon \) (\( \sim 45^\circ \) for
Arend-Roland) is implicit in the matching process, the dust tail
orientation problem was resolved. The explanation here is essentially
a result of the superposition of syndynes (or synchrones) broadened
by the initial particle velocity, together with a specific variation
in the dust ejection rate. Values of several important physical
quantities were derived. The particle size of \( \sim 1 \mu \) was consistent
with other evidence, while the mass emission rates for dust and gas
were approximately equal at \( \sim 7 \times 10^7 \) gm per sec. This high value
(an order of magnitude greater than earlier estimates) confirmed a
prediction by Biermann and Trefftz (1964). An outburst in the dust
emission (revealed by the function \( \dot{N}_d(\theta) \)) was found to have occurred
about 6 days before perihelion.

Due perhaps to the involved nature of its application, the Finson-Probst method has seen very few applications to date. High quality plates are required, on red-sensitive emulsion and with the use of a filter to suppress any Type I features. Sekanina and Miller (1973) have applied the method to Comet Bennett, and again secured a good fit between the observed and calculated intensity profiles. The results obtained confirm in a general way those for Arend-Roland, except that Bennett was richer in smaller particles. In contrast to this, an application of the theory by Jambor (1973) to Comet Seki-Lines 1962c has shown that in this case larger particles, diameters several μ, predominate. This comet was a sun-grazer (perihelion distance 0.03 a.u.) and displayed a split tail, attributed to a sudden increase in the dust production rate near perihelion, followed by a sharp decrease due to vaporisation of the smaller particles by solar heating, and finally a rise to a value somewhat less than the original.

There is clearly a need for wider application of the theory, both as a check on its validity, and as a source of detailed physical information on dust comets. Some reservations regarding the theory may be mentioned. Though it is emphasised by the authors that the theory is devised for pure dust comets, in the three cases mentioned above, the comet was not strictly of this variety. Comet Arend-Roland showed substantial Type I features during at least part of its apparition, and the same is true for Comet Bennett (Sec. 2.8). Any interaction between the two types of tail could cause difficulties; we return to this in Sec. 4.5. Recent observational
studies by Wurm and Mammany (1972) and Wurm (1974) of the sub-
structure in dust emission in the inner coma seem to show that the
emission takes place only in sunward streams, and not in the
symmetric manner assumed in the theory. The authors claim that the
dust emission is always confined within a cone with its aperture
in the sunward direction. The comet studied for the 1972 paper was
in fact Comet Bennett. The 'inner solution' of Probstein has also
been criticised from a theoretical standpoint by Wallis (1974): the
flow is subject to heating processes not previously recognised,
which could affect the deductions made. It would be an interesting
exercise to apply the Finson-Probstein method to a series of plates
of the same comet taken at intervals of \( \sim 10 \) days, and test
critically whether the time-varying functional parameters 'overlap'
in a satisfactory way. (The 'recent history' of the comet is built
in to the tail as observed at a particular time).

There are two main divisions in the numerical implementation of
the Finson-Probstein analysis. The first involves the computation of
the system of syndynes and synchrones relating to the time of
observation, and the second comprises the integration procedure
referred to above. It is essential that the syndyne/synchrone system
be accurate at large distances away from the comet head, which rules
out the Bessel-Bredikhin theory and necessitates the determination
of the particle positions from a knowledge of the orbital elements
of individual particles.

Assuming the comet orbit to be parabolic, the position of the
nucleus in orbital coordinates \((\tau, \theta_c)\) may be found as in Sec. 2.2.
We use here the notation of Finson and Probstein \((\theta \equiv \nu, \) the true
anomaly, \(\tau \equiv \beta, \) the unknown in Barker's equation)
The particles are taken to be emitted with zero velocity. With a parabolic comet orbit and finite radiation pressure, the particle orbit is hyperbolic. Finson and Probstain consider values of \(1 - \mu < 1\), corresponding to diminished gravity and an orbit concave to the Sun. Four constants are required to specify the orbit: \(a_\lambda (\leq 0)\) the semi-major axis, \(e_\lambda (> 1)\) the eccentricity, \(\alpha_\lambda\) the orientation of perihelion with respect to the comet orbit and \(t_{\alpha_\lambda}\) the time of reaching perihelion. Expressions for these are given in terms of \(1/\mu\), \(q_c\) and the values \(\gamma_c\), \(\theta_c\) appropriate to the emission of the particle. The equation of the orbit is

\[
\gamma_\lambda(t) = \frac{q_\lambda(1 - e_\lambda^2)}{1 + e_\lambda \cos(\theta_\lambda - \alpha_\lambda)} .
\]

Kepler's equation for the hyperbola

\[
e_\lambda \sinh F - F = \sqrt{\frac{\mu GM_\lambda}{-a_\lambda^2}} (t - t_{\alpha_\lambda})
\]

is solved to get \(F(t)\), and

\[
\gamma_\lambda(t) = -a_\lambda (e_\lambda \cosh F - 1)
\]

where \(\theta_\lambda(t)\) may be found from

\[
\tan \frac{\theta_\lambda - \alpha_\lambda}{2} = \left(\frac{e_\lambda + 1}{e_\lambda - 1}\right)^{\frac{1}{2}} \tanh \frac{F}{2} .
\]

Finally,

\[
\xi = \gamma_c \cos(\theta_c - \theta_\lambda) - \gamma_c
\]

\[
\eta = \gamma_c \sin(\theta_c - \theta_\lambda)
\]

It is important to generalise the formulae to include \(1 - \mu > 1\), which might be the case for smaller emitted particles; these pursue hyperbolic orbits convex to the Sun. The two cases \(1 - \mu \leq 1\) correspond to the two branches of the standard hyperbola \(x = a \cosh F\), \(y = a \sinh F\). For \(1 - \mu > 1\) we take \(a_\lambda > 0\) (and \(e_\lambda > 1\)), and find that
\[ \gamma_d(t) = \frac{a_d(e_d^2 - 1)}{-1 + e_d \cos(\theta_d - \alpha_d)} \]

\[ e_d \sinh F + F = \sqrt{\frac{\mu GM_\odot}{a_d^3}} (t - t_d) \]

\[ \gamma_d(t) = a_d(e_d \cosh F + 1) \]

\[ \tan \frac{\theta_d - \alpha_d}{2} = \left(\frac{e_d - 1}{e_d + 1}\right)^{\frac{1}{2}} \frac{\tanh F}{2} \]

For the comet nucleus, differentiation respectively of the equations

\[ r = q_c (1 + z^2) \]

\[ \theta = 2 \tan^{-1} z \]

leads to

\[ \frac{dr}{dt} = \sqrt{\frac{GM_\odot}{2q_c}} \sin \theta_c \]

\[ \frac{d\theta}{dt} = \frac{1}{2} \sqrt{2q_c GM_\odot} \]

For the dust particle, differentiation of the equation

\[ r = a(e \cosh F + 1) \]

leads to

\[ \frac{dr}{dt} = \sqrt{\frac{GM_\odot}{a(e^2 - 1)}} e \sin(\theta - \alpha) \]

and differentiation of the equation

\[ -1 + e \cos(\theta - \alpha) = a(e^2 - 1) \]

yields

\[ \frac{d\theta}{dt} = \sqrt{\frac{GM_\odot}{a(e^2 - 1)}} \frac{1}{r} \]

Equating the quantities \( r, \theta, \frac{dr}{dt}, \frac{d\theta}{dt} \) for nucleus and particle at the time of emission \( t_c - \tau \), some reduction produces the following expressions for the dust particle elements:

\[ e_d = \sqrt{1 + \frac{4(1 + u)q_c}{\mu \tau_c}} \]

\[ a_d = \frac{\mu \tau_c}{2(1 + u)} \]

\[ \alpha_d = \theta_c - \sin^{-1}\left(\frac{\sin \theta_c}{\mu e_d}\right) \]

\[ t_{o_d} = (t_c - \tau) - \sqrt{\frac{a_d}{\mu GM_\odot}} (e_d \sinh F(t_c - \tau) + F(t_c - \tau)) \]

where

\[ F = 2 \tan^{-1}\left[\left(\frac{e_d + 1}{e_d - 1}\right)^{\frac{1}{2}} \frac{\tan \frac{\theta_c - \alpha_d}{2}}{2}\right] \]
To fix the correct quadrant for $\phi_c - \varphi_a$, we find

$$\sin (\phi_c - \varphi_a) = \frac{s \sin \phi_c}{\mu \omega a}$$
$$\cos (\phi_c - \varphi_a) = \frac{1}{\omega a} \left( \frac{2 r_c}{\mu \omega c} + 1 \right)$$

The case of $\mu = 1$ (particle under no force) cannot be directly treated by the procedures for $\mu < 1$ or $> 1$; it is clearly advantageous however to be able to include this in the general scheme.

The particle here moves in a straight line after emission, tangent to the orbit of the nucleus. We have for the dust particle

$$\gamma_d^2(t) = \gamma_c^2(t) + s^2 + 2 \gamma_c(t) s \cos \gamma$$

$$= \gamma_c^2 + s^2 + \frac{2 \gamma_c s z}{\sqrt{1 + z^2}}$$

where $s = \gamma_c(t) \cdot \tau = \left( \frac{2 \mu^2}{r_c(t)} \right)^{\frac{1}{2}} \tau$.

