CONFORMING AND NONCONFORMING VIRTUAL ELEMENT METHODS FOR ELLIPTIC PROBLEMS

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Abstract. We present, in a unified framework, new conforming and nonconforming Virtual Element Methods (VEM) for general second order elliptic problems in two and three dimensions. The differential operator is split into its symmetric and non-symmetric parts and conditions for stability and accuracy on their discrete counterparts are established. These conditions are shown to lead to optimal $H^1$- and $L^2$-error estimates, confirmed by numerical experiments on a set of polygonal meshes. The accuracy of the numerical approximation provided by the two methods is shown to be comparable.

1. Introduction

The Virtual Element Method (VEM) was introduced in [6] as a generalisation of the conforming finite element method (FEM), offering great flexibility in utilising meshes with (almost) arbitrary polygonal elements. Unlike the polygonal finite element method (PFEM) [41] and other conforming FEM extensions based on the Generalised Finite Element [4] framework such as the CFE method [28] and the XFEM [26], the VEM handles meshes with general shaped elements in a manner that avoids the explicit evaluation of the shape functions. Indeed, the VEM only requires the knowledge of a polynomial subspace of the local discrete function space to provide stable and accurate numerical methods. This feat is achieved by separating the contribution of the polynomial subspace from that of the remaining non-polynomial virtual subspace through the introduction of suitable projection operators which can be computed just using the VEM degrees of freedom. Correspondingly, the bilinear form is split into two parts: the polynomial consistency terms, responsible for the convergence properties of the method, are computed accurately, while the remaining terms are only required to ensure the stability of the method, and hence they can be roughly estimated from the degrees of freedom. The VEM approach can also be viewed as a variational analogue of the mimetic finite difference (MFD) method; see [10] and the recent review paper [30]. As such, for its analysis we can take advantage of the standard tools of finite element analysis.

An alternative approach is to completely relax the inter-element continuity requirements for the discrete space, so that simple (polynomial) spaces can be used. For instance, discontinuous Galerkin and weak Galerkin methods, where inter-element continuity is weakly imposed, are naturally suited to general meshes; see [18, 20, 36] and the references therein. Other methods that are suited to unstructured polygonal and polyhedral meshes are BEM-based FEM [21, 38, 39] and Trefftz-Discontinuous Galerkin [34].

Here we shall stop just short of that, presenting a general VEM framework which is based on some form of continuity, in the spirit of [23]. The framework is used to introduce a $C^0$-conforming and a nonconforming VEM.

The original VEM in [6] is a $C^0$-conforming method for solving the two-dimensional Poisson equation. A nonconforming counterpart method for solving the same problem was presented in [3]. The extension of these methods to general elliptic problems with variable coefficients in two and three dimensions, is non trivial. Diffusion problems with non-constant diffusion tensors
in two dimensions are treated in [11], where VEMs which incorporate inter-element continuity of arbitrary degree are presented. A crucial step towards the inclusion of low-order differential terms is provided in [1] with the extension of the original \( C^0 \)-conforming VEM to reaction-diffusion problems with constant coefficients in two and three dimensions. This approach is extended to the solution of general elliptic problems in two dimensions in [9]. Concurrently, the VEM framework has been extended to the solution of plate-bending problems [16], linear elasticity problems in two and three spatial dimensions [7, 27], the Steklov eigenvalue problem [35], the simulation of discrete fracture networks [12], and the two-dimensional streamline formulation of the Stokes problem [2].

We present here a conforming and a nonconforming VEM for the numerical treatment of general linear elliptic problems with variable coefficients in two and three spatial dimensions.

Nonconforming finite element spaces were originally developed to approximate the velocity field of the Stokes equations on triangular meshes. The functions in these finite element spaces are piecewise polynomials of degree \( k = 1 \) [23], \( k = 2 \) [25], \( k = 3 \) [22], and \( k > 3 \) [40, 5]. Continuity is required only at a discrete set of points located at element interfaces. These points are the roots of the one-dimensional \( k^{th} \)-order Legendre polynomials defined over each edge, i.e., the nodes of the Gauss-Legendre quadrature rule of order \( k \). This minimal continuity requirement ensures the optimal convergence rate; see, for instance, [23].

Two major issues affect these nonconforming methods. First, the space construction for odd \( k \) differs from that of even \( k \) since the latter normally requires the introduction of a bubble function [25, 40, 5]. Second, the construction of shape functions for the nonconforming formulation in 2D on elements other than triangular and in 3D is usually not a straightforward extension of the 2D construction of triangular elements. A few attempts to address this latter issue are found in [37, 17, 33, 32], where nonconforming finite elements are developed for quadrilaterals, tetrahedra and hexahedra.

The nonconforming virtual formulation that we consider in this work addresses both these issues simultaneously. Indeed, the construction of the nonconforming virtual element space containing polynomials of degree \( k \) is the same regardless of the parity of \( k \) and of the elemental shape, with very general polygonal and polyhedral shapes being allowed.

The accuracy and stability of the two methods is determined in Section 3 through a unified abstract framework, cf. Assumption A1. Here the partial differential operator is split into its symmetric and skew-symmetric parts and the VEM polynomial consistency and stability properties are established for each of these components separately, cf. Assumption A2. This approach is quite natural in that, for instance, it is clear that only the symmetric component is needed for the method’s stability. Indeed a unique stabilisation for all the terms that contribute to the symmetric part (the diffusion, reaction, and symmetric contribution of the convection term) is introduced. The stabilisation automatically adjusts with the relative magnitude of the (symmetric) terms.

To deal with non-constant coefficients and the lower-order terms, we take the approach of [9], rather than that of [11]. A crucial role in the former formulation is played by the \( L^2 \)-projector which maps the functions of the virtual element space and their gradients onto polynomials. In order to have the \( L^2 \)-projection operator computable by using only the degrees of freedom, in Section 4 we generalise a procedure introduced in [1], dubbed VEM enhancement, used here in the context of nonconforming VEM for the first time. In this way a family of virtual element spaces is defined from which a particularly simple choice can be made, cf. Section 8. This approach differs completely from that presented for a non-constant diffusion tensor in [11], which required the construction of a bespoke projection operator dependent on the diffusion tensor. The key advantage of removing this dependence is that lower order terms can be dealt with in an identical manner. Furthermore, we are now easily able to analyse the impact on the method of the approximation of the problem’s coefficients. Here it is important to stress that such approximation is only needed to compute the integrals involved in the polynomial consistency terms of the bilinear form. This fact is discussed in Section 7, with the conclusion that the stability and optimal accuracy of the method based on using polynomials of order up to \( k \) are unaffected by the use of a quadrature scheme to approximate the consistency terms, provided that this is of at least degree \( 2k - 2 \). We stress that this is exactly the same requirement of the finite element methods [19].
The new unified formulation offers some indisputable advantages. From a theoretical viewpoint, it permits us to analyse the conforming and nonconforming VEM in a unified manner, following the standard analyses of finite element methods for elliptic problems. The analysis, detailed in Section 6, ultimately leads to optimal order $H^1$- and $L^2$-error estimates for both methods under the same regularity assumptions on the mesh and the exact solution. By contrast, the analysis of conforming VEMs for the same problem in two space dimensions given in [9] is based on an inf-sup argument relying on the mesh size being small enough. From a practical viewpoint, the implementation of the conforming and nonconforming VEM is formally the same (see Section 8). In fact, the only difference is in the construction of the $L^2$ projection operator for the shape functions and their gradients. This construction depends on the degrees of freedom, which necessarily differ for the conforming and nonconforming VEM.

The conforming and nonconforming VEMs are assessed in Section 9, numerically solving a representative convection-reaction-diffusion problem with variable coefficients in two dimensions. The accuracy of the numerical approximation provided by the two methods is comparable and confirms the optimal convergence rates in the $L^2$ norm established by the theoretical analysis presented in Section 6. Finally, in Section 10 we offer our final conclusions.

Below, we shall use standard notation for the relevant function spaces. For a Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, we denote by $H^s(\Omega)$ the Hilbert space of index $s \geq 0$ of real-valued functions defined on $\Omega$, endowed with the seminorm $| \cdot |_{s, \omega}$ and norm $\parallel \cdot \parallel_{s, \omega}$; further $(\cdot, \cdot)_\omega$ stands for the standard $L^2$ inner-product. When $\omega$ is the whole of the computational domain $\Omega$, we may omit the subscript $\omega$ if it is not ambiguous. Finally, $|\omega|$ denotes the $d$-dimensional Hausdorff measure of $\omega$.

2. THE CONTINUOUS PROBLEM

Consider the boundary value problem

$$-
abla \cdot (\kappa(x)\nabla u) + \beta(x) \cdot \nabla u + \gamma(x)u = f(x) \quad \text{in } \Omega, \quad (2.1a)$$

$$u = 0 \quad \text{on } \partial \Omega, \quad (2.1b)$$

where $\Omega \subset \mathbb{R}^d$ is a polygonal domain for $d = 2$ and a polyhedral domain for $d = 3$. We assume that the coefficients $\kappa_{i,j}(x), \beta_i(x), \gamma(x)$ are in $L^\infty(\Omega)$, and $f \in L^\infty(\Omega)$ is the forcing function. We further suppose that $\kappa(x)$ is a full symmetric $d \times d$ diffusivity tensor and is strongly elliptic, i.e. there exist $\kappa_-, \kappa^* > 0$, independent of $\vec{v}$ and $x$, such that

$$\kappa_+ |\vec{v}(x)|^2 \leq \kappa(x) |\vec{v}(x)|^2 \leq \kappa^* |\vec{v}(x)|^2, \quad (2.2)$$

for almost every $x \in \Omega$ and for any $\vec{v} \in (H^1_0(\Omega))^d$, where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^d$. Finally, we suppose that there exists $\mu_0 \geq 0$ such that

$$\mu(x) := \gamma(x) - \frac{1}{2} \nabla \cdot \beta(x) \geq \mu_0 \geq 0, \quad (2.3)$$

for almost every $x \in \Omega$, and assume that $\nabla \cdot \beta \in L^\infty(\Omega)$.

The variational form of problem (2.1) reads: find $u \in H^1_0(\Omega)$ such that

$$(\kappa \nabla u, \nabla v) + (\beta \cdot \nabla u, v) + (\gamma u, v) = (f, v) \quad \forall v \in H^1_0(\Omega), \quad (2.4)$$

with $(\cdot, \cdot)$ denoting the $L^2$ inner product on $\Omega$. We split the bilinear form on the left-hand side of (2.4) into its symmetric and skew-symmetric parts:

$$a(u, v) := (\kappa \nabla u, \nabla v) + (\mu u, v), \quad (2.5a)$$

$$b(u, v) := \frac{1}{2} \left( [\beta \cdot \nabla u, v] - (u, \beta \cdot \nabla v) \right), \quad (2.5b)$$

and consider discretising the problem written in the equivalent form: find $u \in H^1_0(\Omega)$ such that

$$A(u, v) := a(u, v) + b(u, v) = (f, v) \quad \forall v \in H^1_0(\Omega). \quad (2.6)$$

It is simple to check that the bilinear form $A$ is coercive and bounded, and the variational problem therefore possesses a unique solution by the Lax-Milgram lemma.
Rewriting the variational form in this way would not be necessary for the classical finite element method, but it turns out to be a useful step for the virtual element method to ensure that the discrete framework preserves the properties of the symmetric and skew-symmetric parts of the bilinear form. This is a key point of difference between the conforming method we introduce here and that presented in [9]. There, a virtual element method is developed based on the original variational form (2.4), and the well-posedness of the resulting method relies on the assumption that the mesh size is sufficiently small. This assumption is not necessary for the method we introduce here, as proven in Theorem 1.

3. The Virtual Element Framework

We assume that a virtual element method consists of the following fundamental ingredients:

**Assumption A1.** For any fixed \( h > 0 \) and \( k \in \mathbb{N} \), we have:

- A finite decomposition (mesh) \( \{T_h\} \) of the domain \( \Omega \) into non-overlapping simple polygonal/polyhedral elements with maximum size \( h \). The adjective simple refers to the fact that the boundary of each element in the decomposition must not be self-intersecting. Further, the boundary of every element \( E \in \mathcal{T}_h \) is made of a uniformly bounded number of interfaces: line segments if \( d = 2 \) and planar polygons with a uniformly bounded number of straight edges if \( d = 3 \). Elemental interfaces are either part of the boundary of \( \Omega \) or shared with another element in the decomposition.
- A finite dimensional function space \( V_h \subset H^1(\mathcal{T}_h) \) where
  \[
  H^1(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_E \in H^1(E), \forall E \in \mathcal{T}_h\}
  \]
  (not necessarily a subspace of \( H^1_0(\Omega) \)) to be used as a trial and test space, on which the following Poincaré-Friedrichs inequality for piecewise \( H^1 \) functions holds:
  \[
  \|v_h\|_{0,\Omega}^2 \leq C_{PF}\|v_h\|_{1,E}^2 := C_{PF}\sum_{E \in \mathcal{T}_h} \|\nabla v_h\|_{0,E}^2,
  \]
  hence \( \cdot \|_{1,h} \) is a norm on \( V_h \). We will also use the full broken \( H^1(\Omega) \) norm, defined as
  \[
  \|v_h\|_{1,h}^2 := \|v_h\|_{0,\Omega}^2 + \|v_h\|_{1,h}^2,
  \]
  for \( v_h \in H^1(\mathcal{T}_h) \).
- Further, for each element \( E \in \mathcal{T}_h \), the space \( V_h^E := V_h|_E \) must contain the space \( \mathcal{P}_k(E) \) of polynomials of degree \( k \) on \( E \);
- A bilinear form \( A_h : V_h \times V_h \to \mathbb{R} \), which may be split over the elements in the mesh \( \mathcal{T}_h \) as
  \[
  A_h(u_h, v_h) = \sum_{E \in \mathcal{T}_h} A_h^E(u_h, v_h),
  \]
  for any \( u_h, v_h \in V_h \), where \( A_h^E \) is a bilinear form over the space \( V_h^E \).
- An element \( f_h \in V_h \) approximating the forcing term.

