Abstract

AUGMENTED HOMOTOPOICAL ALGEBRAIC GEOMETRY

Scott Balchin

In this thesis we are interested in extending the theory of homotopical algebraic geometry, which itself is a homotopification of classical algebraic geometry. We introduce the concept of augmentation categories, which are a class of generalised Reedy categories. An augmentation category is a category which has enough structure that we can mirror the simplicial constructions which make up the theory of homotopical algebraic geometry. In particular, we construct a Quillen model structure on their presheaf categories, and introduce the concept of augmented hypercovers to define a local model structure on augmented presheaves.

As an application, we show that a crossed simplicial group is an example of an augmentation category. The resulting augmented geometric theory can be thought of as being equivariant. Using this, we define equivariant cohomology theories as special mapping spaces in the category of equivariant stacks. We also define the $SO(2)$-equivariant derived stack of local systems by using a twisted nerve construction. Moreover, we prove that the category of planar rooted trees appearing in the theory of dendroidal sets is also an augmentation category. The augmented geometry over this setting should be thought of as being stable in the spectral sense of the word. Finally, we show that we can combine the two main examples presented using a categorical amalgamation construction.
Acknowledgements

First and foremost, I would like to thank my supervisor Frank Neumann. Not only has he been an academic mentor throughout my many years at Leicester, but also a friend. Without his continued support and encouragement, I would not have achieved so much. Alongside him, I would like to thank those other members of staff at the University of Leicester, both academic and administrative, who have always made me feel welcome.

The friends that I have made while at Leicester have been a vital part to my survival and happiness. I hope that I have been able to return this emotional kindness in their turmoils.

Finally, I would like to thank family, who have always stood by me and believed in my pursuit for knowledge. In particular, he who is no longer here, and she whom I have found.
Contents

1 Introduction ........................................... 1
  1.1 Overview ........................................ 1
  1.2 Outline of Thesis ................................. 3
  1.3 Notation and Conventions ......................... 4

2 Background Material .................................. 5
  2.1 Simplicial Sets and Simplicial Categories ....... 5
     2.1.1 Simplicial Objects .......................... 5
     2.1.2 Examples of Simplicial Sets ................. 6
        2.1.2.1 The Standard n-Simplex ............... 6
        2.1.2.2 Function Complex ..................... 7
        2.1.2.3 Nerve of a Category .................... 7
        2.1.2.4 Singular Set of a Topological Space .... 8
     2.1.3 Simplicially Enriched Categories .......... 8
  2.2 Quillen Model Structures ....................... 9
     2.2.1 The Homotopy Category of a Model Category .. 11
     2.2.2 New Models From Old ....................... 12
        2.2.2.1 Bousfield Localisation ................. 13
        2.2.2.2 Quillen Transfer ...................... 14
        2.2.2.3 Projective and Injective Models ....... 14
     2.2.3 Examples ................................... 15
        2.2.3.1 Top .................................. 15
        2.2.3.2 \hat{\Delta}_{Kan} ....................... 15
        2.2.3.3 \hat{\Delta}_{Joyal} ..................... 16
        2.2.3.4 sComm .................................. 17
        2.2.3.5 sPr(\mathfrak{C}) ......................... 18
  2.3 Homotopical Algebraic Geometry ................. 19
     2.3.1 Homotopical Stacks .......................... 19
     2.3.2 Properties of Ho(sPr(\mathfrak{C})) .......... 21
        2.3.2.1 Closed Monoidal Structure ............ 21
        2.3.2.2 Cohomology Theories .................... 22
2.3.3 Derived Algebraic Geometry ........................................... 24
2.3.4 Geometric Homotopical Stacks ..................................... 25
  2.3.4.1 Via Iterated Representability ................................. 26
  2.3.4.2 Via $n$-Hypergroupoids ...................................... 27

3 Augmented Homotopical Algebraic Geometry ......................... 30
  3.1 Augmentation Categories .............................................. 30
    3.1.1 Generalised Reedy Categories ................................ 30
    3.1.2 Augmentation Categories ...................................... 33
  3.2 Homotopy of Augmentation Categories ............................. 34
    3.2.1 Normal Monomorphisms ........................................ 35
    3.2.2 Augmented Kan Complexes .................................... 36
    3.2.3 Augmented Homotopy .......................................... 37
    3.2.4 Augmented Trivial Cofibrations ............................... 38
    3.2.5 The Model Structure .......................................... 41
    3.2.6 Properties of $\text{Ho}(\hat{\Psi}_{\text{Kan}})$ .................... 43
  3.3 Augmented Homotopical Algebraic Geometry ...................... 44
    3.3.1 Local Model Structure on Augmented Presheaves .............. 44
    3.3.2 Local Weak Equivalences ..................................... 45
    3.3.3 Enriched Structure .......................................... 46
    3.3.4 Augmented Derived Stacks .................................... 47
    3.3.5 Augmented Geometric Derived Stacks .......................... 47

4 Equivariant Homotopical Algebraic Geometry ....................... 50
  4.1 Crossed Simplicial Groups ........................................... 51
    4.1.1 Cyclic Sets .................................................. 51
      4.1.1.1 Connes’ Cyclic Category ................................ 51
      4.1.1.2 Cyclic Objects ......................................... 52
      4.1.1.3 Geometric Realisation of Cyclic Objects ............... 53
    4.1.2 Crossed Simplicial Groups .................................... 54
      4.1.2.1 Crossed Simplicial Group Objects ........................ 57
      4.1.2.2 $\Delta\mathcal{G}$ is an Augmentation Category .......... 58
      4.1.2.3 Properties of $\text{Ho}(\hat{\Delta\mathcal{G}}_{\text{Kan}})$ .......... 59
  4.2 Equivariant Cohomology Theories .................................. 62
    4.2.1 Equivariant Eilenberg-Mac Lane Spaces ...................... 62
    4.2.2 Equivariant Site Cohomology .................................. 63
  4.3 $SO(2)$-Equivariant Moduli of Local Systems .................... 65
    4.3.1 Lifting Classical Stacks to Equivariant Derived Stacks .... 65
    4.3.2 $SO(2)$-Equivariant Derived Local Systems ................. 68

5 Stable Homotopical Algebraic Geometry .............................. 72
  5.1 The Category of Rooted Trees ....................................... 73
    5.1.1 Operads ..................................................... 73
    5.1.2 Planar Rooted Trees ......................................... 74
    5.1.3 Faces and Degeneracies ...................................... 76
5.2 Dendroidal Sets ....................................................... 77
  5.2.1 Tensor Structure ............................................. 78
  5.2.2 $\Omega$ is an Augmentation Category ......................... 80
  5.2.3 Properties of $\text{Ho}(\Omega_{\text{Kan}})$ .................... 81
  5.2.4 Open Dendroidal Sets ...................................... 83
5.3 Amalgamations ..................................................... 84
  5.3.1 Categorical Pushouts ....................................... 84
  5.3.2 Amalgable Augmentation Categories ......................... 85

6 Conclusion .......................................................... 89

A Planar Crossed Simplicial Groups .................................. 91
  A.1 Planar Crossed Simplicial Groups ............................... 91
    A.1.1 Equivariant Homotopy Theory ............................... 92
    A.1.2 Strong Homotopy Theory .................................... 94
    A.1.3 Intermediate Homotopy Theories ............................ 99
    A.1.4 Coupled Homotopy Theory .................................. 100
      A.1.4.1 Cyclic Sets ............................................ 101
      A.1.4.2 Of Cyclic Type ....................................... 106
      A.1.4.3 Of Dihedral & Quaternionic Type ....................... 107
  A.2 Planar Lie Group Equivariant Presheaves ........................ 109
    A.2.1 Presheaves for the Strong Model ........................... 109
    A.2.2 Presheaves for the Coupled Model ........................ 110
    A.2.3 Strong and Coupled Equivariant Stacks .................... 112

Bibliography .......................................................... 113
Chapter 1

Introduction

In this thesis, we are concerned with the theory of Augmented Homotopical Algebraic Geometry. To do so, we prove a series of results regarding a general framework which extends the theory of ∞-stacks as developed in [106, 107].

1.1 Overview

Classically, algebraic geometry is the study of the zeros of multivariate polynomials, known as varieties. Although this idea is still rooted in modern algebraic geometry, the tools used are now somewhat categorical and abstract.

The exodus towards categorical tools can, of course, be traced back to the school of Grothendieck in 20th century France, with the seminal work of EGA and SGA [48, 50]. Of particular importance is the notion of a scheme, which subsumes the theory of affine varieties [39, 51]. For us, a scheme will be defined via its functor of points, that is, a sheaf

\[ X : \text{Aff}^{\text{op}} \to \text{Set} \tag{1.1.1} \]

where the topology on \( \text{Aff}^{\text{op}} \) is usually taken to be the étale topology [31]. This unassuming definition of a scheme had led to a flourishing field of research fuelled by the emerging theory of categories (both homotopical and non-homotopical). We shall outline the natural modifications of Notation 1.1.1 that have occurred throughout the years, which will then explain the motivation of this thesis.

The first thing to note is how restrictive the category of sets is. The study of moduli problems was marred by the issues arising from isomorphisms associated with the objects in question. To remedy this problem, the category of sets was enlarged to the (2-)category
of groupoids [15, 66]. Therefore the definition of a stack can be worded as a (2-)sheaf

\[ X : \text{Aff}^{op} \to \text{Grpd} \] (1.1.2)

In some settings, the natural notion of equivalence for a geometric object is weaker than that of isomorphism. To encode such information, a homotopical viewpoint is required. The correct way to consider such objects is as an \((\infty, 1)\)-sheaf

\[ X : \text{Aff}^{op} \to \hat{\Delta} \] (1.1.3)

where \(\hat{\Delta}\) is the category of simplicial sets. Such objects are referred to as higher stacks [55, 97].

The final stage of the progression of Notation 1.1.1, unsurprisingly, is an enlargement of the source category of affine schemes. Just as the category of simplicial sets is a homotopic version of sets, we can consider a homotopic version of affine schemes. The resulting category is the \((\infty, 1)\)-category of derived affine schemes. A derived stack is then an \((\infty, 1)\)-sheaf

\[ X : \text{dAff}^{op} \to \hat{\Delta} \] (1.1.4)

Now we can assemble Notations 1.1.1-1.1.4 into the following diagram:

\[
\begin{array}{ccc}
\text{Aff}^{op} & \xrightarrow{A} & \text{Set} \\
\downarrow B & & \downarrow C \\
\text{Grpd} & & \downarrow \Delta \\
\text{dAff}^{op} & \xrightarrow{D} & \hat{\Delta}
\end{array}
\]

\(A\) - Schemes \\
\(B\) - Stacks \\
\(C\) - Higher Stacks \\
\(D\) - Derived Stacks

This is the diagram which appears in [110], which also includes examples of where such objects are required.

Of course, the above discussion is the algebraic setting, in that we work over the algebraic category of (derived) affine schemes. We shall see that we can modify our source category to any (simplicial) site. Classically, of particular significance are the categories \textbf{Top} (resp., \textbf{Mfd}) which give rise to topological (resp., differentiable) stacks [12, 13, 83].

The scope of this thesis is to develop the tools necessary to take the above diagram one step further by enlarging the category of simplices \(\Delta\) to an augmentation category \(\Psi\). By
considering the category of presheaves over an augmentation category, we get an extension of the category of simplicial sets. For some specific examples of augmentation categories, we consider what extra data is encoded within the construction. Just as each modification of Notation 1.1.1 was undertaken to accommodate arising issues, we hope that different augmentations can be considered which are suitable to the problem at hand.

As an example, consider the situation where there is a need to encode an $SO(2)$-action on a derived stack. Such an action cannot be captured by the category of simplicial sets. However, this problem is perfectly suited to the category of cyclic sets $\hat{\Delta}\mathcal{C}$, which is formed as a presheaf category on Connes’ cyclic category $\Delta\mathcal{C}$ [27]. Using the general framework of augmentation categories, we will show that it is possible to adjust Notation 1.1.1 to define $SO(2)$-equivariant derived stacks as certain functors

$$X : d\text{Aff}^{op} \to \hat{\Delta}\mathcal{C}.$$ (1.1.5)

### 1.2 Outline of Thesis

- Chapter 2 will introduce the general concepts from the literature on the theory of simplicial objects and homotopical algebraic geometry.

- Chapter 3 will be concerned with the theory of augmentation categories. We prove the existence of a model structure on the category of presheaves over an augmentation category which extends the homotopy theory of simplicial sets. Using this model we define a local model structure on the category of augmented presheaves which makes up the category of augmented stacks that we will be interested in.

- Chapter 4 introduces the first class of examples of augmentation categories. We prove that the augmentation given by crossed simplicial groups should be considered as an equivariant augmentation. We prove how this augmentation can be used to consider equivariant cohomology theories. Using the construction of twisted nerves, we define an $SO(2)$-equivariant version of the derived stack of local systems, the results of which appear in [6].

- Chapter 5 is the setting for the second example of an augmentation category. We will use the dendroidal category to define augmented homotopical algebraic geometry which should be considered as stable (due to a Quillen equivalence to the category of connective spectra). We combine the constructions done in this chapter with the crossed simplicial groups of Chapter 4 via the theory of categorical amalgamations. In particular, we define what it means for an augmentation category to be amalgable,
and prove that the pushout of an amalgable augmentation category with a crossed simplicial group provides a new augmentation category.

• Chapter 6 is the conclusion of this thesis, summing up all of the results. We also provide some potential questions for further research in the area of augmentation categories.

• Appendix A returns to the setting of crossed simplicial groups. We prove for a specific class of crossed simplicial groups that we can define a stronger model structure, which captures more equivariant structure. Armed with the stronger models, we define categories of strongly equivariant stacks. The results of this appendix appear in [7].

1.3 Notation and Conventions

• For \( \mathcal{C} \) a category, and \( X, Y \) objects of \( \mathcal{C} \), let \( \text{Hom}_\mathcal{C}(X,Y) \) denote the set-valued Hom from \( X \) to \( Y \). We will sometimes write \( \text{Hom}(X,Y) \) if the ambient category \( \mathcal{C} \) is not ambiguous.

• For \( \mathcal{C}, \mathcal{D} \) categories, we let \( \mathcal{D}^\mathcal{C} \) denote the category of functors \( \mathcal{C} \to \mathcal{D} \).

• For \( \mathcal{C} \) a category, we let \( \hat{\mathcal{C}} \) denote the presheaf category \( \text{Set}^{\mathcal{C}^{\text{op}}} \).

• If \( \mathcal{C} \) is a closed category, we will denote by \( \text{hom}(\cdot, \cdot) \) the internal-hom.

• For \( \mathcal{C} \) a suitable category, we will denote by \( \text{hom}^\Delta(\cdot, \cdot) \) the simplicial-hom.

• Let \( \mathcal{C} \) be a category, and \( f: X \to Z, \; g: Y \to Z \) maps in \( \mathcal{C} \). We let \( X \times_Z Y \) denote the pullback

\[
\begin{array}{ccc}
X \times_Z Y & \longrightarrow & Y \\
\downarrow & & \downarrow^g \\
X & \longrightarrow & Z \\
\downarrow^f & & \downarrow \\
\end{array}
\]

• Let \( \mathcal{C} \) be a category, and \( f: X \to Y, \; g: X \to Z \) maps in \( \mathcal{C} \). We let \( Y \sqcup_X Z \) denote the pushout

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow^f & & \downarrow^g \\
Z & \longrightarrow & Y \sqcup_X Z \\
\end{array}
\]

• For \( L: \mathcal{C} \to \mathcal{D} \) a functor with right adjoint \( R \), we will write the adjunct pair with \( L \) on the left:

\[
L : \mathcal{C} \rightleftarrows \mathcal{D} : R
\]
Chapter 2

Background Material

We begin by recalling all of the necessary background material that we will need before we can embark on the construction of augmented homotopical algebraic geometry. The material here is brief, and by no means encapsulates all of the theory required to fully master homotopical algebraic geometry. A thorough, but readable overview of the theory can be found in [104], while the technicalities can be found in [106, 107].

Outline of Chapter 2

(Section 2.1) We recall the theory of simplicial objects, giving important examples of simplicial sets that we will encounter.

(Section 2.2) The theory of Quillen model categories is introduced. After this we look at tools that can be used to manipulate certain model structures.

(Section 2.3) We discuss homotopical algebraic geometry by combining the material of the previous two sections.

2.1 Simplicial Sets and Simplicial Categories

2.1.1 Simplicial Objects

The simplex category and simplicial objects will form a key part of the theory of this thesis, therefore we recall the required details from [71, 88]. We give the combinatorial definition of the simplex category as it is the one most relevant to our work, however it is one of several equivalent definitions.

**Definition 2.1.** The *simplex category* $\Delta$ has for objects finite totally ordered sets $[n] = \{0, 1, \ldots, n\}$ and the morphisms are generated by:
• coface maps $\delta_i: [n-1] \to [n]$ for $n > 0$ and $0 \leq i < n$, which is the injection whose image leaves out $i \in [n]$.

• codegeneracy maps $\sigma_i: [n+1] \to [n]$ for $n > 0$ and $0 \leq i < n$, which is the surjection such that $\sigma_i(i) = \sigma_i(i+1) = i$.

subject to the following simplicial relations:

$$
\delta_j \delta_i = \delta_i \delta_j - 1 \quad 0 \leq i < j \leq n \\
\sigma_j \sigma_i = \sigma_i \sigma_{j+1} \quad 0 \leq i < j < n \\
\sigma_j \delta_i = \begin{cases} 
\delta_j \sigma_{j-1} & 0 \leq i < j < n \\
\text{Id} & 0 \leq j < n \text{ and } i = j \text{ or } i = j + 1 \\
\delta_i \sigma_{j-1} & 0 \leq j \text{ and } j + 1 < i \leq n 
\end{cases}
$$

**Definition 2.2.** Let $\mathcal{C}$ be any category. A simplicial object in $\mathcal{C}$ is a functor $X_\bullet: \Delta^{op} \to \mathcal{C}$. We will denote by $s\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$ with morphisms being the natural transformations between them. In the particular case that $\mathcal{C}$ is the category of sets, we will denote the resulting category of simplicial sets by $\hat{\Delta}$.

Because of our combinatorial definition of $\Delta$, we can also give a combinatorial definition of a simplicial set (or more generally, a simplicial object in $\mathcal{C}$).

**Definition 2.3.** A simplicial set $X_\bullet$ is a collection of sets $X_n$ for all $n \geq 0$, and for each $n$ we have face maps $d_i: X_n \to X_{n-1}$ and degeneracy maps $s_i: X_n \to X_{n+1}$ satisfying the following relations:

$$
d_id_j = d_{j-1}d_i \quad 0 \leq i < j \leq n \\
s_is_j = s_{j+1}s_i \quad 0 \leq i < j < n \\
d_is_j = \begin{cases} 
\sigma_{j-1}d_i & 0 \leq i < j < n \\
\text{Id} & 0 \leq j < n \text{ and } i = j \text{ or } i = j + 1 \\
s_jd_{i-1} & 0 \leq j \text{ and } j + 1 < i \leq n 
\end{cases}
$$

### 2.1.2 Examples of Simplicial Sets

#### 2.1.2.1 The Standard $n$-Simplex

The definition of the standard $n$-simplex is a consequence of the Yoneda embedding. Recall from [72] that for any small category $\mathcal{C}$ and any object $c \in \mathcal{C}$ there is a represented functor

$$
\text{Hom}_\mathcal{C}(-, c): \mathcal{C}^{op} \to \text{Set}
$$
that takes \( b \in \mathcal{C} \) to the set \( \text{Hom}_\mathcal{C}(b,c) \). A morphism \( g: a \to b \) in \( \mathcal{C} \) induces a new function \( g^*: \mathcal{C}(b,c) \to \mathcal{C}(a,c) \) defined via pre-composition. Moreover, a morphism \( f: c \to d \) in \( \mathcal{C} \) defines a natural transformation \( f_*: \text{Hom}_\mathcal{C}(-,c) \to \text{Hom}_\mathcal{C}(-,d) \) which can be constructed point-wise using post-composition with \( f \). The **Yoneda lemma** then states that this defines a functor (the Yoneda embedding) \( \mathcal{C} \hookrightarrow \hat{\mathcal{C}} \) which is fully faithful.

We now return to our setting. For each \( [n] \in \Delta \) we have its image under the Yoneda embedding

\[
\Delta[n] = \text{Hom}_\Delta(-,[n]) = \text{Hom}_{\Delta^\text{op}}([n],-) \in \hat{\Delta}.
\]

We will call \( \Delta[n] \) the **standard \( n \)-simplex**. The Yoneda lemma tells us that for any simplicial set \( X_* \) there is a natural bijection between the map \( \Delta[n] \to X_* \) and the \( n \)-simplices of \( X_* \).

### 2.1.2.2 Function Complex

One of the most important features of \( \hat{\Delta} \) is the existence of function *complexes* or internal hom-objects.

**Definition 2.4** ([46, Section I.5]). Let \( X_* \) and \( Y_* \) be simplicial sets. We define the **function complex** \( \text{hom}(X,Y) \) to be the simplicial set defined by

\[
\text{hom}(X,Y)_n = \text{Hom}_{\hat{\Delta}}(X \times \Delta[n],Y).
\]

We call a category with such an internal-hom *closed*, and we will, in general, denote it by \( \text{hom}(-,-) \).

### 2.1.2.3 Nerve of a Category

Let \( \mathcal{C} \) be a small category. We define a simplicial set called the **nerve** of \( \mathcal{C} \) to be the simplicial set \( N\mathcal{C} \) defined as follows:

- \( N\mathcal{C}_0 = \text{Ob}(\mathcal{C}) \).
- \( N\mathcal{C}_1 = \text{Mor}(\mathcal{C}) \).
- \( N\mathcal{C}_2 = \{ \text{pairs of composable arrows } \to \to \text{ in } \mathcal{C} \} \).
  ...
- \( N\mathcal{C}_n = \{ \text{strings of } n \text{ composable arrows } \to \to \cdots \to \text{ in } \mathcal{C} \} \).

The degeneracy maps \( s_i: N\mathcal{C}_n \to N\mathcal{C}_{n+1} \) take a string of \( n \) composable arrows

\[
c_0 \to c_1 \to \cdots \to c_i \to \cdots \to c_n
\]
and inserts the identity of \(c_i\) in the \(i^{th}\) position. The face maps \(d_i : N \rightarrow N_{n-1}\) compose the \(i^{th}\) and \((i + 1)^{th}\) arrows if \(0 < i < n\), and leave out the first or last arrow for \(i = 0\) or \(n\) respectively.

### 2.1.2.4 Singular Set of a Topological Space

Let \(\textbf{Top}\) denote the category of topological spaces and continuous functions. The final example that we will consider will give us a functor \(S : \textbf{Top} \rightarrow \hat{\Delta}\) called the singular functor. Denote by \(\Delta_n \in \textbf{Top}\) the standard topological \(n\)-simplex:

\[
\Delta_n = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : 0 \leq x_i \leq 1, \sum x_i = 1 \right\}.
\]

**Definition 2.5.** Let \(X\) be a topological space. Then we define its singular simplicial set to be

\[S(X)_n = \text{Hom}_{\textbf{Top}}(\Delta_n, X).\]

Dually to the above functor, we can define \(|-| : \hat{\Delta} \rightarrow \textbf{Top}\) called the geometric realisation. We define the geometric realisation by first defining it on the representable objects \(\Delta[n]\) and then taking a coend construction. We define \(|\Delta[n]|\) to be \(\Delta_n\).

**Definition 2.6.** Let \(X_\bullet\) be a simplicial set, then we define its geometric realisation as

\[|X_\bullet| := \int^n X_n \times \Delta_n.\]

The following proposition gives us the relation between the two functors that we have defined.

**Proposition 2.7 ([76, Theorem 16.1]).** The pair of functors

\[| - | : \hat{\Delta} \leftarrow \textbf{Top} : S(-)\]

forms an adjoint pair, where the singular functor is right adjoint to the geometric realisation.

### 2.1.3 Simplicially Enriched Categories

Now that we have introduced simplicial sets, we can discuss the theory of simplicial categories, which are an example of enriched categories. Therefore we first recall the theory of enriched categories (see for example [63] or [89, Chapter 3]). We will not give all of the technicalities of the definition of an enriched category as it far from the scope of this thesis. However, we do take this convenient opportunity to introduce the concept of monoidal categories.
Definition 2.8.

- A **monoidal** structure on a category $\mathcal{C}$ is a tensor product bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $I \in \mathcal{C}$, a natural associativity isomorphism $a: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, a natural left unit isomorphism $\ell: I \otimes X \to X$, and a natural right unit isomorphism $t: X \otimes I \to X$ such that the triangle and pentagon diagrams commute (see [71]).

- A **monoidal category** is a category together with a monoidal structure.

- A monoidal structure is said to be **symmetric** if there is an isomorphism $X \otimes Y \simeq Y \otimes X$.

If we equip a closed category with a monoidal structure in a compatible way then we get a **closed monoidal category**. In such a setting, we have the notion of **currying**, that is, for objects $X, Y, Z$ in a closed monoidal category $\mathcal{C}$, we have

$$\text{Hom}_{\mathcal{C}}(X, \text{hom}(Y, Z)) \simeq \text{Hom}_{\mathcal{C}}(X \otimes Y, Z).$$

That is, the internal-hom is the right adjoint to the tensor product bifunctor $\otimes$.

**Definition 2.9.** Let $\mathcal{V}$ be a monoidal category. We say that $\mathcal{C}$ is a $\mathcal{V}$-enriched category if it has a collection of objects $\text{Ob}(\mathcal{C})$ and for each pair of objects $X, Y$ a hom-$\mathcal{V}$-object $\text{Hom}(X, Y) \in \mathcal{V}$ with the composition defined using the monoidal structure of $\mathcal{V}$. These objects and homs are subject to the usual categorical axioms.

We will be interested in categories enriched over $\hat{\Delta}$ which will be our choice of model for $\infty$-categories (the technicalities of which are outlined in [70]). The monoidal structure on $\hat{\Delta}$ is the cartesian monoidal structure. That is, for simplicial sets $X_\bullet$ and $Y_\bullet$ their product $(X \times Y)_\bullet$ is defined by sending $[n]$ to $X([n]) \times Y([n])$ with the monoidal structure on $\text{Set}$. Without loss of generality, we assume every category to be simplicially enriched, by letting the hom-object be the obvious discrete simplicial set.

We will refer to a simplicially enriched category as a **simplicial category**. Note that this notion is not the same as an category internal to simplicial sets. However, they are related by the fact that a category internal to simplicial sets with a discrete object set is exactly a simplicially enriched category.

### 2.2 Quillen Model Structures

Here we will cover the theory of Quillen closed model categories, and the necessary tools used to manipulate them. Model categories are important as they are a natural setting for
homotopy theory, and extend the usual homotopy theory of topological spaces. The first section will cover the basic definitions, while Section 2.2.2 will introduce properties and tools that allow us to generate new model structures from old ones. Section 2.2.3 gives a collection of important examples and will make use of the tools presented in Section 2.2.2.

We begin by recalling the axioms for a model structure on a category $\mathcal{M}$ as in [86].

**Definition 2.10.** Given a commutative square of the following form:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow^{i} & & \downarrow^{p} \\
B & \xrightarrow{g} & Y \\
\end{array}
$$

(2.2.1)

a *lift* in the diagram is a map $h: B \to X$ such that $hi = f$ and $ph = g$.

**Definition 2.11.** A *closed Quillen model category* is a category $\mathcal{M}$ with three distinguished classes of maps:

1. Weak equivalences – $W_{\mathcal{M}}$,
2. Fibrations – $\text{Fib}_{\mathcal{M}}$,
3. Cofibrations – $\text{Cof}_{\mathcal{M}}$,

each of which is closed under composition. We will say that a map which is both a fibration (resp., cofibration) and a weak equivalence is a *trivial fibration* (resp., *trivial cofibration*). The classes of maps and the category $\mathcal{M}$ must satisfy the following axioms:

**CM1** $\mathcal{M}$ has all finite limits and colimits. In particular there is a terminal object $\ast$ and initial object $0$.

**CM2** if $f$ and $g$ are maps such that $gf$ is defined and if two of $f$, $g$ and $gf$ are weak equivalences, then so is the third.

**CM3** If $f$ is a retract of $g$ and $g$ is a fibration, cofibration, or a weak equivalence, then so is $f$.

**CM4** Given a commutative diagram of the form 2.2.1, a lift exists when either $i$ is a cofibration and $p$ is a trivial fibration or when $i$ is a trivial cofibration and $p$ is a fibration.

**CM5** Each map $f$ can be factored in two ways:

1. $f = pi$, where $i$ is a cofibration and $p$ is a trivial fibration.
2. \( f = p_i \), where \( p \) is a fibration and \( i \) is a trivial cofibration.

**Definition 2.12.** Let \( \mathcal{M} \) be a closed model category. An object \( X \in \mathcal{M} \) is **fibrant** (resp., **cofibrant**) if the unique map \( X \to \ast \) (resp., the unique map \( 0 \to X \)) is a fibration (resp., cofibration).

**Definition 2.13.** A map \( i: A \to B \) is said to have the **left lifting property** (LLP) with respect to another map \( p: X \to Y \) and \( p \) is said to have the **right lifting property** (RLP) with respect to \( i \) if a lifting exists for any choice of \( f \) and \( g \) making Diagram 2.2.1 commute.

**Proposition 2.14** ([87, Corollary 1.2]). The class of cofibrations are the maps which have the LLP with respect to all trivial fibrations. Dually, the fibrations are the maps which have the RLP with respect to all trivial cofibrations.

Note that by Proposition 2.14 we see that after the weak equivalences have been defined, we can use the fibrations (resp., cofibrations) to define the cofibrations (resp., fibrations). It is also possible to fully describe a model structure by just defining the cofibrations (resp., fibrations) and fibrant (resp., cofibrant) objects as proved in [60, Proposition 1.38]. By similar arguments, one also sees that that fibrations and cofibrations uniquely defines a model structure (but not just the fibrant and cofibrant objects).

### 2.2.1 The Homotopy Category of a Model Category

The goal of Quillen model structures is to study the homotopy theory of the objects of the category. The following definition gives the method which we use to achieve this, along with a description of the way that we compare homotopy theories of different categories.

**Definition 2.15.** Let \( \mathcal{M} \) be a closed model category. Then there is a category \( \text{Ho}(\mathcal{M}) \) called the **homotopy category of the model category** \( \mathcal{M} \) whose:

- Objects are those objects of \( \mathcal{M} \) which are both fibrant and cofibrant.
- Morphisms are the homotopy classes of morphisms of \( \mathcal{M} \).

**Definition 2.16.** Let \( \mathcal{M} \) and \( \mathcal{D} \) be model categories. Let \( F : \mathcal{M} \rightleftarrows \mathcal{D} : G \) be a pair of adjoint functors, with \( F \) left adjoint to \( G \). We say that the pair is a **Quillen adjunction** if \( F \) preserves cofibrations and \( G \) preserves fibrations. Such an adjunction induces a pair of functors

\[
\mathbb{L}F : \text{Ho}(\mathcal{M}) \rightleftarrows \text{Ho}(\mathcal{D}) : \mathbb{R}G
\]

between the homotopy categories. If \( \mathbb{L}F \) and \( \mathbb{R}G \) additionally form an equivalence of categories then we say that \( \mathcal{M} \) and \( \mathcal{D} \) are **Quillen equivalent**. Note that the condition
that $F$ preserves cofibrations and $G$ preserves fibrations is equivalent to the following statements:

- $F$ preserves cofibrations and trivial cofibrations.
- $G$ preserves fibrations and trivial fibrations.
- $F$ preserves trivial cofibrations and $G$ preserves trivial fibrations.

For a proof of this statement, see [53, Proposition 8.5.3].

Before the advent of Quillen model structures, there was still the notion of taking the homotopy category of a category with a given set of weak equivalences. Given a category $\mathcal{C}$ with a set of weak equivalences $W$, we construct its homotopy category $\text{Ho}(\mathcal{C}) = \mathcal{C}[W^{-1}]$ by adding formal inverses to all of the weak equivalences. The problem with this construction lies in set theoretic difficulties. In fact, the new category may not even be locally small. However, using Quillen model structures we avoid such technicalities, with the (co)fibrations providing the explicit description of the objects, and indeed, the two obtained categories are equivalent. Therefore we see that the homotopy category is determined completely by the weak equivalences. We will use the two definitions interchangeably depending on what we are trying to show about our category.

We now give a condition on when two model categories are Quillen homotopy equivalent.

**Theorem 2.17** ([87, Theorem 1.4]). Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a pair of Quillen functors, with $F$ the left adjoint. Suppose that for each $A$ cofibrant in $\mathcal{C}$, and $X$ fibrant in $\mathcal{D}$, a map $f : A \to G(X)$ is a weak equivalence in $\mathcal{C}$ if and only if the corresponding map $f^\flat : F(A) \to X$ is a weak equivalence in $\mathcal{D}$. Then $LF$ and $RG$ form an equivalence of categories.

**Remark 2.18.** It is important to note that a model category consists of the category $\mathcal{M}$ along with the choice of weak equivalences, fibrations and cofibrations. It is possible, as we will see in Section 2.2.3, that a category $\mathcal{M}$ can have multiple model structures which are not equivalent in the sense of Definition 2.16.

### 2.2.2 New Models From Old

In this section we are interested in how we can use an existing model structure on a category $\mathcal{M}$ to generate a new model structure on either $\mathcal{M}$, or another category $\mathcal{D}$. The methods discussed in the following sections require the model structure to have an additional property (which will always be satisfied for the categories that we will consider):
Definition 2.19. A model category $\mathcal{M}$ is cofibrantly generated if there is a set of cofibrations and a set of trivial cofibrations which generate all other (trivial) cofibrations via retracts of transfinite composition of pushouts and coproducts.

2.2.2.1 Bousfield Localisation

(Left) Bousfield localisation is the process of taking the model structure on $\mathcal{M}$ and adding more weak equivalences while keeping the cofibrations constant. An in depth study of the theory here can be found in [53]. We will forgo the explicit definition and give a descriptive definition which will be enough for our purpose.

Definition 2.20. Let $\mathcal{M}$ be a model category. A left Bousfield localisation of $\mathcal{M}$ is another model category $\mathcal{M}_{\text{loc}}$ with the same underlying category and where:

- $W_{\mathcal{M}_{\text{loc}}} \supset W_{\mathcal{M}}$.
- $(\text{Cofibrations of } \mathcal{M}_{\text{loc}}) = (\text{Cofibrations of } \mathcal{M})$.
- Fibrations are defined by the RLP.

