SL\(_2\) -tilings do not exist in higher dimensions (mostly)

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Abstract. We define a family of generalizations of SL\(_2\) -tilings to higher dimensions called \(\epsilon\)-SL\(_2\) -tilings. We show that, in each dimension 3 or greater, \(\epsilon\)-SL\(_2\) -tilings exist only for certain choices of \(\epsilon\). In the case that they exist, we show that they are essentially unique and have a concrete description in terms of odd Fibonacci numbers.

1. SL\(_2\) -tilings of the plane

The aim of this note is to study higher-dimensional analogues of the following object.

Definition 1 ([1]). A bi-infinite array \((a_{ij})_{i,j \in \mathbb{Z}}\) with \(a_{ij} \in \mathbb{Z}_{>0}\) is called an SL\(_2\) -tiling of \(\mathbb{Z}^2\) if the entries satisfy the relation

\[
    a_{i,j+1}a_{i+1,j} - a_{ij}a_{i+1,j+1} = 1. 
\]

A bi-infinite array \((b_{ij})_{i,j \in \mathbb{Z}}\) with \(b_{ij} \in \mathbb{Z}_{>0}\) is called an anti-SL\(_2\) -tiling of \(\mathbb{Z}^2\) if the entries satisfy the relation

\[
    b_{i,j+1}b_{i+1,j} - b_{ij}b_{i+1,j+1} = -1. 
\]

The notion of an anti-SL\(_2\) -tiling is not actually giving anything new as shown by the following lemma, however this notion will be useful for our considerations in higher dimensions.

Lemma 2. If \((a_{ij})_{i,j \in \mathbb{Z}}\) is an SL\(_2\) -tiling, then taking \(b_{ij} = a_{i,-j}\) gives an anti-SL\(_2\) -tiling.

One should think of the difference between SL\(_2\) -tilings and anti-SL\(_2\) -tilings as viewing the lattice \(\mathbb{Z}^2\) “from above” or “from below.” The following result from [1] was our starting point.

Theorem 3 ([1]). There exist infinitely many SL\(_2\) -tilings of \(\mathbb{Z}^2\).

In fact, it is shown in [1] that any admissible frontier of 1’s in the lattice, can be completed into a unique SL\(_2\) -tiling. An interpretation of all possible SL\(_2\) -tilings was later given in [2] in terms of triangulations of a polygon with infinitely many vertices.

The following anti-SL\(_2\) -tiling will be relevant in our higher dimensional analysis. We will call it the staircase anti-SL\(_2\) -tiling of \(\mathbb{Z}^2\).

Example 4. Consider the anti-SL\(_2\) -tiling \((a_{ij})_{i,j \in \mathbb{Z}}\) of \(\mathbb{Z}^2\) with \(a_{ij} = 1\) if \(i + j \in \{0, 1\}\). Using (2) and the well-known recursion \(F_{2r-1}F_{2r+3} = F_{2r+1}^2 + 1\), for the odd Fibonacci numbers, it is easy to see that

\[
    a_{ij} = \begin{cases} 
        F_{2r-1} & \text{if } i + j = r \geq 1; \\
        F_{-2r+1} & \text{if } i + j = r \leq 0; 
    \end{cases}
\]

where we number the Fibonacci numbers as:

\[
\begin{array}{cccccccc}
F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & \ldots \\
1 & 1 & 2 & 3 & 5 & 8 & 13 & \ldots 
\end{array}
\]

The following figure is a portion of this tiling. Note the bolded frontier of 1’s; it is an “infinite staircase.”
2. SL\(_2\)-Tilings in Higher Dimensions

Denote integer vectors by \(i = (i_1, \ldots, i_n)\) and by \(e_k\) the \(k\)-th unit vector. A signature matrix is a symmetric \(n \times n\) matrix \(\epsilon = (\epsilon_{k\ell})\) with \(\epsilon_{k\ell} = \pm 1\) whenever \(k \neq \ell\) and \(\epsilon_{kk} = -1\).

**Definition 5.** Fix a signature matrix \(\epsilon\). An array \((a_i)_{i \in \mathbb{Z}^n}\) with \(a_i \in \mathbb{Z}^n > 0\) is called an \(\epsilon\)-SL\(_2\)-tiling of \(\mathbb{Z}^n\) if for each \(k \neq \ell\) we have

\[
 a_i + e_k a_i + e_\ell - a_i a_i + e_k + e_\ell = \epsilon_{k\ell}. \tag{3}
\]

The requirement on the diagonal entries of signature matrices might seem arbitrary right now because they do not play any role in the above definition; we will see later on that it is indeed a consistent choice.

The situation is now different than the \(n = 2\) case, all the \(\epsilon\)-SL\(_2\)-tilings are not necessarily equivalent, however there do remain relations among them.

