Lyapunov functions and $\mathcal{L}_2$ gain bounds for systems with slope restricted nonlinearities

Matthew C. Turner\textsuperscript{a}, Murray Kerr\textsuperscript{b}

\textsuperscript{a}Department of Engineering, University of Leicester, Leicester, LE1 7RH, UK.
\textsuperscript{b}Deimos Space SL, Madrid, 28760, Spain

Abstract

The stability and $\mathcal{L}_2$ performance analysis of systems consisting of an interconnection of a linear-time-invariant (LTI) system and a static nonlinear element which is Lipschitz, slope restricted and sector bounded is revisited. The main thrust of the paper is to improve and extend an existing result in the literature to enable (i) concise and correct conditions for asymptotic stability of the interconnection and (ii) reasonably tight bounds on the $\mathcal{L}_2$ gain between an exogenous input and a given output to be obtained. Numerical examples indicate that the proposed algorithm performs well compared to competing results in the literature.

Key words: saturation, anti-windup, absolute stability

1 Introduction

This paper considers the system depicted in Figure 1 where the nonlinearity is a sector bounded, slope restricted nonlinearity. The problem addressed is:-

\begin{figure}[h]
\centering
\begin{tikzpicture}
    \node [coordinate] (input) at (0,0) {}; 
    \node [coordinate] (output) at (3,0) {}; 
    \node [coordinate] (lin) at (1.5,0) {$P(s)$}; 
    \node [coordinate] (phi) at (1.5,-1) {$\phi$}; 
    \node [coordinate] (nonlin) at (1.5,-2) {$\Phi(.)$}; 
    \draw [->] (input) -- (lin); 
    \draw [->] (lin) -- (phi); 
    \draw [->] (phi) -- (nonlin); 
    \draw [->] (nonlin) -- (output); 
    \draw [->] (input) -- (phi) node [midway, above] {$w$}; 
    \draw [->] (phi) -- (output) node [midway, above] {$z$}; 
    \draw [->] (nonlin) -- (output) node [midway, above] {$y$}; 
\end{tikzpicture}
\caption{System under consideration}
\end{figure}

Email addresses: mct6@le.ac.uk (Matthew C. Turner), murray.kerr@deimos-space.com (Murray Kerr).
Problem 1  
(1) When \( w(t) \equiv 0 \), find Lyapunov-based conditions which enable global asymptotic stability of the origin of the interconnection to be ascertained.

(2) When \( w(t) \neq 0 \), find conditions, based on the same Lyapunov function used to establish asymptotic stability, which enable the \( L_2 \) gain from the input \( w(t) \) to the output \( z(t) \) to be bounded as tightly as possible.

Variants of this problem have been studied extensively in the literature and, in particular, Problem 1.1 is, in essence, the absolute stability problem which has been treated since at least the 1940s. Popular solutions to Problem 1.1 are the Circle Criterion and the Popov Criterion. Good introductions to both criteria can be found in [11] and [13], and comprehensive treatments of the Lyapunov approach found in [18,8]. Further developments of the Popov/Circle Criteria for multiple equilibria are also given [13], although this is beyond the scope of this paper.

Despite the popularity of the Circle and Popov Criteria, it is well known that when more information, other than sector boundedness, is known about the nonlinearity \( \Phi(.) \), both criteria can be conservative. In particular, when the nonlinearity is slope-restricted various alternative stability criteria can be derived; there are too many to list here but examples may be found in [26,7,3,21,5]. Of particular note are the results of Zames and Falb [26] which provide a very flexible approach to establishing asymptotic stability for single-input-single-output slope-restricted systems. These results were later extended to various classes of multivariable systems by Safonov and colleagues ([20,12,14]). However, for many years they were not widely used due to the complexity of searching for the so-called Zames-Falb multiplier. Recently several results have become available which, to some extent, automate this search [16,15,23,22] and frequently far superior results can be obtained than, for instance, with the Popov Criterion. Despite these improvements, the computational burden associated with, for instance [23,22] tends to be quite high ([2,24]) due to the search for the multiplier not being “quite” convex (it is an LMI-problem plus a line search). For high-order complex systems, this burden can be prohibitive. In addition, the results provided by such IQC-based methods are not intrinsically associated with the construction of Lyapunov functions, which is in an interesting subject in its own right, and also useful if local results are required.

