Improved Circle and Popov Criteria for systems containing magnitude bounded nonlinearities

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Abstract:
This paper presents improved versions of the Circle and Popov Criteria for Lure systems in which the nonlinear element is both sector and magnitude bounded. The main idea is to use the fact that if the nonlinearity is magnitude bounded and the linear system is asymptotically stable, then its state will be ultimately bounded. When the state enters this set of ultimate boundedness, it will satisfy a narrower sector condition which can then be used to prove stability in a wider set of cases than the standard Circle and Popov Criteria. The results are illustrated with some numerical examples.

Keywords: Stability of nonlinear systems, Anti-windup, Robustness analysis, Convex optimization, Absolute stability

1. INTRODUCTION

Many control problems can be expressed as absolute stability problems consisting of a feedback interconnection of a linear system with a static nonlinear element. Despite having been studied for many years, due to its importance in control system analysis and design (Glattfelder and Schaufelberger (2003); Tarbouriech et al. (2011); Zacarian and Teel (2011)), in both the Russian (Leonov et al. (1996); Yakubovich (2002)) and Western literature (Zames and Falb (1968); Chen. and Wen (1995); Haddad and Kapila (1995); Jonsson (1997)), the problem still attracts significant interest today. Examples of recent work include (Safanov and Kulkarni (2000); Mancera and Safanov (2005); Turner et al. (2009); Carrasco et al. (2012, 2014); Altshuller (2011); Dui et al. (2009); Bridgeman and Forbes (2014)); the recent tutorial paper by Carrasco et al. (2016) contains a more comprehensive literature review.

The conservatism present in most absolute stability analyses tends to be related to the assumptions made on the nonlinear element. In the popular Circle and Popov Criteria the nonlinearity is assumed to satisfy sector bounds. For single-input-single-output (SISO) systems these criteria can be checked graphically, or for multi-input-multi-output (MIMO) systems, using linear matrix inequalities (LMI’s). When the nonlinearity satisfies slope restrictions, stronger results can be obtained by way of the Zames-Falb multipliers (Zames and Falb (1968)) which were first introduced by O’Shea (O’Shea (1966); Carrasco et al. (2016)) in the 1960’s. These results have led to several powerful analysis tools based on linear matrix inequalities being developed (Chen and Wen (1995); Turner et al. (2009); Carrasco et al. (2014)). In addition, Lyapunov-based analysis tools have also been developed for the same class of nonlinearities (Park (2002); Turner and Kerr (2014); Valmorbida et al. (2016)).

Most of the papers dedicated to absolute stability are focused on the sector, slope or monotonicity properties of the nonlinearity. This approach has merit because it ensures that the algorithms derived have inherent robustness and, also, that the knowledge required of the nonlinearity is relatively modest. Few papers on absolute stability take advantage of the fact that the magnitude of some nonlinearities is also bounded e.g. the standard saturation nonlinearity sat_u(u) is always bounded by the saturation limits i.e. |sat_u(u)| ≤ u ∀u ∈ R.

The purpose of this paper is to examine the effect of incorporating magnitude bounds into absolute stability analysis of Lure systems. The idea is to use the magnitude bounds to reduce conservatism, whilst making the computational complexity not much greater than that of the standard Circle and Popov Criteria. It is noted that other authors have improved upon the Circle and Popov Criteria with similar ideas in mind. A particularly relevant reference is Materassi et al. (2006) where the notion of a hyperbolic sector was introduced. In particular, Materassi et al. (2006) use their sector bounds to obtain a version of the Circle Criterion guaranteeing boundedness. Nonetheless, the technical conditions on the nonlinearity are not entirely clear in Materassi et al. (2006); in this paper these are clarified. Additionally, in this paper we provide an explicit optimization problem that provides an explicit expression for the ultimate bound of the state: this is carried out in a similar way to Materassi et al. (2006) but the details are different.

