FUSION SYSTEMS ON MAXIMAL CLASS 3-GROUPS OF RANK TWO REVISITED

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ABSTRACT. We complete the determination of saturated fusion systems on maximal class 3-groups of rank two.

1. Introduction

The maximal class 3-groups have been classified by Blackburn in [Bla58]. In this article we revisit the determination of the saturated fusion systems on these groups. We take the presentations for the maximal class 3-groups from [DRV07]. For \( r \geq 4 \), and \( \beta, \gamma, \delta \in \{0, 1, 2\} \), define

\[
B(r; \beta, \gamma, \delta) = \langle s, s_1, \ldots, s_{r-1} \mid R1, R2, R3, R4, R5, R6 \rangle
\]

where the relations are as follows:

- **R1:** \( s_i = [s_{i-1}, s] \) for \( i \in \{2, \ldots, r-1\} \);
- **R2:** \( [s_1, s_i] = 1 \) for \( i \in \{3, \ldots, r-1\} \);
- **R3:** \( s_i^3 s_{i+1}^3 s_{i+2} = 1 \) for \( i \in \{2, \ldots, r-1\} \) \( \text{where} \ s_r = s_{r+1} = 1 \) by definition;
- **R4:** \( [s_1, s_2] = s_{r-1}^\beta \);
- **R5:** \( s_1^3 s_2^3 s_3 = s_{r-1}^\gamma \);
- **R6:** \( s_3^\delta = s_{r-1}^\delta \).

We mostly require that \( r \geq 5 \): note that \( |B(r; \beta, \gamma, \delta)| = 3^r \) and that there are isomorphisms between some of the groups listed. The full list of maximal class 3-groups of order at least \( 3^5 \) is uniquely given up to isomorphism by the requirements:

1. For \( r \) odd,
   \[
   (\beta, \gamma, \delta) \in \{(1, 0, 0), (1, 0, 1), (1, 0, 2), (0, 1, 0), (0, 0, 1), (0, 0, 0)\}.
   \]

2. For \( r \) even,
   \[
   (\beta, \gamma, \delta) \in \{(1, 0, 0), (1, 0, 1), (1, 0, 2), (0, 1, 0), (0, 0, 1), (0, 0, 0), (0, 2, 0)\}.
   \]

Thus, when \( r \geq 5 \), there are six maximal class 3-groups when \( r \) is odd and seven when \( r \) is even.

Recall that for a prime \( p \) a saturated fusion system \( \mathcal{F} \) on a \( p \)-group \( S \) is reduced if and only if \( O^p(\mathcal{F}) = O^p(\mathcal{F}) = \mathcal{F} \) and \( O_p(\mathcal{F}) = 1 \). The fusion system \( \mathcal{F} \) is exotic if \( \mathcal{F} \neq \mathcal{F}_S(G) \) for all finite groups \( G \) with \( S \in \text{Syl}_p(G) \). Our main result is as follows.

**Theorem 1.1.** Suppose that \( S = B(r; \beta, \gamma, \delta) \) is a maximal class 3-group of order at least \( 3^5 \). Assume that \( \mathcal{F} \) is a saturated fusion system on \( S \) and that \( \mathcal{F} \) has at least one \( \mathcal{F} \)-conjugacy class of \( \mathcal{F} \)-essential subgroups. Then either \( \mathcal{F} \) is as described in [DRV07] Theorem 5.10] or \( \beta \neq 0 \) and one of the following holds:

1. \( S = B(r; 1, 0, 0), \langle s, s_{r-1} \rangle \) represents the unique \( \mathcal{F} \)-conjugacy class of \( \mathcal{F} \)-essential subgroups, \( \text{Aut}_\mathcal{F}(<s, s_{r-1}>) \cong SL_2(3) \), \( |\text{Out}_\mathcal{F}(S)| = 2 \) and either
   1. \( r \) is even and \( \mathcal{F} \) is reduced; or

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(b) $r = 2k + 1$ is odd, and $O^3(F)$ is a subsystem of index 3 in $F$ isomorphic to the fusion system of $\text{PSL}_3(q)$ at the prime 3 for some prime power $q$ with $q^3 - 1 = k$.

(ii) $S = B(r; 1, 0, 2)$, $r$ is even and one of the following holds:
(a) $\langle ss_1, s_{r-1} \rangle$ represents the unique $F$-conjugacy class of $F$-essential subgroups, Aut$_F(\langle ss_1, s_{r-1} \rangle) \cong \text{SL}_2(3)$, $|\text{Out}(S)| = 2$ and $F$ is reduced;
(b) $\langle ss_1^2, s_{r-1} \rangle$ represents the unique $F$-conjugacy class of $F$-essential subgroups, Aut$_F(\langle ss_1^2, s_{r-1} \rangle) \cong \text{SL}_2(3)$, $|\text{Out}(S)| = 2$ and $F$ is reduced; or
(c) there are two $F$-conjugacy classes of $F$-essential subgroups represented by $\langle ss_1, s_{r-1} \rangle$ and $\langle ss_1^2, s_{r-1} \rangle$ with $\text{Aut}_F(\langle ss_1, s_{r-1} \rangle) \cong \text{Aut}_F(\langle ss_1^2, s_{r-1} \rangle) \cong \text{SL}_2(3)$, $|\text{Out}(S)| = 2$ and $F$ is reduced.

Furthermore, the fusion systems listed in (i) and (ii) are exotic.
2. Maximal class 3-groups of order at least $3^5$

Let $S = B(r; \beta, \gamma, \delta)$ be a maximal class 3-group with $r \geq 5$ as described in Section 1. We set

$$\gamma_1(S) := \langle s_1, \ldots, s_{r-1} \rangle$$

and, for $i > 1$,

$$\gamma_i(S) = [\gamma_{i-1}(S), S].$$

As $S$ has maximal class, using R2 we obtain

**Lemma 2.1.** For $1 \leq i \leq r-1$, $\gamma_i(S) = \langle s_1, \ldots, s_{r-1} \rangle$ and $S > \gamma_2(S) > \cdots > \gamma_{r-1}(S)$ is the lower central series of $S$. In particular, $|\gamma_{r-1}(S)| = |Z(S)| = o(s_{r-1}) = 3$. □

**Lemma 2.2.** Either $\beta = 0$ and $\gamma_1(S)$ is abelian or $\beta \neq 0$, $\gamma_1(S)$ has centre $\gamma_3(S)$ and derived group $\gamma_{r-1}(S)$. In particular, $\gamma_2(S)$ is abelian and $\gamma_1(S) = C_S(\gamma_{r-2}(S)) = C_S(\gamma_2(S)/\gamma_4(S))$ is characteristic in $S$.

