TWISTED HOCHSCHILD HOMOLOGY AND MACLANE HOMOLOGY

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Abstract. We prove that $H_i(A, \Phi(A)) = 0$, $i > 0$. Here $A$ is a commutative algebra over the prime field $\mathbb{F}_p$ of characteristic $p > 0$ and $\Phi(A)$ is $A$ considered as a bimodule, where the left multiplication is the usual one, while the right multiplication is given via Frobenius endomorphism and $H_\bullet$ denotes the Hochschild homology over $\mathbb{F}_p$. This result has implications in MacLane homology theory. Among other results, we prove that $H_{\text{ML}}\bullet(A, T) = 0$, provided $A$ is an algebra over a field $K$ of characteristic $p > 0$ and $T$ is a strict homogeneous polynomial functor of degree $d$ with $1 < d < \text{Card}(K)$.

1. Introduction

In this short note we study Hochschild and MacLane homology of commutative algebras over the prime field $\mathbb{F}_p$ of characteristic $p > 0$. Let us recall that MacLane homology is isomorphic to the topological Hochschild homology $[13]$ and to the stable $K$-theory as well $[5]$.

Let $A$ be a commutative algebra over the prime field $\mathbb{F}_p$ of characteristic $p > 0$ and let $\Phi(A)$ be denote $A$-$A$-bimodule, which is $A$ as a left $A$-module, while the right multiplication is given via Frobenius endomorphism. We prove that the Hochschild homology vanishes $H_i(A, \Phi(A)) = 0$, $i > 0$. The proof makes use a simple result on homotopy groups of simplicial rings, which says that if $R_\bullet$ is a simplicial ring such that all rings involved in $R_\bullet$ satisfy $x^m = x$, $m \geq 2$ identity then $\pi_i(R_\bullet) = 0$ for all $i > 0$. These results has implications in MacLane homology theory. We extend the computation of Franjou-Lannes and Schwartz $[4]$ of MacLane (co)homology of finite fields with coefficients in symmetric $S^d$ and divided powers $\Gamma^d$ to arbitrary commutative $\mathbb{F}_p$-algebras, provided that $d > 1$. As a consequence of our computations we show that $H_{\text{ML}}\bullet(A, T) = 0$, provided $T$ is a strict homogeneous polynomial functor of degree $d > 1$ and $A$ is an algebra over a field $K$ of characteristic $p > 0$ with $\text{Card}(K) > d$.

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2. When it is too easy to compute homotopy groups

It is well-known that the homotopy groups of a simplicial abelian group $(A_\bullet, \partial_\bullet, s_\bullet)$ can be computed as the homology of the normalized chain complex $(N_\bullet(A_\bullet), d)$, where

$$N_n(A_\bullet) = \{x \in A_n | \partial_i(x) = 0, i > 0\}$$

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and the boundary map $N_n(A_*) \to N_{n-1}(A_*)$ is induced by $\partial_0$. Our first result shows that if $A_*$ has a simplicial ring structure and the rings involved in $A_*$ satisfy extra conditions then homotopy groups are zero in positive dimensions. This fact is an easy consequence of the following result which is probably well-known

**Lemma 1.** Let $R_*$ be a simplicial object in the category of not necessarily associative rings and let $x, y \in N_n(R_*)$ be two elements. Assume $n > 0$ and $x$ is a cycle. Then the cycle $xy \in N_n(R_*)$ is a coboundary.

**Proof.** Consider the element

$$z = s_0(xy) - s_1(x)s_0(y).$$

Then we have

$$\partial_0(z) = xy - (s_0\partial_0(x))y = xy.$$

Moreover,

$$\partial_1(z) = xy - xy = 0,$$

We also have

$$\partial_2(z) = (s_0\partial_1(x))(s_0\partial_1(y)) - x(s_0\partial_1(y)) = 0.$$

Similarly for all $i > 2$ we have

$$\partial_i(z) = (s_0\partial_{i-1}(x))(s_0\partial_{i-1}(y)) - (s_1\partial_{i-1}(x))(s_0\partial_{i-1}(y)) = 0.$$

Hence $z$ is an element of $N_{n+1}(R_*)$ with $\partial(z) = xy$. \hfill $\square$