Notation. Notation is standard. For a square matrix, M, the notation M > 0 (≥ 0) means the matrix is symmetric and positive (semi-) definite. Negative definiteness is defined similarly. The nonlinearity φ(·) : R → R is said to belong to Sector[α, β] in V if the following inequality is satisfied for all positive scalars η > 0

\[(\phi(v) - \beta v)\eta(\phi(v) - \alpha v) \leq 0 \quad \forall v \in V \subset R \quad (1)\]

2. PROBLEM STATEMENT

2.1 Assumptions

In this paper, the stability of the system in Figure 1 will be examined. For simplicity in presentation, only the SISO case is considered, that is u ∈ R. The extension to the multivariable case is straightforward given certain additional assumptions. The plant is described by the following state-space equations

\[P(s) \sim \left\{ \begin{array}{l}
\dot{x} = Ax + B\sigma(u) \\
u = Cx 
\end{array} \right. \quad (2)\]
An odd semi-concave function $\sigma$ are the magnitude boundedness assumption iv), which is central to the results to be derived, and the semi-concavity assumption (iv) is that $|\sigma(u)| \leq \sigma(u)\forall u \in \mathbb{R}, \sigma > 0$. 

Remark 1: The non-standard assumptions on the nonlinearity are the magnitude boundedness assumption iv), which is central to the results to be derived, and the semi-concavity assumption iii) . The latter assumption is convenient but not necessary. An odd semi-concave function $\sigma(u)$ satisfies the following property (Megretski (2001))

$$\frac{\sigma(u)}{u} \text{ monotonically non-increasing} \forall u \in [0, \infty) \quad (3)$$

and the function $\sigma(.)$ itself is also monotonically non-decreasing. Geometrically, this means that the nonlinearity will increase (actually, not decrease) but that its “slope” will decrease (actually, not increase) as its argument increases. The assumption of semi-concavity is stricter than required and relaxations will be discussed later. However, the assumption is in place as it makes it easier to interpret and prove the results. One of the consequences of assumption (iii) and the boundedness assumption (iv) is that

$$\bar{\sigma} = \lim_{|u| \to \infty} |\sigma(u)|$$

Again, a good example of a function satisfying such an assumption is the saturation nonlinearity.

The following assumption is made on the plant.

Assumption 2. The matrix $A + \tau BC$ is Hurwitz for all $\tau \in [0, \beta]$

The assumption requires that the plant is open-loop stable ($\tau = 0$) and also stable for any linear gain in the Sector $[0, \beta]$ which is a necessary logical assumption.

2.2 Main Idea

Before giving a formal proof, the following is a summary of the main idea in the paper. The assumption that $\sigma(.)$ is magnitude bounded and the stipulation that $A$ is Hurwitz together imply that the “large-signal” behaviour of the system will be stable; that is all states will converge, in finite time, to a bounded set in the state-space: it transpires that the state is ultimately bounded within an ellipse $E_0$ - see Figure 2. This ellipse is positively invariant, meaning that if $x(t_1) \in E_0$, then $x(t) \in E_0$ for all $t \geq t_1$. Once the state $x(t)$ enters $E_0$, the argument of $\sigma(.)$ will also be bounded i.e.

$$|u| = |Cx| < u_0 \forall x \in E_0$$

...
This implies that  
\[ \dot{\gamma} > \minimised \text{to enable the smallest ellipsoid} \]
\[ E V (\in \text{finite time and remains there thereafter.} \]

**Proof:** Let \( V(x) = x^TPx \) be a Lyapunov function. Then  
\[ \dot{V}(x) = x'(A'P + PA)x + 2x'PB\sigma(u) \]
\[ \leq x'(A'P + PA + \frac{1}{\gamma}PBB'P)x + \gamma \|\sigma(u)\|^2 \]
\[ \leq x'(A'P + PA + \frac{1}{\gamma}PBB'P)x + \gamma \sigma^2 \]
for all \( \gamma > 0 \). Using inequality (7), we then have  
\[ \dot{V}(x) \leq -\epsilon x^2P + \gamma \sigma^2 \]
\[ = -\epsilon V(x) + \gamma \sigma^2 \]
This implies that \( \dot{V}(x) \) is strictly decreasing for all \( x \) such that  
\[ V(x) > \frac{\epsilon}{\epsilon} \sigma^2 \]
and hence that the state will enter the ellipsoid \( E_0 \) in finite time. Furthermore, because \( E_0 \) is a level set for the Lyapunov function \( V(x) \), it is positively invariant.  
\[ \square \]

