INVERSE-TYPE ESTIMATES ON \( hp \)-FINITE ELEMENT SPACES
AND APPLICATIONS

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Abstract. This work is concerned with the development of inverse-type inequalities for piecewise polynomial functions and, in particular, functions belonging to \( hp \)-finite element spaces. The cases of positive and negative Sobolev norms are considered for both continuous and discontinuous finite element functions. The inequalities are explicit both in the local polynomial degree and the local mesh-size. The assumptions on the \( hp \)-finite element spaces are very weak, allowing anisotropic (shape-irregular) elements and varying polynomial degree across elements. Finally, the new inverse-type inequalities are used to derive bounds for the condition number of symmetric stiffness matrices of \( hp \)-boundary element method discretisations of integral equations, with element-wise discontinuous basis functions constructed via scaled tensor products of Legendre polynomials.

1. Introduction

Inverse-type estimates are widely used in the error analysis of many numerical methods for partial differential equations and integral equations. Classical inverse-type estimates are of the form

\[
\|v\|_{H^s(\tau)} \leq C(s, r_\tau, m_\tau) h_\tau^{-s} \|v\|_{L^2(\tau)},
\]

where \( L^2(\tau) \) and \( H^s(\tau) \) denote the standard (Hilbertian) Lebesgue and Sobolev spaces \((s \geq 0)\), respectively, consisting of functions from a (usually, triangular or quadrilateral) set \( \tau \subset \mathbb{R}^d, d = 2, 3 \), \( v \) being a polynomial function on \( \tau \), \( h_\tau := \text{diam}(\tau) \), and \( m_\tau \) denotes the polynomial degree of \( v \), and \( r_\tau \) the shape-regularity constant. When the domain \( \tau \) is anisotropic, i.e., its size varies substantially in different space directions, the diameter \( h_\tau \) of \( \tau \) does not provide the right scaling for the inverse-type estimate described above. Instead, a sharper estimate reads

\[
\|v\|_{H^s(\tau)} \leq C(s, m_\tau) \rho_\tau^{-s} \|v\|_{L^2(\tau)},
\]

where \( \rho_\tau \) denotes the radius of the largest inscribed circle contained in \( \tau \).

Explicit knowledge of the dependence of the constant \( C(s, m_\tau) \) on the polynomial degree \( m_\tau \) is available. In particular, it is known that

\[
\|v\|_{H^s(\tau)} \leq C(s) m_\tau^{2s} \rho_\tau^{-s} \|v\|_{L^2(\tau)},
\]

(see, e.g., [8]).

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Generalisations of the above inverse-type estimates for functions belonging to finite-element type spaces have been presented in [5, 4]. In particular, in [5] inverse-type inequalities of the form
\[ \| \rho^s u \|_{H^s(\Omega)} \lesssim \| u \|_{L^2(\Omega)} \lesssim \| \rho^{-s} u \|_{H^{-s}(\Omega)}, \]
with \( H^{-s}(\Omega) \) and \( \| \cdot \|_{H^{-s}(\Omega)} \) denoting the negative order Sobolev spaces and the corresponding norms defined by duality; \( \rho \) is a mesh-dependent function representing the local \( \rho_\tau \); here and in the following, \( A \lesssim B \) if and only if there exists a constant \( C \), independent of local mesh parameters and local polynomial degrees, such that \( A \leq CB \).

The results presented in this work, extent the theory developed in [5] in three ways. First, the inverse estimates derived below are explicit in the polynomial degree of the finite element space; second, \( hp \)-finite element spaces are admissible, i.e., the local polynomial degree in each element can vary. Finally, hanging nodes in the mesh are now admissible in the analysis, allowing for meshes of greater generality (e.g., meshes emerging from adaptive algorithms). Throughout this work, the setting and much of the notation of [5] is followed.

The paper is structured as follows. Section 2 introduces the admissible finite element spaces for our analysis. Section 3 contains the main results of this work, i.e., Theorems 3.3, 3.4 and 3.8. Finally, in Section 4, using the results of Section 3, we prove bounds for the condition number of stiffness matrices for the \( hp \)-version (conforming) boundary element method admitting discontinuous basis functions.

## 2. Finite Element Spaces

Let \( \Omega \subset \mathbb{R}^d \) be a bounded \( d \)-dimensional subset of \( \mathbb{R}^3 \), for \( d = 2 \) or \( d = 3 \), i.e., when \( d = 3 \), \( \Omega \subset \mathbb{R}^3 \) is a bounded domain, and when \( d = 2 \), \( \Omega \) is a piecewise smooth Lipschitz manifold in \( \mathbb{R}^3 \) (which may or may not have a boundary). (Of course, the case of \( \Omega \subset \mathbb{R}^2 \) is included through the trivial imbedding into \( \mathbb{R}^3 \)).

We assume the usual notion of a Hilbertian Sobolev space \( H^s(\Omega) \) for positive integer \( s \) (see, e.g., [2]); Sobolev spaces with real positive index \( s \geq 0 \) are defined through the real or \( K \)-method of function space interpolation (see, e.g., [1]), and Sobolev spaces with \( s < 0 \) are defined though duality, in a standard fashion. Since we also consider Sobolev spaces on manifolds, we assume that the pull-backs are also sufficiently smooth.

We consider subdivisions \( T \) of the set \( \Omega \), consisting of pair-wise disjoint elements \( \tau \in T \), with \( \tau \subset \Omega \), constructed in a standard fashion, as mappings
\[ \chi_\tau : \hat{\tau} \to \tau, \]
of a reference element \( \hat{\tau} \) onto \( \tau \). We consider two choices for \( \hat{\tau} : \hat{\tau} := \hat{\sigma}^d \), with \( \hat{\sigma}^d \) denoting the \( d \)-dimensional simplex, or \( \hat{\tau} := \hat{\kappa}^d \), with \( \hat{\kappa}^d = (-1, 1)^d \). The above maps are assumed to be constructed so as to ensure that the union of the closures of the disjoint open elements \( \tau \in T \) forms a covering of the closure of \( \Omega \), i.e., \( \Omega = \bigcup_{\tau \in T} \bar{\tau} \).

Hybrid meshes, i.e., meshes containing both (mapped) simplices and quadrilaterals/hexahedrals are allowed. Moreover, we allow non-conforming meshes containing regular hanging nodes, in the following sense.

**Definition 2.1.** Consider two neighbouring elements \( \tau \) and \( \tau' \), and let \( e \subset \partial \tau \) and \( e' \subset \partial \tau' \) denote the two element edges for which \( e \cap e' \neq \emptyset \). If \( e \neq e' \), then we say
that there exists a hanging node in the subdivision, say $R$ (cf. Figure 1). We say that $R$ is a regular hanging node, if
\begin{equation}
  r^{-1} \leq \frac{\text{meas}_1(e)}{\text{meas}_1(e')} \leq r,
\end{equation}
for $r > 0$ uniformly in $T$; we call $r$ the regularity constant of the subdivision $T$. We shall refer to a subdivision $T$ as regular, if it contains only regular hanging nodes.

