Modelling Tidal Streams from Galactic Satellites

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This thesis is dedicated to
my parents, my wife, Zahra, and Amzar
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Abstract

Astrophysical simulations are very useful for understanding the physical phenomena of the stellar systems. In this simulation, the computational cost is one of the main concern. To tackle this issue, a choice of the time-step criterion is crucial. In this thesis, I study the efficiency of the time-step depending on the tidal force. This quantity follows gauge invariance under adding a constant acceleration where this property is not owned by conventional time-step criteria. Time step function based on the tidal force is more efficient than other time-step criteria. The observed tidal streams are very important to constrain the Galactic potential and the distribution of sub-haloes structure in the Milky-Way. In this thesis, I present a restricted $N$–body model for tidal debris to simulate the formation of tidal streams from Galactic satellites. If the satellite orbits are close to a Galactic disc, the structures of tidal debris do not spread in one dimension like the conventional streams but form two dimensions structures like a ribbon. This phenomenon can be explained analytically by action-angle variables and Hamiltonian formalism. This restricted $N$–body method has fewer ad-hoc assumptions and is more accurate than the alternative (and popular) particle sprinkling techniques at only a modest increase of the computational expense.
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Chapter 1

Introduction

1.1 From Ancient to Modern Astronomy

Human civilization has observed the sky day and night since time immemorial. Humans are amazed at how the moon, planets and stars move. Possibly, one example of sky observation evidence is the Stonehenge monument in England and structures and the writings of the Mayan and Aztec civilizations. The Stonehenge is believed used to mark the time of the summer solstice. Observations of the sky also involve many ancient civilizations such as the Sumeria, China, Egypt, and middle east (Sachs, 1974). The ancient Greek used the geometry in the astronomy and considered it as a branch in mathematics. One example of this is the measurement of the earth radius by using sun shadows in different places.

In an initial investigation of the night sky, the most obvious fact that careful observers catch is that the heavenly body is constantly changing. Not only the daily motion of stars from east to west, but the different stars position observed in the evening sky from different seasons in a year. Another clear feature is the cycle of moon phase in a month. The retrograde motion of some planets is wondering the ancient observer (Carroll and Ostlie, 2006).

The earliest explanation of the movement of the planet and stars in the sky is based on the geometrical model of the earth and the heavenly body system started with an assumption that the earth is fixed and the others are revolving around it. This early model is called the geocentric model and described extensively by Claudius Ptolemy. The retrograde motion of a planet is explained by Hipparchus using the epicycle motion on the large deferent. Not only the wandering motion of the planet is described, the change of the planet brightness is recovered by this method. Later it was found that this method is not adequate to explain the other details from observations. However, Ptolemy revised the epicycle and deferent method by adding the equants. It seems that this theory had a great success, but add more complex descriptions for observed phenomena.
A new model of planet systems emerged to overcome this complexity which stated that the sun is the centre of the system, while the planets and the earth revolve around the sun. This heliocentric system was introduced by Nicolas Copernicus and enables us to arrange the planets in the right order. This theory can explain the maximum position of Mercury and Venus are 28° and 47°, respectively, from the sun and categorize these planets as the inferior planets. Then, the retrograde motion of the planets is simply well described by this theory. For example, the orbital period of Mars is greater than the earth, which makes the earth move faster than Mars and leads to the apparent backward motion of Mars relative to the relatively fixed background stars.

The Copernicus’s model requires the orbit of the planet must be circular which is not suitable for the observation of Tyco Brahe. Later, Johannes Kepler deduced from this observational data that the orbits of the planets are ellipses instead of circles. This becomes the Kepler’s first law. The precise and extensive observation data of Brahe are extracted by Kepler and lead to the three laws of planetary motion (Seaborne 1998).

While Kepler was discovering these laws, Galileo was studying the laws of motion and found the principle of inertia (Feynman et al. 1963). Later, Newton used this principle and introduced the term force on his second law and discovered the classical laws of mechanics. By utilizing the second and third law of Kepler, Newton deduced the law of gravitation which stated that everything pulls everything else with the magnitude inversely proportional to the square distance. The Newton formulation of the laws of mechanics used the new branch in mathematics so-called differential calculus and the notation was developed by Gottfried Wilhelm Leibniz.

The more abstract and yet powerful formulations of the law of mechanics were developed by Joseph Luis Lagrange (Lagrangian Mechanics) and William Rowan Hamilton (Hamiltonian Mechanics). Later, the alternative formulation was developed by Jacobi yielding the Hamilton-Jacobi theory. Additionally, The celestial mechanics was formulated by Pierre Laplace and explains the stability of the solar system on short time scales. Until today, the Newtonian mechanics and others formulations of the mechanics are used in the celestial to galactic systems.

1.2 Galactic Astronomy

The galactic system is one of the stellar systems, gravitationally-bound of the collection of the stars, which has several possible shapes from the ellipse, spiral, and irregular. The sun is in one of these systems so-called Milky way. Dating back to ancient Greek time, the Milky way name came from Via Lactae due to its appearance on the night sky like the
white band stretch across the sky. When Galileo used the telescope to resolve this band, he found that it is actually made of faint stars which look like “celestial fluid” on the naked eye.

The shape of the Milky Way in the sky is explained by Immanuel Kant in his treatise, *General Natural History and Theory of the Heavens*. The Milky Way system structure arises from similar argument to solar system which gravity of the sun bounds the system and the rotation of the planet prevents the planet from collapse, but in this case on a huge scale. The disc-like structure appears on the stellar system if it possesses a systematic rotational motion to balance the inward gravitational pull (Binney and Merrifield, 1998). Starting from an analogy with comets in the solar system, Kant also identified that the small numbers of stars found outside of the disc-plane can not share the ordered rotation of the major component of the system, but must lie on more randomly distributed orbits (Binney and Merrifield, 1998). The possible other similar structures of the Milky Way have been suggested by Kant and these galaxies form islands in the universe.

The detailed study of the structure of the Milky Way continues by using the photometric techniques of William Herschel on star gauging and found that the sun location was near to the centre of the system. This analysis uses an assumption that all the stars have approximately the same intrinsic brightness. The similar conclusion made by Kapteyn after counting the stars in Milky Way showed the equal number of the stars in every direction. From the Kapteyn’s work, the density of stars falls off uniformly from the centre of the system and the sun is located slightly off from the plane. Their analysis did not include the interstellar extinction (dust and gas) that can block the light from far stars like the fog. The dust and gas mostly reside at the disc midplane but very little found in the off plane. The off account of the extinction affects their results because of the limit of stars counting.

The utility of RR Lyrae variable stars as standard candle of the cosmic distance ladder enables Harlow Shapley to map almost hundred of globular clusters which are found distributed throughout the sky, not restricted to disc plane. He also found that the distribution of the globular cluster is not uniform, but more concentrated near the great star clouds in Sagittarius, which shows the centre of the Galaxy was not Sun. His estimation of the distance of the centre of the Milky way from the sun is off by a factor of 2 from the current estimation due to the little knowledge of the interstellar extinction at that time. His conclusion contradicts the early well established Kapteyn universe.

There was a debate about the spiral nebulae, an early term for the spiral galaxy, whether it inside the Milky Way system or outside the system. The discovery and study of Cepheid variable stars in the Magellanic cloud by Henrietta Swan Leavitt enabled astronomers to measure the distance of the spiral nebulae. It was not until Edwin Hubble discovered
Cepheid variables in some of these spiral nebulae that the distances of these nebulae were found to be far away from our system.

It was thought, until the twentieth century, that the galaxies are a building block of the universe because of the brightness contrast between these systems and their surrounding in optical light. But if we turn our eyes to the X-ray spectrum, then the cluster of the galaxy would stand out as the most impressive individual structures.

1.3 The Discovery of Tidal Stream and Its Modelling

Tidal tails consist of the stars that escape from a systems due to interactions with other systems. For example, a close encounter of two galaxies can form a tidal tails or a bridge (Toomre and Toomre, 1972). The formation of this bridge is confirmed by computational models (for example, see Wright (1972)).

Tidal tails also come from satellites orbiting a galaxy. These tidal tails or debris are often called streamers. The discovery of the tidal streams starts from an investigation of the outer structure of globular clusters. Grillmair et al. (1995) examined the outer regions of 12 globular clusters using star count analysis from deep, two-color, photographic photometry. They found that most of their sample show extra tidal wings in their surface density profiles. Using the same analysis, Irwin and Hatzidimitriou (1995) determined the morphology of the dwarf spheroidals (dSphs) and predicted the presence of extra tidal tails up to two tidal radius. Leon et al. (2000) investigated the presence of the tidal tails around the stellar systems using star-count analysis from large field multi-color images of 20 globular clusters. Most of the globular clusters in their sample display strong evidence of tidal interaction with the galactic plane in the form of large and extended deformations. These tidal tails exhibit projected directions preferentially toward the galactic centre (Leon et al., 2000).

Since the streamers have low surface density, the detection of this debris is challenging. For the debris from halo orbit such as M54, Arp2, Terzan 7, and Terzan 8, they form a moving group which have been detected by Ibata et al. (1994) and simulated by Johnston et al. (1995). Johnston et al. (1996) develop a method to detect of presence of tidal debris. The halo stars projected into two dimensions which assumed lies on the great circle and star-count is applied in this surface. This method is useful only for streams that have a smaller region extent.

The most well observed and studied tidal stream originates from low-mass globular cluster Palomar 5 (Pal5) (Odenkirchen et al., 2001; Rockosi et al., 2002; Odenkirchen et al., 2003; Grillmair and Dionatos, 2006; Odenkirchen et al., 2009). The first detection used the
deep five-color photometry on Sloan Digital Sky Surveys (SDSS) data. The discovery of two well-defined tidal tails of Pal5 was used to predict the orbit of Pal5 (Odenkirchen et al., 2001). Moreover, the observed tidal streams show the clumps which formed by enhanced release of stars after strong shock events (Odenkirchen et al., 2001). The total number of stars of Pal5 which contribute to its tails is estimated around 45% (Rockosi et al., 2002). The length of the stream in the sky projection is estimated about 10° (Odenkirchen et al., 2003). Later, the length of this tail was found to be approximately 22° in the sky (Grillmair and Dionatos, 2006).

Odenkirchen et al. (2003) showed that Pal5 is in the process of being tidally disrupted. They also assumed that the tidal tails are parallel to the local orbit path of the cluster. This was later confirmed by Dehnen et al. (2004) using the proper simulations. The other result from their study is that the Pal5 is not a former member of Sagittarius dwarf. The first observational study of the kinematics of debris from disruption of Pal5 was conducted by Odenkirchen et al. (2009). In this study, the streams form a kinematically cold structure in the Galactic halo. Hence, this explains the narrowness of the tails. This study was also used to constrain the local orbit of the Pal5 using their results that the debris have very similar orbits to that of Pal5.