Note that the Riccati-like inequality (7) can be expressed as the following LMI  
\[ \begin{bmatrix} A'P + PA + \epsilon P & PB \\ PT & -\gamma I \end{bmatrix} < 0 \quad (14) \]
This will always have a solution providing \( \epsilon \) is sufficiently large and \( \epsilon \) satisfies the inequality  
\[ \frac{\epsilon}{\epsilon} < -\max(\Re(\lambda_i(A))) \]
The nice feature of solving the LMI (14) is that \( \gamma > 0 \) can be minimised to enable the smallest ellipsoid \( E_0 \) to be calculated.

**Remark 2:** The convergence to the ellipsoid \( E_0 \) is exponential. Equation (13) can be re-written as  
\[ \dot{V}(x) \leq -\epsilon(1 - \theta)V(x) - (\epsilon\theta V(x) - \gamma \sigma^2) \quad \theta \in (0, 1) \]
So if  
\[ V(x) \geq \frac{\gamma}{\epsilon \theta} \sigma^2 \]
it follows that  
\[ V(t) \leq V(0) \exp(-\epsilon(1 - \theta)t) \quad (16) \]

### 3.2 An Improved Circle Criterion

The above lemmas can be used to establish the first main result of the paper.

**Proposition 5.** Consider the system in equation (2) and let Assumptions 1 and 2 be satisfied. Let the matrix \( P > 0 \) and scalars \( \epsilon > 0 \) and \( \gamma > 0 \) be solutions to inequality (7). Then the origin of the system (2) is globally asymptotically stable if there exist a positive definite matrix \( P_0 \) and a positive scalar \( W_0 \) such that the following linear matrix inequality is satisfied:  
\[ \begin{bmatrix} A'P_0 + P_0A - 2\beta_0C'W_0C & P_0B + (\beta C_0C') \quad * \\ PT & -2W_0 \end{bmatrix} < 0 \quad (17) \]
where  
\[ \beta_0 = \sigma_0/u_0 \]
\[ \sigma_0 := \sigma(u_0) \]
\[ u_0 := \sqrt{\gamma/\epsilon \sqrt{CP^{-1}C'}} \]

**Proof:** Application of Lemma 4 to the system (2) implies that if \( x(0) \in \mathbb{R}^n/E_0 \), then the state enters the ellipsoid \( E_0 \) in finite time, \( t_1 \), and, moreover, never leaves it. Thus, consider the evolution of the state in \( E_0 \). The maximum value of \( |u| \) in \( E_0 \) can be calculated as  
\[ \max |u| \quad (21) \]
subject to  
\[ \begin{align*}
\dot{x} &= Cx \\
x(0) &\in E_0 \\
\dot{V} &= x'(A'P + PA)x + 2x'PB\sigma(u) \\
\leq &\ x'(A'P + PA + \frac{1}{\gamma}PBB'P)x + \gamma \|\sigma(u)\|^2 \\
\leq &\ x'(A'P + PA + \frac{1}{\gamma}PBB'P)x + \gamma \sigma^2 \end{align*} \quad (22) \]
By Lemma 3, the maximum value of \( |u| \) is therefore  
\[ u_0 = \sqrt{\gamma/\epsilon \sqrt{CP^{-1}C'}} \]
Thus, noting oddness and monotonicity of \( \sigma(.) \), it follows that for all \( u \in [-u_0, u_0] \), the maximum value of \( \sigma(.) \) is given by  
\[ \sigma_0 = \sigma(u_0) \leq \sigma \quad (24) \]
Calculating \( \beta_0 := \sigma(u_0)/u_0 \), it follows, by semi-concavity of \( \sigma(.) \), that \( \sigma(.) \) belongs to the narrower sector, \( \text{Sector}[\beta_0, \beta] \), where \( 0 < \beta_0 < \beta \), for all \( |u| < u_0 \). This implies that for all states in \( x \in E_0, \sigma(.) \in \text{Sector}[\beta_0, \beta] \).