**Remark 1.** The definition of simple polygons and simple polyhedra is general enough to include, for instance, elements with consecutive co-planar edges/faces, such as those typical of locally refined meshes with hanging nodes, and non-convex elements. Later on, in Assumption (A3) in Section 4.5, we shall introduce some standard mesh regularity assumptions which are required for the approximation properties of the virtual element spaces.
We also place a few more restrictions on the nature of the discrete bilinear form $A_h^E$. As with the continuous bilinear form, we write $A_h^E$ as the sum of a symmetric and a skew-symmetric part:

$$A_h^E(u_h, v_h) = a_h^E(u_h, v_h) + b_h^E(u_h, v_h),$$

and require that they satisfy the following properties:

**Assumption A2.** The bilinear forms $a_h^E$ and $b_h^E$ are assumed to satisfy the properties of polynomial consistency and stability, defined as

- **Polynomial consistency:** If either $u_h \in \mathcal{P}_k(E)$ or $v_h \in \mathcal{P}_k(E)$, the symmetric and skew-symmetric parts of the local virtual element bilinear form must satisfy

$$
a_h^E(u_h, v_h) = \int_{\Omega} \kappa \Pi_{k-1}^0(\nabla u_h) \cdot \Pi_{k-1}^0(\nabla v_h) \, dx + \int_{\Omega} \mu \Pi_{k}^0 u_h \Pi_{k}^0 v_h \, dx,
$$

$$
b_h^E(u_h, v_h) = \frac{1}{2} \int_{\Omega} \beta \cdot [\Pi_{k-1}^0(\nabla u_h) \Pi_k^0 v_h - \Pi_k^0 u_h \Pi_{k-1}^0(\nabla v_h)] \, dx,
$$

where the operator $\Pi_k^0 : L^2(\Omega) \to \mathcal{P}_k(\Omega)$ for $k \geq 2$ denotes the $L^2(\Omega)$-orthogonal projection onto the polynomial space $\mathcal{P}_k(\Omega)$, and is defined for any function $v \in L^2(\Omega)$ as the unique element $\Pi_k^0 v$ of $\mathcal{P}_k(\Omega)$ such that

$$(\Pi_k^0 v, p)_{\Omega} = (v, p)_{\Omega} \quad \forall p \in \mathcal{P}_k(\Omega).$$

- **Stability:** There exist positive constants $\alpha_*, \alpha^*$, and $\beta^*$ independent of $h$ and the mesh element $E$ such that, for all $v_h , w_h \in V_h$, the symmetric part satisfies

$$\alpha_* a_h^E(v_h, v_h) \leq a_h^E(v_h, v_h) \leq \alpha^* a_h^E(v_h, v_h),$$

and the skew-symmetric part satisfies

$$b_h^E(v_h, v_h) = 0 \quad \text{and} \quad b_h^E(v_h, w_h) \leq \beta^* \|v_h\|_{1,E} \|w_h\|_{1,E}.$$

The selection of concrete spaces and bilinear forms of the method is the focus of the next two sections. In particular, the conforming and nonconforming virtual element spaces are developed together in Section 4 with a final definition of the local nonconforming space for $d = 2$ and along with the local conforming space for $d = 2$ in (4.4), while the local conforming space for $d = 3$ is finalised in Section 4.3. The final choices for the bilinear forms $a_h^E$ and $b_h^E$ are given in (5.1) and (5.2) respectively.

However, the simple properties presented above are enough to prove two crucial facts about the behaviour of such a virtual element method: firstly that any such method possesses a unique solution and secondly an abstract Strang-type convergence result, which will be used later on to derive optimal order error bounds in the $H^1$ norm. These results are encapsulated in the following theorems.

**Theorem 1** (Existence and uniqueness of a virtual element solution). Under Assumptions A1 and A2, the problem: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) + b_h(u_h, v_h) = \langle f_h, v_h \rangle \quad \forall v_h \in V_h,$$

possesses a unique solution. Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $V_h$ and its dual $V_h'$.

**Proof.** The stability condition on $a_h^E$ ensures that this bilinear form inherits the coercivity of its counterpart $a_h^E(\cdot, \cdot)$, hence

$$a_h^E(v_h, v_h) \geq \alpha_* \|\nabla v_h\|_{0,E}^2 + \mu_0 \|v_h\|_{0,E}^2.$$  

Summing up the contribution of all elements and using (3.2) we deduce the coercivity bound

$$a_h(v_h, v_h) \geq \frac{\alpha_*}{1 + C_{PF}} \left(\kappa_* |v_h|_{1,h}^2 + (\kappa_* + \mu_0) \|v_h\|_{0,E}^2\right),$$

which remains robust in the case when $\mu_0 = 0$. From inequality (3.4) and the symmetry and bilinearity of $a_h^E$, it follows that $a_h^E$ is an inner product on $V_h^E$. Hence, we can apply the Cauchy-Schwarz inequality and use the right stability inequality of Assumption A2 to prove the continuity.
of $a_h^E$:

$$a_h^E(u_h, v_h) \leq (a_h^E(u_h, u_h))^{\frac{2}{\hat{k}}} (a_h^E(v_h, v_h))^{\frac{2}{\hat{k}}} \leq \alpha^* (a_h^E(u_h, u_h))^{\frac{2}{\hat{k}}} (a_h^E(v_h, v_h))^{\frac{2}{\hat{k}}} \leq \alpha^* \max\{\kappa^*, \|\mu\|_\infty\} \|u_h\|_{1,E} \|v_h\|_{1,E},$$

and the continuity of $a_h$ easily follows.

The stability property for $b_h^E$ means that this term does not feature in the coercivity analysis and imposes continuity with constant $\beta^*$. Hence we may conclude that the problem (3.3) admits a unique solution by the Lax-Milgram lemma.□

**Theorem 2** (Abstract a priori error bound). Under Assumptions A1 and A2, there exists a constant $C > 0$ depending only on the coercivity and the continuity constants such that

$$\|u - u_h\|_{1,h} \leq C \left( \inf_{v_h \in V_h} \|u - v_h\|_{1,h} + \sup_{w_h \in V_h} \frac{|\langle f_h, w_h \rangle - (f, w_h)|}{\|w_h\|_{1,h}} + \inf_{p \in \mathcal{P}_k(T_h)} \left[ \|u - p\|_{1,h} + \sum_{E \in T_h} \sup_{w_h \in V_h} \frac{|A_h^E(p, w_h) - A_h^E(p, w_h)|}{\|w_h\|_{1,E}} \right] + \sup_{w_h \in V_h} \frac{|A(u, w_h) - (f, w_h)|}{\|w_h\|_{1,h}} \right), \quad (3.6)$$

where

$$\mathcal{P}_k(T_h) := \{ p \in L^2(\Omega) : p|_E \in \mathcal{P}_k(E) \quad \forall E \in T_h \}.$$  

The last term on the right-hand side of the above error estimate measures the nonconformity error, i.e., it is non-zero only when $V_h$ is a nonconforming virtual space.

**Proof.** Let $v_h$ be an arbitrary element of $V_h$ and let $w_h = u_h - v_h \in V_h$. Then, the coercivity of $A_h$ implies that

$$\alpha \|u_h - v_h\|_{1,h}^2 \leq A_h(u_h - v_h, w_h) = \langle f_h, w_h \rangle + \sum_{E \in T_h} \left( A_h^E(p - v_h, w_h) - A_h^E(p, w_h) \right)$$

$$= \langle f_h, w_h \rangle + \sum_{E \in T_h} \left[ A_h^E(p - v_h, w_h) - A_h^E(p, w_h) + (A_h^E(p, w_h) - A_h^E(p, w_h)) \right]$$

$$= \langle f_h, w_h \rangle - A(u, w_h) + \sum_{E \in T_h} \left[ A_h^E(p - v_h, w_h) + A_h^E(u - p, w_h) + (A_h^E(p, w_h) - A_h^E(p, w_h)) \right],$$

for any $p \in \mathcal{P}_k(T_h)$. We express the potential nonconformity of the virtual element space $V_h$ as

$$A(u, w_h) = \langle f, w_h \rangle + (A(u, w_h) - (f, w_h)),$$

so that

$$\alpha \|u_h - v_h\|_{1,h}^2 \leq \left[ \|f_h, w_h\| - \langle f, w_h \rangle + \|f_h, w_h\| - A(u, w_h) \right]$$

$$+ \sum_{E \in T_h} \left[ A_h^E(p - v_h, w_h) + A_h^E(u - p, w_h) + (A_h^E(p, w_h) - A_h^E(p, w_h)) \right].$$

Hence, for all $w_h \in V_h \setminus \{0\}$ and $p \in \mathcal{P}_k(T_h)$, applying the continuity of the bilinear forms, multiplying and dividing by $\|w_h\|_{1,h}$, and using the triangle inequality we find that

$$\alpha \|u_h - v_h\|_{1,h} \leq \|f_h, w_h\| - \langle f, w_h \rangle + \|f_h, w_h\| - A(u, w_h)$$

$$+ \|u - v_h\|_{1,h} + 2 \|u - p\|_{1,h} + \sum_{E \in T_h} \frac{|A_h^E(p, w_h) - A_h^E(p, w_h)|}{\|w_h\|_{1,E}}.$$

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4. The Virtual Element Spaces

We introduce two types of spaces in two- and three-dimensions implementing the framework of Section 3: a conforming and a nonconforming virtual element space. With deliberate ambiguity we refer to both spaces as $V_h$ to emphasise the fact that the method is otherwise the same in either case. In all cases the local virtual element space $V_h^E$ must contain the space $\mathcal{P}_k(E)$ of polynomials of degree up to $k$ on $E$, cf. Assumption A1. The complement of $\mathcal{P}_k(E)$ in $V_h^E$ is made up of functions which are deemed expensive to evaluate, although they may be described through a set of (known) degrees of freedom. Central to the virtual element methodology is the idea of computability, defined as follows:

**Definition 1.** A term will be called computable if it may be evaluated using just the degrees of freedom of a function and the polynomial component of the virtual element space.

We require that the polynomial consistency forms of Assumption A2 are computable. Hence the spaces must be constructed in such a way that the projections

$$\Pi_{k-1}^0 \nabla v_h \quad \text{and} \quad \Pi_k^0 v_h,$$

are computable for any function $v_h \in V_h^E$. While the first of these is computable for functions in the original conforming (for $d = 2$) and nonconforming virtual element spaces presented in [6, 3], the second is not. However, a modification to the conforming space was introduced in [1] which renders $\Pi_k^0 v_h$ computable without changing the degrees of freedom used to describe the space. For the construction of the spaces which follow, we consider a generalisation of the process presented in [1].

We first introduce an appropriately scaled basis for $\mathcal{P}_k(E)$, $k \in \mathbb{N}$. Denote by $\mathcal{M}_\ell^k(E)$, $\ell \in \mathbb{N}$, the set of scaled monomials

$$\mathcal{M}_\ell^k(E) := \left\{ \left( \frac{x - x_E}{h_E} \right)^s \middle| s \in \mathbb{N}^d, |s| = \ell \right\},$$

where $h_E$ denotes the diameter of the element $E \in \mathcal{T}_h$ and $x_E$ is its barycentre. The notation $s \in \mathbb{N}^d$ is used for a multi-index with $|s| := \sum_{i=1}^d s_i$ and $x^s := \prod_{i=1}^d x_E^{s_i}$. Further, we define $\mathcal{M}_k(E) := \bigcup_{\ell \in \mathbb{N}} \mathcal{M}_\ell^k(E) =: \{ m_{\alpha} \}_{\alpha \in \Lambda_{d,k}}^{N_{d,k}}$, a basis of $\mathcal{P}_k(E)$, where $N_{d,k} := \dim(\mathcal{P}_k(\mathbb{R}^d)) = \binom{k+d}{k}$, the dimension of the space of polynomials up to total degree $k$ in $d$ dimensions.

**Remark 2.** Scaled monomials are used as a basis for $\mathcal{P}_k(E)$ because they ensure that all degrees of freedom introduced below scale like 1 with respect to the element size $h_E$, cf. Remarks 1.1 and 2.5 in [8]. In turn, this property facilitates the construction of bilinear forms satisfying the stability requirement of Assumption A2, cf. Section 5 below.