**Corollary 2.** Let $R_*$ be a simplicial ring. If the rings involved in $R_*$ satisfy $x^m = x$ identity for $m \geq 2$, then

$$\pi_n(R_*) = 0, \ n > 0.$$

**Proof.** Take a cycle $x \in N_n(R_*)$, $n > 0$. Then the class of $x = xx^{m-1}$ in $\pi_n(R_*)$ is zero. \hfill $\square$

**Remark.** A more general fact is true. Let $\mathbf{T}$ be a pointed algebraic theory [15] and let $X_*$ be a simplicial object in the category of $\mathbf{T}$-models [15]. Then $\pi_1(X_*)$ is a group object in the category of $\mathbf{T}$-models, while $\pi_i(X_*)$ are abelian group objects in the category of $\mathbf{T}$-models for all $i > 1$. Thus $\pi_i(X_*) = 0$, $i \geq 1$ provided all group objects are trivial. This is what happens for the category of rings satisfying the identity $x^m = x$, $m \geq 2$. Another interesting case is the category of Heyting algebras [3].

### 3. Hochschild homology with twisted coefficients

In what follows the ground field is the prime field $\mathbb{F}_p$ of characteristic $p > 0$. All algebras are taken over $\mathbb{F}_p$ and they are assumed to be associative. For an algebra $R$ and an $R$-$R$-bimodule $B$ we let $H_* (R, B)$ and $H^* (R, B)$ be the Hochschild homology and cohomology of $R$ with coefficients in $B$. Let us recall that

$$H_* (R, B) = \text{Tor}_{R \otimes R^{op}} (R, B)$$

and

$$H^* (R, B) = \text{Ext}^{\bullet}_{R \otimes R^{op}} (R, B).$$
Moreover, we let $C_\bullet(R, B)$ be the standard simplicial vector space computing Hochschild homology
$$\pi_\bullet(C_\bullet(R, B)) \cong H_\bullet(R, B).$$
Recall that $C_n(R, B) = B \otimes R \otimes \cdots \otimes R$, while
$$\partial_0(b, r_1, \cdots, r_n) = (br_1, \cdots, r_n),$$
$$\partial_i(b, r_1, \cdots, r_n) = (b, r_1, \cdots, r_i r_{i+1}, \cdots, r_n), \quad 0 < i < n$$
and
$$\partial_n(b, r_1, \cdots, r_n) = (r_n b, r_1, \cdots, r_{n-1}).$$
Here $b \in B$ and $r_1, \cdots, r_n \in R$.

Let $n > 1$ be a natural number and let $A$ be a commutative $\mathbb{F}_p$-algebra. The Frobenius homomorphism gives rise to the functors $\Phi^n$ from the category of $A$-modules to the category of $A$-bimodules, which are defined as follows. For an $A$-module $M$ the bimodule $\Phi^n(M)$ coincides with $M$ as a left $A$-module, while the right $A$-module structure on $\Phi^n(M)$ is given by
$$ma = a p^n m, \quad a \in A, \ m \in M.$$ Having $A$-bimodule $\Phi^n(M)$ we can consider the Hochschild homology $H_\bullet(A, \Phi^n(M))$. In this section we study these homologies. In order to state our results we need some notation. We let $\psi^n(A)$ be the quotient ring $A/(a - a^{p^n})$, $n \geq 1$ which is considered as an $A$-module via the quotient map $A \rightarrow \psi^n(A)$. Thus $\psi^n$ is the left adjoint of the inclusion of the category of commutative $\mathbb{F}_p$-algebras with identity $x^m = x, m = p^n$ to the category of all commutative $\mathbb{F}_p$-algebras.

**Example 3.** Let $n > 1$. If $K$ is a finite field with $q = p^d$ element then $\psi^n(K) = K$ if $n = dt$, $t \in \mathbb{N}$ and $\psi^n(K) = 0$ if $n \neq dt$, $t \in \mathbb{N}$.

**Lemma 4.** Let $A$ is a commutative algebra over a field $K$ of characteristic $p > 0$ with $\text{Card}(K) > p^n$. Then $\psi^n(A) = 0, \ n > 1$.