Thus choosing \( V_0(x) = x^TP_0x \) as a Lyapunov function and considering its derivative in \( E_0 \) we have  
\[ \dot{V}_0(x) = x'(A'P_0 + P_0A)x + 2x'P_0B\sigma(u) \]
\[ \leq x'(A'P_0 + P_0A)x + 2x'P_0B \sigma(u) - 2(\sigma(u) - \beta Cx)'W_0(\sigma(u) - \beta u) \]
\[ = \begin{bmatrix} x & (A'P_0 + P_0A - 2\beta C_0'W_0C' \quad * \\ PT & -2W_0 \end{bmatrix} \begin{bmatrix} x \\ \sigma(u) \end{bmatrix} \quad (27) \]
This inequality arises as a result of the nonlinearity \( \sigma(.) \) being in the the Sector[\beta_0, \beta] for all \( x \in E_0 \) and hence the following inequality holding:  
\[ -(\sigma(u) - \beta u)'W_0(\sigma(u) - \beta u) \geq 0 \quad (28) \]
for all scalars \( W_0 > 0 \). Thus if the matrix inequality in the proposition holds, it implies that \( \dot{V}_0(x) < 0 \) for all \( x \in E_0 \) and hence that the origin is globally asymptotically stable (since all states enter \( E_0 \) in finite time).  
\[ \square \]

![Fig. 4. Bounded, odd but non-semi-concave nonlinearity. Pink \( \cup \) Blue = global sector; Blue = narrow sector inhabited for all \( |u| < u_0 \)](image)
do not satisfy these properties $\sigma(.)$. Figure 4 shows an odd but non semi-concave and not monotonically increasing $\sigma(u)$. However, note that for $u \in [-u_0, u_0]$, it is still true that $\sigma(.) \in \text{Sector}[\beta_0, \beta]$.

With the above in mind, the improved Circle Criterion can be proved under more relaxed conditions where semi-concaveness is dropped.

**Corollary 6.** Consider the system in equation (2) and let items (i), (ii) and (iv) of Assumptions 1 and Assumption 2 be satisfied. Assume also that the feedback interconnection is well-posed. Let the matrix $P > 0$ and scalars $\epsilon$ and $\gamma$ be solutions to inequality (7) and let $u_0$ be defined as in equation (20). Furthermore let
\[
\beta_0 = \min_{u \in [-u_0, u_0]} \frac{\sigma(u)}{u},
\]
(29)
Then the origin of the system (2) is globally asymptotically stable if there exist a positive definite matrix $P_0$ and a positive scalar $W_0$ such that the following linear matrix inequality is satisfied:
\[
\begin{bmatrix}
AP_0 + P_0 A - 2\beta_0 C^T W_0 C & P_0 B + (\beta + \beta_0) C^T W_0 \\
-2W_0 & -2W_0
\end{bmatrix} < 0
\]
(30)

### 3.3 An Improved Popov Criterion

Lemma 3 and 4 can also be used to derive an improved Popov Criterion. Following Park (1997) (see also Heath and Li (2009)) an “indefinite” Popov Criterion is used to provide lower conservatism in the results. Therefore note that a nonlinearity which is locally sector bounded, that is $\sigma(.) \in \text{Sector}[\beta_0, \beta]$ satisfies the following inequalities
\[
\begin{align*}
\sigma(u)u &\geq \beta_0 u^2 & \forall |u| \leq u_0 \\
\beta_0 u^2 &\geq \sigma(u)u & \forall |u| \leq u_0
\end{align*}
\]
(31)
(32)
This implies that the following integral inequalities hold for all $|u| < u_0$ (Park (1997)):
\[
\begin{align*}
\int_{0}^{u} \kappa_1 (\sigma(v) - \beta_0 v)dv &\geq 0 & \kappa_1 > 0 \\
\int_{0}^{u} \kappa_2 (\beta v - \sigma(v))dv &\geq 0 & \kappa_2 > 0
\end{align*}
\]
(33)
(34)
These integral inequalities can be used to prove the second result of the paper.