In [17], a novel Lyapunov function was used in order to obtain less conservative methods for guaranteeing asymptotic stability of the system shown in Figure 1. The Lyapunov function was piecewise quadratic, as in [19,4,10], but also used several integral terms derived from the properties of the nonlinearity. Two main LMI-based results were derived in [17]: Theorem 1 which, although technically correct, featured redundant terms in the LMI’s, thereby causing complications and increasing the computational burden; and Theorem 2, which although simpler, seems to feature a small technical error (at least an assumption of controllability on the LTI part appears to be missing) despite “working” in many cases. In addition the \( L_2 \) gain problem is not addressed in [17]. However, the work of [17] is significant because it generalises the Popov Criterion and appears dramatically less conservative in
many numerical examples, even appearing to out-perform the Zames-Falb multiplier in some cases [2,24] . Our goal in this paper is to present results which are an improved alternative to Theorem 2 in [17], that is they are correct, concise and able to provide reasonably non-conservative $L_2$ gain bounds. The class of system to which the results apply is also extended. The primary motivation for this work is the analysis of the stability of complex systems which can be posed as in Figure 1, which due to their size/dimension/complexity can be difficult to analyse, without excessive conservatism, using standard results.

The paper is structured as follows: in the next section, the problem is formally introduced. The main results are given in the following section; numerical examples in the section after that. Some brief final remarks conclude the paper.

1.1 Notation

Notation is mainly standard. The $L_2$ norm of a vector valued function $x(t)$ is defined as $\|x\|_2 := \left(\int_0^\infty \|x(t)\|^2 dt\right)^{1/2}$ where $\|\cdot\|$ denotes the standard Euclidean norm; any signal whose $L_2$ norm is finite is denoted $x(t) \in L_2$. The nonlinear operator, $T : w \mapsto z$ is said to have $L_2$ gain less than $\gamma$ if $\|z\|_2 < \gamma\|w\|_2 + \beta$ for scalars $\gamma, \beta \geq 0$ and $\forall \ w \in L_2$.

2 Problem description

Consider the system depicted in Figure 1. $P(s)$ denotes a finite-dimensional linear-time-invariant (LTI) system described by the following state-space equations.

$$P(s) \sim \begin{cases} 
\dot{x} = Ax + B_1 w + B_2 \phi \\
z = C_1 x + D_{11} w + D_{12} \phi \\
y = C_2 x + D_{21} w + D_{22} \phi
\end{cases}$$

(1)

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^{n_w}$, $z \in \mathbb{R}^{n_z}$, $y \in \mathbb{R}^{m}$, $\phi = \Phi(y) \in \mathbb{R}^{m}$ and the state-space matrices are dimensioned accordingly. The nonlinear operator $\Phi(.) : \mathbb{R}^m \mapsto \mathbb{R}^m$ is a decentralised globally Lipschitz, sector bounded, slope restricted nonlinearity which satisfies the following assumptions:

**Assumption 2** $\Phi(.) : \mathbb{R}^m \mapsto \mathbb{R}^m$ is decentralised, that is for $\sigma \in \mathbb{R}^m$

$$\Phi(\sigma) = \left[\Phi_1(\sigma_1) \ \Phi_2(\sigma_2) \ \cdots \ \Phi_m(\sigma_m)\right]^T$$

---

1 This appears to be more due to the difficulty in the search for Zames-Falb multipliers than something intrinsic however
and each element, $\Phi_i(.) : \mathbb{R} \mapsto \mathbb{R}$ is globally Lipschitz, zero at zero, and satisfies the following conditions.

$$\frac{\Phi_i(\sigma_i)}{\sigma_i} \in [0, \delta_i] \quad \forall \sigma_i$$

$$\partial \Phi_i(\sigma_i) \in [0, \bar{\delta}_i] \quad \forall \sigma_i$$

for all $i \in \{1, \ldots, m\}$, where $\partial \Phi$ represents the sub-differential of $\Phi$.