Bousfield localisation does not always exist, to this extent we will now give a sufficient condition for localisations of a model category $\mathcal{M}$ to exist.

Definition 2.21. A model category $\mathcal{M}$ is said to be:

- Locally presentable if it is accessible and has all small colimits.
- Combinatorial if it is cofibrantly generated and locally presentable.
- Left proper if the pushout of a weak equivalence along a cofibration is a weak equivalence.

Proposition 2.22 ([8, Theorem 4.7]). A sufficient condition for a model category $\mathcal{M}$ to have a localisation at a set of morphisms is for it to be left proper and combinatorial.

Lemma 2.23. Let $\mathcal{M}$ be a model category, and $\mathcal{M}_{\text{loc}}$ a Bousfield localisation of $\mathcal{M}$. Then:

1. The fibrations of $\mathcal{M}_{\text{loc}}$ are a subset of the fibrations of $\mathcal{M}$.
2. The trivial fibrations of $\mathcal{M}_{\text{loc}}$ are equal to the trivial fibrations of $\mathcal{M}$.

Proof.

1. $\text{Fib}_{\mathcal{M}_{\text{loc}}} = RLP(\text{Cof}_{\mathcal{M}_{\text{loc}}} \cap W_{\mathcal{M}_{\text{loc}}}) \subset RLP(\text{Cof}_{\mathcal{M}_{\text{loc}}} \cap W_{\mathcal{M}}) = \text{Fib}_{\mathcal{M}}$.
2. $\text{Fib}_{\mathcal{M}_{\text{loc}}} \cap W_{\mathcal{M}_{\text{loc}}} = RLP(\text{Cof}_{\mathcal{M}_{\text{loc}}}) = RLP(\text{Cof}_{\mathcal{M}}) = \text{Fib}_{\mathcal{M}} \cap W_{\mathcal{M}}$. 
Finally we look at how the localisation affects the homotopy categories. There are functors on the underlying category such that:

- The identity functor $Id : \mathcal{M} \to \mathcal{M}_{loc}$ preserves cofibrations and weak equivalences.
- The identity functor $Id : \mathcal{M}_{loc} \to \mathcal{M}$ preserves fibrations and trivial fibrations.

giving a Quillen adjunction $\mathcal{M}_{loc} \rightleftarrows \mathcal{M}$ which leads to $\text{Ho}(\mathcal{M}_{loc})$ being a full subcategory of $\text{Ho}(\mathcal{M})$.

### 2.2.2.2 Quillen Transfer

Suppose that we have a model category $\mathcal{C}$ along with another (non-model) category $\mathcal{D}$ which has the same objects as $\mathcal{C}$, but with some extra structure. Quillen transfer is a tool that allows us to consider a model structure on $\mathcal{D}$ induced by the model on $\mathcal{C}$. Proposition 2.24 gives a sufficient condition for us to be able to transfer a structure on $\mathcal{C}$ onto a model structure on $\mathcal{D}$.

**Proposition 2.24** ([29, Theorem 3.3]). Let $\mathcal{C}$ be a cofibrantly generated model category and $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ an adjunction where $L$ is a left adjoint. Say that a morphism in $\mathcal{D}$ is a weak equivalence or fibration whenever its image under $U$ in $\mathcal{C}$ is. Then sufficient conditions for this to define a cofibrantly generated model structure on $\mathcal{D}$ are:

1. $F$ preserves small objects, this holds in particular when $U$ preserves filtered colimits.
2. Any sequential colimit of pushouts of images under $F$ of the generating trivial cofibrations in $\mathcal{C}$ yields a weak equivalence in $\mathcal{D}$. In particular this holds when:
   - $\mathcal{D}$ has a fibrant replacement functor.
   - $\mathcal{D}$ has path objects for fibrant objects (this always holds when $\mathcal{D}$ is a simplicial category).

### 2.2.2.3 Projective and Injective Models

Given a model category $\mathcal{C}$, and $\mathcal{D}$ a small category, it is sensible to ask what model can we put on the functor category $\mathcal{C}^\mathcal{D}$. Subject to some mild assumptions, there are two such models which can be constructed with no additional work.

**Definition 2.25** ([53, §11.6]). Let $\mathcal{C}$ be a combinatorial model category, or simply just cofibrantly generated for the projective case, and $\mathcal{D}$ a small category. We define the following model structures on the functor category $\mathcal{C}^\mathcal{D}$:
the projective model structure $\mathcal{C}^{\text{proj}}$ where the weak equivalences and fibrations are the natural transformations that are object-wise such morphisms in $\mathcal{C}$. The cofibrations are the maps with the LLP with respect to the trivial fibrations.

- the injective model structure $\mathcal{C}^{\text{inj}}$ where the weak equivalences and cofibrations are the natural transformations that are object-wise such morphisms in $\mathcal{C}$. The fibrations are the maps with the RLP with respect to the trivial cofibrations.

Although the projective and injective model structures differ, they have the same homotopy category as their weak equivalences are equal.

The projective and injective model structure enjoy many good properties. A property that we will use throughout is that the projective/injective model structures are left (resp., right) proper when $\mathcal{C}$ is left (resp., right) proper [70, Proposition A.3.3.2]. Moreover, if $\mathcal{C}$ is a simplicial model category, then the projective/injective model structures are simplicial model categories [70, Remark A.2.8.4].

### 2.2.3 Examples

We will now look at examples of Quillen model structures, some of which will use the tools presented in Sections 2.2.2.1, 2.2.2.2 and 2.2.2.3.

#### 2.2.3.1 Top

Recall that Top is the category of topological spaces and continuous maps.

**Proposition 2.26 ([54, Theorem 1.5]).** There is a cofibrantly generated model structure on Top where a map is a:

- Weak equivalences if it is a weak homotopy equivalence.
- Fibration if it is a Serre fibration.
- Cofibration if it is a retract of relative cell complexes.

The cofibrations are generated by the set of boundary inclusions $S^{n-1} \rightarrow D^n \forall n \in \mathbb{N}$.

#### 2.2.3.2 $\hat{\Delta}_{\text{Kan}}$

The Kan-Quillen model structure (sometimes referred to as the classical model) on the category of simplicial sets is arguably the most important model category that we will consider in this thesis. We will begin by introducing the necessary constructions appearing in the definition of the model structure.
**Definition 2.27.** Let $\Delta[n] = \Delta(-, [n])$ be the standard $n$-simplex in $\hat{\Delta}$. For each $0 \leq i \leq n$, the $(n, i)$-horn is the sub-simplicial set $\Lambda^i[n] \hookrightarrow \Delta[n]$ which is the union of all faces except the $i^{th}$ one. We say the horn is *outer* if $k = 0, n$, and *inner* otherwise.

**Definition 2.28.** A map of simplicial sets $f : X \to Y$ is a *Kan fibration* if it has the right lifting property for all horn inclusions. That is, we have the following commutative diagram:

$$
\begin{array}{ccc}
\Lambda^k[n] & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\Delta[n] & \longrightarrow & Y
\end{array}
$$

**Definition 2.29.** A map $f : X \to Y$ in $\hat{\Delta}$ is a *monomorphism* if it is level-wise injective. That is, for all $n \in \mathbb{N}$ there is an injection of sets $f_n : X_n \to Y_n$.

**Proposition 2.30** ([86, Chapter II, §3]). There is a cofibrantly generated model structure on $\hat{\Delta}$, called the Kan-Quillen model structure, denoted $\hat{\Delta}_{Kan}$, which has the following classes of morphisms:

- Weak equivalences are the morphisms whose geometric realisation is a weak homotopy equivalence in $\text{Top}$.
- Fibrations are the Kan fibrations.
- Cofibrations are the monomorphisms.

The cofibrations are generated by the boundary inclusions $\partial \Delta[n] \to \Delta[n]$ and the trivial cofibrations are generated by the horn inclusions $\Lambda^i[n] \to \Delta[n]$. Moreover this model is combinatorial and left proper.

**Theorem 2.31** ([86, Chapter II, §3]). There is a Quillen equivalence

$$
| - | : \hat{\Delta}_{Kan} \rightleftarrows \text{Top} : S(-)
$$

given by the geometric realisation and singular functors.

### 2.2.3.3 $\hat{\Delta}_{Joyal}$

We will now consider a different model structure on $\hat{\Delta}$ which will lead us to an observation involving the theory of Bousfield localisations.

**Definition 2.32.** A simplicial set $X_\bullet$ is a *quasi-category* if all the horns $\Lambda^i[n]$ with $0 < i < n$ (i.e., inner horns) satisfy the lifting condition.
Definition 2.33. A morphism $f: A \to B$ is a weak categorical equivalence if, for all quasi-categories $X$, the induced map of simplicial sets $\text{hom}(B, X) \to \text{hom}(A, X)$ induces an isomorphism when applying the functor $\tau_0$ which takes a simplicial set to the set of isomorphism classes of objects of its fundamental category.

Definition 2.34. A map of simplicial sets $f: X \to Y$ is an inner Kan fibration if it has the right lifting property for all inner horn inclusions. That is, we have the following commutative diagram:

$$
\begin{array}{ccc}
\Lambda^k[n] & \xrightarrow{\sim} & X \\
\downarrow & \searrow f & \\
\Delta[n] & \xrightarrow{\sim} & Y
\end{array}
$$

for $k \neq 0, n$.

Proposition 2.35 ([70, Theorem 2.2.5.1]). There is a cofibrantly generated model structure on $\hat{\Delta}$, called the Joyal model structure, denoted $\hat{\Delta}_{\text{Joyal}}$, which has the following classes of morphisms:

- Weak equivalences are the weak categorical equivalences.
- Fibrations are the inner Kan fibrations.
- Cofibrations are the monomorphisms.

The cofibrations are generated by the boundary inclusions $\partial\Delta[n] \to \Delta[n]$ (there is no known description of the trivial cofibrations).

We now remark that every weak categorical equivalence is a weak homotopy equivalence, and note that $\text{Cof}_{\Delta_{\text{Kan}}} = \text{Cof}_{\hat{\Delta}_{\text{Joyal}}}$. Therefore we can conclude that the Kan-Quillen model structure is in fact a left Bousfield localisation of the Joyal structure, and it is evident from looking at the fibrations that this localisation happens at the outer horn inclusions.

2.2.3.4 $s\text{Comm}$

Denote by $s\text{Comm} := \text{Comm}^{\Delta^{op}}$ the category of simplicial objects in commutative rings. We have a forgetful-free adjunction

$$F: \hat{\Delta} \rightleftarrows s\text{Comm} : U$$

where $F$ is left adjoint to $U$. This adjunction satisfies the assumptions of Proposition 2.24 when $\hat{\Delta}$ is equipped with the Kan-Quillen model structure, which allows us to define the transferred model on $s\text{Comm}$. 
Corollary 2.36. There is a cofibrantly generated model structure on $s\text{Comm}$ where a map $f : X \to Y$ is a weak equivalence (resp., fibration) if $U(f) : U(X) \to U(Y)$ is a weak equivalence (resp., fibration) in $\Delta_{\text{Kan}}$. The cofibrations are defined by the LLP.

2.2.3.5 $s\text{Pr}(\mathcal{C})$

Let $\mathcal{C}$ be a small category. Denote by $s\text{Pr}(\mathcal{C}) := \Delta^{\text{exp}}_{\mathcal{C}}$ the category of simplicial presheaves on the category $\mathcal{C}$. This is equivalently the category of simplicial objects in the ordinary category of presheaves on $\mathcal{C}$. As $\Delta_{\text{Kan}}$ is a combinatorial model category, we can equip $s\text{Pr}(\mathcal{C})$ with both a projective and injective model structure. For now, we will just consider the projective structure, which we know exists because $\Delta_{\text{Kan}}$ is cofibrantly generated.

Definition 2.37. There exists a projective model structure on the category $s\text{Pr}(\mathcal{C})$, denoted $s\text{Pr}_{\text{proj}}(\mathcal{C})$, where a map $f : \mathcal{F} \to \mathcal{F}'$ is a:

- Weak equivalence if for all $X \in \mathcal{C}$ the map $\mathcal{F}(X) \to \mathcal{F}'(X)$ is a weak equivalence in $\Delta_{\text{Kan}}$.
- Fibration if for all $X \in \mathcal{C}$ the map $\mathcal{F}(X) \to \mathcal{F}'(X)$ is a fibration in $\Delta_{\text{Kan}}$.
- Cofibration if it has the LLP with respect to the class of trivial fibrations.

2.2.4 Monoidal Model Categories

Finally, we will discuss the required properties for a monoidal structure to be compatible with a Quillen model structure.

Definition 2.38 ([56, Definition 4.2.6]). A (symmetric) monoidal model category is a model category $\mathcal{C}$ equipped with a closed (symmetric) monoidal structure $(\mathcal{C}, \otimes, I)$ such that the two following compatibility conditions are satisfied:

1. (Pushout-product axiom) For every pair of cofibrations $f : X \to Y$ and $f' : X' \to Y'$, their pushout-product

   \[
   (X \otimes Y') \coprod_{X \otimes X'} (Y \otimes X') \to Y \otimes Y'
   \]

   is also a cofibration. Moreover, it is a trivial cofibration if either $f$ or $f'$ is.

2. (Unit axiom) For every cofibrant object $X$ and every cofibrant resolution $0 \to QI \to I$ of the tensor unit, the induced morphism $QI \otimes X \to I \otimes X$ is a weak equivalence. (Note that this holds automatically in the case that $I$ is cofibrant).
Theorem 2.39 ([56, Theorem 4.3.2]). Let $\mathcal{C}$ be a (symmetric) monoidal model category, then $\text{Ho}(\mathcal{C})$ can be given the structure of a closed (symmetric) monoidal category.

Example 2.40. In [56, Proposition 4.2.8] it is shown that $\hat{\Delta}_{\text{Kan}}$ is a symmetric monoidal model category with respect to $\times$, and therefore $\text{Ho}(\hat{\Delta}_{\text{Kan}})$ can be seen as being enriched over $\text{Ho}(\hat{\Delta}_{\text{Kan}})$.

2.3 Homotopical Algebraic Geometry

We shall now utilise the theory of simplicial objects and Quillen model structures to discuss the theory of homotopical algebraic geometry. We begin by defining what we will mean by a site [3].

Definition 2.41. Let $\mathcal{C}$ be a category with all fiber products. A Grothendieck topology on $\mathcal{C}$ is given by a function $\tau$ which assigns to every object $U$ of $\mathcal{C}$ a collection $\tau(U)$, consisting of families of morphisms $\{\varphi_i: U_i \to U\}_{i \in I}$ with target $U$ such that the following axioms hold:

1. (Isomorphisms) If $U' \to U$ is an isomorphism, then $\{U' \to U\}$ is in $\tau(U)$.

2. (Transitivity) If the family $\{\varphi_i: U_i \to U\}$ in $\tau(U)$ and if for each $i \in I$ one has a family $\{\varphi_{ij}: U_{ij} \to U_i\}_{j \in J}$ in $\tau(U_i)$ then the family $\{\varphi_i \circ \varphi_{ij}: U_{ij} \to U\}_{i \in I, j \in J}$ is in $\tau(U)$.

3. (Base change) If the family $\{\varphi_i: U_i \to U\}_{i \in I}$ is in $\tau(U)$ and if $V \to U$ is any morphism, then the family $\{V \times_U U_i \to V\}$ is in $\tau(V)$.

The families in $\tau(U)$ are called covering families for $U$ in the $\tau$-topology. We call a category $\mathcal{C}$ with such a topology a site, denoting it as $(\mathcal{C}, \tau)$.

Homotopical stacks, (also referred to as $\infty$-stacks), are a homotopification of the classical notion of stacks. Recall that a (1-)stack on a Grothendieck site $(\mathcal{C}, \tau)$ is a pseudo-functor $\mathcal{X}: \mathcal{C}^{op} \to \text{Grpd}$, where $\text{Grpd}$ is the 2-category of groupoids, satisfying some descent conditions. Such objects have been extensively studied, especially in the case when $\mathcal{C} = \text{Aff}$ equipped with the étale topology [4, 30, 45].

2.3.1 Homotopical Stacks

Stacks are used as an answer to moduli problems, and allow one to classify objects along with their isomorphisms. However, as soon as the notion of equivalence changes, we need a different structure to somehow encode this.
As a stack takes values in the 2-category \( \text{Grpd} \), it is a reasonable idea to enlarge this category to one where our new class of equivalence holds. As we have seen, from a homotopical viewpoint, simplicial sets can be used to model topological spaces. Therefore, our goal is to replace \( \text{Grpd} \) with \( \hat{\Delta} \), and to construct the correct descent conditions. Note, as our functors now take values in a simplicially enriched category, we can also assume that our source site is simplicially enriched. We will always make this assumption without ever making it explicit in our definitions.

We note here that in this section, and indeed, throughout the thesis, we shall omit technicalities involved with needing to take fibrant replacements of certain presheaves (\( \infty \)-stackification).

Recall from Section 2.2.3.5 that there is a projective model structure on \( s\text{Pr}(\mathcal{C}) \), where we equip \( \hat{\Delta} \) with its Kan-Quillen model structure. We shall denote this model structure \( s\text{Pr}_{\text{proj}}(\mathcal{C}) \). The issue with this model structure is that it does not reflect the topology of the site. To do this, we add more weak equivalences (in effect, enforcing a sheaf condition).

**Definition 2.42.** Let \( (\mathcal{C},\tau) \) be a site. A map \( f: \mathcal{F} \to \mathcal{F}' \) in \( s\text{Pr}(\mathcal{C}) \) is a local weak equivalence if:

- The induced map \( \pi_0^\tau \mathcal{F} \to \pi_0^\tau \mathcal{F}' \) is an isomorphism of sheaves, where \( \pi_0^\tau \) is the sheafification of \( \pi_0 \).
- Squares of the following form are pullbacks after sheafification:

\[
\begin{array}{ccc}
\pi_n \mathcal{F} & \to & \pi_n \mathcal{F}' \\
\downarrow & & \downarrow \\
\mathcal{F}_0 & \to & \mathcal{F}'_0
\end{array}
\]

**Theorem 2.43** ([57, §3]). Let \( (\mathcal{C},\tau) \) be a site. There exists a cofibrantly generated local model structure on the category \( s\text{Pr}(\mathcal{C}) \) where a map \( f: \mathcal{F} \to \mathcal{F}' \) is a:

- Weak equivalence if it is a local weak equivalence.
- Fibration if it has the RLP with respect to the trivial cofibrations.
- Cofibration if it is a cofibration in \( s\text{Pr}_{\text{proj}}(\mathcal{C}) \).

We will denote this model structure \( s\text{Pr}_\tau(\mathcal{C}) \).

Before continuing, we introduce a different description of the local model structure using Bousfield localisation (\( s\text{Pr}_{\text{proj}}(\mathcal{C}) \) is left proper and combinatorial so such a localisation exists).
Definition 2.44 ([106, Definition 3.1.3]). Let \((\mathcal{C}, \tau)\) be a site. A morphism \(f: \mathcal{F} \to \mathcal{F}'\) in \(\text{Ho}(\mathbf{sPr}_{\text{proj}}(\mathcal{C}))\) is called a \(\tau\)-covering if the induced map \(\pi_0^\tau(\mathcal{F}) \to \pi_0^\tau(\mathcal{F}')\) is an epimorphism of sheaves.

Definition 2.45 ([106, §3.2]). Let \((\mathcal{C}, \tau)\) be a site. A map \(f: X \to Y\) in \(\mathbf{sPr}(\mathcal{C})\) is a hypercovering if for all \(n \in \mathbb{N}\):

\[ X_n \to \text{Hom}_\Delta(\partial \Delta[n], X) \times_{\text{Hom}_\Delta(\partial \Delta[n], Y)} Y_n \]

is a \(\tau\)-covering in \(\text{Ho}(\mathbf{sPr}_{\text{proj}}(\mathcal{C}))\).

Theorem 2.46 ([32, Theorem 6.2]). There is a model structure on \(\mathbf{sPr}(\mathcal{C})\) which is the localisation of \(\mathbf{sPr}_{\text{proj}}(\mathcal{C})\) at the class of hypercovers. Moreover, this model is Quillen equivalent to \(\mathbf{sPr}_\tau(\mathcal{C})\).

Definition 2.47. For a site \((\mathcal{C}, \tau)\) the homotopy category \(\text{Ho}(\mathbf{sPr}_\tau(\mathcal{C}))\) will be referred to as the category of \(\infty\)-stacks on \(\mathcal{C}\). Some may prefer to call \(\text{Ho}(\mathbf{sPr}_\tau(\mathcal{C}))\) simplicial sheaves on \(\mathcal{C}\), in reality this is exactly what we are modelling.

2.3.2 Properties of \(\text{Ho}(\mathbf{sPr}_\tau(\mathcal{C}))\)

We explore some properties of the category of \(\infty\)-stacks on a site. In particular, we will see that it is a closed monoidal category. We will then show how site cohomology can be interpreted as mapping spaces in the category.

2.3.2.1 Closed Monoidal Structure

In this section we prove the existence of a closed monoidal structure on \(\text{Ho}(\mathbf{sPr}_\tau(\mathcal{C}))\), where the monoidal structure is given by direct product. To do so, we show that \(\mathbf{sPr}_\tau(\mathcal{C})\) is a monoidal model category in the sense of Definition 2.38. The proof of this property is non-trivial unless you make a small adjustment to the model that you are considering. The problem that we currently face is that the class of cofibrations in \(\mathbf{sPr}_\tau(\mathcal{C})\) are not easily described. This is because in the projective model they are simply defined via a lifting property. Instead, we choose a model which is homotopy equivalent to the one that we are interested in, and prove the existence of the monoidal model category for this. The following proposition is proved in the same manner as Theorem 2.43.

Proposition 2.48 ([106, Proposition 3.6.1]). There is a model structure on \(\mathbf{sPr}(\mathcal{C})\) called the injective local model where a map \(f: \mathcal{F} \to \mathcal{F}'\) is a:

- Weak equivalence if it is a local weak equivalence.
Fibration if it has the RLP with respect to the trivial cofibrations.

Cofibration if it is a cofibration in the injective model on \(sPr(C)\) (i.e., a point-wise monomorphism).

We shall denote by \(sPr_{inj,\tau}(C)\) this model structure.

**Lemma 2.49.** \(sPr_{inj,\tau}(C)\) is a closed monoidal model category.

**Proof.** We need to show that the pushout-product and unit axiom of Definition 2.38 hold. However, as the cofibrations are monomorphisms, this is trivial. \(\square\)

**Corollary 2.50.** \(Ho(sPr_{inj,\tau}(C)) \simeq Ho(sPr_{\tau}(C))\) is a closed monoidal category. The internal-hom in \(Ho(sPr_{\tau}(C))\) will be denoted \(R\text{Hom}(-,-)\).

We will refer to \(R\text{Hom}(X,Y)\) as the derived mapping stack from \(X\) to \(Y\) for two \(\infty\)-stacks \(X, Y\). One can compute this for each object \(c \in C\):

\[
R\text{Hom}(F,G)(c) := \hom^\Delta (c \times F, R_{inj}G)
\]

where \(R_{inj}\) is the fibrant replacement functor in \(sPr_{inj}(C)\) and \(\hom^\Delta\) is the simplicial-hom of \(sPr(C)\). We will denote by \(R\text{Hom}(-,-)\) the derived simplicial-hom.

### 2.3.2.2 Cohomology Theories

In this section, the link between cohomology theories and \(\infty\)-stacks is explored, as done in [103]. We start by echoing the following:

**Cohomology theories naturally arise as hom-spaces in an \((\infty,1)\)-topos.**

We have not introduced the general concept of an \((\infty,1)\)-topos and we will avoid doing so. We will take \((\infty,1)\)-topos to mean a category of \(\infty\)-stacks \(Ho(sPr_{\tau}(C))\) (which makes sense when taking into consideration that a topos is a category of sheaves). We first begin with the most general definition.

**Definition 2.51.** Let \(X, A\) be two stacks in \(Ho(sPr_{\tau}(C))\). The cohomology of \(X\) with coefficients in \(A\) is defined to be the simplicial set

\[
H(X; A) := R_\tau \text{Hom}(X, A) \in Ho(\hat{\Delta}_{\text{Kan}})
\]

Of course, for a cohomology theory one does not expect a simplicial set as the invariant. Therefore we take \(\pi_0\) of the above construction to obtain \(H^0(-,-)\).

---

1This mantra is quoted from the nLab

https://ncatlab.org/nlab/show/cohomology#ToposTheoryHistoricalAspects.
Definition 2.52. Let $X, A$ be two stacks in $\text{Ho}(s\text{Pr}_\tau(\mathcal{C}))$. The degree-0-cohomology of $X$ with coefficients in $A$ is defined to be

$$H^0(X; A) := \pi_0 \mathbb{R} \text{Hom}(X, A).$$

To get the higher cohomology groups we require some conditions on the coefficient $A$. Recall that an object $A$ is said to have a delooping $B_A$ if the homotopy pullback diagram

$$
\begin{array}{ccc}
A & \to & * \\
\downarrow & & \downarrow \\
* & \to & B_A
\end{array}
$$

exists, with the point $* \to B_A$ being essentially unique.

Definition 2.53. Let $X, A$ be two stacks in $\text{Ho}(s\text{Pr}_\tau(\mathcal{C}))$ such that $A$ is equipped with an $n$-fold delooping $B_n A$. The degree-$n$-cohomology of $X$ with coefficients in $A$ is defined to be

$$H^n(X; A) := \pi_0 \mathbb{R} \text{Hom}(X, B_n A).$$

Definition 2.53 is the one that we will be using from now one when we mean cohomology. However, an even more general setting using abstract $(\infty, 1)$-topoi is given in the book of Lurie [70, Definition 7.2.2.14].

Example 2.54. Let $X$ be a topological space represented by a discrete stack in $\text{Ho}(s\text{Pr}_\tau(\text{Top}))$. Let $A$ be a sheaf of abelian groups on $X$. We construct a stack $K(A, n)$ which sends $Y \in \text{Top}$ to $K(A(Y), n)$ (the Eilenberg-Mac Lane object). We then have

$$H^n(X; A) \simeq \pi_0 \mathbb{R} \text{Hom}(X, K(A, n)).$$

where the left hand cohomology group is ordinary sheaf cohomology of $X$.

Example 2.55. Let $(\mathcal{C}, \tau)$ be a site and $\text{Ho}(s\text{Pr}_\tau(\mathcal{C}))$ the corresponding topos. Let $A$ be a sheaf of abelian groups. We then have

$$H^n(\mathcal{C}; A) \simeq \pi_0 \mathbb{R} \text{Hom}(\ast, K(A, n)).$$

In the case that $\mathcal{C} = \text{Op}(X)$ for a topological space $X$ then we have the small site incarnation of Example 2.54.

We will now describe how Example 2.55 works, following [58, Theorem 8.26]. Begin by noting first of all that $\pi_0 \mathbb{R} \text{Hom}(\ast, -)$ is simply the set of morphisms denoted $[-, -]$. We...
can see the above isomorphism as a chain of natural isomorphisms as follows:

\[
\begin{align*}
[*, K(A, n)] & \simeq [\mathbb{Z}, K(A, n)] \quad \text{(simplicial abelian sheaves)} \\
& \simeq [\mathbb{Z}[0], A[-n]] \quad \text{(sheaves of chain complexes)} \\
& \simeq \text{Ext}_{\mathbb{C}}^n(\mathbb{Z}, A) \quad \text{(2.3.3)} \\
& \simeq H^n(\mathbb{C}; A). \quad \text{(2.3.4)}
\end{align*}
\]

Isomorphism 2.3.1 takes the setting from the homotopy category of simplicial sheaves to the corresponding homotopy category of simplicial abelian sheaves. This step is an isomorphism because of the following result proved by Osdol [109].

**Proposition 2.56** (Illusie Conjecture). The free simplicial abelian presheaf functor \( X \mapsto ZX \) considered as a map \( sPr_{\tau}(\mathbb{C}) \to sPr_{\tau}(\mathbb{C}) \) preserves local equivalences.

Isomorphism 2.3.2 follows from the Dold-Kan correspondence. This takes us into the setting of the homotopy category of simplicial abelian sheaves with morphisms now in the derived category. Finally, isomorphisms 2.3.3 and 2.3.4 can be taken simply as the corresponding definitions.

### 2.3.3 Derived Algebraic Geometry

We will now move from the general setting of homotopical stacks to one where there is a more geometric nature. In particular, we shall introduce the étale site of derived affine schemes. A good overview for the theory discussed in this section can be found at [75].

Recall that \( \mathbf{sComm} \) is the category of simplicial commutative rings and has a transferred model structure from \( \mathbf{Kan} \). The homotopy category \( \text{Ho}(\mathbf{sComm}) \) is the category of derived rings.

There is a fully faithful embedding \( j_0: \mathbf{Comm} \to \text{Ho}(\mathbf{sComm}) \) of the category of commutative rings to the category of derived rings. There is a left adjoint to this functor which we denote \( \pi_0: \text{Ho}(\mathbf{sComm}) \to \mathbf{Comm} \).

**Definition 2.57** ([107, §2.2.2]). A morphism \( A \to B \) is flat (resp., étale, smooth, ...) if and only if it satisfies the following:

- The induced morphism of rings \( \pi_0(A) \to \pi_0(B) \) is flat (resp., étale, smooth, ...).
- For all \( i \geq 0 \) the induced morphism \( \pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B) \) is a bijection.

We now consider the category \( \mathbf{dAff} := \text{Ho}(\mathbf{sComm})^{op} \). For \( A \) a derived ring, we will denote by \( \text{Spec}(A) \) the corresponding object in \( \mathbf{dAff} \). We shall refer to the objects \( \text{Spec}(A) \) as derived affine schemes.
Remark 2.58. Recall that objects in $\textbf{Aff}$ can be described as locally ringed spaces $(X, \mathcal{O}(X))$. A similar construction exists for the objects of $\text{dAff}$. We can represent derived schemes as derived locally ringed spaces, that is, pairs $(X, \mathcal{O}(X))$ where $X \in \text{Top}$ and $\mathcal{O}(X)$ is an $\infty$-stack of rings on $X$. The affine objects are then those pairs $(X, \mathcal{O}(X))$ such that the underlying locally ringed space $(X, \pi_0^0 \mathcal{O}(X))$ is an affine scheme.

We now consider making $\text{dAff}$ into a site by equipping it with the étale topology. We will denote by $\textbf{sA-Mod}$ the category of simplicial $A$-modules. A morphism $f : A \rightarrow B$ of simplicial commutative rings gives an adjunction

$$- \otimes_A B : \textbf{sA-Mod} \rightleftarrows \textbf{sB-Mod} : f^*$$

This adjunction is a Quillen adjunction (which is also a Quillen equivalence if $f$ is an equivalence). We will denote the left derived functor $- \otimes_A B$ as

$$- \otimes^L_A B : \text{Ho}(\textbf{sA-Mod}) \rightarrow \text{Ho}(\textbf{sB-Mod})$$

and call it the base change functor.

Definition 2.59 ([107, Definition 2.2.2.12]). A family of morphisms $\{ \text{Spec}(A_i) \rightarrow \text{Spec}(A) \}_i$ is an étale covering if each morphism $f_i : A \rightarrow A_i$ is an étale morphism of rings and if the family of base-change functors

$$\left\{ - \otimes^L_A A_i : \text{Ho}(\textbf{sA-Mod}) \rightarrow \text{Ho}(\textbf{sA}_i \text{-Mod}) \right\}_i$$

is conservative (that is, a morphism $u$ in $\text{Ho}(\textbf{sA-Mod})$ is an isomorphism if and only if all of the $- \otimes^L_A A_i$ are isomorphisms). The étale topology is then the topology on $\text{dAff}$ for which the covering sieves are generated by these covering families.

Definition 2.60. The category of derived stacks is the category $\text{Ho}(\textbf{sPr}_{\text{et}}(\text{dAff}))$.

2.3.4 Geometric Homotopical Stacks

We begin with a warning. There is a slight discrepancy between definitions in the literature. The two main bodies of work in the literature are that of Toën-Vezzosi [107] and Lurie [69], in which the notions of $n$-geometric stacks are slightly different. From now on we will follow the conventions of [107].

In the study of algebraic stacks, one is not interested in all objects. Instead, one is interested in only those stacks which have a smooth or étale atlas (this can also be worded in terms of internal groupoid objects). We say that stacks with such an atlas are geometric.
In this section we will introduce what we mean by a geometric $\infty$-stack. We do this in two ways, the first uses an atlas representation as in the classical setting. This method, however, is extremely cumbersome. Therefore we introduce the second method which uses the theory of hypergroupoids.

2.3.4.1 Via Iterated Representability

We fix a site $(\mathcal{C}, \tau)$ and a set of covering maps $\mathbf{P}$. The cases that will be interested in are usually $\mathcal{C} = \text{Aff}$ or $\text{dAff}$, the topology will the étale topology and $\mathbf{P}$ being smooth or étale.

**Definition 2.61** ([107, Definition 1.3.3.1]).

1. A stack is $(−1)$-geometric if it is representable.

2. A morphism of stacks $f: F \to G$ is $(−1)$-representable if for any representable stack $X$ and any morphism $X \to G$ the homotopy pullback $F \times^h_G X$ is a representable stack.

3. A morphism of stacks $f: F \to G$ is in $(−1)$-$\mathbf{P}$ if it is $(−1)$-representable and if for any representable stack $X$ and any morphism $X \to G$, the induced morphism $F \times^h_G X \to X$ is a $\mathbf{P}$-morphism between representable stacks.

We now let $n \geq 0$.

1. Let $F$ be any stack. An $n$-atlas for $F$ is a small family of morphisms $\{U_i \to F\}_{i \in I}$ such that

   (a) Each $U_i$ is representable.

   (b) Each morphism $U_i \to F$ is in $(n − 1)\cdot \mathbf{P}$.

   (c) The total morphism $\coprod_{i \in I} U_i \to F$ is an epimorphism.

2. A stack $F$ is $n$-geometric if it satisfies the following conditions:

   (a) The diagonal $F \to F \times^h F$ is $(n − 1)$-representable.

   (b) The stack $F$ admits an $n$-atlas.

3. A morphism of stacks $F \to G$ is $n$-representable if for any representable stack $X$ and any morphism $X \to G$ the homotopy pullback $F \times^h_G X$ is $n$-geometric.

4. A morphism of stacks $F \to G$ is in $n\cdot \mathbf{P}$ if it is $n$-representable and if for any representable stack $X$, any morphism $X \to G$, there exists an $n$-atlas $\{U_i\}$ of $F \times^h_G X$ such that each composite morphism $U_i \to X$ is in $\mathbf{P}$. 
We will say that a stack is geometric if it is $n$-geometric for any $n$.