**Proof.** Since $r \geq 5$, R2 implies $s_1$ centralizes $\gamma_{r-2}(S) = \langle s_{r-2}, s_{r-1} \rangle$ which is abelian. Since $C_S(\gamma_{r-2}(S))$ is normal in $S$ and $s_1 \in C_S(\gamma_{r-2}(S))$, the fact that $S$ has maximal class implies that $C_S(\gamma_{r-2}(S)) = \gamma_1(S)$. Hence $\gamma_{r-2}(S) \leq Z(\gamma_1(S))$. Assume that $\gamma_c(S) \leq Z(\gamma_1(S))$ for some $3 < c \leq r-2$. Then $s_{c-1}$ centralizes $\gamma_c(S)$ and so $\gamma_{c-1}(S)$ is abelian and R2 implies $s_1$ centralizes $\gamma_{c-1}(S)$. Since $S$ has maximal class, this implies $C_S(\gamma_{c-1}(S)) = \gamma_1(S)$. Thus $\gamma_{c-1}(S) \leq Z(\gamma_1(S))$ and we conclude that $\gamma_3(S) \leq Z(\gamma_1(S))$ by induction. Now $\gamma_2(S)$ is abelian and so if $s_1$ and $s_2$ commute, then $\gamma_1(S)$ is abelian, whereas if $s_1$ and $s_2$ do not commute, then $Z(\gamma_1(S)) = \gamma_3(S)$, $\gamma_2(S)$ is abelian and $\gamma_1(S)' = \gamma_{r-1}(S)$. This proves the claim. □

**Lemma 2.3.** We have $\Omega_1(\gamma_1(S)) = \langle s_{r-1}, s_{r-2} \rangle$. In particular, every subgroup of $\gamma_1(S)$ is 2-generated.

**Proof.** By R3, we have $\Omega_1(\gamma_1(S)) \geq \langle s_{r-1}, s_{r-2} \rangle$. Assume that $\Omega_1(\gamma_1(S)) \neq \langle s_{r-1}, s_{r-2} \rangle$. If $\gamma_1(S)$ is abelian, then $\Omega_1(\gamma_1(S))$ has exponent 3 and, as $\gamma_1(S)$ is characteristic in $S$, $\Omega_1(\gamma_1(S)) \geq \gamma_{r-3}(S)$. However, R3 shows that $s_{r-3}^3 = s_{r-1}^2$, a contradiction. Hence, if $\gamma_1(S)$ is abelian, the result holds. Furthermore, in this case we have that every subgroup of $\gamma_1(S)$ is 2-generated.

Suppose that $\gamma_1(S)$ is non-abelian. Then the derived subgroup of $\gamma_1(S)$ is $\gamma_{r-1}(S)$ by Lemma 2.2. Notice that $S = \gamma_{r-1}(S) = B(r - 1; 0, 0, 0)$, Thus $\Omega_1(\gamma_1(S))/\gamma_{r-1}(S) \leq \langle s_{r-3}\gamma_{r-1}(S), s_{r-2}\gamma_{r-1}(S) \rangle$ by applying the previous case to $\gamma_{r-1}(S)$. This shows that $\Omega_1(\gamma_1(S)) \leq \langle s_{r-3}, s_{r-2}, s_{r-1} \rangle$ and as $r \geq 5$, again we see that $s_{r-3}$ has order 9 by R3 and we conclude that $\Omega_1(\gamma_1(S)) = \langle s_{r-1}, s_{r-2} \rangle$ in this case also.

Finally, assume that $A \leq \gamma_1(S)$ is at least 3-generated. If $A$ does not contain $\gamma_{r-1}(S)$, then $A$ is isomorphic to a subgroup of $\gamma_1(S)/\gamma_{r-1}(S)$ and so is 2-generated, a contradiction. Hence $\gamma_{r-1}(S) \leq A$. Furthermore, $A/\gamma_{r-1}(S)$ is 2-generated and $\Omega_1(A/\gamma_{r-1}(S)) = \langle s_{r-2}, s_{r-3} \rangle\gamma_{r-1}(S)$. Since $s_{r-3}^3 = s_{r-1}^2$, we have $A$ is 2-generated a contradiction. □

Because of Lemmas 2.2 and 2.3, we have $\Omega_1(\gamma_1(S))/\Omega_1(S)$ has order $3^2$ when $r \geq 5$ and since $\Omega_i(\gamma_1(S))/\Omega_i(S) = \Omega_{i-1}(\gamma_1(S))/\Omega_1(S)$, by induction $\Omega_i(\gamma_1(S))$ has order at most $3^i$ for each $1 \leq i \leq \lfloor \frac{r}{3} \rfloor$. In particular, $\gamma_1(S)$ has exponent at most $\lfloor \frac{r}{3} \rfloor$.

**Lemma 2.4.** If $x \in S \setminus \gamma_1(S)$, then $C_S(x) = \langle x, s_{r-1} \rangle$ has order 9 and all the elements of the coset $x\gamma_2(S)$ are $S$-conjugate.

**Proof.** Suppose that $x \in S \setminus \gamma_1(S)$. Then $C_S(x) = \langle x \rangle C_{\gamma_1(S)}(x)$. Obviously, $\gamma_{r-1}(S) \leq C_{\gamma_1(S)}(x)$ and we know from Lemma 2.2 that $\gamma_1(S)' \leq \gamma_{r-1}(S)$. This means that $C_{\gamma_1(S)}(x)$ is normal in $\langle x, \gamma_1(S) \rangle = S$. Assume that $C_{\gamma_1(S)}(x) > \gamma_{r-1}(S)$. Then $s_{r-2} \in C_{\gamma_1(S)}(x)$. As $r \geq 5$, $r - 2 \geq 3$ and so $s_{r-2}$ is centralized by $\langle x, s_1 \rangle = S$, and this contradicts $Z(S) = \gamma_{r-1}(S)$. It follows that $C_S(x) = \langle x, s_{r-1} \rangle$ has order 9 and that $|x^3| = 3^{r-2} = |x\gamma_2(S)|$. This proves the result. □
3. Automorphisms of maximal class 3-groups

We continue to assume that $S = B(r; \beta, \gamma, \delta)$ with $r \geq 5$. We will repeatedly use the commutator formulae

$$[xy, z] = [x, z]^y[z, y] = [x, z][x, y][y, z]$$

and

$$[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, z]$$

without mention. In particular, we note the following consequence of these relations in our situation. Suppose that $a, b \in S$, and $v, w \in \gamma_2(S)$. Then, as $\gamma_2(S)$ is abelian by Lemma 2.2, we have

$$[av, bw] = [a, w][a, b][v, b].$$

**Lemma 3.1.** Suppose that $S = B(r; \beta, \gamma, \delta)$, let $d \in \{0, 1, 2\}$ and $e, f \in \{1, 2\}$. The following identities hold.

