Research Article

Optimal Bounds for the Variance of Self-Intersection Local Times

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1. Introduction and Main Results

Let $X, X_1, X_2, \ldots$ be independent, identically distributed, $\mathbb{Z}^d$-valued random variables, and define the random walk $S_0 = 0$, $S_n = \sum_{j=1}^{n} X_j$, for $n \geq 1$. The special case with $\mathbb{P}(X_1 = e) = 1/(2d)$, for all $e \in \mathbb{Z}^d$, with $|e| = 1$, is known as the simple random walk in $\mathbb{Z}^d$ and will be denoted by $(\text{SRW}_n)_{n \in \mathbb{N}_0}$.

Let $l(n, x) = \sum_{j=1}^{n} 1(S_j = x)$ be the local time of $(\text{SRW}_n)_{n \in \mathbb{N}_0}$ at the site $x \in \mathbb{Z}^d$, and define for a positive integer $\alpha$ the $\alpha$-fold self-intersection local time

$$L_n = L_n(\alpha) = \sum_{x \in \mathbb{Z}^d} l(n, x)^\alpha$$

$$= \sum_{i_1, \ldots, i_{\alpha} = 0}^{n} \mathbb{1} \{ S_{i_1} = \cdots = S_{i_{\alpha}} \}. \quad (1)$$

We will denote the corresponding quantities for simple random walk in $\mathbb{Z}^d$ by $L_n^{\text{SRW}}(\alpha, d)$ or simply $L_n^{\text{SRW}}(\alpha)$ when the dimension is clear from the context.

Let $R^+$ and $R^-$ be, respectively, the semigroup and the group generated by the support of $X$,

$$R^+ = \{ x \in \mathbb{Z}^d \mid \mathbb{P}(S_n = x) > 0 \text{ for some } n \geq 0 \},$$

$$R = \{ x \in \mathbb{Z}^d \mid x = y - z \text{ for some } x, y \in R^+ \}. \quad (2)$$

Following Spitzer [1], we call the random variable $X$ and the random walk it generates genuinely $d$-dimensional if the group $R$ is $d$-dimensional.

The quantity $L_n(\alpha)$ has received considerable attention in the literature due to its relation to self-avoiding walks and random walks in random scenery. In particular let the random scenery $\{\xi_x, x \in \mathbb{Z}^d\}$ be a collection of i.i.d. random variables, independent of $(S_n)_{n \in \mathbb{N}_0}$, and define the process $Z_0 = 0$, $Z_n = \sum_{j=1}^{n} \xi_{S_j}$. Then $(Z_n)_{n \in \mathbb{N}_0}$ is commonly referred to as random walk in random scenery and was introduced in Kesten and Spitzer [2], where functional limit theorems were obtained for $Z_{[nt]}$ under appropriate normalization for the case $d = 1$. The case $d = 2$, with $X_1$ centered with nonsingular covariance matrix, was treated in [3] where it
Theorem 1. Let \( X, X_1, X_2, \ldots \) be independent, identically distributed, and genuinely \( d \)-dimensional valued random variables, for any \( d \geq 1 \). Then, there exist positive constants \( C_{\alpha,X} > \epsilon_{\alpha,X} > 0 \), depending on \( \alpha \) and the distribution of \( X \), such that for all \( n \) large enough

\[
\text{var} \left( L_n (\alpha) \right) \leq \epsilon_{\alpha,X} \text{var} \left( L_n^\text{SRW}(\alpha,d) \right) \leq C_{\alpha,X} \text{var}_d(n). \tag{4}
\]

The result was motivated by \cite{1, 10} and improves related results of Becker and König for \( d = 3 \) and \( d = 4 \). Several cases treated in \cite{3, 4, 10–13} can then be obtained as particular cases.

Moreover, we also show the surprising converse. More precisely, we show that the right asymptotic behaviour of \( \text{var}(L_n^\alpha) \) implies that the jumps must have zero mean and finite second moment.

\[
\text{Theorem 2. Let } X, X_1, X_2, \ldots \text{ be independent, identically distributed, and genuinely } d\text{-dimensional with } d \leq 3. \text{ If}
\]

\[
\liminf_{n \to \infty} \frac{\text{var} \left( L_n (\alpha) \right)}{\text{var} \left( L_n^\text{SRW}(\alpha) \right)} > 0, \tag{5}
\]

then \( E|X|^2 < \infty \) and \( EX = 0 \).

As it follows from Theorem 3 given below for \( d = 2, 3 \) and from Theorem 5.2.3 in Chen \cite{12} for \( d = 1 \), if \( EX = 0 \) and \( 0 < E|X|^2 < \infty \), then \( \liminf_{n} \frac{\text{var} \left( L_n (\alpha) \right)}{\text{var} \left( L_n^\text{SRW}(\alpha) \right)} > 0 \).

For any genuinely \( d\)-dimensional random walk with finite second moments and zero mean, the asymptotic behaviour of \( \text{var}(L_n(\alpha)) \) is similar to that of the \( d\)-dimensional simple symmetric random walk. Also, as it follows from our general bounds (see Proposition 4 and Corollary 7) that the asymptotic results for the genuinely \( d\)-dimensional random walk can be reproduced by those of the symmetric one-dimensional random walk with appropriately chosen heavy tails, as was indicated by Kesten and Spitzer \cite{2}. The proofs are based on adapting the Tauberian approach developed in \cite{13}.