**Definition 2.62.** The full subcategory of $n$-geometric stacks of $\text{Ho}(sPr_{\tau}(\mathcal{C}))$ will be denoted $\text{GeSt}_n(\mathcal{C}, P)$. In particular:

- $\text{GeSt}_n(\text{dAff}, \text{sm})$ is the category of derived $n$-Artin stacks.
- $\text{GeSt}_n(\text{dAff}, \text{ét})$ is the category of derived $n$-Deligne-Mumford stacks.

**Proposition 2.63** ([107, Corollary 1.3.3.5]). The category $\text{GeSt}_n(\mathcal{C}, P)$ is stable by homotopy pullbacks and disjoint union.

### 2.3.4.2 Via $n$-Hypergroupoids

In this section, we introduce a second representation of $n$-geometric stacks. This method, via hypergroupoids, is more intuitive than the method presented in Section 2.3.4.1, and is reportedly closer to that envisaged by Grothendieck in [49]. An easy to read overview of the theory can be found in the paper [84], while the full results in the most general setting can be found in [85]. The general idea is that an $n$-geometric stack on the site $\mathcal{C}$ can be resolved by some fibrant object in the category $\mathcal{C}^{\Delta}$.

The impressive thing about this construction is that it completely avoids the need for the local model structure on simplicial presheaves, but is intricately linked to the notion of hypercovers none the less. Note that an $n$-hypergroupoid captures the nerve construction of an $n$-groupoid [33].

**Definition 2.64** ([14, Definition 2.1]). An $n$-hypergroupoid is an object $X \in \hat{\Delta}$ for which the maps

$$X_m = \text{Hom}_{\hat{\Delta}}(\Delta[m], X) \to \text{Hom}_{\hat{\Delta}}(\Lambda^k[m], X)$$

are surjective for all $m, k$, and isomorphisms for all $m > n$ and all $k$. In particular, $X$ is a Kan complex, and therefore fibrant in $\hat{\Delta}_{\text{Kan}}$.

It is possible to characterise $n$-hypergroupoids using the coskeleton construction. Recall that the $m$-coskeleton $\text{cosk}_m X$ is defined to be $(\text{cosk}_m X)_i = \text{Hom}(\Delta[i]_{\leq m}, X_{\leq m})$ where $X_{\leq m}$ is the truncation at $m$.

**Lemma 2.65** ([14, Proposition 3.4]). An $n$-hypergroupoid $X$ is completely determined by its truncation $X_{\leq n+1}$, in fact $X = \text{cosk}_{n+1} X$ (i.e., $X$ is $(n + 1)$-coskeletal). Conversely, a simplicial set of the form $\text{cosk}_{n+1} X$ is an $n$-hypergroupoid if and only if it satisfies the conditions of Definition 2.64 up to level $n + 2$.

**Definition 2.66** ([14, Proposition 2.4]). A morphism $f : X \to Y$ in $\hat{\Delta}$ is a trivial relative $n$-hypergroupoid if the maps

$$X_m \to \text{Hom}_{\hat{\Delta}}(\partial\Delta[m], X) \times_{\text{Hom}_{\hat{\Delta}}(\partial\Delta[m], Y)} Y_m$$
are surjections for all $m$, and isomorphisms for all $m \geq n$. In particular, $f$ is a trivial fibration in $\hat{\Delta}_{\text{Kan}}$.

**Lemma 2.67** ([85, Lemma 2.9]). Let $f: X \to Y$ be a trivial $n$-hypermultipod then $X = Y \times_{\cosk_{n-1} Y} \cosk_{n-1} X$.

We can compare the above definition and property of trivial $n$-hypermultipoids with the definition of a hypercover (Definition 2.45), and observe that a trivial $n$-hypermultipoid can be seen as a truncated or bounded hypercover.

We can move to a geometric setting by considering objects in $\text{sdAff} := \text{dAff}^{\Delta^{op}}$. We also change the surjectivity condition to be surjectivity with respect to a class of covering maps (i.e., smooth or étale).

**Definition 2.68.** A derived Artin (resp., Deligne-Mumford) $n$-hypermultipoid is a simplicial derived affine scheme $X \in \text{sdAff}$ such that the maps

$$X_m = \text{Hom}_\Delta(\Delta[m], X) \to \text{Hom}_\Delta(\Lambda^k[m], X)$$

are smooth (resp., étale) surjections for all $m, k$, and isomorphisms for all $m > n$ and all $k$.

Note that given an (Artin or Deligne-Mumford) $n$-hypermultipoid $X$, we can construct a simplicial presheaf $X$ on $\text{dAff}$ as follows:

$$X: \text{dAff}^{op} \to \hat{\Delta},$$

$$X(A)_n := X_n(A).$$

**Definition 2.69.** A morphism $f: X \to Y$ in $\text{sdAff}$ is a trivial relative derived Artin (resp., Deligne-Mumford) $n$-hypermultipoid if the maps

$$X_m \to \text{Hom}_\Delta(\partial \Delta[m], X) \times_{\text{Hom}_\Delta(\partial \Delta[m], Y)} Y_m$$

are smooth (resp., étale) surjections for all $m$, and isomorphisms for all $m \geq n$ (i.e., are $n$-truncated hypercovers).

A subcategory $\mathcal{D} \to \mathcal{C}$ is said to be wide if $\text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{D})$. Recall from [9] that a relative category is a pair $(\mathcal{C}, W)$ consisting of a category $\mathcal{C}$ and a wide subcategory $W$ whose maps are called weak equivalences. Such a category has a homotopy category by formally inverting all of the weak equivalences.

**Definition 2.70.** The category of derived $n$-Artin (resp., Deligne-Mumford) stacks is obtained from the full subcategory of $\text{sdAff}$ consisting of the relative category of derived
Artin (resp., Deligne-Mumford) $n$-hypergroupoids and the trivial relative derived Artin (resp., Deligne-Mumford) $n$-hypergroupoids as the weak equivalences. We will denote this category $\mathcal{G}^{\text{sm}}_n(\text{dAff})$ (resp., $\mathcal{G}^{\text{ét}}_n(\text{dAff})$).

**Remark 2.71.** There is in fact a full Quillen model structure instead of just a relative category structure. The construction of this model structure is done in [85]. We have not looked at the model structure here as the other classes of maps are extremely technical to describe.

The following theorem is the main result from [85], and proves that we can move freely (up to homotopy) between the $n$-hypergroupoid and $n$-geometric stack constructions.

**Theorem 2.72 ([85, Theorem 4.15]).** There is an equivalence of categories

$$\mathcal{G}^{\text{sm}}_n(\text{dAff}) \simeq \mathcal{G}^{\text{sm}}_n(\text{dAff}),$$

$$\mathcal{G}^{\text{ét}}_n(\text{dAff}) \simeq \mathcal{G}^{\text{ét}}_n(\text{dAff}).$$

In fact, such an equivalence can be formulated for any homotopical algebraic geometric context [107, Definition 1.3.2.13].
Chapter 3

Augmented Homotopical Algebraic Geometry

In this chapter we will develop the theory of augmentation categories. An augmentation category will be one which extends the simplex category in a way that we can develop the model structures required for a generalisation of the theory of homotopical algebraic geometry. We will also be able to use suitable notions of boundary and horns to develop the theory of augmented hypergroupoids to give a combinatorial representation of geometric augmented stacks.

Outline of Chapter 3

(Section 3.1) We introduce the theory of generalised Reedy categories which will form one of the structures for augmentation categories.

(Section 3.2) We fully develop a model structure on the category of presheaves of an augmentation category which reflects the construction of $\hat{\Delta}_{\text{Kan}}$.

(Section 3.3) Using the model structure developed in Section 3.2, we define a local model structure on augmented presheaves. We then show that the homotopy category of this local model structure captures a generalisation of homotopical algebraic geometry.

3.1 Augmentation Categories

3.1.1 Generalised Reedy Categories

A (strict) Reedy category $S$ is a category such that we can equip $\mathcal{E}^{S^{op}}$ with a model structure, for $\mathcal{E}$ a cofibrantly generated model category [90]. The classes of maps in this model structure can be described explicitly using those of $\mathcal{E}$. An example of a strict Reedy
category is the simplex category $\Delta$. One shortcoming of strict Reedy categories is that they do not allow for non-trivial automorphisms on the objects, which occur, for example, in the cyclic category of Connes, introduced in [27]. A generalised Reedy category allows us to capture this automorphism data. Recall that a subcategory $D \subset \mathcal{C}$ is said to be wide in $\mathcal{C}$ if $\text{Ob}(\mathcal{C}) = \text{Ob}(D)$. The following definition appears in [16].

**Definition 3.1** ([16, Definition 1.1]). A generalised Reedy structure on a small category $\mathcal{R}$ consists of wide subcategories $\mathcal{R}^+, \mathcal{R}^-$, and a degree function $d: \text{Ob}(\mathcal{R}) \to \mathbb{N}$ satisfying the following four axioms:

i) Non-invertible morphisms in $\mathcal{R}^+$ (resp., $\mathcal{R}^-$) raise (resp., lower) the degree; isomorphisms in $\mathcal{R}$ preserve the degree.

ii) $\mathcal{R}^+ \cap \mathcal{R}^- = \text{Iso}(\mathcal{R})$.

iii) Every morphism $f$ of $\mathcal{R}$ factors uniquely (up to isomorphism) as $f = gh$ with $g \in \mathcal{R}^+$ and $h \in \mathcal{R}^-$. 

iv) If $\theta f = f$ for $\theta \in \text{Iso}(\mathcal{R})$ and $f \in \mathcal{R}^-$, then $\theta$ is an identity. Moreover, we say that the generalised Reedy structure is dualisable if the following additional axiom holds:

iv') If $f\theta = f$ for $\theta \in \text{Iso}(\mathcal{R})$ and $f \in \mathcal{R}^+$, then $\theta$ is an identity.

A morphism of generalised Reedy categories is a functor $\mathcal{R} \to \mathcal{R}'$ which takes $\mathcal{R}^+$ (resp., $\mathcal{R}^-$) to $\mathcal{R}'^+$ (resp., $\mathcal{R}'^-$) and preserves the degree.

It is possible to generate a large class of generalised Reedy categories using the theory of crossed groups on categories [41].

**Definition 3.2** ([16, Proposition 2.5]). Let $\mathcal{R}, \mathcal{S}$ be categories such that $\mathcal{R} \subseteq \mathcal{S}$ is a wide subcategory. Assume that for all $s \in \mathcal{S}$, there exist subgroups $\mathcal{G}_s \subseteq \text{Aut}_S(s)$ of special automorphisms such that each morphism in $\mathcal{S}$ factors uniquely as a special automorphism followed by a morphism in $\mathcal{R}$. Then $\mathcal{S}$ is a crossed $\mathcal{R}$-group, which we denote $\mathcal{R}\mathcal{G}$. What we call a crossed $\mathcal{R}$-group is sometimes referred to as the total category of a crossed $\mathcal{R}$-group.

There is a compatibility condition appearing in the following proposition which we will not cover as all categories that we will consider will satisfy it [16, Remark 2.9].

**Proposition 3.3** ([16, Proposition 2.10]). Let $\mathcal{R}$ be a strict Reedy category, and $\mathcal{R}\mathcal{G}$ a compatible crossed $\mathcal{R}$ group. Then there is a unique dualisable generalised Reedy structure on $\mathcal{R}\mathcal{G}$ for which the embedding $\mathcal{R} \to \mathcal{R}\mathcal{G}$ is a morphism of generalised Reedy categories.
We shall now use the degree function appearing in the Definition 3.1 to define the notion of (co)skeleton. Denote by $\mathcal{R}_{\leq n}$ the subcategory of $\mathcal{R}$ consisting of objects of degree $\leq n$. Write $t_n : \mathcal{R}_{\leq n} \hookrightarrow \mathcal{R}$ for the corresponding full embedding.

**Definition 3.4** ([16, Definition 6.1]). Let $\mathcal{R}$ be a generalised Reedy category.

- The $n$-skeleton functor is the endofunctor $\text{sk}_n := t_n^* t_n^!$.
- The $n$-coskeleton functor is the endofunctor $\text{cosk}_n := t^*_n t_n^*$.

The class of EZ-categories are a subclass of generalised Reedy categories for which the skeletal filtrations have a nice description. These skeletal filtrations can in turn be described by a corresponding boundary object using notions of face and degeneracy maps analogous to the simplicial case. These boundary objects will allow us to give an explicit description of when objects in the presheaf category are coskeletal.

**Definition 3.5** ([16, Definition 6.6]). An EZ-category (Eilenberg-Zilber category) is a small category $\mathcal{R}$, equipped with a degree function $d : \text{Ob}(\mathcal{R}) \to \mathbb{N}$, such that

1. Monomorphisms preserve (resp., raise) the degree if and only if they are invertible (resp., non-invertible).
2. Every morphism factors as a split epimorphism followed by a monomorphism.
3. Any pair of split epimorphisms with common domain gives rise to an absolute pushout (recall an absolute pushout is a pushout preserved by the Yoneda embedding $\mathcal{R} \hookrightarrow \hat{\mathcal{R}}$).

An EZ-category is a dualisable generalised Reedy category with $\mathcal{R}^+$ (resp., $\mathcal{R}^-$) defined to be the wide subcategory containing all monomorphisms (resp., epimorphisms). Moreover, we will say that an EZ-category $\mathcal{R}$ is symmetric promagmoidal if $\hat{\mathcal{R}}$ has a symmetric tensor product $(\tilde{\mathcal{R}}, \square, I\square)$. Clearly any presheaf category carries the cartesian product, but often the tensor structure that we work with will be different to the cartesian product.

Just as generalised Reedy categories were compatible with taking crossed categories, EZ-structures are also compatible with this operation.

**Lemma 3.6** ([16, §6]). Let $\mathcal{R}$ be a strict EZ-category. Then any compatible crossed group $\mathcal{R} \mathcal{E}$ on $\mathcal{R}$ is an EZ-category.

**Definition 3.7.** Let $\mathcal{R}$ be an EZ-category. Denote by $\mathcal{R}[r]$ the representable presheaf of $r \in \mathcal{R}$ in the topos $\hat{\mathcal{R}}$. The split-epimorphisms will be called the degeneracy operators and the monomorphisms will be called the face operators.
Definition 3.8. Let $\mathbb{R}$ be an EZ-category and $r \in \mathbb{R}$. The boundary, $\partial \mathbb{R}[r] \subset \mathbb{R}[r]$ is the subobject of those elements of $\mathbb{R}[r]$ which factor through a non-invertible face operator $s \to r$. Explicitly,

$$\partial \mathbb{R}[r] = \bigcup_{f : s \to r} f(\mathbb{R}[s]).$$

Lemma 3.9 ([16, Corollary 6.8]). Let $\mathbb{R}$ be an EZ-category and $r \in \mathbb{R}$, then $\partial \mathbb{R}[r] = \text{sk}_{d(r)-1} \mathbb{R}[r]$.

The above definition of boundary coincides exactly with the definition of boundary in the simplicial case. We can now say when an object $X \in \hat{\mathbb{R}}$ is coskeletal.

Lemma 3.10. Let $\mathbb{R}$ be an EZ-category, and $X \in \hat{\mathbb{R}}$. Then the following are equivalent:

1. The unit of the adjunction $X \to \text{cosk}_n(X)$ is an isomorphism.
2. The map $X_r = \text{Hom}(\mathbb{R}[r], X) \to \text{Hom}(\mathbb{R}[r]_{\leq n}, X_{\leq n})$ is a bijection for all $r$ with $d(r) > n$.
3. For all $r$ with $d(r) > n$, and every morphism $\partial \mathbb{R}[r] \to X$, there exists a unique filler $\mathbb{R}[r] \to X$:

$$\begin{array}{c}
\partial \mathbb{R}[r] \longrightarrow X \\
\downarrow \\
\mathbb{R}[r]
\end{array}$$

If $X$ satisfies any of these equivalent definitions we shall say that $X$ is $n$-coskeletal.

Proof. The equivalence of the first two conditions follows from the definition. For the final condition, note that $\text{cosk}_k(X)$ if given by the formula $[n] \mapsto \text{Hom}(\text{sk}_k(\Delta[n]), X)$ by adjointness, then using Lemma 3.9 we see that this is the same as $\text{Hom}(\partial \mathbb{R}[r], X)$ for $d(r) > n$. The unique filler condition is then equivalent to Condition 2.

3.1.2 Augmentation Categories

Definition 3.11 ([16, Proposition 7.2]). Let $\mathbb{R}$ be an EZ-category. A normal monomorphism in $\hat{\mathbb{R}}$ is a map $f : X \to Y$ such that $f$ is monic and for each object $r$ of $\mathbb{R}$ and each non-degenerate element $y \in Y_r \setminus f(X)_r$, the isotropy group $\{ g \in \text{Aut}(r) \mid g^*(y) = y \}$ is trivial.

Definition 3.12. A normal monomorphism in $\hat{\Psi}$ is said to be linear if it is in the saturated class of boundary inclusions $\partial \Psi[\psi] \to \Psi[\psi]$ for $\psi = i_1[n]$. 
We now have all of the necessary tools to introduce what we mean by an augmentation category. In the following sections we will be exploring the properties of these categories alongside developing a geometric framework for them.

**Definition 3.13.** An *augmentation category* is a category $\Psi$ such that:

- (AC1) $\Psi$ is a symmetric promagmoidal EZ-category.
- (AC2) There is a faithful inclusion of EZ-categories $i: \Delta \hookrightarrow \Psi$ such that for any two simplicial sets $X$ and $Y$ we have $i_!(X) \square i_!(Y) \simeq i_!(X \times Y)$, where $\square$ is the tensor product of $\widehat{\Psi}$.
- (AC3) Let $f: A \to B$ and $g: K \to L$ be normal monomorphisms in $\widehat{\Psi}$, then the map $A \square K \sqcup A \square L \sqcup B \square L \to B \square K$

is again a normal monomorphism whenever one of them is linear.

We will usually use $\psi \in \Psi$ for a typical element of an augmentation category.

Clearly $\Delta$ itself is the prototypical example of an augmentation category, and is minimal in the sense that any other augmentation category will factor through it.

### 3.2 Homotopy of Augmentation Categories

From now on, we will assume that all categories in question are augmentation categories. This section will be devoted to proving the existence of a model structure on the presheaf category $\widehat{\Psi}$ for a given augmentation category $\Psi$. In this model, the fibrant objects are generalisations of Kan complexes. The existence of this model structure strongly hinges on the fact that $\widehat{\Psi}$ is *weakly enriched in simplicial sets*. In particular, using the tensor product $\square$ and the compatibility condition of (AC2) define for $K \in \widehat{\Delta}$ and $X, Y \in \widehat{\Psi}$:

$$
\begin{align*}
\hom_{\widehat{\Delta}}(X, Y) &= \Hom_{\widehat{\Psi}}(X \square i_!(\Delta[n]), Y), \\
X \square K &= X \square i_!(K), \\
(Y^K)_\psi &= \Hom_{\widehat{\Psi}}(\Psi[\psi] \square i_!(K), Y).
\end{align*}
$$

The method of constructing the model structure will, in part, utilise the above simplicial compatibility and $\widehat{\Delta}_{\text{Kan}}$ to explicitly describe the weak equivalences.

**Remark 3.14.** The material presented in this section draws heavily on the construction of the stable model structure for dendroidal sets [10], which, in turn, follows the presentation...
of [81] and [52]. In fact, one sees that the definition of the augmentation category is rigid enough that the arguments relating to the model structure developed in [10] are simply altered in a consistent manner, replacing instances of $\Omega$ with $\Psi$.

### 3.2.1 Normal Monomorphisms

In this section we look at the properties of the normal monomorphisms as introduced in Definition 3.11, we recall this definition here using the language of augmentation categories.

**Definition 3.15.** Let $\Psi$ be an augmentation category. A normal monomorphism in $\hat{\Psi}$ is a map $f: X \to Y$ such that $f$ is monic and for each object $\psi$ of $\Psi$ and each non-degenerate element $y \in Y_\psi \setminus f(X)_\psi$, the isotropy group $\{ g \in \text{Aut}(\psi) \mid g^*(y) = y \}$ is trivial.

**Remark 3.16.** Note that if $\Psi$ has a strict Reedy structure, then the class of normal monomorphisms coincides with the class of monomorphisms.

Recall that a class of morphisms is saturated if it is closed under retracts, transfinite compositions and pushouts. Similar to the simplicial case, the normal monomorphisms can be described as the saturated class of boundary inclusions. The following lemma holds for a wider class of categories than just augmentation categories, as proved in [24, Proposition 8.1.35].

**Lemma 3.17.** The class of normal monomorphisms is the smallest class of monomorphisms closed under pushouts and transfinite compositions that contains all boundary inclusions $\partial \Psi[\psi] \to \Psi[\psi]$ for $\psi \in \Psi$.

Using the definition of normal monomorphisms, we will say that an object $A \in \hat{\Psi}$ is normal if $0 \to A$ is a normal monomorphism. From the definition of the normal monomorphisms we get the following trivial property, which leads to an observation of maps between normal objects.

**Lemma 3.18.** A monomorphism $X \to Y$ of $\Psi$-sets is normal if and only if for any $\psi \in \Psi$, the action of $\text{Aut}(\psi)$ on $Y(\psi) - X(\psi)$ is free.

**Corollary 3.19.** If $f: A \to B$ is any morphism of $\Psi$-sets, and $B$ is normal, then $A$ is also normal. If $f$ is a monomorphism and $B$ is normal, then $f$ is a normal monomorphism.

Quillen’s small object argument applied to the saturated class of boundary inclusions yields the following.

**Corollary 3.20.** Every morphism $f: X \to Y$ of $\Psi$-sets can be factored as $f = gh$, $g: X \to Z$, $h: Z \to Y$, where $g$ is a normal monomorphism and $h$ has the RLP with respect to all normal monomorphisms.
Definition 3.21. Let $X \in \hat{\Psi}$, a normalisation of $X$ is a morphism $X' \to X$ from a normal object $X'$ having the RLP with respect to all normal monomorphisms. Note that such a normalisation exists for any $X$ due to the factoring of the map $0 \to X$.

### 3.2.2 Augmented Kan Complexes

We shall now use the boundary objects, along with our definition of degeneracy maps in a general EZ-category to introduce the concept of a horn object.

**Definition 3.22.** Let $f : \psi \to \psi'$ be a face map of $\psi \in \Psi$. The $f$-horn of $\Psi[\psi]$ is the subobject of $\partial \Psi[\psi]$ which excludes the object which factors through $f$. We denote this object $\Lambda^f \Psi[\psi]$. Explicitly,

$$\Lambda^f \Psi[\psi] = \bigcup_{g : \psi' \to \psi} \{g(\mathbb{R}[\psi'])\}$$

**Definition 3.23.** Let $\Psi$ be an augmentation category. A $\Psi$-Kan complex is an object $X \in \hat{\Psi}$ such that it has fillers for all horns. That is there is a lift for all face maps $f$:

$$\begin{array}{ccc}
\Lambda^f \Psi[\psi] & \longrightarrow & X \\
\downarrow & & \downarrow \text{R}
\Psi[\psi]
\end{array}$$

**Remark 3.24.** In the simplicial, and indeed the dendroidal settings, there is the concept of an inner horn. In the general setting of augmentation categories, there seems to be no canonical way to define these objects.

**Definition 3.25.** The smallest saturated class containing all horn extensions $\Lambda^f \Psi[\psi] \to \Psi[\psi]$ will be called the class of anodyne extensions. Therefore an object is $\Psi$-Kan if and only if it has the RLP with respect to all anodyne extensions.

**Proposition 3.26.** Let $Z$ be a $\Psi$-Kan complex and $f : A \to B$ a normal monomorphism, then

$$f^* : \hom^\Delta(B, Z) \to \hom^\Delta(A, Z)$$

is a Kan fibration of simplicial sets.

**Proof.** We need only prove the result when $f$ is also a boundary inclusion $\partial \Psi[\psi] \to \Psi[\psi]$ (as the normal monomorphisms are the saturated class of boundary inclusions by Lemma 3.17). The map $f^*$ has the RLP with respect to the horn inclusion $\Lambda^k[n] \to \Delta[n]$ if and only if $Z$ has the RLP with respect to the map

$$\Lambda^k[n] \square \Psi[\psi] \sqcup \Delta[n] \square \partial \Psi[\psi] \to \Delta[n] \square \Psi[\psi].$$
By (AC3), we have that this map is an anodyne extension, so \( Z \) has the RLP with respect to it.

The following corollary follows by considering the morphism \( f: 0 \to B \).

**Corollary 3.27.** If \( Z \) is a \( \Psi \)-Kan complex, and \( B \) a normal object, then \( \text{hom}^\Delta(B,Z) \) is a Kan complex.

### 3.2.3 Augmented Homotopy

We will use the Kan objects to define a homotopy theory of \( \Psi \)-sets. Due to the compatibility of the tensor structures, we can take \( X \square \Psi[1] \in \hat{\Psi} \) to be a cylinder object of \( X \). It comes from the factorisation of the fold map:

\[
\begin{array}{ccc}
X \sqcup X & \xrightarrow{1_X \sqcup 1_X} & X \\
& \searrow^{i_0 \sqcup i_1} & \downarrow^\varepsilon \\
& X \square \Psi[1] &
\end{array}
\]

One can see that if \( X \) is normal, then \( X \square \Psi[1] \) is normal, and by Corollary 3.19, the map \( i_0 \sqcup i_1 \) is a normal monomorphism.

**Definition 3.28.** Two morphisms \( f, g: X \to Y \) in \( \hat{\Psi} \) are homotopic (\( f \simeq g \)) if there exists \( H: X \square \Psi[1] \to Y \) such that \( f = Hi_0 \) and \( g = Hi_1 \). That is, the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{i_0} & X \square \Psi[1] & \xleftarrow{i_1} & X \\
\downarrow^f & & \downarrow^H & & \downarrow^g \\
Y & & Y & & Y
\end{array}
\]

We will say that \( f: X \to Y \) is a homotopy equivalence if there is a morphism \( g: Y \to X \) such that \( fg \simeq 1_Y \) and \( gf \simeq 1_X \).

**Definition 3.29.** A map \( f: X \to Y \) is a weak equivalence if there exists a normalisation (i.e., cofibrant replacement) \( f': X' \to Y' \) which induces an equivalence of Kan complexes

\[
\text{hom}^\Delta(Y',Z) \to \text{hom}^\Delta(X',Z)
\]

for every \( \Psi \)-Kan complex \( Z \). Note that every homotopy equivalence between normal \( \Psi \)-sets is a weak equivalence.

**Remark 3.30.** As one would expect, if we have a \( \Psi \)-set \( X \), then the corresponding normalisation \( f: X' \to X \) is a weak equivalence.
Lemma 3.31. A morphism of $\Psi$-sets which has the RLP with respect to all normal monomorphisms is a weak equivalence.

Proof. Let $X \rightarrow Y$ be a map between $\Psi$-sets with the RLP with respect to all normal monomorphisms. Denote by $Y' \rightarrow Y$ the normalisation of $Y$. Since $Y'$ is normal we may construct a lift

$$
\begin{array}{c}
0 \\
\downarrow
\end{array}
\quad
\begin{array}{c}
X \\
\downarrow
\end{array}
\quad
\begin{array}{c}
Y
\end{array}
\quad
\begin{array}{c}
Y'
\end{array}

$$

We then factor $s$ as a normal monomorphism $i: X' \rightarrow Y'$ followed by the normalisation $X' \rightarrow X$ to get a lift

$$
\begin{array}{c}
X' \\
\downarrow
\end{array}
\quad
\begin{array}{c}
Y'
\end{array}
\quad
\begin{array}{c}
Y
\end{array}
\quad
\begin{array}{c}
i
\end{array}

$$

The map $f'i = 1_{Y'}$, and since $Y' \rightarrow X'$ is a normal monomorphism, so is $\partial I \square X' \cup I \square Y' \rightarrow I \square X'$. Therefore, there is a lift

$$
\begin{array}{c}
\partial I \square X' \cup I \square Y' \\
\downarrow
\end{array}
\quad
\begin{array}{c}
X'
\end{array}
\quad
\begin{array}{c}
(\partial f'i, 1_{Y'}) \cup i
\end{array}
\quad
\begin{array}{c}
X
\end{array}
\quad
\begin{array}{c}
l_i
\end{array}

$$

We have therefore constructed a homotopy from $lf'$ to $1_{X'}$. Therefore the normalisation $i$ of $f$ is a homotopy equivalence, and therefore induces an equivalence of Kan complexes. \qed

3.2.4 Augmented Trivial Cofibrations

Definition 3.32. A trivial cofibration of $\Psi$-sets is a cofibration which is also a weak equivalence.

Lemma 3.33. A pushout of a trivial cofibration is a trivial cofibration

Proof. Let $f: A \rightarrow B$ be a trivial cofibration and let

$$
\begin{array}{c}
A \\
\downarrow
\end{array}
\quad
\begin{array}{c}
C \\
\downarrow
\end{array}
\quad
\begin{array}{c}
B \\
\downarrow
\end{array}
\quad
\begin{array}{c}
D
\end{array}

$$

be a pushout square. We have that normal monomorphisms are closed under pushouts, therefore $C \rightarrow D$ is a normal monomorphism. We need only show that it is also a weak
equivalence. Assume that $A$ and $B$ are normal. For a $\Psi$-Kan complex $Z$, we have an induced pullback square

$$
\begin{array}{ccc}
\text{hom}^\Delta(D, Z) & \rightarrow & \text{hom}^\Delta(B, Z) \\
\downarrow & & \downarrow \\
\text{hom}^\Delta(C, Z) & \rightarrow & \text{hom}^\Delta(A, Z)
\end{array}
$$

The right side vertical map is a trivial fibration (of simplicial sets) due to the assumption of $A \rightarrow B$ being a trivial cofibration between normal objects. Trivial fibrations are closed under pullbacks and therefore the left vertical map is also a trivial fibration. Therefore we have shown that $C \rightarrow D$ is a trivial cofibration for the case of $A$ and $B$ normal. Now we assume that $A$ and $B$ are not normal. The following method is called the \textit{cube argument} ([10, Lemma 5.3.2]). Let $D' \rightarrow D$ be a normalisation of $D$ and consider the commutative diagram

\[
\begin{array}{ccc}
A' & \rightarrow & C' \\
& \searrow & \downarrow \\
& B' & \rightarrow & D' \\
\downarrow & & & \downarrow \\
A & \rightarrow & C \\
& \searrow & \downarrow \\
& B & \rightarrow & D 
\end{array}
\]

such that the vertical squares are pullbacks. As we are working in a presheaf category, the pullback of a monomorphism is also a monomorphism. We have that $D'$ is normal, and therefore by Corollary 3.19 we have that the maps $A' \rightarrow B'$ and $C' \rightarrow D'$ are normal monomorphisms and as such, all vertical maps are normalisations. The top square is a pushout as pullbacks preserve pushouts in any presheaf category. Therefore $A' \rightarrow B'$ is a trivial cofibration between normal objects, and we have already shown that $C' \rightarrow D'$ is also a trivial cofibration. Therefore we have $C \rightarrow D$ has a normalisation which is a stable weak equivalence, and is therefore itself a stable weak equivalence. \hfill \square

\textbf{Lemma 3.34.} \textit{Anodyne extensions are trivial cofibrations.}

\textit{Proof.} We need only show that every horn inclusion is a weak equivalence. Let $\Lambda^f \Psi[\psi] \rightarrow \Psi[\psi]$ be such a horn inclusion and $\partial \Delta[n] \rightarrow \Delta[n]$ simplicial boundary inclusion. We have by (AC3) that the map

$$
\partial \Delta[n] \square \Psi[\psi] \sqcup \Delta[n] \square \Lambda^f \Psi[\psi] \rightarrow \Delta[n] \square \Psi[\psi]
$$
is an anodyne extension, so every $\Psi$-Kan complex $Z$ has the RLP with respect to it. Therefore the map $\text{hom}^\Delta(\Psi[\psi], Z) \to \text{hom}^\Delta(\Lambda^f\Psi[\psi], Z)$ is a trivial fibration of simplicial sets under the assumption that $Z$ is Kan. Therefore $\Lambda^f\Psi[\psi] \to \Psi[\psi]$ is a weak equivalence and the result is proven. □

**Lemma 3.35.** Every trivial cofibration is a retract of a pushout of a trivial cofibration between normal objects.

**Proof.** Let $u: A \to B$ be a trivial cofibration, and $A' \to A$ a normalisation of $A$. We consider the following commutative diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{u'} & B' \\
\downarrow & & \downarrow \\
A & \xrightarrow{u} & B
\end{array}
\]

constructed by factoring $A' \to B$ as a normal monomorphism $A' \to B'$ followed by a normalisation of $B$. As normalisations are weak equivalences, and weak equivalences satisfy the two out of three property, (as the weak equivalences in $\hat{\Delta}_{\text{Kan}}$ do so), we have that $A' \to B'$ is a trivial cofibration between normal objects. We now consider the pushout

\[
\begin{array}{ccc}
A' & \xrightarrow{u} & B' \\
\downarrow & & \downarrow \\
A & \xrightarrow{v} & P
\end{array}
\]

which provides a map $s: P \to B$. We need to show that $s$ has the RLP with respect to the normal monomorphisms, as this would ensure that $u$ is a retract of $v$ via the lifting

\[
\begin{array}{ccc}
A & \xrightarrow{v} & P \\
\uparrow_{u} & & \uparrow_{s} \\
B & \xrightarrow{\sim} & B
\end{array}
\]

Therefore, we consider the lifting problem

\[
\begin{array}{ccc}
\partial\Psi[\psi] & \longrightarrow & P \\
\downarrow & & \downarrow_{s} \\
\Psi[\psi] & \longrightarrow & B
\end{array}
\]
Using the cube argument again, we pullback along \( \partial \Psi[\psi] \to P \) to form the cube:

\[
\begin{array}{ccc}
E & \to & D \\
\downarrow & & \downarrow \\
C & \to & \partial \Psi[\psi] \\
\downarrow & & \downarrow \\
A' & \to & B' \\
\downarrow & & \downarrow \\
A & \to & P \\
\end{array}
\]

where the horizontal faces are pushouts and the vertical faces are pullbacks. We have that \( E \to C \) is a normalisation, and therefore all the objects in the top face are normal. Therefore \( E \to C \) has a section and therefore so does the pushout \( D \to \partial \Psi[\psi] \). Using this section, we are able to form a commutative diagram

\[
\begin{array}{ccc}
\partial \Psi[\psi] & \to & D \\
\downarrow & & \downarrow \\
\Psi[\psi] & \to & B \\
\end{array}
\]

in which the lift exists. This therefore gives a solution to the required lifting problem.