Note that the second inequality reflects that fact that $\Phi_i(.)$ may not be differentiable everywhere, but from the Lipschitz assumption, means that

$$\partial \Phi_i(\sigma_i) = \frac{d\Phi_i(\sigma_i)}{d\sigma_i} \ a.e.$$  \hspace{1cm} (4)

This is a minor technical difference, compared to the original results of [17], which is easily accommodated in the proofs yet is necessary to treat common slope-restricted nonlinearities such as the saturation and deadzone. In [17] it was shown how the two inequalities (2) and (3) could then be used to derive eight sets of integral inequalities to be used as part of the Lyapunov function. In this work we use only four of those sets of inequalities, but show how the use of such inequalities is able to preserve (and in fact improve upon) Theorem 2 of [17], implying redundancy in those inequalities. For ease of reference we repeat the inequalities used in this paper below, where the $\mu_{I,i}$ are any positive scalars for all $I \in \{1, \ldots, 4\}, \ i \in \{1, \ldots, m\}$.

$$g_{1,i}(x) = \mu_{1,i} \int_0^y \Phi_i(\sigma_i)d\sigma_i \geq 0 \quad \forall y_i, \ \forall i \in \{1, \ldots, m\}$$  \hspace{1cm} (5)

$$g_{2,i}(x) = \mu_{2,i} \int_0^y [\delta_i \sigma_i - \Phi_i(\sigma_i)]d\sigma_i \geq 0 \quad \forall y_i, \ \forall i \in \{1, \ldots, m\}$$  \hspace{1cm} (6)

$$g_{3,i}(x) = \mu_{3,i} \int_0^y [\bar{\delta}_i - \partial \Phi_i(\sigma_i)]\sigma_i d\sigma_i \geq 0 \quad \forall y_i, \ \forall i \in \{1, \ldots, m\}$$  \hspace{1cm} (7)

$$g_{4,i}(x) = \mu_{4,i} \int_0^y \partial \Phi_i(\sigma_i)[\delta_i \sigma_i - \Phi_i(\sigma_i)]d\sigma_i \geq 0 \quad \forall y_i, \ \forall i \in \{1, \ldots, m\}$$  \hspace{1cm} (8)

In addition, from inequality (2) with $\sigma(t) = y(t)$, the standard sector inequality follows:

$$S_\Delta = 2\phi'N_1(\Delta y - \phi) \geq 0$$

$$= 2\phi'N_1(\Delta(C_{21}x + D_{21}w + D_{22}\phi) - \phi) \geq 0$$  \hspace{1cm} (10) \hspace{1cm} (11)

where $\Delta := \text{diag}(\delta_1, \ldots, \delta_m) > 0$ and $N_1 > 0$ is any positive definite diagonal matrix. Also, from inequality (3), at the values of $y$ at which $\Phi(y)$ is differentiable, we have

$$\dot{\phi}_i(\bar{\delta}_i \dot{y}_i - \dot{\phi}_i) \geq 0$$  \hspace{1cm} (12)
Thus, as $\Phi(.)$ is Lipschitz, we have the following inequality

$$S_{\Delta} = 2\phi'N_2(\tilde{\Delta}\dot{y} - \dot{\phi}) \geq 0 \quad \text{a.e.} \quad (13)$$

where $\tilde{\Delta} := \text{diag}(\tilde{\delta}_1, \ldots, \tilde{\delta}_m) > 0$ and $N_2 > 0$ is any positive definite diagonal matrix. Evaluation of the above expression requires knowledge of $\dot{y}$, which in turn requires knowledge of $\dot{w}$, which is generally not assumed. Therefore for the remainder of the paper, we shall assume that $D_{21} \equiv 0$. In this case inequality (13) becomes

$$S_{\Delta} := 2\phi'N_2(\Delta C_2(Ax + B_1w + B_2\phi) + (\Delta D_{22} - I)\dot{\phi}) \geq 0 \quad \text{a.e.} \quad (14)$$

In the next section, it will be necessary to bound the terms $g_{I,j}(x)$ and thus the following lemma will be useful.

**Lemma 3** Assume the feedback interconnection in Figure 1 is well posed$^3$. Then there exist positive scalars $\alpha_{i,j}$ such that $g_{I,j}(x) \leq \alpha_{I,j}\|x\|^2$ for all $I \in \{1, \ldots, 4\}$ and $i \in \{1, \ldots, m\}$.

**Proof:** The proof is straightforward; see the appendix for a sketch.

### 3 Main results

#### 3.1 The result and its derivation

The following standard Lyapunov result is recalled (see [10] for example).