**Proof.** By assumption there exists $k \in K$ such that $k^{p^n} - k$ is an invertible element of $K$. It follows then that the elements of the form $a^{p^n} - a$ generates whole $A$. \hfill \Box

**Theorem 5.** Let $A$ be a commutative $\mathbb{F}_p$-algebra and $n > 1$. Then
$$H_i(A, \Phi^n(A)) = 0$$
for all $i > 0$ and
$$H_0(A, \Phi^n(A)) \cong \psi^n(A).$$

**Proof.** The proof consists of three steps.

**Step 1.** The theorem holds if $A = \mathbb{F}_p[x]$. In this case we have the following projective resolution of $A$ over $A \otimes A = \mathbb{F}_p[x, y]$:
$$0 \rightarrow \mathbb{F}_p[x, y] \xrightarrow{\eta} \mathbb{F}_p[x, y] \xrightarrow{\epsilon} \mathbb{F}_p[x] \rightarrow 0.$$
Here $\epsilon(x) = \epsilon(y) = x$ and $\eta$ is induced by multiplication by $(x - y)$. Hence for any $A$-$A$-bimodule $B$, we have $H_i(A, B) = 0$ for $i > 1$ and
\[
H_0(A, B) \cong \text{Coker}(u) \quad \text{and} \quad H_1(A, B) \cong \text{Ker}(u),
\]
where $u : B \to B$ is given by $u(b) = xb - bx$. If $B = \Phi^n(\mathbb{F}_p[x])$, then $u : \mathbb{F}_p[x] \to \mathbb{F}_p[x]$ is the multiplication by $(x^n - x)$ and we obtain $H_1(A, \Phi^n(A)) = 0$ and $H_0(A, \Phi^n(A)) = \psi^n(A)$.

**Step 2. The theorem holds if $A$ is a polynomial algebra.** Since Hochschild homology commutes with filtered colimits it suffices to consider the case when $A = \mathbb{F}_p[x_1, \ldots, x_d]$. By the Künneth theorem for Hochschild homology (see [10, Theorem X.7.4]) we have $H_*(A, \Phi^n(A)) = H_*(\mathbb{F}_p[x], \Phi^n(\mathbb{F}_p[x]))^{\otimes d}$ and the result follows.

**Step 3. The theorem holds for arbitrary $A$.** We use the same method as used in the proof of [9, Theorem 3.5.8]. First we choose a simplicial commutative algebra $L_\bullet$ such that each $L_n$ is a polynomial algebra, $n \geq 0$ and $\pi_i(L_\bullet) = 0$ for all $i > 0$, $\pi_0(L_\bullet) = A$. Such a resolution exist thanks to [14]. Now consider the bisimplicial vector space $C_*(L_\bullet, \Phi^n(L_\bullet))$. The $s$-th horizontal simplicial vector space is the simplicial vector space $L_\bullet^{\otimes s+1}$. By the Eilenberg-Zilber-Cartier and Künneth theorems it has zero homotopy groups in positive dimensions and $\pi_0(L_\bullet^{\otimes s+1}) = A^{\otimes s+1}$. On the other hand the $t$-th vertical simplicial vector space of $C_*(L_\bullet, \Phi^n(L_\bullet))) is isomorphic to the Hochschild complex $C_*(L_t, \Phi^n(L_t)))$ which has zero homology in positive dimensions by the previous step. Hence both spectral sequences corresponding to the bisimplicial vector space $C_*(L_\bullet, \Phi^n(L_\bullet))$ degenerate and we obtain the isomorphism
\[
H_*(A, \Phi^n(A)) \cong \pi_*(\psi^n(L_\bullet)).
\]
Now we can use Lemma 2 to finish the proof. \qed

**Corollary 6.** Let $A$ be a commutative $\mathbb{F}_p$-algebra, $M$ be an $A$-module and $n > 1$. Then there exist functorial isomorphisms
\[
H_*(A, \Phi^n(M)) \cong \text{Tor}_*^A(\psi^n(A), M), \quad n \geq 0
\]
and
\[
H^*(A, \Phi^n(M)) \cong \text{Ext}^*_A(\psi^n(A), M), \quad n \geq 0.
\]
In particular, if $A$ is a commutative algebra over a field $K$ of characteristic $p > 0$ with $\text{Card}(K) > p^n$, then
\[
H_*(A, \Phi^n(M)) = 0 = H^*(A, \Phi^n(M)).
\]