We continue with some more notation.

**Definition 2.2.** For each $\tau \in T$, we denote by $|\tau|$ its $d$-dimensional measure, by $h_\tau$ its diameter, by $\rho_\tau$ the diameter of the largest inscribed sphere contained in $\bar{\tau}$, and by $m_\tau := \min(1, \tilde{m}_\tau)$, where $\tilde{m}_\tau$ is the polynomial degree of the local finite element basis of $\tau$. Furthermore, if $t \subset \Omega$ is an arbitrary simplex or quadrilateral/hexahedron (not necessarily an element of $T$), the local quantities $h_t$ and $\rho_t$ are defined completely analogously.

Let also $J_\tau$ denote the $3 \times d$ Jacobian of the map $\chi_\tau$, and define the Gram determinant of $\chi_\tau$, by
\[ g_\tau(\hat{x}) := \left( \det(J_\tau^T(\hat{x})J_\tau(\hat{x})) \right)^{1/2}, \]
which appears in the integral change-of-variable formula:
\[ \int_{\tau} f(x)dx = \int_{\hat{\tau}} f(\chi_\tau(\hat{x}))g_\tau(\hat{x})d\hat{x}. \]

Throughout this work, we make the following assumptions (cf. Assumptions 2.2 and 2.6 in [5]).

**Assumption 2.3 (mapping properties).** We have
\[ D^{-1}|\tau| \leq g_\tau(\hat{x}) \leq D|\tau|, \]
\[ E\rho_\tau^2 \leq \lambda_{\min}(J_\tau^T(\hat{x})J_\tau(\hat{x})), \]
uniformly in $\hat{x} \in \hat{\tau}$, with positive constants $D$ and $E$, independent of $\tau$, and $\lambda_{\min}(B)$, $\lambda_{\max}(B)$ denoting the minimum and the maximum eigenvalues of a square matrix $B$. 
Assumption 2.4 (finite element space properties). For some $c_1, c_2, c_3 > 0$ and $M \in \mathbb{N}$, we assume that, for all $\tau, \tau' \in T$ with $\tau \cap \tau' \neq \emptyset$, we have
\[
h_{\tau} \leq c_1 h_{\tau'}, \quad \rho_{\tau} \leq c_2 \rho_{\tau'}, \quad m_{\tau} \leq c_3 m_{\tau'},
\]
\[
\max_{i \in N} \# \{ \tau \in T : x_i \in \bar{\tau} \} \leq M.
\]

The above assumptions on the mesh are very weak, in the sense that they allow very general degenerate (anisotropic) meshes (in particular, power-graded and geometric-graded meshes are allowed); we refer to Section 2 of [5] for a detailed discussion on the admissible meshes under the above assumptions.

Definition 2.5 (finite element spaces). For $m$ and $\hat{\tau} \in \{ \hat{\sigma}^d, \hat{k}^d \}$, we define
\[
\mathbb{P}^m(\hat{\tau}) = \begin{cases} 
polynomials of total degree \leq m on \hat{\tau}, & \text{if } \hat{\tau} = \hat{\sigma}^d; 
\text{polynomials of coordinate degree} \leq m on \hat{\tau}, & \text{if } \hat{\tau} = \hat{k}^d.
\end{cases}
\]
We also define the polynomial degree vector $m = (m_\tau : \tau \in T)$. Then for $i \in \{0,1\}$ and $m_\tau \geq i$, for all $\tau \in T$, we set
\[
S^0_i m(T) = \{ u \in L^\infty(\Omega) : u \circ \chi_\tau \in \mathbb{P}^{m_\tau}(\hat{\sigma}^d), \tau \in T \}
\]
\[
S^i m(T) = \{ u \in C^0(\Omega) : u \circ \chi_\tau \in \mathbb{P}^{m_\tau}(\hat{\tau}), \tau \in T \}.
\]

We present a generalisation of Proposition 2.9 (or Corollary 2.10) in [5].

Proposition 2.6. Let $\tau \in T$ and let $\hat{t}$ be any simplex which is contained in the associated unit element $\hat{\tau} \in \mathbb{R}^d$. Let $\hat{P} \in \mathbb{P}^{m_\tau}(\hat{t})$ denote any $d$-variave polynomial of degree $m_\tau$ on $\hat{t}$ and define $t = \chi_\tau(\hat{t})$, $P = \hat{P} \circ \chi_\tau^{-1}$, where $\chi_\tau$ is assumed to be affine, for simplicity. Then, for all $0 \leq s \leq k$, we have
\[
\|P\|_{H^r(t)} \lesssim m_{\tau}^{2s} \rho_{\tau}^s \|P\|_{L^2(t)}.
\]

Proof. The proof is similar to the proof of Proposition 2.9 in [5]. Here we only note that the crucial difference in the use of the well known inverse estimate
\[
\|v\|_{H^s(t)} \lesssim m_{\tau}^2 \|v\|_{L^2(t)},
\]
for $v = |\hat{P} \circ \nu| \in \mathbb{P}^{m_\tau}(\hat{\tau})$, (see, e.g., (4.6.5) in [8]) with the notation of the proof of Proposition 2.9 in [5]. \qed

3. Inverse estimates

Let $\{x_p : p \in \mathcal{N}\}$ denote the set of nodes of $T$, for some index set $\mathcal{N}$.

Definition 3.1 (Mesh Function). For each $p \in \mathcal{N}$, set $\rho_p = \max\{\rho_\tau : x_p \in \bar{\tau}\}$. The mesh function $\rho$ is the unique function in $S^1(T)$ such that $\rho(x_p) = \rho_p$, for each $p \in \mathcal{N}$.

Definition 3.2 (Polynomial Degree Function). For each $p \in \mathcal{N}$, set $m_p = \max\{m_\tau : x_p \in \bar{\tau}\}$. The polynomial degree function $m$ is the unique function in $S^1(T)$ such that $m(x_p) = m_p$, for each $p \in \mathcal{N}$.

Clearly $\rho$ and $m$ are positive, continuous functions on $\Omega$, and by Assumption 2.4, it follows that if $x \in \tau$, then $\rho(x) \sim \rho_\tau$ and $m(x) \sim m_\tau$, for all $\tau \in T$.