We say that a \( \Psi \)-set \( X \) is countable if each \( X(\psi) \) is a countable set.

**Lemma 3.36** ([10, Proposition 5.3.8]). *The class of trivial cofibrations is generated by the trivial cofibrations between countable and normal objects.*

### 3.2.5 The Model Structure

**Definition 3.37.** A morphism in \( \hat{\Psi} \) is a fibration if it has the RLP with respect to the trivial cofibrations.

**Theorem 3.38.** There is a cofibrantly generated model structure on \( \hat{\Psi} \) with the defined class of weak equivalences (Definition 3.29), fibrations (Definition 3.37) and cofibrations (Definition 3.11). We will denote this model structure \( \hat{\Psi}_{\text{Kan}} \).

**Proof.** We will show that the axioms (CM1)-(CM5) hold. First of all, (CM1) holds automatically as \( \hat{\Psi} \) is a presheaf category. (CM2) holds from the definition of weak equivalences and the fact that the weak equivalences in the Quillen model on \( \hat{\Delta} \) satisfy this property. Axiom (CM3) also holds without much concern. The first non-trivial axiom to show is (CM5). The fact that every map can be factored as a cofibration followed...
by a trivial fibration follows from the small object argument for the set of all boundary inclusions and Lemma 3.31. The factorisation as a fibration followed by a trivial cofibration follows from the small object argument for the set of all trivial cofibrations between normal countable Ψ-sets by Lemma 3.36.

One half of (CM4) holds from the definition of the fibrations (Definition 3.37). Assume we have the following commutative diagram:

\[
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow{i} & & \downarrow{p} \\
B & \rightarrow & Y
\end{array}
\]

for \(i\) a cofibration and \(p\) a trivial fibration. We need to produce a lift \(B \rightarrow X\). We factor \(p: X \rightarrow Y\) as a cofibration \(Y \rightarrow Z\) followed by a map \(Z \rightarrow X\) having the RLP with respect to all cofibrations. Then \(Z \rightarrow X\) is a weak equivalence and by two out of three, so is \(Y \rightarrow Z\). We first find the lift in

\[
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow{i} & & \downarrow{p} \\
B & \rightarrow & Y
\end{array}
\]

and then in

\[
\begin{array}{ccc}
X & \rightarrow & X \\
\downarrow{p} & & \downarrow{p} \\
Z & \rightarrow & Y
\end{array}
\]

The composition of the two lifts gives us the necessary lift \(B \rightarrow X\). The generating cofibrations are given by the boundary inclusions and the set of generating trivial cofibrations are the trivial cofibrations between normal and countable Ψ-sets.

Proposition 3.39. The fibrant objects in \(\widehat{\Psi}_{\text{Kan}}\) are the Ψ-Kan complexes.

Proof. Let \(Z\) be fibrant in \(\widehat{\Psi}_{\text{Kan}}\), by Lemma 3.34 we have that the anodyne extensions are exactly the trivial cofibrations, therefore \(Z\) has the RLP with respect to the anodyne extensions and is therefore Ψ-Kan. Conversely, let \(Z\) be Ψ-Kan, and \(A \rightarrow B\) a trivial cofibration between normal objects, then the map \(\text{hom}^\Delta(B, Z) \rightarrow \text{hom}^\Delta(A, Z)\) is a trivial fibration of simplicial sets. We have that the trivial fibrations in \(\Delta_{\text{Kan}}\) are surjective on vertices, and we can deduce that \(Z\) has the RLP with respect to the map \(A \rightarrow B\). Lemma 3.35 then implies that every Ψ-Kan complex has the RLP with respect to all trivial cofibrations.
3.2.6 Properties of $\text{Ho}(\hat{\Psi}_{\text{Kan}})$

In this section we briefly list some of the required properties of $\hat{\Psi}_{\text{Kan}}$. The first property that we prove is left properness which will be essential when considering augmented presheaves.

**Proposition 3.40.** $\hat{\Psi}_{\text{Kan}}$ is left proper.

**Proof.** Consider the following pushout

$$
\begin{array}{ccc}
A & \rightarrow & C \\
\downarrow & & \downarrow \\
B & \rightarrow & D
\end{array}
$$

with $A \rightarrow B$ a weak equivalence and $A \rightarrow C$ a cofibration. We can reduce to the case where all objects are normal via the cube argument. We have an induced diagram

$$
\begin{array}{ccc}
\text{hom}^\Delta(D,Z) & \rightarrow & \text{hom}^\Delta(B,Z) \\
\downarrow & & \downarrow \\
\text{hom}^\Delta(C,Z) & \rightarrow & \text{hom}^\Delta(A,Z)
\end{array}
$$

which is a pullback for any $Z \in \hat{\Psi}$. If $Z$ were a $\Psi$-Kan complex, then all simplicial sets in the above diagram are also Kan complexes and the right vertical map is an equivalence of simplicial sets. The left vertical map is an equivalence. This follows as trivial fibrations are stable under pullbacks, and so by Ken Brown’s Lemma [22], all weak equivalences between fibrant objects are stable under pullback.

The following lemma holds from the construction of compatible Kan complex objects.

**Lemma 3.41.** There is a Quillen adjunction

$$i_! : \hat{\Delta}_{\text{Kan}} \leftrightarrows \hat{\Psi}_{\text{Kan}} : i^*$$

**Proof.** The right adjoint $i^*$ sends fibrations (resp., trivial fibrations) of $\hat{\Psi}_{\text{Kan}}$ to fibrations (resp., trivial fibrations) of $\hat{\Delta}_{\text{Kan}}$ by construction, and therefore is part of a Quillen pair, for which $i_!$ is the left adjoint.

Recall that we do not always have the assumption that $\hat{\Psi}_{\text{Kan}}$ satisfies the pushout-product axiom, therefore we cannot hope for a closed monoidal structure in full generality. However, what does hold is the fact that $\text{Ho}(\hat{\Psi}_{\text{Kan}})$ is a simplicial category. Using the tools presented in [10, Proposition 5.4.4] and [52, Lemma 3.8.1] we can formally prove the
following, although the result is non-surprising due to the way we have constructed the classes of maps in our model structure.

**Lemma 3.42.** The category $\text{Ho}(\hat{\Psi}_{\text{Kan}})$ is enriched over $\text{Ho}(\hat{\Delta}_{\text{Kan}})$.

**Remark 3.43.** If $\Psi$ is a strict EZ-category then the model $\hat{\Psi}_{\text{Kan}}$ is a Cisinski type model structure defined in [24]. This follows as for a strict EZ-category we have by Remark 3.16 that the normal monomorphisms are then just the monomorphisms.

### 3.3 Augmented Homotopical Algebraic Geometry

#### 3.3.1 Local Model Structure on Augmented Presheaves

We will now build a local model structure on $\Psi\text{-Pr}(\mathcal{C}) := \hat{\Psi}^{\text{op}}$ which reflects the local model structure on simplicial presheaves. First of all, we consider the projective point-wise model structure on $\Psi\text{-Pr}(\mathcal{C})$, again denoted $\Psi\text{-Pr}_{\text{proj}}(\mathcal{C})$. Note that as $\hat{\Psi}$ is left-proper and combinatorial (for it is accessible and cofibrantly generated), we have that the projective point-wise model is also left-proper and combinatorial.

We will now construct the analogue of hypercovers. In the simplicial case we introduced hypercovers using the boundary construction, which was done as we can simply edit the necessary components to get a class of augmented hypercovers.

**Definition 3.44.** Let $\Psi$ be an augmentation category and $(\mathcal{C}, \tau)$ a site. A map $f : X \to Y$ in $\Psi\text{-Pr}(\mathcal{C})$ is a *hypercovering* if for all $\psi \in \Psi$ the map

$$X_\psi \to \text{Hom}_{\hat{\Psi}}(\partial \Psi[\psi], X) \times_{\text{Hom}_{\hat{\Psi}}(\partial \Psi[\psi], Y)} Y_\psi$$

is a $\tau$-covering in $\text{Ho}(\Psi\text{-Pr}_{\text{proj}}(\mathcal{C}))$. Note that by using Lemma 3.10, this is equivalent to asking the same condition for the class of maps

$$X_\psi \to (\cosk_{d(\psi) - 1} X)_r \times_{(\cosk_{d(\psi) - 1} X)} Y_\psi$$

**Definition 3.45.** The *local model structure* on $\Psi\text{-Pr}(\mathcal{C})$ is the left Bousfield localisation of $\Psi\text{-Pr}_{\text{proj}}(\mathcal{C})$ at the class of hypercoverings. We shall denote this model structure $\Psi\text{-Pr}_\tau(\mathcal{C})$.

**Definition 3.46.** For a site $(\mathcal{C}, \tau)$, we will call the homotopy category $\text{Ho} (\Psi\text{-Pr}_\tau(\mathcal{C}))$ the category of $\Psi$-*augmented stacks* on $\mathcal{C}$. We could just as well consider this homotopy category to be the category of augmented sheaves.

**Lemma 3.47.** For $(\mathcal{C}, \tau)$ a site, there is a Quillen adjunction

$$i_! : s\text{Pr}_\tau(\mathcal{C}) \rightleftarrows \Psi\text{-Pr}_\tau(\mathcal{C}) : i^*$$
Proof. We will prove the statement for the respective injective local models and then we can compose with the identity functor to the projective local case, which will prove the result. In the local injective models, the cofibrations are the point-wise cofibrations. By Lemma 3.41, we know that $i_!$ sends cofibrations of $\hat{\Delta}_{\text{Kan}}$ to cofibrations of $\hat{\Psi}_{\text{Kan}}$. Therefore, we now need only show that $i_!$ preserves the trivial cofibrations now. This follows as if $f: X \to Y$ is a (non-augmented) hypercover, then $i_!f: i_!X \to i_!Y$ is an augmented hypercover. □

Remark 3.48. We have introduced the above theory for the case when $\mathcal{C}$ is a simplicial site. However for $\Psi$-presheaves, it would also make sense to allow $\Psi$-sites. That is, categories $\mathcal{C}$ enriched over $\hat{\Psi}$ such that there is a Grothendieck topology on $\pi_0(\mathcal{C})$.

3.3.2 Local Weak Equivalences

We now describe what the weak equivalences in the local model structure should look like. We introduce the concept of augmented homotopy groups, resembling the simplicial case, using the augmented homotopy of Section 3.2.3.

Definition 3.49. Let $X \in \hat{\Psi}$ be fibrant in $\hat{\Psi}_{\text{Kan}}$, and $\psi \in \Psi$. Denote by $\pi_\psi(X, x_0)$ the set of equivalence classes of morphisms $\alpha: \Psi[\psi] \to X$ which fit into the following commutative diagram in $\hat{\Psi}$:

$$
\begin{array}{ccc}
\partial \Psi[\psi] & \longrightarrow & \Psi[0] \\
\downarrow & & \downarrow x_0 \\
\Psi[\psi] & \alpha \longrightarrow & X
\end{array}
$$

where the equivalence relation is given by homotopy equivalence of Definition 3.28.

Remark 3.50. Although we will not pursue it here, one would hope that the object $\pi_\psi(X, x_0)$ is a group which is abelian for $d(\psi) \geq 2$.

Definition 3.51. Let $(\mathcal{C}, \tau)$ be a site. A map $f: \mathcal{F} \to \mathcal{F}'$ in $\Psi\text{-Pr}(\mathcal{C})$ is a local weak equivalence if:

- The induced map $\pi_0^\tau \mathcal{F} \to \pi_0^\tau \mathcal{F}'$ is an isomorphism of sheaves.
- Squares of following form are pullbacks after sheafification:

$$
\begin{array}{ccc}
\pi_\psi \mathcal{F} & \longrightarrow & \pi_\psi \mathcal{F}' \\
\downarrow & & \downarrow \\
\mathcal{F}_0 & \longrightarrow & \mathcal{F}'_0
\end{array}
$$
Conjecture 3.52. A map \( f : \mathcal{F} \to \mathcal{F}' \) is a local weak equivalence if and only if it is a hypercover.

If the above conjecture is true, which seems likely due to the combinatorial nature of augmentation categories in relation to the simplex category, then we would get the following result, mirroring Theorem 2.43.

Corollary 3.53. Let \((\mathcal{C}, \tau)\) be a site. There exists a cofibrantly generated model structure on the category \( \Psi\text{-Pr}(\mathcal{C}) \) where a map \( f : \mathcal{F} \to \mathcal{F}' \) is a:

1. Weak equivalence if it is a local weak equivalence.
2. Fibration if it has the RLP with respect to the trivial cofibrations.
3. Cofibration if it is a cofibration in the point-wise projective model.

Moreover this model is Quillen equivalent to the local model \( \Psi\text{-Pr}_\tau(\mathcal{C}) \).

3.3.3 Enriched Structure

In this section we will show that the local model structure \( \Psi\text{-Pr}_\tau(\mathcal{C}) \) is enriched over the local model structure on simplicial presheaves, and is therefore a simplicial category. In the case that \( \hat{\Psi} \) is a closed monoidal category we can go further and show that \( \Psi\text{-Pr}_\tau(\mathcal{C}) \) is in fact a closed monoidal model category. To do this, we will use a trick used in [106, §3.6], which considers instead the local injective model structure. Denote by \( \Psi\text{-Pr}_{\text{inj}}(\mathcal{C}) \) the injective model structure. From the properties of \( \hat{\Psi}_{\text{Kan}} \) we have that this model is left proper and cofibrantly generated, and therefore a left Bousfield localisation exists. We localise at the set of hypercovers once again and retrieve the model category \( \Psi\text{-Pr}_{\text{inj}, \tau}(\mathcal{C}) \). Clearly we have an equivalence of categories \( \text{Ho}(\Psi\text{-Pr}_{\text{inj}, \tau}(\mathcal{C})) \simeq \text{Ho}(\Psi\text{-Pr}_\tau(\mathcal{C})) \) as the weak equivalences in each model are the same. As the cofibrations in the local injective model are simply the normal monomorphisms, and finite products preserve local weak equivalences, anything we have proved regarding the enriched structure of \( \hat{\Psi}_{\text{Kan}} \) can be carried over to this setting. Using this justification, we state the two following lemmas and corresponding definitions.

Lemma 3.54. The category \( \text{Ho}(\Psi\text{-Pr}_\tau(\mathcal{C})) \) is enriched over \( \text{Ho}(s\text{Pr}_\tau(\mathcal{C})) \), and subsequently over \( \text{Ho}(\hat{\Delta}_{\text{Kan}}) \). In particular, \( \Psi\text{-Pr}_\tau(\mathcal{C}) \) is a simplicial model category.

Definition 3.55. The simplicial presheaf enrichment in \( \text{Ho}(\Psi\text{-Pr}_\tau(\mathcal{C})) \) will be denoted

\[
\Psi\text{-}\rightharpoonup_{\tau} \text{Hom}(\cdot, \cdot) : \text{Ho}(\Psi\text{-Pr}_\tau(\mathcal{C})) \times \text{Ho}(\Psi\text{-Pr}_\tau(\mathcal{C})) \to \text{Ho}(s\text{Pr}_\tau(\mathcal{C}))
\]
The corresponding simplicial enrichment will be denoted

\[ \Psi^-\mathcal{R}^\Delta \text{Hom}(-, -): \text{Ho}(\Psi^-\text{Pr}_\tau(\mathcal{E})) \times \text{Ho}(\Psi^-\text{Pr}_\tau(\mathcal{E})) \to \text{Ho}(\hat{\Delta}_{\text{Kan}}) \]

**Lemma 3.56.** If \( \hat{\Psi} \) satisfies the pushout-product axiom, then the category \( \text{Ho}(\Psi^-\text{Pr}_\tau(\mathcal{E})) \) is a closed monoidal category, and is subsequently enriched over \( \text{Ho}(\hat{\Psi}_{\text{Kan}}) \). In particular, \( \Psi^-\text{Pr}_\tau(\mathcal{E}) \) is a \( \Psi \)-model category.

**Definition 3.57.** The internal-hom in \( \text{Ho}(\Psi^-\text{Pr}_\tau(\mathcal{E})) \) will be denoted

\[ \Psi^-\mathcal{R}_\tau \text{Hom}(-, -): \text{Ho}(\Psi^-\text{Pr}_\tau(\mathcal{E})) \times \text{Ho}(\Psi^-\text{Pr}_\tau(\mathcal{E})) \to \text{Ho}(\Psi^-\text{Pr}_\tau(\mathcal{E})) \]

The corresponding augmented enrichment will be denoted

\[ \Psi^-\mathcal{R}_\tau \text{Hom}(-, -): \text{Ho}(\Psi^-\text{Pr}_\tau(\mathcal{E})) \times \text{Ho}(\Psi^-\text{Pr}_\tau(\mathcal{E})) \to \text{Ho}(\hat{\Psi}_{\text{Kan}}) \]

### 3.3.4 Augmented Derived Stacks

We shall now use the model structures developed in Section 3.3.1 to discuss the theory of augmented stacks, which will be the main objects of interest in this thesis. Following the simplicial setting, we make the following definition.

**Definition 3.58.** Let \( (\mathcal{E}, \tau) \) be a site, and \( \Psi \) an augmentation category.

- The category \( \text{Ho}(\Psi^-\text{Pr}_\tau(\mathcal{E})) \) will be called the category of \( \Psi \)-augmented \( \infty \)-stacks.
- An object \( F \in \text{Ho}(\Psi^-\text{Pr}_\tau(\mathcal{E})) \) will be referred to as a \( \Psi \)-augmented \( \infty \)-stack.
- For two stacks \( F, G \in \text{Ho}(\Psi^-\text{Pr}_\tau(\mathcal{E})) \), we will call the object \( \Psi^-\mathcal{R}_\tau \text{Hom}(-, -) \) (resp., \( \Psi^-\mathcal{R}_\tau \text{Hom}(-, -) \)) the \( \Psi \)-mapping stack (resp., mapping stack) from \( F \) to \( G \). Note that the \( \Psi \)-mapping stack does not exist in full generality.

### 3.3.5 Augmented Geometric Derived Stacks

We now introduce augmented derived geometric stacks through a modified \( n \)-hypergroupoid construction. Of course, all that we do in this setting can be reformulated for when the site in question is not \( \text{dAff} \), but any homotopical algebraic geometric context.

**Definition 3.59.** A derived Artin (resp., derived Deligne-Mumford) \( (\Psi, n) \)-hypergroupoid is an object \( X \in \text{dAff}^{\Psi^n} \) such that the maps

\[ X_\psi = \text{Hom}_\Psi(\Psi[\psi], X) \to \text{Hom}_\Psi(\Lambda^f \Psi[\psi], X) \]
are smooth (resp., étale) surjections for all objects $\psi$ and face maps $f$, and are isomorphisms for all $\psi$ with $d(\psi) > n$. Note that a derived $(\Psi, n)$-hypergroupoid is an augmented $\Psi$-Kan complex. Moreover, using Lemma 3.10 we see that a $(\Psi, n)$-hypergroupoid is $(n + 1)$-coskeletal.

**Definition 3.60.** A derived Artin (resp., derived Deligne-Mumford) trivial $(\Psi, n)$-hypergroupoid is a map $f: X \to Y$ in $\text{dAff}^{\Psi^{\text{op}}}$ such that the maps

$$X_\psi \to \text{Hom}_{\hat{\Psi}}(\partial \Psi[\psi], X) \times_{\text{Hom}_{\hat{\Psi}}(\partial \Psi[\psi], Y)} Y_\psi$$

are smooth (resp., étale) surjections for all $\psi, f$ and are isomorphisms for all $\psi$ with $d(\psi) > n$. In particular, a derived Artin trivial $(\Psi, n)$-hypergroupoid is a trivial fibration in $\hat{\Psi}_{\text{Kan}}$.

**Lemma 3.61.** Let $f: X \to Y$ be a trivial $(\Psi, n)$-hypergroupoid then $X = Y \times_{\text{cosk}_{n-1} Y} \text{cosk}_{n-1} X$.

**Proof.** This follows from comparing the definition to the general result about augmented coskeletal objects in Lemma 3.10. Note that this shows that $(\Psi, n)$-hypergroupoids can be seen as $n$-truncated $\Psi$-hypercovers.

**Definition 3.62.** A model for the $\infty$-category of strongly quasi-compact $(\Psi, n)$-geometric derived Artin (resp., Deligne-Mumford) stacks is given by the relative category consisting of the derived Artin (resp., Deligne-Mumford) $(\Psi, n)$-hypergroupoids and the class of derived trivial Artin (resp., Deligne-Mumford) $(\Psi, n)$-hypergroupoids. We will denote the homotopy category as

$$\Psi-G_{n}^\text{sm}(\text{dAff}) \quad \text{(resp., } \Psi-G_{n}^\text{ét}(\text{dAff}) \text{)}$$

**Remark 3.63.** We could have also (equivalently) formulated the theory of $(\Psi, n)$-geometric derived Artin or Deligne-Mumford stacks by using a representability criteria. However, using the hypergroupoid construction highlights the beauty of the construction with the intertwining of the hypercovering conditions.

We have listed the following as a conjecture, as relative categories do not have the notion of Quillen equivalence, and therefore we can only talk about the derived adjunction that may exist between the homotopy categories. To prove the following conjecture, one would need to construct the full Quillen model structure, and prove that there is a Quillen adjunction. It could also be the case that this result does not hold in all cases, but only for a subclass of augmentation categories.
Conjecture 3.64. Let $\Psi$ be an augmentation category, then there is an adjunction

$$\mathbb{L}i_! : \mathcal{G}_n^{sm}(d\text{Aff}) \rightleftarrows \Psi \cdot \mathcal{G}_n^{sm}(d\text{Aff}) : R^i$$

resp., $L_i : \mathcal{G}_n^{d}(d\text{Aff}) \rightleftarrows \Psi \cdot \mathcal{G}_n^{d}(d\text{Aff}) : R^i$)

We finish this section by outlining a potential direction for future research of $(\psi, n)$-hypergroupoids. One of the key tools used in derived algebraic geometry is that of quasi-coherent complexes as defined in [69, §5.2], of particular interest is the cotangent complex which controls infinitesimal deformations. In [85] an alternative, homotopy equivalent definition of quasi-coherent complexes was given using $n$-hypergroupoids, giving a cosimplicial object. This definition would be adjustable to the augmented setting, giving rise to a particular coaugmented object. The question is then what type of deformation would such an augmented cotangent complex control?
Chapter 4

Equivariant Homotopical Algebraic Geometry

In this chapter we will look at crossed simplicial groups, of which the cyclic category of Connes is one of the most well known examples. We will prove that all crossed simplicial groups are augmentation categories, and therefore give the existence of the Kan model structure. We will explore the homotopy type of this model structure and see that it has an equivariant flavour. We will use this to discuss equivariant cohomology theories and equivariant local systems, the latter of which are equivariant 1-geometric derived stacks.

Outline of Chapter 4

(Section 4.1) We introduce crossed simplicial groups and explore some of their properties. In particular, we prove that all crossed simplicial groups are augmentation categories, and therefore the machinery developed in Chapter 3 can be applied.

(Section 4.2) Using the idea of cohomology theories as derived mapping spaces, we construct equivariant Eilenberg-MacLane spaces to define equivariant cohomology theories. As a specific example we show that we can retrieve the notion of $SO(2)$-equivariant cohomology using a chain of Quillen equivalences.

(Section 4.3) We introduce the geometric derived stack of local systems. We then adjust the machinery for this stack to the equivariant setting and calculate the $SO(2)$-equivariant stack of local systems on the Hopf sphere ($S^3$ with an action of $SO(2)$).
4.1 Crossed Simplicial Groups

In this section we will look at the theory of crossed simplicial groups, which are an extension of the simplex category. We will begin by looking at Connes’ cyclic category which is the prototypical example of a crossed simplicial group, and we will discuss cyclic sets. Cyclic sets are an object of interest in their own right, they were first introduced as a gadget involved in the definition of cyclic homology in [27], making up a core of invariants in the theory of non-commutative geometry (see [19, 28]).

4.1.1 Cyclic Sets

A motto for cyclic sets could be “simplicial sets with a circle action”. Just as simplicial theory allows for a discrete, combinatorial model of topological spaces, cyclic theory is a discrete, combinatorial model of topological spaces with an action of $SO(2)$.

4.1.1.1 Connes’ Cyclic Category

There have been various formulations of Connes’ cyclic category, some of which are discussed in [28, Appendix A.β]. The first that we will consider is a representation in generators and relations (reflecting our definition of $\Delta$).

**Definition 4.1.** The cyclic category $\Delta \mathcal{C}$ has for objects $[n] \in \Delta$ and morphisms generated by the relations of the category $\Delta$ along with the additional generator $\tau_{n} : [n] \rightarrow [n]$ subject to the following relations:

\[
\begin{align*}
\tau_{n} \delta_{i} &= \delta_{i-1} \tau_{n-1} \quad \text{for } 1 \leq i \leq n, \\
\tau_{n} \delta_{0} &= \delta_{n}, \\
\tau_{n} \sigma_{i} &= \sigma_{i-1} \tau_{n+1} \quad \text{for } 1 \leq i \leq n, \\
\tau_{n} \sigma_{0} &= \sigma_{n} \tau_{n+1}^{2}, \\
\tau_{n+1} &= \text{id}_{n}.
\end{align*}
\]

**Remark 4.2.** In the literature the cyclic category is usually denoted by $\Lambda$, however we use the notation $\Delta \mathcal{C}$ to keep consistent with the rest of the notation of crossed simplicial groups.

From Definition 4.1 we can observe that the object $[n]$ has an action of $C_{n+1}$, the cyclic group of order $n + 1$. Also we can see from the relations that any map $[m] \rightarrow [n]$ in $\Delta \mathcal{C}$ can be described as a power of $\tau_{m}$ followed by a series of face and degeneracy maps. This gives us an alternative description of $\Delta \mathcal{C}$. 

Definition 4.3. The cyclic category is the category $\Delta C$ equipped with an embedding $i: \Delta \to \Delta C$ such that:

1. The functor $i$ is bijective on objects (i.e., $\text{Ob}(\Delta C) = \text{Ob}(\Delta)$).

2. Any morphism $u: i[m] \to i[n]$ in $\Delta C$ can be uniquely written as $i(\phi) \circ g$ where $\phi: [m] \to [n]$ is a morphism in $\Delta$ and $g \in C_{n+1}$. We call this decomposition the canonical decomposition.

Example 4.4. Here we will give an example of what a morphism in $\Delta C$ looks like. It is natural to consider the object $[n]$ of $\Delta C$ as a circle with $(n + 1)$ nodes and directed arrows between them. Below is a morphism $f: [2] \to [8]$ adapted from an example in [74]:

![Figure 4.1](image_url)

The following proposition gives us an insight into how the cyclic category will act when we use it as a presheaf category. Recall from [80] that the classifying space of a category $\mathcal{C}$ is the geometric realisation of its nerve.

Proposition 4.5 ([68, Proposition 7.2.7]). The classifying space $B\Delta C$ is a classifying space of $SO(2)$.

4.1.1.2 Cyclic Objects

Definition 4.6. Let $\mathcal{C}$ be a category. A cyclic object in $\mathcal{C}$ is a functor $X: \Delta \mathcal{C}^{\text{op}} \to \mathcal{C}$. We will denote by $\Delta \mathcal{C} \mathcal{C}$ the category of all cyclic objects in a category $\mathcal{C}$. As usual, we will write $\hat{\Delta} \mathcal{C}$ for the category of cyclic sets.

Using the definition of $\Delta \mathcal{C}$ from Definition 4.1, we can easily describe cyclic objects in terms of generators and relations.
Lemma 4.7. For objects over the cyclic category, we have a simplicial object along with the additional generators $t_n : [n] \to [n]$ subject to the following relations:

\[
\begin{align*}
t_{n+1} & = id : [n] \to [n], \\
d_i t_n & = t_{n-1} d_{i-1} : [n] \to [n-1], \\
s_i t_n & = t_{n+1} s_{i-1} : [n] \to [n+1], \\
d_0 t_n & = d_n : [n] \to [n-1], \\
s_0 t_n & = t_{n+1}^2 s_n : [n] \to [n+1].
\end{align*}
\]

Our first example of a cyclic object in a category $\mathcal{C}$ will be given by a free functor construction. This free construction is the left adjoint to the induced forgetful functor $i^*: \hat{\Delta}\mathcal{C} \to \hat{\Delta}$.

Definition 4.8. Let $X_\bullet$ be a simplicial object, define a $\Delta \mathcal{C}$-object $i_!(X)$ whose $n$-simplices are given by $C_{n+1} \times X_n$ with the $C_{n+1}$ action defined to be left multiplication on $C_{n+1}$. The face and degeneracy maps are given by:

\[
\begin{align*}
d_i (g, x) & = (d_i(g), d_{g^{-1}(i)}(x)), \\
s_i (g, x) & = (s_i(g), s_{g^{-1}(i)}(x)).
\end{align*}
\]

Just as we have standard simplicial sets $\Delta[n]$, we also have standard $\Delta \mathcal{C}$-sets.

Definition 4.9. The standard $\Delta \mathcal{C}$-sets are given by:

\[
\Delta \mathcal{C}[n] = \text{Hom}_{(\Delta \mathcal{C})^\text{op}}([n], -).
\]

From the properties of adjoint functors, we see that $\Delta \mathcal{C}[n] = i_!(\Delta[n])$.

By the Yoneda lemma we see that the standard $\Delta \mathcal{C}$-set has the following universal property:

\[
\text{Hom}_{\Delta \mathcal{C}}(\Delta \mathcal{C}[n], X) \simeq \text{Hom}_{\hat{\Delta}}(\Delta[n], i^* X) \simeq X_n.
\]

4.1.1.3 Geometric Realisation of Cyclic Objects

The key point that we have been seeing about cyclic objects is that they are somehow simplicial objects with a circular action. Just as we have a realisation functor $|-| : \hat{\Delta} \to \text{Top}$ we have a cyclic realisation functor. We will denote by $\text{Top}^{SO(2)}$ the category of topological spaces with an $SO(2)$-action.
Proposition 4.10 ([34, Proposition 2.8]). There exists a cyclic realisation functor
\[ |-|_C : \widehat{\Delta}^C \rightarrow \text{Top}^{SO(2)} \]
such that the following diagram commutes up to a natural isomorphism.

\[
\begin{array}{ccc}
\text{Top}^{SO(2)} & \xrightarrow{|-|_C} & \text{Top} \\
\downarrow u & & \downarrow u \\
\widehat{\Delta}^C & \xrightarrow{|i^*|} & \text{Top} \\
\end{array}
\]

where \( u \) is the forgetful functor which forgets the circle action, and \(|i^*|\) is the normal realisation of the underlying simplicial set.

Also, analogous to the singular functor \( S(-) : \text{Top} \rightarrow \widehat{\Delta} \) which is the right adjoint to the realisation functor, we can define a right adjoint to the cyclic realisation functor.

Proposition 4.11 ([34, Theorem 4.2]). The cyclic realisation functor \(|-|_C\) has a right adjoint, the cyclic singular functor. Let \( X \in \text{Top}^{SO(2)} \) then \( S_C(X)_n = \text{Hom}(\Delta^C[n], X) \).

4.1.2 Crossed Simplicial Groups

Crossed simplicial groups are a generalisation of the simplex category \( \Delta \) to allow group actions, just like the cyclic category carries actions of the cyclic groups. They were introduced as tools for use in functor homology [68], but have recently seen other uses in the theory of structured surfaces [38]. We will begin by giving the basic definitions and properties of crossed simplicial groups before looking at some examples. Crossed simplicial groups are crossed groups in the sense of Definition 3.2.

Definition 4.12 ([43, Definition 1.1]). A crossed simplicial group is a category \( \Delta \mathcal{G} \) equipped with an embedding \( i : \Delta \rightarrow \Delta \mathcal{G} \) such that:

1. The functor \( i \) is bijective on objects.

2. Any morphism \( u : i[m] \rightarrow i[n] \) in \( \Delta \mathcal{G} \) can be uniquely written as \( i(\phi) \circ g \) where \( \phi : [m] \rightarrow [n] \) is a morphism in \( \Delta \) and \( g \) is an automorphism of \( [m] \) in \( \Delta \mathcal{G} \). We call this decomposition the canonical decomposition.

We will leave the notation of the functor \( i \) implicit, and just refer to objects of \( \Delta \mathcal{G} \) as \([n], n \geq 0\). To every crossed simplicial group \( \Delta \mathcal{G} \) we can assign a sequence of groups \( \mathcal{G}_n = \text{Aut}_{\Delta \mathcal{G}}([n]) \). The sequence of groups \( \mathcal{G}_n \) assembles in to a simplicial set, which we denote \( \mathcal{G} \).
Example 4.13. Any simplicial group is an example of a crossed simplicial group, with the obvious action of \( G_n \) on \( \text{Hom}_\Delta([m],[n]) \).

Example 4.14. The cyclic category \( \Delta \mathcal{C} \) as discussed in Section 4.1.1 is an example of a crossed simplicial group.

Before we consider more examples, we will now define the Weyl crossed simplicial group (referred to as the hyper-octahedral crossed simplicial group in [43]) which will lead to a classification theorem for crossed simplicial groups.

Definition 4.15 ([43, Theorem 3.3]). There is a crossed simplicial group \( \Delta \mathcal{W} \) called the Weyl crossed simplicial group where \( G_n = W_{n+1} = C_2 \wr S_{n+1} \), the Weyl group of the \( B_n \) root system, sometimes also referred to as the hyper-octahedral group [5].

The importance of \( \Delta \mathcal{W} \) is outlined in the following theorem.

Theorem 4.16. Let \( \Delta \mathcal{G} \) be a crossed simplicial group.

1. There is a canonical functor \( \pi: \Delta \mathcal{G} \to \Delta \mathcal{W} \).

2. For every \( n \geq 0 \), there is an induced short exact sequence of groups

\[
1 \to \mathcal{G}'_n \to \mathcal{G}_n \to \mathcal{G}''_n \to 1
\]

where \( \mathcal{G}'_n \) is the kernel and \( \mathcal{G}''_n \) is the image of the homomorphism \( \pi_n: \mathcal{G}_n \to W_{n+1} \).