\[
\text{Theorem 3. Let } d = 1, 2, 3, \text{ and suppose that for } t \in \Gamma := [-\pi, \pi]^d \text{ one has}
\]

\[
f(t) = 1 - g|t| + R(t), \quad \text{for } d = 1,
\]

\[
or f(t) = 1 - \langle \Sigma t, t \rangle + R(t), \quad \text{for } d = 2, 3,
\]

where \( \Sigma \) is a nonsingular covariance matrix and \( R(t) = o(|t|) \) for \( d = 1 \) and \( o(|t|^2) \) for \( d = 2, 3 \) as \( t \to 0 \). Then

\[
\text{var} \left( L_n (\alpha) \right) \sim \begin{cases} \frac{\pi^2 + 6}{12} \frac{(\alpha!)^2 (\alpha - 1)!}{(\pi^2)^{\alpha-2}} n^2 \log(n)^{2\alpha-4}, & \text{for } d = 1, \\
\frac{1}{2} \frac{(\alpha!)^2 (\alpha - 1)!}{(2\pi \sqrt{2})^d} n^d \log(n)^{3\alpha-4} (\kappa + 1), & \text{for } d = 2, \\
(\kappa_1 + \kappa_2) n \log(n), & \text{for } d = 3, \alpha = 2,
\end{cases}
\]

where

\[
\kappa = \int_0^{\infty} dk \int_0^{\infty} ds \left[ (1 + r) (1 + s) \sqrt{1 + r + s}^2 - 4rs \right]^{-1}
\]

\[
\frac{\pi^2}{6},
\]

and \( \kappa_1 \) and \( \kappa_2 \) are defined in (58) and (63), respectively.

Moreover, if \( L_n(\alpha) \) is the self-intersection local time of another random walk, independent of \( (S_n)_n \), whose characteristic function also satisfies (6), then \( \text{var} \left( L'_n(\alpha) \right) = \text{var} \left( L_n(\alpha) \right)(1 + o(1)) \).

2. Proofs

2.1. General Bounds. We first develop a technique to treat random walks with independent but not necessarily identically distributed increments.
Proposition 4 (general upper bound). Assume that $X_1, X_2, \ldots$ are independent $\mathbb{Z}^d$-valued random variables and let $S_{u,v} := X_u + \cdots + X_{u+v}$. Suppose further that for all $n \in \mathbb{N}$ and integers $a, b, v \geq 0$, with $a + u \leq b$ and any $x \in \mathbb{Z}^d$, one has

$$
P(S_{a,u} \pm S_{b,v} = x) \leq \phi(u + v), \quad \text{(A)}
$$

$$
P(S_{a,u} = 0) = \phi(S_{a,u} + S_{b,v} = 0) \leq \psi(u, v), \quad \text{(B)}
$$

where $\phi(u)$ is nonincreasing and $\psi(u, v)$ is nonincreasing in $u$ and is nondecreasing and subadditive in $v$ in the sense that $\psi(u, v + w) \leq A_{\psi}[\psi(u, v) + \psi(u, w)]$, for some constant $A_{\psi}$ independent of $u$, $v$, and $w$. Then, for some constant $K = cA_{\psi}(1 + A_{\psi})^{\alpha - 2}$ depending only on $\alpha$,

$$
\var(\lambda_n(\alpha)) \leq Kn \left( \sum_{i=0}^{n-1} \phi(i) \right)^{2\alpha - 4} \cdot \sum_{i, j = 0}^{n-1} [\phi(j_1 \vee i) \phi(k \vee i) + \phi(j_1 \psi(i + k, j))].
$$

Proof of Proposition 4. We first write out the variance as a sum

$$
\var(\lambda_n(\alpha)) = (\alpha!)^2 \sum_{k_1, \ldots, k_{n-1} \leq -k_a \quad l_1, \ldots, l_{n-1} \leq k_a} \left[ \prod_{t=2}^{n-1} \sum_{\gamma(t)} \right] \frac{\prod_{i=1}^{n} \phi(i)}{2^{n-1}} \cdot \sum_{k_1 \leq \cdots \leq k_a \quad l_1 \leq \cdots \leq l_a} \sum_{w_1, \ldots, w_a} \sup_{u, v} P(S_{w_1, u} = \delta_1 y) \cdot \sum_{w_1, \ldots, w_a} \sum_{u, v} \sup_{w_1} P(S_{w_1, u} = \delta_1 y).
$$

Summing over the free index $m_{ij}$, it is clear that

$$
I_n \leq (n + 1) \sum_{m_1, \ldots, m_{2\alpha - 1}} \sum_{y \in \mathbb{Z}^d} \sum_{\delta(\delta) \leq 3} \prod_{t=1}^{2\alpha - 1} \sup_{w_t} P(S_{w_t, m_t} = \delta_1 y).
$$

For any $\delta = (\delta_1, \ldots, \delta_{2\alpha - 1})$ with $\nu(\delta) = \nu$, exactly $u := 2\alpha - 1 - \nu$ elements are equal to 0, and therefore by Assumption (A) with $x = 0$ we have

$$
I_n \leq C (n + 1) \sum_{t=3}^{\alpha} \sum_{i=0}^{n} \phi(i) \left( x^{2\alpha - 1 - \nu} \right) \cdot \sum_{j_1, \ldots, j_{\nu} \in \mathbb{Z}^d} \sum_{\iota \in \{-1, 1\}^\nu} \prod_{t=1}^{\nu} \sup_{w_t} P(S_{w_t, j_t} = \delta_1 y).
$$

An important role is played by the manner in which the two sequences are interlaced, since, for example, if $k_{\alpha} \leq l_1$ or $l_\alpha \leq k_1$, the term vanishes by the Markov property.

We will treat the sum over indices with $k_1 \leq l_1$. The sum over the remaining index set with $k_1 > l_1$ can be treated in a similar fashion and will contribute a constant factor. Therefore, we assume that $k_1 \leq l_1$ and we arrange the two sequences in an ordered sequence of combined length $2\alpha$ which we denote as $(p_1, \ldots, p_{2\alpha})$; we also define $(e_1, \ldots, e_{2\alpha})$ where $e_i = 0$ if $p_i$ came from $k = (k_1, \ldots, k_\alpha)$ and $e_i = 1$ if $p_i$ came from $I = (l_1, \ldots, l_{\alpha})$. Finally we define two new sequences $m_0, m_1, \ldots, m_{2\alpha - 1}$, and $\delta_1, \ldots, \delta_{2\alpha - 1}$, where $m_0 = p_1, m_1 = p_{1+1} - p_1$, and $\delta_i = e_i - e_j, i = 1, \ldots, \alpha$. Notice that since we assume that $k_1 \leq l_1$, we have $p_1 = k_1$ and $e_i = 0$. Let $\nu(\delta) := \sum_{i=1}^{2\alpha - 1} |\delta_i|$ denote the interlacement index. The terms with $\nu = 1$ vanish, while the terms with $\nu = 2$ will be considered separately.