The next result is a generalisation of Theorem 3.2 in [5].
Theorem 3.3. Let \( 0 \leq s \leq 1, -\infty < \alpha < \pi < \infty, \) and \(-\infty < \beta < \beta < \infty. \) Then,

\[
\| \frac{\rho^\alpha}{m^{\beta}} u \|_{H^s(\Omega)} \lesssim \| \frac{\rho^{\alpha-s}}{m^{\beta/2}} v \|_{L^2(\Omega)},
\]

uniformly in \( \alpha \in [\alpha, \pi] \) and \( \beta \in [\beta, \beta], u \in S_1^m(T). \)

Proof. We have

\[
\nabla \left( \frac{\rho^\alpha}{m^{\beta}} u \right) = \alpha \frac{\rho^{\alpha-1}}{m^{\beta}} \nabla \rho - \beta u \frac{\rho^{\alpha}}{m^{\beta+1}} \nabla m + \frac{\rho^\alpha}{m^{\beta}} \nabla u,
\]

For \( v \in P^m(\tau), \) we recall Markov’s inequality

\[
\| \nabla v \|_{L^\infty(\tau)} \leq C m^2 \rho^{-1}_\tau \| v \|_{L^\infty(\tau)},
\]

where \( C \) is a constant that depends only on the shape of \( \tau, \) but not its size (see, e.g., in Theorem 4.76 in [8] for \( d = 2; \) for \( d = 3 \) the proof is analogous). Using this together with Assumption 2.4 and Proposition 2.6, we have, respectively,

\[
\| \nabla \left( \frac{\rho^\alpha}{m^{\beta}} u \right) \|_{L^2(\tau)}^2 \lesssim \| \frac{\rho^{\alpha-1}}{m^{\beta}} \nabla \rho \|_{L^2(\tau)}^2 + \| \frac{\rho^\alpha}{m^{\beta+1}} \nabla m \|_{L^2(\tau)}^2 + \| \frac{\rho^\alpha}{m^{\beta}} \nabla u \|_{L^2(\tau)}^2 + \| \frac{\rho^\alpha}{m^{\beta}} u \|_{L^2(\tau)}^2,
\]

and the proof for \( s = 1 \) follows by summation over \( \tau \in T. \) The proof for \( s \in (0,1) \) follows by interpolation. \( \square \)

Before considering the case of \( h_p \)-inverse estimates for functions belonging to \( S_0^m(T), \) we present the following auxiliary result.

Lemma 3.4. Let \( \tau \) denote the reference (triangular or quadrilateral) element, and let \( \hat{\tau} : \tau \rightarrow \mathbb{R} \) be a polynomial of degree \( m. \) Let also function \( \hat{\eta}_\delta : \hat{\tau} \rightarrow \mathbb{R} \) defined as

\[
\hat{\eta}_\delta(x) = \begin{cases} 
1, & \text{if } \text{dist}(\hat{x}, \partial \hat{\tau}) \geq \delta; \\
\delta^{-1} \text{dist}(\hat{x}, \partial \hat{\tau}), & \text{otherwise.}
\end{cases}
\]

Then \( \hat{\eta}_\delta \hat{u} \in H_0^1(\hat{\tau}), \) and the following estimates hold:

\[
\| \hat{\eta}_\delta \hat{u} \|_{L^2(\hat{\tau})} \leq \| \hat{u} \|_{L^2(\hat{\tau})}, \\
\| (1 - \hat{\eta}_\delta) \hat{u} \|_{L^2(\hat{\tau})} \leq \delta m^2 \| \hat{u} \|_{L^2(\hat{\tau})}, \\
\| \nabla (\hat{\eta}_\delta) \hat{u} \|_{L^2(\hat{\tau})} \leq \delta^{-1} m^2 \| \hat{u} \|_{L^2(\hat{\tau})}.
\]

The proof Lemma 3.4 is given in the Appendix for reasons of clarity of the presentation.

The next result is a generalisation of Theorem 3.4 in [5].
Theorem 3.5. Let $0 \leq s < \frac{1}{2}$, $-\infty < \alpha < \pi < \infty$, and $-\infty < \beta < \beta_{\sigma} < \infty$. Then,
\[
\| \frac{\rho^s}{m^\alpha} u \|_{H^s(\Omega)} \lesssim \| \frac{\rho^{3-s}}{m^{\beta-2s}} \|_{L^2(\Omega)},
\]
uniformly in $\alpha \in [\alpha, \pi]$ and $\beta \in [\beta_{\sigma}, \beta_{\pi}]$, $u \in S^{m}_{0}(T)$.

Proof. We consider $s > 0$, for otherwise, the proof is trivial. For brevity, here we denote $\tilde{u} = \frac{m^{\alpha}}{\rho^s} u$.

We shall define the fractional order Sobolev space $H^s(\Omega)$ using the K-method of function space interpolation (see, e.g., [1]): $H^s(\Omega)$ is defined as the interpolation space between $L^2(\Omega)$ and $H^1(\Omega)$ with interpolation parameter $\theta = s$. The norm $\| \tilde{u} \|_{H^s(\Omega)}$ can be expressed through the, so-called, K-functional $K(t, \tilde{u})$, as
\[
\| \tilde{u} \|_{H^s(\Omega)} := \left( \int_{0}^{\infty} t^{-2s} K^2(t, \tilde{u}) \frac{dt}{t} \right)^{\frac{1}{2}},
\]
where
\[
K(t, \tilde{u}) := \inf_{u_0 \in L^2(\Omega), u_1 \in H^1(\Omega)} \left\{ \|u_0\|_{L^2(\Omega)}^2 + t^2 \|u_1\|_{H^1(\Omega)}^2 \right\}^{\frac{1}{2}}.
\]