Below, $s$ denotes a $d - 1$ dimensional mesh interface (either an edge when $d = 2$ or a face when $d = 3$) of the mesh element $E$, and the set of all mesh interfaces in $\mathcal{T}_h$ will be denoted by $\mathcal{S}_h$. This set is divided into the set of boundary edges $\mathcal{S}_h^{bdry} := \{ s \in \mathcal{S}_h : s \subset \partial \Omega \}$ and internal edges $\mathcal{S}_h^{int} := \mathcal{S}_h \setminus \mathcal{S}_h^{bdry}$. Bases for polynomial spaces defined on an interface $s$ can be similarly constructed; the same notation will be used. We denote by $\nu_E$ the number of interfaces $s \in \partial E$.

4.1. The Local Spaces. The construction of the local nonconforming virtual element space is formally the same for $d = 2$ or 3, while the construction of the conforming space is hierarchical in the space dimension. Because of this, while we simultaneously provide here a definition of the nonconforming space for $d = 2$ or 3, we initially only consider the conforming space for $d = 2$, postponing the construction for $d = 3$ until the end of the section.

We first introduce the two sets of degrees of freedom used in [6, 3] to describe the original local virtual element spaces. The final spaces presented later in the section will be described using exactly the same degrees of freedom, so we record them here.

**Definition 2.** The degrees of freedom for the local conforming and nonconforming spaces are

(a) for the conforming space
• the value of $v_h$ at each vertex of $E$;

• for $k > 1$, the moments of $v_h$ of up to order $k - 2$ on each mesh interface $s \subset \partial E$

$$\frac{1}{|s|} \int_s v_h m_\alpha \, ds \quad \forall m_\alpha \in M_{k-2}(s);$$

for the nonconforming space, the moments of $v_h$ of up to order $k - 1$ on each mesh interface $s \subset \partial E$

$$\frac{1}{|s|} \int_s v_h m_\alpha \, ds \quad \forall m_\alpha \in M_{k-1}(s);$$

(b) for either space, when $k > 1$, the moments of $v_h$ of up to order $k - 2$ inside the element $E$

$$\frac{1}{|E|} \int_E v_h m_\alpha \, dx \quad \forall m_\alpha \in M_{k-2}(E).$$

We impose an (arbitrary, but fixed) ordering on these degrees of freedom, and use the notation $dof_r(w)$ to denote the $r$th degree of freedom of the sufficiently smooth function $w$.

A counting argument shows that the cardinality of the above sets of degrees of freedom is $n_E = \nu_{E} + \nu_{E} N_{1,k-2} + N_{2,k-2}$ for the conforming case and $n_E = \nu_{E} N_{d-1,k-1} + N_{d,k-2}$ for the nonconforming case. In particular, we note that for fixed $k$ and $d = 2$ the number of local degrees of freedom is the same in either case. Of course, their geometric arrangement means that this will not be true in general for the number of global degrees of freedom. The degrees of freedom for a hexagonal element are represented in Figure 1 and Figure 2 for the nonconforming and conforming case, respectively. The nonconforming degrees of freedom for a cubic element are shown in Figure 3.

We first define an auxiliary superspace of the original conforming and nonconforming spaces in which both terms in (4.1) are easily computable. Again with deliberate ambiguity, we refer to the enlarged space in each case as $W_{E}^{E}$. For the enlarged nonconforming space on the element $E$, we define

$$W_{E}^{E} := \left\{ v_h \in H^1(E) : \Delta v_h \in \mathcal{P}_k(E) \text{ and } \frac{\partial v_h}{\partial n} \in \mathcal{P}_{k-1}(s) \forall s \subset \partial E \right\} .$$

For the conforming space, we first introduce the boundary space

$$B_{k}(\partial E) := \left\{ v \in C^0(\partial E) : v|_s \in \mathcal{P}_{k}(s) \text{ for each interface } s \text{ of } \partial E \right\} ,$$

and define $W_{E}^{E}$ as

$$W_{E}^{E} := \left\{ v_h \in H^1(E) : \Delta v_h \in \mathcal{P}_k(E) \text{ and } v_h|_{\partial E} \in B_{k}(\partial E) \right\} .$$

It is clear that, in either case, we have $\mathcal{P}_k(E) \subset W_{E}^{E}$. Further, the space $W_{E}^{E}$ may be described using the combination of the degrees of freedom of Definition 2 and the extra degrees of freedom:

**Definition 3.** The extra degrees of freedom are taken as the moments of $v_h$ of order $k$ and $k - 1$ inside the element $E$

$$\frac{1}{|E|} \int_E v_h m_\alpha \, dx \quad \forall m_\alpha \in \mathcal{M}_k^*(E) \cup \mathcal{M}_{k-1}^*(E).$$
The proof that the combined set of degrees of freedom is unisolvent for the conforming space is given in [1], and for the nonconforming space follows exactly as the original unisolvence proof in [3]. It is based on the observation that each space uniquely defines its elements as those functions which solve a particular class of Poisson problem, with piecewise polynomial Dirichlet and Neumann boundary conditions in the conforming and nonconforming case, respectively, specified by the degrees of freedom. Similarly, we can easily prove the crucial fact that any \( p_k \in \mathcal{P}_k(E) \) is uniquely determined by the original degrees of freedom of Definition 2, cf. [6, 3]. Indeed, if \( p_k \in \mathcal{P}_k(E) \) and all the original degrees of freedom of \( p_k \) are zero, then\[
abla p_k, \nabla p_k)_E = (-\Delta p_k, p_k)_E + \int_{\partial E} \frac{\partial p_k}{\partial n} p_k \, ds = 0.
\]The first term is zero as \( \Delta p_k = 0 \) if \( k = 1 \) and as \( \Delta p_k \in \mathcal{P}_{k-2}(E) \) if \( k > 1 \) and hence the first term is a linear combination of the (zero) internal degrees of freedom of \( p_k \) in this case. The second term is also zero because it is a linear combination of the boundary degrees of freedom of \( p_k \) in the nonconforming case, while in the conforming case we simply have \( p_k|_{\partial E} = 0 \) due to the fact that the boundary degrees of freedom are unisolvent for a polynomial on each edge. Hence, \( p_k = \text{constant in } E \). The fact that \( p_k = 0 \) now follows from the fact that \( p_k \) is zero on the boundary of \( E \) in the conforming case, and from \( \int_E p_k \, ds = 0 \) for any \( s \in \partial E \) in the nonconforming case.

Using collectively the degrees of freedom of Definitions 2 and 3, it is possible to compute both of the projections in (4.1) for any \( v_h \in \mathcal{W}_h^E \). Calculating \( \Pi^0_k v_h \) requires solving the variational problem: find \( \Pi^0_k v_h \in \mathcal{P}_k(E) \) such that\[
(\Pi^0_k v_h, m_\alpha)_E = (v_h, m_\alpha)_E \quad \forall m_\alpha \in \mathcal{M}_k(E),
\]which is computable as the quantities on the right-hand side are the internal degrees of freedom of \( v_h \). Similarly, computing \( \Pi^0_{k-1} \nabla v_h \) requires finding the polynomial \( \Pi^0_{k-1} \nabla v_h \in (\mathcal{P}_k(E))^2 \) such that\[
(\Pi^0_{k-1} \nabla v_h, m_\alpha)_E = (\nabla v_h, m_\alpha)_E \quad \forall m_\alpha \in (\mathcal{M}_{k-1}(E))^2.
\]This is possible because, after integration by parts, the right-hand side is found to just depend on polynomials on each edge of the boundary and internal degrees of freedom of \( v_h \):\[
(\nabla v_h, m_\alpha)_E = \int_{\partial E} n \cdot m_\alpha v_h \, ds - (v_h, \nabla \cdot m_\alpha)_E.
\]Note that this time only the original degrees of freedom of Definition 2 are required, even in the enlarged space.

However, in each case this enlarged space requires an extra \( \text{card}(\mathcal{M}_k^*(E)) + \text{card}(\mathcal{M}_{k-1}^*(E)) \) degrees of freedom (namely the extra internal moments in Definition 3) compared with the original spaces introduced in [6] and [3], which are described by the degrees of freedom of Definition 2 only. To reduce the number of degrees of freedom, we adopt a generalisation of the procedure introduced in [1], producing a family of different subspaces of \( \mathcal{W}_h^E \) spanned by the original sets of degrees of freedom.
freedom of Definition 2, yet in which we can still compute the required projections in (4.1). The procedure consists of the following three steps.

1) We introduce an equivalence relation \( \sim \) on \( W^E_h \), defining \( v_h \sim w_h \) if all of the original degrees of freedom of \( v_h \) and \( w_h \) are equal, and consider the quotient space \( W^E_h / \sim \) which, by construction, is spanned by the original degrees of freedom of Definition 2.

2) Since \( P_k(E) \subset W^E_h \) and any \( p_k \in P_k(E) \) is uniquely determined by the original degrees of freedom, we may conclude that any equivalence class \([v_h]\) contains at most one polynomial. Then, we may unambiguously associate \( P_k(E) \) with the resulting ‘polynomial’ subspace of \( W^E_h / \sim \). Hence, we can introduce any projection operator \( \Pi^*_k : W^E_h / \sim \to P_k(E) \subset W^E_h / \sim \) which associates a polynomial to each equivalence class \([v_h]\). \footnote{Alternatively, one may think of this as associating a polynomial to each combination of the original degrees of freedom in Definition 2, independent of the extra degrees of freedom in Definition 3.}

3) The local virtual space \( V^E_h \) is then defined by selecting a specific representative from each equivalence class in \( W^E_h / \sim \). For each \([v_h] \in W^E_h / \sim \) we take the function \( w_h \in [v_h] \) such that the extra degrees of freedom (in Definition 3) of \( w_h \) are equal to those of \( \Pi^*_k[v_h] \). Note in particular that \( P_k(E) \subset V^E_h \).

Remark 3. This is a generalisation of the idea introduced for the conforming space in [1], where only the \( H^1(\Omega) \)-orthogonal projection of \( v_h \) into \( P_k(E) \) was considered for \( \Pi^*_k \). The space resulting from this choice is well defined because the \( H^1(\Omega) \)-orthogonal projection of \( v_h \) is computable using just the original degrees of freedom. However, the freedom in choosing \( \Pi^*_k \) is something we wish to exploit to produce a more computationally efficient method, particularly when \( d = 3 \). We explore more possible choices of \( \Pi^*_k \) in Section 4.4, although for now we leave this choice open.

In more concrete terms, given a projector \( \Pi^*_k \), we define the local virtual element spaces to be

\[
V^E_h := \left\{ v_h \in W^E_h : (v_h - \Pi^*_kv_h, p)_E = 0 \quad \forall p \in \mathcal{M}_k^s(E) \cup \mathcal{M}_{k-1}^s(E) \right\},
\]

where \( W^E_h \) denotes either the enlarged conforming or nonconforming space. Clearly, we can use the original degrees of freedom of Definition 2 to describe \( V^E_h \).

Computing \( \Pi^*_k v_h \) for each \( v_h \in V^E_h \) is now possible, since the terms on the right-hand side of (4.2) are either degrees of freedom of \( v_h \) or moments of \( \Pi^*_k v_h \).

### 4.2. The Global Spaces.

The global virtual element space in each case is constructed as a subspace of an infinite dimensional space \( V \), defined differently for the conforming and nonconforming methods. For the conforming method, we simply take \( V := H^1(\Omega) \). For the nonconforming method, we introduce the subspace \( H^1_{\text{nc}}(\mathcal{T}_h) \) of the nonconforming broken Sobolev space \( H^1(\mathcal{T}_h) \) defined in (3.1), by imposing certain weak inter-element continuity requirements such that

\[
V := H^1_{\text{nc}}(\mathcal{T}_h) = \left\{ v \in H^1(\mathcal{T}_h) : \int_s [v] \cdot n_s q \, ds = 0 \quad \forall q \in \mathcal{P}_{k-1}(s), \forall s \in \mathcal{S}_h \right\}.
\]

The jump operator \([\cdot]\) across a mesh interface \( s \in \mathcal{S}_h \) is defined as follows for \( v \in H^1(\mathcal{T}_h) \). If \( s \in \mathcal{S}^\text{int}_h \), then there exist \( E^+ \) and \( E^- \) such that \( s \subset \partial E^+ \cap \partial E^- \). Denote by \( v^\pm \) the trace of \( v|_{E^\pm} \) on \( s \) from within \( E^\pm \) and by \( n^\pm_\ast \) the unit outward normal on \( s \) from \( E^\pm \). Then, \([v] := v^+ n^+_\ast + v^- n^-_\ast \). If, on the other hand, \( s \in \mathcal{S}^\text{bdry}_h \), then \([v] := v n_s \), with \( v \) representing the trace of \( v \) from within the element \( E \) having \( s \) as an interface and \( n_s \) is the unit outward normal on \( s \) from \( E \).

It may be seen that the broken Sobolev norm \( |\cdot|_{1,\Omega} \) is a norm on \( H^1_{\text{nc}}(\mathcal{T}_h) \) and hence the same will be true for any virtual element subspace \( V_h \), as required by Assumption A1, cf. [13].