Terms with $\nu \geq 3$. We first consider the sum $I_n$ over the terms with $\nu \geq 3$ for which we drop the negative part and obtain the bound

$$
\sum_{\nu \geq 3} \sum_{m_1, \ldots, m_{2\alpha - 1}} \sum_{y \in \mathbb{Z}^d} \sum_{\delta(\delta) \leq 3} \prod_{t=1}^{2\alpha - 1} \sup_{w_t} P(S_{w_t, m_t} = \delta_1 y).
$$

Letting $(\bar{S}_{w, n \in \mathbb{N}})$ denote an independent copy of the random walk $(S_n)_{n \in \mathbb{N}}$ and assuming without loss of generality that $j_1 \leq \cdots \leq j_\nu$, we have that for any $\delta \in \{-1, 1\}^\nu$

$$
\sup_{w_1, w_2} P(S_{w_1, j_1} = \delta_1 y) \leq \left( \sum_{t=2}^{\nu-1} \sup_{y \in \mathbb{Z}^d} P(S_{w_1, j_t} = y) \right) \cdot \sup_{w_1, w_2} P(S_{w_1, j_1} - \delta_1 \bar{S}_{w_1, j_0} = 0) \cdot \left( \prod_{t=2}^{\nu-1} \phi(j_t) \right) \cdot \phi(j_1 + j_\nu) \leq \nu \cdot \left( \prod_{t=2}^{\nu-1} \phi(j_t) \right) \cdot \phi(j_1 + j_\nu).
$$

(10)
Let $G_n := \sum_{i=0}^{n} \phi(i)$. Since $\phi$ is nonincreasing we have that
\[
\Delta_{n,v} := \sum_{0 \leq j \leq \lfloor s \rfloor - 1} \prod_{t=2}^{v} \phi(j \lor j_t)
\leq \sum_{j=0}^{n-1} \phi(j) \sum_{v \leq j \leq \lfloor s \rfloor - 1} \prod_{t=2}^{v} \phi(j \lor j_t)
= G_n \Delta_{n,v-1},
\]
and iterating this procedure, for $v \geq 3$, we have that $\Delta_{n,v} \leq \Delta_{n,3} G_{n}^{2v-3}$. Combining the two bounds and summing over $v = 3, \ldots, 2\alpha - 1$, we have that
\[
I_n \leq \sum_{v=3}^{2\alpha-1} c(\alpha) n \Gamma_{n,v} \leq c(\alpha) n \Gamma_{n,2\alpha-1} \Delta_{n,3}
\leq c(\alpha) n \Gamma_{n,2\alpha-1} \Delta_{n,3}
\]
where $c(\alpha)$ is a constant depending only on $\alpha$.

Terms with $v = 2$. Next we consider the sum $J_n$ over the terms with $v = 2$, which occurs when, for some $j$, the indices $i_1, \ldots, i_a$ all lie in $[k, k+1]$. Then it is easy to see that this sum $J_n$ is bounded above by
\[
J_n \leq Cn \sup_{w \in \Gamma \setminus N} \sum_{m_{t-1} \geq 1} \sum_{t=0}^{\alpha-2} \prod_{t=1}^{\alpha-1} \phi(m_t) \psi(m_0 + m_{t-1}, m_1 + \cdots + m_{\alpha-1})
\leq Cn \Gamma_{n,2\alpha-1} \Delta_{n,2\alpha-1} \Delta_{n,3}
\leq c(\alpha) n \Gamma_{n,2\alpha-1} \Delta_{n,3}
\]

Corollary 5. Assume that the conditions of Proposition 4 are satisfied with $\phi(m) = T/m$ and $\psi(m, k) = T m^{r-1} (k \land m)$. Then,
\[
\var(L_n(\alpha)) \leq c_n T^{2\alpha-2}
\]

It is straightforward to see that Corollary 5 includes random walks with mean zero and finite second moment; for example, $d = 2$ corresponds to $r = 1$ and $d = 3$ to $r = 3/2$. Therefore several relevant results in [3, 7–13] are obtained as a special case of Corollary 5 and extended to the case of independent but not necessarily identically distributed variables, for example, by applying the local limit theorem, as conducted in [8].

Also when the random walk increment $X$ is in the domain of attraction of the one-dimensional symmetric Cauchy law [13, 14] or in the case of planar random walk with second moments [3, 7–9, 11], it is well known that the conditions of Proposition 4 are satisfied with $\phi(m) = T/m$ and $\psi(m, k) = T m^{r-1} (k \land m)$.

However, we can do better for symmetric variables and show that condition (A) implies (B), which together with the comparison technique motivates the following results. For a real number $\alpha$, we write $[x]$ for the integer part of $x$.