**Lemma 6.** Let \(\epsilon = (\epsilon_{k\ell})\) be any signature matrix and write \(\epsilon^{(r)}\) for the matrix obtained from \(\epsilon\) by changing the sign of all the entries in row \(r\) and column \(r\), leaving the diagonal entries fixed. That is, \(\epsilon^{(r)} = (\epsilon'_{k\ell})\) where \(\epsilon'_{k\ell} = -\epsilon_{k\ell}\) if exactly one of \(k\) and \(\ell\) equals \(r\) and \(\epsilon'_{k\ell} = \epsilon_{k\ell}\) otherwise. If \((a_i)_{i \in \mathbb{Z}^n}\) is an \(\epsilon\)-SL\(_2\)-tiling, then taking \(b_i = a_i - 2i e_r\) gives an \(\epsilon^{(r)}\)-SL\(_2\)-tiling.

**Definition 7.** If \(\epsilon\) is a signature matrix such that \(\epsilon_{k\ell} = 1\) (resp. \(\epsilon_{k\ell} = -1\)) whenever \(k \neq \ell\), we refer to an \(\epsilon\)-SL\(_2\)-tiling as an SL\(_2\)-tiling (resp. anti-SL\(_2\)-tiling) of \(\mathbb{Z}^n\).

**Lemma 8.** Let \(n \geq 3\) and assume \((a_i)_{i \in \mathbb{Z}^n}\) is either an SL\(_2\)-tiling or an anti-SL\(_2\)-tiling of \(\mathbb{Z}^n\). Then for any \(r \in \mathbb{Z}\) the set \(\{a_i : \sum_{j=1}^n i_j = r\}\) consists of a single element.

**Proof.** Pick any three distinct indices \(j, k, \ell \in [1, n]\). To prove our claim we compute \(a_i + e_j + e_k + e_\ell\) in terms of \(a_i, a_i + e_j, a_i + e_k, a_i + e_\ell\) in three different ways. For simplicity of notation we set:

\[
 \epsilon_{jk} = \epsilon_{kj} = \epsilon_{k\ell} = \epsilon, \quad a_i = a, \quad a_i + e_j = x, \quad a_i + e_k = y, \quad a_i + e_\ell = z.
\]

The following picture will be useful.

![Diagram](image.png)

Using (3) three times we get

\[
 a_i + e_j + e_k = \frac{xy - \epsilon}{a}, \quad a_i + e_k + e_\ell = \frac{yz - \epsilon}{a}, \quad a_i + e_j + e_\ell = \frac{xz - \epsilon}{a}.
\]
Then applying (3) three more times gives

\[ a_i+e_i, a_i-e_i = a_i a_{i+e_i} + a_{i+e_i} - a_i^2 + 1 = a_i^2 + 1, \]

where we applied Lemma 5 in the last equality. If \( a_i > 1 \), this implies \( a_{i+e_i} < a_i \) or \( a_{i-e_i} < a_i \), contradicting minimality, so we must have \( a_i = 1 \). In turn, again leveraging Lemma 5 this implies \( \{a_i+e_i, a_i-e_i\} = \{1, 2\} \).

Without loss of generality we will assume \( a_{i+e_i} = 2 \) and \( \sum_{j=1}^n i_j = 1 \). Then applying (3) repeatedly shows that \( a_{i'} \) with \( \sum_{j=1}^n i_{j'} = r \geq 1 \) is exactly the \( r \)-th odd Fibonacci number \( F_{2r-1} \) (see Example 4). Similarly one sees that \( a_{i'} \) with \( \sum_{j=1}^n i_{j'} = r \leq 0 \) is the odd Fibonacci number \( F_{-2r+1} \).

**Theorem 9.** For \( n \geq 3 \), there exists a unique (up to translation) anti-SL\(_2\)-tiling of \( \mathbb{Z}^n \). Any of its “two dimensional slices” obtained by fixing all but two of the coordinates of \( i \) is a translation of the staircase anti-SL\(_2\)-tiling of \( \mathbb{Z}^2 \) from Example 7. In particular, all the integers appearing are odd Fibonacci numbers.

**Proof.** Assume \((a_i)_{i \in \mathbb{Z}^n}\) is a anti-SL\(_2\)-tiling of \( \mathbb{Z}^n \). Pick \( i \) with \( a_i \) minimal. Applying (3) gives

\[ a_{i+e_i} a_{i-e_i} = a_i a_{i+e_i} + a_{i+e_i} - a_i^2 + 1 = a_i^2 + 1, \]

where we applied Lemma 5 in the last equality. If \( a_i > 1 \), this implies \( a_{i+e_i} < a_i \) or \( a_{i-e_i} < a_i \), contradicting minimality, so we must have \( a_i = 1 \). In turn, again leveraging Lemma 5 this implies \( \{a_i+e_i, a_i-e_i\} = \{1, 2\} \).