**Proposition 7.** Consider the system in equation (2) and let Assumptions 1 and 2 be satisfied. Let the matrix $P > 0$ and scalars $\epsilon$ and $\gamma$ be solutions to inequality (7). Then the origin of the system (2) is globally asymptotically stable if there exist a positive definite matrix $P_0$, a positive scalar $W_0$ and an indefinite scalar $\kappa$ such that the following linear matrix inequality is satisfied:
\[
\begin{bmatrix}
AP_0 + P_0 A - 2\beta_0 C^T W_0 C & P_0 B + (\beta + \beta_0) C^T W_0 + \kappa A^T C^T \\
-2W_0 + \kappa (CB + B^T C^T) & -2W_0
\end{bmatrix} < 0
\]
(35)
where
\[
\begin{align*}
\beta_0 &= \sigma_0 / u_0 \\
\sigma_0 &= \sigma(u_0) \\
u_0 &= \sigma \sqrt{\gamma / \epsilon} \sqrt{C P^{-1} C^T}
\end{align*}
\]
(36)
(37)
(38)

**Proof:** The first part of the proof is identical to the proof of Proposition 5: Lemma 4 means that the state of the system enters the ellipsoid $E_0$ after finite time and never leaves it; Lemma 3 then can be used to show that in $E_0$, the signal $u(t)$ is such that $|u(t)| < u_0$ and that the nonlinearity is such that $\sigma(u) \in \text{Sector}[\beta_0, \beta]$ for all $x \in E_0$.

The proof is completed by choosing the Lyapunov function
\[
V_0(x) = V_1 (x) + V_2 (x) + V_3 (x)
\]
(39)
where
\[
V_1 (x) = x' P_1 x
\]
(40)
\[
V_2 (x) = \int_{0}^{u} \kappa_1 (\sigma(v) - \beta_0 v)dv
\]
(41)
\[
V_3 (x) = \int_{0}^{u} \kappa_2 (\beta v - \sigma(v))dv
\]
(42)
Differention of this Lyapunov function along the trajectories of the system (2) in the ellipsoid $E_0$ yields
\[
\begin{align*}
V'_1 (x) &= x' (A' P_1 + P_1 A) x + 2x' P_1 B \sigma(u) \\
V'_2 (x) &= 2 \kappa_1 (\sigma(u) - \beta_0 C x) (C (Ax + B \sigma(u))) \\
V'_3 (x) &= 2 \kappa_2 (\beta C x - \sigma(u)) (C (Ax + B \sigma(u)))
\end{align*}
\]
(43)
(44)
(45)
Combining these expressions and incorporating the sector inequality (28) yields the LMI (46) at the top of the next page. where $\kappa = \kappa_1 - \kappa_2$ and $\eta = \kappa_2 \beta - \kappa_1 \beta_0$. It can then be shown (Park (1997); Heath and Li (2009)) that the matrix $P_1 + \eta C^T C$ can be replaced by a positive definite matrix $P_0$ and, moreover that $P_0 > 0$ implies $P_0 > 0$ together with the integral inequalities (33) and (34), this then shows that the Lyapunov function (39) is positive definite. Satisfaction of the LMI (35) then implies that inequality (46) is satisfied which implies that $V_0 (x) < 0 \forall x \in E_0$. □