We now construct a suitable splitting $u_0 + u_1 = \tilde{u}$. For $\tau \in T$, we define $\eta_\tau^0$ by $\eta_\tau^0 := \tilde{u}_\tau \circ \chi_\tau^{-1}$, where $\tilde{u}_\tau$ as in the statement of Lemma 3.4. We consider the splitting $\tilde{u} = u_0 + u_1$, defined element-wise by
\[
u_0|_\tau := \begin{cases} (1 - \eta_\tau^0) \tilde{u}|_\tau, & 0 \leq t \leq \gamma; \\ \tilde{u}|_\tau, & t > \gamma, \end{cases} \quad \text{and} \quad u_1|_\tau := \begin{cases} \eta_\tau^0 \tilde{u}|_\tau, & 0 \leq t \leq \gamma; \\ 0, & t > \gamma, \end{cases}
\]
with $\gamma := \rho_{\tau}/m^{\alpha}_{\tau}$. From Lemma 3.4, we know that $u_1 \in H^1(\Omega)$. Then, we have
\[
K^2(t, \tilde{u}) \leq \|u_0\|_{L^2(\Omega)}^2 + t^2 \|u_1\|_{H^1(\Omega)}^2,
\]
and, therefore,
\[
\| \tilde{u} \|_{H^s(\Omega)}^2 \leq \int_{0}^{\infty} t^{-2s-1} \left( \|u_0\|_{L^2(\Omega)}^2 + t^2 \|u_1\|_{H^1(\Omega)}^2 \right) dt
\]
\[
= \sum_{\tau \in T} \int_{0}^{\infty} t^{-2s-1} \left( \|u_0^\tau\|_{L^2(\tau)}^2 + t^2 \|u_1^\tau\|_{H^1(\tau)}^2 \right) dt
\]
\[
\leq \sum_{\tau \in T} \int_{0}^{\gamma} t^{-2s-1} \left( \| (1 - \eta_\tau^0) \tilde{u} \|_{L^2(\tau)}^2 + t^2 \| \eta_\tau^0 \tilde{u} \|_{H^1(\tau)}^2 \right) dt
\]
\[
+ \sum_{\tau \in T} \int_{\gamma}^{\infty} t^{-2s-1} \| \tilde{u} \|_{L^2(\tau)}^2 dt
\]
\[
(3.5)
\]
For a function $f : \tau \rightarrow \mathbb{R}$, we denote $\tilde{f} : \tilde{\tau} \rightarrow \mathbb{R}$, with $\tilde{f} := f \circ \chi_\tau^{-1}$. Then, the term $\| (1 - \eta_\tau^0) \tilde{u} \|_{L^2(\tau)}^2$ can be bounded as follows:
\[
\| (1 - \eta_\tau^0) \tilde{u} \|_{L^2(\tau)}^2 \leq \frac{\rho^{2\alpha}_{\tau}}{m^{\frac{\alpha}{2}}_{\tau}} \| (1 - \eta_\tau^0) u \|_{L^2(\tau)}^2 \leq \frac{\rho^{2\alpha}_{\tau}}{m^{\alpha}_{\tau}} \| (1 - \eta_\tau^0) \tilde{u} \|_{L^2(\tilde{\tau})}^2
\]
\[
\leq \frac{\rho^{2\alpha}_{\tau}}{m^{\frac{\alpha}{2}}_{\tau} - 2\delta} \| \tilde{u} \|_{L^2(\tau)}^2 \lesssim \frac{\rho^{2\alpha}_{\tau}}{m^{\alpha}_{\tau} - 2\delta} \| u \|_{L^2(\tau)}^2 \lesssim m^{2\alpha}_{\tau} \delta \| \tilde{u} \|_{L^2(\tau)}^2
\]
\[
(3.6)
\]
where in the third inequality we have made use of (3.3), and in the second and fourth inequality, we made use of Assumption 2.3.
Proof. This is a standard mollifier argument. We can choose $\eta^\sharp \tilde{u}$ to be the characteristic function of a set $\Omega$, having the same shape as $\bar{\tau}$ and faces parallel to the faces of $\bar{\tau}$, such that $\text{dist}(\bar{\tau}, \partial \bar{\tau}) \geq \epsilon$, for some $\epsilon > 0$ (i.e., $\bar{\tau}$ is $\epsilon$-away from the boundary of $\bar{\tau}$). Then, there exists a function $P_\ell \in H^k(\mathbb{R}^d)$, such that

$$P_\ell \equiv 0 \quad \text{on} \quad \mathbb{R}^d \setminus \bar{\tau}, \quad 0 \leq P_\ell(x) \leq 1 \quad \text{for all} \quad x \in \bar{\ell}, \quad \frac{1}{2} \leq P_\ell(x) \leq 1 \quad \text{for all} \quad x \in \bar{t},$$

and $\|D^n P_\ell\|_{L^\infty(\bar{\tau})} \lesssim \epsilon^{-1}$, for $i = 1, 2, \ldots, k$.}

Proof. This is a standard mollifier argument. We can choose $P_\ell = A_\ell^t \text{char}(\ell)$, where $A_\ell^t$ the standard mollification operator (see, e.g., [2]), and char($\omega$) denotes the characteristic function of a set $\omega$.

We continue with the following technical result (cf. Lemma 3.5 in [5]).

**Lemma 3.6.** Let $\bar{\tau}$ be the reference element, and consider $\bar{i} \subset \bar{\tau}$ a set $\bar{i} \subset \bar{\tau}$, having the same shape as $\bar{\tau}$ and faces parallel to the faces of $\bar{\tau}$ such that $\text{dist}(\bar{\ell}, \partial \bar{\tau}) \geq \epsilon$, for some $\epsilon > 0$ (i.e., $\bar{\tau}$ is $\epsilon$-away from the boundary of $\bar{\tau}$). Then, there exists a function $\eta^\sharp \tilde{u}$ in $H^k(\mathbb{R}^d)$, such that

$$\|\eta^\sharp \tilde{u}\|_{H^k(\bar{\tau})} \lesssim \sum_{\tau \in T} \left(\int_0^1 t^{-2s-1} \left(\frac{m^2 \gamma^{-2s+2}}{\rho^2 - 2s + 2} + \frac{m^4 \gamma^{-2s+2}}{2s}\right)dt\right)\|\tilde{u}\|^2_{L^2(\tau)}.$$

Recalling that $\gamma = \rho^2/m^2$, the result follows. (We note that, with this choice of $\gamma$, we have $\delta \equiv (4\rho^2)^{-1} \leq (2m^2)^{-1}$, for all $\tau \in T$, yielding $\eta^\sharp \tilde{u} \in H^k(\bar{\tau})$.)

We continue with some intermediate results of technical nature.

**Lemma 3.7.** Let $\bar{\tau}$ be as in Definition 2.5. Then, for each $u \in \mathbb{P}^m(\bar{\tau})$, there exists a set $\bar{i} \subset \bar{\tau}$, having the same shape as $\bar{\tau}$ and faces parallel to the faces of $\bar{\tau}$, such that

$$\rho_\ell \approx 1 \quad \text{and} \quad \|u\|_{L^2(\bar{i})} \geq \frac{1}{2} \|u\|_{L^2(\bar{\tau})}.$$
Figure 2. Splitting of $\hat{\tau}$

Proof. Consider a set $\hat{t} \subset \hat{\tau}$ whose faces are parallel to the faces of $\hat{\tau}$ at a distance $\epsilon$ (see Figure 2 for a geometric representation of the setting when $d = 2$), and subdivide $\hat{\tau} \setminus \hat{t}$ into 4 subsets $\hat{t}_i$, $i = 1, 2, 3, 4$ as shown in Figure 2. To simplify the presentation consider the case $d = 2$. Then for $\hat{t}_1$, we have

$$\|u\|^2_{L^2(\hat{t}_1)} = \int_{-1+\epsilon}^{1} \int_{1-\epsilon}^{1} u^2(x, y) dx dy = \int_{-1+\epsilon}^{1} \|u(\cdot, y)\|^2_{L^2(\hat{t}_1, y)} dy$$

$$\leq \int_{-1+\epsilon}^{1} \epsilon \|u(\cdot, y)\|^2_{L^\infty(\hat{t}_1, y)} dy \leq \int_{-1}^{1} \epsilon \|u(\cdot, y)\|^2_{L^\infty(\hat{t}, y)} dy$$

$$\leq \int_{-1}^{1} 2\epsilon Cm^2 \|u(\cdot, y)\|^2_{L^2(\hat{t}, y)} dy = 2\epsilon Cm^2 \|u\|^2_{L^2(\hat{t})},$$

where in the third inequality we have used Bernstein’s inequality (3.8)

$$\|v\|_{L^\infty(\tau)} \lesssim m^d |\tau|^{-\frac{1}{2}} \|v\|_{L^2(\tau)}$$

with $d = 1$ (see, e.g., Theorem 3.92 in [8] for $d = 1$, Theorem 4.76 in [8] for $d = 2$; for $d = 3$ the proof is analogous). Quite similarly, we can deduce analogous bounds for $\hat{t}_i$, $i = 2, 3, 4$, or when $\hat{\tau}$ is a simplex, to obtain

$$\|u\|^2_{L^2(\hat{\tau} \setminus \hat{t})} = \sum_{i=1}^{4} \|u\|^2_{L^2(\hat{t}_i)} \leq 8\epsilon Cm^2 \|u\|^2_{L^2(\hat{t})}. $$