The other prominent observed tidal tail is NGC5466. Odenkirchen and Grebel (2004) analyzed the spatial distribution of the stars in the outer parts of this cluster and found the positive tidal perturbation of the cluster and eventual mass loss due to the Galactic tidal field. The length of the arc of the tidal tails is approximately 4° in the sky corresponding to roughly 1 kpc in the projected length (Belokurov et al., 2006a). The reported average width of this tidal tails is about 1.4° (Grillmair and Johnson, 2006).

A simpler feature of the extended tidal tails is found at the outer region of NGC5053 (Lauchner et al., 2006). The length of this tidal tails is approximately 1.7 kpc. The outer region of the M92 (NGC 6431) also show the extra tidal halo extended out to roughly 0.5° from cluster centre (Testa et al., 2000). NGC 4147 also exhibits a tidal stream and was found by mapping the streams with color-magnitude diagram (Martínez Delgado et al., 2004). The analyzed spatial distribution of the stars surrounding 5 metal-poor globular cluster in the Galactic halo (M15, M30, M53, NGC5053, and NGC5466) show the extra-tidal features from their radial surface densities and azimuthal number density profiles (Chun et al., 2010). The tidal tails are also potentially expected to be detected around NGC 5904 and Pal14 (Jordi and Grebel, 2010). The search of the extra-tidal features is still ongoing at present with wider and deeper large scale digital sky surveys.

The modelling of the tidal streams is needed to understand their formation and their observable in the sky. Combes et al. (1999) carried out about a dozen of simulations using
Fast Fourier Transform (FFT) algorithm of the tidal interactions between a globular cluster and the Galaxy. The results of their study are:

(1) all clusters are always surrounded by tidal tails and with density falls with a power law of a function of radius with the slope around -4 (see also McGlynn, 1990),

(2) these tails are preferentially composed of the low-mass stars, since they are coming from the external radii of the cluster, due to mass segregation built up by two-body relaxation near the core,

(3) the mass loss is enhanced for a cluster in direct rotation with respect to its orbit,

(4) stars are not distributed homogeneously through the tails, but form clumps.

The disruption of Pal5 and the formation of its tidal tails is studied by Dehnen et al. (2004) using \(N\)-body simulations with Fast Multipole Method (FMM) (Dehnen, 2014). The morphology of the tidal streams varies between thin and long near perigalacticon and short and thick near apogalacticon due to variation of the orbital velocity and hence the separation between tails stars and cluster (Dehnen et al., 2004). Their study also demonstrated that near apogalacticon, the tidal tails show a streaky structure which each streak correspond to a swarm of stars that has been set loose by the same disc shock.

The clumps in the tidal tails, as previously found in the simulation by Combes et al. (1999), were investigated by Capuzzo Dolcetta et al. (2005). They argue that the clumpy structure is not related to the tidal shock with Galactic substructure. Some authors believe that the clumps is formed due to epicycle motion of the stars (Küpper et al., 2008; Just et al., 2009; Küpper et al., 2010b). These methods are based on an assumption that the stars will be lost at the two Lagrange points.

The stars lost from satellites form tidal streams orbiting the Galaxy and can be used to infer the gravitational potential of Galaxy itself. An analogy to this method is an estimation of the mass of the sun from planet orbits or mass of black hole from orbiting stars (Ghez et al., 2005). The use of tidal stream trajectory to probe the potential is based on the similarity of its trajectory with its progenitor’s orbit (McGlynn, 1990; Johnston et al., 1996, 1999; Binney, 2008). However, the tidal stream do not delineate the orbit (Choi et al., 2007; Eyre and Binney, 2009, 2011).

The number of the small-scale dark matter of sub-haloes predicted by standard Λ Cold Dark Matter (CDM) is more than observed Galactic satellites orbiting Galaxy (e.g. Klypin et al., 1999; Moore et al., 1999; Diemand et al., 2007, 2008; Springel et al., 2008). The existence of the dark matter sub-haloes can be tested by their interactions with observed
stellar streams (Ibata et al., 2002; Yoon et al., 2011; Erkal et al., 2016; Bovy et al., 2017). Since the stellar streams from halo orbit are generally narrow in space and kinematically cold, they are vulnerable to such a massive dark matter sub-halo. The interaction between such streams and dark matter sub-haloes can open the gaps within the debris or even destroy them. Hence, the tidal streams can also be used to constrain the distribution of dark matter sub-haloes in the Milky Way halo (e.g. Johnston et al., 2005; Koposov et al., 2010; Law and Majewski, 2010).

1.4 Some Observational Works of the Tidal Stream

A tidal stream can be defined as an overdensity in a halo, or as a collection of stars with similar location and velocities. There are three main methods to find a substructure or stellar streams in Galaxy, i.e. (a) standard candles and photometric parallax, (b) match filter technique and (c) statistical photometric parallax (Grillmair and Carlin, 2016).

One of the most extents of the tidal stream is debris from Sagittarius (Sgr) dwarf galaxy orbiting Milky Way which covering full 3600 of the sky projection. Ibata et al. (1994) detected a large group of co-moving stars in the direction of the Galactic center which belongs to a Sagittarius dwarf. The radial velocities for all the target stars derived from cross-correlation each target spectrum with radial velocity standard star templates. The radial velocity obtained from the spectroscopic data of 3.9 m Anglo-Australian Telescope (AAT). In their study, they used the UK Schmidt Telescope (UKST) survey to perform color-magnitude diagram analysis. They also showed that the Sgr dwarf is in tidal disruption process and elongated toward Galactic plane. The estimated metallicity of the Sgr dwarf is [Fe/H] = −1.4 ± 0.3 (Ibata et al., 1994).

Alard (1996) studied the field around 80 from the Sgr dwarf (centered at $l = 3^0$ and $b = -7^0$). He used the data of B and R photographic Schmidt plate from DUO project, a long project monitoring around 10 million stars in a field toward the Galactic bulge. A structure associated with Sgr identified in his study in a sample of RR Lyrae stars of Bailey type ab (RRab). This structure is consistent with an extension structure found by Ibata et al. (1994) (Alard, 1996).

The field around Sgr dwarf become an interesting spot to detect the tidal stream. Mateo et al. (1998) suggested that the outer portions of Sgr resembling a stream rather than an extension of the ellipsoidal inner regions of the galaxy. This study used the photometric data from 4 m telescope at Cerro Tololo Inter- American Observatory (CTIO).

Martínez-Delgado et al. (2004) reported the two new detections of the tidal streams in the northern cap of the Sgr, located at 450 and 550 from the center of the galaxy. This study
used the analysis of wide field and deep color-magnitude diagram from Sloan Digital Sky Survey (SDSS) with 2.5 m Isaac Newton Telescope (INT).

Belokurov et al. (2006b) revealed the tidal stream of the Sgr dwarf spheroidal galaxy and the other substructures from a color cut ($g-r < 0.4$) of SDSS Data Release (DR) 5. This tidal stream was clearly visible from an RGB composite image composed of magnitude slices of the stellar density of the stars.

Koposov et al. (2012) analyzed the structure of the tidal streams using SDSS DR8. They found two streams (a brighter and a fainter stream) where both have similar distance gradient but have different morphological properties and stellar population. The brighter stream is broader, contains more metal-rich stars (also dominated by more than one stellar population), and has a richer color-magnitude diagram as compared to the fainter one (Koposov et al., 2012).

Palomar 5 (Pal 5) is a very low mass, large core radius, and low central concentration of the globular cluster. These properties make Pal 5 is vulnerable to tidal disruption process. Therefore, a tidal stream is expected to arise from this cluster.

Odenkirchen et al. (2001) firstly detected massive tidal tails around Pal 5 with SDSS commissioning data. The tidal tails were detected from the color-magnitude diagram with selection criteria. The area covered in this study is approximately 15 deg$^2$. The length of the observed tidal tails is 2.6$^0$ on the sky projection (Odenkirchen et al., 2001).

Rockosi et al. (2002) used the matched filtering technique to optimize the detection of the tidal stream of Pal 5 in SDSS data. This technique optimally extracts the signal from noise. The area covered in this study is around 41 deg$^2$. The match filtering technique was also applied in the detection of the tidal tails of Pal 5 by (Odenkirchen et al., 2003) and (Grillmair and Dionatos, 2006) using SDSS data.

Odenkirchen et al. (2009) studied the kinematics of the debris from Pal 5. The radial velocity was extracted from data using high-resolution spectroscopy on the Very Large Telescope (VLT) of the European Southern Observatory (ESO). The measured velocity dispersion is 4.7 km/s (Odenkirchen et al., 2009).

Another example of the tidal stream is NGC 5466. NGC 5466 is a metal-poor Galactic globular cluster that lies approximately 16 kpc away from the Sun. The Galactic coordinate of this cluster is $l = 42.2^0$ and $b = 73.6^0$.

Belokurov et al. (2006a) detected approximately 4$^0$ length tidal tails around NGC 5466. This study used the neural network to reconstruct the probability distribution of the cluster star in ugriz space. The five bands u, g, r, i, and z were extracted from SDSS data.
The tidal stream of NGC 5466 was also detected by Grillmair and Johnson (2006) with a larger extent. The method in this study to detect the stream is an optimal contrast filtering technique. The data was extracted from SDSS data.

1.5 The History of N-Body Simulations

The employed method in the numerical experiment or N–body simulations generally can be divided by two classes, the ‘collisional’ and ‘collisionless’, depending on the astrophysical systems. If in a system close encounters between stars are considerably important, the system should be treated as collisional and must be modelled as accurately as possible. An example of this system is a dense star cluster and the galactic centre. On the other hand, if the close interactions between stars in a systems are negligible, then this system can be treated as collisionless. In this case, the gravitational potential experienced by a star in the system as a sum of many individual point mass potential is well approximated by a smooth mean potential. The examples of this system are galaxies and large scale structure.

Since both systems have different challenges in their simulations, I will explain the historical development of the N–body simulations separately. First, I focus on the collisional systems and then on the collisionless systems.

The study of the collisional stellar dynamics using N–body model was pioneered by von Hoerner (1960). The total number of particles at that moment was up to \( N = 16 \). Later, the number of the particle increased to \( N = 25 \) (von Hoerner, 1963). The number of particles in their simulation was small because of the low-performance of the computer at that time. In his study, such simulation showed the formation of a close pair of particle or binary and escapee from a cluster.

The number of particles in N–body model increased up to \( N = 100 \) when Aarseth (1963) studied the clusters in the galaxies in his simulation. Terlevich (1980) studied the dynamical evolution of open star cluster with simulation implying the number of particles from 250 to 1000. The performance of the computer always approximately double every two years along with the number of transistors in the integrated circuit of the computer (this is called Moore’s law, e.g. Brock, 2006). This is also same for the case of the number of particles if the cost scale \( \propto N \) (Dehnen and Read, 2011).