**Lemma 4** Assume $V(x)$ is globally Lipschitz in $x$ and that there exist positive scalars $c_1$, $c_2$ and $c_3$ such that

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2 \quad \forall x \neq 0 \quad (15)$$

$$\frac{dV(x)}{dt} \leq -c_3\|x\|^2 \quad \text{a.e.} \quad (16)$$

then $x(0)$ is a globally exponentially stable equilibrium point

The main result of the paper is the following theorem.

**Theorem 5** Consider the interconnection depicted in Figure 1 where $P(s)$ has state-space realisation given by equation (1) and $\Phi(.)$ satisfies Assumption 2. Assume $D_{21} = 0$, then the following statements are true.

1. When $w(t) \equiv 0$, if there positive definite symmetric matrices $X \in \mathbb{R}^{n \times n}$ and $Z \in \mathbb{R}^{m \times m}$, positive definite diagonal matrices $M_1, M_2, M_3, M_4, N_1, N_2 \in \mathbb{R}^{m \times m}$

---

$^2$ This assumption is not required with the Zames-Falb multiplier approach; see [1]

$^3$ A feedback interconnection is said to be well-posed if unique solutions exist to the feedback equations
\( \mathbb{R}^{m \times m} \) and a matrix \( Y \in \mathbb{R}^{n \times m} \) such that the following matrix inequalities are satisfied

\[
L_1 := \begin{bmatrix} L_{11} + L'_{11} & L_{12} & L_{13} \\ * & L_{22} + L'_{22} & L_{23} \\ * & * & L_{33} + L'_{33} \end{bmatrix} < 0 \tag{17}
\]

\[
L_2 := \begin{bmatrix} X & Y \\ * & Z \end{bmatrix} > 0 \tag{18}
\]

where

\[
L_{11} := (X + C'_2(\Delta M_2 + \tilde{\Delta} M_3)C_2)A \tag{19}
\]

\[
L_{12} := (X + C'_2(\Delta M_2 + \tilde{\Delta} M_3)C_2)B_2 + A'Y + A'C'_2(M_1 + M_2(\Delta D_{22} - I) + \Delta M_3 D_{22}) + C'_2 A N_1 \tag{20}
\]

\[
L_{13} := Y + C'_2(M_2 D_{22} + M_3(\Delta D_{22} - I) + \Delta M_4) + A'C'_2 \tilde{\Delta} N_2 \tag{21}
\]

\[
L_{22} := B'_2Y + N_1(\Delta D_{22} - I) + (M_1 + (\Delta D_{22} - I)M_2 + D_{22} \tilde{\Delta} M_3)C_2B_2 \tag{22}
\]

\[
L_{23} := Z + (M_1 + (\Delta D_{22} - I)'M_2) D_{22} + D'_{22} M_3 (\Delta D_{22} - I) + (\Delta D_{22} - I)'M_4 + B'_2 C'_2 \tilde{\Delta} N_2 \tag{23}
\]

\[
L_{33} := N_2(\Delta D_{22} - I) + (\Delta D_{22} - I)'N_2 \tag{24}
\]

then interconnection is well-posed and globally exponentially stable.

(2) When \( x(0) = 0 \), if there positive definite symmetric matrices \( X \in \mathbb{R}^{n \times n} \) and \( Z \in \mathbb{R}^{m \times m} \), positive definite diagonal matrices \( M_1, M_2, M_3, M_4, N_1, N_2 \in \mathbb{R}^{m \times m} \) and a matrix \( Y \in \mathbb{R}^{n \times m} \) such that the following matrix inequalities are satisfied

\[
\begin{bmatrix} L_1 & L_3 \\ * & L_4 \end{bmatrix} < 0 \tag{25}
\]

\[
L_2 > 0 \tag{26}
\]

where

\[
L_3 = \begin{bmatrix} (X + C'_2(\Delta M_2 + \tilde{\Delta} M_3)C_2)B_1 & C'_1 \\ Y'B_1 + (M_1 + (\Delta D_{22} - I)M_2 + D_{22} \tilde{\Delta} M_3)C_2B_1 & D'_{12} \tag{27}
\end{bmatrix}
\]

\[
L_4 = \begin{bmatrix} -\gamma I & D'_{11} \\ * & -\gamma I \tag{28}
\end{bmatrix}
\]

then \( ||z||_2 < \gamma ||w||_2 \) for some real scalar \( \gamma > 0 \)
Proof:

Item 1

The idea of the proof is to choose a Lyapunov function candidate satisfying the conditions of Lemma 4. Thus consider the following Lyapunov function candidate

$$V(x) = \begin{bmatrix} x \\ \phi \end{bmatrix}' \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} \begin{bmatrix} x \\ \phi \end{bmatrix} + 2 \sum_{I=1}^{4} \sum_{i=1}^{m} g_{I,i}(x)$$  \hspace{1cm} (29)

Lipschitz continuity of $\Phi(y)$ implies continuity of $V(x)$. It follows that $V(x)$ is radially unbounded if the following matrix is positive definite.

$$P := \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} > 0$$  \hspace{1cm} (30)

Further note that $P > 0$ implies that

$$V(x) \geq \lambda_{\min}(P) \left\| \begin{bmatrix} x \\ \phi \end{bmatrix} \right\|^2 + 2 \sum_{I=1}^{4} \sum_{i=1}^{m} g_{I,i}(x)$$  \hspace{1cm} (31)

$$\geq \lambda_{\min}(P) \|x\|^2$$  \hspace{1cm} (32)

Thus $V(x) \geq c_1 \|x\|^2$. To see that $V(x) \leq c_2 \|x\|^2$ note that, if the system is well-posed (as will be proven), there exists a $\beta \geq 0$ such that $\|y\| \leq \beta \|x\|$. Thus it follows that

$$V(x) \leq \lambda_{\max}(P) \left\| \begin{bmatrix} x \\ \phi \end{bmatrix} \right\|^2 + 2 \sum_{I=1}^{4} \sum_{i=1}^{m} g_{I,i}(x)$$  \hspace{1cm} (33)

$$\leq \lambda_{\max}(P) (1 + \|\Delta\|^2 \beta^2) \|x\|^2 + 2 \sum_{I=1}^{4} \sum_{i=1}^{m} \alpha_{I,i} \|x\|^2$$  \hspace{1cm} (34)

where Lemma 3 has been used to bound the terms $g_{I,i}(x)$. It therefore remains to prove that $\dot{V}(x) \leq -c_3 \|x\|^2$ almost everywhere. Thus, note that

$$\dot{V}(x) = 2 \left[ \begin{bmatrix} \dot{x} \\ \dot{\phi} \end{bmatrix}' \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} \begin{bmatrix} x \\ \phi \end{bmatrix} \right] + 2 \left\{ \phi'M_1 \dot{y} + (\Delta y - \phi)'M_2 \dot{y} + (\Delta M_3 \dot{y} - \dot{\phi})'M_3 y + (\Delta y - \phi)'M_4 \phi \right\}$$  \hspace{1cm} (35)
almost everywhere, where

\[
\begin{align*}
M_1 &= \text{diag}(\mu_{2,1}, \ldots, \mu_{2,m}) \quad (36) \\
M_2 &= \text{diag}(\mu_{3,1}, \ldots, \mu_{3,m}) \quad (37) \\
M_3 &= \text{diag}(\mu_{5,1}, \ldots, \mu_{5,m}) \quad (38) \\
M_4 &= \text{diag}(\mu_{6,1}, \ldots, \mu_{6,m}) \quad (39)
\end{align*}
\]

Noting \( \dot{y} = C_2 \dot{x} + D_{22} \dot{\phi} \) almost everywhere, and using the state-space realisation in equation (1) and the sector/slope inequalities, (11) and (14), then gives, after some tedious algebra

\[
\dot{V}(x) = \begin{bmatrix} x \\ \phi \\ \dot{\phi} \end{bmatrix}' \begin{bmatrix} L_{11} + L_{11}' & L_{12} & L_{13} \\ * & L_{22} + L_{22}' & L_{23} \\ * & * & L_{33} + L_{33}' \end{bmatrix} \begin{bmatrix} x \\ \phi \\ \dot{\phi} \end{bmatrix} \quad \text{a.e.} \quad (40)
\]

This inequality is satisfied if matrix inequality (17) in the theorem is satisfied. In turn this implies that

\[
\dot{V}(x) \leq -\lambda_{\text{min}}(L_1) \left\| \begin{bmatrix} x \\ \phi \\ \dot{\phi} \end{bmatrix} \right\|^2 \quad \text{a.e.} \quad (41)
\]

\[
\Rightarrow \dot{V}(x) \leq -c_3 \|x\|^2 \quad \text{a.e.} \quad (42)
\]

Thus the system is globally exponentially stable.