Selecting $\epsilon = (\frac{\epsilon}{2Cm^2})^{-1}$, we deduce $\|u\|^2_{L^2(\hat{\tau} \setminus \hat{t})} \leq \frac{3}{4} \|u\|^2_{L^2(\hat{\tau})}$. Using this, we have, respectively,

$$\|u\|^2_{L^2(\hat{t})} = \|u\|^2_{L^2(\hat{\tau})} - \|u\|^2_{L^2(\hat{\tau} \setminus \hat{t})} \geq \|u\|^2_{L^2(\hat{\tau})} - \frac{3}{4} \|u\|^2_{L^2(\hat{\tau})} = \frac{1}{4} \|u\|^2_{L^2(\hat{\tau})},$$

Thus, $\hat{t} \subset \hat{\tau}$ has the same shape as $\hat{\tau}$ (and faces parallel to those of $\hat{\tau}$, too) with $\text{dist}(\hat{t}, \partial \hat{\tau}) \gtrsim \epsilon \sim m^{-2}$. Hence, $1 \geq \rho t \gtrsim 1 - \epsilon \sim 1$, and the proof is complete for $d = 2$. The above argument remains completely analogous for $d \geq 3$, and hence the result holds for any $d \in \mathbb{N}$. The proof for reference triangular domains follows completely analogously. 

Combining the two lemmata above, we have the following result.
Proposition 3.8. Let \( u \in \mathbb{P}^{m}(\hat{t}), \hat{t} \subset \tilde{t} \) as in Lemma 3.7 with \( \text{dist}(\hat{t}, \partial \Omega) \gtrsim m^{-2} \), and \( P_{t} \) as in Lemma 3.6. Then, we have
\begin{equation}
\|uP_{t}\|_{H^{k}(\hat{t})} \lesssim m^{2k}\|u\|_{L^{2}(\hat{t})}.
\end{equation}

Proof. We prove the result for \( k = 1 \); then, the result for general \( k \in \mathbb{N} \) follows by induction.

We have, respectively,
\begin{align*}
\|\nabla (uP_{t})\|_{L^{2}(\hat{t})} & \leq \|\nabla u\|_{L^{2}(\hat{t})} + \|u\nabla P_{t}\|_{L^{2}(\hat{t})} \\
\|P_{t}\|_{L^{\infty}(\hat{t})}\|\nabla u\|_{L^{2}(\hat{t})} + \|\nabla P_{t}\|_{L^{\infty}(\hat{t})}\|u\|_{L^{2}(\hat{t})} & \lesssim m^{2}\|u\|_{L^{2}(\hat{t})} + m^{2}\|u\|_{L^{2}(\hat{t})} \sim m^{2}\|u\|_{L^{2}(\hat{t}),}
\end{align*}
where in the last inequality we have made use of Proposition 2.6 and the \( L^{\infty} \)-bounds of \( P_{t} \) and its first derivative, from Lemma 3.6.

The next result is a generalisation of Theorem 3.6 in [5].

Theorem 3.9. Let \( i \in \{0, 1\}, 0 \leq s \leq k, -\infty < \alpha < \beta < \infty, -\infty < \beta < \beta \lesssim \infty \), and assume that \( m_{\beta} \geq i \), for all \( \tau \in \mathcal{T} \). Then,
\begin{equation}
\rho^{s+\alpha} \langle \frac{\beta^{k} \rho^{s}}{m^{2k+\beta}} u \rangle_{L^{2}(\tau)} \lesssim \| \rho^{s} \|_{H^{-\beta}(\tau)}^{2},
\end{equation}
uniformly in \( \alpha \in [\alpha, \beta] \) and \( \beta \in [\beta, \beta] \), \( u \in \mathcal{S}^{m}(\mathcal{T}) \).

Proof. The result is clear for \( s = 0 \). We prove it for \( s = k \in \mathbb{N} \); the result then follows by interpolation. Without loss of generality we assume \( u \neq 0 \), for otherwise the result is trivial. We want to construct \( w \in H^{k}(\Omega) \), such that
\begin{equation}
\| \rho^{s} \|_{H^{-\beta}(\tau)} \lesssim \| \frac{\beta^{k} \rho^{s}}{m^{2k+\beta}} u \|_{L^{2}(\tau)}^{2},
\end{equation}
and
\begin{equation}
\| w \|_{H^{k}(\tau)} \lesssim \| \frac{\beta^{k+\alpha}}{m^{2k+\beta}} u \|_{L^{2}(\tau)};
\end{equation}
then the result follows immediately from the definition of the dual norm.

To construct \( w \), we work as follows. For any \( \tau \in \mathcal{T} \), we have \( \hat{u} := u \circ \chi_{\tau}^{-1} \in \mathbb{P}^{m}(\hat{t}) \), and by Lemma 3.7, there exists \( t(\tau) \subset \hat{t} \) such that
\begin{equation}
\rho_{t(\tau)} \sim 1 \quad \text{and} \quad \| \hat{u} \|_{L^{2}(\hat{t})} \gtrsim \frac{1}{2} \| \hat{u} \|_{L^{2}(\hat{t})},
\end{equation}
Scaling yields
\begin{equation}
\rho_{t(\tau)} \sim \rho_{\tau} \quad \text{and} \quad \| u \|_{L^{2}(t(\tau))} \gtrsim \frac{1}{2} \| u \|_{L^{2}(\tau)},
\end{equation}
for \( t(\tau) := t(\tau) \circ \chi_{\tau}^{-1} \). Let \( \tau \subset \Omega \) be an element, and consider \( t(\tau) \subset \tau \) as above. Then, from the proof of Lemma 3.7, we have \( \text{dist}(t(\tau), \partial \tau) \gtrsim m_{\beta}^{-2} \). Now, using Lemma 3.6 and Proposition 3.8, there exists a function \( P_{t(\tau)} \in H^{k}(\mathbb{R}^{d}) \), such that
\begin{equation}
P_{t(\tau)} \equiv 0 \quad \text{on} \quad \mathbb{R}^{d} \setminus \tau, \quad 0 \leq P_{t(\tau)}(\mathbf{x}) \leq 1 \quad \text{for all} \quad \mathbf{x} \in \tau,
\end{equation}
and
\begin{equation}
\| u P_{t(\tau)} \|_{H^{k}(\tau)} \lesssim m^{2k}\rho_{t(\tau)}^{-k}\| u \|_{L^{2}(\tau)},
\end{equation}
with the last inequality resulting from (3.9) through a standard scaling argument.
We are now in position to define
\[ w = \sum_{\tau \in T} m_{\tau}^{-4k-\beta} \rho_{\tau}^{2k+\alpha} u_{\tau}. \]