3. The above short exact sequence assembles to a sequence of functors

\[
\Delta \mathcal{G}'_n \to \Delta \mathcal{G}_n \to \Delta \mathcal{G}''_n
\]

where \( \Delta \mathcal{G}'_n \) is a simplicial group and \( \Delta \mathcal{G}''_n \subset \Delta \mathcal{W} \) is a crossed simplicial subgroup of \( \Delta \mathcal{W} \) (i.e., is a crossed simplicial group and also a subcategory of \( \Delta \mathcal{W} \)).

Proof. Theorem 1.7 of [38] proves this theorem in this form, however the proof can also be found in [43] and [64].

As a consequence of Theorem 4.16, we see that the classification of crossed simplicial groups reduces to the classification of crossed simplicial subgroups of \( \Delta \mathcal{W} \). The following corollary gives these subgroups.

Corollary 4.17. For any crossed simplicial group \( \Delta \mathcal{G} \), there is a short exact sequence of functors

\[
\Delta \mathcal{G}'_n \to \Delta \mathcal{G}_n \to \Delta \mathcal{G}''_n
\]
such that $\Delta \mathcal{G}'$ is a simplicial group and $\Delta \mathcal{G}''$ is one of the following seven crossed simplicial groups:

- $\Delta$ - The trivial crossed simplicial group.
- $\Delta \mathcal{C}$ - The cyclic crossed simplicial group, $\mathcal{C}_n = C_{n+1}$.
- $\Delta \mathcal{S}$ - The symmetric crossed simplicial group, $\mathcal{S}_n = S_{n+1}$.
- $\Delta \mathcal{R}$ - The reflexive crossed simplicial group, $\mathcal{R}_n = C_2$.
- $\Delta \mathcal{D}$ - The dihedral crossed simplicial group, $\mathcal{D}_n = D_{n+1}$.
- $\Delta \mathcal{T}$ - The reflexosymmetric crossed simplicial group, $\mathcal{T}_n = T_{n+1} = C_2 \rtimes S_{n+1}$.
- $\Delta \mathcal{W}$ - The Weyl crossed simplicial group, $\mathcal{W}_n = W_{n+1} = C_2 \wr S_{n+1}$.

We will say that $\Delta \mathcal{G}$ is of type $\Delta \mathcal{G}''$.

**Definition 4.18.** We will call the above seven crossed simplicial groups the *simple crossed simplicial groups*. Note that these crossed simplicial groups have the following inclusion structure in their groups $\mathcal{G}_n$:

![Diagram of inclusion structure]

**Example 4.19.** One of the main examples of a crossed simplicial group which is not simple uses the braid groups. We denote by $B_n$ the braid group on $n$ braids. There is a surjection $\mu: B_n \to S_n$ which has kernel $P_n$ which is the pure braid group. The family of braid groups $\{B_{n+1}\}_{n\geq 0}$ is a crossed simplicial group $\Delta \mathcal{B}$ which is given by the extension via the classification theorem:

$$\Delta \mathcal{P} \to \Delta \mathcal{B} \to \Delta \mathcal{G}$$

where $\Delta \mathcal{P}$ is the simplicial group of pure braids. This object has seen use in the theory of braid representations, see for example [113].
Example 4.20. The last example that we will consider arises from the quaternion groups $Q_n$. These groups are dicyclic (an extension of a cyclic group by $C_2$), therefore we can see that there is a crossed simplicial group $\Delta \Omega = \{Q_{n+1}\}_{n \geq 0}$ due to the following extension:

$$\Delta \mathcal{R} \to \Delta \Omega \to \Delta \mathcal{D}.$$ 

The dihedral (resp., quaternionic) crossed simplicial groups are the object of study in dihedral (resp., quaternionic) homology as discussed in [67, 99].

4.1.2.1 Crossed Simplicial Group Objects

Definition 4.21. Let $\Delta \mathcal{G}$ be a crossed simplicial group and $\mathcal{C}$ be any category. A $\Delta \mathcal{G}$-object in $\mathcal{C}$ is defined to be a functor $X : (\Delta \mathcal{G})^{\text{op}} \to \mathcal{C}$. We shall denote such a functor as $X_\bullet$ with $X_n$ being the image of $[n]$. If $\lambda : [m] \to [n]$ is a morphism in $\Delta \mathcal{G}$ we shall denote by $\lambda^* : X_n \to X_m$ the associated morphism $X(\lambda)$. We shall denote the category of all such objects as $\Delta \mathcal{G} \cdot \mathcal{C}$. The category of set valued presheaves will be denoted $\hat{\Delta} \mathcal{G}$.

Sometimes it is more convenient to consider a $\Delta \mathcal{G}$-object as a simplicial object with some extra structure.

Proposition 4.22 ([43, Lemma 4.2]). A $\Delta \mathcal{G}$-object in a category $\mathcal{C}$ is equivalent to a simplicial object $X_\bullet$ in $\mathcal{C}$ with the following additional structure:

- Left group actions $\mathcal{G}_n \times X_n \to X_n$.
- Face relations $d_i(gx) = d_i(g) \left(d_{g^{-1}(i)}x\right)$.
- Degeneracy relations $s_i(gx) = s_i(g) \left(s_{g^{-1}(i)}x\right)$.

In particular a $\Delta \mathcal{G}$-map $f_\bullet : X_\bullet \to Y_\bullet$ is the same thing as a simplicial map such that each of the $f_n : X_n \to Y_n$ is $\mathcal{G}_n$-equivariant.

As a consequence of Proposition 4.22, we can give concrete combinatorial definitions of some crossed simplicial group objects, just as we did in the cyclic case.

Definition 4.23. For objects over the dihedral category $\Delta \mathcal{D}$, we have a simplicial object along with the additional generators $w_n, t_n : [n] \to [n]$ subject to the following relations:

$$w_n^2 = t_n^{n+1} = \text{id} : [n] \to [n]$$

$$(t_nw_n)^2 = \text{id} : [n] \to [n]$$

$$d_it_n = t_{n-1}d_{i-1} : [n] \to [n-1], \ s_it_n = t_{n+1}s_{i-1} : [n] \to [n+1] \text{ for } 1 \leq i \leq n$$
\[
d_{i}w_{n} = w_{n-1}d_{n-i} : [n] \to [n-1], \quad s_{i}\omega_{n} = w_{n+1}s_{n-i} : [n] \to [n+1] \text{ for } 1 \leq i \leq n
\]
\[
d_{0}t_{n} = d_{n} : [n] \to [n-1], \quad s_{0}t_{n} = t_{n+1}s_{n} : [n] \to [n+1] \text{ for } n \geq 1
\]

As in the cyclic case, the left adjoint \(i_{!}\) to \(i^{*} : \widehat{\Delta \mathfrak{G}} \to \widehat{\Delta}\) is the obvious free construction which can be explicitly written by changing the cyclic groups appearing in Definition 4.8 to \(\mathfrak{G}_{n}\). Finally, we note that \(\widehat{\Delta \mathfrak{G}}\) is a closed monoidal category with respect to the cartesian product on simplicial sets, this allows us to define the internal-hom as follows.

**Definition 4.24.** Let \(X_{\bullet}\) and \(Y_{\bullet}\) be \(\Delta \mathfrak{G}\)-sets. We define the function complex \(\text{hom}(X, Y)\) to be the \(\Delta \mathfrak{G}\)-set defined by \(\text{hom}(X, Y)_{n} = \text{Hom}_{\widehat{\Delta \mathfrak{G}}}(X \times \Delta \mathfrak{G}[n], Y)\).

### 4.1.2.2 \(\Delta \mathfrak{G}\) is an Augmentation Category

We now prove the main result about crossed simplicial groups, that they are, in fact, examples of augmentation categories in the sense of Definition 3.13.

**Theorem 4.25.** The category \(\Delta \mathfrak{G}\) is an augmentation category.

**Proof.**

(AC1) From Lemma 3.6 we see that \(\Delta \mathfrak{G}\) is an EZ-category. A tensor structure is given by just taking the cartesian product.

(AC2) The inclusion \(\Delta \hookrightarrow \Delta \mathfrak{G}\) and the compatibility of the cartesian structures follows by definition of crossed simplicial groups.

(AC3) In Lemma 4.32 below, we prove that the normal monomorphisms have the pushout-product property in full generality, which is a stronger result. We forgo the proof of (AC3) and instead refer to the proof of Lemma 4.32.

**Corollary 4.26.**

- There is a Quillen model structure on the category of \(\Delta \mathfrak{G}\)-sets, denoted \(\widehat{\Delta \mathfrak{G}}_{\text{Kan}}\), where the fibrant objects are the \(\Delta \mathfrak{G}\)-Kan complexes, and the cofibrations are the normal monomorphisms. Moreover there is a Quillen adjunction

\[
\begin{align*}
i_{!} : \widehat{\Delta}_{\text{Kan}} & \rightleftarrows \widehat{\Delta \mathfrak{G}}_{\text{Kan}} : i^{*}
\end{align*}
\]
• There is a local model structure on the category of $\Delta G$-presheaves, denoted $\Delta G$-$Pr_r(C)$ obtained as the left Bousfield localisation of the point-wise Kan model at the class of $\Delta G$-hypercovers. Moreover there is a Quillen adjunction

$$i^*_! : sPr_r(C) \leftrightarrows \Delta G$-$Pr_r(C) : i^*$$

• The category of $(\Delta G,n)$-geometric derived Artin stacks, denoted $\Delta G$-$G_{n}^{sm}(dAff)$, is given as the homotopy category of derived $(\Delta G,n)$-hypergroupoids with respect to the class of trivial $(\Delta G,n)$-hypergroupoids.

**Definition 4.27.** The augmented homotopical algebraic geometry theory arising from $\Delta G$ will be referred to as $G$-equivariant. For example, the category $\text{Ho}(\Delta G$-$Pr_r(C))$ will be referred to as the category of $G$-equivariant $\infty$-stacks.

### 4.1.2.3 Properties of $\text{Ho}(\widehat{\Delta G}_{\text{Kan}})$

We will now explore the Kan model structure on $\widehat{\Delta G}$, which will lead to the justification of using the term $G$-equivariant to describe the augmentation. We shall mainly fix our attention to the cyclic category $\Delta C$, however, the homotopy of arbitrary crossed simplicial groups is discussed in Appendix A.

The Kan model structure on $\widehat{\Delta C}$ was in fact the first model structure to be developed for cyclic sets by Dwyer, Hopkins and Kan [34].

**Proposition 4.28 ([34, Theorem 3.1]).** The category $\widehat{\Delta C}$ has a cofibrantly generated model structure where a map $f : X \to Y$ is a:

- Weak equivalence if $i^*(f) : i^*(X) \to i^*(Y)$ is a weak equivalence in $\widehat{\Delta}_{\text{Kan}}$.
- Fibration if $i^*(f) : i^*(X) \to i^*(Y)$ is a fibration in $\widehat{\Delta}_{\text{Kan}}$.
- Cofibration if it has the LLP with respect to the trivial fibrations.

We shall denote this model $\widehat{\Delta C}_{\text{DHK}}$.

By comparison of the structure of the fibrations [34, Proposition 3.2] and cofibrations [34, Proposition 3.5], we see that the model structure of Proposition 4.28 is exactly that of $\widehat{\Delta C}_{\text{Kan}}$ (recall that the fibrations and cofibrations are enough to uniquely define a model structure).

**Corollary 4.29.** There is an equivalence $\widehat{\Delta C}_{\text{Kan}} \simeq \widehat{\Delta C}_{\text{DHK}}$. 
Recall from Proposition 4.10 that there is a cyclic realisation functor \(|-|_\mathcal{C}: \hat{\Delta}\mathcal{C} \to \text{Top}^{SO(2)}\) along with its right adjoint \(S_\mathcal{C}(-)\). We can now use the theory of [34] to describe the homotopy type of \(\hat{\Delta}\mathcal{C}_{\text{Kan}}\).

**Proposition 4.30 ([35, Theorem 2.2]).** There is a model structure on \(\text{Top}^{SO(2)}\) where a map \(f: X \to Y\) is a:

- Weak equivalence if the underlying map of topological spaces is a weak equivalence in \(\text{Top}\).
- Fibration if the underlying map of topological spaces is a fibration in \(\text{Top}\).
- Cofibration if it has the LLP with respect to the trivial fibrations.

**Proposition 4.31 ([34, Corollary 4.3]).** There is a Quillen equivalence

\[
\hat{\Delta}\mathcal{C}_{\text{Kan}} \rightleftarrows \text{Top}^{SO(2)}
\]

with the equivalence furnished by the cyclic realisation and singular functors.

Using the fact that a cyclic object is the same as a simplicial objects along with extra datum (Lemma 4.7), we see that a cyclic \(\infty\)-stack can be viewed as an \(\infty\)-stack with extra datum. In light of Proposition 4.31, we therefore see that objects of \(\text{Ho}(\Delta\mathcal{C}\text{-Pr}_{\tau}(\mathcal{C}))\) can be viewed as \(\infty\)-stacks along with an \(SO(2)\)-action. Hence the terminology of equivariant.

One important property of the Kan model structure on crossed simplicial groups is that the pushout-product axiom holds, as a consequence we have that \(\Delta\mathcal{C}\text{-Pr}_{\tau}(\mathcal{C})\) is a closed monoidal model category. We now provide a proof of this claim, adapted from [95, Lemma 2.2.15].

**Lemma 4.32.** For \(\Delta\mathcal{G}\) a crossed simplicial group, \(\hat{\Delta}\mathcal{G}_{\text{Kan}}\) is a monoidal model category.

**Proof.** We first show that the pushout-product axiom holds. We need to show that given any pair of cofibrations \(f: X \to Y\) and \(f': X' \to Y'\), their pushout-product

\[
f \boxtimes f': (X \times Y') \bigsqcup_{X \times X'} (Y \times X') \to Y \times Y'
\]
is a cofibration that is trivial whenever $f$ or $f'$ is. The condition amounts to considering the following pushout diagram

$$
\begin{array}{ccc}
X \times X' & \xrightarrow{f_\ast} & X \times Y' \\
\downarrow{f_\ast} & & \downarrow{f'_\ast} \\
Y \times X' & \xrightarrow{f'_\ast} & P \\
\end{array}
$$

By the universal condition of pushouts, we have that $P$ is represented by pairs $(y,x') \in Y \times X'$ and $(x,y') \in X \times Y'$ subject to the relation $(f(x),x') \sim (x,f'(x'))$. We first show that $f_\ast$ and $f'_\ast$ are cofibrations. We already have that both of these maps are monomorphisms, furthermore, if $(y,x')$ is not in the image of $f_\ast$, then $y$ is not in the image of $f$, which is the normality condition (this shows that $f_\ast$ is a cofibration, $f'_\ast$ follows similarly). Next we will show that $f \boxtimes f'$ is a cofibration. It is clearly a monomorphism as both $f$ and $f'$ are monomorphisms. Let $p \in Y \times Y'$ be an element such that it is not in the image of $f \boxtimes f'$. Therefore $p$ is represented by an element not in the image of either $f_\ast$ or $f'_\ast$. However, since we have that these maps are cofibrations, the $\mathfrak{S}_n$ acts freely on $p$ and $f \boxtimes f'$ is therefore a cofibration.

All that is left to show is the condition about the trivial cofibrations. However, Diagram 4.1.1 induces a pushout diagram in $\hat{\Delta}_{Kan}$ via the forgetful functor $i^\ast$. As $\hat{\Delta}_{Kan}$ itself is a monoidal model category, we can conclude that the pushout-product is a weak equivalence if $f$ or $f'$ is such.

We now need to show that the unit axiom holds. Let $X$ be a normal object, then we need to show that $\Delta\mathfrak{S}[0]' \times X \to \Delta\mathfrak{S}[0] \times X$ is a weak equivalence, where $\Delta\mathfrak{S}[0]'$ is the normalisation of the unit. This is true when $i^\ast(\Delta\mathfrak{S}[0]' \times X) \to i^\ast(\Delta\mathfrak{S}[0] \times X)$ is a weak equivalence of simplicial sets, and as right adjoints preserve products, this is equivalent to asking that $i^\ast(\Delta\mathfrak{S}[0]') \times i^\ast(X) \to i^\ast(\Delta\mathfrak{S}[0]) \times i^\ast(X)$ is a weak equivalence of simplicial sets. In particular, we only require that $i^\ast(\Delta\mathfrak{S}[0]'') \to i^\ast(\Delta\mathfrak{S}[0])$ is a weak equivalence. Consider the composition $0 \to \Delta\mathfrak{S}[0]' \to \Delta\mathfrak{S}[0]$, this is a horn inclusion and therefore is a trivial cofibration, and in particular a weak equivalence. Consequently, by two-out-of-three, the map $i^\ast(\Delta\mathfrak{S}[0]'') \to i^\ast(\Delta\mathfrak{S}[0])$ is a weak equivalence as required. Therefore the unit axiom holds.
4.2 Equivariant Cohomology Theories

In this section, we will show that the category of $\mathcal{G}$-equivariant $\infty$-stacks is a setting in which one can do $\mathcal{G}$-equivariant cohomology. To do this, we will follow the presentation of Section 2.3.2.2, where we introduced cohomology theories as mapping spaces between $\infty$-stacks. Note that we can make the following definition as we have shown that $\widehat{\mathcal{G}}_{\text{Kan}}$ is monoidal in Lemma 4.32.

**Definition 4.33.** Let $X, A$ be two stacks in $\text{Ho}(\Delta \mathcal{G}-\text{Pr}_r(\mathcal{C}))$. The $\mathcal{G}$-equivariant cohomology of $X$ with coefficients in $A$ is defined to be the $\Delta \mathcal{G}$-set

$$H_\mathcal{G}(X; A) := \Delta \mathcal{G}-\mathbb{R}\text{Hom}_{\text{Ho}(\widehat{\mathcal{G}}_{\text{Kan}})}(X, A)$$

4.2.1 Equivariant Eilenberg-Mac Lane Spaces

In this section we define equivariant Eilenberg-Mac Lane spaces, this will allow us in the next section to define equivariant site cohomology. To do this we first revisit the construction of classical Eilenberg-Mac Lane spaces via the linearisation of spheres [93, Example 1.14].

We begin by recalling that the simplicial circle is the simplicial set defined to be $S^1_\Delta := \Delta[1]/\partial\Delta[1]$ and the simplicial $n$-sphere is defined to be the $n$-fold smash product:

$$S^n_\Delta := S^1_\Delta \wedge \cdots \wedge S^1_\Delta.$$  

We can then consider for any abelian group $A$ the space $A \otimes \mathbb{Z}[S^n_\Delta]$ where the second term is the free abelian group generated by the level-wise non-basepoints of $S^n_\Delta$. That is, we can see the points of $A \otimes \mathbb{Z}[S^n_\Delta]$ to be finite sums of points in $S^n_\Delta$ with coefficients in $A$ modulo the relation that all $A$-multiples of the basepoint are zero. One can equip this space with the induced quotient topology where $A$ is given the discrete topology.

**Proposition 4.34** ([1, Corollary 6.4.23]). For any abelian group $A$ the space $A \otimes \mathbb{Z}[S^n_\Delta]$ is a $K(A, n)$ space. Note that for $n \geq 1$ the space has a unique connected component and therefore we do not need to specify a basepoint.

As we have defined the above using a simplicial construction, it can very easily be modified to a $\Delta \mathcal{G}$-setting. We begin by defining the analogue of the simplicial sphere.

**Definition 4.35.** The $\Delta \mathcal{G}$-circle is the $\Delta \mathcal{G}$-set defined to be $S^1_\mathcal{G} := \Delta \mathcal{G}[1]/\partial \Delta \mathcal{G}[1]$. The $\Delta \mathcal{G}$-$n$-sphere, denoted $S^n_\mathcal{G}$, is the the $n$-fold smash product of $S^1_\mathcal{G}$. 
Definition 4.36. Let $A$ be an abelian group, we define the $\Delta \Theta$-$K(A,n)$ object to be

$$K^{\Theta}(A,n) := A \otimes \mathbb{Z}[S^1_{\Theta}],$$

where $\mathbb{Z}[S^1_{\Theta}]$ is the abelian group freely generated by the level-wise non-basepoint elements.

4.2.2 Equivariant Site Cohomology

We now look at altering the site cohomology from the simplicial case (Example 2.55) to the equivariant case.

Definition 4.37. Let $A$ be a sheaf of abelian groups. We define the $\Theta$-equivariant cohomology of a site $(\mathcal{C}, \tau)$ to be

$$H^n_{\Theta}(\mathcal{C}; A) := \pi_0 \left( \Delta \Theta-\mathbb{R}-\text{Hom}(\ast, K^{\Theta}(A,n)) \right).$$

We will unpack this definition, with the focus on the cyclic setting. First of all we prove at an equivariant version of the Illusie conjecture (Proposition 2.56), for which we need to consider the category $\Delta \mathcal{C}$-$\text{Rmod}$ of $\Delta \mathcal{C}$-objects in the category of $R$-modules. There is a free functor $C: \Delta \mathcal{C}$-$\text{Set} \to \Delta \mathcal{C}$-$\text{Rmod}$ which is the left adjoint to the forgetful functor. The category $\Delta \mathcal{C}$-$\text{Rmod}$ can be equipped with a model structure via the transfer theorem (i.e., a map is a weak equivalence (resp., fibration) if the underlying map of cyclic sets is a weak equivalence (resp., fibration) in $\hat{\Delta} \mathcal{C}_{\text{Kan}}$). We can clearly promote this free functor $C$ onto $\Delta \mathcal{C}$-sheaves by applying it sectionwise. Using this construction we can move from the setting of $\Delta \mathcal{C}$-sheaves to $\Delta \mathcal{C}$-abelian sheaves $[\ast, K^{\Theta}(A,n)] \to [C\ast, K^{\Theta}(A,n)]$ and in fact this association is an isomorphism due to the following lemma:

Lemma 4.38 (Equivariant Illusie Conjecture). The free $\Delta \mathcal{C}$-abelian presheaf functor $X \mapsto CX$ considered as a map $\Delta \mathcal{C}$-$\text{Pr}^r(\mathcal{C}) \to \Delta \Theta$-$\text{Pr}^r(\mathcal{C})$ preserves local equivalences.

Proof. This follows from the fact that the weak equivalences in $\hat{\Delta} \mathcal{C}_{\text{Kan}}$ are those maps whose underlying simplicial map is a weak equivalence of simplicial sets. As the Illusie conjecture holds in the simplicial case, it therefore also holds here. $\square$

We have now replicated the first step from the simplicial case. Recall that the next step was the isomorphism $[\tilde{Z} \ast, K(A,n)] \simeq [\tilde{Z}[0], A[-n]]$ from the maps in the homotopy category of simplicial abelian sheaves to maps between sheaves of chain complexes which...
was facilitated using the Dold-Kan correspondence. In the general equivariant case, such an isomorphism is unaccessible as we would need some Dold-Kan style theorem between the abelian objects and certain “chain complexes”. However, in the cyclic case such a construction exists as a consequence of work on so-called duchain complexes (see [36, 37, 101]). We will not introduce the full theory here, but just give the relevant result. First we introduce what we mean by cyclic chain complexes. From now on, we will denote by $R$ a ring with $0 \neq 1$.

**Definition 4.39** ([36, §1.1]). A duchain complex over $R$ is a diagram of $R$-modules

$$X_0 \xleftarrow{\partial_1} X_1 \xrightarrow{\partial_2} X_2 \xleftarrow{\partial_3} \cdots$$

such that $\partial^2 = \delta^2 = 0$, but otherwise the $\partial$'s and $\delta$'s are independent. A map of duchain complexes is a set of maps $f_i: X_i \to Y_i$ such that all the obvious squares commute.

Denote by $f_i(t) = (1 + (-1)^i t)^{i+1}$ for $i \geq 0$. A cyclic chain complex is a duchain complex such that

- $f_{n-1} f_n (\partial \partial) x = x$, for all $x \in X_n$, $n > 0$.
- $f_0 (\partial \partial) x = x$, for all $x \in X_0$.

a morphism of cyclic chain complexes is a map of the underlying duchain complexes.

We can now use this notion of cyclic complexes to give the statement of the “cyclic Dold-Kan theorem”.

**Proposition 4.40** ([37, Proposition 2.2]). The category of cyclic chain complexes over $R$ is equivalent to the category of $\Delta \mathcal{C}$-$R$-modules.

Using the above proposition, we see that up to homotopy, we can move to a mapping between sheaves of cyclic complexes.

Dwyer and Kan proved in the paper [37] that the category of cyclic $R$-modules equipped with the transferred Kan model is Quillen equivalent to another, more manageable category. Namely the category of mixed complexes (referred to as commutative duchain complexes in the aforementioned reference).

**Definition 4.41** ([37, §1.3]). A mixed complex is a duchain complex such that $[\delta, \partial] = \partial \delta + \delta \partial = 0$.

If we were to apply the above discussion to $[C^\ast, K^\mathcal{C}(A, n)]$, then we see we would be considering a hom-set in the category of sheaves of mixed chain complexes (as we are only
working up to a homotopical setting). This observation then allows us to finally justify our
definition. In the simplicial case we had

\[[\hat{\mathbb{Z}}[0], A[-n]] \simeq \text{Ext}^n_{\mathbb{C}}(\mathbb{Z}, A) \simeq H^n(\mathbb{C}; A)\].

We have a very similar situation in the cyclic case. In the paper of Kassel (see [61]) it is
shown that cyclic cohomology can be computed as derived functors on a mixed complex
(in fact he computes the cyclic homology, we take the analogous dual construction here).

**Proposition 4.42 ([61, Proposition 1.3 (Dual)]).** Given a cyclic \(R\)-module \(E\), we have
the associated mixed complex \(\hat{E}\). There is an isomorphism

\[HC^n(E) \simeq \text{Ext}^n_{\Delta \mathbb{R}}(E, R) \simeq \text{Ext}^n_{\Delta \mathbb{C}}(\hat{E}, R)\].

Therefore we have heuristically shown, by the fact that we have a construction analogous
to classical cyclic cohomology, that our definition of equivariant site cohomology is indeed
a sensible one. Note that from the results of [59], we can in fact see the cyclic cohomology
as Borel \(SO(2)\)-equivariant cohomology.

The advantage of our definition is that we do need results about Dold-Kan type
correspondences to compute equivariant cohomology for a general crossed simplicial group.

### 4.3 \(SO(2)\)-Equivariant Moduli of Local Systems

We will finish this chapter by giving a construction of a specific geometric derived
\(SO(2)\)-equivariant stack. We choose to look at an equivariant version of the derived moduli of
local systems. We shall only discuss the twisted cyclic nerve of a category below. However,
more general constructions have been done for the simple crossed simplicial groups in [6].

#### 4.3.1 Lifting Classical Stacks to Equivariant Derived Stacks

Given any (classical) stack \(\mathcal{X}\), we can construct a derived stack \(j(N\mathcal{X})\). That is, we first
take the nerve of each groupoid in the stack to get a higher stack, and then we can take
a fibrant replacement \(j\), coming from the inclusion \(j_0: \text{Aff}^p \to \text{dAff}^p\), to get a derived
stack (see [107, §2.2.6.1] for the discussion of this machinery in the case that we will be
interested in). These ideas can be summed up using the following diagrams:

![Diagram](image)

What we shall now do is for any groupoid $\mathcal{G}$, construct a twisted cyclic nerve $\tilde{N}^c\mathcal{G}$ which renders the following diagram commutative:

$$
\begin{array}{ccc}
\text{Grpd} & \xrightarrow{N} & \tilde{\Delta} \\
\downarrow & & \downarrow j^* \\
\tilde{N}^c & \xrightarrow{j^*} & \Delta^c
\end{array}
$$

**Definition 4.43.** Let $\mathcal{G}$ be a groupoid. Its twisted cyclic nerve $\tilde{N}^c\mathcal{G}$ is defined to be the simplicial object such that in degree $n$ we have the $n$ maps in a diagram of the form:

$$
x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_1} \cdots \xrightarrow{a_n} x_n
$$

with the cyclic operator $\tau_n$ being defined as follows:

$$
x_n \xrightarrow{(a_1 \cdots a_n)^{-1}} x_0 \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} x_{n-1}
$$

The face and degeneracy maps are those of the ordinary simplicial nerve (defined in Section 2.1.2.3).

**Theorem 4.44.** Let $\mathcal{G}$ be a groupoid, then its twisted cyclic nerve is a cyclic set.

**Proof.** We shall show that the choice of cyclic operator $\tau_n$ satisfies the conditions of Lemma 4.7. In particular, we must show the following:

1. $\tau_n^{n+1} = \text{id}: [n] \rightarrow [n],$
2. $d_i \tau_n = \tau_{n-1} d_{i-1}: [n] \rightarrow [n-1],$
3. $s_i \tau_n = \tau_{n+1} s_{i-1}: [n] \rightarrow [n+1],$
4. $d_0 \tau_n = d_n: [n] \rightarrow [n-1],$
5. $s_0 \tau_n = \tau_{n+1} s_n: [n] \rightarrow [n+1].$
We begin with property (1). Note that

\[
\tau_n^2 \left( x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} x_n \right) = x_n - 1 \left( (a_1 \cdots a_n)^{-1} a_1 \cdots a_{n-1} \right) \xrightarrow{\left( (a_1 \cdots a_n)^{-1} a_1 \cdots a_{n-1} \right)^{-1}} x_0 \xrightarrow{a_n-2} \cdots \xrightarrow{a_n} x_n - 2
\]

Therefore, one sees that

\[
\left( (a_1 \cdots a_n)^{-1} a_1 \cdots a_{n-1} \right)^{-1} = (a_n^{-1} \cdots a_1^{-1} a_1 \cdots a_{n-1})^{-1}
\]

and by the property of groupoids, this becomes \((a_n^{-1})^{-1} = a_n\). For this map to get back to its original position it must be shifted \(n - 1\) times, for a total of \(n - 1 + 2 = n + 1\) applications of \(\tau_n\). Generalising this idea to the other maps, it is clear that \(\tau_{n+1} = \text{id}\) as required.

We now go on to show the compatibility of the face and degeneracy maps (2)-(5). Consider the action of \(\tau_n\):

\[
x_n \xrightarrow{(a_1 \cdots a_n)^{-1}} x_0 \xrightarrow{a_1} \cdots \xrightarrow{a_n} x_{n-1}
\]

We wish to apply face maps to this diagram. If \(i = 0\), \(d_0\) removes \(x_n\) from the sequence (as it is in the 0th position. Clearly this is the same as just applying \(d_n\) on the original diagram, and we see that (4) holds. Now assume that \(i \neq 0\), then \(d_i\) composes the morphisms at the \(i^{th}\) object, which is \(x_{i-2}\). This is the same as removing the \((i - 1)^{st}\) object in the original diagram (which is \(x_{i-2}\)), and then applying the cyclic operator. This shows the required property (2). The degeneracy properties are shown in an analogous fashion.
The twisted cyclic nerve construction allows us to extend the diagrams above to the following:

\[
\begin{array}{c}
\text{Aff}^{\text{op}} \xrightarrow{j_0} \text{Stacks} \xrightarrow{N} \text{Higher Stacks} \\
\text{dAff}^{\text{op}} \xrightarrow{\Delta} \text{Derived Stacks} \xrightarrow{i^*} \Delta\mathcal{C}\text{-Derived Stacks} \\
\Delta\mathcal{C}\text{-Stk}(\text{Aff}) \xrightarrow{j} \Delta\mathcal{C}\text{-Stk}(\text{dAff})
\end{array}
\]

Of course, it would be of interest to consider what happens with the other categorical nerves introduced in [6]. We have chosen to work with the cyclic case as it is the simplest to describe.

### 4.3.2 $SO(2)$-Equivariant Derived Local Systems

We now use the discussion from the previous section to construct the moduli of equivariant derived local systems on spaces with $SO(2)$-action. To do this, we of course, first need to introduce the derived stack of local systems. A very readable overview of this derived stack, along with the motivation for its derived structure can be found at [23].

**Definition 4.45.** Let $G$ be an algebraic group defined over a field $k$. The classifying stack $\mathcal{B}G$ assigns to a scheme $U$ the groupoid whose objects are principal $G$-bundles $\pi: \mathcal{E} \to U$, and the morphisms being isomorphisms of principal $G$-bundles. We will simplify notation and write $\mathcal{B}G$ for $j(N\mathcal{B}G)$, the corresponding derived stack.

**Definition 4.46** ([104, Definition 6.1.1]). Let $\mathcal{B}G$ be the derived classifying stack of an algebraic group and $X$ a topological space. The *derived stack of $G$-local systems on $X$* is the stack

$$\mathbb{R}\text{Loc}(X,G): \text{dAff}^{\text{op}} \to \hat{\Delta}$$

$$U \mapsto \text{Map}(X,|\mathcal{B}G(U)|)$$

That is, $\mathbb{R}\text{Loc}(X,G)(U)$ is the simplicial set of continuous maps from the space $X$ to the space $|\mathcal{B}G(U)|$. 
Lemma 4.47. The stack $\mathbb{R}\text{Loc}(X, G)$ is a derived 1-algebraic stack.

Proof. Assume that $X$ is contractible. Then $\mathbb{R}\text{Loc}(X, G) \simeq \mathbb{R}\text{Loc}(\ast, G) \simeq BG$ is 1-algebraic. We then prove by induction that $\mathbb{R}\text{Loc}(S^r, G)$ is algebraic for all $r \geq 0$. The case of $r = 0$ follows from the argument of $X \simeq \ast$. For $r > 0$ there is a homotopy pushout

\[
\begin{array}{ccc}
S^{r-1} & \longrightarrow & D^r \\
\downarrow & & \downarrow \\
D^r & \longrightarrow & S^r 
\end{array}
\]

which induces a homotopy pullback in the category of derived stacks

\[
\begin{array}{ccc}
\mathbb{R}\text{Loc}(S^r, G) & \longrightarrow & \mathbb{R}\text{Loc}(D^r, G) \\
\downarrow & & \downarrow \\
\mathbb{R}\text{Loc}(D^r, G) & \longrightarrow & \mathbb{R}\text{Loc}(S^{r-1}, G) 
\end{array}
\]

By induction hypothesis we have that $\mathbb{R}\text{Loc}(D^r, G)$ and $\mathbb{R}\text{Loc}(S^{r-1}, G)$ are 1-algebraic. By the stability of algebraic derived stacks by homotopy pullback we have that $\mathbb{R}\text{Loc}(S^r, G)$ is 1-algebraic.