Finally, the global space is constructed in either case from the local spaces presented above as

\[
V_h := \left\{ v_h \in V : v_h|_E \in V^E_h \quad \forall E \in \mathcal{T}_h \right\}.
\]

As global degrees of freedom we take the equivalents of those in Definition 2, namely, for each function \( v_h \in V^E_h \),

(a) for the conforming space

- the value of \( v_h \) at each internal vertex of \( \mathcal{T}_h \);
Figure 3. Degrees of freedom of the nonconforming VEM for a cubic mesh element for \( k = 1, 2, 3, 4 \); face moments are marked by a hexagon; internal moments are marked by a square. Only the internal degrees of freedom and those of the visible faces are marked; the numeric labels indicate the number of degrees of freedom when they are more than 1.

Figure 4. Degrees of freedom of the conforming VEM for a cubic mesh element for \( k = 1, 2, 3, 4 \); vertex values and edge moments are marked by a circle; face moments are marked by an hexagon; internal moments are marked by a square. Only the internal degrees of freedom and those of the visible faces and edges are marked; the numeric labels indicate the number of degrees of freedom when they are more than 1.

- for \( k > 1 \), the moments of \( v_h \) of up to order \( k - 2 \) on each mesh interface \( s \in S_h^{\text{int}} \)
  \[ \frac{1}{|s|} \int_s v_h m_\alpha \, ds \quad \forall m_\alpha \in M_{k-2}(s); \]
  for the nonconforming space, the moments of \( v_h \) of up to order \( k - 1 \) on each mesh interface \( s \in S_h^{\text{int}} \)
  \[ \frac{1}{|s|} \int_s v_h m_\alpha \, ds \quad \forall m_\alpha \in M_{k-1}(s); \]

(b) for either space, when \( k > 1 \), the moments of \( v_h \) of up to order \( k - 2 \) inside each element \( E \in T_h \)
  \[ \frac{1}{|E|} \int_E v_h m_\alpha \, dx \quad \forall m_\alpha \in M_{k-2}(E). \]

The local degrees of freedom corresponding to boundary vertices and edges \( s \in S_h^{\text{bdry}} \) are fixed as zero in accordance with the definition of the ambient spaces. The unisolvency of these degrees of freedom follows from the definition of the relevant ambient space in each case and the unisolvency of the local degrees of freedom.

4.3. The Conforming Space for \( d = 3 \). The construction of the local conforming virtual element space for \( d = 3 \) is based recursively on the space just detailed for \( d = 2 \). Define \( V_h^{\partial E} \) as the 2-dimensional conforming virtual element space of order \( k \) constructed over the polygonal interfaces making up \( \partial E \). Then, we define the space \( W_h^E \) in this case to be

\[ W_h^E := \{ v_h \in H^1(E) : v_h|_{\partial E} \in V_h^{\partial E} \text{ and } \Delta v_h \in P_k(E) \}. \]
The degrees of freedom that we take for each function $v_h \in V_h^E$ are then
(a) the degrees of freedom of $V_h^E$;
(b) for $k > 1$, the moments of $v_h$ of up to order $k - 2$ inside the element $E$
\[
\frac{1}{|E|} \int_E v_h m_\alpha \, dx \quad \forall m_\alpha \in \mathcal{M}_{k-2}(E).
\]
plus the extra degrees of freedom of Definition 3.

This allows us to construct the space $V_h^E$ from $W_h^E$ in exactly the same manner detailed above. As before, the space $V_h^E$ is spanned by just the first sets of degrees of freedom given in (a) and (b) above. The proof that these degrees of freedom are unisolvent is again given in [1]. Also, it is clear that the dimension of the local space for $d = 3$ is $n_E = \nu_k^E + \nu_k^E N_{1,k-2} + \nu_k^E N_{2,k-2} + N_{3,k-2}$ where $\nu_k^E$ and $\nu_k^E$ denote, respectively, the number of vertices and edges of $E$. The degrees of freedom for a cubic element are shown in Figure 4.

Computing $\Pi_h^E v_h$ is just the same as for $d = 2$, since the terms on the right hand side of (4.2) are either degrees of freedom of $v_h$ or moments of $\Pi_h^E v_h$. To compute $\Pi_h^E \nabla v_h$, we must compute the face terms in (4.3). Using the $L^2(s)$-orthogonal projection on the face $s$, these may be rewritten as
\[
\int_s n \cdot m_\alpha \Pi_h^{s,*} v_h \, ds \quad \forall s \subset \partial E.
\]
The face projection $\Pi_h^{s,*} v_h$ is computable using the degrees of freedom of $v_h$ on the face $s$ since $v_h|_s \in V_h^s$, where $V_h^s$ denotes the (local) VEM space on the face $s$, and consequently this term is also computable. This means that the degrees of freedom in this space allow us to compute both of the required terms in (4.1).

Finally, the global space and the set of global degrees of freedom for $d = 3$ are constructed from the local ones in the obvious way, completely analogously to the case for $d = 2$.

4.4. The projection $\Pi_h^*$. We now discuss possible choices for the projection operator $\Pi_h^*$, used to define the spaces in the preceding section in such a way that the $L^2(E)$-orthogonal projection is computable. As discussed above, the projection $\Pi_h^* v_h$ of any $v_h \in V_h^E$ needs to be computable using only the degrees of freedom of $v_h$ or moments of $\Pi_h^* v_h$. We will then be used to define the degree $k$ and $k - 1$ moments of $v_h$.

The original choice introduced in [1] was to take $\Pi_h^*$ to be the elliptic projection. The computability of this projection for either the conforming or nonconforming space is widely discussed in, for example, [6, 3, 1, 8]. Since this projection played a pivotal role in the original conforming and nonconforming methods for the Poisson problem, this choice made sense since the projection operator would already have to be computed. However, since this projection is no longer needed for the current methods, they can be made more efficient by avoiding computing unnecessary (and expensive) projection operators.

A projection $\Pi_h^* : W_h^E/\sim \to \mathcal{P}_k(E)$ may be defined using any computable inner product $B_E(\cdot, \cdot) : W_h^E/\sim \times W_h^E/\sim \to \mathbb{R}$ as the solution $\Pi_h^* v_h \in \mathcal{P}_k(E)$ for any $v_h \in W_h^E/\sim$ of
\[
B_E(\Pi_h^* v_h, m_\alpha) = B_E(v_h, m_\alpha) \quad \forall m_\alpha \in \mathcal{P}_k(E).
\]
For instance, the elliptic projection used in [6, 3] results from taking $B_E(\cdot, \cdot)$ as the $H^1$ inner product. A simple alternative choice is to take $B_E(\cdot, \cdot)$ as the Euclidean inner product on the space $\mathbb{R}^{n_E}$ of vectors of degrees of freedom of functions in $W_h^E/\sim$.

With these specific examples in mind, we first discuss how to compute the projection for a general $B_E(\cdot, \cdot)$. Let $\{\psi_i\}_{i=1}^{n_E}$ be the Lagrangian basis functions of $W_h^E/\sim$ with respect to the original degrees of freedom in Definition 2, and define the matrix $D$ as
\[
(D)_{i\alpha} = \text{dof}_i(m_\alpha) \quad (4.6)
\]
for $\alpha = 1, \ldots, N_{d,k}$ and $i = 1, \ldots, n_E$ and $\text{dof}_i(m_\alpha)$ is the $i$-th degree of freedom of $m_\alpha$ (see Definition 2). Since $W_h^E/\sim$ is finite dimensional, $B_E(\cdot, \cdot)$ can be written as the symmetric positive definite matrix $B = (B_E(\psi_i, \psi_j))$. Then, we can write $(B_E(m_\alpha, \psi_i)) = D^T B \text{ and } (B_E(m_\alpha, m_\beta)) = D^T B D$. 

Define the action of $\Pi_k^* \psi$ on the shape functions $\{\psi_i\}_{i=1}^{N_{d,k}}$ through the matrix $\Pi_k^G$, where

$$\Pi_k^* \psi_i = \sum_{\alpha=1}^{N_{d,k}} m_\alpha (\Pi_k^G)_{\alpha i},$$

so that the $j$th column of $\Pi_k^G$ contains the coefficients of the expansion of the polynomial $\Pi_k^* \psi_j$ in the monomial basis $\{m_\alpha\}_{\alpha=1}^{N_{d,k}}$. Consequently, the projection problem (4.5) can be written in the matrix form

$$\Pi_k^G = (D^T BD)^{-1} D^T B$$

The matrix $D^T BD$ is invertible due to the assumption that $B_E(\cdot, \cdot)$ is an inner product on $W_h^E \sim \mathcal{P}_h(E)$. In an implementation, one could either compute $B$ (if possible, for instance for the Euclidean projection above) and use this alongside $D$ to evaluate the other terms, or compute $D^T B$ directly (for instance for the elliptic projection) and use this.

Finally, it becomes apparent that choosing the bilinear form $B$ to fix the projection is equivalent to picking any symmetric positive definite matrix $B$. Taking $B = I$, we obtain the simple choice

$$\Pi_k^G = (D^T D)^{-1} D^T$$

which relates to the Euclidean projection mentioned above.

**Remark 4.** Note that the computability of $B$ for a specific choice of inner product $B_E(\cdot, \cdot)$ is not an issue here. In fact, the only matrices which are needed are $D$ and $(B_E(m_\alpha, \psi_i)) = D^T B$. In an implementation, one could either compute $B$ (if possible, for instance for the Euclidean projection above) and use this alongside $D$ to evaluate the other terms, or compute $D^T B$ directly (for instance for the elliptic projection) and use this.

### 4.5. Approximation Properties

Both the conforming and nonconforming spaces presented above satisfy optimal approximation results for the approximation of sufficiently smooth functions. Since these results will be used throughout the remainder of the paper, we collect them together here. They rely on the following assumption on the regularity of the mesh $T_h$.

**Assumption A3.** (Mesh regularity). Let $h_E$ and $h_s$ denote the diameter of a $d$ dimensional mesh element $E$ and a $d − 1$ dimensional mesh interface $s$ respectively. We assume the existence of a constant $\rho > 0$ such that

1. for every element $E$ of $T_h$ and every interface $s$ of $E$, $h_s \geq \rho h_E$
2. every element $E$ of $T_h$ is star-shaped with respect to a ball of radius $\rho h_E$
3. for $d = 3$, every interface $s \in S_h$ satisfies (1) and (2) above as a 2-dimensional element.

We note that all of the results and proofs presented in this paper continue to hold in the more general setting of elements which consist of a finite union of star-shaped domains, although we will not pursue this struggle for generality.

**Theorem 3** (Approximation using polynomials). Suppose that Assumption A3 is satisfied. Let $E \in T_h$ and let $\Pi^G_\ell : L^2(E) \rightarrow \mathcal{P}_\ell(E)$, for $\ell \geq 0$, denote the $L^2(E)$-orthogonal projection onto the polynomial space $\mathcal{P}_\ell(E)$. Then, for any $w \in H^m(E)$, with $1 \leq m \leq \ell + 1$, it holds

$$\|w - \Pi^G_\ell w\|_{0,E} + h_E \|w - \Pi^G_\ell w\|_{1,E} \leq C h_E^{m-1} \|w\|_{m,E}.$$  

Let $s$ be an interface shared by $E^+, E^- \in T_h$ and let $\Pi^{0,s}_\ell : L^2(s) \rightarrow \mathcal{P}_s(s)$, for $s \geq 0$, denote the $L^2(s)$-orthogonal projector onto the polynomial space $\mathcal{P}_s(s)$. Then, for every $w \in H^m(E^+) \oplus H^m(E^-)$, with $1 \leq m \leq \ell + 1$, it holds

$$\|w - \Pi^{0,s}_\ell w\|_{0,s} + h_s \|w - \Pi^{0,s}_\ell w\|_{1,s} \leq C h_s^{m-1/2} \left(\|w\|^2_{m,E^+} + \|w\|^2_{m,E^-}\right)^{1/2}.$$  

In both instances, the positive constant $C$ depends only on the polynomial degree $\ell$ and the mesh regularity.

This theorem may be proven using the standard theory in [14], see also Section 1.6 in [10] for the details. We further require the following result regarding the approximation of sufficiently smooth functions from the virtual element space.
Theorem 4 (Approximation using virtual element functions). Suppose that Assumption A3 is satisfied and let $V_h$ denote either the conforming or nonconforming virtual element space. Let $m$ be a positive integer such that $2 \leq m \leq k + 1$. Then, for any $w \in H^m(\Omega)$, there exists an element $w_1 \in V_h$ such that

$$
\|w - w_1\|_0 + h|w - w_1|_1 \leq C h^m |w|_m
$$

where $C$ is a positive constant which depends only on the polynomial degree $k$ and the mesh regularity.

Virtual element basis have the correct uniform scaling. This is shown by a generalised scaling argument [15] to account for the variability of the elements’ geometric properties (e.g. a different number of interfaces). The proof is given in Appendix A for completeness.

Lemma 1 (Scaling of basis functions). Suppose that Assumption A3 is satisfied. Let $E \in T_h$ and \( \{\phi_i\}_{i=1}^{n_E} \) be the Lagrangian basis functions of either the conforming or the nonconforming space $V_h^E$ with respect to the local set of degrees of freedom. For $r \in \{0, 1\}$, it holds that

$$
C^{-1} h^{\frac{4}{3} - r} \leq |\phi_i|_{r,E} \leq C h^{\frac{4}{3} - r}
$$

for all $i = 1, \ldots, n_E$, where the constant $C$ depends only on the polynomial degree $k$ and the mesh regularity. Here, $|\cdot|_{0,E} := |||\cdot|||_{0,E}$.