Also,

$$\sin \alpha = \frac{s \sin \gamma}{\gamma_d} = \frac{s}{\gamma_d \sqrt{1 + z^2}}$$

Thus the coordinates of the dust particle are given by

$$\gamma_d(t) = \sqrt{\gamma_c^2(t_e) + s^2 + 2 \gamma_c(t_e) s z / \sqrt{1 + z^2}}$$

$$\phi_d(t) = \phi_c(t_e) + \tan^{-1} \left( \frac{s / \gamma_d^2(t_e) (1 + z^2) - s^2}{1 - s \gamma_d(t_e) (1 + z^2)} \right)$$
where \( r_c = q_c (1 + z^2) \); \( z = \ell \cos \frac{\theta_c}{2} \).

The equation of the trajectory is

\[
\frac{2q_e}{r_a(t)} = \cos \left\{ \theta_a(t) - \theta_c(t) \right\} + \cos \theta_a(t).
\]

Actually, the angle \( \alpha \) may exceed 90°; \( \cos \alpha = \frac{r^2 + r_a^2 - s^2}{2r r_a} \) and \( \theta_A \) must be increased by \( \pi \) if \( \cos \alpha \) is negative).

Thus, given the orbital elements of the nucleus, the \((\ell, \eta)\) coordinates of a dust particle may be computed for a particular time of observation \( t_c \), having assigned the values of \( 1-\mu \) and the instant of emission \( t_c - \tau \). The two parameters \( 1-\mu \) and \( \tau \) may be varied so that a syndyne locus (\( 1-\mu \) fixed, \( \tau \) varying) or a synchrone locus (\( \tau \) fixed, \( 1-\mu \) varying) are obtained. The program given in Appendix 3 allows this to be done, and if desired, the curves may be produced directly on a graph plotter. The necessary data being available (Sec. 2.3), the projection \((M, N)\) of each point on the photographic plane may also be determined, so that the program provides a very convenient means of obtaining syndyne and synchrone configurations, in tabular and graphical form. These are referred to in Sec. 4.6.
4.5 Mixed tails and coupling

We have already noted that tails of the pure Type II variety are few in number; many more comets are of the 'mixed' kind, displaying dust and gas tails simultaneously. The term 'mixed' must be interpreted with care, for a comet may be mixed at certain times and pure Type I or II at others. One tail often persists while features of the other type appear intermittently: usually a persistent dust tail with occasional Type I features, e.g., Comet Arend-Roland. Comet Bennett had a strong dust tail, but prominent Type I features were seen over most of the period around perihelion (Sec. 2.8). This would certainly merit description as a mixed comet. The relative strengths (in an optical sense) of the two components, when present, are obviously an important factor in deciding whether a comet should be classified as mixed.

Comet Mrkos 1957d possessed strong gas and dust tails throughout its apparition, and is often cited as a standard example of the mixed category. At the time when interest in tail orientation was growing, it was noticed (Belton, Brandt and Hodge, 1963) that the Type I and Type II tails of Comet Mrkos were tangent at the head, i.e., they had a common value of $\epsilon$. This was in fact $\sim 5^\circ$, characteristic of the values generally found for Type I tails. The authors assumed (despite the result obtained by Liller mentioned below) that the dust particles are influenced by the interplanetary plasma (solar wind), so that the dust tail orientation was not that which would have resulted had the particles been subject only to solar gravity and radiation pressure. The interaction might, however, be with the ions comprising the Type I tail, which themselves interact with the solar wind. Alternatively, the dust particles might
not be involved in any such interaction, the common value of $\epsilon$ being coincidental. According to Brandt (1968), the phenomenon is widespread, though exceptions (e.g. Comet Perrine 1895c) are known where the respective orientations are widely different. Brandt also mentions that the tangency is particularly noticeable on plates obtained with short exposure. We suggest that this may be due to the fact that on such plates, the fainter, more diffuse part of the dust tail is suppressed, so that the (usually) brighter 'leading edge' of the dust tail is seen to coincide with the straight Type I tail near the head. With Comet Mrkos, the two tails apparently become de-coupled within 10 km of the nucleus. Presumably this means that they are then visually distinct. This separation of the dust and gas tails would occur in the absence of any coupling, due to the greater inherent curvature of the former.

On the theoretical side, Liller (1960) considered the coupling between cometary dust and the solar wind plasma. For gas-kinetic interaction only, along the relative velocity vector of the comet and the solar wind, the resulting acceleration of the particles was calculated as 0.03 cm per sec - negligible in comparison with that due to radiation pressure. (The assumed solar wind velocity of $10^3$ km per sec and density $10^3$ per cm$^3$ were furthermore much too high).

Belton (1965) and Notni (1966) considered coupling between dust particles and tail ions (taken to be O$^+$) on the basis of gas-kinetic and Coulomb drag, concluding that it could occur and be significant. However, the involved parameters relating to the ions and dust particles are very poorly known. In connection with the apparent decoupling, Belton points out that the cross-section for momentum transfer in Coulomb collisions is proportional to $V^+$ for the range of
velocities involved; as the ions accelerate, the momentum transferred
to the dust rapidly diminishes.

No significant advance seems to have been made subsequently on
the theoretical or observational study of this matter. Finson and
Probstein (1968) state that "there would appear to be little difficulty
with the resolution of the problem of mixed tail orientation once
more detailed estimates are available for the state of the plasma
component, the dust particle charge, magnetic field configuration, etc'.
It has not been established that any problem exists. The fact that
dust and gas tails have (to within observational error) the same
orientation, albeit in a number of cases, does not prove that the
dust tail is affected by the ion tail; the dust tail might have the
same orientation in the absence of the other. We return to this point
below.
4.6 Tail analysis by syndynes and synchrones: time trends

Syndyne curves have been used in the investigation of comet tails for many years, the fitting of such a curve giving some information on the particles in the tail (Sec. 4.2 and 4.3). Until comparatively recently, the application of synchrones has been limited to the modelling of certain tails as 'complete synchrones' and the rather doubtful interpretation of the streaks seen near the ends of a few dust tails as 'synchronic bands'. With the aid of automatic computers, accurate syndyne and synchrone curves may readily be constructed, and if desired, projected into the photographic plane (Appendix 3). The setting up of a syndyne-synchrone regime to cover the apparent field of an observed tail is the basis of the Finson-Probstein method of analysis. Fig. 28 shows syndynes and synchrones in the photographic plane for Comet Arend-Roland on 1957 April 27.8, produced by the graph plotter actuated by the program of Appendix 3. Some further curves are given in Appendix 5.

Though in the Finson-Probstein method the purpose of these diagrams is to form a basis for the integration stage, they can be extremely useful in their own right. In particular, a synchrone lying on the field of a tail shows that all the particles on the curve were emitted at a single (known) instant. An explanation of anomalous tails is immediately forthcoming. In Fig. 28, there is a concentration of 'older' synchrones near to the $\overline{+}M$ direction. The 'youngest' synchrones are also close together, but those of intermediate age are thinned out by the projection effect. This accounts for the apparent division of the tail into normal and anomalous parts. From the diagram, the approximate time of emission of the particles forming the antitail (found to precede the time of perihelion) may be deduced, and also an upper bound on the parameter
Fig. 28
for these particles. Any synchrone is intersected by consecutive syndynes, each characterised by a value of \( 1-\mu \). The sydyne passing through the observed limit of the tail indicates the maximum value of \( 1-\mu \). It is evident from Fig. 28 that an antitail of moderate length will have a value of \( 1-\mu \) much smaller than 0.1, i.e., the antitail is composed of comparatively large particles. These results seem to be true in general for anomalous tails. Sekanina (1974) has listed conditions, involving the use of synchrones, under which an antitail may be observed, and has expressed the view that this type of analysis may become standard practice for every dust comet. The program of Appendix 3 will be useful for the purpose. It is able to deal with any value of \( 1-\mu \) (Sec. 4.4).

Some properties of sydyne and synchrone curves may be recapitulated. Syndynes are always tangent to the \( \xi \)-axis at the origin (i.e., at the nucleus); they always exhibit appreciable curvature; distance along them (measured away from the origin) corresponds to an earlier time of emission, i.e., a greater value of \( \tau \). Synchrones are less curved than syndynes, and make a finite angle \( \epsilon \) with the \( \xi \)-axis at the origin; the significance of distance along them has been mentioned above. The angle \( \epsilon \) increases with the age of the synchrone. The time-behaviour of syndynes and synchrones has an obvious relevance to change of dust tail orientation angles with time. Finson and Probstin (1968) have plotted syndynes over a range of dates on either side of perihelion for a fixed value of \( 1-\mu \) (Comet Arend-Roland). The syndynes are of course all tangent to the \( \xi \)-axis at the origin, but the 'rate of bending' away from the \( \xi \)-axis is taken as an indication of general tail direction. This increases to a maximum at the time of perihelion, and diminishes afterwards.
A plot of the observed tail orientations of the comet shows the same trend: the scatter of points is large, but $\varepsilon$ clearly increases up to perihelion and diminishes afterwards.

The program of Appendix 3, which calculates the sydyne and synchrone loci rigorously, may be used to compute directly the initial orientation angles of synchrones ($\tan \varepsilon = \Delta \eta / \Delta \xi$ near the origin). For syndynes, a measure of the 'general tail direction' may be defined as $\tan \varepsilon = \eta / \xi$, where $\sqrt{\xi^2 + \eta^2} \approx \epsilon$, a chosen distance from the nucleus. $\epsilon$ is fixed here as $5.10^5$ km, a typical coma radius. Using Comet Arend-Roland as the standard example, the following time-changes have been derived, and are here shown diagrammatically.