We now finish the proof for general spaces $X$. Let $X_k$ be the $k$ skeleton of $X$. $X$ is a finite CW-complex so there exists some $k$ such that $X = X_k$. For any $k$ there is a homotopy pushout

\[
\begin{array}{ccc}
\coprod S^{k-1} & \longrightarrow & \coprod D^k \\
\downarrow & & \downarrow \\
X_{k-1} & \longrightarrow & X_k 
\end{array}
\]

which induces a homotopy pullback in the category of derived stacks

\[
\begin{array}{ccc}
\mathbb{R}\text{Loc}(X_k, G) & \longrightarrow & \mathbb{R}\text{Loc}(X_{k-1}, G) \\
\downarrow & & \downarrow \\
\coprod^h \mathbb{R}\text{Loc}(D^k, G) & \longrightarrow & \coprod^h \mathbb{R}\text{Loc}(S^{k-1}, G) 
\end{array}
\]

By stability of algebraic derived stacks by finite limits, we can deduce that $\mathbb{R}\text{Loc}(X_k, G)$ is 1-algebraic by induction. 

Definition 4.48. Let $BG^\infty := j(\tilde{N}^\infty(BG))$ be the cyclic derived classifying stack of an algebraic group and $X$ a topological space with an action of $SO(2)$. The $SO(2)$-equivariant
derived stack of local systems is the stack

\[ R\text{Loc}^E(X, G) : \text{dAff}^{op} \to \Delta \mathcal{C} \]

\[ U \mapsto \text{Map}^{SO(2)}(X, |BG^e(U)|_e) \]

That is, \( R\text{Loc}^E(X, G)(U) \) is the cyclic set of continuous maps in \( \text{Top}^{SO(2)} \) from the space \( X \) to the space \( |BG^e(U)|_e \).

**Remark 4.49.** We can adjust the above theory for the twisted dihedral nerve by using the fact that there is a pair of adjoint functors \(|-| : \Delta \mathcal{D}\text{-Set} \rightleftarrows \text{Top} : S_\mathcal{D}(-)\) between the categories of dihedral sets and topological spaces with \( O(2) \)-action.

The following theorem explains our choice of terminology, the fact the above construction really is doing something equivariant.

**Theorem 4.50.** Let \( X \in \text{Top}^{SO(2)} \) be a topological space with an action of \( SO(2) \). Denote by \( X/\text{SO}(2) \) the orbit space of \( X \), i.e., the space obtained by identifying points of \( X \) in the same orbit. Then

\[ R\text{Loc}^e(X, G) \simeq R\text{Loc}(X/\text{SO}(2), G). \]

**Proof.** We can prove this by looking at each element \( \text{Map}^{SO(2)}(X, |BG^e(U)|_e) \). First of all we use a result from Loday [68, §7.3.5] which states that the cyclic realisation of the twisted nerve construction of a group \( G \) has trivial \( SO(2) \)-action when twisting by the identity element. As every groupoid is equivalent to the disjoint union of groups, we can conclude that the action of \( SO(2) \) on \( |BG^e(U)|_e \) is also trivial. Due to the action being trivial, a general result about \( SO(2) \)-spaces, such as in [79, §1.1], allows us to move from mapping spaces in \( \text{Top}^{SO(2)} \) to \( \text{Top} \) in the following manner:

\[ \text{Map}^{SO(2)}(X, |BG^e(U)|_e) \simeq \text{Map}(X/\text{SO}(2), |BG(U)|). \]

The result then follows from this observation.

**Corollary 4.51.** If \( X \in \text{Top}^{SO(2)} \) has trivial \( SO(2) \)-action then

\[ R\text{Loc}^e(X, G) \simeq R\text{Loc}(X, G). \]

**Example 4.52.** To conclude, we compute the \( SO(2) \)-equivariant derived stack on a non-trivial example. Consider the \( SO(2) \)-space \( S^3_{\text{Hopf}} \) to be the 3-sphere along with the action of the Hopf map (i.e., scalar multiplication). The orbit space \( S^3_{\text{Hopf}}/\text{SO}(2) \) is homotopic to
$S^2$. Therefore by Theorem 4.50 we get:

$$\mathbb{R} \mathbf{Loc}^C(S^3_{\text{Hopf}}, G) \simeq \mathbb{R} \mathbf{Loc}(S^2, G) \simeq [\text{Spec } \text{Sym}_k(\mathfrak{g}^*[1])/G]$$

with $\mathfrak{g}^*[1]$ being the Koszul resolution of the identity element of $G$, and $\text{Sym}_k$ being the corresponding $k$-symmetric algebra. The final equivalence is computed in the literature, for example, [105, p. 200].
Chapter 5

Stable Homotopical Algebraic Geometry

In this chapter, we will explore the second example of an augmentation category. Namely, the category of finite rooted trees $\Omega$ studied in [82], and further properties explored in [2, 11, 25, 52, 112]. The category of dendroidal sets, $\hat{\Omega}$, are to $(\infty, 1)$-operads as simplicial sets are to $(\infty, 1)$-categories. After we have proved that $\Omega$ is an augmentation category, we will explore the properties of $\hat{\Omega}_{\text{Kan}}$, which from the work of [10], is a model for connective spectra. Therefore, the augmented algebraic geometry in this setting should be seen as a stable augmentation.

We then go further and show that it is possible to combine the theory of crossed simplicial groups with that of the planar rooted tree category. We prove that the pushout $\Delta \mathcal{G} \sqcup_{\Delta} \Omega$ is an augmentation category, and discuss the possible homotopy type of the Kan model structure on presheaves over $\Delta \mathcal{G} \sqcup_{\Delta} \Omega$. This framework extends to any augmentation category which has the property of being amalgable (Definition 5.41).

Outline of Chapter 5

(Section 5.1) We introduce the category $\Omega$ and describe its structure by using the theory of operads, and via construction of suitable face and degeneracy maps.

(Section 5.2) We look at the category of dendroidal sets, and study the tensor structure induced by the Boardman-Vogt tensor product of operads (as opposed to the cartesian product). We use this to prove that $\Omega$ is an augmentation category. We then go on to explore the homotopy theory of $\hat{\Omega}_{\text{Kan}}$. It is sensible to think that the pushout-product axiom would hold in this case, but there would be no operad related data in the monoidal structure.
(Section 5.3) We use the categorical theory of amalgamations to discuss a set of sufficient conditions for when the pushout of an augmentation category with a crossed simplicial group again produces an augmentation category. We then prove that $\Omega$ has this amalgamation property.

5.1 The Category of Rooted Trees

We begin by introducing the category of planar rooted trees. This category, as the name suggests, has objects being rooted trees with a planar structure. The simplex category can be seen as a subcategory of this category by viewing its objects as linear trees.

5.1.1 Operads

We begin by recalling the theory of operads over a category as in [65, 77]. Operads will give us a convenient way to describe the morphisms in $\Omega$. We let $\mathcal{E}$ be a cocomplete symmetric monoidal category with tensor product $\otimes$ and unit $I$. We also denote the symmetric group on $n$ elements by $\Sigma_n$.

**Definition 5.1** ([77, Definition 1.1]). A (non-symmetric) operad in $\mathcal{E}$ consists of objects $P(n)$ of $\mathcal{E}$ for $n \in \mathbb{N}$ together with the following:

- A **unit**, which is given by a morphism $I \to P(1)$.
- A **composition product** given by morphisms

  $$P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) \to P(k)$$

  for every $n, k_1, \ldots, k_n$ and $k = \sum_{i=1}^{n} k_i$.

Moreover we will call $\mathcal{E}$ symmetric if the following holds:

- A **permutation of variables** given by a right action of $\Sigma_n$ on $P(n)$ for every $n$.

A map of operads $f: P \to Q$ is given by morphisms $f_n: P(n) \to Q(n)$ for every $n$ which are compatible with the composition, unit and action of the symmetric group (if $P$ and $Q$ are symmetric).

There is an obvious generalisation of an operad by letting it be an operad with multiple objects, which we will call colours.

**Definition 5.2.** Let $C$ be a set whose elements we call colours. A **$C$-coloured operad** in $\mathcal{E}$ consists of the following:
For each sequence \(c_1, \ldots, c_n, c\) of elements of \(C\), an object \(P(c_1, \ldots, c_n; c)\) of \(E\). The object represents the set of operations which takes input colours \(c_1, \ldots, c_n\) and outputs colour \(c\).

- **Units** given by a morphism \(I \to P(c; c)\) for every \(c\) in \(C\).

- For every \((n + 1)\)-tuple of colours \((c_1, \ldots, c_n; c)\), and \(n\) given tuples \((d_{1,1}, \ldots, d_{1,k_1}; c_1), \ldots, (d_{n,1}, \ldots, d_{n,k_n}; c_n)\)
  
  a composition product given by morphisms
  
  \[
P(c_1, \ldots, c_n; c) \otimes P(d_{1,1}, \ldots, d_{1,k_1}; c_1) \otimes \cdots \otimes P(d_{n,1}, \ldots, d_{n,k_n}; c_n)
  \]
  
  \[
  \to P(d_{1,1}, \ldots, d_{1,k_1}, \ldots, d_{n,1}, \ldots, d_{n,k_n}; c)
  \]

- **Permutations** given by an action of the symmetric group. An element \(\sigma\) of \(\Sigma_n\) gives a map \(\sigma^* : P(c_1, \ldots, c_n; c) \to P(c_{\sigma(1)}, \ldots, c_{\sigma(n)}; c)\).

We also require that the composition product is compatible with the action of the symmetric groups and subject to associativity and unitary compatibility relations.

A map of coloured operads from a \(C\)-coloured operad \(P\) to a \(D\)-coloured operad \(Q\) is given by a map \(f : C \to D\) of colours and maps

\[
\varphi_{c_1, \ldots, c_n; c} : P(c_1, \ldots, c_n; c) \to Q(f(c_1), \ldots, f(c_n); f(c))
\]

We will denote by \(\text{Operad}(\mathcal{E})\) the category whose objects are coloured symmetric operads in \(\mathcal{E}\), and whose morphisms are the maps of coloured operads. However in the case that \(\mathcal{E} = \hat{\Delta}\), we will write \(\text{sOperad}\), and in the case \(\mathcal{E} = \text{Set}\), we will write \(\text{Operad}\).

### 5.1.2 Planar Rooted Trees

**Definition 5.3.** A finite rooted tree is a finite connected graph with no loops and a chosen outer edge which we call the root. The remaining outer edges will be called leaves. A planar rooted tree is a rooted tree \(T\) together with a linear ordering of \(\text{in}(v)\) for each vertex \(v\) of \(T\).

Every planar rooted tree \(T\) gives rise to a non-symmetric operad, denoted \(\Omega_p(T)\). The set of colours of \(\Omega_p(T)\) is the set of edges of \(T\), and the operations are generated by the vertices of the tree. That is for each vertex \(v\) with input edges \(e_1, \ldots, e_n\) and output edge \(e\), we have an operation \(v \in \Omega_p(T)(e_1, \ldots, e_n; e)\). The other operations are the unit operations,
or compositions of the above. This operad has the property that for all combinations \( e_1, \ldots, e_n; e \), the set of operations \( \Omega_p(T)(e_1, \ldots, e_n; e) \) contains at most one element.

**Example 5.4.** Consider the following planar tree

```
    e
   / \n  β   f
 /   / \
 b   c   γ
   \   |
    \ α
     a
```

The operad \( \Omega_p(T) \) has six colours given by \( a, b, c, d, e \) and \( f \). Then there are generators

- \( \alpha \in \Omega_p(T)(b, c, d; a) \)
- \( \beta \in \Omega_p(T)(e, f; b) \)
- \( \gamma \in \Omega_p(T)(; d) \)

and the units \( 1_a, 1_b, \ldots, 1_f \). We then have the other operations which are obtained by composition, namely:

- \( \alpha \circ_1 \beta \in \Omega_p(T)(e, f, c, d; a) \).
- \( \alpha \circ \gamma \in \Omega_p(T)(b, c; a) \).
- \( \alpha(\beta, 1_c, \gamma) = (\alpha \circ_1 \beta) \circ_4 \gamma = (\alpha \circ_3 \gamma) \circ_1 \beta \in \Omega_p(T)(e, f; c, a) \).

These operations completely determine the operad \( \Omega_p(T) \).

**Definition 5.5.** The category \( \Omega_p \) is the category where the objects are given by the finite rooted trees and the morphisms \( S \to T \) are given by the morphism between the induced non-symmetric operads.

**Definition 5.6 ([111, Definition 2.2.4]).** There is a category \( \Omega \) which is obtained from \( \Omega_p \) by equipping all objects \( T \) with all possible non-planar tree automorphisms (that is, we do not fix a planar representation of the trees). A morphism \( S \to T \) in \( \Omega \) is given by the map of the corresponding symmetric operads. We shall sometimes refer to \( \Omega \) as the dendroidal category.

**Remark 5.7.** Note, that by construction, the category \( \Omega \) is a crossed \( \Omega_p \)-group in the sense of Definition 3.2.
As we are interested in $\Omega$ in an augmentation category, we highlight the relations between $\Delta$ and $\Omega$ (and therefore also $\Omega_p$). We can identify the objects of $\lfloor n \rfloor \in \Delta$ with the linear trees of Figure 5.1.

This gives a fully faithful inclusion $i: \Delta \hookrightarrow \Omega$. In fact, if we denote by $\eta$ the tree with one edge and no vertices, then we get $\Omega/\eta = \Delta$. The other property of the inclusion $i$ that will be fundamental later on is that it exhibits $\Delta$ as a sieve in $\Omega$.

**Definition 5.8.** Let $S$ be a fully faithful subcategory of $\mathcal{C}$, then we say that $S$ is a:

- **Sieve** in $\mathcal{C}$ if for every morphism $f: c \to s$ in $\mathcal{C}$ with $s \in S$, $c$ and $f$ are also in $S$.
- **Cosieve** in $\mathcal{C}$ if for every morphism $f: s \to c$ in $\mathcal{C}$ with $s \in S$, $c$ and $f$ are also in $S$.

### 5.1.3 Faces and Degeneracies

We now give a set of morphisms that will generate all of the morphisms in the category $\Omega_p$ (resp., $\Omega$). These are generalisations of the face and degeneracy maps which appear in $\Delta$, and indeed if we restrict to the linear trees $L_n$ then we retrieve these classical maps. Of course, these are exactly the face and degeneracy maps which will make up the EZ-category structure.

**Definition 5.9 ([111, §2.2.1]).** Let $T$ be a planar rooted tree, with $b$ an inner edge. If we denote by $T/b$ the tree that we get by contracting $b$, then there is a natural map $\partial_b: T/b \to T$ in $\Omega_p$ which we call the *inner face* map associated to $b$. 
**Definition 5.10** ([111, §2.2.1]). Let $T$ be a planar rooted tree and $v$ a vertex of $T$ with exactly one inner edge. Denote by $T/v$ the tree obtained from $T$ by removing the vertex $v$ and all outer edges. We denote the face map $\partial_v: T/v \to T$, and refer to it as an *outer face* of $T$.

**Definition 5.11** ([111, §2.2.1]). Let $T$ be a tree and $v$ a vertex of valence one, and denote by $T\setminus v$ the tree obtained from $T$ by removing vertex $v$ and merging the two incident edges. There is a map $\sigma_v: T \to T\setminus v$ which we call the *degeneracy map* associated to $v$.

All of these morphisms satisfy some identities. We will not list them here, but the details can be found in [111, §2.2]. Finally, we note that any morphism in $\Omega_p$ decomposes as a series of face and degeneracy maps [82, Lemma 3.1].

### 5.2 Dendroidal Sets

**Definition 5.12** ([111, Definition 2.4.1]). The presheaf category $\hat{\Omega}$ will be called the category of *dendroidal sets*. Similarly the presheaf category $\hat{\Omega}_p$ will be the category of *planar dendroidal sets*.

From this definition we see that a dendroidal set $X$ is given by a set $X(T)$, which we shall denote $X_T$ for every tree $T$. This comes together with a map $\alpha^*: X_T \to X_S$ for each morphism $\alpha: S \to T$ in $\Omega$.

A *morphism of dendroidal sets* $f: X \to Y$ is given by maps $f: X_T \to Y_T$ for each tree $T$, commuting with the structure maps. That is if $\alpha: S \to T$ is a morphism in $\Omega$ and $x \in X_T$, then $f(\alpha^*x) = \alpha^*f(x)$. 

[Diagram of tree topology and face maps]
If $Y$ and $X$ are dendroidal sets, then $Y$ is a dendroidal subset of $X$ if $Y_T \subseteq X_T$ for all trees $T$, and if the inclusion map $Y \hookrightarrow X$ is a morphism of dendroidal sets.

We begin by highlighting that there are canonical inclusions and restriction functors between the three categories in question, these assemble into the two following commutative diagrams.

\[
\begin{array}{ccc}
\Delta & \xrightarrow{u} & \Omega_p \\
\downarrow & & \downarrow \\
\Omega & & \Omega
\end{array}
\quad
\begin{array}{ccc}
\hat{\Delta} & \xrightarrow{u^*} & \hat{\Omega}_p \\
\downarrow & & \downarrow \\
\Omega & & \hat{\Omega}
\end{array}
\]

these functors have left and right adjoints which gives us the following diagrams of relations

\[
\begin{array}{ccc}
\hat{\Delta} & \xrightarrow{u} & \hat{\Omega}_p \\
\downarrow & & \downarrow \\
\Omega & & \Omega \\
\end{array}
\quad
\begin{array}{ccc}
\hat{\Delta} & \xrightarrow{u^*} & \hat{\Omega}_p \\
\downarrow & & \downarrow \\
\Omega & & \Omega \\
\end{array}
\]

which are given by the corresponding Kan extensions. For example $i^*$ sends a dendroidal set $X$ to the simplicial set $i^*(X)_n = X_i([n])$ and its left adjoint $i_!: \hat{\Delta} \to \hat{\Omega}$ is extension by zero, which sends a simplicial set $X$ to the dendroidal set defined by

\[
i_!(X)_T = \begin{cases} X_n & \text{if } T \simeq i([n]) \\ \emptyset & \text{otherwise} \end{cases}
\]

5.2.1 Tensor Structure

We shall now construct the tensor product on dendroidal sets. This is the tensor structure that we will equip $\hat{\Omega}$ with when proving it is an augmentation category.

**Definition 5.13** ([111, Definition 3.1.1]). We have for each tree $T \in \Omega$ a coloured operad $\Omega(T)$ in Operad, this induces an adjunction

\[
\tau_d : \hat{\Omega} \rightleftarrows \text{Operad} : N_d
\]

The functor $N_d$ is called the dendroidal nerve. For any operad $P$ the dendroidal nerve of $P$ is the dendroidal set

\[
N_d(P)_T = \text{Hom}_{\text{Operad}}(\Omega(T), P)
\]
The dendroidal nerve is fully faithful and we have $N_d(\Omega(T)) = \Omega[T]$ for all trees $T$ in $\Omega$. It extends the usual categorical nerve, if $\mathcal{E}$ is a monoidal category with associated coloured operad $\mathcal{E}$, then $i^*(N_d(\mathcal{E})) = N(\mathcal{E})$.

The left adjoint $\tau_d(X)$ is called the operad generated by $X$. Given a dendroidal set $X$, the set of colours $\text{col}(\tau_d(X))$ is equal to $X_\eta$, where $\eta$ is the tree with one edge and no vertices. The operations are generated by the elements of $X_{C_n}$ with respect to some relations. Here $C_n$ is the $n$-th corolla, which is the tree with one vertex and $n$ leaf edges.

Similarly to $N_d$, $\tau_d$ extends the functor $\tau: \hat{\Delta} \to \text{Cat}$. This gives us a diagram (not fully commutative) of adjoint functors. There is, of course, another central column in this diagram which gives relations between planar dendroidal sets and non-symmetric coloured operads.

\[
\begin{array}{ccc}
\hat{\Delta} & \xrightarrow{i_t} & \hat{\Omega} \\
N & \downarrow{\tau} & \downarrow{\tau_d} \\
\text{Cat} & \xleftarrow{i} & \text{Operad}
\end{array}
\]

**Definition 5.14** ([18]). Let $\mathcal{P}$ be an operad with a set of colours $C$, and $\mathcal{Q}$ an operad with a set of colours $D$. Their Boardman-Vogt tensor product $P \otimes_{BV} Q$ is the operad whose set of colours is $C \times D$ and whose operations are given by generators and relations. There is one generating operations for every pair $(p, d)$ with $p \in \mathcal{P}(c_1, \ldots, c_n; c)$ and $d \in D$, denoted $p \otimes d \in \mathcal{P} \otimes_{BV} \mathcal{Q}((c_1, d), \ldots, (c_n, d); (c, d))$

and for each pair $(c \otimes q)$ with $c \in C$ and $q \in \mathcal{Q}(d_1, \ldots, d_n)$, denoted $(c \otimes q) \in \mathcal{P} \otimes_{BV} \mathcal{Q}((c, d_1), \ldots, (c, d_n); (c, d))$

for all $n \in \mathbb{N}$. These are subject to the following relations:

- The tensor product $c \otimes (-)$ with $c \in C$ respects composition in $\mathcal{Q}$, dually, the tensor product $(-) \otimes d$ respects composition in $\mathcal{P}$. Moreover both respect the action of the symmetric group on the operations.

- The operations in $\mathcal{P}$ and $\mathcal{Q}$ distribute over each other in $\mathcal{P} \otimes_{BV} \mathcal{Q}$.

**Proposition 5.15.** [112, Theorem 2.22] The category Operad is a closed symmetric monoidal category with respect to $\otimes_{BV}$.

We can use this tensor product and the dendroidal nerve to define a tensor product on dendroidal sets. As $\hat{\Omega}$ is a presheaf category and each object is the colimit of representables,
we can give the construction simply by defining it on the representable objects. For representables \( \Omega[S] \) and \( \Omega[T] \) we define

\[
\Omega[S] \otimes \Omega[T] := N_d(\Omega(S) \otimes_{BV} \Omega(T)).
\] (5.2.2)

Note that this tensor product is not part of a monoidal structure as it is not strictly associative \([26]\). However, this tensor structure agrees with the monoidal structure on \( \hat{\Delta} \) in the sense that \( i_!(X \times Y) \simeq i_!(X) \times i_!(Y) \) for \( X, Y \) simplicial sets ([82, Proposition 5.3]).

### 5.2.2 \( \Omega \) is an Augmentation Category

We shall now prove that \( \Omega \) is an augmentation category. In the process, we also show that \( \Omega_p \) is an augmentation category.

**Theorem 5.16.** The category \( \Omega \) is an augmentation category.

**Proof.**

(AC1) First we require that \( \Omega \) is an EZ-category. This is given in [16, Example 7.6(c)], but we will elaborate on some of the details here. The degree function \( d: \Omega \to \mathbb{N} \) is given by \( d(T) = \# \{ \text{vertices of } T \} \). Every morphism in \( \Omega \) can be decomposed as an automorphism (arising from considering different planar structures) followed by a series of face and degeneracy maps [82, Lemma 3.1]. There is a category \( \Omega_{\text{planar}} \) which fixes a planar representation of the trees, and is a strict EZ-category. By the description of the morphisms in \( \Omega \), we see that it is a crossed \( \Omega_{\text{planar}} \) group, and by Lemma 3.6 is a generalised EZ-category. The degeneracy and face operators that appear in the morphism structure of \( \Omega \) are exactly those we use in the EZ-structure. We also require a tensor product \( \Box \) which gives \( \Omega \) the structure of a symmetric promagmoidal EZ-category. In this case \( \Box = \otimes \), the Boardman-Vogt tensor product on dendroidal sets as described in Equation 5.2.2

(AC2) Next, we require an inclusion \( i: \Delta \hookrightarrow \Omega \) which is compatible with the monoidal structure. The inclusion is given by considering the object \([n]\) as the linear tree \( L_n \) which has \( n + 1 \) vertices and \( n \) edges. The compatibility of the tensor product with the monoidal structure on simplicial sets is given in [82, Proposition 5.3].

(AC3) The proof of AC3 is highly technical, relying on the shuffle of trees. However, the property is proved as the main result of [26], and we will not concern ourself with the details here.

\( \square \)
Corollary 5.17.

- There is a Quillen model structure on the category of dendroidal sets, denoted $\hat{\Omega}_{\text{Kan}}$, where the fibrant objects are the dendroidal Kan complexes, and the cofibrations are the normal monomorphisms. Moreover there is a Quillen adjunction
  \[ i^! : \hat{\Delta}_{\text{Kan}} \rightleftarrows \hat{\Omega}_{\text{Kan}} : i^* \]

- There is a local model structure on the category of dendroidal presheaves, denoted $\Omega \text{-Pr}_{\tau}(\mathcal{C})$ obtained as the left Bousfield localisation of the point-wise Kan model at the class of dendroidal hypercovers. Moreover there is a Quillen adjunction
  \[ i^! : s\text{Pr}_{\tau}(\mathcal{C}) \rightleftarrows \Omega \text{-Pr}_{\tau}(\mathcal{C}) : i^* \]

- The category of $(\Omega, n)$-geometric derived Artin stacks, denoted $\Omega \text{-G}_{n}^{\text{sm}}(d\text{Aff})$, is given as the homotopy category of derived $(\Omega, n)$-hypergroupoids with respect to the class of trivial $(\Omega, n)$-hypergroupoids.

Remark 5.18. As $\Omega_p$ is also an augmentation category, it is possible to replace all symbols $\Omega$ by $\Omega_p$ in Corollary 5.17. Note that also because $\Omega_p$ is a strict EZ-category, this model structure will be of Cisinski type.

Definition 5.19. The augmented homotopical algebraic geometry theory arising from $\Omega$ will be referred to as stable. For example, the category $\text{Ho}(\Omega \text{-Pr}_{\tau}(\mathcal{C}))$ will be referred to as the category of stable $\infty$-stacks.

Warning 5.20. Unfortunately, unlike the simplicial and crossed simplicial cases, the model structure $\hat{\Omega}_{\text{Kan}}$ is not a closed monoidal category. This produced several errors in the existing literature, as it was assumed that the pushout-product axiom holds. This error was spotted and corrected in the errata of [25], with details appearing in [26]. In Section 5.2.4, we will see that there is a full subcategory of $\Omega$ in which the pushout-product axiom does hold. One could also consider using the cartesian tensor product on the category of dendroidal sets which is cartesian closed as it is a presheaf category.

5.2.3 Properties of $\text{Ho}(\hat{\Omega}_{\text{Kan}})$

We finish this section by discussing the homotopy type of $\hat{\Omega}_{\text{Kan}}$. This section will lead to the choice of the quantifier stable to describe $\Omega$-augmentation. The results in this section are built on [10], where the model $\hat{\Omega}_{\text{Kan}}$ was explicitly described. Note that in the aforementioned reference, the Kan model structure is called the stable model structure.
We first highlight a difference to what one may find in the literature. The model structure on \( \hat{\Omega} \) that one usually encounters is the operadic model structure (sometimes referred to the Cisinski-Moerdijk structure). The fibrant objects in this model structure are the inner Kan complexes. This model structure is Quillen equivalent to a certain model structure on \( sOperad \). It is shown in [10] that the Kan model structure is a localisation of this model structure at the collection of outer horns. One can easily see the duality between this construction and the relationship between \( \hat{\Delta}^{Kan} \) and \( \hat{\Delta}^{Joyal} \).

Recall from [47] that a spectrum \( E = \{ E_i \}_{i \in \mathbb{Z}} \) is a sequence of based spaces \( E_n \) and based homeomorphisms \( E_i \simeq \Omega E_{i+1} \). We say that a spectrum is connective if \( E_i = 0 \) for \( i < 0 \). We can view a connective spectrum as an infinite loop space via delooping machinery as in [78], or equally as a \( \Gamma \)-space as in [94]. We will denote by \( \text{ConSp}(\text{Top}) \) the category of connective spectra.

**Proposition 5.21** ([20], Proposition 2.2). There is a model structure on \( \text{ConSp}(\text{Top}) \), called the stable model structure, where a map \( f : X \to Y \) is a:

- Weak equivalence if \( f_* : \pi_* X \simeq \pi_* Y \) where \( \pi_* X = \lim_{\to k} \pi_{*+k} X_n \).
- Fibration if it has the RLP with respect to trivial cofibrations.
- Cofibration if \( f_n : X_n \to Y_n \) is a cofibration of spaces for all \( n \geq 0 \).

**Proposition 5.22** ([11], Theorem 5.4). The Kan model structure on dendroidal sets is Quillen equivalent to the stable model structure on connective spectra.

From Proposition 5.22 we see that we can consider \( \hat{\Omega}^{Kan} \) as modelling connective spectra. Although \( \hat{\Omega}^{Kan} \) is not a closed monoidal category, we can still consider cohomology theories by using the simplicial mapping space which exists even without the closed monoidal structure. With this concept in mind, we make a definition of stable cohomology using the theory of dendroidal sets.

**Definition 5.23.** Let \( X, A \) be two objects in \( \text{Ho}(\hat{\Omega}^{Pr}(\mathcal{C})) \). The stable cohomology of \( X \) with coefficients in \( A \) is defined to be the simplicial set

\[
H^{Stab}(X; A) := \Omega \cdot \Delta^{\hat{\Delta}} \text{Hom}(X, A) \in \text{Ho}(\hat{\Delta}^{Kan})
\]

In this setting we can consider a sensible replacement of Eilenberg-Mac Lane spaces by Eilenberg-Mac Lane spectrums.

**Definition 5.24.** Let \( A \) be an abelian group. The Eilenberg-Mac Lane spectrum of \( A \), denoted \( HA \), is the connected spectrum with component spaces \( (HA)_n = A \otimes \mathbb{Z}[S^n] \). By
Proposition 5.22, we can find a dendroidal set, which we also denote $HA$, which is homotopy equivalent to the Eilenberg-Mac Lane spectrum.

**Definition 5.25.** Let $(\mathcal{C}, \tau)$ be a site. Let $A$ be a sheaf of abelian groups. We define the *stable cohomology* of $\mathcal{C}$ to be

$$H^{\text{Stab}}(\mathcal{C}; A) := \mathbb{R}^\Delta \text{Hom}(\ast, HA).$$

**Question 5.26.** Can we compute and interpret $H^{\text{Stab}}(\mathcal{C}; A)$ in some specific cases? It seems related to the ordinary case in that it somehow collects together all possible cohomology groups. These could possibly be retrieved by applying $\pi_i$.

### 5.2.4 Open Dendroidal Sets

We finish this section by introducing the category of *open dendroidal sets*. We do this as the category of open dendroidal sets is a closed monoidal model category, and therefore it goes some of the way to repair the lack of internal-hom objects in $\text{Ho}(\hat{\Omega}_{\text{Kan}})$.

**Definition 5.27** ([26, Definition 2.1]). A tree $T \in \Omega$ is said to be *open* if it contains no stumps (that is, vertices with no incoming edges). Denote by $\Omega_o$ the full subcategory of $\Omega$ consisting of the open trees.

**Lemma 5.28.** The category $\Omega_o$ is an augmentation category.

*Proof.* As $\Omega_o$ is a full subcategory of $\Omega$, it inherits the symmetric promagmoidal EZ-structure. The only thing we need to prove is the compatibility with the simplex category. Note that the linear trees $L_n$ of Figure 5.1 are open trees. Therefore there is a suitable inclusion $i: \Delta \hookrightarrow \Omega_o$ as required. □

**Lemma 5.29** ([26, Corollary 2.5]). The pushout-product axiom holds in $\hat{\Omega}_{o\text{Kan}}$.

**Corollary 5.30.** There is a closed monoidal model structure on the category of open dendroidal sets, denoted $\hat{\Omega}_{o\text{Kan}}$, where the fibrant objects are the open dendroidal Kan complexes, and the cofibrations are the normal monomorphisms. Moreover there is a Quillen adjunction

$$i_!: \hat{\Delta}_{\text{Kan}} \rightleftarrows \hat{\Omega}_{o\text{Kan}}: i^*$$

**Remark 5.31.** Although the fact that $\hat{\Omega}_{o\text{Kan}}$ is a closed monoidal model category, using it for augmented homotopical algebraic geometry is still unfeasible. The reason for this is that there is currently no work on what the homotopy type of $\hat{\Omega}_{o\text{Kan}}$ is.
5.3 Amalgamations

To complete this chapter, and indeed, the main body of this thesis, we now describe a way to combine the examples of $\Delta G$ and $\Omega$ into a new augmentation category using the theory of categorical pushouts. Usually, taking a categorical pushout leads to an unwieldy result, however, we will show that it is of the form of an amalgamation.

5.3.1 Categorical Pushouts

It is known that if you take the algebraic free products with amalgamations of groups, then the original groups embed into this new group [91]. An amalgamation in an arbitrary category is a pushout along two monic maps.

**Definition 5.32.** A category is said to have the amalgamation property if amalgamations exists for all diagrams of monic maps $B \to A \to C$.

It can be shown that such diagrams share the analogous embedding property of free products with amalgamations of groups. Although the category of small categories, $\mathbf{Cat}$, does not satisfy the amalgamation property for all pushouts, there are some sufficient conditions which ensure the amalgamation property, this will be our next topic of focus.

Recall that for any functors $F_X: \mathcal{W} \to \mathcal{X}$ and $F_Y: \mathcal{W} \to \mathcal{Y}$ we can form the pushout category $\mathcal{Z}$ which makes the following universal diagram of functors commutative

\[
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{F_Y} & \mathcal{Y} \\
F_X \downarrow & & \downarrow G_Y \\
\mathcal{X} & \xrightarrow{G_X} & \mathcal{Z}
\end{array}
\]

For the rest of this section we will assume that $\mathcal{W}$ is a subcategory of both $\mathcal{X}$ and $\mathcal{Y}$ and that the functors $F_X$ and $F_Y$ are embeddings.

**Definition 5.33.** We will say that the pushout category $\mathcal{Z}$ is an amalgamation if the functors $G_X$ and $G_Y$ are embeddings.

Not all pushouts of this form are an amalgamation, some assumptions are needed on the embeddings $F_X$ and $F_Y$. This was first considered by Trnková in [108], but a more general condition was proved by MacDonald and Scull in [73].

**Definition 5.34** ([73, Definition 3.1]). A class of morphisms $\mathcal{M}$ of $\mathcal{X}$ has the 3-for-2 property when if $f, g$ and $h = g \circ f$ are morphisms in $\mathcal{X}$, if any two of $f, g$ and $h$ are in $\mathcal{M}$, then the third is also in $\mathcal{M}$.

**Definition 5.35** ([73, Definition 3.2]). A functor $F_Y: \mathcal{W} \to \mathcal{Y}$ has the 3-for-2 property if the set of image morphisms $\mathcal{F} := \{F_Y(\omega) \mid \omega \text{ a morphism in } \mathcal{W}\}$ satisfies the 3-for-2 property.
Theorem 5.36 ([73, Theorem 3.3]). If the functors $F_X: W \to X$ and $F_Y: W \to Y$ are embeddings which both satisfy the 3-for-2 property, then the induced functors $G_X: X \to Z$ and $G_Y: Y \to Z$ are also embeddings. Therefore $Z$ is an amalgamation.