Proposition 6 (bounds via comparison with characteristic function of symmetric random variables). Let $X_1, X_2, \ldots$, be independent $\mathbb{Z}^d$-valued random variables and let $f_i(\Gamma) := E \exp(itX_i)$. Assume that there exist a measurable function $f : \Gamma \to [0, 1]$ and a positive nonincreasing sequence $\{\phi(m)\}_{m \in \mathbb{N}_0}$, such that
\[
|1 - f_i(\Gamma)| \leq T f_i(\Gamma),
\]
\[
|f_i(\pm t)| \leq f_i(t),
\]
\[
\int_{\Gamma} f_i(t)^m \, dt \leq \phi(m),
\]
for all integers $i, m \geq 0$, all $t \in \Gamma$, and some positive constant $T$. Then there exists another positive constant $K = c(\alpha, d, T)$ such that
\[
\var(L_n(\alpha)) \leq Kn \left( \sum_{i=0}^{n-1} \phi\left( \left[ \frac{i}{2} \right] \right) \right)^{2\alpha-4} \sum_{j=0}^{n} \phi\left( \left[ \frac{j}{2} \right] \right) \sum_{k=j}^{\infty} \phi\left( \left[ \frac{k}{2} \right] \right).
\]
Proof of Proposition 6. Using the notation of Proposition 4, for positive integers \(a, u, b, v\), with \(a + u \leq b\), and any \(x \in \mathbb{R}^d\)

\[
P(S_{a,u} + e \cdot S_{b,v} = x)
\leq \frac{1}{(2\pi)^d} \int f(t)^{au} \prod_{j=\lfloor a+u \rfloor}^{\lfloor b+v \rfloor} |f_j(\epsilon_j, t)| \, dt
\leq \frac{1}{(2\pi)^d} \int f(t)^{au} \, dt \leq \frac{1}{(2\pi)^d} \phi(u + v).
\]

(21)

To find \(\psi(u, v)\), notice that since \(f(t) \geq 0\),

\[
\phi(m) \geq \int f(t)^m [1 - f(t)^m] \, dt
= \sum_{j=0}^{m-1} \int f(t)^{mj} (1 - f(t)) \, dt
\geq m \int f(t)^m (1 - f(t)) \, dt = mQ(2m)
\]

whence \(Q(m) \leq 2\phi([m/2])/m\). Therefore,

\[
|\mathbb{P}(S_{a,u} = 0) - \mathbb{P}(S_{a,u} + S_{b,1} = 0)|
\leq \frac{1}{(2\pi)^d} \int \left| \sum_{j=\lfloor a+u \rfloor}^{\lfloor b+1 \rfloor} f_j(\epsilon_j, t) \right| \, dt
\leq CT \int f(t)^m [1 - f(t)] \, dt \leq \frac{CT\phi([u/2])}{u}.
\]

(23)

A telescoping argument implies that

\[
|\mathbb{P}(S_{a,u} = 0) - \mathbb{P}(S_{a,u} + S_{b,v} = 0)| \leq CT\phi\left(\left\lfloor \frac{u}{2}\right\rfloor\right) \frac{v}{u}.
\]

(24)

On the other hand for \(u \leq v\) we can obtain a tighter bound through

\[
\mathbb{P}(S_{a,u} = 0) - \mathbb{P}(S_{a,u} + S_{b,v} = 0) \leq \mathbb{P}(S_{a,u} = 0)
\leq \phi(u).
\]

(25)

Combining the two bounds above it follows that (B) is satisfied with \(\psi(u, v) = \phi([u/2]) \min(u, v)/u\). Thus all conditions of Proposition 4 are satisfied and the result follows.

The following corollary allows for the case where \(\phi(m)\) is regularly varying.

Corollary 7. Assume that the conditions of Proposition 6 are satisfied with \(\phi(m) = \varphi(m)m^{-r}\), \(r \geq 1\), where \(\varphi(\cdot)\) is slowly varying at \(\infty\). Then,

\[
\var{L_n(\alpha)} \leq K\Delta_n(\alpha, \phi)
\leq c_4 T^{2r-2} \left\lfloor n^{2r^{-2} \var{\text{var}(X)}} \right\rfloor
\leq c_3 \var{\text{var}(X)}^2 \var{\text{var}(X)}^2.
\]

(26)

Several results in [3, 7–13] are obtained as a special case of Corollary 7 and can be extended to dependent variables, for example, a random walk driven by a hidden Markov chain. In addition, following [2], we can construct a one-dimensional symmetric random walk with characteristic function \(f(t) = 1 - c|t|^{1/r} + o(|t|^{1/r})\), where \(r = 2/d\) for \(d = 2, 3\) and \(r = 1/2\) for \(d \geq 4\), whose asymptotic behaviour is similar to that of genuinely \(d\)-dimensional random walk.

The following example of genuinely \(2\)-dimensional recurrent walk with infinite variance was motivated by Spitzer [1, pp. 87].

Example 8. Let \(X_1, X_2, \ldots\) be independent, identically distributed, \(\mathbb{Z}^2\)-valued random variables, such that \(P(|X_1| = k) = c/(k^2 \log(k)^g)\), for \(k \geq 4\) and \(g \in [0, 1)\). Let \((S_n)_{n \in \mathbb{N}_0}\) be the corresponding random walk in \(\mathbb{Z}^2\). Then we have

\[
\var{L_n(\alpha)} \leq c n^2 \max\left\{ \log n, \log n \right\}^{2r-4} \log n^{2(1-g)}.
\]

(27)

for \(n \geq 10\). Under these assumptions we have that \(P(S_n = 0) \leq c/n \log n^{1-g}\), which is in the critical range, where the random walk is recurrent, without second moment. To see why, we note that by a lengthy but straightforward calculation it can be shown that the characteristic function of \(X\) satisfies

\[
\phi(n) = \frac{c}{n \log (e \vee n)^{1-g}},
\]

where \(h(r) = \left[ 1 + \log\left( \frac{1}{r} \right) \right]^{1-g} \cdot").