**Synchrones** (i) Initial tail angle for synchrones emitted at a fixed time ($20^d$, $4^d$) before the date of observation.

![Fig. 29](image)
The curves in Fig. 29 are symmetrical, but not about perihelion; the earlier the emission time before observation, the later the occurrence of the maximum after perihelion.

(ii) Time variation of the initial tail angle for one particular synchrone: in contrast to (i), the development of the same synchrone is followed.
The curves in Fig. 30 show the well-known property of increase of $\varepsilon$ with time, this being most rapid about the time of perihelion. The increase however is not monotonic: all the curves have a maximum value of $\varepsilon$, followed by a slow decrease. The earlier the emission time, the greater the maximum value of $\varepsilon$ attained (this can exceed $90^\circ$).

**Syndynes**

Fig. 31 shows the 'general tail direction', as defined above, for syndynes of different $1-\mu$. The shape of the curves is similar to (i) above, but in this case they are all symmetrical about perihelion. Smaller chosen values of $\ell$ would of course result in generally greater values of $\varepsilon$. 
These orientation-time properties of the loci do not seem to have been given in full previously. The detailed Finson-Probstein analysis of dust tails applies to a single observation, all features of the tail on this date being accounted for by fixing the time-varying parameters up to the time of observation. Thus the orientation on any date is decided by the values of these functions over a finite time before that date. Prediction of future orientations would require extrapolation of the functions, which is risky. The general trend only of dust tail orientations can therefore be stated. This is perhaps best based on the synchrone orientations in Fig. 29. This diagram (for the $20^d$ synchrone) agrees quite well with the trend of the observations.

In Sec. 4.3, it was described how attempts to understand the orientation of distant comet tails by the fitting of syndynes was considered to be unsatisfactory. Sekanina (1973, 1974) has compared the observed orientations with computed synchrone orientations, and given diagrams of the variation of the computed position angle $\theta$ with time. The program of Appendix 4 was prepared to enable this procedure to be followed; however, the sweeping changes in $\theta$ apparent in the diagrams are mainly the result of the changing Earth-comet configuration, and diagrams of $\epsilon$ against time are more readily interpreted. Replacing Sekanina's diagram for Comet Haro-Chavira, we have Fig. 32. (The diagram for Comet Baade shows similar features). The plotted points (observations) show an increasing trend in $\epsilon$; the orientations of the synchrone for $500^d$ before perihelion produce a curve which is fairly central with respect to the points, and shows the expected increase of $\epsilon$ with time. Sekanina adds further curves for 200 and 2000 days before perihelion, which are (roughly) envelopes
of the plotted points, seeming to claim that what appear as error bounds are indicative of continuous particle emission between these times, with a 'cut-off' at 200°. This proceeding seems doubtful to the writer. However, the approximate measured length of the tails corresponds to particles larger than about 0.1 mm in diameter (ρ = 1), and Sekanina proposes that these may be clathrate grains. It may be noticed that the original syndyne fitting of Donn (1963) indicated particles of precisely this size: it is only because this particle size was unacceptable at the time that any 'problem of the distant comets' arose.

We return finally to the question of mixed tails. The time trend of the orientation of a Type I tail, based on an assumed (constant) solar wind velocity was described in Sec. 3.6. The general shape of the curve (Fig. 20) is very similar to the curves for synchrones and syndynes in Figs. 29 and 31. Type I and Type II tails present simultaneously thus tend to show the same kind of time variation in their orientations. Their tangency tends to be maintained over a range of time, making the supposition of a physical coupling between them somewhat less needful. With Comet Mrkos, the observed orientations, when plotted, show a wide scatter, but are not inconsistent with the curve of Fig. 20. On the other hand, dynamical aberration could not
possibly account for the trend in observed orientations of Comet Arend-Roland. Apart from the unacceptably low solar wind velocity of 60 km per sec required to bring the computed curve near the observations, the former is then very asymmetrical and no kind of fit is possible.

If actual coupling of tails exists, the application of the Finson-Probststein method in such cases can hardly be valid. A different initial orientation could be impressed on the dust tail before de-coupling occurs, and this would certainly affect the deductions made. Comet Bennett would seem to qualify as a case of mixed tails, but has been made the subject of a Finson-Probststein analysis (Sekanina and Miller, 1973). The value of $\epsilon$ implied by a diagram given by these authors is some $15^\circ$. This was on 1970 March 18, only 2$^d$ before perihelion, and is consistent with the values of $\sim 9^\circ$ obtained after perihelion (Sec.2.8), when $\epsilon$ is normally diminishing. The apparent common $\epsilon$ of the tails could be circumstantial, the finer dust particles producing the moderate values of $\epsilon$ observed. If so, this doubt of the validity of the Finson-Probststein analysis would be removed.
CHAPTER 5

CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK
5.1 Conclusions

The importance of a systematic treatment of tail observation has been demonstrated in Chapter 2; the new formulae enable the photographic plane - orbital plane transformation to be conveniently carried out and give an improved treatment of foreshortening factors, etc. Methods of defining tail orientation have been critically reviewed and the geometrical treatment of this quantity given special attention. Calculation of the error in the transformation to the orbital plane has been dealt with, and the effect of this error given due emphasis. As well as treating parabolic orbits, formulae for elliptic orbits have also been developed.

Perspective error in tail orientation due to the use of the tail bisection method has been investigated, with derivation of approximate analytical formulae. It is found that this error is generally negligible, thus supporting the use of this convenient method of measuring tail orientation.

The methods presented here have been applied in the study of Comet Bennett. In the course of this study, a rapid change in the angle \( \psi \) (P.A. of the negative velocity vector) was noted and explained, and a method suggested for using star trails for determining the P.A. of comet tails. The derived values of the tail orientation for Comet Bennett, while being too few to confirm any definite trend, are not inconsistent with the trend mentioned below. The observations are indicative of considerable fluctuation.
in the solar wind velocity.

Oscillation of Type I comet tails has been considered, and a comparison of the cases of Comet Burnham and Comet Halley lends some support to the view that the oscillations are due to an external cause, probably the effect of the solar wind.

On the assumption that Type I tail orientation is explained by dynamical aberration, formulae have been derived to show the expected time variation in the tail orientation as the comet moves in its orbit. Change of tail orientation with time has not received much attention hitherto: the value of this type of study has been stressed.

In the case of mechanical theory of Type II tails, the desirability of extending the accurate calculation of syndynes and synchrones for $1-\mu \gg 1$ has been pointed out, and this has been implemented. Tail analysis by means of syndynes and synchrones (particularly the latter) has been shown to be useful. Time trends in the initial orientation of these curves are derived, with discussion of the application to comets for which tail orientation data are available. The general similarity of time trends for Type I and Type II tails has not been previously emphasised.

The problem of mixed tails and possible coupling has been re-examined with particular reference to time changes in tail orientation. The reality of the coupling is questioned, and implications for certain Finson-Probststein analyses of dust tails are pointed out.
5.2 Suggestions for further work

(i) The analysis of cometary observations for tail orientation is of continuing importance. The procedures and program described here will facilitate this work. Any cometary plate on which a tail is sufficiently distinct should be analysed to add to the orientation data already accumulated. Long runs of observations on one comet are specially valuable. Plates taken at intervals of a day or so are needed to detect tail oscillations; if these are confirmed in a particular comet, exposures at intervals of a few hours are very desirable.

(ii) Addition of the integration stage to the sydyne/synchrone program already developed would enable Finson-Probstein analyses to be performed if suitable red-sensitive plates of dust comets become available. There is a great need for more applications of this theory to be made.

(iii) It has been shown that the assumption of a flat tail lying in the orbital plane is a well-justified one. However, the effect on the geometrical formulae of a tail having appreciable 'thicknes' might be investigated, together with the consequences of departures from the orbital plane.

(iv) It would be of interest to investigate the effect on the Finson-Probstein theory of non-uniform dust emission from the nucleus, and the effect of a hypothetical coupling of the dust tail to an accompanying gas tail.
(v) Studies of the time variation in tail orientation should be extended, particularly in the case of mixed tails as a potential source of information on possible coupling between the tails.

(vi) Since dynamical analysis of Type II comet tails cannot yield information on the particle density $\rho$ and diameter $d$ independently, observational techniques that can do this are particularly valuable. The only existing observations are from scattering theory and the infra-red observation of silicate spectral features.

(vii) A more thorough treatment of light scattering by comet tails is desirable.
APPENDICES

General

Automatic computing is indispensable in performing the calculations described in this work. The accurate calculation of syndynes and synchrones (Sec. 4.4) in particular would be quite impracticable by hand.

The programs given here are written in FORTRAN, and have been developed and run on an IBM 1130 computer. Two of the programs may actuate a CALCOMP graph plotter if desired, and so produce graphical results directly. Design and details of the programs will not be discussed here, but mention may be made of the following: (i) in all the programs, Barker's equation and/or Kepler's equation in one of its forms must be solved. The former is a cubic, and the latter a transcendental equation. Each is solved efficiently by the Newton-Raphson iteration method. (ii) A pitfall in the preparation of the programs was the placing of calculated angles in the wrong quadrant. Special attention is called for on this point.

Appendix 1 (p. 174)

Geometrical analysis for comet observations: all relevant geometrical quantities are computed for a parabolic or elliptic orbit, including the reduction of observed tail orientation and errors involved. Dynamical aberration is also treated.