Note that any full functor is automatically 3-for-2. The following lemma gives the structure of the category $Z$.

Lemma 5.37 ([44, Lemma 10.2]). Let $Z$ be an amalgamation of $X \leftarrow W \rightarrow Y$ such that the map $W \rightarrow X$ is full. Then:

- $Ob(Z) = Ob(Y) \bigsqcup (Ob(X) \setminus Ob(W))$.
- The morphisms of $Z$ have two forms:
  1. A morphism $X_0 \overset{f}{\rightarrow} X_1$ with $f \in Mor(X) \setminus Mor(W)$.
  2. A path $X_0 \overset{f_1}{\rightarrow} Y_1 \overset{d}{\rightarrow} Y_2 \overset{f_2}{\rightarrow} X_2$ where $d$ is a morphism in $Y$, and $f_1, f_2 \in Mor(X) \setminus Mor(W) \cup \{identities on Ob(Z)\}$. If $f_1$ is non-trivial then $Y_1 \in W$. If $f_2$ is non-trivial, then $Y_2 \in W$.

5.3.2 Amalgable Augmentation Categories

We shall now discuss when an augmentation category has the amalgamation property. We will then prove that for an amalgable augmentation category $Ψ$, the pushout $ΔS \sqcup ΔΨ$ is an augmentation category. First, recall that any morphism in $Δ$ can be decomposed in the form $SD$ where $S$ is a composition of degeneracy maps, and $D$ is a composition of face maps. Furthermore, any morphism in a crossed simplicial group $ΔS$, by definition, can be written in the form $TSD$ for $SD$ as above and $T$ a composition of morphisms of some $S_n$. We can then use Proposition 4.22 to rearrange this in to the form $S' T' D'$.

Definition 5.38. An augmentation category $Ψ$ is amalgable if $Δ$ is a sieve in $Ψ$.

Proposition 5.39. Let $Ψ$ be an amalgable augmentation category and $ΔS$ a crossed simplicial group. Then the category $ΨS := ΔS \sqcup ΔΨ$ is an amalgamation.

Proof. By assumption, the inclusion $Δ \rightarrow Ψ$ is a sieve, and consequently full. Therefore we need only check that the map $j: Δ \rightarrow ΔS$ is 3-for-2. As this map is not full, we must check it explicitly.

We are interested in the image set $ℐ := \{j(ω) \mid ω \text{ is a morphism in } Δ\}$. Recall that $ℐ$ has the 3-for-2 property if when if two of $f, g, h = g \circ f$ are in $ℐ$ then the third is also in $ℐ$. Assume that $f, g \in ℐ$ then $f = S_f D_f$, $f = S_g D_g$ and $h = g \circ f = S_f D_f S_g D_g \cong S_h D_h$ for some composition of face and degeneracy maps (we can always do the last step due to
the relations between face and degeneracy maps in \( \Delta \). Therefore \( h \in F \). Now without loss of generality assume that \( f, h \in F \), we must show that \( g \in F \). This follows again from the unique decomposition property, if \( g \neq F \) then \( g = S_g T_g D_g \) for some non-trivial composition \( T_g \). Therefore \( g \circ f = S_f D_f S_g T_g D_g = S_h T_h D_h \) for some composition of face, automorphism and degeneracy maps. Therefore \( g \circ f = h \notin F \) which is a contradiction.

Using Lemma 5.37, and the fact that \( \Delta \) is a sieve in \( \Psi \), we can explicitly describe the structure of \( \Psi G \).

**Lemma 5.40.** The category \( \Psi G \) has the following structure:

- \( \text{Ob}(\Psi G) = \text{Ob}(\Psi) \).

- The morphisms of \( \Psi G \) have the forms:
  1. A morphism \( S \xrightarrow{f} T \) with \( f \in \text{Mor}(\Psi) \setminus \text{Mor}(\Delta) \).
  2. A path \( [m] \xrightarrow{d} [n] \xrightarrow{f_2} T \) with \( d \in \text{Mor}(\Delta G) \) and \( f_2 \in \text{Mor}(\Psi) \setminus \text{Mor}(\Delta) \cup \{ \text{identities on \text{Ob}(\Psi G)} \} \).

**Proof.**

- We had for a general pushout that \( \text{Ob}(\mathcal{G}) = \text{Ob}(\Delta G) \bigsqcup (\text{Ob}(\Psi) \setminus \text{Ob}(\Delta)) \). Using the fact \( \text{Ob}(\Delta) = \text{Ob}(\Delta G) \), (as \( \Delta \) is wide in \( \Delta G \)), we get the result for the objects of \( \Psi G \).

- The morphisms come directly from Lemma 5.37 with the only difference being we do not have a map \( S \xrightarrow{f_1} [m] \) in the path. This is due to the fact that \( \Delta \) is a sieve in \( \Psi \) so no such map exists.

**Definition 5.41.** Let \( \mathcal{C} \) be a category and \( \mathcal{D} \) a subcategory. We will say that \( \mathcal{C} \) is \( \mathcal{D} \)-strict if for all objects \( d \in \mathcal{D} \) we have \( \text{Aut}_\mathcal{C}(d) \) trivial.

**Corollary 5.42.** Let \( \Psi \) be an amalgable augmentation category, then the pushout \( \Psi G := \Delta G \sqcup \Delta \Psi \) is a crossed \( \Psi \)-group.

**Proof.** Using Lemma 5.40, we have that \( \Psi \) is wide in \( \Psi G \). Next, note that because the inclusion is full, \( \Psi \) is \( \Delta \)-strict. Therefore we have that the morphisms \( [m] \xrightarrow{d} [n] \xrightarrow{f_2} T \) can be decomposed as an automorphism of \( [m] \) (namely \( \text{Aut}_{\Delta G}([m]) \)) followed by a map in \( \Psi \).
Finally, we show our desired result, which shows the compatibility between amalgamations and augmentation structures.

**Theorem 5.43.** Let $\Psi$ be an amalgable augmentation category, then the pushout $\Psi \mathcal{G}$ is an augmentation category.

**Proof.**

(AC1) We have shown in Corollary 5.42 that $\Psi \mathcal{G}$ has the structure of a crossed $\Psi$-group and is therefore an EZ-category. The tensor structure is provided by the tensor structure of $\Psi$.

(AC2) We have an inclusion $i: \Delta \hookrightarrow \Psi \mathcal{G}$ coming from the fact that the pushout has the amalgamation property. Moreover this inclusion is compatible with the tensor product by construction.

(AC3) The required pushout-product property holds as we have proved that it does in $\Psi$ and $\Delta \mathcal{G}$.

\[\square\]

**Example 5.44.** The category $\Omega$ is amalgable (we have already noted that $\Delta$ is a sieve in $\Omega$). Therefore for any crossed simplicial group $\Delta \mathcal{G}$ we can form an augmentation category $\Omega \mathcal{G}$. Just as $\hat{\Omega}_{\text{Kan}}$ is not a monoidal model category, we cannot expect $\hat{\Omega} \mathcal{G}_{\text{Kan}}$ to be monoidal. However, what is sensible to expect is that $\text{Ho}(\hat{\Omega} \mathcal{G}_{\text{Kan}})$ is enriched over $\text{Ho}(\Delta \mathcal{G}_{\text{Kan}})$.

**Question 5.45.** What is the homotopy type of $\hat{\Omega} \mathcal{G}_{\text{Kan}}$? Using all of the theory regarding $\hat{\Omega}_{\text{Kan}}$ and $\Delta \mathcal{G}_{\text{Kan}}$, one could sensibly conjecture that it will be of type $\text{ConSp} \left( \text{Top}^{\mathcal{G}_{\text{Kan}}} \right)$ where the model structure is the one that forgets the group action on the topological spaces.

**Remark 5.46.** We can carry forward the classification of crossed simplicial groups (Theorem 4.16) to amalgamations of $\Psi$ in an obvious way. Consider the following diagram of pushouts:
where the top sequence is the decomposition appearing in the classification theorem of crossed simplicial groups. We then note that we get an induced series of functors on the pushouts by commutativity of the squares (bottom sequence).
Chapter 6

Conclusion

This thesis introduced the general framework of augmentation categories, resulting in the theory of augmented homotopical algebraic geometry. To do so, we proved that for any augmentation category $\Psi$, the category of presheaves $\hat{\Psi}$ possesses a model structure where the fibrant objects are defined using a suitable notion of Kan complexes and the cofibrations are the normal monomorphisms. Due to the careful construction of this model category, we have the existence of a Quillen adjunction between $\hat{\Delta}_{\text{Kan}}$ and $\hat{\Psi}_{\text{Kan}}$. We then went on to develop local models on the category of augmented presheaves using the boundary object construction to define augmented hypercovers. The development of the augmented hypercovers also interplayed with the construction of the category of $n$-geometric augmented stacks using a modified $n$-hypergroupoid construction.

Equipped with the general framework, we considered two different examples of augmentation categories. The first example was that of crossed simplicial groups. We explored how using a crossed simplicial group in the machinery of augmented homotopical algebraic geometry led to an equivariant setting. Using this mindset we defined equivariant site cohomology as mapping spaces in the category of equivariant stacks over the site. By defining a crossed simplicial group version of the nerve construction, we showed that any 1-stack can be lifted to an equivariant stack. We used this construction to explore the $SO(2)$-equivariant version of the derived stack of local systems, explicitly computing the result in the specific case of $S^3_{\text{Hopf}}$.

The second example that we considered was that of the dendroidal category. After proving that it was in fact an augmentation category, we discussed how the Kan model structure captures the homotopy of connective spectra. We then used this example as a catalyst for discussing categorical amalgamations of augmentation categories. We proved that under some mild assumptions the pushout of an augmentation category with a crossed simplicial group yields a new augmentation category. Conjecturally, the augmentation
arising from such an amalgamation should be an equivariant version of the original augmentation.

**Future Work**

In the body of the thesis we have already outlined some potential directions for further research (Questions 5.26 and 5.45, Conjectures 3.52 and 3.64). We take this final opportunity to discuss two more questions which would be of interest to consider.

**Classification of Augmentation Categories**

We have already seen in Remark 5.46 that it is possible to classify the amalgamations of a single amalgable augmentation category $\Psi$ by using the classification of crossed simplicial groups. However, what would be more useful is a classification of the collection of (amalgable) augmentation categories. Such a classification would of course have a homotopical link with the different possible ways of extending the theory of topological spaces to other categories in different homotopical contexts.

**Test Category Structure**

It seems evident that there should be a strong link between the theory of augmentation categories and that of test categories. Recall that a test category is one that can be used to model the homotopy theory of $\text{Top}$ [49]. In particular, for a category $A$ to be a test category, there exists a model structure on $\hat{A}$ which is Quillen equivalent to $\text{Top}$. It was proved in [2] that the dendroidal category is in fact a test category. The crossed simplicial groups however are not test categories, the reason being the non-trivial automorphisms on the objects $[n]$ (recall that for $\Omega$ we have $\text{Aut}([n]) = \emptyset$). It seems sensible to suggest that for a $\Delta$-strict augmentation category, it is possible to define a test category model structure by localising the Kan model structure at the representable objects (compare this claim with [2, Theorem 6.5]).

We can extend this idea further still. For $G$ a compact Lie group, say that a category $A$ is a $G$-test category if there exists a model structure on $\hat{A}$ which is Quillen equivalent to the weak model structure on $\text{Top}^G$. For example, the cyclic category $\Delta\mathcal{C}$ is the canonical example of a $SO(2)$-test category. One would expect that for $\Psi$ an amalgable augmentation category, the augmentation $\Psi\mathcal{G}$ should be a $|\mathcal{G}|$-test category by again localising at the representable objects. The above discussion can be phrased as the following conjecture.

**Conjecture 6.1.** Let $\Psi$ be a $\Delta$-strict augmentation category, then $\Psi$ is a test category. Moreover, if $\Psi$ is amalgable, then $\Psi\mathcal{G}$ is a $|\mathcal{G}|$-test category.
Appendix A

Planar Crossed Simplicial Groups

In this appendix we will consider model structures on \( \hat{\Delta G} \) for a particular class of crossed simplicial groups which are not Quillen equivalent to \( \hat{\Delta G}_{\text{Kan}} \). These model structures are stronger in an equivariant homotopy sense. We will use these model structures to introduce stronger models on \( \hat{\Delta G}^{\text{eq}} \). We expect that the outcome of this will give access to Bredon style cohomology theories as opposed to the Borel type cohomology theories which arises from \( \hat{\Delta G}_{\text{Kan}} \).

Outline of Appendix A

(Section A.1) We look at planar crossed simplicial groups, and develop two new model structures on them. We provide a Quillen equivalence between these models and certain models arising from equivariant homotopy theory.

(Section A.2) We develop local model structures for the presheaf categories using the two above model structures in the projective model as opposed to \( \hat{\Delta G}_{\text{Kan}} \).

A.1 Planar Crossed Simplicial Groups

We begin by introducing planar crossed simplicial groups. These were first specifically studied in [38] and further in [102], they are important as they correspond to Lie groups that appear as structure groups of surfaces. For us they will be of interest as their geometric realisation are compact Lie groups (in all but two exceptional cases) which arise in the theory of equivariant homotopy theory. We will use this link to construct new model structures on \( \hat{\Delta G} \).

Definition A.1. A planar crossed simplicial group is a crossed simplicial group which is of type \( \Delta \mathcal{C} \) or \( \Delta \mathcal{D} \) in the classification theorem (Theorem 4.16).
Planar crossed simplicial groups are an interesting thing to study as their geometric realisations are planar Lie groups, which can be used to add extra structure to surfaces as done in [38].

**Definition A.2** ([38, Definition 1.29]). A morphism of Lie groups \( \rho: \tilde{G} \rightarrow G \) is a connective covering if \( \rho \) is a covering of its image and \( \rho^{-1}(G_e) \) coincides with \( \tilde{G}_e \). A planar Lie group is a connective covering of \( O(2) \).

Planar Lie groups have a full classification, and this classification matches up exactly with the planar crossed simplicial groups [38, Proposition I.32.].

Table A.1 gives all of the planar crossed simplicial groups along with their geometric realisations, where we consider all \( M, N \in \mathbb{N} \).

| \( \Delta \mathcal{G} \)           | \( \mathcal{G}_n \) | \( |\mathcal{G}| \) | Compact? |
|----------------------------------|---------------------|--------------------|----------|
| Cyclic - \( \Delta \mathcal{C} \) | \( \mathbb{Z}/(n+1) \) | \( SO(2) \)          | Y        |
| Dihedral - \( \Delta \mathcal{D} \) | \( D_{n+1} \)        | \( O(2) \)          | Y        |
| Paracyclic - \( \Delta \mathcal{C}_\infty \) | \( \mathbb{Z} \)     | \( \widetilde{SO}(2) \) | N        |
| Paradihedral - \( \Delta \mathcal{D}_\infty \) | \( D_\infty \)       | \( \widetilde{O}(2) \) | N        |
| \( N \)-Fold Cyclic - \( \Delta \mathcal{C}_N \) | \( \mathbb{Z}/N(n+1) \) | \( \text{Spin}_N(2) \) | Y        |
| \( N \)-Fold Dihedral - \( \Delta \mathcal{D}_N \) | \( D_{N(n+1)} \)     | \( \text{Pin}^+_N(2) \) | Y        |
| \( M \)-Fold Quaternion - \( \Delta \mathcal{Q}_M \) | \( Q_{M(n+1)} \)     | \( \text{Pin}^-_{2M}(2) \) | Y        |

**Table A.1:** The classification of planar crossed simplicial groups.

The groups \( \widetilde{SO}(2) \) and \( \widetilde{O}(2) \) are the universal covers of \( SO(2) \) and \( O(2) \) respectively.

**Remark A.3.** The planar Lie groups have the structure as indicated in Figure A.1, which is mirrored by functors in the realm of crossed simplicial groups.

For the rest of this appendix we will only be interested in the planar crossed simplicial groups which have a compact group as its realisation. The only two planar crossed simplicial groups which are not compact are the paracyclic and paradihedral ones as \( \widetilde{SO}(2) \simeq \mathbb{R} \). We will refer to the remaining planar crossed simplicial groups as the compact planar crossed simplicial groups.

### A.1.1 Equivariant Homotopy Theory

Equivariant homotopy theory is the study of homotopy categories of topological spaces with an action of a compact Lie group (see [40, 79]). To such a category we can assign a
The whole collection of models, which look at the fixed point data of subgroups of the acting Lie group.

**Theorem A.4** ([92, Appendix A, Proposition 2.10]). Let \( G \) be a compact Lie group, and \( \mathbf{Top}^G \) the category of spaces with \( G \) action. Let \( \mathcal{F} \) be any family of closed subgroups of \( G \), then there is a model structure on \( \mathbf{Top}^G \) in which a map \( f : X \to Y \) is a:

- **Weak equivalence** if the induced maps \( f^H : X^H \to Y^H \) are weak equivalences for all \( H \in \mathcal{F} \).
- **Fibration** if the induced maps \( f^H : X^H \to Y^H \) are fibrations for all \( H \in \mathcal{F} \).
- **Cofibration** if it has the LLP with respect to trivial fibrations.

The following definition is the generalisation of the cyclic realisation and singular functors that we introduced in Section 4.1.1.3. They allow us to move between the discrete and topological settings.

**Definition A.5.** Let \( \Delta \mathcal{E} \) be a planar crossed simplicial group, then there is a realisation functor \( |−|_\mathcal{E} : \Delta \mathcal{E}\text{-Set} \to \mathbf{Top}^{[\mathcal{E}]} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\Delta \mathcal{E}\text{-Set} & \xrightarrow{|−|_\mathcal{E}} & \mathbf{Top}^{[\mathcal{E}]}
\end{array}
\]

where \( |i^*−| \) is the realisation of the underlying simplicial set and \( u \) is the forgetful functor. This functor admits a right adjoint called the singular functor \( S_{[\mathcal{E}]} \) defined as

\[
S_{[\mathcal{E}]} (X)_n = \text{Hom}_{\mathbf{Top}^{[\mathcal{E}]}} ([\mathcal{E}] \times \Delta_n, X).
\]

We have already seen from the cyclic case, that \( \Delta \mathcal{E}_{\text{Kan}} \) is Quillen equivalent to the model structure arising from Theorem A.4 when \( \mathcal{F} = \{\emptyset\} \) as we only consider the underlying
simplicial object. The goal of the following sections will be to equip $\hat{\Delta}\mathfrak{G}$ with model structures Quillen equivalent to model structures where:

- $\mathcal{F} = \{\text{all closed finite subgroups of } G\}$. We will denote this model $\hat{\Delta}\mathfrak{G}_{\text{Str}}$ and call it the strong model. The corresponding topological model will be denoted $\text{Top}^{\mathcal{C}^G}$.

- $\mathcal{F} = \{\text{all closed subgroups of } G\}$. We will denote this model $\hat{\Delta}\mathfrak{G}_{\text{Cpld}}$ and call it the coupled model. The corresponding topological model will be denoted $\text{Top}^{\leq G}$.

We will also consider a collection of intermediate model structures which can be accessed between $\hat{\Delta}\mathfrak{G}_{\text{Kan}}$ and $\hat{\Delta}\mathfrak{G}_{\text{Str}}$.

**A.1.2 Strong Homotopy Theory**

We now look at the strong model structure which was introduced by Spaliński for $\Delta\mathcal{C}$, $\Delta\mathcal{D}$ and $\Delta\Omega$ ([98, 99]). We will begin by reviewing the cyclic case, extending it to the $N$-fold cyclic category. The $N$-fold cases follow almost immediately from the $N = 1$ case, and could be considered a corollary of the work of Spaliński. The following terminology and theorem appears in [98, §2].

Let $\mathcal{D}$ be a category closed under coproducts, $I$ an arbitrary indexing set and suppose we have a family of adjoint functors $\Psi_i : \hat{\Delta} \rightleftarrows \mathcal{D} : \Phi_i$. We denote by $\mathcal{F}$ the set of horn inclusions in $\hat{\Delta}$ and $\mathcal{G}$ the set of boundary inclusions. For an $f \in \mathcal{F} \cup \mathcal{G}$, let $X_f = \text{Domain}(f)$ and $Y_f = \text{Range}(f)$. Let $E$ be the set of elements $(e, j, f, g)$ where $e$ is an index, $j \in I$ and $g : \Psi_j(X_f) \to Z$. A $\Psi_\ast$-regular pushout is a pushout of the following form:

\[
\begin{array}{c}
\prod_{(e,j,f,g) \in E} \Psi_j X_f \\
\prod_{\Psi_j f} \Psi_j f
\end{array}
\xrightarrow{\Sigma_g} Z
\]

The morphism $h$ is said to be $\Psi_\ast$-induced from $\mathcal{F} \cup \mathcal{G}$.

We say that an object $A$ of $\mathcal{D}$ is $\Psi_\ast$-sequentially small with respect to $\mathcal{F} \cup \mathcal{G}$ if $\text{Hom}_\mathcal{D}(A, -)$ commutes with sequential colimits of diagrams in which all morphisms are $\Psi_\ast$-induced from $\mathcal{F} \cup \mathcal{G}$.

**Theorem A.6** ([98, Theorem 2.14]). Let $\mathcal{D}$ be a category closed under coproducts, $I$ an arbitrary indexing set, and $\mathcal{H} = \{(\Psi_i, \Phi_i) \mid i \in I\}$ a family of adjoint functors $\Psi_i : \hat{\Delta} \rightleftarrows \mathcal{D} : \Phi_i$ such that:

1. $\mathcal{D}$ has all finite limits and arbitrary small colimits.
2. For all horn inclusions \( f \), all \( i, j \in I \), \( \Phi_i \Psi_j(f) \) is a trivial cofibration.

3. For \( X \) a simplicial horn or boundary, and \( j \in I \), the object \( \Psi_j X \) is \( \Psi_* \)-sequentially small with respect to all horn and boundary inclusions.

4. For all \( i \in I \) the functor \( \Phi_i \):
   
   (a) Preserves coproducts.
   
   (b) Takes \( \Psi_* \)-regular pushouts to homotopy pushout diagrams.
   
   (c) Preserves sequential colimits in \( D \) in which the morphisms are \( \Psi_* \)-induced from horn and boundary inclusions.

Then there is a model structure on \( D \) in which a functor \( f : X \to Y \) is a:

- **Weak equivalence** if and only if for all \( i \in I \), the map \( \Phi_i(f) : \Phi_i(X) \to \Phi_i(Y) \) is a weak equivalence in \( \hat{\Delta}_{Kan} \).

- **Fibration** if and only if for all \( i \in I \), the map \( \Phi_i(f) : \Phi_i(X) \to \Phi_i(Y) \) is a fibration in \( \hat{\Delta}_{Kan} \).

- **Cofibration** if it has the LLP with respect to trivial fibrations.

To use Theorem A.6 for model structures on \( \hat{\Delta} \Theta \) we need only define the family of adjoint functors. We will only define one side of the adjunction, namely the \( \Phi_i \) family, as the \( \Psi_i \) are simply constructed from these in a slightly technical way. We begin with the cyclic case.

**Definition A.7.** Let \( X \) be a simplicial set, define the \( r \)-fold edgewise subdivision of \( X \) to be the simplicial set \( \text{sd}_r X \) where \( \text{sd}_r X_n = X_{r(n+1)-1} \), and the face and degeneracy maps are described in terms of the face and degeneracies of \( X \) in the following way:

\[
\begin{align*}
d'_i &= d_i \cdot d_{i+(n+1)} \cdots d_{i+(r-1)(n+1)} : X_{r(n+1)-1} \to X_{rn-1}, \\
s'_i &= s_i \cdot s_{i+(r-1)(n+1)} \cdots s_{i+(n+1)} \cdot s_i : X_{rn-1} \to X_{r(n+1)-1}.
\end{align*}
\]

**Remark A.8.** Let \( X \in \hat{\Delta}C_N \) be an \( N \)-fold cyclic set, let \( \text{sd}_{Nr} X \) denote the subdivision of the underlying simplicial set. Then \( \text{sd}_{Nr} X \) carries a natural \( \mathbb{Z}/Nr \) action given by

\[
\theta_n : \text{sd}_{Nr} X_n \to \text{sd}_{Nr} X_n \\
x \mapsto \tau^{n+1}_{Nr(n+1)-1}(x)
\]
where \( \tau_n: [n] \to [n] \) is the cyclic operator. Therefore for every \( r \geq 1 \) there is a functor

\[
\Phi_{Nr}: \hat{\Delta}C_N \to \hat{\Delta}
\]

\[
X \mapsto (\text{sd}_{Nr} X)^{\mathbb{Z}/Nr}
\]

**Proposition A.9.** The category \( \hat{\Delta}C_N \) of \( N \)-fold cyclic sets has a cofibrantly generated model category structure in which a map \( f: X \to Y \) is a:

- **Weak equivalence** if and only if for all \( r \geq 1 \), the map \( \Phi_{Nr} (f): \Phi_{Nr}(X) \to \Phi_{Nr}(Y) \) is a weak equivalence in \( \hat{\Delta}_{Kan} \).

- **Fibration** if and only if for all \( r \geq 1 \), the map \( \Phi_{Nr} (f): \Phi_{Nr}(X) \to \Phi_{Nr}(Y) \) is a fibration in \( \hat{\Delta}_{Kan} \).

- **Cofibration** if it has the LLP with respect to trivial fibrations.

We call the above model structure on \( \hat{\Delta}C_N \) the strong model structure and we will denote it by \( \hat{\Delta}C_{N Str} \). Moreover there is a Quillen equivalence \( \hat{\Delta}C_{N Str} \rightleftarrows \text{Top}^{\text{Spin}_N(2)} \).

**Proof.** The case of \( N = 1 \) is given in [98, Theorem 3.10], where it is shown that the functors \( \Phi_r \) are part of an adjunction which satisfy the conditions of Theorem A.6. For \( N > 1 \), we are looking at a sub-collection of the functors for \( N = 1 \), which therefore still have the desired properties. Therefore we have the existence of the model structure. For the Quillen adjunction, the detailed case of \( N = 1 \) is given in [98, Theorem 5.1], where it is shown using the cyclic realisation/singular adjunction. As will be a common theme in this appendix, due to the properties shared by the cyclic and \( N \)-fold cyclic realisations, these conditions trivially hold for the \( N \)-fold case, and the result follows. \( \square \)

We now move on to consider the dihedral and quaternionic cases. In this case we need to be able to access the fixed point data of the reflexive part of the groups, so to do this we introduce another subdivision functor.
Definition A.10. Let $X$ be a simplicial set, define the *Segal subdivision* of $X$ to be the simplicial set $\text{sq}X$ where $\text{sq}X_n = X_{2n+1}$, and the face and degeneracy maps are described in terms of the face and degeneracies of $X$ in the following way:

$$
\begin{align*}
    d'_i &= d_i \cdot d_{2n+1-i} : X_{2n+1} \to X_{2n-1}, \\
    s'_i &= s_{2n-1} \cdot s_i : X_{2n-1} \to X_{2n+1}.
\end{align*}
$$

Figure A.3: Segal Subdivision of $\Delta[2]$.

Remark A.11. If $X \in \mathcal{D}_N$ is an $N$-fold dihedral set, let $\text{sq}X$ denote the Segal subdivision of the underlying simplicial set. Then $\text{sq}X$ carries a natural $\mathbb{Z}/2$ action given by

$$
\rho_n : \text{sq}X \to \text{sq}X \\
    x \mapsto \omega_{2n+2}(x)
$$

where $\omega_n : [n] \to [n]$ is the reflexive operator. Therefore there is a functor

$$
\Gamma_1 : \mathcal{D}-\text{Set} \to \hat{\Delta} \\
    X \mapsto (\text{sq}X)^{\mathbb{Z}/2}
$$

Using a combination of the two subdivision functors that we have introduced we can now access the fixed point data for any dihedral group.

Definition A.12. For $X$ a simplicial set and $r \geq 1$, define the *$r$-dihedral subdivision functor* $\text{sbd}_r$ to be

$$
\text{sbd}_r X = \text{sq}(\text{sd}_r(X))
$$

Which by Remarks A.8 and A.11 gives a functor

$$
\Gamma_{Nr} : \mathcal{D}_N \to \hat{\Delta} \\
    X \mapsto (\text{sbd}_r X)^{DNr}
$$
Proposition A.13. The category $\hat{\Delta} \mathcal{D}_N$ has a cofibrantly generated model structure where a map $f : X \to Y$ is a:

- Weak equivalence if for all $r \geq 1$ the maps
  \[ \Phi_{Nr}(f) : \Phi_{Nr}(X) \to \Phi_{Nr}(Y), \Gamma_{Nr}(f) : \Gamma_{Nr}(X) \to \Gamma_{Nr}(Y) \]
  are weak equivalences in $\hat{\Delta}_{\text{Kan}}$.

- Fibrations if for all $r \geq 1$ the maps
  \[ \Phi_{Nr}(f) : \Phi_{Nr}(X) \to \Phi_{Nr}(Y), \Gamma_{Nr}(f) : \Gamma_{Nr}(X) \to \Gamma_{Nr}(Y) \]
  are fibrations in $\hat{\Delta}_{\text{Kan}}$.

- Cofibration if it has the LLP with respect to the class of trivial fibrations.

We call the above model structure on $\hat{\Delta} \mathcal{D}_N$ the strong model structure and we will denote it by $\hat{\Delta} \mathcal{D}_N^{\text{Str}}$. Moreover there is a Quillen equivalence $\hat{\Delta} \mathcal{D}_N^{\text{Str}} \rightleftharpoons \text{Top}^{<\text{Pin}_R^{(2)}}$.

Proof. The existence of the model for $N = 1$ is given in [99, Theorem 4.3], as we already have the functors $\Phi_r$ satisfying the conditions of Theorem A.6, what is shown is that the family $\Gamma_r$ also satisfies these assumptions. The Quillen equivalence for $N = 1$ is given in [99, Theorem 4.3]. The case for $N > 1$ follows as discussed in the proof of Proposition A.9. \( \square \)

We complete this section with a model for $\hat{\Delta} \mathcal{Q} M$ which follows very easily from the dihedral case. We see that the group $Q_r$ also acts on $\text{sbd}_r X$ where $X$ is a quaternionic set. Therefore for $r \geq 1$ we define the functor

\[ \Gamma^q_r : \Delta \Omega \text{-Set} \to \hat{\Delta} \]

\[ X \mapsto (\text{sbd}_r X)^{Q_r} \]

Lemma A.14 ([99, Theorem 5.2, Theorem 5.3]). The category $\hat{\Delta} \mathcal{Q} M$ has a cofibrantly generated model structure where a map $f : X \to Y$ is a:

- Weak equivalence if for all $r \geq 1$ the maps
  \[ \Phi_{Mr}(f) : \Phi_{Mr}(X) \to \Phi_{Mr}(Y), \Gamma^q_{Mr}(f) : \Gamma^q_{Mr}(X) \to \Gamma^q_{Mr}(Y) \]
  are weak equivalences in $\hat{\Delta}_{\text{Kan}}$. 
• Fibrations if for all \( r \geq 1 \) the maps

\[
\Phi_{Mr}(f) : \Phi_{Mr}(X) \to \Phi_{Mr}(Y), \Gamma^q_{Mr}(f) : \Gamma^q_{Mr}(X) \to \Gamma^q_{Mr}(Y)
\]

are fibrations in \( \hat{\Delta}_{Kan} \).

• Cofibration if it has the LLP with respect to the class of trivial fibrations.

We call the above model structure on \( \hat{\Delta}_M \) the strong model structure and we will denote it by \( \hat{\Delta}_{M\text{Str}} \). Moreover there is a Quillen equivalence \( \hat{\Delta}_{M\text{Str}} \simeq \text{Top}^{<\text{Spin}_{M}(2)} \).

### A.1.3 Intermediate Homotopy Theories

By restricting to a finite \( k \in \mathbb{N} \), we can access models which live between the Kan and strong model structures. In this section we will only discuss the appropriate theory for the \( N \)-fold cyclic case, but it works mutatis mutandis for all of the compact planar crossed simplicial groups. The follow proposition follows from the discussion in the previous section.

**Proposition A.15.** For all \( k > 0 \), the category \( \hat{\Delta}_C \) of \( N \)-fold cyclic sets has a cofibrantly generated model category structure in which a map \( f : X \to Y \) is a:

• Weak equivalence if and only if for all \( 1 \leq r \leq k \), the map \( \Phi_{Nr}(f) : \Phi_{Nr}(X) \to \Phi_{Nr}(Y) \) is a weak equivalence in \( \hat{\Delta}_{Kan} \).

• Fibration if and only if for all \( 1 \leq r \leq k \), the map \( \Phi_{Nr}(f) : \Phi_{Nr}(X) \to \Phi_{Nr}(Y) \) is a fibration in \( \hat{\Delta}_{Kan} \).

• Cofibration if it has the LLP with respect to trivial fibrations.

We shall call the above model structure on \( \hat{\Delta}_C \) the \( k \)-intermediate model structure and we will denote it by \( \hat{\Delta}_{C\text{Str}} \). Moreover there is a Quillen equivalence

\[
\hat{\Delta}_{C\text{Str}} \simeq \text{Top}^{<\text{Spin}_{N}(2)}
\]

where \( \text{Top}^{<\text{Spin}_{N}(2)} \) is the model structure on \( \text{Top}^{\text{Spin}(2)} \) with \( F = \{ C_{Ni} \mid i \leq k \} \).

We can see that as we increase \( k \), the conditions for a map to be a weak equivalence or fibration become more strict. Due to this, the set of cofibrations will be increasing. This idea is summed up in the following lemma.
Lemma A.16.

\[
\begin{align*}
W^\text{\text{-}k+1}_{\Delta N_{\text{Str}}} & \subset W^\text{-}k_{\Delta N_{\text{Str}}} , \\
\text{Fib}^\text{\text{-}k+1}_{\Delta N_{\text{Str}}} & \subset \text{Fib}^\text{-}k_{\Delta N_{\text{Str}}} , \\
\text{Cof}^\text{\text{-}k+1}_{\Delta N_{\text{Str}}} & \supset \text{Cof}^\text{-}k_{\Delta N_{\text{Str}}}. 
\end{align*}
\]

Corollary A.17.

\[
\text{Ho}\left(\Delta^\text{-}k+1_{N_{\text{Str}}}\right) \subset \text{Ho}\left(\Delta^\text{-}k_{N_{\text{Str}}}\right).
\]

A.1.4 Coupled Homotopy Theory

We now move onto a different collection of model categories. In this case we will develop discrete models with respect to all closed subgroups of \(G\), not just the finite subgroups. Note that the action, for example, of \(SO(2)\) on any cyclic set is discrete, in [42] the fixed point set is given as

\[
\text{Fix} = \{x \in X_0 \mid s_0 x = s_1 x = t_1 s_0 x\}.
\]

Therefore to capture this extra data, we must construct a new category which extends \(\Delta\) in such a way we can keep track of the infinite group action.