(i) $w^3[w, s]^3[w, s, s] = w^3[w, s^2]^3[w, s^2, s^2] = 1$ for all $w \in \gamma_2(S)$.

(ii) $[[s_j^e, s_j^f], s_j^d] = s_{r-1}^{2e+f}.$

(iii) $(s_j^{e_j})^3(s_j^{f_j}, s_j^{e_j})^3(s_j^{d_j}, s_j^{e_j}, s_j^{f_j}) = s_{r-1}^{f_j}.$

**Proof.** Write $w = s_2^{e_2} \ldots s_{r-1}^{e_{r-1}}$ for suitable $e_j \in \{0, 1, 2\}$. Then, as $\gamma_2(S)$ is abelian,

$$[w, s] = \prod_{j=2}^{r-1} [s_j, s]^{e_j} = \prod_{j=2}^{r-1} s_j^{e_j}$$

and

$$[w, s, s] = \prod_{j=2}^{r-1} [s_j+1, s]^{e_j} = \prod_{j=2}^{r-1} s_j^{e_j}.$$ 

Therefore

$$(3.1) \quad w^3[w, s]^3[w, s, s] = \prod_{j=2}^{r-1} s_j^{3e_j} \prod_{j=2}^{r-1} s_j^{e_j} \prod_{j=2}^{r-1} s_j^{e_j} \prod_{j=2}^{r-1} s_j^{e_j} = \prod_{j=2}^{r-1} s_j^{3e_j} s_j^{e_j} s_j^{e_j} = 1. $$

Now we calculate

$$[w, s^2] = [w, s]^2[w, s]$$

and, by using Equation 3.1 with $[w, s, s]$ in place of $w$ for the final equality,

$$[w, s^2, s^2] = [[w, s]^2[w, s, s], s^2]$$

$$= [w, s, s^2]^2[w, s, s, s^2]$$

$$= (w, s, s^2)^2[w, s, s, s]w, s, s, s$$

$$= [w, s, s]^{-4}[w, s, s, s]^{-4}[w, s, s, s]$$

$$= [w, s, s][w, s, s].$$

We obtain:

$$w^3[w, s^2]^3[w, s^2, s^2] = w^3([w, s]^6[w, s, s]^3][w, s][w, s, s]$$

$$= w^3[w, s]^3[w, s, s][w, s]^3[w, s, s]$$

$$= 1.$$
This proves (i). We also calculate
\[
[[s_1^t, s^c], s_1^d] = \begin{cases}
[s_2, s_1^d] = s_r^{2d}
\quad \text{if } e = f = 1; \\
[s_2^{2r}, s_1^d] = s_r^{2d}
\quad \text{if } e = 1, f = 2; \\
[s_2^d, s_1^d] = s_r^{2d}
\quad \text{if } e = 2, f = 1; \\
[s_2^d, s_1^{d_1}] = [s_2, s_1^d] = s_r^{2d}
\quad \text{if } e = f = 2.
\end{cases}
\]

Hence (ii) holds.

For part (iii), the case \( e = f = 1 \) is immediate using \( \textbf{R2} \) and \( \textbf{R5} \). Suppose that \( e = 1 \) and \( f = 2 \). Then, as \([s_1, s, s_1] \in \gamma_1(S)^t \leq \gamma_{r-1}(S) \) which has order 3, \([s_1, s, s_1]^3 = 1\) and so
\[
(s_1^2)^3[s_1^2, s]^3[s_1^2, s, s] = s_1^6([s_1, s][s_1, s_1][s_1, s])^3[[s_1, s][s_1, s_1][s_1, s], s] \]
\[
= s_1^6[s_1, s]^6([[s_1, s][s_1, s_1][s_1, s]], s) = s_1^6s_2^6s_3^2 \quad \text{if } e = 1, f = 2.
\]

Now we calculate when \( e = 2 \) and \( f = 2 \),
\[
s_1^3[s_1, s^2]^3[s_1^2, s^2, s^2] = s_1^3(s_2^3s_3)^3[s_2^3s_3, s^2] = s_1^3s_2^3s_3^4s_4s_5 = s_1^3s_2^3 = s_r^{-1}.
\]

Finally, assume that \( e = f = 2 \). Then
\[
(s_1^2)^3[s_1^2, s^2]^3[s_1^2, s^2, s^2] = s_1^6[s_1, s]^6[s_1^2, s, s]^3[s_1^2, s^2, s^2] \]
\[
= s_1^6[s_1^2, s]^6[s_1^2, s, s]^3[[s_1^2, s]^2[s_1^2, s, s], s] \]
\[
= s_1^6([s_1, s][s_1, s_1][s_1, s])^6[[s_1^2, s]^2[s_1^2, s, s], s] = s_1^6s_2^6s_3^2 \quad \text{if } e = 2, f = 2.
\]

This establishes (iii). \( \square \)

For \( v, w, \in \gamma_2(S), d \in \{0, 1, 2\} \) and \( e, f \in \{1, 2\} \) define \( \theta_{e,d,f,v,w} : S \to S \) by
\[
\theta_{e,d,f,v,w} : s \mapsto s^e s_1^d v, s_1 \mapsto s_1^f w.
\]

Suppose that \( \theta = \theta_{e,d,f,v,w} \) is one of these maps. We shall investigate the restrictions on \( e, d, f, v \) and \( w \) required to ensure that \( \theta \) is an automorphism of \( S \).

Define \( t := s \theta = s^e s_1^d v \) and \( t_1 := s_1 \theta = s_1^f w \). For \( j > 1 \), set
\[
t_j := [t_{j-1}, t].
\]

Note that for \( k \geq r, t_k = 1 \) and \( t_j \in \gamma_j(S) \) for all \( j \geq 1 \). Since \( \gamma_3(S) \leq Z(\gamma_1(S)) \), \([t_1, t_i] = 1\) for \( i \in \{3, \ldots, r-1\} \). Thus \( \textbf{R1} \) and \( \textbf{R2} \) are satisfied.

Notice that
\[
[t_2, t] = [t_2, s^e s_1^d v] = [t_2, s^e][t_2, s_1^d v]
\]
and

$$[t_2, t, t] = [t_2, s^e, s^e].$$

Since $$[t_2, s^d v] \in \gamma_{r-1}(S)$$, $$[t_2, s^d v]^3 = 1$$. Hence, by Lemma 3.1(i),

$$t_2^3 t_3^3 t_4 = t_2^3 ([t_2, s^e] [t_2, s^d v]^3 [t_2, s^e, s^e] = t_2^3 [t_2, s^e]^3 [t_2, s^e, s^e] = 1.$$ 

Suppose that $$j \geq 3$$. Then $$t_j \in \gamma_3(S) \leq Z(\gamma_1(S))$$ and so $$[t_j, t] = [t_j, s^e]$$ and $$[t_j, t, t] = [t_j, s^e, s^e]$$. Therefore we can apply Lemma 3.1 again to obtain

$$t_j^3 t_{j+1}^3 t_{j+2} = 1.$$ 

Thus $${\bf R3}$$ is satisfied.