Appendix 2 (p. 181)

A program to compute the time change in orientation angle on the basis of dynamical aberration, for a comet with a parabolic orbit.
Appendix 3 (p. 182)

This program computes the forms of syndynes and synchrones for a comet with a parabolic orbit, in either the orbital plane ($\xi, \eta$) or the photographic plane ($M, N$), and for any value of $1-M^2$. Output may be tabular and/or graphical.

Appendix 4 (p. 187)

An extended version of Appendix 3, which computes the initial orientation angles of synchrones in the ($\xi, \eta$) and ($M, N$) planes, giving also the position angle $\theta$ on the plane of the sky.
APPENDIX 1

PAGE 1

// JG B T.

LOG DRIVE CART SPEC CART AVAL PHY DRIVE
GOOD 0100 0100 0000

V2 MOYA ACTUAL 16K CONFIG 16K

// FOR
* LIST SOURCE PROGRAM
* ARITHMETIC TRACE
* EXTENDED PRECISION
* I/OCS (CARD,1132:PRINTER,TYPewriter,KEYBOARD,DISK)
* I/OCS (CARD,1403:PRINTER)

REAL LE, LH, LB, LEP, LHP, LD, LAM1, LAM2, LAM3, LAMP, LAM2P, LAM3P
REAL LG, LGEP, M, N
10 FORMAT(F9.4,5X,3(F9.5,5X),F9.6)
11 FORMAT(1,10X,F9.4,5X,3(F9.5,5X),F9.6)
12 FORMAT(A2,5X,F9.4,5X,F5.1)
13 FORMAT(3(F15.7,1X,F7.6,1X,F6.0,1X))
14 FORMAT(1,10X,A2,5X,F9.4,5X,F5.1,12,2X,12,2X,12,2X,F4.1,5X,13,2X,F4.1,5X,
1F1G.6)
15 FORMAT(1,10X,F5.2,3X,F5.1,3X,2(F6.4,3X),F5.2,5(3X,F5.1),5X,F6.4)
16 FORMAT(1)
17 FORMAT(3X,F9.4,5X,3(F9.5,5X),F9.6,5X,F7.5)
18 FORMAT(10X,F9.4,5X,3(F9.5,5X),F9.6,5X,F7.5)
20 FORMAT(10X,F9.4,5X,3(F9.5,5X),F9.6,5X,F7.5)

C1=5.29577995
C2=0.1745129252
P1=3.1415926536
P2=6.2831853072
SEPS=0.39788118
CEPS=0.91743695
CPH=C2*74.36666667
PSC=C2*7.25
SPH=SIN(PHC)
CPS=COS(PSC)

C1=C2*74.36666667
C2=C2*7.25

100 PAUSE 3
CALL DATSW(10,IP1)
READ(2,16) NO
IF(IP1-1)90,90,91
90 READ(2,19) T,MSD,OPCD,RI,ECC
WRITE(5,20)T,MSD,OPCD,RI,ECC
AX=Q/(1.-FCE)
M=SQRT(0.97142264/AX**3)
GO TO 92
91 READ(2,10) T,MSD,OPCD,RI,ECC
WRITE(5,11) T,MSD,OPCD,RI,ECC
92 QMS=C2*QMS
CMC = C2 * NMC
I = C2 * R1
SOMC = SIN(OMC)
COMC = COS(OMC)
SMC = SIN(MC)
COMC = COS(OMC)
SI = SIN(I)
CI = COS(I)
EX1 = SOMC * SOMC
EX2 = SOMC * COMC
PZ1 = SOMC * SI
EX4 = COMC * SOMC
EX5 = COMC * COMC
QZ1 = COMC * SI
QY1 = EX5 * CI - EX1
PY1 = EX2 * CI + EX4
PX1 = EX5 - EX1 * CI
P = PX1
QX1 = EX2 - EX4 * CI
QX = QX1
PY = PY1 * CEP5 - PZ1 * SEPS
QY = QY1 * CEP5 - QZ1 * SEPS
PZ = PZ1 * CEP5 + PY1 * SEPS
QZ = QZ1 * CEP5 + QY1 * SEPS
RX = PY * CZ - PZ * QY
RY = PZ * QX - PX * QZ
RZ = PX * QY - PY * QX
RX1 = RX
RY1 = -COMC * SI
RZ1 = CI
PXH = CPH
PYH = -SPH * CPS
PZH = SPH * SPH
QXH = SPH
QYH = CPH * CPS
QZH = CPH * SPH
RXH = 0.0
RYH = SPH
RZH = CPS
D0 = 50 J = 1, N0
READ(2, 121) NBN, T0BS, THETA
READ(2, 131) XC0, XD1, XD2, YC0, YD1, YD2, ZC0, ZD1, ZD2
TSUN = T0BS - 0.5
LI = TSUM
P = TSUN - L1
B2 = P * (P - 1) * 0.25
XC = XCG + (P * XD1 + B2 * XD2) * 1.0E-7
YC = YCG + (P * YD1 + B2 * YD2) * 1.0E-7
ZC = ZCG + (P * ZD1 + B2 * ZD2) * 1.0E-7
TM = T0BS - T
IF(IP1 = 1) 62, 62, 93
62 M = 1 * FMT * C2
E = 0.0
94 E1 = E - (E - M - ECC * SIN(E)) / (1.0 - ECC * COS(E))
IF(ABS(E1 - E) < 1.0E-95, 95, 96
96 E = E1
GO TO 94
95 R = AX * (1.0 - ECC * COS(E))
BETA = SQRT((1.0 + ECC) / (1.0 - ECC)) * SIN(0.5 * E) / COS(0.5 * E)
SINE = SIN(E)
C^SE=C^SE(E)
B=Sqrt(Ax*Ax*(1.-ECC**2))
RP=Ax*COS-ECC)
RC=Ax*SME
G0 TO 97
93 BETA=0.0
AK=(RK*TMT)/(RT2*0**1.5)
51 BETA=BETA-(BETA-3+3*BETA-AK)/3*(BETA**2+1))
IF (ABS(BETA-BETA)-1.E-9)52,52,53
53 BETA=BETA1
G0 TO 51
52 R=Q*(1+BETA**2)
RP=Q*(1-BETA**2)
RC=Q*BETA
TA=2*ATAN(BETA)
97 TA=2.*ATAN(BETA)
TAP=C1*TA
XE=PX1*RP+PX1*RP
YE=PY1*RP+PY1*RP
ZE=PX1*RP+PX1*RP
XI=PX1*X+XH+YE
YH=PX1*XH+XH+YE+RY+ZE
ZH=PX1*XH+XH+YE+RY+ZE
XB=PX1*XH+XH+YE
YB=PY1*ZH+QY+YB
ZB=ZH
BE=ATAN2(XE/SQRT(XE**2+YE**2))
LE=ANGLE(YE,XE)
BH=ATAN2(ZH/SQRT(XH**2+YH**2))
LI=A*GLC(YH,XH)
LB=ATAN2(ZB/SQRT(XB**2+YB**2))
LB=A*GLC(YB,XB)
BEP=C1*EB
LEP=C1*EL
BHP=C1*BH
LHP=C1*HL
BBP=C1*B
LBBP=C1*LBB
XS=PX*RP+PX*RP
YS=PY*RP+PY*RP
ZS=PX*RP+PX*RP
XI=XS*XG
ETA=YS*YG
ZETA=ZS*ZG
YHC=SQR(TX**2+ETA**2+ZETA**2)
PS=ATAN4(ABS(ETA/XI))
IF (ETA)134,134,135
134 ACAL=PI
G0 TO 141
135 ACAL=0.
G0 TO 141
131 IF (XI)136,141,138
136 ACAL=PI+PS
G0 TO 141
138 ACAL=PI-PS
G0 TO 141
133 IF (XI)139,140,140
139 ACAL=PI-PS
G0 TO 141
PAG E 140 ACAL=PS
141 EX=ZETA/RHO
DCAL=ATAN(EX/SORT(1-EX**2))
RA=ACAL*3.197186344
IRA=RA
RMT=(RA-IRA)*60.
IMT=RMT
RST=(RMT-IMT)*60.+0.05
ODELTO=DCAL*C1
IDEL-DFLTO
RM=ABS((DELT-DEL)*60.1+0.05
C ASPECT DATA
XEA=-XC
YEA=-YC
ZEA=-ZC
UEA=XEA
VEA=YeA*SEPS+SEPS*ZEA
XEC=PX*UEA+PY*VEA+PZ*ZEA
YEC=PX*UEA+PY*VEA+PZ*ZEA
ZEC=PX*UEA+PY*VEA+PZ*ZEA
LAM1=ATAN(ZE'/SQRT(RH**2-ZE'^*2))
TAE=ATAN(ABS(XE'/XEO))
IF(YEO)160,161,161
161 IF(XEO)162,163,163
162 TAE=PI-TAE
GO TO 163
160 IF(XEO)164,165,165
164 TAE=PI
GO TO 163
165 TAE=-TAE
163 TAE=C1*TAE
LAM2=TAE-1A
RE=SORT(XC*+2+YC*+2+ZC*+2)
LAM3=ATAN(ZE'/SQRT(RE**2-ZE'^*2))
LAM1P=C1*LAM1
LAM2P=C1*LAM2
LAM3P=C1*LAM3
LGE=ANGLE(VEA,UEA)
LDEP=C1*LGE
CPHA=(RH**2+R**2-RE**2)/(2*R*RHO)
SPHA=SORT((1-CPHA**2))
PHA=C1*ANGLE(SPHA,CPHA)
C RADIUS VECTOR
SIMA=SIN(ACAL)
COSA=COS(ACAL)
SIMD=SIN(DCAL)
COSD=COS(DCAL)
CRV=SIMA*XS+COSA*YS
DRV=COSA*SIMD*XS-SIMA*SIMD*YS+COSD*ZS
THI=Crv/Drv
PHI=ANGLE(CRV,Drv)
C NEGATIVE VELOCITY VECTOR
IF(IP1=1)70,70,71
70 VP=AX*SIME
VG=-B*COSG
GO TO 72
71 VP=0*Beta
VG=0
72 VX=PX*VP+QX*VO
$V_Y = PY + V_P + Q_Y + V_Q$
$V_Z = P_Z + V_P + Q_Z + V_Q$
$AVV = SINA + VX + COSA + V_Y$
$DVV = COSA + SIM + VX - SINA + SIM + V_Y + COSD + V_Z$
$TPS = AVV / BVV$
$PSI = \text{ANGLE}(AVV, BVV)$
$THETC = C_2 + \text{THETA}$
$THH = \text{SIM}((THETC) / \text{COS}(THETC))$
$A = (AVV - BVV * TTH) / (DRV * TTH - CRV)$
$IF (IPI = 1) 73, 73, 74$