Definition A.18 ([17, Definition 1.2]). Let \(\mathcal{C}\) and \(\mathcal{D}\) be categories, and \(F: \mathcal{C} \to \mathcal{D}\) a functor. Then the category \(\mathcal{C}^\cdot \mathcal{D}\) has:

1. Objects given by triples \((A, B, FA \to B)\) with \(A \in \mathcal{C}\) and \(B \in \mathcal{D}\).
2. Morphisms specified by maps \(f_1: A \to A'\) and \(f_2: B \to B'\) such that the following diagram commutes:

\[
\begin{array}{ccc}
FA & \longrightarrow & B \\
\downarrow_{Ff_1} & & \downarrow_{f_2} \\
FA' & \longrightarrow & B'
\end{array}
\]

Remark A.19. The construction of Definition A.18 can be seen as the comma category \((f/1_D)\) [71]. Therefore it can also be seen as a presentation of the lax limit of \(F: A \to B\) in \(\mathfrak{Cat}\) [62].

Definition A.20. Let \(\mathcal{C}\) and \(\mathcal{D}\) be model categories. A functor \(F: \mathcal{C} \to \mathcal{D}\) is \textit{Reedy admissible} if \(F\) preserves colimits and \(F\) has the property that given a morphism

\[
(A, B, FA \to B) \to (A', B', FA' \to B')
\]
Planar Crossed Simplicial Groups

\[ A' \to A \text{ is a trivial cofibration in } \mathcal{C} \text{ and} \]
\[ FA' \bigcup_{FA} B \to B' \]
is a trivial cofibration in $\mathcal{C}$ then $B \to B'$ is a weak equivalence in $\mathcal{D}$. For example, any left Quillen functor is Reedy admissible.

**Theorem A.21** ([17, Theorem 1.1]). Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and $F : \mathcal{C} \to \mathcal{D}$ a Reedy admissible functor. Then $\mathcal{C}_F \mathcal{D}$ admits a model structure where a map
\[ (A, B, FA \to B) \to (A', B', FA' \to B') \text{ in } \mathcal{C}_F \mathcal{D} \]
is a:

- **Weak equivalence** if $A \to A'$ is a weak equivalence in $\mathcal{C}$ and $B \to B'$ is a weak equivalence in $\mathcal{D}$.

- **Fibration** if $A \to A'$ is a fibration in $\mathcal{C}$ and $B \to B'$ is a fibration in $\mathcal{D}$.

- **Cofibration** if $A \to A'$ is a cofibration in $\mathcal{C}$ and $FA' \bigcup_{FA} B \to B'$ is a cofibration in $\mathcal{D}$.

We now give two results about how cofibrant generation and properness interact with the model structure of Theorem A.21.

**Lemma A.22.** If $\mathcal{A}$, $\mathcal{B}$ are cofibrantly generated model categories and $F : \mathcal{A} \to \mathcal{B}$ is Reedy admissible, then the category $\mathcal{A}_F \mathcal{B}$ is cofibrantly generated.

**Proof.** We know that a map in $\mathcal{A}_F \mathcal{B}$ is a cofibration if and only if $A \to A'$ is a cofibration in $\mathcal{A}$ and $FA' \bigcup_{FA} B \to B'$ is a cofibration in $\mathcal{B}$. We also know that $\mathcal{A}$ and $\mathcal{B}$ have generating sets of cofibrations. Therefore the generating cofibrations of $\mathcal{A}_F \mathcal{B}$ is some $C \subset A \times B$. \( \square \)

**Lemma A.23** ([17, Lemma 7.10]). If $\mathcal{A}$, $\mathcal{B}$ are left proper model categories and $F : \mathcal{A} \to \mathcal{B}$ is Reedy admissible, then the category $\mathcal{A}_F \mathcal{B}$ is left proper.

A.1.4.1 Cyclic Sets

We begin with the cyclic case as proved by Blumberg in [17]. We will reproduce his proofs here, as this will allow us to describe the changes needed for the $N$-fold cyclic case. We begin by constructing a Reedy admissible functor $\nabla : \hat{\Delta}_{\text{Kan}} \to \hat{\Delta}_{\text{Str}}$ such that $|X|^H \simeq |\nabla X|^H$ for all finite $H < SO(2)$. 

**Definition A.24.** Let $\nabla_n = S\xi(\Delta|n|)$. Then we have that $\nabla^\bullet$ is a cosimplicial cyclic set. We then define

$$\nabla : \hat{\Delta} \to \widehat{\Delta\xi},$$

$$\nabla X = X \otimes_{\Delta_{op}} \nabla^\bullet.$$

This functor has a right adjoint $A$ given by $A(Y)_n = \text{Hom}_{\widehat{\Delta\xi}}(\nabla_n, Y)$.

We consider the category $\mathcal{P} := \hat{\Delta}\nabla\widehat{\Delta\xi}$ where $\hat{\Delta}$ is equipped with the Kan model structure and $\widehat{\Delta\xi}$ is equipped with the strong model structure.

**Lemma A.25 ([17, Lemma 4.2]).** The functor $(-)^G$ on based $G$-spaces preserves pushouts of diagrams where one leg is a closed inclusion.

**Lemma A.26 ([17, Lemma 4.5]).** The functor $\nabla$ is Reedy admissible.

**Proof.** To show this we need to show that given a map

$$(A, B, \nabla A \to B) \to (A', B', \nabla A' \to B')$$

in $\mathcal{P}$ such that $A \to A'$ is a trivial cofibration and

$$\nabla A' \bigsqcup_{\nabla A} B \to B'$$

is a trivial cofibration, then the map $B \to B'$ is a weak equivalence. We begin by noting that the map $B \to B'$ is the composite

$$B \to \nabla A' \bigsqcup_{\nabla A} B \to B'$$

and we know by assumption that the second map is a weak equivalence. Therefore it will suffice just to show that the first map is a weak equivalence. By taking the realisation this is equivalent to showing that

$$|B|_\xi^H \to \left| \nabla A' \bigsqcup_{\nabla A} B \right|_\xi^H$$

is a weak equivalence of spaces for all finite $H \subset SO(2)$. We then use the fact that the geometric realisation is a colimit to see that

$$\left| \nabla A' \bigsqcup_{\nabla A} B \right|_\xi^H \simeq \left( \left| \nabla A' \right|_\xi \bigsqcup \left| B \right|_\xi \right)^H.$$
Since $A \to A'$ is a cofibration of simplicial sets, it is an inclusion and $|\nabla A|_\varepsilon \to |\nabla A'|_\varepsilon$ is a closed inclusion. By Lemma A.25 the fixed-point functor commutes with the pushout so we get

$$
\left( |\nabla A'|_\varepsilon \bigsqcup |B|_{\Delta \varepsilon} |C \right)^H \simeq |\nabla A'|_\varepsilon^H \bigsqcup |B|_\varepsilon^H.
$$

Therefore we see that $|B|_\varepsilon^H \to |\nabla A'|_\varepsilon^H \bigsqcup |\nabla A|_\varepsilon^H |B|_\varepsilon^H$ is a pushout of a trivial cofibration and therefore itself a trivial cofibration.

**Corollary A.27 ([17, Corollary 4.7]).** There exists a cofibrantly generated model structure on $\mathcal{P}$ in which a map is a:

- Weak equivalence if $A \to A'$ is a weak equivalence of simplicial sets and $B \to B'$ is a weak equivalence of cyclic sets.
- Fibration if $A \to A'$ is a fibration of simplicial sets and $B \to B'$ is a fibration of cyclic sets.
- Cofibration if $A \to A'$ is a cofibration of simplicial sets and $\nabla A' \bigsqcup \nabla A B \to B'$ is a cofibration of cyclic sets.

We now must construct a bijection between our two categories of interest, $\mathcal{P}$ and $\text{Top}^{SO(2)}$.

**Definition A.28.** The functor $L: \mathcal{P} \to \text{Top}^{SO(2)}$ takes a triple $(A, B, \nabla A \to B)$ to the pushout in the diagram

$$
\begin{array}{ccc}
|\nabla A|_\varepsilon & \longrightarrow & |B|_\varepsilon \\
\zeta \downarrow & & \downarrow \\
|A| & \longrightarrow & X
\end{array}
$$

where the map $\zeta: |\nabla A|_\varepsilon \to |A|$ is the natural map which induces weak equivalences on passage to fixed point subspaces for all finite subgroups of $SO(2)$. A morphism $(A, B, \nabla A \to B) \to (A', B', \nabla A' \to B')$ gives rise to a commutative diagram:

$$
\begin{array}{ccc}
|A| & \longrightarrow & |\nabla A|_\varepsilon \\
\downarrow & & \downarrow \\
|A'| & \longrightarrow & |\nabla A'|_\varepsilon \\
\end{array}
\hspace{1cm}
\begin{array}{ccc}
|B|_\varepsilon & \longrightarrow & |B'|_\varepsilon \\
\downarrow & & \downarrow \\
|B'|_\varepsilon & \longrightarrow & |B'|_\varepsilon
\end{array}
$$

**Definition A.29.** The functor $R: \text{Top}^{SO(2)} \to \mathcal{P}$ takes $X$ to the triple

$$
\left( S \left( X^{SO(2)} \right), S_\varepsilon (X), \nabla S \left( X^{SO(2)} \right) \to S_\varepsilon (X) \right)
$$
Here the map $\nabla S \left( X^{SO(2)} \right) \to S_\mathcal{E}(X)$ is the adjoint to the composite

$$\nabla S \left( X^{SO(2)} \right) \big|_\mathcal{E} \to S \left( X^{SO(2)} \right) \to X^{SO(2)} \hookrightarrow X.$$ 

A map $X \to Y$ in $\text{Top}^{SO(2)}$ induces maps $S \left( X^{SO(2)} \right) \to S \left( Y^{SO(2)} \right)$ and $S_\mathcal{E}(X) \to S_\mathcal{E}(Y)$ which fit in the following commutative diagram:

\[
\begin{array}{ccc}
\nabla S \left( X^{SO(2)} \right) & \longrightarrow & S_\mathcal{E}(X) \\
\downarrow & & \downarrow \\
\nabla S \left( Y^{SO(2)} \right) & \longrightarrow & S_\mathcal{E}(Y)
\end{array}
\]

**Theorem A.30** ([17, Section 6]). The functors $L$ and $R$ specify a Quillen equivalence between $\mathcal{P}$ equipped with the model structure given in Corollary A.27 and $\text{Top}^{\leq SO(2)}$.

To prove this we first of all need to show that we get a Quillen adjunction from the adjoint pair $(L, R)$.

**Lemma A.31** ([17, Proposition 5.14]). The adjoint pair $(L, R)$ is a Quillen adjunction between $\mathcal{P}$ and $\text{Top}^{\leq SO(2)}$.

**Proof.** We just need to show that $R$ preserves fibrations and trivial fibrations as in the case for $L$ preserving cofibrations and trivial cofibrations is completely analogous. If $f : X \to Y$ is a (trivial) fibration, then $S(X) \to S(Y)$ and $S_\mathcal{E}(X) \to S_\mathcal{E}(Y)$ are (trivial) fibrations in $\hat{\Delta}_{\text{Kan}}$ and $\hat{\Delta}_{\text{Str}}$ respectively as $S(-)$ and $S_\mathcal{E}(-)$ are right Quillen adjoints in their own right. \qed

We now move towards proving the condition for the adjoint pair to be an equivalence. First, we recall some lemmas from [17]

**Lemma A.32** ([17, Lemma 6.3]). If $X \to Y$ is a cofibration of cyclic sets, then the induced map $|X|^{SO(2)}_\mathcal{E} \to |Y|^{SO(2)}_\mathcal{E}$ is a homeomorphism.

**Corollary A.33.** If $X$ is a cofibrant cyclic set, then $|X|^{SO(2)}_\mathcal{E}$ is empty.

**Lemma A.34** ([17, Lemma 6.5]). Let $(A, B, \nabla A \to B)$ be a cofibrant object in $\mathcal{P}$ and define

$$Z = |A|_s \bigsqcup_{|\nabla A|_\mathcal{E}} |B|_\mathcal{E}.$$ 

Then $Z^{SO(2)} \simeq |A|_s$ and for all finite $H \subset SO(2)$, $Z^H \simeq |B|^H_\mathcal{E}$.

**Proof of Theorem A.30** ([17, Section 6]). All that is left to show is that given a cofibrant object $(A, B, \nabla A \to B)$ in $\mathcal{P}$ and a fibrant object $X$ of $\text{Top}^{\leq SO(2)}$, a map

$$f : (A, B, \nabla A \to B) \to RX$$
is a weak equivalence in $\mathcal{P}$ if and only if the adjoint $\tilde{f} : L(A,B,\nabla A \to B) \to X$ is a weak equivalence in $\text{Top}^{\leq SO(2)}$. If we unpack the definitions of $L$ and $R$, we see that what needs verifying is

$$(A, B, \nabla A \to B) \to \left( S \left( X^{SO(2)} \right), S_\epsilon(X), \nabla S \left( X^{SO(2)} \right) \to S_\epsilon(X) \right)$$

is a weak equivalence if and only if

$$|A|_s \bigsqcup_{|\nabla A|_\epsilon} |B|_\epsilon \to X$$

is a weak equivalence.

We begin by assuming that $|A|_s \bigsqcup_{|\nabla A|_\epsilon} |B|_\epsilon \to X$ is a weak equivalence in $\text{Top}^{\leq SO(2)}$. This implies that the induced map

$$\left( |A|_s \bigsqcup_{|\nabla A|_\epsilon} |B|_\epsilon X \right)^{SO(2)} \to X^{SO(2)}$$

is a weak equivalence. Using Lemma A.34 we have that the map

$$|A|_s \to \left( |A|_s \bigsqcup_{|\nabla A|_\epsilon} |B|_\epsilon \right)^{SO(2)}$$

is a weak equivalence, and therefore their composition is a weak equivalence. Therefore we have that the adjoint $A \to S(X^{SO(2)})$ is a weak equivalence of simplicial sets.

Now for any finite $H \subset SO(2)$ we have that the map

$$\left( |A|_s \bigsqcup_{|\nabla A|_\epsilon} |B|_\epsilon X \right)^H \to X^H$$

is a weak equivalence by assumption, and again by Lemma A.34 we have

$$|B|_\epsilon^H \to \left( |A|_s \bigsqcup_{|\nabla A|_\epsilon} |B|_\epsilon \right)^H$$

is a weak equivalence, and therefore their composition is a weak equivalence also. This implies that the adjoint $B \to S_\epsilon(X)$ is a weak equivalence of cyclic sets.
Now we assume the converse, i.e., that the adjoint

\[(A, B, \nabla A \to B) \to \left( S \left( X^{SO(2)} \right), S_\varepsilon(X), \nabla S \left( X^{SO(2)} \right) \to S_\varepsilon(X) \right)\]

is a weak equivalence. This implies that \(|A|_s \to X^{SO(2)}| is a weak equivalence of simplicial sets and that \(|B|_\varepsilon \to X| is a weak equivalence of simplicial sets. Using the above discussion and the two of of three property of model categories, we get that \(|A|_s \sqcup |\nabla A|_\varepsilon \to X| is a weak equivalence.

For model structures of this type, we will frequently abuse notation and just denote

\[\overline{\Delta \mathcal{C}_{Cpld}} := \mathcal{P} := \overline{\Delta \nabla \Delta \mathcal{C}_{Str}}\]

and call it the \textit{coupled model} on cyclic sets. However we should remember that in fact \(\overline{\Delta \mathcal{C}_{Cpld}}\) has a different underlying category than \(\overline{\Delta \mathcal{C}}\).

\[\text{A.1.4.2 Of Cyclic Type}\]

We can now consider changing the above machinery to work in the case of \(\overline{\Delta \mathcal{C}}_N\). All of the following statements follow almost instantly from the machinery introduced by Blumberg which we reproduced in the previous section, and by using the study of the strong model structure from Section A.1.2.

First of all we must think of what is the correct underlying category to use. We will be considering the category \(\mathcal{P}_N := \overline{\Delta \nabla \Delta \mathcal{C}}_{NStr}\), where

\[\nabla^N_n = S_{\varepsilon_N} (|\Delta[n]|), \]
\[\nabla^N X = X \otimes_{\Delta^N} \nabla^N_* X.\]

**Lemma A.35.** \(\nabla^N\) is Reedy admissible \(\forall N \in \mathbb{N}\). Therefore there is a model structure on \(\mathcal{P}_N\) given by Theorem A.21.

**Proof.** If we look at the proof of Lemma A.26, we see that the only tool that we required was the fact that the geometric realisation was a colimit combined with Lemma A.25 which works for all \(G\). We have that the realisation \(|\varepsilon_N|\) is a colimit by definition, and therefore the proof is identical mutatis mutandis.

**Theorem A.36.** There is a Quillen equivalence

\[L : \mathcal{P}_N \rightleftarrows \text{Top}^{\leq \text{Spin}_N(2)} : R.\]
where the adjunctions $L$ and $R$ are the obvious ones obtained from the cyclic case.

**Proof.** As above, we can prove this statement using the proofs from Section A.1.4.1. The properties of $\Delta\mathcal{E}$-sets that are used to prove the statement have been proved in the case of $\Delta\mathcal{E}_N$-sets. In particular in Section A.1.2 we proved the Quillen equivalence

$$\hat{\Delta}\mathcal{E}_{N\text{Str}} \simeq \text{Top}^{\text{Spin}_N(2)}.$$

Again, we will ease notation and denote

$$\hat{\Delta}\mathcal{E}_{N\text{Cpld}} := \mathcal{P}_N := \hat{\Delta}N\Delta\mathcal{E}_{N\text{Str}}$$

and call it the **coupled model** on $N$-fold cyclic sets.

### A.1.4.3 Of Dihedral & Quaternionic Type

The question now is how can we extend this to other planar crossed simplicial groups. Spaliński answered this question with respect to $\hat{\Delta}\mathcal{D}$ in relation to $\text{Top}^{O(2)}$ [100]. We outline the adjustments needed from the above to facilitate this.

In the case of $O(2)$, we have two infinite subgroups, namely $SO(2)$ and $O(2)$ itself. Therefore we need some way to take into account the $C_2$ action which relates $SO(2)$ and $O(2)$. Denote by $\Delta\mathcal{R}$ the crossed simplicial group where $\mathcal{G}_n = C_2$ for all $n$. We can put a model structure on the category $\hat{\Delta}\mathcal{R}$ which has a Quillen adjunction to the equivariant model on $\text{Top}^{C_2}$ (see [100, §3] for details).

**Definition A.37.** Let $\Xi_n = S_D(\Delta^n_{C_2})$ where $\Delta^n_{C_2} := C_2 \times \Delta[n]$. Then we have that $\Xi_*$ is a cosimplicial dihedral set. We then define

$$\Xi: \hat{\Delta}\mathcal{R} \to \hat{\Delta}\mathcal{D}_{\text{Str}},$$

$$\Xi X = X \otimes_{\Delta^{N\text{op}}} \Xi_*.$$

We consider the category $\mathcal{K} := \hat{\Delta}\mathcal{R} \subseteq \hat{\Delta}\mathcal{D}_{\text{Str}}$. We see that this is related to the previous case, all that we have done is replaced $\hat{\Delta}$ with $\hat{\Delta}\mathcal{R}$ which will track the $C_2$ action. As Spaliński points out in [100], the proofs involving the category $\mathcal{K}$ are analogous to the case proved by Blumberg due to the properties of the realisation functor.

**Lemma A.38** ([100, Lemma 4.5]). The functor $\Xi: \hat{\Delta}\mathcal{R} \to \hat{\Delta}\mathcal{D}_{\text{Str}}$ is Reedy admissible.

**Corollary A.39.** There exists a model structure on $\mathcal{K}$ in which a map is a:
• Weak equivalence if $A \to A'$ is a weak equivalence of $\Delta_R$-sets and $B \to B'$ is a weak equivalence of dihedral sets.

• Fibration if $A \to A'$ is a fibration of $\Delta_R$-sets and $B \to B'$ is a fibration of dihedral sets.

• Cofibration if $A \to A'$ is a cofibration of $\Delta_R$-sets and $\Xi A' \cup_{\Xi A} B \to B'$ is a cofibration of dihedral sets.

We now must construct a bijection between our two categories of interest, $\mathcal{K}$ and $\text{Top}^{O(2)}$. We can do this similar to the cyclic case, and we get the following result.

**Theorem A.40** ([100, Section 5]). The functors $L$ and $R$ specify a Quillen equivalence between $\mathcal{K}$ equipped with the model structure given in Corollary A.39 and $\text{Top}^{\leq O(2)}$.

Now, again we modify to allow it to work for $\Delta \mathcal{D}_N$, the proof of which is obtainable from the theory discussed above and using the ideas from the strong model structures. We consider the category $\mathcal{K}_N := \widehat{\Delta_R \Xi N \Delta \mathcal{D}_{N \text{Str}}}$ where

\[
\Xi_N^N = S_\mathcal{D}_N \left( \Delta_{\mathbb{Z}_2}^N \right),
\]

\[
\Xi_N X = X \otimes_{\Delta_R \text{op}} \Xi_N^N.
\]

**Lemma A.41.** $\Xi_N^N$ is Reedy admissible $\forall N \in \mathbb{N}$, therefore there is a model structure on $\mathcal{K}_N$.

**Theorem A.42.** There is a Quillen equivalence

\[
L : \mathcal{K}_N \rightleftarrows \text{Top}^{\leq \text{Pin}^+ N(2)} : R.
\]

Denote by

\[
\Delta \mathcal{D}_N \text{Cpld} := \mathcal{K}_N := \widehat{\Delta_R \Xi N \Delta \mathcal{D}_{N \text{Str}}}
\]

the coupled model on $N$-fold dihedral sets.

Finally, we look at the $\Delta \Omega_M$ case. Its properties follow after the obvious changes have been made. We consider the category $\Omega_M := \widehat{\Delta_R \Psi M \Delta \Omega_{M \text{Str}}}$ where

\[
\Psi_M^N = S_{\Omega_M \left( \Delta_{\mathbb{Z}_2}^N \right)},
\]

\[
\Psi_M X = X \otimes_{\Delta_R \text{op}} \Psi_M^N.
\]

**Lemma A.43.** $\Psi_M^N$ is Reedy admissible $\forall M \in \mathbb{N}$, therefore there is a model structure on $\Omega_M$. 

Theorem A.44. There is a Quillen equivalence

\[ L : \Omega_M \rightleftharpoons \text{Top}^{<\text{Pin}^\perp(2)} : R. \]

Denote by

\[ \widehat{\Omega}_{MCpld} := \widehat{\Omega}_{qM} \widehat{\Omega}_{MStr} \]

the coupled model on \( M \)-fold quaternionic sets.

A.2 Planar Lie Group Equivariant Presheaves

Finally, in this section, we will combine the theory of simplicial presheaves and the models for planar crossed simplicial group sets. We shall define these model structures by giving the relevant local weak equivalences.

Definition A.45. Let \( \Delta \mathfrak{G} \) be a compact planar crossed simplicial group and \((\mathcal{C}, \tau)\) a small Grothendieck site. A \( \Delta \mathfrak{G} \)-presheaf on \( \mathcal{C} \) is a functor \( F : \mathcal{C}^{op} \to \Delta \mathfrak{G}\text{-Set} \). We will denote by \( \Delta \mathfrak{G}\text{-Pr}(\mathcal{C}) \) the category of \( \Delta \mathfrak{G} \)-presheaves on \( \mathcal{C} \).

A.2.1 Presheaves for the Strong Model

We will begin by looking at presheaves over the strong model. This was developed in the cyclic case by Seteklev and Østvær in [95, 96].

As in the simplicial case, we being by putting a point-wise model structure on \( \Delta \mathfrak{G}\text{-Pr}(\mathcal{C}) \) which does not look at all at the topology on \( \mathcal{C} \) (the existence of which is given by the fact \( \widehat{\Delta \mathfrak{G}}_{Str} \) is cofibrantly generated).

Theorem A.46 ([96, Theorem 4.2]). There is a projective point-wise model structure on \( \Delta \mathfrak{G}\text{-Pr}(\mathcal{C}) \), denoted \( \Delta \mathfrak{G}\text{-Pr}^{\text{Str}}(\mathcal{C}) \), where the weak equivalences are the point-wise strong weak equivalences, and the fibrations are the strong point-wise fibrations. The cofibrations are those maps which have the LLP with respect to the acyclic fibrations. Moreover this model structure is combinatorial, where the generating set of cofibrations is given by \( C \otimes i \) where \( i \) is a generating cofibration of \( \widehat{\Delta \mathfrak{G}}_{Str} \) and \( C \in \mathcal{C} \).

We will now construct the local model structure on the \( \Delta \mathfrak{G} \)-presheaves. To do so, we will explicitly describe the weak equivalences (this construction appears in [95] for the cyclic case). We shall outline the construction for the cyclic case, with the other cases following with the correct replacement of subdivision functors.

First of all note that the functors \( \Phi_r : \widehat{\Delta \mathfrak{C}} \to \widehat{\Delta} \) induce a family of functors

\[ \Phi_r : \widehat{\Delta \mathfrak{C}}_* \to \widehat{\Delta}_* \]
between the pointed versions of these categories. For $X$ a fibrant cyclic set and $x \in X_0$ we define $\pi^n_r(X, x) := \pi_n \Phi_r s(X, x)$.

**Definition A.47.** A map $f: \mathcal{F} \to \mathcal{F}'$ in $\Delta \mathcal{C} \text{-Pr} (\mathcal{C})$ is a *strong local weak equivalence* if for all $r \geq 1$:

- the induced map $\tilde{\pi^n_0} \mathcal{F} \to \tilde{\pi^n_0} \mathcal{F}'$ is an isomorphism of sheaves.
- the induced squares

\[
\begin{array}{ccc}
\pi^n_r \mathcal{F} & \rightarrow & \pi^n_r \mathcal{F}' \\
\downarrow & & \downarrow \\
\Phi_r \mathcal{F}_0 & \rightarrow & \Phi_r \mathcal{F}'_0
\end{array}
\]

are pullbacks after sheafification.

**Remark A.48.** From the definition of $\pi^n_r(X, x)$ it is easy to see that a map $f: \mathcal{F} \to \mathcal{F}'$ in $\Delta \mathcal{C} \text{-Pr} (\mathcal{C})$ is a strong local weak equivalence if and only if $\Phi_r (f)$ is a local weak equivalence of simplicial presheaves for all $r \geq 1$. Therefore we see that the class of strong local weak equivalences have the required properties to be a class of maps in a model structure. In fact, it seems that it may be possible to prove the result of Spalinski, Theorem A.6, in the case of adjunctions between the categories of presheaves.

**Definition A.49.** The *strong local model* on $\Delta \mathcal{C} \text{-Pr} (\mathcal{C})$ is the model structure where a map is a:

- Weak equivalences if it is a strong local weak equivalence.
- Fibration if it has the RLP with respect to the trivial cofibrations.
- Cofibration if it is a cofibration in the point-wise projective model.

We will denote this model $\Delta \mathcal{C} \text{-Pr}^{\text{Str}}_r (\mathcal{C})$.

### A.2.2 Presheaves for the Coupled Model

We are now interested in presheaves with values in coupled categories. We will consider this from two different viewpoints. Let $\mathcal{X}$ be any site. We define two categories

$$
P(\mathcal{X}) := \text{Fun}(\mathcal{X}^{\text{op}}, \mathcal{C}_F \mathcal{D})$$

$$
Q(\mathcal{X}) := \text{Fun}(\mathcal{X}^{\text{op}}, \mathcal{C}) \ast \text{Fun}(\mathcal{X}^{\text{op}}, \mathcal{D})
$$

where $F^* : \text{Fun}(\mathcal{X}^{\text{op}}, \mathcal{C}) \to \text{Fun}(\mathcal{X}^{\text{op}}, \mathcal{D})$ is the induced functor on the presheaf categories by composition with $F$. 
Lemma A.50. If the functor $F$ is Reedy admissible, then so is $F^*$ between the point-wise model structures on the presheaf categories.

Proof. As we will only be concerned with the resulting homotopy theory, it is enough to prove this for the injective point-wise model. Assume that $A \to A'$ is a trivial cofibration in $\text{Fun}(X^{\text{op}}, C)$ and $F^* A' \amalg_{F^* A} B \to B$ a trivial cofibration in $\text{Fun}(X^{\text{op}}, D)$. We need to show that $B \to B'$ is a weak equivalence in $\text{Fun}(X^{\text{op}}, D)$. Recall that in the injective point-wise model structure, $A \to A'$ is a trivial cofibration if and only if $\forall x \in X$ the induced map $A(x) \to A'(x)$ is a trivial cofibration in $C$. Likewise we consider $\forall x \in X$ the maps $F^* A'(x) \amalg_{F^* A(x)} B(x) \to B(x)$ be trivial cofibrations in $D$. However note that $F^* A'(x) = F(A(x))$. By assumption on $F$ it then follows that $\forall x \in X$ the maps $B(x) \to B'(x)$ are weak equivalences in $D$. Therefore $B \to B'$ is a weak equivalence in the point-wise model structure for $\text{Fun}(X^{\text{op}}, D)$.

Proposition A.51. The categories $P(X)$ and $Q(X)$ are isomorphic. Moreover, there is a Quillen equivalence between the point-wise model structures.

Proof. An object of $Q(X)$ is of the form $p = (A : X^{\text{op}} \to C, B : X^{\text{op}} \to D, F^* A \to B)$. For $x \in X$ we can look at $p(x) = (A(x), B(x), F(A(x)) \to B(x))$ which can be trivially viewed as an object of $C_{F,D}$. This assignment for each $x \in X$ allows us to construct a functor $\alpha : Q(X) \to P(X)$. This functor can be seen as being fully faithful and essentially surjective due to its constructive nature. The Quillen equivalence follows from comparing the classes of weak equivalences and fibrations in each category and showing that they coincide with the functor $\alpha$.

Taking into account Proposition A.51, we arrive at the following definition for presheaves with values in the coupled model structure. We begin as always with the cyclic case.

Definition A.52. A $\Delta C_N$-coupled presheaf on a small Grothendieck site $(C, \tau)$ is an object of

$$s\text{Pr}(C)_{(\tau_N)}, \Delta C_N^{\text{Cpld}} \text{-Pr}^{\text{Str}}(C)$$

We will denote by $\Delta C_N^{\text{Cpld}} \text{-Pr}^{\text{Cpld}}(C)$ the category of all $\Delta C_N$ coupled presheaves on $C$.

Now we wish to equip this category with a local model structure. To do this we will just change the presheaf categories appearing in the definition by their local models. We list the following as a corollary since the existence of the model structures follows from Theorem A.21 and a slight modification of Lemma A.50.
Corollary A.53. There is a cofibrantly generated model structure on $\Delta \mathcal{C}_N \cdot \Pr(\mathcal{C})$ (resp., $\Delta \Omega_N \cdot \Pr(\mathcal{C})$, $\Delta \Omega_M \cdot \Pr(\mathcal{C})$) defined as

$$s\Pr_*(\mathcal{C})_{(\mathcal{C}_N)} \cdot \Delta \mathcal{C}_N \cdot \Pr^{Str}(\mathcal{C})$$

(resp., $\Delta \mathcal{R}_N \cdot \Pr_*(\mathcal{C})_{(\mathcal{C}_N)} \cdot \Delta \Omega_N \cdot \Pr^{Str}(\mathcal{C})$)

(resp., $\Delta \mathcal{R}_M \cdot \Pr_*(\mathcal{C})_{(\mathcal{C}_M)} \cdot \Delta \Omega_M \cdot \Pr^{Str}(\mathcal{C})$)

We will denote this model by $\Delta \mathcal{C}_N \cdot \Pr^{Cpld}(\mathcal{C})$ (resp., $\Delta \Omega_N \cdot \Pr^{Cpld}(\mathcal{C})$, $\Delta \Omega_M \cdot \Pr^{Cpld}(\mathcal{C})$) and call it the coupled local model structure.

A.2.3 Strong and Coupled Equivariant Stacks

Finally, we simply comment that it is possible to construct categories of stacks corresponding to the strong and coupled models.

Definition A.54. Let $\mathcal{C}$ be a site. The category of strong (resp., coupled) $\Delta \mathcal{G}$-stacks is the homotopy category $\text{Ho}(\Delta \mathcal{G} \cdot \Pr^{Str}(\mathcal{C})$) (resp., $\text{Ho}(\Delta \mathcal{G} \cdot \Pr^{Cpld}(\mathcal{C})$).

Remark A.55. In the Kan model structure we showed that we could retrieve equivariant cohomology. This cohomology was of Borel type. A strong level of equivariance can be achieved by using Bredon cohomology, where the fixed point data of closed subgroups is taken into account [21]. By considering the homotopy type of the strong and coupled models, it is not unnatural to assume that taking the mapping spaces in the above categories should result in certain Bredon cohomology theories.

We can define the $n$-geometric strong and coupled stacks by using the $n$-hypergroupoid construction. The change that we need to make is to incorporate the subdivision functors to encode the equivariant structure. As usual, we will work in the cyclic case, and we will only consider the strong model.

Definition A.56. A morphism $f: X \to Y$ of derived cyclic affine schemes is a trivial strong cyclic relative Artin (resp., Deligne-Mumford) $n$-hypergroupoid if $\Phi_r(f): \Phi_r(X) \to \Phi_r(Y)$ is a trivial relative Artin (resp., Deligne-Mumford) $n$-hypergroupoid for all $r \geq 1$.

Definition A.57. The homotopy category of strong cyclic $n$-geometric derived Artin (resp., Deligne-Mumford) stacks is the full subcategory of $\text{dAff}^{\Delta \mathcal{C}_N \cdot \Pr^{Str}(\mathcal{C})}$ consisting of the $\Delta \mathcal{G}$-derived Artin (resp., Deligne-Mumford) $n$-hypergroupoids and inverting the trivial strong cyclic relative Artin (resp., Deligne-Mumford) $n$-hypergroupoids.
Bibliography