The sequence \(\phi(m)\) is identified via Fourier inversion, polar coordinates, and a Laplace argument,

\[
\int_{f(t)n} f(t) \, dt \leq c \int_0^\infty e^{-nr\left( 1 + \log\left( \frac{1}{r} \right) \right)^{-g}} + O(e^{-n}) \leq \frac{c}{n \log (e \vee n)^{1-g}} = \phi(n).
\]

2.2. Bounds for Identically Distributed Variables

Proposition 9 (general upper bound for i.i.d.). Let \(X, X_1, X_2, \ldots\) be independent, identically distributed,
$\mathbb{Z}^d$-valued random variables. Suppose that for any $x \in \mathbb{Z}^d$ and all positive integers $a, u, b,$ and $v,$ with $a + u \leq b,$ it holds that
\begin{equation}
\mathbb{P}(S_{a,u} + S_{b,v} = x) \leq \phi(u + v),
\end{equation}
where $\phi(m)_{m \in \mathbb{N}_0}$ is a nonincreasing sequence. Then for some constant $K = c(\alpha)$ we have that
\begin{equation}
\var(L_n(\alpha)) \leq Kn\left(\sum_{j=0}^{n-1} \phi(j) \sum_{j=0}^{n} \phi\left(\frac{k}{\alpha}\right)\right),
\end{equation}
Proof of Proposition 9. By inspecting the proof of Proposition 6, we notice that we only need to bound the term $J_n.$ Consider typical ordering
\begin{equation}
0 \leq i_1 \leq \cdots \leq i_k \leq \cdots \leq i_{k+1} \leq \cdots \leq i_n \leq n,
\end{equation}
and let us change variables to $(m_0, \ldots, m_{2n})$ such that $m_0 + \cdots + m_{2n} = n.$ Then the contribution to $J_n$ is given by
\begin{equation}
\sum_{m_0, m_{2n}} \prod_{j \leq k, k+1} \mathbb{P}(S_{m_j} = 0) \cdot [\mathbb{P}(S_{m_k, m_{2n}} = 0) - \mathbb{P}(S_{m_k + m_{2n}} = 0)].
\end{equation}
We keep $m_1$ fixed for $j \neq \alpha, k + \alpha$ and we sum over $m = m_k + m_{k+1}$ from 0 to some $M = M(n, m_j)_{j \neq k, k+1}.$ Then for given $m_k, \ldots, m_{k+1},$ the term in the sum is
\begin{equation}
\sum_{m=0}^{M} (m + 1) \left[\mathbb{P}(S_m = 0) - \mathbb{P}(S_{m+q} = 0)\right],
\end{equation}
where $q = m_k + \cdots + m_{k+1}.$ Then since $M \leq n - q,$ it is an easy exercise to show that this sum is bounded above by
\begin{equation}
\sum_{m=0}^{M} (m + 1) \left[\mathbb{P}(S_m = 0) - \mathbb{P}(S_{m+q} = 0)\right] \leq \sum_{m=0}^{q-1} (m + 1) \mathbb{P}(S_m = 0) + q! (n - q \geq q) \sum_{m=q}^{m} \mathbb{P}(S_m = 0) \leq \sum_{m=0}^{m} \mathbb{P}(S_m = 0),
\end{equation}
where $m^* = \max|m_k, \ldots, m_{k+1}|.$ The result follows by summing over all indices apart from $m^*$ and changing the order of summation.

2.3. Proofs of Main Results

Proof of Theorem 1. We apply a comparison argument found to be useful in many areas (e.g., Montgomery-Smith and Pruss [15], and Lefèvre and Utev [16]). More specifically we bound the quantity $\var(L_n)$ by the corresponding quantity for the symmetrised random walk.

Following Spitzer's argument we notice that with $f(t) = \mathbb{E}[\exp(\mathbb{R} \cdot X)]$
\begin{equation}
\mathbb{P}(S_{a,u} + eS_{b,v} = x) \leq e \int \left|f(t)ight|\left|f(-t)\right| \mathbb{E}[\exp(-\frac{\lambda (u + v)}{2} |t|^2) dt.
\end{equation}
Since $|f(t)|^2$ is the characteristic function of a symmetric random variable in $\mathbb{Z}^d,$ for some positive $\lambda,$ we have $1 - |f(t)|^2 \geq \lambda |t|^2,$ and, hence,
\begin{equation}
\mathbb{P}(S_{a,u} + eS_{b,v} = x) \leq c \left[\exp\left(-\frac{\lambda (u + v)}{2} |t|^2\right) dt.
\end{equation}
The result follows from Proposition 9 applied with $\phi(m) = m^{-d/2}.$

The proof of Theorem 2 will be based on the following lemma.

Lemma 10. Assume $X, X_1, X_2, \ldots$ are independent, identically distributed, genuinely $d$-dimensional random variables such that $\mathbb{E}[X] = \infty.$ Then there exists a monotone, slowly varying sequence $(h_n)_{n \in \mathbb{N}_0},$ such that $h_n \to 0$ as $n \to \infty$ and
\begin{equation}
\sup_{x \in \mathbb{Z}^d} \mathbb{P}(S_n = x) \leq c_d \int \mathbb{E}[e^{\mathbb{R} \cdot X}] \mathbb{E}[\exp(-\frac{\lambda (u + v)}{2} |t|^2) dt \leq h_n m^{-d/2}.
\end{equation}
Proof of Lemma 10. Without loss of generality we assume that $X$ is symmetric. Let $\sigma_{\varepsilon} := \mathbb{E}[e \cdot X] I(|X| \leq L).$ Following Spitzer, since $X$ is genuinely $d$-dimensional, we may assume that there exist positive constants $c, W,$ such that for any unit vector $|e| = 1$ we have that $\sigma_{\varepsilon} \geq c$ and $1 - f(t) \geq c|t|^2$ for all $t \in \Gamma.$ Let $\lambda_d$ be the $d$-dimensional Lebesgue measure on $\mathbb{R}^d$ and let $\mu_d$ be the Lebesgue-Haar measure on $\mathbb{S}^{d-1} := \{e \in \Gamma : |e| = 1\}.$ Notice that since $\mathbb{E}[X] = \infty,$ for any $K,$ we have $\mu_d(e : \sigma_{\varepsilon} \geq K) = 0.$