$H = A_1 + SQRT((1 - ECC * COSE) / (1 + ECC * COSE))$
$VELP = 365.25 * RK * SIN((SQRT(D)) * (1 + ECC * COSE))$
$VELQ = 365.25 * RK * COS((SQRT(D)) * (1 - ECC * COSE))$
$V = SQRT((2 / (2 - 1 + (AX)^2))$
$GAM = ATAN((1 + ECC * COS(TA)) / (ECC * SIM(TA)))$

$G = \text{SQRT}(1 + (BETA)^2)$
$VELC = 365.23 * RK / (SQRT((AX)^3) * (1 + ECC + SE))$
$VELP = -BETA * VELQ$
$V = \text{SQRT}((1773.06 / (\pi / BETA))$

$75 IF (GAM) 80, 81, 81$
$80 \text{GAM} = \text{GAM} + \text{EPSI}^*$
$TEPS = \text{SP' (GAM)} / (H - \text{DP})$
$EPSN = \text{ATAH (TEPS)}$
$W = V^H$

$VRA = V^S + V^S (GAM) * W$
$VTRA = V^S + V^S (GAM) * W$
$H = SIM (GAM + EPSI^* / SIM (EPSN))$
$GAM = C_1 * GAM$
$EPSN = C_1 * EPSN$
$PHI = C_1 * PHI$
$PSI = C_1 * PSI$
$VELX = PX + VELP + QX + VELQ$
$VELY = PY + VELP + QY + VELQ$
$VELZ = PZ + VELP + QZ + VELQ$
$VK = 3.74554 * SQRT((VELX**2 + VELY**2 + VELZ**2))$
$TVP = COS(TA - EPSN)$
$TVQ = SIM(TA - EPSN)$
$TVX = PX * TVP + UX * TVQ$
$TVY = PY * TVP + QX * TVQ$
$TVZ = PZ * TVP + QZ * TVQ$

$FA = \text{SPH (TPH - TPSI) / LCOS (THETC) ** 2} * (TTH - TPSI) * (TPHI - TTH)$
$DWIDTH = WFAMG$
$DETH = FANG / (LCOS(GAM) + EPSN) / SIN(GAM + EPSN) / COS(EPSN) / SIN(EPSN)$
$DWT = ABS(C2 - DETH) / 0.5$
$DETH = ABS(DWT) / 0.05$
$WRITE (5, 14) C000, C000, IRA, INT, RST, IDEL, RMA, BETA$
$WRITE (5, 15) H1, H2, H3, V, GAM, EPSN, THETA, PHI, PSI, PHA$
$WRITE (5, 17) BPM, BPM, BPM, BPM, BPM, BPM, BPM, BPM, BPM, BPM$
$110, AG, PHA$

$WRITE (5, 18) DWT, DETH, VRA, VTRA$
$XIC = XEP * COS(TA) + YEP * SIN(TA) - R$
$ETAC = XEP * SIM(TA) - YEP * COS(TA)$
$ZETAC = ZEN$

$KHS = SQRT((XIC**2 + ZETAC**2 + ZETAC**2) / (XIC**2 + ZETAC**2 + ZETAC**2))$

$ALC = C1 + ATAM(RH - ZETAC) / (ETAC**2 + ZETAC**2 + ZETAC**2) / (TPS - XIC + ETAC)$

$ALCV = C1 + ATAM(RH - ZETAC) / (ETAC**2 + ZETAC**2 + ZETAC**2) / (BETA * XIC + ETAC)$

50 CONTINUE
GO TO 100
101 CONTINUE
CALL EXIT
END

UNREFERENCED STATEMENTS
101

FEATURES SUPPORTED
ARITHMETIC TRACE
EXTENDED PRECISION
I/OCS

CORE REQUIREMENTS FOR
COMMON 0 VARIABLES 690 PROGRAM 3062

END OF COMPILATION
// XEQ
(Subroutine ANGLE)

SUBROUTINE ANGLE

// JOB

LOG DRIVE  CART SPEC  CART AVAIL  PHY DRIVE
0000  0100  0100  0000

V2 MOS A ACTUAL 16K CONFIG 16K

// FOR

*LIST SOURCE PROGRAM
*EXTENDED PRECISION

FUNCTION ANGLE(S,C)
ANGLE=ATAN(S/C)
IF(S) 80,81,81
  80 IF(C) 82,85,85
  81 IF(C) 82,93,93
  85 ANGLE=ANGLE+6.2831853072
  GO TO 93
  82 ANGLE=ANGLE+3.1415926536
  93 RETURN
END

FEATURES SUPPORTED
EXTENDED PRECISION

CORE REQUIREMENTS FOR ANGLE
COMM'''' 6 VARIABLES 6 PROGRAM 64

RELATIVE ENTRY POINT ADDRESS IS 0000 (HEX)

END OF COMPILATION

// DUP

*DELETE ANGLE
CART ID 0100  DB ADDR 548C  DB CNT 0006

*STORE WS UA ANGLE
CART ID 0100  DB ADDR 548C  DB CNT 0006
APPENDIX 2

PAGE 1

// JOB T

LOG DRIVE CART SPEC CART AVAIL PHY DRIVE
0000 0100 0100 0000

V2 M09A ACTUAL 16K CONFIG 16K

// FOR

*LIST SOURCE PROGRAM
*EXTENDED PRECISION
*ARITHMETIC TRACE
*IOCS(CARD,1403 Printer)

20 FORMAT(I3,4X,F4.1,2X,F6.1)
21 FORMAT(F5.1,3X,F7.5)
22 FORMAT(10X,F5.1,5X,F7.5,10X,F6.1,5X,F6.2)

RK=0.017202099
RT2=1.41421336

11 READ(2,21)N,Q
READ(2,20)NO,DT,T
W1=C.0005775*H
EX=W1*SQRT(0.3*Q)/RK
TA=0.0

DO 10 J=1,N
A=(RK*T)/(RT2*Q**1.5)

52 TAI=TA-(TA**3+3*(TA-A1))/(3*(TA**2+1))
IFI_ABS(TAI-TA)-1.E-8)30,51,51

51 TAI=TA1
GO TO 52

50 EPS=ATAN(1./(EX*(1.+TA**2)-TA))+57.29577951
WRITE(5,22)W,Q,T,EPS

10 T=T+DT
PAUSE
GO TO 11

12 CALL EXIT
END

UNREFERENCED STATEMENTS

12

FEATURES SUPPORTED
ARITHMETIC TRACE
EXTENDED PRECISION
IOC S

CORE REQUIREMENTS FOR
COMMON 0 VARIABLES 52 PROGRAM 272

END OF COMPILATION

// XEQ
APPENDIX 3

PAGE 1

> JOB T

LOG DRIVE CART SPEC CART AVAIL PHY DRIVE
0000 0100 0100 0000

V2 M09A ACTUAL 16K CONFIG 16K

// FOR
*LST SOURCE PROGRAM
*ONE WORD INTEGERS
*ARITHMETIC TRACE
*EXTENDED PRECISION
*IDCS (CARD, TYPEWRITER, KEYBOARD, 1403 PRINTER, DISK)

REAL T, MCM, MCMK, MCMKM, NCMK
80 FORMAT(F9.7, 3X, F10.5)
81 FORMAT(F10.5)
82 FORMAT(F4.2)
83 FORMAT(F9.2)
84 FORMAT(10X, F5.3, 5X, F6.3, 5X, 2(F9.5, 3X), 2X, 2(F9.5, 3X))
85 FORMAT(///)
86 FORMAT(F10.5, 5X, 3(F9.5, 5X), F9.6)
87 FORMAT(3(F10.7, 1X, F7.0, 1X, F6.0, 1X))