**Proof.** Observe that $C_*(A, \Phi^n(A))$ is a complex of left $A$-modules. By Theorem 5 it is a free-resolution of $\psi^n(A)$ in the category of $A$-modules. Hence it suffices to note that
\[
C_*(A, \Phi^n(M)) \cong M \otimes_A C_*(A, \Phi^n(A)),
\]
\[
C^*(A, \Phi^n(M)) \cong \text{Hom}_A(C_*(A, \Phi^n(A)), M),
\]
where $C^*$ denotes the standard complex for Hochschild cohomology. The last assertion follows from Lemma 4. \qed
Example 7. It follows for instance that $H^i(A, \Phi^n(M)) = 0$, $i > 0$, provided $M$ is an injective $A$-module and $n > 1$. In particular $H^i(A, \Phi^n(A)) = 0$ if $A$ is a self-injective algebra. On the other hand if $A = \mathbb{F}_p[x_1, \ldots, x_d]$ then $H^i(A, \Phi^n(A)) = 0$, $i \neq d$, $n > 1$ and $H^d(A, \Phi^n(A)) = \psi^n(A)$, $n > 1$.

4. Application to MacLane cohomology

We recall the definition of MacLane (co)homology. For an associative ring $R$ we let $\mathcal{F}(R)$ be the category of finitely generated free left $R$-modules. Moreover, we let $\mathcal{F}(R)$ be the category of all covariant functors from the category $\mathcal{F}(R)$ to the category of all $R$-modules. The category $\mathcal{F}(R)$ is an abelian category with enough projective and injective objects. By definition [8] the MacLane cohomology of $R$ with coefficient in a functor $T \in \mathcal{F}(R)$ is given by

$$H_{\text{ML}}^\bullet(R, T) := \text{Ext}^\bullet_{\mathcal{F}(R)}(I, T),$$

where $I \in \mathcal{F}(R)$ is the inclusion of the category $\mathcal{F}(R)$ into the category of all left $R$-modules. One defines MacLane homology in a dual manner (see [13, Proposition 3.1]). For an $R$-$R$-bimodule $B$, one considers the functor $B \otimes_R (-)$ as an object of the category $\mathcal{F}(R)$. For simplicity we write $H_{\text{ML}}^\bullet(R, B)$ instead of $H_{\text{ML}}^\bullet(R, B \otimes_R (-))$. There is a binatural transformation

$$H_{\text{ML}}^\bullet(R, B) \to H^\bullet(R, B)$$

which is an isomorphism in dimensions 0 and 1.

In the rest of this section we consider MacLane (co)homology of commutative $\mathbb{F}_p$-algebras.

Lemma 8. For any commutative $\mathbb{F}_p$-algebra $A$ one has an isomorphism

$$H_{\text{ML}}^{2i}(A, \Phi^n(A)) = \psi^n(A), \ i \geq 0, n > 1,$$

and

$$H_{\text{ML}}^{2i+1}(A, \Phi^n(A)) = 0, \ i \geq 0, n > 1.$$

Proof. According to [12, Proposition 4.1] there exists a functorial spectral sequence

$$E_{pq}^2 = H_p(A, H_{\text{ML}}_q(\mathbb{F}_p, B)) \Rightarrow H_{\text{ML}}_{p+q}(A, B).$$

Here $B$ is an $A$-$A$-bimodule. By the well-known computation of Breen [2], Bökstedt [1] (see also [4]) we have

$$H_{\text{ML}}^{2i}(\mathbb{F}_p, B) = B$$

and

$$H_{\text{ML}}^{2i+1}(\mathbb{F}_p, B) = 0.$$

Now we put $B = \psi^n(A)$ and use Theorem 5 to get $E_{pq}^2 = 0$ for all $p > 0$. Hence the spectral sequence degenerates and the result follows. \qed
We now consider MacLane cohomology with coefficients in strict polynomial functors [6]. Let us recall that the strict homogeneous polynomial functors of degree \(d\) form an abelian category \(\mathcal{P}_d(A)\) and there exist an exact functor \(i : \mathcal{P}_d(A) \rightarrow \mathcal{F}(A)\) [7]. For an object \(T \in \mathcal{P}_d(A)\) we write \(\text{HML}_\bullet(A, T)\) instead of \(\text{HML}_\bullet(A, i(T))\). Projective generators of the category \(\mathcal{P}_d\) are tensor products of the divided powers, while the injective cogenerators are symmetric powers. Let us recall that the \(d\)-th divided power functor \(\Gamma^d \in \mathcal{F}(A)\) and \(d\)-th symmetric functors \(S^n\) are defined by

\[
\Gamma^d(M) = (M^{\otimes d})_{\Sigma_d}, \quad S^n(M) = (M^{\otimes d})_{\Sigma_d}.
\]