Inagaki (1986) examined the behaviour of the core of the globular cluster by \( N = 1000 \) and \( N = 3000 \) body simulations. He showed that the core density of the cluster varied significantly. The number of particles in the simulation doubled in the year 1993 when Aarseth and Heggie studied the evolution of the primordial binaries and unequal mass stars inside the cluster. Three years later, Spurzem and Aarseth (1996) simulated \( N = 10^4 \) body
of Plummer model to investigate the core collapse\(^1\) in cluster with equal mass stars. At the same year, Makino (1991) studied the possibility of a gravothermal oscillation\(^2\) in the \(N = 2^{15}\) body systems using a parallelized special purpose computer GRAPE (GRAvity PipE)-4. The advanced generation of special purpose computer GRAPE-6 is used to run the \(N\)-body simulation for studying the evolution of multi-mass star clusters in an external tidal radius (Baumgardt and Makino, 2003). In the last decade, the number of particles to be computed in the simulation using special purpose supercomputer, like GRAPE-6A, can reach \(4 \times 10^6\) particles (e.g. Harfst et al., 2007).

The study of collisionless systems with the numerical simulations of \(N\)-body systems was started by Holmberg (1941) in his 47 light bulbs experiment to study the tidal deformation of two models of stellar systems by an interaction at a certain distance. Computer simulations with \(N = 300\) were conducted by Peebles (1970) to model the coma cluster in galaxies. A significant increase of total number of particles involving in the simulation was carried out by Efstathiou and Eastwood (1981) with \(N = 1000\) and \(N = 20000\). The purpose of this simulation is to investigate the clustering of particles in the Friedmann model of the universe. Frenk et al. (1985) studied the structure of the galactic haloes with in the size of 14 Mpc from a redshift of 6 to present day. The model in this simulation contained \(2 \times 10^{14} \, M_\odot\), which is represented by \(2^{15} \approx 32000\) particles in the universe dominated by CDM. The numerical experiment with large number of particles up to \(N = 3 \times 10^5\) was conducted by Dubinski and Carlberg (1991) to study the density profiles and shape of dark halos. A ten times larger resolution simulation \((N = 3 \times 10^6)\) has been conducted by Moore et al. (1998) to study the structure of CDM halos. The resolution is slightly increased with \(N = 5 \times 10^6\) in the simulation of Ghigna et al. (2000). Springel et al. (2005) increase the number of particles to \(N \approx 10^{10}\) in their cosmological simulation to study the evolution of cluster, galaxies, and the large scale structure in CDM model. A \(N = 7 \times 10^{10}\) simulation of dark matter particles has been performed by Teyssier et al. (2009) using the RAMSES code based on an Adaptive Mesh Refinement (AMR) technique (see e.g. Teyssier, 2002). The significant increase in total number of particles in the simulation in the last two decades was driven by the usage of highly parallelized computers.

---

1 Core collapse is a dramatic increase of the central density of cluster. This is due to a mass segregation, a phenomenon such that the heavier stars sink toward the core and the low-mass stars is pushed to the outskirt, which is driven from the accumulated many close encounter between stars.

2 After a core collapse (contraction), a hard binary (binary that has larger binding kinetic energy than the typical kinetic energy of surrounding stars) can become an input energy to expansion. If the energy production is unstable, then a gravothermal oscillation can happen (e.g. Sugimoto and Bettwieser, 1983; Bettwieser and Sugimoto, 1984).
1.6 Outline of this Thesis

This thesis consists of five chapters: the history and science introduction, time-step in \( N \)-body simulation, modelling tidal streams, tidal ribbon, and conclusion. Here I summarize the science chapters focusing on the new time-step in gravitational \( N \)-body simulations and modelling the tidal tail from the galactic satellite.

In the second chapter, I review the categorization of collisional and collisionless systems in order to place the right and effective simulation of a particular system. I also review the time integration to track the dynamics of the system for small step sizes. I introduce the time step criterion to depend on the tidal force that closely follows the dynamical time of the system. This time step function would be altered by the dynamics of the system and in contrast to other time steps which depend on potential and acceleration. Finally, I show the efficiency of this time step criterion applied to single orbit on the smooth distribution of mass and the self-gravity isolated spherical system.

In the third chapter, I review the action-angle formalism from the canonical transformation and its application to the tidal streams. This formalism reveals the structure of the tidal streams (especially the dimensionality of the stream). The formation of a circular band stream, called tidal ribbon, formed from the numerical simulation of satellite in disc orbit. I review this structure and the connection between the mathematical insight of action-angle variable. Moreover, I explore the effect of some parameters in the simulation, such as the vertical profile of the disc density and cluster potential. Most of contents in this chapter are based on the paper Dehnen and Hasanuddin (2018).

In the fourth chapter, I present technical details of the model for tidal streams in the Galactic system like the Milky Way. There are several components in this model: the satellite and Milky Way potential, the distribution function of stars in the satellite, and methods of sampling inside the satellite. The purpose of this model is to reduce the computational cost, better than those traditional \( N \)-body simulations, meanwhile preserving the nature of the stripping process of the tidal streams. I also compare this model with the ad-hoc model of the particle sprinkling methods.

The final chapter concludes the several results from the previous chapters. I also list the open questions and possible future works.
Chapter 2

New Time-step Criterion for $N$-Body Simulations

2.1 Introduction

The physical phenomena of stellar systems can be understood by $N$-body simulations. The stars or particles interact with each other by gravitational force. The stars velocity and position are updated by solving the Newton equation.

The dynamical time or orbital time of a particle in the simulation usually varies in a wide range. For this reason, we need to assign an individual time-step for a particle. The selection of the individual time-step is based on a criterion or a time-step function. The criteria can be a function of any combination of the particle quantities such as position, velocity, potential, acceleration, etc. as long as it has a unit time.

The continuous time step is not practicable for the large number of particles in the $N$-body simulation. It is advantageous to make discrete time-steps with a factor of two. This method allows the simultaneous updating of the particle velocity and position of the same step-size. This scheme is called the block time-step.

In this chapter, I will give a short review of the state-of-the-art for modelling the stellar systems, both for "collisional" and "collisionless" $N$-body simulation. I also study the new time step criterion for the simulations.

2.2 $N$-Body Simulation

The $N$-body simulation is a numerical method to solve $N$ set of Newton’s equations for the evolution of astrophysical systems containing $N$-particles which are interacting each-other by gravity. Gravitational $N$-body simulations are classified into two kinds, the collisional and the collisionless. The term collisional is not merely the direct collision, but refers to
which close interactions of each individual particles are considerably important so that it
must be modelled as accurately as possible. On the other hand, the collisionless systems
refer to which close individual interactions are unimportant and it can be treated as in
statistical sense. The quantity that can separate the two classifications is the "two-body
relaxation" time. This timescale can be approximated by (e.g. Dehnen and Read, 2011)

$$t_{relax} \approx \frac{N}{8 \ln \Lambda} t_{dyn},$$

(2.1)

where $t_{dyn}$ is the dynamical time, $\ln \Lambda = \ln(b_{\text{max}}/b_{\text{min}})$ is called the Coulomb logarithm,
and $b_{\text{max}}$ and $b_{\text{min}}$ are the maximum and the minimum impact parameters for the system.
The value $\Lambda$ is approximately in order of $N$. The equation (2.1) can be written as (Binney
and Tremaine, 2008)

$$t_{relax} \approx \frac{0.1 N}{\ln N} t_{cross},$$

(2.2)

where $t_{cross} \equiv R/v$ is the time needed for a typical star to cross the galaxy once. For
example, galaxies have typically $N \approx 10^{11}$ stars and have a relaxation time $\approx 10^9$
crossing time. Meanwhile, their lifetime is only a few hundred of crossing time. Hence, in these
systems, the stellar encounters are unimportant, except in the central part where the density
is highest. For a globular cluster with typically $1$ Myr crossing time and number of stars
$10^5$, after the $10$ Gyr of its lifetime, the relaxation process is important and affects their
structures. The other examples of collisional systems are compact planetary systems.

The stars in a collisionless system are well approximated to move under the influence
of gravitational field generated by the smooth distribution of mass rather than a collection
of mass points. The majority of stellar systems are collisionless including presumably
non-baryonic dark matter. Binaries are also collision-less, because their orbit remains un-
changed (unless they interact with other celestial bodies, or tidal effects play a role).

In this chapter, I shall consider both collisional and collisionless systems. In the re-
main ing chapters, I focus on the collisionless systems.

### 2.2.1 Time Integration

The basic equation to be solved in the $N$-body simulations is a set of $N$ of Newton Equation
or equivalently the Hamilton equation

$$\dot{p}_i = -\frac{\partial H}{\partial x_i},$$

(2.3)

The dynamical or crossing time is a characteristic orbital time scale: the time required for a significant
fraction of an orbit. An approximation of the dynamical time is $t_{dyn} = (G\rho)^{-1/2}$
and
\[ \dot{x}_i = \frac{\partial H}{\partial p_i}, \]  
\[ (2.4) \]
where \( x_i \) is the position, \( p_i \) is the momentum, \( \dot{p}_i = dp/dt = m_i a_i \) is the force exerted to \( i \)-th particle, \( m_i \) is the mass of the \( i \)-th particle, \( a_i \) is the acceleration of the \( i \)-th particle, \( \dot{x}_i = dx/dt = v \) is the velocity of the \( i \)-th particle, and \( H \) is the Hamiltonian of the systems.

\[ H = \sum_i \frac{p_i^2}{2m_i} - \Phi, \]  
\[ (2.5) \]
where \( \Phi \) is the total potential energy. For the collisional systems, the \( \Phi \) is calculated by
\[ \Phi = -G \sum_i \sum_{j>i} \frac{m_i m_j}{|x_i - x_j|}. \]  
\[ (2.6) \]
For the collisionless systems, the \( \Phi \) is the potential energy due to smooth distribution of the mass. The relation of \( \Phi \) and the density (\( \rho \)) is written via Poisson’s equation
\[ \nabla^2 \Phi = 4\pi G \rho. \]  
\[ (2.7) \]

One of the numeric methods to solve the position and velocity of particle at advance time with time-step size \( \Delta t \) via Taylor expansion is
\[ x(t + \Delta t) = x(t) + v(t)\Delta t + \frac{1}{2}a(t)\Delta t^2 + O(\Delta t^3) \]  
\[ (2.8) \]
and
\[ v(t + \Delta t) = v(t) + a(t)\Delta t + \frac{1}{2}\dot{a}(t)\Delta t^2 + O(\Delta t^3). \]  
\[ (2.9) \]
We can obtain the simplest method by taking this expansion to first order such that no derivative of the acceleration needed. This numerical method is called Euler method. In fact, the Euler method is not time reversible.