RK = 0.017202099
RT2 = 1.41421356
CVF = 14.9674
PF = 2.0
C2 = 0.1745329252
SEPS = 0.39708118
CEPS = 0.91743695
READ(2, 86) T, OMS0, OMSD, RI, QC
101 READ(2, 81) TCJ
READ(2, 61) XCO, XD1, XD2, YCO, YD1, YD2, ZCO, ZD1, ZD2
OMS = C2 * OMS0
MCM = C2 * MCMD
I = C2 * RI
SOMS = SIN(OMS)
CMCM = COS(OMS)
SOMC = SIN(MCM)
CMCM = COS(MCM)
SI = SIN(I
CI = COS(I
PX1 = OMS * COSC + SOMS * SOMC * CI
PY1 = OMS * COSC + SOMS * CMCM * CI
PZ1 = OMS * SI
QX1 = SOMS * CMCM - OMS * SOMC * CI
QY1 = SOMS * SOMC + OMS * CMCM * CI
QZ1 = OMS * SI
RX1 = SOMC * SI
RY1 = CMCM * SI
RZ1 = CI
CALL DATSW(10, IP10)
IF(IP1C - 1) 45, 45, 46
45 CALL PLOT(0., -13., -3)
CALL PLOT(0., 5., 5., -3)
CALL PLOT(0., -4., 3)
CALL PLOT(0., 4., 2)
CALL PLOT(0., 0., 3)
CALL PLOT(12., 0., 3)
CALL PLOT(0., 0., 3)
46 TC = TCJ - T
$$TSUN=TCJ-0.5$$
$$L1=TSUM$$
$$P=TSUM-L1$$
$$B2=P*(P-1)*0.25$$
$$XC=XC0+(P*XD1+B2*XD2)*1.E-7$$
$$YC=YC0+(P*YD1+B2*YD2)*1.E-7$$
$$ZC=ZC0+(P*ZD1+B2*ZD2)*1.E-7$$
$$XEA=-XC$$
$$YEA=-YC$$
$$ZEA=-ZC$$
$$UEA=XEA$$
$$VEA=CEPS*YEA+SEPS*ZEA$$
$$WEA=-SFPS*YEA+CEPS*ZEA$$
$$Xe^=PX1*UEA+PY1*VEA+PZ1*WEA$$
$$YEH=QX1*UEA+CY1*VEA+GZ1*WEA$$
$$ZEH=RX1*UEA+RY1*VEA+KZ1*WEA$$
$$Z=0.0$$
$$AK=(RK*TC)/(RT2*QC**1.5)$$
$$Z1=Z-(Z**3+3*(Z-\text{AK}))/(3*(Z**2+1))$$
$$\text{IF}(ABS}(Z1-Z)-1.0-8) 52,57,53$$
$$Z=Z1$$

10  \text{GO TO 51}$$
52  RC=QC*(1.0+Z**2)$$
\text{THC=2.0*ATAN}(Z)$$
\text{PAUSE 7}$$
$$X=XC*(\text{COS}(\text{THC})+\text{SFAC}(\text{THC})-RC$$
$$ETAC=XED*\text{SIN}(\text{THC})-YED*\text{COS}(\text{THC})$$
$$ZETAC=ZEED$$
$$DELT=\text{SQRT}(X**2+ETAC**2+ZETAC**2)$$
\text{PAUSE 7}$$
$$FX1=\text{SQRT}(ETAC**2+ZETAC**2)$$
$$\text{TAU}=0.0$$
$$\text{MU}=0.0$$
$$J=M=1$$
\text{CALL DATSW(1,IP1)}$30  \text{IF}(IP1-1) 10,10,11$$
11  \text{READ}(6,83) \text{MU}$$
\text{READ}(6,83) TD$$
\text{GO TO 24}$$
10  \text{READ}(6,83) TAU$$
\text{READ}(6,83) DMU$$
24  \text{WRITE(5,85)}$$
\text{IF}(IP10-1) 47,47,48$$
47  \text{CALL PLOT(0.,0.,3)}$$
48  \text{IF}(IP1-1) 12,12,13$$
13  \text{TAU}=\text{TAU+TD}$$
\text{GO TO 15}$$
12  \text{MVU}+0.0\text{DMU}$$
15  \text{TE}=TC-\text{TAU}$$
Z=0.0$$
AK=(RK*TE)/(RT2*QC**1.5)$$
61  Z1=Z-(Z**3+3*(Z-\text{AK}))/(3*(Z**2+1))$$
\text{IF}(ABS}(Z1-Z)-1.0-8) 62,62,63$$
63  Z=Z1$$
\text{GO TO 61}$$
62  RC1=QC*(1.0+Z**2)$$
\text{THC1}=2.0*\text{ATAN}(Z)$$
RMU=1.0\text{DMU}$$
\text{IF}(ABS}(RMU)-1.0-2) 110,110,111$$
111  \text{IF}(\text{RMU}) 112,100,113
PAGE 3

110 D=SQRT(2*RC1**2/RCl)*TAU
RD=SQRT((RCl**2+D**2+2*RCl*D)/SQRT(1+Z**2))
SIMPS=RD/SQRT(RCl**2+0**2+2*RCl*D*Z/SQRT(1+Z**2))
CCPS=(RD**2+RCl**2-0**2)/2*RD*RCl
PS=ATAN(SINPS/CCPS)
IF(CCPS)292,293,293
PS=PS+3.1415926536
THD=THCl+PS
IF(THD)290,295,295
TEST=2*QC/RD-CCPS(THID-THCl)-CCPS(THD)
GO TO 200

112 ED=SQRT((1+4*RMU*QC/(RC1*RMU**2))
RMU=ABS(RMU)
AD=RMU/RCl/(2*OMU)
SAN=SIM(THCl)/(RMU*ED)
CA=(2*QC/(RMU*RCl)+1)/ED
AN=ATAN(SAN/CAN)
IF(SAM)190,190,191
190 IF(CAN)192,195,195
191 IF(CAM)192,193,193
193 AN=AN+6.283185307
GO TO 193

192 AN=AN+3.1415926536

193 ALD=THCl-AN
EX1=SQRT(((ED+1)/(ED-1))*{SIN(0.5*AN)/COS(0.5*AN)}
FD=ALOG((1+EX1)/(1-EX1))
HYP=SQRT((AD**3/RMU))/RK
THD=TE-HYP*(ED*0.5*{EXP(FD)-EXP(-FD)})+FD
HYP1=(1./HYP)*(FX-TC)
F=0.0
172 FL={ED*0.5*{EXP(F)-EXP(-F)}-F-HYP1}/(ED*0.5*{EXP(F)+EXP(-F)}+1.)
IF(ABS(F1-F))170,170,171
171 F=F1
GO TO 172

170 RD=AD*(ED*0.5*{EXP(F)-EXP(-F)}+1.)
DIFF=ED*0.5*{EXP(F)-EXP(-F)}-F-HYP1
EX2=EXP(0.5*F)
EX3=EXP(-0.5*F)
THD=ALD+2*ATAN(SQRT((FD-1)/(ED+1))*{EX2-EX3}/(EX2+EX3))
GO TO 200

113 ED=SQRT((1+4*RMU*QC/(RC1*RMU**2))
AD=RMU/RCl/(2*OMU)
SAN=SIM(THCl)/(RMU*ED)
CA=(2*QC/(RMU*RCl)-1)/ED
AN=ATAN(SAN/CAN)
IF(SAM)90,91,91
90 IF(CAN)92,95,95
91 IF(CAM)92,93,93
95 AN=AN+6.283185307
GO TO 93
92 AN=AN+3.1415926536

93 ALD=THCl-AN
EX1=SQRT(((ED-1)/(ED+1))*{SIN(0.5*AN)/COS(0.5*AN)}
FD=ALOG((1+EX1)/(1-EX1))
HYP=SQRT(-AD**3/RMU)/RK
THD=TE-HYP*(ED*0.5*{EXP(FD)-EXP(-FD)})-FD
HYP1=(1./HYP)*(FX-TC)
F=0.0
72 FL={ED*0.5*{EXP(F)+EXP(-F)}-F-HYP1}/(ED*0.5*{EXP(F)+EXP(-F)}+1.)
IF(ABS(F1-F))70,70,71
71 F=F1
GO TO 72

70 RD=ALD*(ED*0.5*(EXP(F)+EXP(-F))-1.)
DIFF=ED*0.5*(EXP(F)-EXP(-F))-F-HYPL
EX2=EXP(0.5*F)
EX3=EXP(-0.5*F)
THD=ALD+2*ATAM(SQRT((ED+1)/(ED-1))*((EX2-EX3)/(EX2+EX3)))

200 XI=RD*COS(THC-THD)-RC
ETA=RD*SINTM(THC-THD)
FX2=DELT**2-XIC*XI-ETAC*ETA
MC=M=DELT*(FX1*XI-XIC*ETAC*ETA/FX1)/FX2
MCN=DELT**2*ZETAC+ETA/(FX1+FX2)
MCNKM=CVF*XI
ETAKM=CVF*ETA
MCNKM=CVF*MC
CALL DATSW(10, IP10)
CALL DATSW(11, IP11)
IF(IP10-1) 41, 41, 42
41 CALL DATSW(6, IP6)
IF(IP6-1) 73, 73, 74
73 PP1=PF*NCMKM
PP2=PF*MC
CLE 75
74 PP1=PF*ETAKM
PP2=PF*XICKM
75 CALL PLOT(PP1, PP2, 2)
42 IF(IP1-1) 43, 43, 44
43 WRITE(5, 84) NMU, TAU, XIKM, ETAKM, MCNKM, NCNKM
44 CONTINUE
IF(IP1-1) 16, 16, 17
17 CALL DATSW(2, IP2)
IF(IP2-1) 18, 18, 13
16 CALL DATSW(4, IP6)
IF(IP4-1) 18, 18, 12
18 IF(IP1-1) 19, 19, 20
20 PAUSE 3
CALL DATSW(3, IP3)
IF(IP3-1) 21, 21, 22
22 TAU=0.0
GO TO 11
19 PAUSE 5
CALL DATSW(5, IP5)
IF(IP5-1) 21, 21, 23
23 NMU=0.0
GO TO 10
21 IF(JN) 31, 31, 32
31 J=N0
IF(IP1-1) 28, 28, 29
29 IP1=1
GO TO 33
28 IP1=2
33 PAUSE 7
TAU=0.0
DMU=0.0
GO TO 30
31 PAUSE 15
IF(IP10-1) 295, 295, 101
295 CALL PLOT(15, 0, -3)
GO TO 101
100 CALL EXIT
END