Fix a small positive $x$ such that $\sqrt{c/\lambda} \geq 2W,$ and for any $e > 0$ let $K = K(e) = e^{-d/2}.$ Then there exists $L = L(e) > 0$ small enough so that $\mu_d(e : \sigma_{\varepsilon} < K) \leq e^{d/2}.$ We partition $\mathbb{S}^{d-1}$ in two sets
\begin{equation}
A_{L,K} = \{e \in \mathbb{S}_d : \sigma_{\varepsilon} \geq K\},
\end{equation}
\begin{equation}
\overline{A}_{L,K} = \{e \in \mathbb{S}_d : \sigma_{\varepsilon} < K\},
\end{equation}
so that, for any direction $e \in \overline{A}_{L,K},$
\begin{equation}
\{z \in \mathbb{R} : 1 - f(z e) \leq x) \subseteq \{z : cz^2 \leq x\}
\end{equation}
\begin{equation}
\subseteq \left\{z : |z| \leq \sqrt{\frac{x}{c}}\right\}.
\end{equation}
Hence, using $d$-dimensional spherical coordinates,

$$
\lambda_d \left \{ (z, e) \in \mathbb{R} \times A_{L,K} : 1 - f(iz) \leq x \right \} \leq \mu_d \left \{ A_{L,K} \right \} \left( \frac{x}{c} \right)^{d/2} \left( \frac{1}{d} \right) \leq e^{d/2} \left( \frac{x}{c} \right)^{d/2} \left( \frac{1}{d} \right).
$$

(41)

On the other hand, for any $t$,

$$
1 - f(t) = 2 \sum_{k \in \mathbb{Z}^d} \sin \left( \frac{|t| \cdot k}{2} \right) P(X = k)
\geq \left( \frac{1}{4} \right) E \left( |t \cdot X|^2 \right) I \left( |t \cdot X| \leq \frac{1}{2} \right)
\geq \left( \frac{|t|^2}{4} \right) \sigma_{d/|t|,1/2} |t|.
$$

(42)

Now, assume that $\sqrt{c/x} \geq 2L$. Then for any direction $e \in A_{L,K}$, by choice of $x$ and since $\sigma_{d/2}$ is increasing in $L$, for $ce^2 \leq 1 - f(iz) \leq x$ or $|z| \leq \sqrt{x/c}$, it must be the case that

$$
x \geq 1 - f(iz) \geq \left( \frac{z^2}{4} \right) \sigma_{d/2} \geq \left( \frac{z^2}{4} \right) \sigma_{d/2x}.
$$

(43)

implying that, on the set $A_{L,K}$, it must be that $|z| \leq 2\sqrt{x/K}$. Changing to $d$-dimensional polar coordinates, we find that

$$
\lambda_d \left \{ (z, e) \in \mathbb{R} \times A_{L,K} : 1 - f(iz) \leq x \right \} \leq \int_{A_{L,K}} \int_{|r| \leq \sqrt{x/K}} r^{d-1} dr \, de \leq C_d e^{d/2} x^{d/2}.
$$

(44)

Overall, for $x \leq c/4L^2$, $\lambda_d |t - 1 - f(t) | \leq c_d (xe^{d/2})^2$, and hence $|t | \in \Gamma : 1 - f(t) \leq x$ has Lebesgue measure $o(x^{d/2})$.

Let $F(x)$ be the cumulative distribution function of the random variable log$(1/f(\cdot))$ defined on the probability space $\Gamma$ with normalised Lebesgue measure. Then $F$ is continuous at $x = 0$ and supported on $\mathbb{R}^+$. Moreover, we have that $F(x) = o(x^{d/2})$ as $x \downarrow 0$. Therefore, for some positive sequence $(e_n)_{n \in \mathbb{N}}$ with $e_n \to 0$, we have that

$$
\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(t)^n \, dt = \int_0^\infty e^{-nx} \, dF(x)
= n \int_0^\infty e^{-nx} F(x) \, dx \leq n^{-d/2} e_n.
$$

(45)

It remains to show that there exists a positive, monotone, slowly varying sequence $(h_n)_{n \in \mathbb{N}}$, such that $e_n \leq h(n) \to 0$ as $n \to \infty$. Let $\delta_n \leftarrow \sup_{f \in \mathbb{R}} \delta_f$ and $a_0 = 0$ and for $n \geq 1$ define $a_n$ recursively by $a_n = \min(2a_{n-1}, 1/\delta_n)$, for $2^{n-1} < n \leq 2^n$, so that $a_n \to 0$ is monotone, $a_{2^n} \leq 2a_{2^{n-1}}$ implying that $a_n \leq 4a_n$, and $1/a_n \geq \delta_n \geq e_n$. Finally, take $h_n = 1/\max(a_n, \log a_n)$.

Proof of Theorem 2. Assume that $\mathbb{E}|X|^2 = \infty$ and $d = 2$ or $d = 3$. Then, by Lemma 10 there exists a slowly varying sequence $h_n \to 0$ as $n \to \infty$ such that $\int_{\mathbb{R}^d} |X|^d \, d\mu \leq h_n n^{-d/2}$. Applying Corollary 7 with $r = 1$ and $r = 3/2$ we, respectively, find that

$$
\var(L_n(\alpha)) \leq \left\{ \begin{array}{ll}
Kn^2 & \text{for } d = 2, \\
Kn \sum_{k=1}^n h(k)^2 & \text{for } d = 3.
\end{array} \right.
$$

(46)

Finally assume that $\mathbb{E}|X|^2 < \infty$ and $\mathbb{E}|X| = \mu \neq 0$. Then $P(S_n = 0) = P(S_n = -n\mu)$ whence it follows that $P(S_n = 0) = o(n^{d/2})$ (see, e.g., [17, Theorem 2.3.10]). Then inspecting the proof of Proposition 4, one can readily obtain the desired bound for the $f_n$ term, while with slight modification the bound for the $L_n$ term also follows.