We start to investigate $${\bf R4}$$. We have $$t_1 = s_1^j w$$, and for some integer $$n_2$$,

$$t_2 = [t_1, t] = [s_1^j w, s^e s_1^d v] = [s_1^j, v] [s_1^f, s^e s_1^d] [w, s^e s_1^d] = [s_1^j, v] [s_1^f, s^e, s_1^d] [w, s^e s_1^d] = [s_1^j, v] [s_1^f, s^e] [s_1^f, s^e, s_1^d] [w, s^e, s_1^d] = [s_1^f, s^e] [w, s^e] s_{r-1}^{n_2}.$$ 

Therefore,

$$t_3 = [t_2, t] = [s_1^f, s^e] [w, s^e] s_{r-1}^{n_2}, s^e s_1^d v] = [s_1^f, s^e, s_1^d v] [w, s^e] s_{r-1}^{n_2}, s^e] = [s_1^f, s^e] [s_1^f, s^e, s_1^d] [w, s^e, s^e] = [s_1^f, s^e] [s_1^f, s^e, s_1^d] [w, s^e, s^e].$$

Note that $$t_2 = [s_1^f, s^e] [w, s^e] s_{r-1}^{n_2} = s_2^j g_3$$ for some $$g_3 \in \gamma_3(S)$$. Similarly, $$t_3 = s_3^{j-1} g_4$$ for some $$g_4 \in \gamma_4(S)$$. Continuing in this manner we see that $$t_j = s_{j-1}^{j-1} g_{j+1}$$ for some $$g_{j+1} \in \gamma_{j+1}(S)$$. In particular,

$$(3.2) \quad t_{r-1} = s_{r-1}^{r-2 j}.$$ 

Again we calculate

$$[t_1, t_2] = [s_1^f w, [s_1^f, s^e] [w, s^e] s_{r-1}^{n_2}] = [s_1^f w, [s_1^f, s^e] [w, s^e]] = [s_1^f w, [s_1^f, s^e]] [s_1^f w, [s_1^f, s^e]] [w, s^e] = [s_1^f w, [s_1^f, s^e]] [s_1^f w, [s_1^f, s^e]] = [s_1^f w, [s_1^f, s^e]] = s_{r-1}^{\beta_2 \beta}.$$ 

where the last equality follows from Lemma 3.1(ii) and the fact that $$s_{r-1}$$ has order 3.

Therefore, for $${\bf R4}$$ to hold we must have

$$e f \beta \equiv e^{r-2} \beta \pmod{3}.$$ 

Now, recalling that $$\gamma_2(S)$$ is abelian and $$\gamma_{r-1}(S)$$ has order 3, we calculate
$t_1^2t_2t_3 = (s_1^f w)^3 ([s_1^f, s^e] [w, s^e] s_{r-2}^n)^3 ([s_1^f, s^e], s_1^d) [[s_1^f, s^e], s^e] [w, s^e, s^e]$

$= (s_1^f w)^3 [s_1^f, s^e]^3 [w, s^e] [s_1^f, s^e], s^e] [w, s^e, s^e] s_{r-1}^{2def\beta}$

$= (s_1^f)^3 w^3 [w, s_1^f]^3 [s_1^f, s^e]^3 [w, s^e] [s_1^f, s^e] [w, s^e, s^e] s_{r-1}^{2def\beta}$

$= (s_1^f)^3 [s_1^f, s^e]^3 [s_1^f, s^e] w^3 [w, s^e] [s_1^f, s^e] s_{r-1}^{2def\beta}$

$= s_{r-1}^{f+2def\beta}$

where the last equality follows from Lemma 3.1(iii).

Therefore for $\mathbf{R5}$ to hold we require

$$2dec\beta \equiv \gamma(e^{r-2} - 1) \pmod{3}.$$}

We now determine $t^3$ (and this calculation will be used later in a slightly different setting). First of all notice that all the elements of the coset $t\gamma_2(S)$ are $S$-conjugate by Lemma 2.4 and $t^3 \in \gamma_1(S) \cap C_S(t) = \gamma_{r-1}(S)$ and so $(t^3)^b = t^3$ for all $b \in S$. Thus to investigate $\mathbf{R6}$, we may adjust $t$ by conjugacy in $S$ and rather than consider $s^e s_1^d w$, we cube $s^e s_1^d$.

$$(s^e s_1^d)^3 = s^e s_1^d s^e s_1^d s^e s_1^d = (s_1^d)^2 s_1^d [s_1^d, s^e] s^e s_1^d [s_1^d, s^e] s_1^d$$

$$= (s^e)^3 s_1^d [s_1^d, s^e] s_1^d [s_1^d, s^e] s_1^d [s_1^d, s^e] s_1^d$$

$$= s_{r-1}^{\gamma} s_{r-1}^{\gamma} s_{r-1}^{\gamma} s_1^d [s_1^d, s^e] s_1^d [s_1^d, s^e] s_1^d [s_1^d, s^e] s_1^d$$

$$= s_{r-1}^{\gamma} s_{r-1}^{\gamma} s_1^d [s_1^d, s^e] s_1^d [s_1^d, s^e] s_1^d [s_1^d, s^e] s_1^d$$

$$= s_{r-1}^{\gamma} s_{r-1}^{\gamma} s_1^d [s_1^d, s^e] s_1^d [s_1^d, s^e] s_1^d [s_1^d, s^e] s_1^d$$

Thus for $\mathbf{R6}$ to be satisfied we require

$$(3.3) \quad e(d^2 \beta + \delta) + d\gamma \equiv e^{r-2} f\delta \pmod{3}.$$}

We have proved

**Proposition 3.2.** The map $\theta_{e,d,f,v,w} : S \rightarrow S$ is an automorphism of $S = B(r; \beta, \gamma, \delta)$ if and only if the following hold:

1. $e\beta \equiv e^{r-2} \beta \pmod{3}$.
2. $2dec\beta \equiv \gamma(e^{r-2} - 1) \pmod{3}$.
3. $e(d^2 \beta + \delta) + d\gamma \equiv e^{r-2} f\delta \pmod{3}$.

\[ \square \]

**Proposition 3.3.** The following hold:

1. Suppose that $r$ is even.
   - $\text{Aut}(B(r; 0, 0, 0)) = \{\theta_{e,d,f,v,w} \mid v, w, \in \gamma_2(S), d \in \{0, 1, 2\}, e, f \in \{1, 2\}\}$ has order $2^2 \cdot 2^{2r-3}$.
   - $\text{Aut}(B(r; 0, 1, 0)) = \{\theta_{e,0,f,v,w} \mid v, w, \in \gamma_2(S), e, f \in \{1, 2\}\}$ has order $2^2 \cdot 2^{2r-4}$.
   - $\text{Aut}(B(r; 0, 2, 0)) = \{\theta_{e,0,f,v,w} \mid v, w, \in \gamma_2(S), e, f \in \{1, 2\}\}$ has order $2^2 \cdot 2^{2r-4}$.

\[ \square \]
(4) $\text{Aut}(B(r; 0, 0, 1)) = \{\theta_{e,d,e,v,w} \mid v, w, \in \gamma_2(S), d \in \{0, 1, 2\}, e \in \{1, 2\}\}$ has order $2 \cdot 3^{2r-3}$.