UNREFERENCED STATEMENTS
80 82

FEATURES SUPPORTED
ARITHMETIC TRACE
ONE WORD INTEGERS
EXTENDED PRECISION
IOCS

CORE REQUIREMENTS FOR
COMMON O VARIABLES 358 PROGRAM 2504

END OF COMPILATION
// XEQ
APPENDIX 4

PAGE 1

// JOB T

LOG DRIVE CART SPEC CART AVAIL PHY DRIVE
0000 0100 0100 0000

V2 M09A ACTUAL 16K CONFIG 16K

// FOR
*LIST SOURCE PROGRAM
*ONE WORD INTEGERS
*ARITHMETIC TRACE
*EXTENDED PRECISION
*IMCS (CARD,TYPewriter,KEYBOARD,1403 PRINTER,DISK)

REAL 1,MC,MC1,MC2

80 FORMAT(F9.7,3X,F10.5)
81 FORMAT(10.5)
82 FORMAT(F4.2)
83 FORMAT(F8.3)
84 FORMAT(7X,F9.5,3X,2(F9.5,3X),2X,2(F9.5,3X),3X,4(F8.3,1X))
85 FORMAT(///)
86 FORMAT(F10.5,5X,3(F9.5,5X),F9.6)
87 FORMAT(3(F10.7,1X,F7.0,1X,F6.0,1X))

RX=0.017202099
RT2=1.41421356
CVF=14.9674
PF=2.0
CI=57.29577951
C2=0.01745329252
PI=3.1415926536
PZ=6.2831853072
SEPS=0.39788118
CEPS=0.1736959

READ(2,86) T,OMSD,OMCD,RI,UC

101 READ(2,81) TCJ

READ(2,81) XCO,XD1,XD2,YCO,YD1,YD2,ZCO,ZD1,ZD2

OMS=C2*OMSD
OMG=C2*OMCD
I=C2*RI
SOMS=SIN(OMS)
COMS=COS(OMS)
SOMC=SIN(OMC)
COMC=COS(OMC)
SI=SIN(I)
CI=COS(I)

PX1=COMS*COMC-SOMS*SOMC*CI
PY1=COMS*COMC+SOMS*COSC*CI
PZ1=SOMS*SI
QX1=-SOMS*COMC-COMS*SOMC*CI
QY1=-SOMS*COMC+COSC*SOMC*CI
QZ1=COMS*SI
RX1=SOMC*SI
RY1=-COMC*SI
RZ1=CI

PX=PX1
PY=PY1*CEPS-PZ1*SEPS
PZ=PZ1*CEPS+PY1*SEPS
QX=QX1
QY=QY1*CEPS-QZ1*SEPS
QZ=QZ1*CEPS+QY1*SEPS

CALL DATSW(10,IP10)
IF (PIO-1) 45,45,46
45 CALL PLOT(0.,-13.,-3)
CALL PLOT(0.,5.,-3)
CALL PLOT(0.,-9.,-3)
CALL PLOT(0.,14.,2)
CALL PLOT(0.,0.-3)
CALL PLOT(12.,0.,2)
CALL PLOT(0.,0.,3)
46 TC=TCJ-T
TSUM=TCJ-0.5
L1=TSUM
P=TSUM-L1
B2=P*(P-1)*0.25
XC=XC0+(P*X1)+B2*X2)*E-7
YC=YCO+(P*Y1)+B2*Y2)*E-7
ZC=ZCO+(P*Z1)+B2*Z2)*E-7
XEA=-XC
YEA=-YC
ZEA=-ZC
UEA=XEA
VEA=CEPS*YEA+SEPZ*ZEAA
WEA=SEPZ*YEA+CEPS*ZEAA
XE0=PX1*UCEA+PY1*VEA+PZ1*WEA
YEO=QY1*UCEA+QY1*VEA+Qu1*WEA
ZEO=RZ1*UCEA+RZ1*VEA+RZ1*WEA
Z=R,0
AK=(RK*TC)/(RT2*Q0**1.5)
51 Z1=Z-(Z**3*(Z-1.5))/(3*(Z**2+1))
IF (ABS(Z1-Z)-.5) 52,52,53
52 RC=GC**1.25
THC=2*ATAN(Z)
S**1=ST(THA)
RP=QC**1.25
RQ=2*Q**2
XS=PX*RP+OX*RQ
YS=PY*RP+OY*RQ
ZS=PZ*RP+OZ*RQ
XIS=x5*XG
ETAS=YS*YC
ZETAS=75+ZC
PS=ATAN(ABS(ETAS/XS))
RHS=SORT(XIS*2+ETAS*2+ZETAS*2)
IF (ETAS) 131,132,133
132 IF (X1S) 134,141,135
134 ACAL=PI
GO TO 141
135 ACAL=0
GO TO 141
131 IF (X1S) 136,141,138
136 ACAL=PI+PS
GO TO 141
138 ACAL=PI+PS
GO TO 141
133 IF (X1S) 139,140,140
139 ACAL=PI-PS
GO TO 141
140 ACAL=PS
EXS = ZETAS / RHOS

DCAL = ATAN( EXS / SQRT(1 - EXS**2) )
SINA = SIN(ACAL)
C0SA = COS(ACAL)
SI' = S'** (ACAL)
COSD = COS( DCAL)
CRV = -S'** XS + COSA*YS
DRY = COSA*SIG**XS - SIN**XS + COS*YS + COSD*ZS
PHI = PI**ANGLE(CRV, DRY)
XIC = XE**COS**THC**YEO*S'**THC**RC
ETAC = XE**COS**THC**YEO*COS**THC
ZETAC = ZE**0
DELT = SQRT( XIC**2 + ETAC**2 + ZETAC**2 )
FXL = SQRT( ETAC**2 + ZETAC**2 )
TAU = 0.0
OMU = 0.0
JN = 0