Then,
\[
\left| \left( \frac{\partial^{\rho}}{m^{\beta}} u, w \right) \right| \geq \sum_{\tau \in T} m_{\tau}^{-4k-2\beta} \rho_{\tau}^{2k+2\alpha} \int_{\tau} u^{2} P_{\tau}^{(r)} \geq \sum_{\tau \in T} m_{\tau}^{-4k-2\beta} \rho_{\tau}^{2k+2\alpha} \frac{1}{2} \int_{\tau} u^{2} \geq \sum_{\tau \in T} m_{\tau}^{-4k-2\beta} \rho_{\tau}^{2k+2\alpha} \frac{1}{8} \int_{\tau} u^{2}
\]
\[
\geq \| \rho_{k+\alpha} \|_{L^{2}(\Omega)}^{2} \| w \|_{H^{k}(\Omega)}^{2},
\]
where in the third step we used the fact that \( P_{\tau}^{(r)} \geq \frac{1}{2} \) on \( t(\tau) \), and on the fourth step we made use of (3.10).

Also, we have
\[
\| w \|_{H^{k}(\Omega)}^{2} = \sum_{\tau \in T} \| w \|_{H^{k}(\tau)}^{2} \leq \sum_{\tau \in T} m_{\tau}^{-8k-2\beta} \rho_{\tau}^{4k+2\alpha} \| u_{\tau} \|_{L^{2}(\tau)}^{2},
\]
and, using (3.11), we obtain
\[
\| w \|_{H^{k}(\tau)}^{2} \leq \sum_{\tau \in T} m_{\tau}^{-8k-2\beta} \rho_{\tau}^{4k+2\alpha} m_{\tau}^{4k} \rho_{\tau}^{-2k} \| u \|_{L^{2}(\tau)}^{2} \leq \| \rho_{k+\alpha} \|_{L^{2}(\Omega)}^{2},
\]
and the result now follows. \( \square \)

Remark 3.10. In [5], where the polynomial degree in the estimates proven therein is uniform and constant, the relation
\[
\| u \|_{L^{\infty}(\tau)} \| \tau \|^{\frac{1}{2}} \sim \| u \|_{L^{2}(\tau)}
\]
was used extensively in the proofs presented in that work. When the explicit dependence on the polynomial degree of \( u \) is required, as it is the case in the present work, (3.12) becomes
\[
\| u \|_{L^{\infty}(\tau)} \| \tau \|^{\frac{1}{2}} m^{d} \leq \| u \|_{L^{2}(\tau)} \leq \| u \|_{L^{\infty}(\tau)} \| \tau \|^{\frac{1}{2}},
\]
using Berstein’s inequality (3.8). Note that the lower bound involves the polynomial degree raised to the negative power equal to the dimension of the computational domain. However, the dimension of the computational domain is not present in conjunction with the polynomial degree in the classical \( hp \)-inverse estimates (for polynomials) of the form
\[
\frac{h^{s}}{m^{d}} \| u \|_{H^{s}(\tau)} \leq \| u \|_{L^{2}(\tau)} \leq \frac{m^{2s}}{h^{s}} \| u \|_{H^{-s}(\tau)},
\]
and, thus, the same expected for the inverse estimates on \( hp \)-finite element spaces. Hence, we systematically avoided using \( \| u \|_{L^{\infty}(\tau)} \) in the proofs of Theorems 3.3 and 3.9 above, making use instead of tensor-product-type constructions.

Next, we present an application of the above developments, regarding the estimation of the condition number of stiffness matrices arising in \( hp \)-boundary element methods.
4. The Conditioning of \(hp\)-Boundary Element Matrices Emerging from Discontinuous Subspaces

Most boundary integral equations arising from elliptic partial differential equations have a variational formulation in a low order Sobolev space. Typical example is the classical single layer potential for the Laplacian, whose solutions belong to \(H^{-1/2}(\Gamma)\), where \(\Gamma\) is either a domain in \(\mathbb{R}^d\) or \(d\)-dimensional manifold in \(\mathbb{R}^{d+1}\), satisfying the same assumptions as \(\Omega\) in Section 2. We are interested in approximating the solutions to such integral equations using the \(hp\)-version boundary element method and to study the conditioning of the resulting stiffness matrices.

For \(-1 \leq s < 1/2\), we consider the general linear integral equation

\[
(\lambda I + K)u_{an}(x) := \lambda u_{an}(x) + \int_{\Gamma} k(x, y)u_{an}(y)dy = f(x), \quad x \in \Gamma,
\]

for some given scalar \(\lambda \in \mathbb{R}\), kernel function \(k\), analytic solution \(u_{an}\) and sufficiently smooth right-hand side \(f\). The corresponding weak form is

Find \(u_{an} \in H^s(\Gamma)\) such that \(a(u_{an}, v) := ((\lambda I + K)u_{an}, v) = (f, v) \quad \forall v \in H^s(\Gamma)\).

Remark 4.1. The spaces \(H^s(\Gamma)\) (see, e.g., [7]) are often used in the analysis of integral equations. Since for \(-1 \leq s < 1/2\), we have \(H^s(\Gamma) \cong \mathcal{H}^s(\Gamma)\), we shall be working with classical Sobolev spaces instead.

Let \(T\) be a subdivision of \(\Gamma\), consisting of quadrilateral elements, satisfying the assumptions in Section 2. Since \(s < 1/2\), we can choose element-wise discontinuous finite-dimensional subspace \(S^m(T)\), constructed from tensor-product Legendre-polynomial local basis functions.

The corresponding discrete problem reads

\[
(\lambda I + K)\hat{u}_{an}(x) := \lambda \hat{u}_{an}(x) + \int_{\Gamma} k(x, y)\hat{u}_{an}(y)dy = \hat{f}(x), \quad x \in \Gamma,
\]

where \(\hat{u}\) is the \(hp\)-boundary element approximation to \(u_{an}\).