We will seek a method that has the property of time reversible and preserves certain conserved quantities exactly such as the phase-space volume and the Jacobi constants. The numerical method based on the maps that preserves the symplecticity (canonical nature) is called the symplectic integrator.

Let the set of all coordinates and momenta be denoted by \( w = (x_i, p_i) \). The time evolution for systems with Hamiltonian function \( H \) is a canonical transformation governed by the equation
\[ \frac{dw}{dt} = \hat{H} w \equiv \{w, H\}, \]  
\[ (2.10) \]
with $\{,\}$ the Poisson bracket. The operator $\hat{H}$ is a Lie operator of function $H$. The formal solution of the equation (2.10) is

$$w(t + \Delta t) = e^{\Delta \hat{H}}w(t),$$

(2.11)

where operator $e^{\Delta \hat{H}}$ acting on $w$ is a symplectic map from $t$ to $t + \Delta t$. If no exact solution to equation (2.10) exists, then the operator $e^{\Delta \hat{H}}$ has no finite expression and the numeric solution is required. If $H$ can be split as $H = T + V$, where $T$ and $V$ are kinetic and potential energy operators, then the map $e^{\Delta \hat{H}}$ can be approximate by the composition of the maps $e^{\Delta \hat{T}}$ and $e^{\Delta \hat{V}}$ (Dehnen and Hernandez 2017). The maps $e^{\Delta \hat{T}}$ and $e^{\Delta \hat{V}}$ work on the $w$ by the following expressions

$$e^{\Delta \hat{T}} \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} x + p\Delta t \\ p \end{bmatrix}$$

(2.12)

and

$$e^{\Delta \hat{V}} \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} x \\ p - \nabla V(x)\Delta t \end{bmatrix}.$$  

(2.13)

These operations are known as 'drift' and 'kick' operations, since they only change either the position (drift) or velocity (kick) (Dehnen and Read 2011). The simplest splitting operator method is

$$e^{\Delta \hat{H}} \approx e^{\Delta \hat{V}} e^{\Delta \hat{T}}.$$  

(2.14)

This symplectic method is called the modified Euler method. The evolution of system is

$$x(t + \Delta t) = x(t) + v(t)\Delta t$$

(2.15)

and followed by

$$v(t + \Delta t) = v(t) + a(t + \Delta t)\Delta t.$$  

(2.16)

The error of this method can be analyzed using Campbell-Baker-Housdorff formula (Campbell 1896, 1897; Baker 1902, 1905; Hausdorff 1906, Dynkin 1947)

$$\ln(e^X e^Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \ldots,$$

(2.17)

with $[X, Y] = XY - YX$ as the commutator, $X$ and $Y$ are operators (matrices). Using the Jacobi identity, it can be shown that the commutator of two Lie operators is a Lie operator
as well[4] Lie operator for the functions of the phase space variable is a Poisson bracket of Lie operators.

From formula (2.17), the method (2.14) evolve the system under the approximate Hamiltonian \( \tilde{H} = H + H_{\text{err}} \), and

\[
H_{\text{err}} = \frac{\Delta t}{2} \{ \hat{T}, \hat{V} \} + \frac{\Delta t^2}{12} \left( \{ \hat{T}, \{ \hat{T}, \hat{V} \} \} + \{ \hat{V}, \{ \hat{V}, \hat{T} \} \} \right) + \mathcal{O}(\Delta t^3),
\]
(2.18)
The error made in energy is \( \mathcal{O}(\Delta t) \).

The another of the splitting method is leapfrog integrator or Verlet method

\[
e^{\Delta t \hat{H}} \approx e^{\Delta t \hat{T}} e^{-\Delta t \hat{V}} e^{\Delta t \hat{T}} / 2.
\]
(2.19)

Operation of these operators on the function \( w(x, v) \) resulting

\[
v(t + \Delta t/2) = v(t) + a(t) \Delta t/2,
\]
(2.20)

\[
x(t + \Delta t) = x(t) + v(t + \Delta t) \Delta t,
\]
(2.21)

\[
v(t + \Delta t) = v(t + \Delta t/2) + a(t + \Delta t) \Delta t/2.
\]
(2.22)

This method is called leapfrog Kick-Drift-Kick (KDK). The another version is leapfrog Drift-Kick-Drift (DKD)

\[
e^{\Delta t \hat{H}} \approx e^{\Delta t \hat{V}} e^{\Delta t \hat{K}} e^{\Delta t \hat{V}} / 2,
\]
(2.23)

will evolve the system

\[
x(t + \Delta t/2) = x(t) + v(t) \Delta t/2,
\]
(2.24)

\[
v(t + \Delta t) = v(t) + a(t + \Delta t/2) \Delta t,
\]
(2.25)

\[
x(t + \Delta t) = v(t + \Delta t) + v(t + \Delta t) \Delta t.
\]
(2.26)

The KDK version is preferable in our simulation because the accelerations are known at the second-order accurate position and can be used to control the time step. This also facilitates

\[
\{ \hat{A}, \hat{B} \} f = \hat{A} \hat{B} f - \hat{B} \hat{A} f
\]

\[
= [\hat{A}, \{ \hat{B}, f \}] - [\hat{B}, \{ \hat{A}, f \}]
\]

\[
= [\hat{A}, \{ \hat{B}, f \}] + [\hat{B}, \{ \hat{A}, f \}]
\]

\[
= -[f, [\hat{A}, \hat{B}]]
\]

\[
= \{ [\hat{A}, \hat{B}], f \}
\]

\[4\text{Let } \hat{A} \text{ and } \hat{B} \text{ are two Lie operators that can work on a function } f.\]
the synchronization of force computations for all active particles in the block-step scheme (see section 2.2.2).

The leapfrog integrator evolves the system under the approximate Hamiltonian with error

\[ H_{\text{err}} = \frac{\Delta t^2}{24} \left( 2\{\dot{T}, \{\dot{T}, \dot{V}\}\} - \{\dot{V}, \{\dot{V}, \dot{T}\}\} \right) + O(\Delta t^4). \]  

(2.27)

There are only even power of \( \Delta t \) in the error which gives the leapfrog method a time reversible property.

The intermediate velocity \( v(t + \Delta t/2) \) is used as an auxiliary quantity and can be eliminated to form a Taylor expansion of position and velocity to second order accuracy

\[ x(t + \Delta t) = x(t) + v(t)\Delta t + \frac{1}{2}a(t)\Delta t^2 \]  

(2.28)

and

\[ v(t + \Delta t) = v(t) + \frac{1}{2}(a(t) + a(t + \Delta t))\Delta t \]  

(2.29)

For more explanations about the leapfrog scheme, see the recent reviews (Dehnen and Read, 2011; Dehnen and Hernandez, 2017).

2.2.2 Time Stepping Criteria

For most N-body simulations, the dynamical time may vary between days and giga-years. For example, the stars orbiting its satellite have the orbital period much smaller than satellite period orbiting Galaxy. This requires individual time step (Aarseth, 2003). Moreover, the speed of the star on an eccentric orbit varies significantly between pericenter and apocenter. Employing a short constant time-step in this situation will require enormous force calculations near to apocenter and the long one will make big error near to pericenter. Because of these arguments, we need a variable individual time-step.

Due to the variation of time-step \( \Delta t \) in space and time, the symplectic nature of the leapfrog is broken (Dehnen and Read, 2011). But, we can do almost as well as symplectic by ensuring the scheme remains time symmetric (Quinlan and Tremaine, 1990). One route to time symmetry is to solve the implicit leapfrog equations (2.28) and (2.29) using time-step

\[ \Delta t = \frac{1}{2}[T(x_0, v_0, a_0, ...) + T(x_1, v_1, a_1, ...)], \]  

(2.30)

where \( T \) is the time step function or time step criterion, which depends on the position, velocity, potential, acceleration, etc. The value of \( \Delta t \) in the equation (2.30) can be calculated if the force at \( t + \Delta t \) is known. Hence, this scheme is implicit and requires an iterative
solution (Hut et al., 1995). Since each iteration needs a force computation and all iterations need more extra force calculations, this implicit schemes are not used in practice.

A simple example of adaptive time step is by taking the geometric mean of the old and new time-step.

$$\sqrt{\Delta t_{\text{old}} \Delta t_{\text{new}}} = T(x, v, a, ...)$$  \hspace{1cm} (2.31)

where $\Delta t_{\text{old}}$ and $\Delta t_{\text{new}}$ are the time-step used to evolve "from" and "to" the arguments of $T$. Obviously, this scheme is explicit.

Due to large number of particles in the $N$-body simulation, it is advantageous to make a discrete time-step instead of continuous time-step, allowing a group of particles to be advanced at the same time. The procedure is to arrange the particles in a hierarchy of time-steps organized in powers of two, with reference to a "base time-step" $\Delta t_0$ (Makino, 1991),

$$\Delta t_k = \frac{\Delta t_0}{2^k},$$  \hspace{1cm} (2.32)

for a given time step rung $k$. In principle, any rung $k$ may be prescribed. However, it is rare for more than about 12 rungs to be populated in a realistic simulation with $N \leq 10^3$, increasing by a few rung for $N \simeq 10^4$ (Aarseth, 2003).

The hierarchy organization above is called the block time-stepping scheme. This scheme leads to significant efficiency savings because particles on the same rungs are evolved simultaneously by using parallelization. The particles can move to the higher rungs (shorter time-step) whenever they like, but they may only move to the lower rungs (longer time-step) at synchronization points where the end of the longer and shorter time steps coincide.

Figure 2.1: Schematic diagram of a block time-stepping scheme. The blue horizontal arrows denote the time-steps. The red vertical arrows denote the exchange of particles at synchronization points.
This also leads to asymmetry in time steps and presents some challenges. An attempt to construct a near time symmetric block time-step scheme is obtained by Makino et al. (2006) with some iterations to determine the time step.

The following statements are the application of equation (2.31) on the block time-stepping. We only have three possibilities to assign the new time step based on the known function $T$ since the time-step is discrete in block time-stepping scheme. They are $2\Delta t$, $\Delta t$, and $\frac{1}{2}\Delta t$. The principle is to let the equation (2.31) nearly hold at all times, i.e.

$$T^2 \leq \Delta_{old}\Delta_{new} \leq 2T^2. \quad (2.33)$$

If we try to double the time-step and the multiplication of the new time step and the old time step is less still than $T^2$, then we must really increase the old time step by factor of 2. If we try to keep the time-step and the multiplication of the new time-step and the old time-step is still larger than $T^2$, then we must decrease the time step by a factor $\frac{1}{2}$. The above statements can be written as

$$\Delta_{new} = \begin{cases} 2\Delta_{old}, & \text{if } 2\Delta_{old}^2 \leq T^2 \\ \frac{1}{2}\Delta_{old}, & \text{if } \Delta_{old} \geq T \\ \Delta_{old}, & \text{otherwise.} \end{cases} \quad (2.34)$$

Although method (2.34) also suffers from the growth of energy error in block-step scheme (I will review latter in this section), we use it for the sake of simplicity and investigation of the new time step criterion only. For the recent attempts towards time symmetry integration, one can refer to some advanced explicit schemes by Dehnen (2017).