5. THE BILINEAR FORMS

In this section, we introduce a choice of the virtual element bilinear forms $a_h^E$ and $b_h^E$ that satisfy the abstract properties presented in Section 3. We remark that the bilinear forms we pick are exactly the same regardless of whether we are considering the conforming or the nonconforming method. Moreover, as described in Section 8, the implementation of the two methods differs only in the practical construction of the $L^2$-projection operators due to the different choice of the degrees of freedom.

Definition 4. Let $E \in T_h$. A computable (see Definition 1) bilinear form $S^E : V_h^E / P_k(E) \times V_h^E / P_k(E) \to \mathbb{R}$ is said to be a local admissible stabilising bilinear form if it is symmetric, positive definite and it satisfies

$$
c_0 a^E(v_h, v_h) \leq S^E(v_h, v_h) \leq c_1 a^E(v_h, v_h) \quad \forall v_h \in V_h^E / P_k(E),
$$

for some constants $c_0$ and $c_1$ independent of $E$ and $h$.

Given an admissible stabilising bilinear form $S^E(\cdot, \cdot)$, we can then define

$$
a_h^E(u_h, v_h) := (\kappa \Pi_{k-1}^u u_h, \Pi_{k-1}^u v_h)_E + (\mu \Pi_k^u u_h, \Pi_k^v v_h)_E + S^E((I - \Pi_k^0) u_h, (I - \Pi_k^0) v_h), \quad (5.1)
$$

and

$$
b_h^E(u_h, v_h) := \frac{1}{2} \left[ (\beta \cdot \Pi_{k-1}^u u_h, \Pi_k^v v_h)_E - (\Pi_k^u u_h, \beta \cdot \Pi_{k-1}^v v_h)_E \right]. \quad (5.2)
$$

It is clear that any bilinear form $A_h^E$ resulting from (5.1) and (5.2) with $S^E$ admissible satisfies Assumption A2, cf. [6]. Moreover, all of the terms in this bilinear form are computable since the construction and degrees of freedom of the space allow us to compute $\Pi_k^u u_h$ and $\Pi_{k-1}^u v_h$ for any $v_h \in V_h^E$, while $S^E$ is computable by assumption.

Remark 5. The use of the $L^2(E)$-orthogonal projection $\Pi_k^0$ in the stabilising term in (5.1) is by no means the only possibility. Indeed, one could instead use any computable projection $\Pi_k : V_h \to P_k(E)$, since all we need is to ensure that the operator $(I - \Pi_k)$ is a projection onto $V_h^E / P_k(E)$. Alternative choices could be to use the elliptic projection, as in [6], or the projection $\Pi^*_k$ used in the definition of the space, since this needs to be computed already to compute the $L^2$ projection. Another option could be to choose $\Pi_k$ in conjunction with the stabilising term in such a way as to incorporate important new features into the virtual element method, such as monotonicity, positivity and maximum/minimum principles for the numerical solutions, like the mimetic finite difference stabilising terms described in [31]. This topic, however, is beyond the scope of the present paper and will be considered in future works.
Lemma 2. Suppose that Assumptions A1-A3 are satisfied and let 
In particular, different admissible (cf. Definition 4) stabilisation terms could be used. We start 
bilinear forms satisfying Assumption A1, and not only to the specific choice given in Section 5. 
methods introduced above. We stress that the analysis applies in general for any set of VEM 

We now introduce a choice of admissible stabilising bilinear form. This is essentially a scaled version of the one already used in \[6\].

Proposition 1. Let \( \kappa_E \) and \( \mu_E \) be some (positive) constant approximations of \( \kappa \) and \( \mu \) over \( E \) respectively. Then, the bilinear form

\[
S^E(v_h, w_h) := (\kappa_E h_E^{d-2} + \mu_E h_E^d) \sum_{r=1}^{n_E} \text{dof}_r(v_h) \text{dof}_r(w_h),
\]

for \( v_h, w_h \in V_h^E/ \mathcal{P}_k(E) \), where \( n_E \) denotes the number of degrees of freedom of \( V_h^E \), is admissible and \( \text{dof}_i(w_h) \) is the \( i \)-th degree of freedom of \( w_h \) (see Definition 2).

Proof. The stabilising term \( S^E \) is positive definite because it is simply the Euclidean inner product over the finite dimensional space \( \mathbb{R}^{n_E - \dim(\mathcal{P}_k(E))} \) of vectors of degrees of freedom, which is isomorphic to \( V_h^E/ \mathcal{P}_k(E) \), and due to the assumptions on the problem coefficients. Since \( a^E \) is an inner product on \( V_h^E \) and thus also on \( V_h^E/ \mathcal{P}_k(E) \), the existence of the constants \( c_0 \) and \( c_1 \) of Definition 4 follows from the equivalence of the norms induced by these inner products.

The fact that these constants are independent of \( h \) is due to the fact that \( S^E \) scales the same as \( a^E \). It is clear that the \( H^1 \) part of \( a^E \) scales like \( \kappa_E h_E^{d-2} \) while the \( L^2 \) term (\( \mu v_h, w_h \) \( E \) scales like \( \mu_E h_E^d \). Since the degrees of freedom are specifically chosen to scale like 1 (cf. \([8]\)), the coefficient at the front of \( S^E \) ensures that the term has the correct scaling even when one of the coefficients \( \kappa \) or \( \mu \) locally degenerates. Further, the constants \( c_0 \) and \( c_1 \) can be shown to be independent of the shape of \( E \) due to Lemma 1 and the fact that \( S^E(v_h, v_h) \) depends on the choice of the values of the degrees of freedom, but not on the element \( E \).

Remark 6. The stabilising term in Proposition 1 is just one of a family of admissible stabilising terms, defined as appropriately scaled inner products on the subspace of the degrees of freedom relating to functions in \( V_h^E/ \mathcal{P}_k(E) \). Here we have chosen the Euclidean inner product for simplicity.

6. Error Analysis

We are now in a position to prove optimal order error bounds in the \( H^1 \) and \( L^2 \)-norm for the methods introduced above. We stress that the analysis applies in general for any set of VEM bilinear forms satisfying Assumption A1, and not only to the specific choice given in Section 5. In particular, different admissible (cf. Definition 4) stabilisation terms could be used. We start the analysis with an estimate of the nonconformity error introduced by using the nonconforming virtual element space.

Lemma 2. Suppose that Assumptions A1-A3 are satisfied and let \( u \in H^{m+1}(\Omega) \) for some positive integer \( m \geq 1 \) be the solution to (2.4). Define \( r = \min(k, m) \) and suppose that the coefficients \( \kappa, \beta, \mu \in W^{r+1, \infty}(\Omega) \). Then, there exists a positive constant \( C \) independent of \( h \) and \( u \) such that

\[
\sup_{w_h \in V_h} \frac{|A(u, w_h) - (f, w_h)|}{\|w_h\|_{1,h}} \leq C h^r \|u\|_{r+1},
\]

where \( V_h \) is the nonconforming virtual element space described in Section 4.

Proof. We apply the definition of the jump operator \( [\n] \) and the Green’s identity under the assumption that \( m \geq 1 \), which implies that \( u \in H^2(\Omega) \), and use the fact that \( V_h \subset H_0^{1, \text{nc}}(T_h) \) to obtain

\[
|A(u, w_h) - (f, w_h)| = \left| \sum_{s \in S_h} \int_s (\kappa \nabla u - \frac{1}{2} u \beta) \cdot [w_h] \, ds \right|
= \left| \sum_{s \in S_h} \int_s ((\kappa \nabla u - \frac{1}{2} u \beta) \cdot \Pi_{k-1}^s (\kappa \nabla u - \frac{1}{2} u \beta)) \cdot ([w_h] - \Pi_0^s [w_h]) \, ds \right|.
\]
Treating each term in this summation using the Cauchy-Schwartz inequality and then applying the approximation estimates of Theorem 3, we obtain, cf. [3] or [23],

$$|A(u, w_h) - (f, w_h)| \leq C h^r \sum_{s \in S_h} \|u\|_{r+1, E_s^+ \cup E_s^-} \left(|w_h|_{1, E_s^+}^2 + |w_h|_{1, E_s^-}^2\right)^{\frac{1}{2}}$$

where for each side $s$ the symbols $E_s^+$ and $E_s^-$ denote the two elements sharing that side, and consequently the lemma holds.

**Theorem 5 (H\(^1\) error bound).** Suppose that Assumptions A1-A3 are satisfied. Let $k \geq 1$ be a positive integer and let $u \in H^{m+1}(\Omega)$ be the true solution to problem (2.4) for some positive integer $m$. Define $r = \min(k, m)$ and suppose that the coefficients $\kappa, \beta, \mu \in W^{r+1, \infty}(\Omega)$ satisfy (2.2) and (2.3) and assume that $f \in H^{r-1}(\Omega)$. Let $(f_h, v_h) := \sum_{E \in T_h} (f_h, v_h)_E$, with $f_h|_E := \Pi_{\text{max}(k-2,0)} f|_E$. Denote by $u_h \in V_h$ the corresponding virtual element solution to problem (3.3) where $V_h$ is either the conforming or the nonconforming virtual element space presented in Section 4. Then, there exists a constant $C$ independent of $h$ and $u$ such that

$$\|u - u_h\|_{1,h} \leq C h^r (\|u\|_{r+1} + \|f\|_{r-1})$$

**Proof.** We prove the theorem by separately bounding the terms of the Strang-type abstract convergence result of Theorem 2. The first term on the right-hand side of (3.6), i.e. $\inf_{w_h \in V_h} \|u - v_h\|_{1,h}$, is easily bounded by introducing any interpolant $u_h \in V_h$ of $u$ as in Theorem 4. For the second term, first we suppose that $k \geq 2$. In this case we may apply the definition of the $L^2$-projection of $f$ to find that

$$\sup_{w_h \in V_h} \left| \frac{\sum_{E \in T_h} (\Pi_{k-2} f - f)(w_h - \Pi_0 w_h) \, dx}{\|w_h\|_{1,h}} \right| \leq C h^r \|f\|_{r-1}$$

having used the bounds of Theorem 3. A similar argument applies with $f_h = \Pi_0 f$ when $k = 1$.

Turning now to the infimum over the polynomial subspace, we first observe that

$$\inf_{p \in P_k(T_h)} \left[ \|u - p\|_{1,h} + \sum_{E \in T_h} \sup_{w_h \in V_h} \frac{|A^E(p, w_h) - A^E_h(p, w_h)|}{\|w_h\|_{1,E}} \right] \leq \|u - \Pi_k^0 u\|_{1,h} + \sum_{E \in T_h} \sup_{w_h \in V_h} \frac{|A^E(\Pi_k^0 u, w_h) - A^E_h(\Pi_k^0 u, w_h)|}{\|w_h\|_{1,E}}.$$

To treat the second term on the right-hand side, we recall the splitting of the bilinear forms into their symmetric and skew-symmetric parts, and bound each part separately. Here we only detail the bounding of the difference between the symmetric parts since the skew-symmetric parts can be treated analogously. From the definition of $a^E$ and the polynomial consistency property of $a^E_h$ (cf., Assumption (A2)) it follows that

$$|a^E(\Pi_k^0 u, w_h) - a^E_h(\Pi_k^0 u, w_h)| \leq \int_E \kappa \nabla \Pi_k^0 u \cdot (1 - \Pi_k^0) \nabla w_h \, dx + \int_E \gamma \Pi_k^0 u (1 - \Pi_k^0) w_h \, dx \leq \|w_h\|_{1,E} \left( \|(1 - \Pi_k^0) (\kappa \nabla \Pi_k^0 u)\|_{0,E} + \|(I - \Pi_k^0) (\gamma \Pi_k^0 u)\|_{0,E} \right).$$
and now the results of Theorem 3 and the regularity assumption on $\kappa$ and $\mu$ implies
\[
\sup_{w_h \in V_h^k} \left| a^E(\Pi_k^0 u, w_h) - a^E(\Pi_k^0 u, w_h) \right| \leq Ch^r_E\|u\|_{r+1,E}.
\]
A similar bound holds for $|b^E(\Pi_k^0 u, w_h) - b^E(\Pi_k^0 u, w_h)|$ due to the regularity assumption on $\beta$. Combining these bounds we therefore obtain
\[
\sup_{w_h \in V_h^k} \left| A^E(\Pi_k^0 u, w_h) - A^E(\Pi_k^0 u, w_h) \right| \leq Ch^r_E\|u\|_{r+1,E}.
\]
The result then follows by observing that Lemma 2 provides an optimal order bound for the remaining term relevant to the nonconforming case only. $\square$

**Theorem 6 ($L^2$ error bound).** Suppose that Assumptions A1-A3 are satisfied, and further assume that the domain $\Omega$ is convex. Let $k \geq 1$ be a positive integer and let $u \in H^{m+1}(\Omega)$ be the solution to the problem (2.4) for some positive integer $m$. Define $r = \min(k, m)$ and suppose that the coefficients $\kappa, \beta, \mu \in W^{r+1}\cap H^{\infty}(\Omega)$ satisfy (2.2) and (2.3) and assume that $f \in H^{r-1}(\Omega)$. Let $(f_h, v_h) := \sum_{E \in T_h} (f_h, v_h)_E$, with $f_h|_E := \Pi_k^0 f|_E$. Denote by $u_h \in V_h$ the corresponding virtual element solution to problem (3.3) where $V_h$ is either the conforming or the nonconforming virtual element space presented in Section 4. Then, there exists a constant $C$ independent of $h$ and $u$ such that
\[
\|u - u_h\|_0 \leq Ch^{r+1}\|u\|_{r+1}.
\]