CALL DATSH (1, IP1)
30 IF (IP1 - 1) 10, 10, 11
10 READ(6, 83) OMU
READ(6, 83) TD
GO TO 24
10 READ(6, 83) TAU
READ(6, 83) DMU
24 WRITE(5, 85)
IF (IP1 - 1) 47, 47, 48
47 CALL PLOT (0.0, 0.0, 0.0)
48 IF (IP1 - 1) 12, 12, 13
13 TAU = TAU + TD
GO TO 15
12 OMU = OMU + OMU
15 TE = TC - TAU
Z = 0.0
AK = (XK**TE)/(RT2**CC**1.5)
61 Z1 = Z - (Z**3 + 3*Z**2) / (3*Z**2 + 1)
IF (ABS(Z1 - Z) - 1.0.E-8) 62, 62, 63
63 Z = Z1
GO TO 61
62 RC1 = XC**1.0**Z**2
THC1 = 2*ATAN(Z)
RMU = 1.0 - OMU
IF (ABS(RMU) - 1.0.E-2) 110, 110, 111
11 IF (RMU) 112, 110, 113
110 D = SQRT( 2**RC1**2 + D**2 )
RD = SQRT( RC1**2 + D**2 )
SIMPS = D / (RD**SQRT(1**Z**2))
COSPS = (RD**2 + RC1**2 - D**2 ) / (2**RD**RC1)
PS = ATAN( SIMPS / COSPS )
IF (COSPS) 292, 293, 293
292 PS = PS + 1.0.E+9
293 THD = THC1 + PS
TEST = 2**QC**RD - COS(THD - THC1) - COS(THD)
GO TO 200
112 ED = SQRT( 1**4**OMU**QC / (RC1**RMU**2) )
RMU = A55 / (RMU)
AD = RMU**RC1 / (2**OMU)
SAN = SIN(THC1) / (RMU*ED)
CAN = (2**QC / (RMU*RC1) ) / ED
AN = ATAN( SAN / CAN )
IF (SAN) 190, 190, 191
190 IF(CAN) 192, 195, 195
191 IF(CAM) 192, 193, 193
195 AN=AN+6.283185307
GO TO 193
192 ALD=THC1-AN
EX1=SQRT(((ED+1)/(ED-1))*(SIN(0.5*A)/COS(0.5*A)))
FD=ALC((1+EX1)/(1-EX1))
HYP=SQRT(AD*3/RMU)/DK
TOD=TE-HYP*(ED*0.5*(EXP(FD)-EXP(-FD)))-FD)
HYPL=(1.7/HYP)* (TC-TOD)
F=0.0
172 FI=F-(ED*0.5*(EXP(F)-EXP(-F)))-F-HYP1)/(ED*0.5*(EXP(F)+EXP(-F))+1.)
IF(AOS(F1-F)-E-6) 70, 70, 71
71 F=F1
GO TO 72
170 R0=AD*(ED*0.5*(EXP(F)+EXP(-F))+1.)
DIFF=ED*0.5*(EXP(F)-EXP(-F)))-F-HYP1
EX2=EXP(0.5*F)
EX3=EXP(-0.5*F)
THD=ALD+2*ATAN(SQRT((ED-1)/(ED+1)))* ((EX2-EX3)/(EX2+EX3)))
GO TO 200
113 ED=SQRT((1+A/CM)/(RC1+RMU**2))
AD=-RMU*RC1/(2*O-F-U)
SA=(2^C/RC1/RMU-1)/ED
AM=ATAN(SA/CAN)
IF(SAN) 90, 91, 91
90 IF(CAN) 72, 75, 75
91 IF(CAM) 92, 93, 93
95 AN=AN+6.283185307
GO TO 93
92 AN=AN+3.1415926536
93 ALD=THC1-AN
EX1=SQRT(((ED-1)/(ED+1))*(SIN(0.5*A)/COS(0.5*A)))
FD=ALC((1+EX1)/(1-EX1))
HYP=SQRT(AD*3/RMU)/DK
TOD=TE-HYP*(ED*0.5*(EXP(FD)-EXP(-FD)))-FD)
HYPL=(1.7/HYP)* (TC-TOD)
F=0.0
72 FI=F-(ED*0.5*(EXP(F)-EXP(-F)))-F-HYP1)/(ED*0.5*(EXP(F)+EXP(-F))-1.)
IF(AOS(F1-F)-E-7) 70, 70, 71
71 F=F1
GO TO 72
70 R0=AD*(ED*0.5*(EXP(F)+EXP(-F))-1.)
DIFF=ED*0.5*(EXP(F)-EXP(-F))-F-HYP1
EX2=EXP(0.5*F)
EX3=EXP(-0.5*F)
THD=ALD+2*ATAN(SQRT((ED-1)/(ED+1)))* ((EX2-EX3)/(EX2+EX3)))
200 XI=RD*COS(THC-THD)-RC
ETA=RD*SIN(THC-THD)
EPS=C1*ATAN(ETA/XI)
IF(EPS) 230, 231, 231
230 EPS=EPS+180.0
231 FX2=DELT**2-XIC*XIC+1-ETAC*ETA
MCN=DELT(*FX1*XIC-XIC*ETA+ETAC*ETA/FX1)/FX2
HCH=DELT**2+2*ETAC*ETA/FX1*FX2
EPS1=C1*ATAN(4*RC*C*TAN((3*RC*SQRT(2+QC)+4*RC*SQRT(2+QC+4*RC*SQRT(2+QC+RC)*TAU))))
EPBO=C1*ATAN(4*RC*C*TAN((3*RC*SQRT(2+QC)+4*RC*SQRT(2+QC+4*RC*SQRT(2+QC+RC)*TAU))))
\[ \theta = \text{FPSS} + \phi \]
\[ \text{IF} (\theta - 360.0) < 150, 151, 151 \]
\[ \theta = \theta - 360.0 \]
\[ \text{C O N T I N U E} \]
\[ x_{ikm} = CVF \times x \]
\[ \eta_{k} = CVF \times \eta \]
\[ s = \text{SORT} (x_{ikm}^2 + \eta_{k}^2) \]
\[ m_{cmk} = CVF \times m \]
\[ n_{ckm} = CVF \times n \]
\[ \text{CALL DATSW}(10, \text{IP10}) \]
\[ \text{CALL DATSW}(11, \text{IP11}) \]
\[ \text{IF} (\text{IP10} - 1) = 41, 41, 42 \]
\[ \text{CALL DATSW}(6, \text{IP6}) \]
\[ \text{IF} (\text{IP6} - 1) = 73, 73, 74 \]
\[ p_{p1} = PF \times n_{cmk} \]
\[ p_{p2} = PF \times n_{cmk} \]
\[ \text{G O T O} 75 \]
\[ p_{p1} = PF \times \eta_{k} \]
\[ p_{p2} = PF \times x_{ik} \]
\[ \text{CALL PLOT} (p_{p1}, p_{p2}, 2) \]
\[ \text{IF} (\text{IP11} - 1) = 43, 43, 44 \]
\[ \text{WRITE}(5, 64) \]
\[ \text{OMU}, \tau_{0}, x_{ik}, \eta_{k}, m_{cmk}, n_{cmk}, \eta_{k}, \epsilon_{s}, s \]
\[ \text{T A G} = \text{Cl} + \tau \]
\[ \text{F O R} (\tau_{0}, 2 (5, F8.3)), 2 (5, F8.3) \]
\[ \text{C O N T I N U E} \]
\[ \text{IF} (\text{IP1} - 1) = 16, 16, 17 \]
\[ \text{CALL DATSW}(2, \text{IP2}) \]
\[ \text{IF} (\text{IP2} - 1) = 18, 18, 13 \]
\[ \text{CALL DATSW}(4, \text{IP4}) \]
\[ \text{IF} (\text{IP4} - 1) = 18, 18, 12 \]
\[ \text{CALL DATSW}(3, \text{IP3}) \]
\[ \text{IF} (\text{IP3} - 1) = 19, 19, 20 \]
\[ \text{P A U S E} 3 \]
\[ \text{CALL DATSW}(5, \text{IP5}) \]
\[ \text{IF} (\text{IP5} - 1) = 21, 21, 22 \]
\[ \tau_{0} = 0.0 \]
\[ \text{G O T O} 11 \]
\[ \text{P A U S E} 5 \]
\[ \text{CALL DATSW}(5, \text{IP5}) \]
\[ \text{IF} (\text{IP5} - 1) = 21, 21, 23 \]
\[ \text{OMU} = 0.0 \]
\[ \text{G O T O} 10 \]
\[ \text{IF} (\text{JN}) = 31, 31, 32 \]
\[ \text{JN} = 0 \]
\[ \text{IF} (\text{IP1} - 1) = 28, 28, 29 \]
\[ \text{IP1} = 1 \]
\[ \text{G O T O} 33 \]
\[ \text{IP1} = 2 \]
\[ \text{P A U S E} 7 \]
\[ \tau_{0} = 0.0 \]
\[ \text{OMU} = 0.0 \]
\[ \text{G O T O} 30 \]
\[ \text{P A U S E} 15 \]
\[ \text{G O T O} 101 \]
\[ \text{C A L L} \text{EXIT} \]
\[ \text{END} \]

UNREFERENCED STATEMENTS
80 82 88

FEATURES SUPPORTED
ARITHMETIC TRACE
ONE WORD INTEGERS
EXTENDED PRECISION
ICCS

CORE REQUIREMENTS FOR
COMMON 0 VARIABLES 470 PROGRAM 3086

END OF COMPILATION
APPENDIX 5

Syndyne curves for various types of particle

The program given in Appendix 3 may readily be used to obtain syndyne curves appropriate to particles of a given density $\rho$ and diameter $d$. The value of $1-\mu$ is calculated from

$$1-\mu = \frac{1.19 \times 10^+}{\rho d} \quad (q_r \sim 1)$$

Taking three typical densities, $\rho = 1, 3, 7$, corresponding to clathrate, silicate and iron particles respectively, and particle diameters of 0.3, 1.0, 3.0, 10.0 (microns) in each case, the syndyne curves are presented here for Comet Arend-Roland on 1957 April 27.8. They are plotted in the orbital plane ($\xi, \gamma$ coordinates).
\( \rho = 3 \)

\[ \begin{align*}
S & \times 10^{-6} \text{ km} \\
\hline
5 & 10 & 15 & 20 & 25 \\
\end{align*} \]

\[ \begin{align*}
\eta & \times 10^{-6} \text{ km} \\
\hline
5 & 10 & 15 & 20 & 25 \\
\end{align*} \]

- \( d = 0.3 \)
- \( 1 - \mu = 0.32 \)
- \( d = 1.0 \)
- \( 1 - \mu = 0.4 \)
- \( d = 3.0 \)
- \( 1 - \mu = 0.13 \)
- \( d = 10.0 \)
- \( 1 - \mu = 0.04 \)
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This work deals with the geometrical problems inherent in the observation of cometary tails, and with attempts at explanation of their forms.

Following the introduction, a review of the geometrical aspects of tail observation is presented in Chapter 2. New formulae are developed for projection from photographic to orbital plane, and for foreshortening etc. Tail orientation is fully treated and expressions are derived for orientation error in this projection. Equations for an elliptic orbit are developed in addition to those for the usual assumption of a parabolic orbit. Perspective error resulting from bisection of apparent tail images is investigated, and appropriate formulae derived. A set of observations of Comet Bennett 1969i is analysed and the results compared with those of other workers.

The influence of the solar wind on Type I tails is reviewed in Chapter 3 with special reference to dynamical aberration. Some results for Comet Bennett are presented. Oscillation of Type I tails is then discussed. A comparison made between the cases of Comet Burnham 1960 II and Comet Halley 1835 III lends some support to the view that the oscillations are due to external influences on the tail. Formulae are developed for the progressive change in Type I tail orientation to be expected as the comet pursues its orbit.

Mechanical theories of Type II tails are reviewed in Chapter 4, special attention being given to tail orientation. The accurate calculation of syndyne and synchrone curves is treated, and the formulae are extended to allow of any value of $\mu$. Tail analysis by syndynes and synchrones is considered; results are presented for the time variation in the initial orientation of these curves. Implications for mixed tails and possible coupling between tails are discussed.

Finally, Chapter 5 briefly summarises the conclusions and gives suggestions for further work.