Now, we turn to how to arrange the particles in the rungs. We need some criteria to decide which rung a particle should be. These criteria are called time-step functions $T$. There are many possible functions of properties (e.g., local density, potential, softening-length, acceleration, time derivative of acceleration, velocity, etc.) that can be choices to make the criteria as long as the functions have the dimension of time (Quinn et al., 1997; Power et al., 2003; Springel, 2005).

In this thesis, we compare the two available time-step criteria with the new time-step criterion which is suggested by Dehnen and Read (2011). The first time step criterion is the function depend on the acceleration and potential, i.e.

$$T = \eta \sqrt{\frac{\Phi}{|a|}}, \quad (2.35)$$

where $\eta$ is a dimensionless accuracy parameter. This criterion is suitable to use together with a low-order integrator like the leapfrog, which has only acceleration to play with. Applying this criterion to Kepler problem,

$$\Phi = \frac{GM}{r}, \quad (2.36)$$
Figure 2.2: The comparison of free fall time, dynamical time, Kepler criterion, and softening criterion for Hernquist (1990) (left panel), i.e. \( \rho(r) = \frac{M_a}{2\pi r (r+a)^3} \) and Navarro-Frenk-White (1996) (right panel), i.e. \( \rho(r) = \frac{\rho_0}{(r/a)^2 (1+r/a)^2} \) profiles with \( a \) is a scale radius. Both time and radius unit are adjusted such that \( G = M = 1 \).

\[
|a| = \frac{GM}{r^2}, \tag{2.37}
\]

with \( G \) universal gravitational constant, \( M \) total mass point, and \( r \) radius of particle, will give

\[
T = \frac{\eta}{\sqrt{GM}} r^{3/2}, \tag{2.38}
\]
i.e. exactly proportional to the dynamical time of the system. From now on, we call this criterion as Kepler time-step criterion. The fundamental issue arise here for time-step depends on the potential since the transformation \( \Phi \to \Phi + \text{constant} \) has no dynamical effect except to alter the time-steps. In other words, this transformation is gauge invariant but the time-step criterion is not.

The other time-step criterion is the time-step which is a function that depends on the softening force softening length \( \epsilon \), which is usually employed in cosmological simulations. This time-step criterion is written as

\[
T = \eta \sqrt{\frac{\epsilon}{|a|}}, \tag{2.39}
\]

From now on we call this criterion the softening criterion because it depends on the softening length, although it also depends on the magnitude of the acceleration. This criterion

\[\text{The softening length, } \epsilon, \text{ is used to soften the force at close encounter between two particles as the force approaches infinity if the distance between particles goes to zero. Force softening is a method used in collisionless } N \text{-body models to mitigate the unwanted effects of close interactions (collisions), which are artefacts of the simulations (as } N \text{ is much smaller in the model than in the galaxy and the typical close interaction acceleration decreases with } N \text{ like } N^{-1/3}. \]
Figure 2.3: The comparison of free fall time, dynamical time, and Kepler criterion for Kepler potential. All of these lines are overlapping each other. Both time and radius unit are adjusted such that $G = M = 1$.

comes from the free fall time $t_{ff} \propto \sqrt{r/|a|} > \sqrt{\epsilon/|a|}$ which leads the time-step to become too small. The Softening criterion seems success in cosmological simulation. For example, this criterion is used in GADGET-2 (Springel, 2005). However, this criterion will yield infrequent changes in time-step so that it is more like that of fixed time-step than adaptive time-step (see figure 2.2). Moreover, it is not directly related to the dynamical time in several problems (see figure 2.3). For example, in the case of Kepler problem, $T \propto r$, the time-step will be too small at large radii, and too large at small radii. Similar to Kepler criterion, the softening criterion is also not gauge invariant. This is because adding a constant to all acceleration does not change the dynamics (another gauge transformation), but does change the time step.

2.3 The Choice of Time-step Criteria

Both of the time-step criteria, Kepler and softening criteria, appear to work well in a wide range of the applications, at least for the special problems for which they were designed. However, there are some unsatisfying factors for these type of criteria. The change of potential by adding a constant, $\Phi \rightarrow \Phi + \text{constant}$, would alter the time-step but has no
dynamical effect. Adding a constant uniform acceleration, generated for example by an external agent, to star cluster does not influence their internal dynamics but does alter the time steps. This transformation is gauge invariant but the time-step is not. The gauge invariant property of transformation holds for the gradient of the acceleration or tidal force. The suggested time-step criterion is (Dehnen and Read, 2011)

\[ T = \frac{\eta}{\sqrt{||\nabla a||}}, \]  

(2.40)

where \(\nabla a\) is the gradient of acceleration (tensor) and \(|| \cdot ||\) denotes the matrix norm. From now on, this time-step criteria will be called as tidal criterion. Notably, if we apply the tidal criterion to a Kepler problem, it will reduce exactly proportional to dynamical time and hence recovers the Kepler criterion (see figure 2.3). Moreover, for the isolated systems with power-law mass profiles, it is very similar to the natural time-step criterion proposed by Zemp et al. (2007). Their time step criterion is

\[ T = \frac{\eta}{\sqrt{G\rho_{\text{enc}}(r)}}, \]  

(2.41)

where

\[ \rho_{\text{enc}} \equiv \frac{M(r)}{r^3} \]  

(2.42)

is the enclosed density within the radius \(r\) and \(M(r)\) is the total mass within radius \(r\) from the centre of attraction around which the particle in question is orbiting. In a general simulation, this is not trivial to estimate. The determination of the enclosed density is quite straightforward for a particle orbiting a spherical symmetric system. Unfortunately, for the \(N\)-body systems, the calculation of the enclosed density is challenging. However, Zemp et al. (2007) have developed an algorithm to determine the enclosed density within a tree-based gravity code[6].

In the following section, I study the efficiency of the tidal criterion comparing to the Kepler criterion and softening criterion.

### 2.3.1 Test of the New Time-step Criterion

In this section, I will test the efficiency of the simulation using the tidal time step criterion and compare it with the criterion of the Kepler time step and the Softening time step criterion.

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2.3.1.1 The Application to Orbit Integration

I run the simulation of a test particle orbit in a spherical potential like Kepler potential \((\Phi = -GM/r)\). The simulation starts with a test particle in the apocenter position and orbit in the eccentricity of 0.98. In this simulation, the orbit is integrated using leapfrog KDK.

The relative energy error on the single orbit is not symmetric with respect to pericenter time where the increase of error from apocenter to pericenter is more than the decrease of error on the second half of orbit (see fig. 2.4). The reason is with adaptive time-stepping, the orbit is integrated more accurately at pericenter (due to smallest step size) than at apocenter (due to largest step size). The consequence is the growth of relative energy error over long period of orbits (see fig. 2.5). The residual error after one period and a constant number of steps is \(1.83 \times 10^{-5}\) for tidal criterion and \(-9.31 \times 10^{-5}\) for softening criterion. Of course, the softening criterion make many fewer changes of step size in one orbit and hence seem suffer less from the asymmetry issues than that tidal criterion. However, the number of steps for softening criterion increase in one orbit. For the same number of steps for both criteria and in one orbit, the softening criterion need to make more larger step size and hence result a higher relative energy error. The final relative energy error of tidal criterion scheme is smaller than those softening criterion for the same steps. The simulation using the tidal criterion is more efficient than the softening criterion for Kepler orbit.

I also studied the propagation of energy error of a test particle in the Hernquist potential \(^\text{[Hernquist 1990]}\) with \(G = M = 1\) and the scale radius 1. The test particle starts at apocenter initially and orbits the center with eccentricity 0.8. The comparison of the relative energy error for three time step criteria is described in fig. 2.6.
Figure 2.5: The comparison of the relative energy error for tidal criterion (left panel) and softening criterion (right panel). The accuracy parameters for both time-step criteria are adjusted such that they have same number of steps $10^7$ in $10^4$ orbits.

Figure 2.6: The comparison of the relative energy error for tidal (red), kepler (green), and softening (blue) criterion in the Hernquist potential for 2000 orbits. The accuracy parameters are adjusted so that each simulation has same number of steps.
2.3.1.2 The Application to N-Body Simulation

To test the tidal time step in the N-body simulation, I used the leapfrog integrator and direct summation applied to a $N = 1000$ isotropic Plummer model (Plummer, 1911). I allowed the particles in the rungs with minimum rung $k = 0$ and maximum rung $k = 12$ and employed the geometric mean adaptive scheme. The Plummer type of the force softening is also used in this test with softening length $\epsilon = 0.01$.

I generate the isolated Plummer model as the initial condition. The unit used is adjusted such that $G = M = b = 1$, where $G$ is the universal gravitational constant, $M$ is the total mass of the system, and $b$ is the scale radius of the Plummer model. The virial radius of this system is 1.732. The dynamical time at the virial radius is 3.266. The initial total energy of the system is -0.151 and the kinetic and potential energy follows the virial theorem.

The simulations of the isolated system run for about 300 dynamical times at the virial radius. We monitor the relative energy error which is defined as the deviation of the total energy from initial total energy per absolute value of the initial total energy. The propagation of the relative energy error can be seen in the fig. 2.7. The total numbers of force computa-

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7The virial radius $r_{\text{vir}}$ is defined for radius at which the velocity dispersion is a maximum, i.e. $\frac{GM(r)}{r_{\text{vir}}} \approx \sigma_{\text{max}}^2$ with $M(r)$ the mass inside radius $r$. 

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Figure 2.8: The total numbers of forces (left panel) and CPU hours (right panel) vs maximum of relative energy error. Red triangle, green circle, and blue square points represent the simulations using the tidal, Kepler, and softening criterion, respectively.

The growth of the energy error is due to fact that the block-step scheme breaks the time-symmetry in the forward adaptive scheme. However, I am only interesting in the efficiency of the time step functions (criterion).