The argument which proves well-posedness is not trivial, but has been dealt with in many other papers (e.g. [25,4,6]). Effectively, well-posedness follows if it is possible to prove that the Jacobian of \( F(y) = y - D_{22} \Phi(y) \) is nonsingular and a member of a compact, convex set almost everywhere. Following the arguments in [25,6], this follows if there exists a positive definite diagonal matrix \( V_s \in \mathbb{R}^{m \times m} \) such that

\[
(I - D_{22} \bar{\Delta})' V_s + V_s (I - D_{22} \bar{\Delta}) > 0
\]

This condition is exactly that appearing in inequality (17), via \( L_{33} + L_{33}' < 0 \); hence the system is well-posed.

**Item 2**

This is proved in much the same way as the first item so the proof is just sketched. It is well known that the \( L_2 \) gain of a system of the form depicted in Figure 1 is bounded by \( \gamma \) if the following inequality holds.

\[
\dot{V}(x) + \frac{1}{\gamma} \|z\|^2 - \gamma \|w\|^2 < 0 \quad \text{a.e.} \quad (43)
\]
Using the same form of $V(x)$ as suggested earlier, it can be shown that inequality (43) holds (almost everywhere) if inequality (25) holds.

\[ \square \square \]

3.2 \hspace{1em} \textit{Comparison to the Popov Criterion}

It is fruitful to compare Theorem 5 to the Popov Criterion for the typical case of strictly proper systems, that is when $D_{22} \equiv 0$. In this case, inequality (17) in Theorem 5 reads, with $\tilde{\mathbf{X}} = \mathbf{X} + C'_2(\Delta M_2 + \Delta M_3)C_2$,

\[
\begin{bmatrix}
\tilde{\mathbf{X}}A + A'\tilde{\mathbf{X}} & \tilde{\mathbf{X}}B_2 + A'\mathbf{Y} + A'C'_2(M_1-M_2) + C'_2\Delta N_1 & \mathbf{Y} + C'_2(\Delta M_4 - \Delta M_3) + A'C'_2\Delta N_2 \\
* & \mathbf{Y}B_2 + B'_2\mathbf{Y}' - 2N_1 + (M_1-M_2)C_2B_2 + B'_2C'_2(M_1-M_2) & \mathbf{Z} - M_4 + B'_2C'_2\Delta N_2 \\
* & * & -2N_2
\end{bmatrix} < 0
\]

With the special choices $\mathbf{Y} = 0$, $\mathbf{Z} = M_4$ and $M_3 = \Delta M_4$, inequality (44) then reduces to

\[
\begin{bmatrix}
\tilde{\mathbf{X}}A + A'\tilde{\mathbf{X}} & \tilde{\mathbf{X}}B_2 + A'C'_2\Lambda P + C'_2\Delta N_1 & A'C'_2\Delta N_2 \\
* & \Lambda P C_2B_2 + B'_2C'_2\Lambda P' - 2N_1 & B'_2C'_2\Delta N_2 \\
* & * & -2N_2
\end{bmatrix} < 0
\]

where $\Lambda P = M_1 - M_2 \in \mathbb{D}^{m \times m}$ is an indefinite matrix. For small enough $N_2$, this inequality will hold if the following inequality is satisfied

\[
\begin{bmatrix}
A'\tilde{\mathbf{X}} + \tilde{\mathbf{X}}A & \tilde{\mathbf{X}}B_2 + A'C'_2\Lambda P + C'_2\Delta N_1 \\
* & \Lambda P C_2B_2 + B'_2C'_2\Lambda P - 2N_1
\end{bmatrix} < 0
\]

This matrix inequality is precisely the one which arises from the Popov Criterion (see [2]), and thus the Popov Criterion can be seen to be a special case of Theorem 5. In fact, for the special case of the saturation/deadzone functions, it was shown in [4] that the first term of the Lyapunov function:

\[
V(x) = \begin{bmatrix} x \\ \phi \end{bmatrix} \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y}' & \mathbf{Z} \end{bmatrix} \begin{bmatrix} x \\ \phi \end{bmatrix}
\]

actually generalises the integral term found in the Lur’e type Lyapunov function used to prove a variant of the Popov Criterion ($\Lambda P > 0$); in other words the piecewise quadratic part of the Lyapunov function means the integral terms of the Popov Criterion are unnecessary.
3.3 Comparison to Park’s results