**Proof.** Let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution to the dual problem
\[
-\nabla \cdot (\kappa \nabla \psi) - \beta \cdot \nabla \psi + (\gamma - \nabla \cdot \beta) \psi = u - u_h.
\]
Then, due to the convexity of $\Omega$, $\psi$ satisfies the regularity bound
\[
\|\psi\|_2 \leq C\|u - u_h\|_0,
\]
and consequently for any interpolant $\psi_1$ as in Theorem 4, we have
\[
\|\psi - \psi_1\|_{1,h} \leq Ch\|\psi\|_2 \leq Ch\|u - u_h\|_0.
\]
Multiplying (6.1) by $u - u_h$ and integrating, we find that
\[
\|u - u_h\|_0^2 = (u - u_h, -\nabla \cdot (\kappa \nabla \psi) - \beta \cdot \nabla \psi + (\gamma - \nabla \cdot \beta) \psi) = A(u - u_h, \psi) + \sum_{s \in S_h} \int_s \left(\kappa \nabla \psi - \frac{1}{2} \beta \psi\right) \cdot [u - u_h] \, ds,
\]
and the edge-wise term may be bounded by arguing as in Lemma 2, to find that
\[
\sum_{s \in S_h} \int_s \left(\kappa \nabla \psi - \frac{1}{2} \beta \psi\right) \cdot [u - u_h] \, ds \leq Ch\|\psi\|_2\|u - u_h\|_{1,h} \leq Ch^{r+1}\|u\|_{r+1}\|u - u_h\|_0.
\]
To bound the other term, we add and subtract appropriately,
\[
A(u - u_h, \psi) = A(u - u_h, \psi - \psi_1) + A(u - u_h, \psi_1) = A(u - u_h, \psi - \psi_1) + (A(u, \psi_1) - (f, \psi_1)) + (A_h(u_h, \psi_1) - A(u_h, \psi_1)) + ((f, \psi_1) - (f_h, \psi_1)),
\]
obtaining terms which relate to those of the original abstract error bound. We label these as $T_1$, $T_2$, $T_3$ and $T_4$ respectively, and bound them separately.

We first observe that $T_1$ may be bounded using the continuity of the variational form and the $H^1$-norm error bound of Theorem 5 as
\[
T_1 := A(u - u_h, \psi - \psi_1) \leq C\|u - u_h\|_{1,h}\|\psi - \psi_1\|_{1,h} \leq Ch^{r+1}\|u\|_{r+1}\|u - u_h\|_0.
\]
The term $T_2$ measures the nonconformity of the method and may be bounded using Lemma 2 as
\[ |T_2| := |A(u, \psi_1) - (f, \psi_1)| \leq Ch^r \|u\|_{r+1} \|\psi - \psi_1\|_{1,h} \]
\[ \leq \frac{Ch^{r+1}}{2} \|u\|_{r+1} \|u - u_h\|_0. \]
Using the definition of the $L^2$-projection, we can rewrite $T_f$ as
\[ T_f := (f, \psi_1) - (f_h, \psi_1) = \sum_{E \in \mathcal{T}_h} (f - \Pi_{k-1}^0 f, \psi_1)_E = \sum_{E \in \mathcal{T}_h} (f - \Pi_{k-1}^0 f, \psi_1 - \Pi_0^0 \psi_1)_E \]
\[ \leq \sum_{E \in \mathcal{T}_h} \|f - \Pi_{k-1}^0 f\|_{0,E} \|\psi_1 - \Pi_0^0 \psi_1\|_{0,E} \]
\[ \leq C \sum_{E \in \mathcal{T}_h} h^r \|f\|_{r,E} h \|\psi_1\|_{1,E} \leq Ch^{r+1} \sum_{E \in \mathcal{T}_h} \|f\|_{r,E} \|\psi\|_{2,E} \]
\[ \leq Ch^{r+1} \|f\|_{r+1} \|u - u_h\|_0. \]
Finally we turn to the inconsistency term $T_3$, namely
\[ T_3 := A_h(u_h, \psi_1) - A(u_h, \psi_1) = \sum_{E \in \mathcal{T}_h} A^E_h(u_h, \psi_1) - A^E(u_h, \psi_1) \]
\[ = \sum_{E \in \mathcal{T}_h} (A^E_h(u_h - \Pi_0^0 u, \psi_1 - \Pi_0^0 \psi) - A^E(u_h - \Pi_0^0 u, \psi_1 - \Pi_0^0 \psi)) + \]
\[ + (A^E_h(\Pi_k^0 u, \psi_1) - A^E(\Pi_k^0 u, \psi_1)) + (A^E_h(u_h, \Pi_0^0 \psi) - A^E(u_h, \Pi_0^0 \psi)) \]
The first difference can then easily be bounded using the fact that both the variational form and the VEM bilinear form are continuous in the $H^1$ norm, so
\[ A^E_h(u_h - \Pi_0^0 u, \psi_1 - \Pi_0^0 \psi) - A^E(u_h - \Pi_0^0 u, \psi - \Pi_0^0 \psi) \leq C \|u_h - \Pi_0^0 u\|_{1,E} \|\psi_1 - \Pi_0^0 \psi\|_{0,E} \]
\[ \leq Ch^{r+1}_E \|u\|_{r+1,E} \|u - u_h\|_{0,E}. \]
The bound for the other two differences is obtained by splitting each bilinear form up into its constituent terms and applying the definition of polynomial consistency. For the diffusion terms, the polynomial consistency property means we consider
\[ a^E_h(\Pi_k^0 u, \psi_1) - a^E(\Pi_k^0 u, \psi_1) = \int_E \kappa \nabla \Pi_k^0 u \cdot (\Pi_{k-1}^0 - I) \nabla \psi_1 \, dx \]
\[ = \int_E (\Pi_{k-1}^0 - I) \left( \kappa \nabla \Pi_k^0 u \right) \cdot \nabla (\psi_1 - \psi) \, dx + \]
\[ + \int_E (\Pi_{k-1}^0 - I) \left( \kappa \nabla \Pi_k^0 u \right) \cdot \nabla (\psi - \Pi_0^0 \psi) \, dx \]
\[ \leq Ch^{r+1}_E \|u\|_{r+1,E} \|u - u_h\|_{0,E}, \]
having applied the Cauchy-Schwarz inequality and the polynomial approximation bounds in the final step. The second difference is similarly treated by adding and subtracting terms and applying the Cauchy-Schwarz inequality and the polynomial approximation bounds along with the regularity of the dual solution $\psi$ and the $H^1$-norm error bound
\[ a^E_h(u_h, \Pi_0^0 \psi) - a^E(u_h, \Pi_0^0 \psi) = \int_E \kappa \left( \Pi_{k-1}^0 - I \right) \nabla u_h \cdot \nabla \Pi_0^0 \psi \, dx \]
\[ = \int_E \kappa \left( \Pi_{k-1}^0 - I \right) \nabla (u_h - u) \cdot \nabla (\Pi_0^0 - I) \psi \, dx + \]
\[ + \int_E \kappa \left( \Pi_{k-1}^0 - I \right) \nabla u \cdot \nabla (\Pi_0^0 - I) \psi \, dx + \]
\[ + \int_E \nabla (u_h - \Pi_k^0 u) \cdot (\Pi_{k-1}^0 - I) (\kappa \nabla \psi) \, dx \]
\[ \leq Ch^{r+1}_E \|u\|_{r+1,E} \|u - u_h\|_{0,E}. \]
The bounds for the other components of the bilinear form in these differences are treated completely analogously. Consequently, we may combine these individual bounds to determine the optimal order bound in the statement of the theorem. □

7. The Effects of Numerical Integration

In any practical implementation of the method, the coefficients $\kappa$, $\beta$, and $\mu$, have to be approximated, meaning that the polynomial consistency properties of Assumption A2 will hold in an approximate way in general. One possibility, which we assess here, is to utilise numerical quadratures. Crucially, such numerical quadratures will only affect the consistency terms. These have precisely the same structure of standard finite elements terms (integral products of polynomial trial and test functions weighted by the coefficients), and as such the variational crime introduced by their approximation can be assessed using the classical finite element analysis. Indeed, within this section we show that the actual implemented methods retain the stability and optimal accuracy properties of the theoretical virtual element methods proposed above, provided that the quadrature scheme used is, at least, of polynomial order $2k - 2$. We emphasise that this is exactly the same requirement as for the classical finite element methods used to solve the same problem (cf. [19]).

In more concrete terms, suppose that we are approximating integrals over the element $E$ using a quadrature rule $Q^E_m$ of degree $m$, so

$$
\int_E g \, dx \approx Q^E_m(g) := \sum_{\ell=1}^{L} \omega_{\ell} g(q_{\ell}),
$$

for a finite set of quadrature points $\{q_{\ell}\}_{\ell=1}^{L}$ and associated weights $\{\omega_{\ell}\}_{\ell=1}^{L}$. Thus in practice, the implementation of the method will be based on the perturbed bilinear form

$$
a^E_h(u_h, v_h) := \sum_{\ell=1}^{L} \omega_{\ell} \kappa(q_{\ell}) (\Pi^0_{k-1} \nabla u_h)(q_{\ell}) \cdot (\Pi^0_{k-1} \nabla v_h)(q_{\ell}) + \sum_{\ell=1}^{L} \omega_{\ell} \mu(q_{\ell}) (\Pi^0_{k-1} u_h)(q_{\ell}) (\Pi^0_{k-1} v_h)(q_{\ell})
$$

$$
+ (\kappa_{E} h_{E}^{d-2} + \mu_{E} h_{E}^{d}) \sum_{r=1}^{n_E} \text{dof}_{r}((I-\Pi_{k})u_h) \text{dof}_{r}((I-\Pi_{k})v_h),
$$

and similarly for the skew-symmetric part $b^E_h$. Once more, we note that the use of quadrature only affects the consistency term. The following two theorems show that the use of an appropriate quadrature rule does not affect either the stability or the accuracy of the method.

**Theorem 7.** Suppose the quadrature scheme $Q^E_m$ with $m \geq 2k - 2$ has strictly positive weights and is exact for the space $P_{2k-2}(E)$ and/or the set $\{q_{\ell}\}_{\ell=1}^{L}$ of quadrature points contains a $P_{k-1}(E)$ unisolvent subset. Let $\mathfrak{A}_h$ denote the bilinear form $A_h$ with the polynomial consistency integrals approximated using the quadrature scheme $Q^E_m$. Then, $\mathfrak{A}_h$ satisfies the stability property of Assumption A2.

**Proof.** We first wish to show that the bilinear form $a^E_h$ defines a norm on $V^E_h$, or on $V^E_h/P_0(E)$ when $\mu \equiv 0$, equivalent to the norm imposed by $a^E$ in either case. Suppose $\mu \neq 0$. Then $a^E_h$ is clearly already a semi-norm and all that remains to be shown is that $a^E_h(v_h, v_h) = 0 \Rightarrow v_h = 0$.

Let $a^E_h(v_h, v_h) = 0$. Then, we must have

$$
Q^E_m((\Pi^0_{k-1} \nabla v_h) \cdot (\kappa\Pi^0_{k-1} \nabla v_h)) = 0.
$$

By the assumptions on $Q^E_m$ and the strong ellipticity of $\kappa$ (see (2.2)), this implies that $\Pi^0_{k-1} \nabla v_h = 0$, and consequently we may deduce that either (a), $v_h \in P_0(E)$ or (b), $v_h \in V^E_h/P_k(E)$ where, with a slight abuse of notation, we associate $V^E_h/P_k(E)$ with the non-polynomial subspace of $V^E_h$.\[\square\]
Suppose case (a) holds, so \( v_h \in \mathcal{P}_0(E) \). Then we also have
\[
0 = Q_m^E(\mu \Pi_k^0 v_h)^2 = Q_m^E(\mu v_h^2),
\]
since \( \Pi_k^0 \) is the identity on \( \mathcal{P}_0(E) \), and thus we deduce that \( v_h = 0 \).

Alternatively, suppose that case (b) holds, so \( v_h \in \mathcal{V}_h^E/\mathcal{P}_k(E) \). Then, since \( a_h^E(v_h, v_h) = 0 \), it follows that
\[
S^E((1 - \Pi_k)v_h, (1 - \Pi_k)v_h) = 0,
\]
and we may deduce that \( v_h = 0 \).

From this, we may conclude that \( (a_h^E(\cdot, \cdot))^\frac{1}{2} \) is a norm on \( \mathcal{V}_h^E \). Moreover, \( (a_h^E(\cdot, \cdot))^\frac{1}{2} \) is also a norm on \( \mathcal{V}_h^E \) and since this is a finite dimensional subspace of \( H^1(E) \), the resulting norms are equivalent. As with the bilinear form \( a_h^E \), the constants in the equivalence are independent of \( h \) due to the correct scaling of \( a_h^E \) and can be made independent of \( E \) as in the proof of Proposition 1.