Note that for $d = 1$ the situation is much simpler since then var$(L_n(\alpha)) \sim C E L_n^SRW(\alpha, d)^2$ and if $\mathbb{E}|X|^2 = \infty$ or $\mathbb{E}|X| \neq 0$, $E L_n^SRW(\alpha, d) = o(n^{(1+\alpha)/2})$.

Proof of Theorem 3. We first give the proof for the case $d = 1$. As in the proof of Proposition 4 we begin from expression (10) and define the sequences $p_i$ and $\delta_i$ for $i = 1, \ldots, 2n - 1$, and the quantity $v(\delta) = \sum_{i=1}^{n-1} |\delta_i|$. Recall that $v(\delta)$ measures the interlacement of the two sequences $k_1, \ldots, k_d$ and $l_1, \ldots, l_d$. For example, $v(\delta) = 1$ occurs when either $k_1 \leq l_1$ or $l_1 \leq k_1$, in which case the contribution vanishes by the Markov property. On the other hand $v(\delta) = 2$ when, for example, $l_1, \ldots, l_n \in [k_1, k_{n+1}]$ for some $i$. Finally $v(\delta) = 3$ occurs when, for example,

$$
k_1 \leq \cdots \leq k_3 \leq l_1 \leq \cdots \leq l_2 \leq k_{n+1} \leq \cdots \leq k_n \leq l_n
$$

(47)

From the proof of Proposition 4, and using the bound $P(S_n = 0) \leq c/n$, the terms of the sum are bounded above by $n^d \log(n)^{2x-1-\nu(\delta)}$, and thus the leading term appears when either $v(\delta) = 2, 3$, with other terms giving strictly lower order. We will therefore analyze these two situations in detail in order to derive the exact asymptotic constants. When $v = 3$, the two terms in the difference individually give the correct order and will be treated by the classical Tauberian theory. However for $v = 2$, the two terms only give the correct order when considered together. This however forbids the use of Karamata’s Tauberian theorem since the monotonicity restriction would require roughly that $X_i$ is symmetric. Thus the complex Tauberian approach, as developed in [13], is required to justify the answer.

Case 1 ($v(\delta) = 3$). Assume that part of the sequence $i = [l_1, \ldots, l_n]$ lies between $k_2$ and $k_{n+1}$ and the rest between $k_1$ and $k_{n+1}$. Then using the change of variables
\[ i_1 = m_0, \]
\[ i_2 = m_0 + m_1, \]
\[ \vdots \]
\[ i_r = m_0 + \cdots + m_{r-1}, \]
\[ j_1 = m_0 + \cdots + m_r, \]
\[ j_2 = m_0 + \cdots + m_{r+1}, \]
\[ \vdots \]
\[ j_s = m_0 + \cdots + m_{r+s-1}, \]
\[ \vdots \]
\[ j_a = m_0 + \cdots + m_{a+s-1}, \]
\[ j_{a+1} = m_0 + \cdots + m_{a+s}, \]
\[ j_{a+2} = m_0 + \cdots + m_{a+s+1}, \]
\[ \vdots \]
\[ j_{n-1} = m_0 + \cdots + m_{2a-1}, \]
\[ i_{r+1} = m_0 + \cdots + m_{r+s}, \]
\[ i_{r+2} = m_0 + \cdots + m_{r+s+1}, \]
\[ \vdots \]
\[ i_{\alpha} = m_0 + \cdots + m_{\alpha+s-1}, \]
\[ j_{s+1} = m_0 + \cdots + m_{\alpha+s}, \]
\[ j_{s+2} = m_0 + \cdots + m_{\alpha+s+1}, \]
\[ \vdots \]
\[ j_{\alpha} = m_{2a}, \]
\[ n = m_0 + \cdots + m_{2a}, \]

we rewrite the positive term in (10) as

\[
\sum_{n=0} a(n) = \sum_{S(m_1) = \cdots = S(m_r); S(j_1) = \cdots = S(j_a)} \prod_{j=r+1}^{a+s} P(S(m_j) = 0) \prod_{t=r+1}^{a+s} P(S(m_t) = 0) \prod_{s=r+1}^{a+s} P(S(m_s) = 0). \tag{49}
\]

Notice that from [13] we have that \( \sum_{n=0} \lambda^n P(S_n = 0) \sim \log(1/(1 - \lambda))/\pi \gamma. \)

Let
\[ a(\lambda) = (1 - \lambda)^{-3} \left[ - \log(1 - \lambda) \right]^{2a-4}, \]
\[ c_\gamma = (\pi \gamma)^{-2a+4}. \tag{50} \]

Then, by direct calculations and Fourier inversion formula

\[
\sum_{n=0} \lambda^n a(n) = c_\gamma (1 - \lambda) a(\lambda) \sum_{x \in \mathbb{Z}, k, k' \geq 0} \lambda^{k+k'+k'} P(S_k = x) P(S_k' = -x) \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \frac{dt \, ds}{(1 - \lambda f(t))(1 - \lambda f(s)(1 - \lambda f(t + s))} \sim c_\gamma (1 - \lambda) a(\lambda) \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \frac{dxdy}{(1 + |x|)(1 + |y|)(1 + |x+y|)} \sim \left( \frac{1}{4\gamma^2} \right) c_\gamma a(\lambda). \tag{51}
\]

Next we consider the negative term in (10)

\[
b(n) = \sum_{m_1, \ldots, m_{2a-1}} P(S(m_1) = \cdots = S(m_{r-1}) = S(m_r) + \cdots + S(m_{r+s}) = 0) \prod_{t=r+1}^{a+s} P(S(m_t) = 0) \prod_{s=r+1}^{a+s} P(S(m_s) = 0) \prod_{t=r+1}^{a+s} P(S(m_t) = 0). \tag{52}
\]