**Remark 3:** The proof of the improved Popov Criterion has followed a slightly different route from the conventional proof of the Popov Criterion. In both Park (1997) and Heath and Li (2009), it is assumed that the nonlinearity $\sigma(.)$ inhabits the Sector$[0, \beta]$, which can be achieved by loopshifting arguments. In our case, because Lemma 4 is applied first to narrow the sector from Sector$[0, \beta]$ to Sector$[\beta_0, \beta]$, it would be tedious to then apply loopshifting to bring the nonlinearity into Sector$[0, \beta_0, \beta]$. Hence the proposition has been proved directly with the Sector$[\beta_0, \beta]$. □

**Remark 4:** There is some uncertainty over the correctness of the results of Heath and Li (2009); Park (1997) in the general multivariable case Heath (2015), but they appear to be correct in the single-input-single-output case considered here.

### 4. APPLICATION TO SATURATION FUNCTION

The work here was inspired by the analysis of feedback loops containing saturation-like nonlinearities so it is fruitful to consider how the results fare when being applied to such nonlinearities. The saturation nonlinearity under consideration is
\[
\sigma(u) = \text{sat}_u(\beta u) = \text{sign}(u) \times \min \{ \beta |u|, \bar{u} \}
\]
(47)
Note that, clearly the saturation satisfies Assumption 1. In particular, for all $u \in \mathbb{R}$, $\text{sat}_u(\beta u) \in \text{Sector}[0, \beta]$ and $\text{sat}_u(\beta u) / u$ is monotonically non-decreasing as required by semi-concavity. In this case the crucial parameters in Proposition 5 are therefore
\[
\begin{align*}
\bar{u} &= \bar{u} \\
u_0 &= \sqrt{\gamma / \epsilon} \sqrt{C P^{-1} C^T} \bar{u} \\
\sigma_0 &= \min \{ \beta_0, \bar{u} \}
\end{align*}
\]
(48)
(49)
(50)
where the matrix $P > 0$ and the scalar $\gamma > 0$ is obtained from the LMI (14) and $\epsilon$ is set at
\[
\epsilon / 2 = 0.99 \max (\Re(\lambda_i (A)))
\]
The examples overleaf show the maximum value of $\beta$ which can be applied to the system in Figure 1 for which various criteria predict stability. The list of plants $P(s)$ tested are given in Table 1.

Table 2 shows the maximum value of $\beta$ predicted using various stability criteria, such that stability is guaranteed. All examples clearly show that the the Circle and Popov criteria can be improved using the results in this paper. Note that in Example 5 the improved Circle Criterion actually guarantees stability for a larger value of $\beta$ than the standard Popov Criterion. All results obtained here are conservative compared to results taking into account slope restrictions - see Turner et al. (2009); Carrasco et al. (2014, 2016), but of course this information is not used by the improved Circle and Popov Criteria - boundedness is used instead.

5. NORM-BOUNDED INEQUALITIES

Lemma 4 has established that, under Assumptions 1 and 2 that the state of the system (2) converges to an ellipsoid $E_0$. Furthermore, this then implies that the nonlinearity $\sigma(\cdot)$ then satisfies tighter sector bounds, that is $\sigma(\cdot) \in \text{Sector}[\beta_1, \beta]$. However, instead of considering reduced sector bounded, one could consider norm bounds on the nonlinearity.

Thus, assume that $\sigma(\cdot)$ satisfies Assumptions 1 i), iii) and iv), but instead of the sector condition, assume that following norm-bounded inequalities are satisfied:

$$|\sigma(u)| \leq \beta |u| \quad \forall u \in \mathbb{R} \quad (51)$$

$$|\sigma(u)| \geq \beta_0 |u| \quad \forall u \in \mathbb{R} \quad (52)$$

These then imply the quadratic inequalities for any positive scalars $\eta_1$ and $\eta_2$:

$$\eta_1 (\beta^2 u^2 - \sigma^2(u)) \geq 0 \quad \forall u \in \mathbb{R} \quad (53)$$

$$\eta_2 (\sigma^2(u) - \beta_0^2 u^2) \geq 0 \quad \forall u \in \mathbb{R} \quad (54)$$

These inequalities are implied by the sector bound (Assumption 1 ii)) but the converse is not true. With these inequalities in mind, the following proposition can be proved in a similar manner to Proposition 5.