Let \(L_i\) denote the Legendre polynomial of degree \(i\) defined on \((-1, 1)\). It is known that the Legendre polynomials form an orthogonal basis of \(L^2(-1, 1)\). Hence, for any \(\hat{u} \in \mathbb{P}^m(-1, 1) \subset L^2(-1, 1)\), there exist \(\hat{U}_i \in \mathbb{R}, i \in \mathbb{N}_0\), such that

\[
\hat{u} = \sum_{i=0}^m \hat{U}_i L_i;
\]

the corresponding Paserval’s identity reads

\[
\|\hat{u}\|^2_{L^2(-1, 1)} = \sum_{i=0}^m \hat{U}_i^2 \frac{2}{2i + 1}.
\]

On the reference element \(\hat{\tau} \equiv \hat{\mathbb{R}}^d := (-1, 1)^d, \ d = 2, 3\), we consider the the tensor-product polynomial basis

\[
\text{span}\{L_{i_1} \ldots L_{i_d} : 0 \leq i_j \leq m, \ i_j \in \mathbb{N}_0, j = 1, \ldots, d\}.
\]

Thus, every polynomial \(\hat{u} : \hat{\tau} \rightarrow \mathbb{R}\) of degree at most \(m\) in each variable can be expressed in terms of the above basis, i.e., there exist \(\hat{U}^\tau_{i_1, \ldots, i_d} \in \mathbb{R}\) such that

\[
\hat{u}(\hat{x}_1, \ldots, \hat{x}_d) = \sum_{0 \leq i_j \leq m} \hat{U}^\tau_{i_1, \ldots, i_d} \prod_{j=1}^d L_{i_j}(\hat{x}_j),
\]
Using (4.2), along with Fubini’s Theorem, we deduce

\begin{equation}
(4.3) \quad \|\hat{u}\|_{L^2(\tau)}^2 = \sum_{0 \leq i_j \leq m} (U_{i_1, \ldots, i_d}^\tau)^2 \prod_{j=1}^d \frac{2}{2i_j + 1}.
\end{equation}

For a typical \( u \in S^m(T) \), we have \( u|_{\tau} := \hat{u}_\tau \circ \chi_{\tau}^{-1} \), where

\[ \hat{u}_\tau(x_1, \ldots, x_d) = \sum_{0 \leq i_j \leq m} U_{i_1, \ldots, i_d}^\tau \prod_{j=1}^d L_j^\tau(\hat{x}_j), \]

for some real numbers \( U_{i_1, \ldots, i_d}^\tau \) with \( 0 \leq i_j \leq m, j = 1, \ldots, d, \tau \in \mathcal{T} \).

Let \( A \) denote the symmetric stiffness matrix of the discrete problem (4.1). Then, for \( U := (U_{i_1, \ldots, i_d}^\tau : 0 \leq i_j \leq m, j = 1, \ldots, d, \tau \in \mathcal{T}) \), we have

\[ \mathbf{U}^T A \mathbf{U} = a(u, u) \sim \|u\|_{H^s(\Gamma)}^2 \quad \text{and} \quad \mathbf{U}^T \mathbf{U} = \sum_{\tau \in \mathcal{T}} \sum_{0 \leq i_j \leq m} (U_{i_1, \ldots, i_d}^\tau)^2. \]

Thus, if we show the bounds

\[ \Lambda_{\min} \sum_{\tau \in \mathcal{T}} \sum_{0 \leq i_j \leq m} (U_{i_1, \ldots, i_d}^\tau)^2 \lesssim \|u\|_{H^s(\Gamma)}^2 \lesssim \Lambda_{\max} \sum_{\tau \in \mathcal{T}} \sum_{0 \leq i_j \leq m} (U_{i_1, \ldots, i_d}^\tau)^2, \]

we shall immediately have \( \lambda_{\min}(A) \lesssim \Lambda_{\max} \) and \( \lambda_{\min}(A) \gtrsim \Lambda_{\min} \).

**Lemma 4.2.** If \( 0 \leq s < 1/2 \), we have

\[ \lambda_{\min}(A) \gtrsim \min_{\tau \in \mathcal{T}} \{ m_{\tau}^{-d} \|\tau\| \} \quad \text{and} \quad \lambda_{\max}(A) \lesssim \max_{\tau \in \mathcal{T}} \{ m_{\tau}^{4s} \|\tau\| \rho_{\tau}^{-2s} \} \]

**Proof.** Let \( d = 2 \). We have

\begin{equation}
(4.4) \quad \|u\|_{L^2(\tau)}^2 \leq \|u\|_{H^s(\Gamma)}^2 \lesssim \|\frac{\rho^{-s}}{m_{\tau}^{-2s}} u\|_{L^2(\tau)}^2,
\end{equation}

where the first inequality is trivial, and the second inequality follows from Theorem 3.5. Paserval’s identity (4.3), along with scaling, yields

\[ \|u\|_{L^2(\tau)}^2 = \sum_{\tau \in \mathcal{T}} \|u\|_{L^2(\tau)}^2 \sim \sum_{\tau \in \mathcal{T}} |\tau| \|\hat{u}\|_{L^2(\tau)}^2 = \sum_{\tau \in \mathcal{T}} |\tau| \sum_{i,j=0}^{m_{\tau}} (U_{i,j}^\tau)^2 \frac{2}{2i + 1} \frac{2}{2j + 1}. \]

Hence, in view of (4.4), we have

\[ \|u\|_{H^s(\Gamma)}^2 \gtrsim \|u\|_{L^2(\tau)}^2 \gtrsim \sum_{\tau \in \mathcal{T}} |\tau| \sum_{i,j=0}^{m_{\tau}} (U_{i,j}^\tau)^2 \quad \text{and} \quad \lambda_{\min}(B) \]

and, thus, the lower bound on \( \lambda_{\min}(B) \) follows.
On the other hand, Paserval’s identity gives

\[
\| \frac{\rho^{-s}}{m^{-2s}} u \|^2_{L^2(\Gamma)} = \sum_{\tau \in \mathcal{T}} \| \frac{\rho^{-s}}{m^{-2s}} u \|^2_{L^2(\tau)} \sim \sum_{\tau \in \mathcal{T}} m_\tau^{4s} \rho_\tau^{-2s} \| u \|^2_{L^2(\tau)} \\
\sim \sum_{\tau \in \mathcal{T}} m_\tau^{4s} \rho_\tau^{-2s} |\tau| \| \hat{u} \|^2_{L^2(\hat{\tau})} \\
= \sum_{\tau \in \mathcal{T}} m_\tau^{4s} \rho_\tau^{-2s} |\tau| \sum_{i,j=0}^{m_\tau} (U_{i,j}^\tau)^2 \frac{2}{2i+1} \frac{2}{2j+1} \\
\lesssim \sum_{\tau \in \mathcal{T}} m_\tau^{4s} \rho_\tau^{-2s} |\tau| \sum_{i,j=0}^{m_\tau} (U_{i,j}^\tau)^2 \\
\leq \max_{\tau \in \mathcal{T}} \{ m_\tau^{4s} \rho_\tau^{-2s} |\tau| \} \sum_{\tau \in \mathcal{T}} \sum_{i,j=0}^{m_\tau} (U_{i,j}^\tau)^2.
\]

Hence, in view of (4.5), the upper bound on \( \lambda_{\max}(B) \) is shown.

The corresponding bounds for \( d = 3 \) follow in a completely analogous fashion.