The results here improve on the results of [17] in several useful ways: (i) non-smooth nonlinearities, which are often encountered in practice (anti-windup analysis for instance), are admitted; (ii) algebraic loops are admitted ($D_{22}$ is not set to zero) whereas [17] simply considers strictly proper systems; and (iii) $L_2$ gain bound bounds are provided. Finally a crucial contribution is that the LMI’s given in Theorem 5 are simpler to solve than those given in Theorem 1 of Park (less matrix variables are required for same-sized LMI’s) but they are correct and more flexible than those given in Theorem 2 of [17]. In particular, Theorem 2 [17] requires at least the extra assumption that the matrix pair $(A, B_2)$ be controllable (the stabilisability assumption effectively made in [17] is not sufficient). Furthermore, Theorem 2 in [17] features fewer free parameters than Theorem 5: effectively the freedom is selecting the diagonal matrices $M_1, M_2, M_3$ and $M_4$ is lost. In other words the first part of Theorem 5 is a correct and flexible alternative to Theorem 2 in [17]; the second part extends this to $L_2$ gain properties.

4 Numerical examples

This section presents some numerical examples which are used to compare the performance of the algorithm with several others available in the literature. The algorithms which we compare to are:

1. The Popov Criterion given, as presented in [16] and computed using [15].
2. The Zames-Falb multiplier results of [22]. The algorithm given in [22] is an LMI-problem combined with a line search so in general is more computationally intensive than a pure LMI problem. For the results given below, a crude line-search was used so improved results could be obtained using a finer line search. We note that some numerical errors appeared in [22] due to an incorrect LMI relaxation noted in [2]; this relaxation is not used in the calculations performed here. Alternatively, $L_2$ gain bounds could be calculated using [15], but this requires the Zames-Falb multiplier to be fixed a priori, or a search over this multiplier to be performed. This is again computationally intensive so we prefer to use the results of [22].
3. The results of [4]. Although these results are derived on the basis of the non-linearity being a saturation or a deadzone, they are an interesting comparison because the results of this paper (and the Zames-Falb results) are also applicable to these nonlinearities. It was shown in [4] that the numerical results compared favourably to other recent results in the area, notably [9], and can be considered state-of-the-art.

For the numerical examples, a very simple structure of $P(s)$ is considered, viz:
\[ P(s) = \begin{bmatrix} G(s) & -G(s) \\ G(s) & -G(s) \end{bmatrix} \sim \begin{bmatrix} A & B-B \\ C & D-D \\ C & D-D \end{bmatrix} \]  

For simplicity, the nonlinearities are all scalar with sector and slope bounds equal to unity, i.e. \( \Delta = 1, \bar{\Delta} = 1 \). The linear part for each example is given in Table 1 and the \( \mathcal{L}_2 \) gain bounds given by the different criteria are given in Table 2. The Mathworks LMI solver from Matlab 7.3 and available in the Robust Control Toolbox, was used as the LMI solver in all cases. From the latter table it is evident that in all of the examples tested, Theorem 5 produces \( \mathcal{L}_2 \) gain bounds as least as small as the Popov Criterion - as expected since the proof of Theorem 5 shows that the Popov parameters are actually redundant. Examples 3 and 4 show that, while in some cases Theorem 5 gives somewhat more conservative \( \mathcal{L}_2 \) gain estimates than those based upon Zames-Falb multipliers ([22]), they are improved compared to the Popov Criterion. Again it is noted that the \( \mathcal{L}_2 \) gain results based on the Zames-Falb multipliers require a considerably greater computational burden for systems of moderate to high order, although in the cases of Examples 3 and 4, the Zames-Falb results are undoubtedly less conservative. Examples 5,6 and 7 show that in some cases Theorem 5 can actually provide the lowest \( \mathcal{L}_2 \) gain bound, outperforming results based on both the Zames-Falb multipliers, the Popov Criterion and even results dedicated to the saturation and deadzone [9]. We note that in principle, for a finer line search, the Zames-Falb multiplier results of [22] may actually be able to provide an equally tight bound, but again, the computational overhead would be much greater.