Without loss of generality we will assume \( a_{i+e_i} = 2 \) and \( \sum_{j=1}^n i_j = 1 \). Then applying (3) repeatedly shows that \( a_{i'} \) with \( \sum_{j=1}^n i_{j'} = r \geq 1 \) is exactly the \( r \)-th odd Fibonacci number \( F_{2r-1} \) (see Example 4). Similarly one sees that \( a_{i'} \) with \( \sum_{j=1}^n i_{j'} = r \leq 0 \) is the odd Fibonacci number \( F_{-2r+1} \).

**Proposition 10.** There does not exist any anti-SL\(_2\)-tiling of \( \mathbb{Z}^n \) for \( n \geq 3 \).

**Proof.** It suffices to show that there is no SL\(_2\)-tiling of \( \mathbb{Z}^3 \). Assume \((a_i)_{i \in \mathbb{Z}^3}\) is an SL\(_2\)-tiling of \( \mathbb{Z}^3 \). Pick \( i \) with \( a_i \) minimal. Applying (3) gives

\[ a_{i+e_i} a_{i-e_i} = a_i a_{i+e_i} + a_{i+e_i} - a_i^2 - 1 = a_i^2 - 1, \]

where we applied Lemma 5 in the last equality. But this implies \( a_{i+e_i} < a_i \) or \( a_{i-e_i} < a_i \), contradicting minimality.

**Corollary 11.** For \( n = 3 \), there are precisely 4 signature matrices \( \mathbf{e} \) for which there exists an \( \mathbf{e} \)-anti-SL\(_2\)-tiling. For such \( \mathbf{e} \), this \( \mathbf{e} \)-anti-SL\(_2\)-tiling is unique (up to translation). More precisely, an \( \mathbf{e} \)-anti-SL\(_2\)-tiling exists if and only if \( \epsilon_{12} \epsilon_{13} \epsilon_{23} = -1 \).

**Proof.** The claim follows immediately from the observation that any signature matrix for \( n = 3 \) is either one of the two satisfying \( \epsilon_{12} = \epsilon_{13} = \epsilon_{23} \) or is obtained from one of these with a single application of Lemma 6.

We are finally ready to classify all \( \mathbf{e} \)-anti-SL\(_2\)-tilings for any \( n \geq 3 \).

**Theorem 12.** For \( n \geq 3 \), there are precisely \( 2^{n-1} \) signature matrices \( \mathbf{e} \) for which there exists an \( \mathbf{e} \)-anti-SL\(_2\)-tiling of \( \mathbb{Z}^n \). They are precisely the signature matrices obtainable from the anti-SL\(_2\)-signature matrix by repeated application of Lemma 6. Whenever an \( \mathbf{e} \)-anti-SL\(_2\)-tiling exists, it is unique up to translation.

**Proof.** Let \((a_i)_{i \in \mathbb{Z}^n}\) be an \( \mathbf{e} \)-anti-SL\(_2\)-tiling of \( \mathbb{Z}^n \). Fixing all but any three distinct entries of \( i \) gives a tiling of \( \mathbb{Z}^3 \). Therefore, it follows from Corollary 11 that we have an inclusion \( E \subset E' \), where \( E \) is the set of \( n \times n \) signature matrices \( \mathbf{e} \) which admit an \( \mathbf{e} \)-anti-SL\(_2\)-tiling, and \( E' \) is the set of \( n \times n \) signature matrices \( \mathbf{e} \) satisfying \( \epsilon_{ij} \epsilon_{kl} \epsilon_{kj} = -1 \) for any triple of distinct indices \( j, k, l \).

Any row (or equivalently any column) of a matrix \( \mathbf{e} \) in \( E' \) determines uniquely all the remaining entries of \( \mathbf{e} \), therefore \( E' \) is in bijection with \( \{\pm 1\}^{n-1} \) and \#\( E' = 2^{n-1} \).

Using Lemma 5 there is an action of \( (\mathbb{Z}/2\mathbb{Z})^{n-1} \) on \( E \) given by \( \mathbf{e} \mapsto \mathbf{e}^r \) for \( 1 \leq r \leq n-1 \). This action is free; indeed the only element of \( (\mathbb{Z}/2\mathbb{Z})^{n-1} \) leaving invariant the last column of any given matrix of \( E \) is the identity. Thanks to Theorem 9, \( E \) is not empty and so we compute \#\( E \geq 2^{n-1} = \#E' \geq \#E \) and deduce that \( E = E' \).

The uniqueness claim also follows immediately from Corollary 11 by fixing all but any three distinct entries of \( i \).
Note that the claim of Theorem 12 could be rephrased by saying that, up to fixing the origin and choosing the orientation of each of the coordinate axes, there is a unique tiling of \( \mathbb{Z}^n \) for \( n \geq 3 \).

**Remark 13.** It is now clear why we choose the diagonal entries of \( \epsilon \) to be equal to \(-1\): any \( \epsilon \)-\( SL_2 \)-tiling consists of odd Fibonacci numbers and \( (3) \) is satisfied also for \( k = \ell \).

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