(5) $\text{Aut}(B(r; 1, 0, 0)) = \{\theta_{e,0,e,v,w} \mid v, w, \in \gamma_2(S), e \in \{1, 2\}\}$ has order $2 \cdot 3^{2r-4}$.

(6) $\text{Aut}(B(r; 1, 0, 1)) = \{\theta_{e,0,e,v,w} \mid v, w, \in \gamma_2(S), e \in \{1, 2\}\}$ has order $2 \cdot 3^{2r-4}$.

(7) $\text{Aut}(B(r; 1, 0, 2)) = \{\theta_{e,0,e,v,w} \mid v, w, \in \gamma_2(S), e \in \{1, 2\}\}$ has order $2 \cdot 3^{2r-4}$.

(ii) Suppose that $r$ is odd.

(1) $\text{Aut}(B(r; 0, 0, 0)) = \{\theta_{e,d,f,v,w} \mid v, w, \in \gamma_2(S), d \in \{0, 1, 2\}, e, f \in \{1, 2\}\}$ has order $4 \cdot 3^{2r-3}$.

(2) $\text{Aut}(B(r; 0, 1, 0)) = \{\theta_{1,0,f,v,w} \mid v, w, \in \gamma_2(S), f \in \{1, 2\}\}$ has order $2 \cdot 3^{2r-4}$.

(3) $\text{Aut}(B(r; 1, 0, 0)) = \{\theta_{e,0,0,v,w} \mid v, w, \in \gamma_2(S), e \in \{1, 2\}\}$ has order $2 \cdot 3^{2r-4}$.

(4) $\text{Aut}(B(r; 1, 0, 1)) = \{\theta_{e,0,1,v,w} \mid v, w, \in \gamma_2(S), e \in \{1, 2\}\}$ has order $2 \cdot 3^{2r-4}$.

(5) $\text{Aut}(B(r; 1, 0, 2)) = \{\theta_{e,0,1,v,w} \mid v, w, \in \gamma_2(S), e \in \{1, 2\}\}$ has order $2 \cdot 3^{2r-4}$.

Proof. We have $\gamma_1(S) = C_5(\gamma_{r-2}(S))$ is characteristic in $S$ by Lemma 2.2. Hence every automorphism of $S$ is of the form $\theta_{e,d,f,v,w}$ for suitable $e, d, f, v$ and $w$. The result now follows from Proposition 3.2. □

Lemma 3.4. We have $s_{r-1}\theta_{e,d,f,v,w} = s_{r-1}^{e^{-2}f}$. □

Proof. Let $\theta = \theta_{e,d,f,v,w}$. Then, setting $t = s\theta$, $t_1 = s_1\theta$ and $t_j = [t_{j-1}, t]$ for $j \geq 2$, we have

$$t_j = [t_{j-1}, t] = [s_{j-1}\theta, s\theta] = [s_{j-1}, s]\theta = s_j\theta.$$ 

Now applying Equation (3.2) yields $s_{r-1}\theta = t_{r-1} = s_{r-1}^{e^{-2}f}$. □

4. Fusion systems on maximal class 3-groups

Suppose that $F$ is a saturated fusion system on $S$, where $S$ is one of the groups $B(r; \beta, \gamma, \delta)$.

Lemma 4.1. Suppose that $E$ is an $F$-essential subgroup. If $E \leq \gamma_1(S)$, then $E = \gamma_1(S)$ is abelian and $\text{Aut}_F(E) \cong \text{SL}_2(3)$ or $\text{GL}_2(3)$.

Proof. If $\gamma_1(S)$ is abelian, then $E = \gamma_1(S)$ and we have nothing to do. So suppose that $\gamma_1(S)$ is non-abelian. Since $E$ is $F$-centric, $E > \gamma_3(S) = Z(\gamma_1(S))$. Suppose that $E \neq \gamma_1(S)$. Then $|\gamma_1(S) : E| = 3$ and $E$ is abelian. Furthermore, $E$ is normalized by $\gamma_1(S)$. By Lemma 2.3 $|E/\Phi(E)| = 9$ and $\Omega_1(\gamma_1(S)) = \langle s_{r-1}, s_{r-2} \rangle$. Since $E$ is $F$-essential, $[E, \gamma_1(S)] \not\subseteq \Phi(E)$ and so we deduce that $\gamma_{r-1}(S) \not\subseteq \Phi(E)$. It follows that $E$ is cyclic and therefore $E$ is not essential, a contradiction. Hence $E = \gamma_1(S)$. Since $|E/\Phi(E)| = 9$, we now have $\text{Out}_F(E) \cong \text{SL}_2(3)$ or $\text{Out}_F(E) \cong \text{GL}_2(3)$. Since the Sylow 2-subgroup of $\text{Aut}_F(E)$ has to act faithfully on $\Omega_1(E) = \langle s_{r-1}, s_{r-2} \rangle$, we deduce that $\gamma_{r-1}(S) \neq \gamma_1(S)'$ and this is a contradiction. □

Lemma 4.2. Suppose that $E$ is an $F$-essential subgroup of $S$ and $E \nleq \gamma_1(S)$. The following hold:

(i) $E\gamma_2(S)/\gamma_2(S)$ has order 3;

(ii) $\text{Out}_F(E) \cong \text{SL}_2(3)$ or $\text{Out}_F(E) \cong \text{GL}_2(3)$; and

(iii) $E$ is either extraspecial of order $3^3$ or elementary abelian of order $3^2$.

Furthermore, if $F$ is $F$-essential with $E\gamma_2(S) = F\gamma_2(S)$, then $E$ and $F$ are $S$-conjugate. In particular, $F$ has at most four $S$-classes of $F$-essential subgroups.