By direct calculations and (6),

\[
\sum_{n} \lambda^n b(n) = \left( \frac{1}{\pi \gamma} \log \left( \frac{1}{1 - \lambda} \right) \right)^{2a-4} (1 - \lambda)^{-2} \sum_{m_1, \ldots, m_{2a-1}} P(S(m_1) = \cdots = S(m_{r-1}) = S(m_r) + \cdots + S(m_{r+s}) = 0) \prod_{t=r+1}^{a+s} P(S(m_t) = 0) \prod_{s=r+1}^{a+s} P(S(m_s) = 0) \prod_{t=r+1}^{a+s} P(S(m_t) = 0). \tag{53}
\]

and using Fourier inversion and (6) the internal sum behaves as

\[
(2\pi)^{-2a+r+s} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} (1 - \lambda \phi(x))^{-1} (1 - \lambda \phi(y))^{-1} (1 - \lambda \phi(y'))^{-1} \frac{1}{(1 - \lambda \phi(y''))} dxdy \sim (\pi \gamma)^{-2a+r+s} (1 - \lambda)^{-1} \tag{54}
\]
Then, we have $\sum_{n} \lambda^{n}b(n) \sim (\pi^{2}/6(\pi\gamma)^{\frac{3\alpha-2}{2}})\alpha(\lambda)$, whence the Tauberian theorem implies that $a(n) - b(n) \sim n^{2}\log(n)^{2\alpha-4}/24\pi^{2}\gamma_{2}r^{2\alpha-2}$. Most importantly we see that the lengths and locations of the chains, $r$ and $s$, do not affect the asymptotic behaviour. Noting that if $1 \leq r, s \leq \alpha - 1$, we can partition $2\alpha = r + s + (\alpha - r) + (\alpha - s)$, in $(\alpha - 1)^{2}$ ways, and thus overall the total contribution from terms with $\nu = 3$ is

$$\left[ \frac{2\alpha(\alpha - 1)^{2}}{12\pi^{2}4\gamma_{2}r^{2\alpha-2}} \right] n^{2}\log(n)^{2\alpha-4}. \quad (55)$$

**Case 2** ($\nu(\delta) = 2$). The typical term $c(n)$ was introduced in (33) in the proof of Proposition 9. Now we let $\lambda \in \mathbb{C}$, with $|\lambda| < 1$. By lengthy but direct calculations we can derive an expression of the form

$$\sum_{n} \lambda^{n}c(n) = \frac{\alpha - 1}{(\gamma\pi)^{2\alpha-2}}\alpha(\lambda) + o(\alpha(\lambda)), \quad \lambda \rightarrow 1. \quad (56)$$

The approach developed in [13] can then be used to bound the error terms and show that $c(n) \sim [(\alpha - 1)/2(\nu\gamma)^{\frac{3\alpha-2}{2}}n^{2}\log(n)^{2\alpha-4}].$

Finally taking into account the fact that $l_{1}, \ldots, l_{\alpha}$ can be in any of the $\alpha - 1$ intervals $[k_{i}, k_{i+1}]$, for $i = 1, \ldots, \alpha - 1$, the results follow the overall contribution of terms with $\nu(\delta) = 2$

$$\frac{(\alpha - 1)^{2}}{2\gamma(\pi)^{2\alpha-2}}n^{2}\log(n)^{2\alpha-4}. \quad (57)$$

The case for $d = 2$ is very similar, so we move on to the case $d = 3$.

**Case 3** ($d = 3$ and $\alpha = 2$). Using the same notation as before, we have three terms to consider $a(n), b(n)$, and $c(n)$. We first consider $c(n)$. Letting $K := \epsilon/\sqrt{1 - \lambda}$ and using the usual power series construction and spherical coordinates

$$\sum_{n} \lambda^{n}c(n) = (1 - \lambda)^{-2}(2\pi)^{-6} \int_{\theta_{1},\theta_{2}=0}^{\pi} \sin(\theta_{1}) \sin(\theta_{2}) d\theta_{1} d\theta_{2} \int_{\phi_{1},\phi_{2}=0}^{2\pi} \sin(\theta_{1}) \sin(\theta_{2}) d\phi_{1} d\phi_{2} \int_{r,s=0}^{2\pi} r^{2} s^{2} \sin(\theta_{1}) \sin(\theta_{2}) d\theta_{1} d\theta_{2} d\phi_{1} d\phi_{2} dr ds.

The other integral is slightly easier

$$I_{2}(\lambda) \sim |\Sigma|^{-1} \pi/2 \log K \int_{\phi_{1},\phi_{2}=0}^{2\pi} \sin(\theta_{1}) \sin(\theta_{2}) d\phi_{1} d\phi_{2} d\theta_{1} d\theta_{2}, \quad (62)$$

and thus overall we must have that

$$I_{1} - I_{2}(\lambda) \sim \frac{1}{2}(2\pi)^{-6} |\Sigma|^{-1}(1 - \lambda)^{-2} \log\left(\frac{1}{1 - \lambda}\right) \int_{\theta_{1},\theta_{2}=0}^{\pi} \frac{\arccos(A)}{\sqrt{1 - A^{2}}} \frac{\pi}{2} \sin(\theta_{1})$$
\[
\cdot \sin(\theta_2) \, d\phi_1 \, d\phi_2 \, d\theta_1 \, d\theta_2 = \kappa_2 (1 - \lambda)^{-2} \log \left( \frac{1}{1 - \lambda} \right),
\]
whence it follows that \( \text{var}(L_n(2)) \sim (\kappa_1 + \kappa_2)n \log n \).

To prove the last claim let \( S'_n = X_1' + \cdots + X'_n \) be another random walk, independent of \( S_n \), such that its characteristic function \( \hat{f}'(t) = E[\exp(i t X'_1)] \) also satisfies the expansion (6). Then using [13, Lemma 3.1], one can adapt the proof of [13, Theorem 2.1] to show that \( L'_n(\alpha) = L_n(\alpha) + o(L_n(\alpha)) \). \( \square \)

### Competing Interests

The authors declare that they have no competing interests.

### References