Here tensor products are taken over \(A\), \(\Sigma_d\) is the symmetric group on \(d\)-letters, which acts on the \(d\)-th tensor power by permuting of factors, \(M \in \mathcal{F}(A)\) and \(X^G\) (resp. \(X_G\)) denotes the module of invariants (resp. coinvariants) of a \(G\)-module \(X\), where \(G\) is a group.

For a functor \(T \in \mathcal{F}(A)\) we let \(\widetilde{T} \in \mathcal{F}(\mathbb{F}_p)\) be the functor defined by

\[
\widetilde{T}(V) = T(V \otimes A).
\]

According to [13, Theorem 4.1] the groups \(\text{HML}_i(\mathbb{F}_p, \widetilde{T})\) have an \(A\)-\(A\)-bimodule structure. The left action comes from the fact that \(T\) has values in the category of left \(A\)-modules, while the right action comes from the fact that \(T\) is defined on \(\mathcal{F}(A)\). In particular it uses the action of \(T\) on the maps \(l_a : X \rightarrow X\), where \(a \in A\), \(X \in \mathcal{F}(A)\) and \(l_a\) is the multiplication on \(a\). Since \(T(l_a) = l_{a^t}\) if \(T\) is a strict homogeneous polynomial functor of degree \(d\) [6], the bimodule \(\text{HML}_i(\mathbb{F}_p, \widetilde{T})\) is of the form \(\Phi^n(M)\) provided \(d = p^n\).

**Theorem 9.** Let \(d > 1\) be an integer and let \(A\) be a commutative \(\mathbb{F}_p\)-algebra. Then \(\text{HML}_\bullet(A, \Gamma^d) = 0\) if \(d\) is not a power of \(p\). If \(d = p^n\) and \(n > 0\), then

\[
\text{HML}_i(A, \Gamma^d) = 0 \quad \text{if } i \neq 2p^n t, \quad t \geq 0
\]

and

\[
\text{HML}_i(A, \Gamma^d) = \psi^n(A) \quad \text{if } i = 2p^n t, \quad t \geq 0.
\]

In particular \(\text{HML}_\bullet(A, \Gamma^d) = 0\) provided \(A\) is an algebra over a field \(K\) of characteristic \(p > 0\) with \(\text{Card}(K) > d\).

**Proof.** According to [13, Theorem 4.1], [12] there exists a functorial spectral sequence:

\[
E^2_{pq} = H_p(A, \text{HML}_q(\mathbb{F}_p, \widetilde{T})) \Longrightarrow \text{HML}_{p+q}(A, T).
\]

For \(T = \Gamma^n_A\) one has \(\widetilde{T} = \Gamma^n_{\mathbb{F}_p} \otimes A\). Here we used the notation \(\Gamma^n_A\) in order to emphasize the dependence on the ring \(A\). By the result of Franjou, Lannes and Schwartz [4] \(\text{HML}_i(\mathbb{F}_p, \widetilde{T})\) vanishes unless \(d = p^n\) and \(i = 2p^n t, \quad t \geq 0\). Moreover in these exceptional cases \(\text{HML}_i(\mathbb{F}_p, \widetilde{T})\) equals to \(\Phi^n(A)\) (as an \(A\)-\(A\)-module). Hence the spectral sequence together with Theorem [5] gives the result. \(\square\)

**Corollary 10.** Let \(A\) be a commutative algebra over a field \(K\) of characteristic \(p > 0\) with \(\text{Card}(K) > d > 1\). If \(T\) is a strong homogeneous polynomial functor of degree \(d\). Then

\[
\text{HML}_\bullet(A, T) = 0 = \text{HML}^\bullet(A, T).
\]
Proof. We already proved that the result is true if $T$ is a divided power. By the well-known vanishing result [11] the result is also true if $T = T_1 \otimes T_2$ with $T_1(0) = 0 = T_2(0)$. Since any object of $\mathcal{P}_d$ has a finite resolution which consists with finite direct sums of tensor products of divided powers [6] the result follows. \qed

References