Remark 3: Although the results here appear to provide at least as small \( \mathcal{L}_2 \) gain bounds as those of [4], they cannot be applied in the case that \( D_{21} \neq 0 \). This is due to the proof of Theorem 5 requiring the derivative of \( y(t) \) to be evaluated; if \( D_{21} \neq 0 \) this would mean a term involving \( w \) would appear, upon which no assumption is made (\( w \in \mathcal{L}_2 \) does not imply \( \dot{w} \in \mathcal{L}_2 \)). Hence we are forced to make the assumption that \( D_{21} = 0 \); no such stipulation is made in [4]. \( \square \)

5 Conclusion

This paper has proposed a new Lyapunov based algorithm for analysing the performance and stability properties of systems containing slope restricted nonlinearities. The results are based on a Lyapunov function proposed by [17] but the results have greater scope, allowing the treatment of systems containing algebraic loops, non-differentiable Lipschitz nonlinearities and also incorporating \( \mathcal{L}_2 \) performance. Numerical examples have shown that the results compare favourably to the state-of-the-art results available in the literature.
Table 1
Table of transfer functions $G(s)$

<table>
<thead>
<tr>
<th>Example</th>
<th>$G(s)$</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$G_1(s) = -\frac{s+1}{s^2+0.1s+1}$</td>
<td>[15] - Popov</td>
</tr>
<tr>
<td>2</td>
<td>$G_2(s) = \frac{1}{2}G_1(s)$</td>
<td>[15] - Sector</td>
</tr>
<tr>
<td>3</td>
<td>$G_3(s) = \left(\frac{2s^2+s+2}{(s^2+3s+20)(s+1)^2}\right)^2$</td>
<td>[15] (modified)</td>
</tr>
<tr>
<td>4</td>
<td>$G_4(s) = 0.015\left(\frac{2s^2+0.1s+0.02}{(s+1)^2(s^2+0.5s+0.2)}\right)^2$</td>
<td>New</td>
</tr>
<tr>
<td>5</td>
<td>$G_5(s) = \frac{0.2s}{s^2+0.2s^2+6s^2+0.1s+1}$</td>
<td>Ex 4 [23] (scaled)</td>
</tr>
<tr>
<td>6</td>
<td>$G_6(s) = \frac{-0.2s}{s^2+0.4s^3+6s^2+0.1s+1}$</td>
<td>New</td>
</tr>
<tr>
<td>7</td>
<td>$G_7(s) = -G_6(s)$</td>
<td>New</td>
</tr>
</tbody>
</table>

Table 2
$L_2$ gain bounds of example systems

<table>
<thead>
<tr>
<th>Example</th>
<th>$L_2$ gain bound</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Popov ([15])</td>
</tr>
<tr>
<td>2</td>
<td>7.0742</td>
</tr>
<tr>
<td>3</td>
<td>5.7645</td>
</tr>
<tr>
<td>4</td>
<td>$\infty$</td>
</tr>
<tr>
<td>5</td>
<td>2.363</td>
</tr>
<tr>
<td>6</td>
<td>9.0312</td>
</tr>
<tr>
<td>7</td>
<td>7.0155</td>
</tr>
</tbody>
</table>

References


A Proof of Lemma 3

The proof proceeds on a case-by-case basis. Note that when \( w = 0 \), the well-posedness assumption on Figure 1 implies that there exists a positive scalar \( \beta_i \) such that \( \| y_i \| \leq \beta_i \| x \| \) for all \( i \in \{1, \ldots, m\} \). First consider \( g_{1,i}(x) \); then we have for all \( i \in \{1, \ldots, m\} \)

\[
g_{1,i}(x) \leq \mu_{1,i} \int_0^u |\Phi_i(\sigma_i)| d\sigma_i \]  
\[
\leq \mu_{1,i} \int_0^u \delta |\sigma_i| d\sigma_i \]  
\[
\leq \mu_{1,i} \delta_i \| \sigma_i \|_\infty \]  
\[
\leq \mu_{1,i} \delta_i \beta_i \| x \| ^2 \]  

Bounds for \( g_{I,i}(x) \) when \( I = 2, 3, 4 \) follow similarly. □□