The number of forces for the tidal time-step criterion is less than those of time-step criteria for same maximum relative energy error. This means that the tidal criterion gives the best efficiency in the N-body simulation (see fig. 2.8). This is because it satisfies the dynamical time of the system. The dynamical time in the Plummer system is

\[ t_{\text{dyn}} = \sqrt{\frac{4\pi}{3GM}} \left( r^2 + b^2 \right)^{3/4}. \]  

The norm of the gradient of the acceleration can be obtained by using the equation (A.4) in the appendix A. The resulting norm of the gradient of the acceleration in the Plummer system is

\[ ||\nabla a|| = \sqrt{3GM} \left( \frac{2r^4 + b^4}{r^2 + b^2} \right)^{1/2}. \]  

Thus, the tidal step criterion (2.40) becomes

\[ T_{\text{tidal}} = \frac{\eta}{3^{1/4} \sqrt{GM}} \left( r^2 + b^2 \right)^{5/4}, \]  

(2.45)
The time-step criterion is proportional to the dynamical time at near or far from center in the spherical potential (see fig. 2.9).

This is different from the other time-step criteria. The Kepler and Softening time-step criterion are

\[ T_{\text{Kepler}} = \frac{\eta \sqrt{GM}}{r} \left( \frac{r^2 + b^2}{r} \right)^{5/4}, \tag{2.46} \]

and

\[ T_{\text{softening}} = \frac{\eta \sqrt{\epsilon \sqrt{GM}}}{r^{1/2}} \left( \frac{r^2 + b^2}{r^{1/2}} \right)^{3/4}. \tag{2.47} \]

The Kepler time-step criterion only follows the dynamical time at regions further away from the center and is not suitable at the region near the center. Meanwhile, the softening time step does not resemble the dynamical time step at both regions.

The application of the tidal step criterion in N-body simulation still suffers from the fluctuation of the time-step in the high density region due to the graininess of the mass (see fig. 2.9). This limitation can perhaps be reduced by increasing the softening length, but also suffers the deviation from the true tidal step criterion. To sum up, the tidal time-step criterion is very promising in the future.
2.3.2 Summary

The tidal force based time-step function follows closely to the dynamical time. The transformation $\Phi \rightarrow \Phi + \text{const}$ and $a \rightarrow a + \text{const}$ does not alter the internal dynamics of the system but alter significantly the potential and acceleration based time step criteria. This is not the case for the tidal force based time-step criterion. Hence, this time step follows the internal dynamics of the system.

This time-step criterion scheme is more efficient than the other time step criteria for both orbit integration and $N$-body simulations. The only disadvantage is the need for the gradient of acceleration calculation. The improvement of tidal time step criterion is still needed to reduce the fluctuation of the tidal time step function in the high density region of the system.
Chapter 3

Tidal Ribbons

3.1 Introduction

Tidal tails stars are generated by the stripping process of stars from the galactic satellites by the tidal field of their host galaxy. The tidal field acting on the galactic satellite will stretch it toward the galactic center and compress it in tangential directions. This stretching increases the opportunity of stars to be stripped especially when the satellite crosses the disc of galaxy. This is caused by the increase of internal kinetic energy during disc shock (Spitzer, 1958). Hence, less bound stars are vulnerable from stripping. The subsequent release of the stars from the satellite commonly forms long one-dimensional structures within six-dimensional phase space. Tidal tails consist of two tails, leading and trailing. The leading tail has longer orbital period than the trailing tail.

With wider and deeper large scale digital sky surveys (such as SDSS[1], 2MASS[2] and WISE[3]), the number of known stellar streams in the Galactic halo is more than a dozen. These stellar streams are mainly detected using the photometric technique alone which relies on the separation in color magnitude space of relatively low-metallicity halo stars from the far larger population of nearby foreground stars [Grillmair and Carlin, 2016]. The number of observed tidal streams is most likely to increase. The distance and velocity data from Gaia mission will uncover stellar streams undetectable from photometry alone (due

1SDSS (Sloan Digital Sky Survey) is a large area sky survey which provide both photometry and spectroscopy data. The survey uses a 2.5 m telescope equipped with a CCD (Charged-Coupled Device) camera to image the sky in five optical bands and the two digital spectographs. The wavelength of the optical bands are u’=3550 Å, g’=4770 Å, r’=6230 Å, i’=7620 Å, and z’=9130 Å (York et al., 2000).

22MASS (Two Micron All Sky Survey) is a sky survey for imaging the celestial sphere in the near infrared J(1.25 μm), H(1.65 μm), and K(2.16 μm) bandpasses. The surveys uses two 1.3 m diameter telescopes located at Mount Hopkins, Arizona and Cerro Tololo, Chile (Skrutskie et al., 2006).

3WISE (Wide-field Infrared Survey) is a sky survey for imaging sky in the infrared with passband W1, W2, W3, W4 centered at wavelength 3.4, 4.6, 12.0, and 22.0 μm using a 40 cm spacecraft telescope (Wright et al., 2010).
to low surface density).

The observed tidal streams are generally thin, dynamically cold, and hardly phase-mixed. Due to these features, tidal tails are vulnerable to disruption by dark matter sub-haloes, such as predicted by Λ Cold Dark Matter (CDM) theory. Hence, these features of tidal tails have been proposed to test the existence of small-scale dark matter sub-haloes and to constrain its distribution in the Milky-Way halo (Yoon et al., 2011; Erkal et al., 2016; Bovy et al., 2017).

In the section 3.2, I introduce and review the action angle variable concept and use this formalism to explain the long thin structure of streams (Tremaine, 1999). I also use the action angle variable concept for tidal streams in the next section to analyze the tidal streams from the result of the numerical simulations. The technical details of the simulation using the restricted N-body model for tidal streams from the Galactic satellite will be presented in the chapter 4.

In this study, I show that debris from progenitors on Galactic disc orbits forms two-dimensional structures: vertically extended ribbons around the Galaxy instead of one-dimensional structures. Such debris ribbons can be very useful for constraining the vertical disc potential, i.e. for inferring the Oort limit and the total mass distribution vertically across the disc, because they trace the full vertical extent at each visited \((R, \phi)\) position within the disc, which may included the Solar neighbourhood.

### 3.2 Action Angle Variables

The geometry of tidal streams stripped from a satellite can be described by an action angle variable formulation, first pointed out by Tremaine (1999) and Helmi and White (1999). Here, I will review the action angle and its application to the tidal stream.

Let consider the canonical transformation from a phase space \((x, p)\) to an action angle \((\theta, J)\). Here, the actions are constant. Hence, from the Hamilton equation, the Hamiltonian depends only on action (Goldstein et al., 2000),

\[
\mathcal{H} = \mathcal{H}(J) \quad \text{(3.1)}
\]

and the Hamilton equation is

\[
\frac{\partial \mathcal{H}}{\partial J} = \dot{\theta}. \quad \text{(3.2)}
\]

The equation (3.2) can be easily solved because the left hand side is constant and has a simple solution

\[
\theta = \theta_0 + \Omega t, \quad \text{(3.3)}
\]
where $\theta_0$ is initial angle and $\Omega = \frac{\partial H}{\partial J}$ is frequency vector. The angle increase linearly with time, while action and frequency remain constant.

The initial angle and action between stripped stars and progenitor are small. Because the action is constant during the orbits, the offset between debris stars and satellite, $\Delta J$, remains constant. The evolution of angle offset, $\Delta \theta$, depends on the initial angle offset, $\Delta \theta_0$, and frequency offset, $\Delta \Omega$, via the following equation.

$$\Delta \theta = \Delta \theta_0 + \Delta \Omega t.$$  \hfill (3.4)

A comparison with equation (3.3) shows that the debris behaves like an orbit, but with the angles and frequencies, $\theta$ and $\Omega$, replaced with the respective offsets, $\Delta \theta$ and $\Delta \Omega$, between debris star and progenitor. For large $t$, the second term of equation (3.4) become dominant over initial angle offset, and this equation becomes

$$\Delta \theta = \Delta \Omega t.$$  \hfill (3.5)

If $\Delta J \ll J_{\text{sat}}$, the satellite’s action, then we may approximate the frequency offset using Taylor expansion around $J_{\text{sat}}$ as

$$\Delta \Omega \approx H \cdot \Delta J,$$  \hfill (3.6)

where $H$ is a Hessian of Hamiltonian which has components

$$H_{ik} = \frac{\partial^2 H}{\partial J_i \partial J_k}.$$  \hfill (3.7)

Since $H$ is a real-valued symmetric matrix, it can be diagonalized (Tremaine, 1999). As matrix $H$ is diagonalizable, there exists a matrix $P$ such that

$$P H P^{-1} = D(\lambda),$$  \hfill (3.8)

where $D_{ij}(\lambda) = \lambda_i \delta_{ij}$, $\lambda = \{\lambda_i\}$ are the eigenvalues of $H$. Since $H$ is symmetric, then $P$ is orthogonal and $P^T = P^{-1}$. It is easy to see from equation (3.8) that $P$ is a matrix that transform the basis vector to eigen-vectors basis. Each rows of matrix $P$ is an eigen-vector of matrix $H$. If we change the basis of vector $\Delta \theta$ and $\Delta J$ to basis of eigen-vectors of $H$ via

$$\Delta \theta' = P \Delta \theta$$ and $$\Delta J' = P \Delta J,$$  \hfill (3.9)

the equation (3.5) becomes

$$\Delta \theta' = D \cdot \Delta J'.$$  \hfill (3.10)

Equation (3.10) show that the structure of the stream depends on the ratio of eigenvalues. If the modulus of one eigenvalue, $|\lambda_1|$, is much larger than that of the others two, then the
stream will expand predominantly along the direction of the associated eigenvector and forms a thin one dimensional structure. In the case of one eigenvalue is comparable to another one, while the third one is negligible small, then the debris can expand to two dimensional structure (Tremaine 1999).

With the knowledge of Hamiltonian as a function of actions only, we can determine the dimensionality of the tails. For the special case where a satellite orbiting in the Kepler potential, the Hamiltonian of the system can be written as (Goldstein et al. 2000)

\[ H = E = -\frac{2\pi^2 (GM)^2}{(J_r + L)^2}, \]  

(3.11)

with \( J_r \) the radial action and \( L = J_\theta + J_\phi \) the total angular momentum (sum of angular momentum in \( X - Y \) and \( Z \) direction). The motion is completely degenerate, i.e. all of the frequencies are equal (Goldstein et al. 2000). The frequency of the motion is

\[ \Omega = \frac{\partial H}{\partial J_r} = \frac{\partial H}{\partial L} = \frac{4\pi^2 (GM)^2}{(J_r + L)^3} = \frac{(-E)^{3/2}}{\pi GM}. \]  

(3.12)

Hence, the frequency vectors of the streams including the satellite are uniform (see Fig. 3.1).

The general case for the Kepler potential is a system with the Hamiltonian as a function of linear combination of the actions.

\[ H = f(\omega \cdot J), \]  

(3.13)

with constant vector \( \omega \) and arbitrary function \( f \). In this situation, the energy surfaces in actions space are plane parallel, the frequency \( \Omega = f' \omega \), and the Hessian of the Hamiltonian

\[ H_{jk} = f''(J_r)\omega_k. \]  

(3.14)

This has one non-zero eigenvalue, \( f'' \omega^2 \) and the corresponding eigenvector is \( \omega \), which is same direction as frequency vector. Since \( \omega \) is constant, then the frequency offset of streams have same directions and form one-dimensional structure aligned with an orbit.