On the other hand, when \( \mu \equiv 0 \) we find that \( (a_h^E(\cdot, \cdot))^\frac{1}{2} \) and \( (\mu a_h^E(\cdot, \cdot))^\frac{1}{2} \) are both norms on \( \mathcal{V}_h^E/\mathcal{P}_0(E) \) and both zero on \( \mathcal{P}_0(E) \). Again, the fact that \( \mathcal{V}_h^E/\mathcal{P}_0(E) \) is finite dimensional allows us to deduce that the two norms are equivalent.

The stability property for \( b_h^E \) also holds because \( b_h^E(v_h, v_h) = b_h^E(v_h, v_h) = 0 \) and it is straightforward to check that \( b_h^E(u_h, v_h) \leq C\|\beta\|_{E} \|u_h\|_{1,E} \|v_h\|_{1,E} \).

The next result addresses the questions about the accuracy of the method when the quadrature is employed to evaluate the integrals of the virtual bilinear forms, and should be compared with Theorem 5.

**Theorem 8.** Suppose that Assumptions A1-A3 are satisfied. Let \( k \geq 1 \) be a positive integer and let \( u \in H^{r+1}(\Omega) \) be the true solution to problem (2.4) for some positive integer \( s \). Define \( r = \min(k, s) \) and suppose that the coefficients \( \kappa, \beta, \mu \in W^{r+1,\infty}(\Omega) \), satisfying (2.2) and (2.3). Suppose that the right-hand side function \( f \in H^{-1}(\Omega) \) is approximated by \( \tilde{f}_h := \Pi_{\max(k-2,0)}^0 f \).

Suppose that the quadrature scheme \( Q_m^E \) with \( m \geq 2k - 2 \) is exact for the space \( \mathcal{P}_{2k-2}(E) \). Let \( \mathfrak{A}_h \) denote the virtual element bilinear form \( \mathfrak{A}_h \) obtained by approximating the integrals using the quadrature scheme \( Q_m^E \) and \( u_h \in \mathcal{V}_h \) be the solution obtained from this scheme. Then, there exists a positive constant \( C \), independent of \( h \) and \( u \) such that
\[
\|u - u_h\|_{1,h} \leq C h^r (1 + h) \|u\|_{r+1,E} + C h^r \|f\|_{r-1,E}.
\]

**Proof.** Expanding as in the proof of Theorem 2, it may be shown that the only extra term depending on the quadrature scheme which arises in the abstract error bound is
\[
\sum_{E \in T_h} \sup_{w_h \in \mathcal{V}_h^E} \frac{|A_h^E(u, w_h) - \mathfrak{A}_h^E(u, w_h)|}{\|w_h\|_{1,E}},
\]
where \( u := \Pi_k^0 u \). As usual, we split this term into the different components of the bilinear form and bound them separately. We give here the bound for \( a_h^E(u, w_h) - a_h^E(u, w_h) \); the bound for the skew-symmetric term follows analogously. Since the stabilising term is unaffected by the quadrature, we only need to bound
\[
\sup_{w_h \in \mathcal{V}_h^E} \frac{|Q_m^E((\kappa \nabla u_h) \cdot \Pi_k^0 \nabla w_h) - \int_E (\kappa \nabla u_h) \cdot \Pi_k^0 \nabla w_h dx|}{\|w_h\|_{1,E}} + \sup_{w_h \in \mathcal{V}_h^E} \frac{|Q_m^E((\mu u_h) \cdot \Pi_k^0 w_h) - \int_E (\mu u_h) \cdot \Pi_k^0 w_h dx|}{\|w_h\|_{1,E}}.
\]

Arguing as in Theorem 4.1.4 in [19] and using the stability of the \( L^2(E) \) projector, we find that
\[
|Q_m^E((\kappa \nabla u_h) \cdot \Pi_k^0 \nabla w_h) - \int_E (\kappa \nabla u_h) \cdot \Pi_k^0 \nabla w_h dx| \leq C h^r \|u\|_{r+1,E} \|w_h\|_{1,E},
\]
and
\[ Q^E_m((\mu u_\tau) \cdot \Pi^0_k w_h) - \int_E (\mu u_\tau) \Pi^0_k w_h \, dx \leq C \left| h^E ||u||_{r,E} \right| ||w_h||_{1,E}. \]

The theorem then follows by treating the skew-symmetric term similarly and combining the result with the original $H^1$-norm bound of Theorem 5.  

A similar analysis can be carried out to control the error in the numerical approximation $f_h$ of the forcing function $f$, and to recover the optimal order of convergence in the $L^2$-norm, see always [19].

8. Implementation

In this section we detail the implementation of the elemental $L^2$-projection operators appearing in the local bilinear forms of Section 5. Once these are at hand, the terms appearing in the VEM take the form typical of standard FEM, namely integrals of problem data times polynomials, which can be computed with a quadrature formula according to Section 7. The stabilisation terms are an exception but, as we shall see, these are straightforward to compute.

To ease the notation, we use Roman fonts to index the degrees of freedom and the corresponding shape functions and keep using Greek indices as in the previous sections to index the scaled monomials generating $\mathcal{P}_h(E)$.

Let $V^E_h$ be either the local conforming or nonconforming virtual element space. We explicitly construct the matrices representing the projections in (4.1) with respect to the Lagrangian basis $\{\phi_i\}_{i=1}^{n_E}$ of $V^E_h$ associated with the degrees of freedom in Definition 2.

The polynomial $\Pi^0_k \phi_i$ is the solution of the projection problem:
\[ (m_\alpha, \Pi^0_k \phi_i)_E = (m_\alpha, \phi_i)_E \quad \forall \alpha = 1, \ldots, N_{d,k}. \tag{8.1} \]

Each polynomial $\Pi^0_k \phi_i$ for $i = 1, \ldots, n_E$ can be expanded on the monomials $m_\alpha$ generating $\mathcal{P}_h(E)$. We collect the coefficients of all these expansions in the columns of matrix $\Pi^0_k$, so that
\[ \Pi^0_k \phi_i = \sum_{\alpha=1}^{N_{d,k}} m_\alpha (\Pi^0_k)_{\alpha i}, \tag{8.2} \]

and we reformulate the (local) projection problem (8.1) in matrix form as
\[ H_k \Pi^0_k = \mathbf{C}. \]

with
\[ (H_k)_{\alpha \beta} = (m_\alpha, m_\beta)_E, \tag{8.3} \]
\[ C_{\alpha i} = (m_\alpha, \phi_i)_E, \]

for $i = 1, \ldots, n_E$, and $\alpha, \beta = 1, \ldots, N_{d,k}$. Since the space $V^E_h$ does not use the extra degrees of freedom in Definition 3, matrix $\mathbf{C}$ must be constructed in two parts according to the definition of the space and the projection $\Pi^*_k$ in (4.7):
\[ C_{\alpha i} = \begin{cases} (m_\alpha, \phi_i)_E & \text{if } m_\alpha \in \mathcal{M}_{k-2}(E), \\ (m_\alpha, \Pi^*_k \phi_i)_E = (H(D^T D)^{-1} D^T)_{\alpha i} & \text{if } m_\alpha \in \mathcal{M}_{k-1}(E) \cup \mathcal{M}_k(E), \end{cases} \tag{8.4} \]

with matrix $D$ given by (4.6). The matrix $\mathbf{C}$ is fully computable from (8.4) as the moments of $\phi_i$ are simply degrees of freedom.

The polynomial vector $\Pi^0_{k-1} \nabla \phi_i$ is the solution of the projection problem:
\[ (\Pi^0_{k-1} \nabla \phi_i, m_\alpha)_E = (\nabla \phi_i, m_\alpha)_E \quad \forall m_\alpha \in (\mathcal{M}_{k-1}(E))^d \]
\[ = \int_{\partial E} m_\alpha \cdot n \phi_i \, ds - (\phi_i, \nabla \cdot m_\alpha)_E. \tag{8.5} \]
We expand the $\ell = 1, \ldots, d$ components of each polynomial vector $\Pi_{k-1}^\ell \nabla \phi_i$ on the monomials $m_\alpha$ and we collect all the expansion coefficients in the columns of the matrices $\Pi_{k-1}^{\ell, x_i}$, so that:

$$
\Pi_{k-1}^{\ell} \frac{\partial \phi_i}{\partial x_l} = \sum_{\alpha=1}^{N_{d,k-1}} m_\alpha (\Pi_{k-1}^{\ell, x_i})_{\alpha i}.
$$

(8.6)

The projection problem (8.5) can be reformulated in matrix form as

$$
H_{k-1} \Pi_{k-1}^{\ell, x_i} = R_{x_i},
$$

with the matrix $H_{k-1}$ defined analogously to $H_k$, cf. (8.3), and $R_{x_i}$, $\ell = 1, \ldots, d$, given by

$$
(R_{x_i})_{\alpha i} = \sum_{s \subseteq \partial E} m_\alpha n_\ell \phi_i \text{ ds} - \left( \phi_i, \frac{\partial m_\alpha}{\partial x_l} \right)_E,
$$

(8.7)

for $\alpha = 1, \ldots, N_{d,k}$ and $i = 1, \ldots, n_E$.

**Remark 7** (Computability). For the nonconforming method, the first integral term in (8.7) is the side moment of $\phi_i$ against the monomial $m_\alpha$, and coincides with a degree of freedom of $\phi_i$ for both $d = 2, 3$. For the conforming method and $d = 2$, the first integral term in (8.7) is computable after reconstructing the trace of $\phi_i$ on $s$, which is a univariate polynomial of degree $k$, and multiplying by the trace of $m_\alpha$, which is also a known polynomial. For the conforming method and $d = 3$, the first integral must be computed recursively using the $L^2$-orthogonal projection of $\phi_i$ on each face $s \subseteq \partial E$ as

$$
\int_{\partial E} m_\alpha \cdot n \phi_i \text{ ds} = \sum_{s \subseteq \partial E} \int_{s} m_\alpha \cdot n \Pi_{k}^{0,s} \phi_i \text{ ds}.
$$

The projection $\Pi_{k}^{0,s}$ can be computed on each face of $E$ exactly as when $d = 2$. The second term in (8.7) is simply a combination of the internal degrees of freedom of $\phi_i$, which are the same for both the conforming and the non-conforming virtual element method.

Having the matrices $\Pi_{k}^0$ and $\Pi_{k-1}^{0, x_i}$, $l = 1, \ldots, d$, at our disposal, we are able to implement the VEM in each element. Indeed, for the term of $a_h^E$ that contains the diffusion coefficient $\kappa_{\ell n}$ we have:

$$
(\kappa_{\ell n} \Pi_{k-1}^0 \frac{\partial \phi_i}{\partial x_l} \Pi_{k-1}^0 \frac{\partial \phi_j}{\partial x_n} )_E = \sum_{\alpha, \beta = 1}^{N_{d,k-1}} (\kappa_{\ell n} m_\alpha, m_\beta)_E (\Pi_{k-1}^{0, x_i})_\alpha (\Pi_{k-1}^{0, x_n})_\beta,
$$

for the reaction term in $a_h^E$, we easily find that:

$$
(\mu \Pi_{k-1}^0 \phi_i, \Pi_{k-1}^0 \phi_j)_E = \sum_{\alpha, \beta = 1}^{N_{d,k}} (\mu m_\alpha, m_\beta)_E (\Pi_{k-1}^0)_{\alpha i} (\Pi_{k-1}^0)_{\beta j}.
$$

For the skew-symmetric bilinear form $b_h^E$, first notice that $\beta \cdot \Pi_{k-1}^0 \nabla \phi_j = \sum_{l=1}^d \beta_l \Pi_{k-1}^0 \frac{\partial \phi_j}{\partial x_l}$, where $\beta_l$ is the $l$-th component of $\beta$. Therefore, the first term of $b_h^E$ is given by:

$$
\int_E \beta \Pi_{k-1}^0 \phi_i \cdot \Pi_{k-1}^0 \nabla \phi_j \text{ dx} = \sum_{l=1}^d \sum_{\alpha = 1}^{N_{d,k}} \sum_{\beta = 1}^{N_{d,k-1}} (\beta_l m_\alpha, m_\beta)_E (\Pi_{k-1}^0)_{\alpha i} (\Pi_{k-1}^{0, x_l})_{\beta j},
$$

and a similar expression is found for the second term by exchanging $i$ and $j$.