Proposition 8. Consider the system in equation (2) and let Assumptions 1 i), iii) and iv) and 2 be satisfied. Assume also that inequalities (53) and (54) are satisfied. Let the matrix $P > 0$ and scalars $\epsilon$ and $\gamma$ be solutions to inequality (7). Then the origin of the system (2) is globally asymptotically stable if there exist a positive definite matrix $P_0$ and a positive scalars $\eta_1$ and $\eta_2$ such that the following linear inequality is satisfied:

$$\begin{bmatrix} A'P_0 + P_0A + (\eta_1 \beta^2 - \eta_2 \beta_0^2)C'C \quad P_0B \\ \star \quad -(\eta_1 - \eta_2) \end{bmatrix} < 0 \quad (55)$$

where

$$\beta_0 = \sigma_0/u_0 \quad (56)$$

$$\sigma_0 := \sigma(u_0) \quad (57)$$

$$u_0 := \sigma_{\infty}/\sqrt{\epsilon} \sqrt{C'P^{-1}C} \quad (58)$$

This proposition seems potentially less conservative than the standard small gain theorem, where only an upper bound on the norm of the nonlinearity is used, but fact is this not the case.

To see this, note that the matrix inequality associated with the standard small gain theorem would be

$$\begin{bmatrix} A'P_0 + P_0A + \beta^2 C'C \quad P_0B \\ \star \quad -\tau_1 \end{bmatrix} < 0 \quad (59)$$

Next note that inequality (55) in Proposition 8 could be written as

$$\begin{bmatrix} A'P_0 + P_0A + (\eta_1 - \eta_2) \beta^2 C'C \quad P_0B \\ \star \quad -(\eta_1 - \eta_2) \end{bmatrix} < 0 \quad (60)$$

Defining $\tau_2 := \eta_1 - \eta_2 > 0$, then yields

$$\begin{bmatrix} A'P_0 + P_0A + \tau_2 \beta^2 C'C \quad \eta_2 (\beta^2 - \beta_0^2)C'C \quad P_0B \\ \star \quad -\tau_2 \end{bmatrix} < 0 \quad (61)$$

This is of a similar form to inequality (59) except the (1,1) element features an extra term

$$\eta_2 (\beta^2 - \beta_0^2)C'C$$

which, because $\beta > \beta_0 > 0$, is always positive semi-definite and, therefore, potentially, makes inequality (61) more difficult to satisfy than (59). As a consequence Proposition 8 can not provide improved results over the standard Small Gain Theorem. This implies that boundedness of the nonlinearity $\sigma(\cdot)$, on its own, will not necessarily reduce conservatism in analysis: this property must be used in conjunction with sector boundedness, or another such condition.

6. CONCLUSION

This paper has proposed versions of the Popov and Circle Criteria which, in addition to sector boundedness, use information about the magnitude bound of the nonlinearity in order to establish asymptotic stability of the origin. The LMI’s associated with establishing stability are of similar complexity to those of the standard Circle or Popov Criteria, making the results computationally appealing. Further work should enable the magnitude bounds on the nonlinearity to be combined with slope restrictions so the results of Park (2002) and Zames and Falb (1968) can also be improved.

REFERENCES


1.7636 1.0828 4.0273 5.4057 31.3071 10.4974

1.9961 1.0891 4.6029 6.3079 36.9016 11.2076

Table 1. Table of transfer functions $P(s)$

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Ex 1</th>
<th>Ex 2</th>
<th>Ex 3</th>
<th>Ex 4</th>
<th>Ex 5</th>
<th>Ex 6</th>
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<td>1.6903</td>
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Table 2. Maximum bound on $\beta$ obtainable using various stability criteria