Proof. Since $\Phi(S) = \gamma_2(S)$, $E \nleq \gamma_1(S)$ and $E \neq S$, we have $|E\gamma_2(S)/\gamma_2(S)| = 3$ which is (i). Using [VL91] Lemma 1.2 with Lemma 2.2 yields that $E\gamma_2(S)$ has maximal class and we know $\gamma_2(S)$ is abelian. We may repeat this argument until we obtain $E = E\gamma_i(S)$ for some $i \geq 2$ has maximal class. In particular, $|E/\Phi(E)| = 9$ and so, as $E$ is $F$-essential, either $\text{Out}_F(E) \cong \text{GL}_2(3)$
Lemma 4.3. Suppose that $|E| \geq 3^4$. We obtain a contradiction by showing that $\text{Aut}_F(E)$ does not possess a subgroup of order 8. This follows immediately from Proposition 3.3 when $|E| \geq 3^5$. The case $|E| = 3^4$ is a straightforward computation (for example using MAGMA [BCP97]). Hence $|E| \in \{3^2, 3^3\}$ and $E$ has exponent 3. This demonstrates (iii) holds.

Notice that $E \geq Z(S) = \gamma_{r-1}(S) \geq \gamma_1(S)'$ and so $E \cap \gamma_1(S)$ is normalized by $S$. Therefore, as $\Omega_1(\gamma_1(S)) = \langle s_{r-2}, s_{r-1} \rangle$ by Lemma 2.3

$$E \cap \gamma_1(S) = \begin{cases} \gamma_{r-1}(S) & \text{if } |E| = 9; \text{ and} \\ \gamma_{r-2}(S) & \text{if } |E| = 27. \end{cases}$$

Suppose $F$ is an $\mathcal{F}$-essential subgroup with $E < F$. Then $E$ is elementary abelian of order 9 and $F$ is extraspecial of order 27. Since $\text{Aut}_F(F)$ acts transitively on the maximal subgroups of $F$, $E$ is $\text{Aut}_F(F)$-conjugate to $\gamma_{r-2}(S)$. This contradicts the fact that $E$ is fully $\mathcal{F}$-normalized and we conclude that there is no such containment.

Suppose that $E\gamma_2(S) = F\gamma_2(S)$. We may assume that $E \cap \gamma_1(S) \leq F \cap \gamma_1(S)$. Let $x \in E \setminus \gamma_2(S)$ and $y \in F \setminus \gamma_2(S)$ be such that $x\gamma_2(S) = y\gamma_2(S)$. Then by Lemma 2.4 $x$ and $y$ are $S$-conjugate. Hence we may suppose that $x \in E \cap F$. Then $E = \langle x \rangle(E \cap \gamma_2(S))$ and $F = \langle x \rangle(F \cap \gamma_2(S))$. Hence $E \leq F$ and thus $E = F$ as claimed.

For $x \in S \setminus \gamma_1(S)$ to be contained in an elementary abelian subgroup of order 9 or an extraspecial subgroup of order 27, it suffices that $x$ has order 3 since then $\langle x \rangle \gamma_{r-1}(S)$ and $\langle x \rangle \gamma_{r-2}(S)$ are such subgroups. Using Equation (3.3) we see that $ss_3^1\gamma_2(S)$ has order 3 if and only if $d^2\beta + d\gamma \equiv 0 \pmod{3}$.

Table 1 lists the groups $B(r; \beta, \gamma, \delta)$ and cosets $ss_3^1\gamma_2(S)$ which consist of elements of order 3.

<table>
<thead>
<tr>
<th>$S$</th>
<th>$ss_3\gamma_2(S)$</th>
<th>$ss_1\gamma_2(S)$</th>
<th>$ss_3^2\gamma_2(S)$</th>
<th>$S$</th>
<th>$ss_3\gamma_2(S)$</th>
<th>$ss_1\gamma_2(S)$</th>
<th>$ss_3^2\gamma_2(S)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(r; 0, 0, 0)$, $r$ even</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>$B(r; 0, 0, 0)$, $r$ odd</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>$B(r; 0, 1, 0)$, $r$ even</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>$B(r; 0, 1, 0)$, $r$ odd</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B(r; 0, 2, 0)$, $r$ even</td>
<td>✓</td>
<td></td>
<td></td>
<td>$B(r; 0, 0, 1)$, $r$ odd</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B(r; 0, 0, 1)$, $r$ even</td>
<td>✓</td>
<td></td>
<td></td>
<td>$B(r; 1, 0, 0)$, $r$ odd</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B(r; 1, 0, 0)$, $r$ even</td>
<td>✓</td>
<td></td>
<td></td>
<td>$B(r; 1, 0, 1)$, $r$ odd</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B(r; 1, 0, 1)$, $r$ even</td>
<td>✓</td>
<td></td>
<td></td>
<td>$B(r; 1, 0, 2)$, $r$ odd</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B(r; 1, 0, 2)$, $r$ even</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 1. Elements of order 3 in the designated cosets of $S = B(r; \beta, \gamma, \delta)$

At this stage, we can confirm that if $\beta = 0$, then all the potential fusion systems have been discovered by [DRV07, Theorem 5.10].

Lemma 4.3. Suppose that $E$ is $\mathcal{F}$-essential. Then

$$N_{\text{Aut}_F(E)}(\text{Aut}_S(E)) = \{\beta|_E \mid \beta \in \text{Aut}_F(N_S(E)), E\beta = E\} = \{\alpha|_E \mid \alpha \in \text{Aut}_F(S), E\alpha = E\}.$$ 

Proof. Recall the definition of $H_E$ from [AKO11, Proposition I.3.3]. Then, as $E$ is $\mathcal{F}$-essential, $H_E/\text{Inn}(E)$ is strongly $p$-embedded in $\text{Out}_F(E)$. Since, by Lemmas 4.1 and 4.2 (ii), $\text{Out}_F(E) \cong \text{SL}_2(3)$ or $\text{GL}_2(3)$, $H_E = N_{\text{Aut}_F(E)}(\text{Aut}_S(E))$. Let $\theta$ be a generator of $H_E$. Then, by definition, there exists $R > E$ and $\psi \in \text{Hom}_F(R, S)$ with $E\psi = E$ such that $\theta = \psi|_E$. Thus $\psi|_{\text{N}_R(E)} \in$
Hom(N_R(E), S). Since |N_S(E)/E| = 3, N_R(E) = N_S(E) and so \( \psi\big|_{N_S(E)} \in \text{Aut}_F(N_S(E)) \) and \( \theta = \psi\big|_E \). It follows that
\[
H_E = \{ \psi\big|_E \mid \psi \in \text{Aut}_F(N_S(E)), E\psi = E \}
\]
and so the first equality holds.