For any spherical potential, the action \( J_\theta \) and \( J_c \) can always be combined into one action, \( L \), which is the total angular momentum. Hence, the Hamiltonian depends only on \( J_r \) and \( L \). For example, a galaxy potential with power law circular velocity,

\[ v_{\text{circ}}(r) = v_0(r/r_0)^\beta, \]  

(3.15)

with \( r_0 \) and \( v_0 \) are the scale radius and velocity at scale radius, respectively, the radial action can be approximated by (Dehnen 1999)

\[ J_r = \varphi^{-1}(L_c - L)\left(1 - \zeta L_c^{-1}(L_c - L)\right), \]  

(3.16)
Figure 3.1: The sketch of frequency vector (black arrows) of tidal streams and its progenitor in the Kepler potential. The frequency vectors are computed from equation (3.12). The largest blue circle denotes progenitor and others blue circle denote the debris. The red arrows represent the frequency offset from its progenitor. The black lines are the energy contours from equation (3.11).
Figure 3.2: The orbit for 47 Tuc over the last Gyr in the $X$-$Y$ and $R$-$Z$ (meridional) planes (note the different scales). $X$, $Y$, and $Z$ are the axes of the Galactic coordinate system. $X$ points the Sun location. $Z$ points the Galactic north pole. $Y$ direction is the right angle of $X$ and $Z$. The 47 Tuc current position is $(X, Y, Z) = (-6.63, -2.31, -2.84)$ kpc. For orbit computation, the velocity of 47 Tuc in this coordinate system is $(V_X, V_Y, V_Z) = (-76.24, 160.91, 32.67)$ km/s.

with $\zeta \equiv \frac{1}{2}(2-\varphi)(\varphi-1)\varphi^{-2}$, where $\varphi = \kappa/\omega = \sqrt{2(\beta+1)}$ is the ratio of epicycle frequencies and $L_c = L_c(E)$ the angular momentum of the circular orbit at given energy. The expression for the angular momentum of circular orbit for these power-law models is

$$L_c = \frac{r_0}{(v_0)^{1/(2\beta)}} \left( \frac{2\beta E}{1 + \beta} \right)^{(1+\beta)/(2\beta)}. \quad (3.17)$$

The implied Hamiltonian is in the form of

$$H = \frac{v_0^2}{2(1-2\varphi^-2)} (r_0 I)^{2/(1-2\varphi^-2)}, \quad (3.18)$$

with the function $I$ [Dehnen and Hasanuddin, 2018]

$$I = \frac{1}{2(1-\zeta)} \left[ \varphi J_r + (1-2\zeta)L + \sqrt{\varphi^2 J_r^2 + 2\varphi(1-2\zeta)J_r L + L^2} \right]. \quad (3.19)$$

The deviation of $H$ from the form $(3.13)$ in spherical potential implies that the Hessian matrix has a second non-zero eigenvalue ($\lambda_2$) and also that the angle between progenitor frequency and eigenvector of largest eigenvalue ($\lambda_1$) are not parallel, but deviate by some angle $\theta$. For near flat rotation curve of spherical galaxy, the ratio of second eigenvalue and the largest eigenvalue ($\lambda_2/\lambda_1$) $\lesssim 0.03$ and angle $\theta \lesssim 3^\circ$ [Dehnen and Hasanuddin, 2018]. Hence, the stream is still considering as long stream that only slightly misaligned from an orbit.
3.3 Tidal Tails from a Cluster on a Disc Orbit

I run a simulation of the tidal debris from a progenitor with a mass of \(10^6 M_\odot\) orbiting an axisymmetric potential of the Milky Way (Allen and Santillan [1991]) from 5 Gyr in the past to present time on the orbit of the globular cluster 47 Tuc. The Allen and Santillan (1991) potential has three components (bulge, disc, and halo). The bulge and disc component are modelled by Plummer model and Miyamoto-Nagai model, respectively. The halo component is represented by spherical isothermal potential. In this potential, the circular speed at the sun galactic radius is 220 km/s. The halo potential is (Allen and Santillan [1991])

\[
\Phi_h(r) = -\frac{M(r)}{r} - \frac{M_h}{1.02 a_h} \left[ -\frac{1.02}{1 + (r/a_h)^{1.02}} + \ln(1 + (r/a_h)^{1.02}) \right]^{100},
\]

where the function \(M(r)\) is

\[
M(r) = \frac{M_h(r/a_h)^{2.02}}{1 + (r/a_h)^{2.02}}.
\]

\(M_h\) is the total mass of halo, and \(a_h\) is the scale radius of halo.

47Tuc (NGC104) is a second brightest globular cluster after \(\omega\) Cen in the constellation of Tucanae. It can be observed near the Small Magellanic Cloud in the sky projection. Its center has current position (J2000) of R.A. = 00°.24′.05.67″ and dec = −72°.04′.52.6″ in the sky projection with distance 4.5 kpc from Sun (Harris, 1996). The proper motion of 47Tuc is \((\mu_\alpha \cos \delta, \mu_\delta) = (7.26 \pm 0.03, -1.25 \pm 0.03)\) mas/yr and it was derived from best-fitting line among multi epoch near infrared observations of VISTA survey of the Magellanic cloud (Cioni et al., 2016). The mass of the cluster is around \(1 \times 10^6 M_\odot\) and it was derived from a monte carlo simulation of Giersz and Heggie (2011). The core radius of the cluster is 0.36 arcmin with King model central concentration \(c = 2.07\) (Harris, 1996). The age of this cluster was estimated around 13.06 Gyr old (Marín-Franch et al., 2009; Forbes and Bridges, 2010). The metallicity of this cluster is \([Fe/H] = -0.72\) (Harris, 1996).

The cluster is modelled as Plummer sphere with scale radius 5 pc. The orbit has eccentricity 0.12 and has inclination angle 23° from disc mid-plane. The resulting orbit over the last Gyr is shown in Fig. [3.2]. Debris are represented as test particles initially sampled onto weakly bound orbits within the progenitor, thereby correctly modelling the stripping process. The numerical methods and initial conditions are described in detail in Chapter 4.

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3.3.1 The Structure of the Tidal Tails

The spatial structure of the simulated tidal tails is presented in Fig. 3.3 for the $X-Y$ and $R-Z$ projections and the top two panels of Fig. 3.4 for $R-\phi$ and $Z-\phi$ projections. In this simulation, the sun is located at $X, Y, Z = (-8.33, 0, 0)$ kpc. In the Galactic-plane projection ($X - Y$ or $R - \phi$), the debris fills a narrow region near the cluster orbit but the azimuthal extent of one radial period is slightly shorter than that of cluster orbit. This condition due to $\Delta \Omega_R / \Delta \Omega_\phi$ is approximately same for all tail stars, but slightly greater than $\Omega_R / \Omega_\phi$ for the cluster orbit in the equation (3.5).

On the other hand, the appearance of tails in vertical extents are completely different as shown in the $R-Z$ (right panel of Fig. 3.3) and $Z-\phi$ (the second panel from top of Fig. 3.4), which do not follow an orbit at all. Instead, the tails diverge out and fill a vertical band with similar extent to that of cluster. Within that band, the distribution of tails stars is not uniform and the centroid of the distribution has $\Delta \Omega_Z / \Delta \Omega_\phi \sim 1.3(\Omega_Z / \Omega_\phi)_{cl}$. The formation of this band corresponds to a widening of the distribution of the ratios $\Delta \Omega_Z / \Delta \Omega_\phi$. This can be understood as a consequence of the fact that the disc potential is vertically highly anharmonic, such that a small difference in the vertical action or energy translates into a large difference in the vertical frequency. I will investigate this point in more detail in section 3.3.2 and 3.3.3.

The tidal tails show very little internal substructure or clumping, except very close to

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4 The galactic coordinate system use axes $X, Y, Z$ with $R = \sqrt{X^2 + Y^2}$. Meanwhile, the cluster centered coordinate system use axes $x, y, z$. The azimuthal angle is defined as $\phi = \arctan(Y/X)$

5 The subscript cl denotes the cluster.
the cluster. This can be seen in the distribution over $\Delta \phi$ (shown in the bottom panel of Fig. 3.4), which also shows a smooth variation caused by a higher particle density near apo-centre (note that outside the thin vertical lines in Fig. 3.4 the model is incomplete, missing stars lost prior to the start of the simulation). The streaky structures can be seen near to the cluster (clearly shown in trailing tails). This streaky structure has been reported previously in the simulation by Dehnen et al. (2004). Each of streak related to the stars that have lost at the same disc shock. This streaky feature occasionally emerges when the satellite at the apogalacticon Dehnen et al. (2004). Unsurprisingly, the escape of stars is enhanced at crossings of the disc mid-plane, where the tides are strongest (see the right panel of Fig. 3.4).

Fig. 3.5 displays the distributions of the tail stars over the offsets of energy and actions relative to the cluster orbit. The trailing and leading tails are clearly distinguished by their energy: $\Delta E < 0$ for the leading and $\Delta E > 0$ for the trailing tail. The offset in the actions can have either sign for both the trailing and the leading tail, though for $\Delta J_Z$ and $\Delta J_\phi$ there is a clear trend of negative (positive) offsets for the leading (trailing) tail. However, the offset in the combined action $J_\phi + J_Z$ (which generalizes the total angular momentum) is much better correlated with $\Delta E$ than either $\Delta J_Z$ or $\Delta J_\phi$ (bottom panel in Fig. 3.5).

Remarkably, there are some extra-tidal particles with orbital energy very similar to that of the cluster ($\Delta E \sim 0$ in Fig. 3.5). Most of these particles are still within $\sim 0.7$ kpc from the cluster, which influences their orbits and affects $\Delta \mathbf{J}$ including the possibility of re-capture. Such extra-tidal particles with $\Delta E \sim 0$ have not been previously reported. This is only found in simulations of tidal disruption by disc shocks along low-inclination orbits, when the predominant tidal acceleration is perpendicular to both the orbit and the direction to the Galactic centre.