**Lemma 4.3.** If \(-1 \leq s \leq 0\), we have

\[
\lambda_{\min}(A) \gtrsim \min_{\tau \in \mathcal{T}} \{ m_\tau^{4s-d} |\tau| \rho_\tau^{-2s} \} \quad \text{and} \quad \lambda_{\max}(A) \lesssim \max_{\tau \in \mathcal{T}} |\tau|
\]

**Proof.** Let \( d = 2 \). We have

(4.5) \[
\| \frac{\rho^{-s}}{m^{-2s}} u \|^2_{L^2(\Gamma)} \lesssim \| u \|^2_{H^s(\Gamma)} \leq \| u \|^2_{L^2(\Gamma)},
\]

where the first inequality follows from Theorem 3.9, and the second inequality follows from the dual imbedding. Paserval’s identity (4.3), along with scaling, yields

\[
\| \frac{\rho^{-s}}{m^{-2s}} u \|^2_{L^2(\Gamma)} = \sum_{\tau \in \mathcal{T}} \| \frac{\rho^{-s}}{m^{-2s}} u \|^2_{L^2(\tau)} \\
\sim \sum_{\tau \in \mathcal{T}} m_\tau^{4s} \rho_\tau^{-2s} \| u \|^2_{L^2(\tau)} \sim \sum_{\tau \in \mathcal{T}} m_\tau^{4s} |\tau| \rho_\tau^{-2s} \| \hat{u} \|^2_{L^2(\hat{\tau})} \\
= \sum_{\tau \in \mathcal{T}} m_\tau^{4s} |\tau| \rho_\tau^{-2s} \sum_{i,j=0}^{m_\tau} (U_{i,j}^\tau)^2 \frac{2}{2i+1} \frac{2}{2j+1} \\
\gtrsim \sum_{\tau \in \mathcal{T}} m_\tau^{4s} |\tau| \rho_\tau^{-2s} \sum_{i,j=0}^{m_\tau} (U_{i,j}^\tau)^2 m_\tau^{-2}.
\]

Hence, in view of (4.5), we have

\[
\| u \|^2_{H^s(\Gamma)} \geq \| \frac{\rho^{-s}}{m^{-2s}} u \|^2_{L^2(\Gamma)} \gtrsim \min_{\tau \in \mathcal{T}} \{ m_\tau^{4s-2} |\tau| \rho_\tau^{-2s} \} \sum_{\tau \in \mathcal{T}} \sum_{i,j=0}^{m_\tau} (U_{i,j}^\tau)^2,
\]

and, thus, the lower bound on \( \lambda_{\min}(B) \) follows.
On the other hand, Paserval’s identity gives
\[
\|u\|_{L^2(\Gamma)}^2 = \sum_{\tau \in T} \|u\|_{L^2(\tau)}^2 \sim \sum_{\tau \in T} |\tau| \|\hat{u}\|_{L^2(\hat{\tau})}^2 = \sum_{\tau \in T} |\tau| \sum_{i,j=0}^{m_\tau} (U_{i,j}^{\tau})^2 \frac{2}{2i+1} \frac{2}{2j+1}
\leq \sum_{\tau \in T} |\tau| \sum_{i,j=0}^{m_\tau} (U_{i,j}^{\tau})^2 \leq \max_{\tau \in T} \{ |\tau| \} \sum_{\tau \in T} \sum_{i,j=0}^{m_\tau} (U_{i,j}^{\tau})^2.
\]
Hence, in view of (4.5), the upper bound on \(\lambda_{\max}(B)\) is shown.

The corresponding bounds for \(d = 3\) follow in a completely analogous fashion. \(\square\)

Remark 4.4. A different upper bound for \(\lambda_{\max}\) is presented in Lemma 4.3 of [6], for nodal finite element bases using arguments involving dual Sobolev embedding. It would be interesting to investigate further the sharpness of each bound with respect to different meshes.

Finally, recalling that the condition number \(\text{cond}(A)\) of a symmetric positive definite matrix \(A\), is given by
\[
\text{cond}(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)},
\]
an upper bound for the condition number of \(A\) now follows.

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Appendix: Proof of Lemma 3.4

Proof. Since \(\hat{\eta}_{\delta} \in H_0^1(\hat{\tau})\) and \(\hat{u} \in C^\infty(\hat{\tau})\), we have \(\hat{\eta}_{\delta}\hat{u} \in H_0^1(\hat{\tau})\). The proof of the estimate (3.2) is trivial.

Next, we prove the bounds (3.3) and (3.4) in two dimensions via a tensor-product construction. The proof for three dimensions is analogous. Let \(\hat{\tau} = (-1, 1)^2\), and consider the splitting of \(\hat{\tau}\) into 5 subregions as drawn in Figure 3. Consider \(\hat{\eta}_{\delta}\) on \(\hat{\tau}_1 \subseteq \hat{\tau}\). It is immediate that
\[
\hat{\eta}_{\delta}(x, y) = \delta^{-1}(x + 1), \quad \text{for} \ (x, y) \in \hat{\tau}_1.
\]
Thus
\[
\| (1 - \hat{\eta}_h) \hat{u} \|_{L^2(\hat{\tau}_i)}^2 \leq \int_{-1}^{1} \int_{-1}^{-1+\delta} (1 - \delta^{-1}(x + 1))^2 \hat{u}^2 \, dx \, dy
\]
\[
\leq \int_{-1}^{1} \| \hat{u} \|_{L^2((-1,-1+\delta) \times (y))}^2 \int_{-1}^{-1+\delta} (1 - \delta^{-1}(x + 1))^2 \, dx \, dy
\]
\[
\leq \int_{-1}^{1} \delta \| \hat{u} \|_{L^2((-1,-1+\delta) \times (y))}^2 \, dy \leq \int_{-1}^{1} \delta \| \hat{u} \|_{L^2((-1,1) \times (y))}^2 \, dy
\]
\[
\leq \int_{-1}^{1} \delta m^2 \| \hat{u} \|_{L^2((-1,1) \times (y))}^2 \, dy = \delta m^2 \| \hat{u} \|_{L^2(\hat{\tau})}^2,
\]
where in the fifth inequality we made use of Bernstein’s inequality (3.8) in one dimension. Also, we have
\[
\| \nabla (\hat{\eta}_h) \hat{u} \|_{L^2(\hat{\tau}_i)}^2 = \| \frac{\partial \hat{\eta}_h}{\partial x} \hat{u} \|_{L^2(\hat{\tau}_i)}^2 = \| \delta^{-1} \hat{u} \|_{L^2(\hat{\tau}_i)}^2 \leq \int_{-1}^{1} \int_{-1}^{-1+\delta} \delta^{-2} \hat{u}^2 \, dx \, dy
\]
\[
\leq \int_{-1}^{1} \delta^{-1} \| \hat{u} \|_{L^2((-1,-1+\delta) \times (y))}^2 \, dy
\]
\[
\leq \int_{-1}^{1} \delta^{-1} \| \hat{u} \|_{L^2((-1,1) \times (y))}^2 \, dy
\]
\[
\leq \int_{-1}^{1} \delta^{-1} m^2 \| \hat{u} \|_{L^2((-1,1) \times (y))}^2 \, dy = \delta^{-1} m^2 \| \hat{u} \|_{L^2(\hat{\tau})}^2.
\]

The proof for \( \hat{\tau}_i, i = 2, 3, 4 \) is completely analogous, as is the proof for the case of triangular elements.

\[\square\]

References

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