By Lemmas [4.1 and 4.2] no \( F \)-essential subgroup properly contains \( E \), so every element of Aut_\( F \)(N_S(E)) is the restriction of an element of Aut_\( F \)(S) by Alperin’s Theorem [AKO11, Theorem I.3.5]. This provides the second asserted equality. \( \square \)

**Lemma 4.4.** Assume that \( \gamma_1(S) \) is \( F \)-essential. Then \( r \) is odd and \( (\beta, \gamma, \delta) \in \{(0, 0, 0), (0, 1, 0)\} \).

**Proof.** Let \( A = \gamma_1(S) \) and assume that \( A \) is \( F \)-essential. Then \( A \) is abelian and Aut_\( F \)(A) contains a normal subgroup isomorphic to SL_2(3) by Lemma [4.1]. We conclude that all the elements of \( A \setminus \Phi(A) \) have the same order and so \( |A| = 3^{2k} \) for some \( k \geq 2 \) and \( r \) is odd. Let \( \tau_A \) be an element of order 2 in Aut_\( F \)(A) which corresponds to the centre of SL_2(3). Then \( \tau_A = \sigma|_A \) for some \( \sigma \in \text{Aut}_F(S) \) and \( \sigma \) centralizes \( S/A \) and inverts \( A/\gamma_2(S) \). Now consulting Proposition 3.3 delivers the conclusion \( (\beta, \gamma, \delta) \in \{(0, 0, 0), (0, 1, 0)\} \) and we recover the results from [DRV07, Theorem 5.10]. \( \square \)

**Lemma 4.5.** Let \( F \) be a saturated fusion system on \( B(r; 0, \gamma, \delta) \) with at least one class of \( F \)-essential subgroups. Then \( F \) is as described in [DRV07, Theorem 5.10].

**Proof.** Lemma [4.2] and Table [1] indicate that the groups \( B(r; 0, 0, 1) \) have no \( F \)-essential subgroups which are not contained in \( \gamma_1(S) \). This shows that all candidates for \( S \) have been considered in [DRV07, Theorem 5.10]. Here we note that Table [1] indicates that \( E_0 \) and \( V_0 \) have exponent 3 when \( S = B(r; 0, 1, 0) \). If \( r \) is even, we obtain the examples in [DRV07, Theorem 5.10] whereas when \( r \) is odd there are no examples. To understand this observe that \( E_0\gamma_2(S) = V_0\gamma_2(S) = \langle s \rangle \gamma_2(S) \). If \( E \) is one of \( E_0 \) or \( V_0 \) and is \( F \)-essential, then \( \text{Out}_F(E) \) contains a subgroup isomorphic to SL_2(3) and hence an element \( \sigma \) of order 2 which inverts \( E/\Phi(E) \) and so also \( \langle s \rangle \Phi(E)/\Phi(E) \). By Lemma 4.3 \( \sigma \) is the restriction of an automorphism \( \sigma^* \) of \( S \). However Proposition 3.3 (ii)(3) shows that every automorphism of \( S \) which normalizes \( E/\gamma_2(S) \) actually centralizes \( \langle s \rangle \gamma_2(S)/\gamma_2(S) \) and so we have no candidates for \( \sigma^* \). Thus we recover the results from [DRV07, Theorem 5.10] in this case. \( \square \)

From now on we assume that \( \beta \neq 0 \). For \( d \in \{0, 1, 2\} \) define
\[
V_d = \langle ss_1^d, s_{r-1} \rangle \quad \text{and} \quad E_d = \langle ss_1^d, s_{r-1}, s_{r-2} \rangle.
\]
Hence \( V_d \) is abelian of order 9 and \( E_d \) is extraspecial of order 27. In addition, \( V_d \) and \( E_d \) have exponent 3 if and only if \( ss_1^d \) has order 3. Thus using Table [1] we obtain the following table of possible \( F \)-essential subgroups up to \( S \)-conjugacy.

<table>
<thead>
<tr>
<th>( S )</th>
<th>( E_0 )</th>
<th>( E_1 )</th>
<th>( E_2 )</th>
<th>( V_0 )</th>
<th>( V_1 )</th>
<th>( V_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B(r; 1, 0, 0) )</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( B(r; 1, 0, 1) )</td>
<td></td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( B(r; 1, 0, 2) )</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 2.** Candidates for the \( F \)-essential subgroups

We record the following result:

**Lemma 4.6.** \( B(r; 1, 0, 1) \) is resistant.

**Proof.** By Lemma [4.4] \( \gamma_1(S) \) is not \( F \)-essential and there are no other candidates for essential subgroups by Table [2]. \( \square \)
Lemma 4.7. Suppose that $S = B(r; 1, 0, 0)$ and $\mathcal{F}$ is a saturated fusion system on $S$ which has at least one $\mathcal{F}$-conjugacy class of $\mathcal{F}$-essential subgroups. Then $V_0$ represents the unique such class, $\text{Aut}_\mathcal{F}(V_0) \cong \text{SL}_2(3)$, $|\text{Out}_\mathcal{F}(S)| = 2$ and either

(i) $r$ is even and $\mathcal{F}$ is reduced; or
(ii) $r = 2k + 1$ is odd, and $\mathcal{O}^3(\mathcal{F})$ is a subsystem of index 3 isomorphic to the fusion system of $\text{PSL}_3(q)$ for some prime power $q$ with $v_3(q - 1) = k$.

Proof. Let $D$ be an $\mathcal{F}$-essential subgroup. Using Table 2 we see that up to $S$-conjugacy $D = V_0$ or $D = E_0$. By Lemma 4.3 $|\text{Out}_\mathcal{F}(S)| \geq 2$ so we conclude from Proposition 3.3 that $|\text{Out}_\mathcal{F}(S)| = 2$. Thus without loss of generality, we define $\theta \in \text{Aut}_\mathcal{F}(S)$ via:

$$\theta := \begin{cases} 
\theta_{2,0,2,1,1} & \text{if } r \text{ is even;} \\
\theta_{2,0,1,1,1} & \text{if } r \text{ is odd.}
\end{cases}$$

We see that $\theta$ inverts $s$ and from Lemma 3.4

$$s_{r-1}\theta = \begin{cases} s_{r-1}\theta_{2,0,2,1,1} = s_{r-1}^2 = s_{r-1}^{-1} & \text{if } r \text{ is even;} \\
s_{r-1}\theta_{2,0,1,1,1} = s_{r-1}^2 = s_{r-1}^{-1} & \text{if } r \text{ is odd.}
\end{cases}$$

By Lemma 4.3 we have $\text{Out}_\mathcal{F}(D) \cong \text{SL}_2(3)$ and the central involution in $\text{Out}_\mathcal{F}(D)$ is the image of $\theta_D$. If $D = E_0$, then $\theta|_D$ centralizes $Z(D) = \gamma_{r-1}(S)$ so we conclude that $D = V_0$. Now $N_S(D)\langle \theta | N_S(D) \rangle$ is isomorphic to the normalizer of the Sylow 3-subgroup $M$ of a group $P$ isomorphic with $3^2: \text{SL}_2(3)$ and so the amalgamated product $G = P*M \gamma_{r-1}(S)$ realizes $\mathcal{F} = \mathcal{F}_S(G)$. Since $D$ is minimal among all $\mathcal{F}$-centric subgroups of $S$, $\mathcal{F}$ is saturated by [Sem13, Theorem C].