For the stabilisation term, we first note that, for $i = 1, \ldots, n_E$, the polynomial $\Pi_{k}^0 \phi_i$ is also a function of the virtual element space $V_h^E$, and, thus, can be expanded using the shape functions $\phi_j$ generating $V_h^E$. We collect the coefficients of these expansions in the columns of the matrix $\Pi_{k}^{0, \phi}$, so that:

$$
\Pi_{k}^0 \phi_i = \sum_{j=1}^{n_E} \phi_j (\Pi_{k}^{0, \phi})_{ji}.
$$

(8.8)
Comparing (8.2) and (8.8) and recalling the definition of $D$ yields the matrix relation $\Pi^0_k \phi = D \Pi^0_k$. Finally, according to the expressions given in Proposition 1, the stabilising term is given by

$$S^E((I - \Pi^0_k)\phi_i, (I - \Pi^0_k)\phi_j) = (\kappa_E h^d E_k^d + \bar{\gamma}_E h^d E_k^d) \left( (I - \Pi^0_k \phi)^T (I - \Pi^0_k \phi) \right)_{ij},$$

since $\text{dof}_v((I - \Pi^0_k)\phi_i) = (1 - \Pi^0_k \phi)_{iv}$. Similarly, since we have the projector $\Pi^0_k$ at our disposal, we might as well use $f_h := \Pi^0_k f|_E$ to approximate $f|_E$. In this case the right-hand side vector is given by:

$$(f_h, \phi_j)_E = \int_E \Pi^0_k f \phi_j \, dx = \int_E f \Pi^0_k \phi_j \, dx = \sum_{\alpha=1}^{N_{d,k}} (f, m_{\alpha})_E \left( \Pi^0_k \right)_{\alpha i}.$$

In practice, the $L^2$-products on the right-hand side of the above formulas have to be somehow approximated in accordance with the theory presented in Section 7.

9. Numerical Results

All the numerical experiments presented in this section are obtained using (8.9) for the approximation of the right-hand side, and the choice given by (4.7) for $\Pi^0_k$. However, a comparison with the implementation using the elliptic projection for $\Pi^0_k$ (cf. [1]) did not reveal any significant difference in the behaviour of the method. We give evidence of this fact within the reaction-dominated test case below.

The numerical experiments are aimed to confirm the a priori analysis developed in the previous sections. In a preliminary stage, the consistency of both the conforming and nonconforming VEM, i.e. the exactness of these methods for polynomial solutions, has been tested numerically by solving the elliptic equation with boundary and source data determined by $u(x, y) = x^m + y^m$ on different set of polygonal meshes and for $m = 1$ to 4. In all the cases, we measure an error whose magnitude is of the order of the arithmetic precision, thus confirming this property.

To study the accuracy of the method we solve the convection-reaction-diffusion equation on the domain $\Omega = (0, 1) \times (0, 1)$. The variable coefficients of the equation are given by

$$\kappa(x, y) = \begin{pmatrix} 1 + y^2 & -xy \sin(2\pi x) \sin(2\pi y) \\ -xy \sin(2\pi x) \sin(2\pi y) & 1 + x^2 \end{pmatrix},$$

$$\beta(x, y) = \begin{pmatrix} -2 (x + 2y^2 - 1) \\ 3 (3x^2 - 2y + 3) \end{pmatrix},$$

$$\gamma(x, y) = x^2 + y^3 + 1.$$

The forcing term and the Dirichlet boundary condition are set in accordance with the exact solution

$$u(x, y) = \sin(2\pi x) \sin(2\pi y) + x^5 + y^5.$$

The performance of the methods presented above is investigated by evaluating the rate of convergence on three different sequences of five meshes, labeled by $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$, respectively. The top panels of Fig. 5 show the first mesh of each sequence and the bottom panels show the mesh of the first refinement. The meshes in $\mathcal{M}_1$ are built by partitioning the domain $\Omega$ into square cells and relocating each interior node to a random position inside a square box centered at that node. The sides of this square box are aligned with the coordinate axis and their length is equal to 0.8 times the minimum distance between two adjacent nodes of the initial square mesh. The meshes in $\mathcal{M}_2$ are built as follows. First, we determine a primal mesh by remapping the position $(\tilde{x}, \tilde{y})$ of the nodes of an uniform square partition of $\Omega$ by the smooth coordinate transformation $[29]$

$$x = \tilde{x} + (1/10) \sin(2\pi \tilde{x}) \sin(2\pi \tilde{y}),$$

$$y = \tilde{y} + (1/10) \sin(2\pi \tilde{x}) \sin(2\pi \tilde{y}).$$

The corresponding mesh of $\mathcal{M}_2$ is built from the primal mesh by splitting each quadrilateral cell into two triangles and connecting the barycentres of adjacent triangular cells by a straight segment. The mesh construction is completed at the boundary by connecting the barycentres of the triangular cells close to the boundary to the midpoints of the boundary edges and these latters
Figure 5. First (top) and second (bottom) mesh of the three mesh families.

<table>
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<th>$N_s$</th>
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Table 1. Mesh data for the meshes in $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$; $N\_E$, $N\_s$ and $N\_v$ are the numbers of mesh elements, interfaces and vertices, respectively, and $h$ is the mesh size parameter.

to the boundary vertices of the primal mesh. The meshes in $\mathcal{M}_3$ are obtained by filling the unit square with a suitably scaled non-convex octagonal reference cell.

All the meshes are parametrised by the number of partitions in each direction. The starting mesh of every sequence is built from a $5 \times 5$ regular grid, and the refined meshes are obtained by doubling this resolution. Mesh data for each refinement level, i.e., numbers of mesh elements, number of edges, number of vertices, are reported in Table 1.

Approximation errors are measured by comparing the polynomial quantities $\Pi^0_0 u_h$ and $\Pi^0_{k-1} \nabla u_h$, which are obtained by a post-processing of the numerical solution, with the exact solution $u$ and solution’s gradient $\nabla u$. The relative errors for the approximation of solution $u$ and its gradient as a function of the mesh size $h$ are shown in the log-log plots of Fig. 6 for the mesh sequence $\mathcal{M}_1$, Fig. 7 for the mesh sequence $\mathcal{M}_2$, and Fig. 8 for the mesh sequence $\mathcal{M}_3$. The values of the measured error are marked with a circle for the conforming VEM and with a triangle for the nonconforming VEM. The plots on the left show the relative errors for the approximation of the solution, while the plots on the right show the relative errors for the approximation of the solution’s gradient. The expected slopes are shown for each error curve directly on the plots. The
numerical results confirm the theoretical rate of convergence. The conforming and nonconforming VEMs provide very close results on any fixed mesh, with the conforming method slightly out performing the nonconforming VEM in few cases. Similar results (not shown) are observed when comparing the two methods with respect to the respective number of degrees of freedom. Indeed, for each mesh shown, the difference on the number of degrees of freedom does not depend on the polynomial degree $k$ and is about equal to the number of elements, in favour of the conforming VEM.
9.1. Reaction-dominated test case. On $\Omega = (0,1) \times (0,1)$, we solve a reaction-dominated problem with homogeneous boundary condition, forcing term $f_h = 1$, and coefficients $\kappa = \epsilon$ with $\epsilon = 10^{-4}$, $\beta = 0$ and $\gamma = 1$. As in [1], we use the virtual element space containing polynomials of degree up to $k = 2$. To prove the robustness of the VEM with respect to the choice of the projection operator $\Pi_k^*$ and the stabilization term, we consider the following two combinations (corresponding to the method proposed here and the method of [9]): (i) $\Pi_k^*$ given by (4.7) for the space construction and $\Pi_0^k$ for the stabilization term; (ii) the local elliptic projection [1, 9] $\Pi_k^F$ for both the space construction and the stabilization term.

The polynomial projection of the numerical solution provided by the conforming VEM for these two different implementations is shown in the left panels of Fig. 9, while the right panels show the same for the nonconforming VEM. These plots are substantially in agreement with those reported in [1]. In particular, we see that the two different choices for $\Pi_k^*$ and the stabilization term mentioned above produce substantially identical results. Instead, the solutions provided by the conforming and the nonconforming virtual element are different, as it can be reasonably expected. The (projected) nonconforming solution provides a better approximation near the corners of the domain than that obtained by using the conforming VEM. This behaviour is expected as is due to the flexibility of the nonconforming approximation at the four internal nodes of the four corner elements.

10. Conclusion

We have introduced a unified abstract framework for the Virtual Element Method, through which conforming and nonconforming VEMs for solving general second order elliptic convection-reaction-diffusion problems with non-constant coefficients in two and three dimensions are defined, analysed, and implemented in a largely identical manner. We have shown that both methods produce solutions which converge to the true solution at the optimal rate in the $H^1$- and $L^2$-norms, supported by numerical experiments on a variety of different mesh topologies including non-convex polygonal elements.

The framework is based on assuming that the $L^2$-projector onto the polynomial subspace of the virtual element space is computable, in this respect following the approach of [1, 9]. By generalising the process considered in [1], we have introduced families of new possible conforming and nonconforming virtual element spaces in which the $L^2$-projection is indeed exactly computable.
directly from the degrees of freedom used to describe the space. From this family we have de-
tailed a particular space for which the implementation of the $L^2$-projection takes a simple form independent of the method, the polynomial degree, and the space dimension. It also becomes apparent that since the accurate approximation of the problem’s data is only needed to evaluate the polynomial consistency part of the bilinear form, the variational crime theory of classical finite element methods applies in the virtual element setting. Extensions of the present framework to include stabilisation techniques for convection-dominated diffusion problems and the design of virtual element methods for Stokes problems will be considered in future works.
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Appendix A. Proof of Lemma 1

Define $\hat{E}$ as a reference element obtained by scaling each space variable by $h_E$, and let $\hat{\phi}_i$ denote the scaled $\phi_i$. Then,

$$|\phi_i|_{r,E} = h_E^{\frac{d}{2} - r} |\hat{\phi}_i|_{r,E},$$

and it remains to prove that there exists a constant $C$, independent of $\hat{E}$, such that

$$C^{-1} \leq |\hat{\phi}_i|_{r,E} \leq C. \quad (A.1)$$

The constant $C$ should also depend on the basis function index $i$, although this has been omitted as, eventually, this dependence can be absorbed by taking the maximum/minimum over the finite number of indices.

It remains to prove that the constant is also independent of $\hat{E}$.

As the number of vertices of each element is assumed to be bounded, we may restrict the analysis to the family of all admissible reference elements with a given number of vertices, say $n$. Let $\Sigma \subset \mathbb{R}^{dn}$ denote the set of all possible configurations of the $n$ vertices forming admissible reference elements. Clearly, $\Sigma$ is bounded due to the requirement that $h_E = O(1)$. Furthermore, the mesh regularity requirements of Assumption A3 ensure that degenerate situations (collapsing faces, edges, or vertices) are excluded. This fact prevents an element from getting arbitrarily close to inadmissible configurations, and hence $\Sigma$ is also closed and thus compact.

Now consider $\hat{\phi}_i = \hat{\phi}_i(X)$ as a function of the configuration $X \in \Sigma$ of the $n$ vertices of $\hat{E}$. Recall that every $w_h \in W^E_h$, and hence every $v_h \in V^E_h$, is the solution of a Poisson problem, with Dirichlet or Neumann boundary conditions in the conforming or nonconforming case, respectively. It may be shown that for a fixed configuration $X$ the choice of degrees of freedom corresponds uniquely and continuously to a choice of data for the Poisson problem defining the space, in a manner which depends on the configuration $X$. The proof of this fact is given in full details in [6] for the original VEM space and is an immediate consequence of the degrees of freedom’s unisolvence proof given in [1] and [3] for the present VEM spaces. Moreover, this mapping between degrees of freedom and problem data is also continuous with respect to changes in the configuration $X$, due to the continuous dependence of the solution of the Poisson problem on its domain, cf. [24, Chapter II, §6.5].

Recall that the degrees of freedom of $\hat{\phi}_i$ are fixed, and let $X_\lambda$ be a family of configurations depending continuously on the real parameter $\lambda$. Then, from the considerations above, the corresponding function $\lambda \rightarrow \hat{\phi}_i(X_\lambda)$ associating $\lambda$ to the corresponding solution is continuous. It follows that the function $X \rightarrow |\hat{\phi}_i(X)|_{r,E}$ is continuous on the compact set $\Sigma$. As such, this function is bounded, and we can conclude that $|\hat{\phi}_i|_{r,E}$ is bounded above by a constant which is independent of $\hat{E}$, i.e. the upper bound of (A.1).

To prove the lower bound in (A.1), we essentially need to show that $\hat{\phi}_i$ is sufficiently far from constant. To do this, we will prove that the width of the range of values assumed by $\hat{\phi}_i$ on $\hat{E}$ is independent of the configuration $X$. Then, the mesh regularity assumptions ensure that $|\hat{\phi}_i|_{r,E}$ must also be uniformly bounded below and the theorem follows.
If \( \hat{\phi}_i \) is associated with a vertex degree of freedom in the conforming space, it is clear that the range of values obtained by \( \hat{\phi}_i \) must contain at least \([0, 1]\), by definition.

Now suppose that \( \hat{\phi}_i \) is associated with a moment degree of freedom on the facet (element, face, or edge) \( s \). From the definition of the degrees of freedom, we can be sure that there exists at least one point \( x \in \partial E \) such that \( \hat{\phi}_i(x) = 0 \). Moreover, for some \( m_\alpha \in M_{k-2}(s) \),

\[
1 = \frac{1}{|s|} \int_s \hat{\phi}_i m_\alpha \, dx \leq \frac{\sup_{x \in s} |\hat{\phi}_i(x)|}{|s|} \int_s |m_\alpha| \, dx \leq \sup_{x \in s} |\hat{\phi}_i(x)|,
\]

having used the definition of the scaled monomial \( m_\alpha \). Consequently, we may conclude that the width of the range of \( \hat{\phi}_i \) on \( E \) is at least 1, independently of the configuration of \( E \).

References