If $r$ is even then $S = [S, \theta]$ so $\text{foc}(\mathcal{F}) = S$ and $\mathcal{F}$ is reduced as $\theta|_D \in \mathcal{O}^3(\text{Aut}_\mathcal{F}(D))$. If $r$ is odd, then, since $\theta$ centralizes $s_1$,

$$[S, \theta] = \langle s, \gamma_2(S) \rangle = \langle s, s_2, s_3, \ldots, s_{r-1} \rangle \cong B(r - 1; 0, 0, 0).$$

Since $D$, $D^{s_1}$ and $D^{e_1}$ are $\mathcal{O}^3(\mathcal{F})$-essential and not fused in $\mathcal{O}^3(\mathcal{F})$, the result follows from [DRV07, Tables 2,4].

Lemma 4.8. Suppose that $S = B(r; 1, 0, 2)$ and $\mathcal{F}$ is a saturated fusion system on $S$ which has at least one $\mathcal{F}$-conjugacy class of $\mathcal{F}$-essential subgroups. Then $r$ is even and one of the following holds:

(i) $V_1$ represents the unique $\mathcal{F}$-conjugacy class of $\mathcal{F}$-essential subgroups, $\text{Aut}_\mathcal{F}(V_1) \cong \text{SL}_2(3)$, $|\text{Out}_\mathcal{F}(S)| = 2$ and $\mathcal{F}$ is reduced;
(ii) $V_2$ represents the unique $\mathcal{F}$-conjugacy class of $\mathcal{F}$-essential subgroups, $\text{Aut}_\mathcal{F}(V_2) \cong \text{SL}_2(3)$, $|\text{Out}_\mathcal{F}(S)| = 2$ and $\mathcal{F}$ is reduced; or
(iii) there are two $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-essential subgroups represented by $V_1$ and $V_2$ with $\text{Aut}_\mathcal{F}(V_1) \cong \text{Aut}_\mathcal{F}(V_2) \cong \text{SL}_2(3)$, $|\text{Out}_\mathcal{F}(S)| = 2$ and $\mathcal{F}$ is reduced.

Proof. Let $D$ be an $\mathcal{F}$-essential subgroup. By Table 2 we have $D \in \{V_1, V_2, E_1, E_2\}$ up to $S$-conjugacy. Arguing as in Lemma 4.7 (using Lemma 4.3 and Proposition 3.3) we see that $|\text{Out}_\mathcal{F}(S)| = 2$. If $r$ is odd then $\text{Aut}_\mathcal{F}(S)$ permutes $\{(s_{s_1})\gamma_2(S), (s_{s_1})\gamma_2(S)\}$ transitively by Proposition 3.3. In particular no element of $\text{Aut}_\mathcal{F}(S)$ of order 2 normalizes an element of $\{V_1, V_2, E_1, E_2\}$. Using Lemma 4.3 we deduce that $r$ is even. Set

$$\theta := \begin{cases} 
\theta_{2,0,2,1,s_2s_{r-1}} & \text{if } D \in \{V_1, E_1\} \\
\theta_{2,0,2,s_{r-1}^2} & \text{if } D \in \{V_2, E_2\}.
\end{cases}$$

Then $s_{r-1}\theta = s_{r-1}^{-1}$ and so as $\text{Out}_\mathcal{F}(D) \cong \text{SL}_2(3)$ we see as before that $D \in \{V_1, V_2\}$. Observe that if $D = V_1$, then

$$(s_{s_1})\theta = s\theta s_1 \theta = s^2s_1^2s_{r-1}^{-1} = (ss_1)^{-1}$$
and, if $D = V_2$, then 
\[(ss_1^2)\theta = s\theta(s_1\theta)^2 = s^2s_2^2s_1^4 = (ss_1^2)^{-1}.
\]
Thus $\theta$ inverts $D$ as required.

Again observe that $S = [S, \theta]$ so $\text{foc}(\mathcal{F}) = S$ and $\mathcal{F}$ is reduced as $\theta|_D \in O^3(\text{Aut}_D(D))$. For $i=1,2$, $NS(V_i)(\theta|_{NS(V_i)})$ is isomorphic to the normalizer of the Sylow 3-subgroup $M_i$ of a group $P_i$ isomorphic with $3^2 : \text{SL}_2(3)$. Hence there is an amalgamated product $G_i = P_i \ast_{M_i} S(\theta)$ which realizes a fusion system $\mathcal{F}_i = \mathcal{F}_S(G_i)$ satisfying the conditions in (i) and (ii) respectively. Since $V_i$ is minimal among all $\mathcal{F}_1$-centric subgroups of $S$, $\mathcal{F}_i$ is saturated by $[\text{Sem14}, \text{Theorem C}]$. In particular there are unique fusion systems satisfying these conditions. Now, since $V_2$ is a fully $\mathcal{F}_1$-normalized subgroup which is minimal among all $\mathcal{F}_1$-centric subgroups of $S$, $[\text{Sem14}, \text{Theorem C}]$ also implies that $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$ is saturated. This is the unique fusion system described by (iii). $\square$

**Lemma 4.9.** The fusion systems described in Lemmas 4.7 and 4.8 are exotic.

**Proof.** Suppose that $\mathcal{F}$ represents one of the fusion systems of interest and let $S = B(r; 1, 0, 0)$ or $B(r; 1, 0, 2)$. If $C$ is a non-trivial strongly $\mathcal{F}$-closed subgroup of $S$, then $C$ is normal in $S$ and so $s_{r-1} \in C$. Thus $C \cap V \neq 1$ where $V$ is an $\mathcal{F}$-essential subgroup. It follows that $V \leq C$ and then $\langle V^S \rangle = V\gamma_2(S) \leq C$. In fact $V\gamma_2(S)$ is strongly $\mathcal{F}$-closed if $S = B(r; 1, 0, 0)$ or $S = B(r; 1, 0, 2)$ and $\mathcal{F}$ has only one $\mathcal{F}$-class of $\mathcal{F}$-essential subgroups. Since $\langle V^S \rangle \cong B(r-1; 0, 0, 0)$, $[\text{DRV07}, \text{Proposition 2.19}]$ applies to say that if $\mathcal{F}$ is realised by a finite group $G$, then it is realized by an almost simple group. Now the arguments in $[\text{DRV07}$ page 1751 (a), (b) and (c)] prove the result. $\square$

**References**


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