### 3.3.2 Effect of varying the vertical disc mass profile

As discussed above, the vertical smearing of the tidal debris implies a large spread in $\Delta \Omega_Z$, which can be understood in terms of the highly anharmonic vertical motion caused by the strong vertical concentration of the Galactic disc. In order to test this hypothesis, I have simulated tidal tails for Galactic models with various vertical concentrations by varying the parameters of the model for the disc potential

$$\Phi_d(R, Z) = -\frac{GM_d}{\sqrt{R^2 + (a + \sqrt{b^2 + Z^2})^2}}$$

(Miyamoto and Nagai, 1975). For $a = 0$, this obtains a spherical Plummer (1911) model with scale radius $b$, while $b = 0$ yields a razor-thin Kuzmin (1956) disc with scale radius $a$.37
Figure 3.4: Tidal tails formed from our model (grey distribution) for a cluster on the orbit (blue line) and at the present location of 47 Tuc. The top three panels show 2D distributions of tail stars over azimuthal offset $\Delta \phi$ from the cluster and, from top to bottom, cylindrical radius $R$, vertical height $Z$ above the disc, and time $t_{\text{esc}}$ of escape from the cluster. The thin horizontal lines indicate disc crossings of the cluster and the blue curves in the top two panels its orbit. The 1D histograms on the bottom and right give the distribution of tail stars over $\Delta \phi$ and $t_{\text{esc}}$. Owing to the finite simulation time, the model is only complete for $\Delta \phi$ between the thin vertical lines.
Figure 3.5: Distributions of tail stars over offsets from the cluster in orbital energy (with \( \Delta E < 0 \) and \( > 0 \) for the leading and trailing tail, respectively), actions \((L_Z = J_\phi, J_Z = J_\theta, J_R)\), and the combined angular action \(|J_\phi| + J_\theta\) (which generalizes total angular momentum) for the same simulation as Figs. 3.4. \( J_R \) and \( J_Z \) have been estimated via the Galpy code (Bovy, 2015) using the Stäckel approximation (Binney, 2012), giving 4% and 0.4% rms deviations from their respective orbital means.
Figure 3.6: The spatial distribution of tails stars (as in the top two panels of Fig. 3.4) for different values of the parameters $a$ and $b$ of the disc potential (3.22) as indicated (Fig. 3.4 corresponds to $b = 0.25$) at constant radial scale $a + b = 5.5678$ kpc. The smaller $b$ the thinner the disc, while $a = 0$ gives a spherical Plummer potential.
In any case, the potential in the disc midplane \((Z = 0)\) is identical and only depends on the sum \(a + b\), which acts as scale radius. The nominal Galaxy model uses \(b = 0.25\) kpc and \(a = 5.3178\) kpc (Allen and Santillan [1991]).

Fig. 3.6 shows the appearance of the tidal tails around 47 Tuc for different choices of \(b\) at the same \(a + b\) (and hence the same rotation curve). There are two trends with thinner discs (decreasing \(b\)): the vertical fanning of the tails increases and the tail centroid deviates more strongly from the cluster orbit. At the same time, the appearance of the tails in the plane (the \(R\) vs. \(\phi\) plots) is only mildly affected by the choice of \(b\). In other words, the mean and spread of \(\Delta \Omega_Z/\Delta \Omega_\phi\) is increased but hardly that of \(\Delta \Omega_R/\Delta \Omega_\phi\).

This points to a larger spread in \(\Delta \Omega_Z\) with decreasing \(b\), which is expected from

\[
\Delta \Omega_Z \approx \frac{\partial \Omega_Z}{\partial J} \cdot \Delta J \sim \frac{\partial \Omega_Z}{\partial J_Z} \Delta J_Z \tag{3.23}
\]

because of a larger \(\partial \Omega_Z/\partial J_Z\) for smaller \(b\). For example, a one-dimensional analysis (assuming the vertical motion decouples from the horizontal motion) obtains \(\partial \Omega_Z/\partial J_Z \propto J_Z^{-4/3}\) (see section 3.3.3) for a razor-thin disc. This implies strong fanning for tails from satellites on orbits with large vertical extent compared to the disc scale height. For a spherical galaxy, on the other hand, \(\Omega_Z = \Omega_\phi\), such that in \(Z\) vs. \(\phi\) plots the tail should resemble an orbit, in agreement with the bottom panel of Fig. 3.6.

We can also see more substructure near the cluster (at \(\Delta \phi \sim 0\)) for rounder (larger \(b\)) models. Another trend with increasing \(b\), i.e. thicker discs, is a smaller number of tail stars with \(\Delta J \sim 0\) (see Fig. 3.5), i.e. orbits very close to that of the cluster.

### 3.3.3 Action in thin disc potential

The disc-potential in the simulation can be categorized as a thin disc. On near-circular disc orbits, the vertical gravitational force is dominated by the attraction of the local Galactic disc. If we approximate the disc as razor thin, then the potential at fixed \(R\) is

\[
\Phi(R, z) \approx \Phi(R, 0) + 2\pi G \Sigma(R)|z|, \tag{3.24}
\]

with \(\Sigma(R)\) the surface mass density of the disc. For the one-dimensional vertical motion in this potential, the energy in vertical motion

\[
E_z = \frac{1}{2}(3\pi^2 G \Sigma J_z)^{2/3} \tag{3.25}
\]

and the frequency in \(Z\)-direction

\[
\Omega_z = \frac{dE_z}{dJ_z} = \frac{1}{3}(3\pi^2 G \Sigma)^{2/3} J_z^{-1/3}. \tag{3.26}
\]
The most important property of these relations is the non-linear dependence on the vertical action $J_z$. Even if the assumptions made here are not fully correct, the basic fact that $\Omega_z$ is a strong function of $J_z$ for orbits with vertical extent in the range $\sim 0.4-4$ kpc remains valid (Dehnen and Hasanuddin, 2018). The Hamiltonian for a thin disc can be approximated to be of the form

$$H = f(\varphi J_R + |J_\phi| + j^{1/3} J_z^{2/3}),$$

with $j$ an appropriate constant. If we ignore the slight curvature of the energy surfaces in the $J_R-J_\phi$ plane, we expect from this Hamiltonian with appropriate constant $j$, that the Hessian is approximately

$$H_{ik} \propto \Omega_i \Omega_k + \alpha_i \alpha_k,$$

with $\alpha \sim \Omega_z \hat{z}$. For $\Omega_r : \Omega_\phi : \Omega_z = 1.5 : 1 : 4$, this Hessian obtains $|\lambda_2/\lambda_1| = 0.25$ and $\hat{e}_2 \cdot \Omega/|\Omega| = 0.46$, i.e. a substantial second eigenvalue with eigenvector far from orthogonal to $\Omega$, and hence likely to be projecting onto most debris’ $\Delta J$ (Dehnen and Hasanuddin, 2018). The distribution of frequency vectors of debris in the actions space can be seen in Fig. 3.7.

In this simulation, the numeric values of actions and frequency of progenitor (estimated via the Bovy GALPY code) are

$$J_R, J_\phi, J_z \approx (12.73, 1242.79, 179.95) \text{ kpc kms}^{-1}$$

Figure 3.7: The sketch of frequency vector (black arrows) of tidal streams and its progenitor on thin disc. The red arrows represent the frequency offset from its progenitor. The black lines are the energy contours from equation (3.27).
Given the above actions and frequencies for progenitor and debris, we can calculate the Hessian matrix via linear fit from equation (3.6). The numeric eigenvalues of the resulting \( \mathbf{H} \) are

\[
\lambda_1, \lambda_2, \lambda_3 \approx (-0.106, -0.033, 0.0011) \text{ kpc}^{-2}
\]  

and the corresponding eigenvectors are

\[
\hat{e}_1 \approx \begin{pmatrix} 0.43 \\ 0.37 \\ 0.82 \end{pmatrix}, \quad \hat{e}_2 \approx \begin{pmatrix} 0.73 \\ 0.38 \\ -0.56 \end{pmatrix}, \quad \hat{e}_3 \approx \begin{pmatrix} 0.52 \\ -0.84 \\ 0.11 \end{pmatrix}
\]  

The projection of eigenvectors to the direction of progenitor’s frequency is

\[
|\hat{e}_{1,2,3} \cdot \hat{\Omega}| \approx 0.95, 0.3, 0.03.
\]  

These values are not far from our simple model based on the thin disc approximation.

In addition, the projection of frequency offset onto eigenvectors in this simulation can be seen in the Fig. 3.8. In this figure, the debris form two sheet-like regions, one for leading and another for the trailing stream, in three dimensional \( \Delta \Omega \) space, most likely perpendicular to \( \hat{e}_3 \). As a consequence, the debris spread out vertically instead of thin stream. On contrary, the debris form thin structure and stretch in direction \( \hat{e}_1 \) in the spherical potential (bottom panel of Fig. 3.8).

We can analyze quantitatively the dimensional structure of debris by using the Principal component analysis (PCA) method (see Appendix B for more description). Applying this method to the frequency data alone in the thin disc potential for 47Tuc orbit, the ratio of the variances of the first and second principal component is only 57:1. Meanwhile, for the debris in the spherical potential, the ratio of the variances of the first and second principal component is 62 times larger than for the thin disc potential. From these values, it can be deduced that the frequency vectors strongly point to one preferable direction in the case of spherical potential compared to a thin-disc potential. Hence, the debris spreads essentially in only one direction for a spherical potential, but not for a disc orbit in a thin-disc potential.
Figure 3.8: The projection of the frequency offset of debris onto the eigenvectors $e_1$, $e_2$, and $e_3$ of the Hessian matrix for Hamiltonian in a thin-disc (top panel) and spherical potential (bottom panel).
Figure 3.9: As the top two panels of Fig. 3.4 (or the $b = 0.25$ kpc case in Fig. 3.6) but for a cluster five times less massive.

### 3.3.4 The Cluster Potential

Plummer model for the cluster has two parameters: total mass $M_{cl}$ and core radius. A change in core radius has little effect on the cluster potential at the tidal radius, but directly affects the number of stars at that radius and hence liable to be tidally stripped. A more concentrated cluster has fewer stars in the tail, but their distribution is very nearly the same as for a less concentrated cluster.

A change in the cluster mass, on the other hand, affects the tidal radius and the offsets $\Delta E$ and $\Delta J$ of orbital energy and actions between tails stars and cluster. Since

$$\Delta J \cdot \Omega \approx \Delta E \approx r_{\text{tid}} |\nabla \Phi_{\text{MW}}| \propto r_{\text{tid}} \propto M_{cl}^{1/3},$$

(Johnston [1998], Dehnen et al. [2004], Sanders and Binney [2013a]) the offsets $\Delta E$ and $\Delta J$ are scaled by factor $M_{cl}^{1/3}$. I have run a model identical to that presented in Section 3.3.1, except for a cluster mass of $M_{cl} = 2 \times 10^5 M_\odot$, five times less than before. The distributions in $\Delta E$ and $\Delta J$ are indeed essentially identical to those of the more massive cluster (Fig. 3.5) except scaled down by $5^{-1/3} \approx 0.6$, as expected (see Fig. 3.10).

The resulting spatial distributions of the tail stars, shown in Fig. 3.9, are very similar to those for the more massive cluster, except that they don’t extend as far in $\Delta \phi$. This is exactly what one expects from the relation $\Delta \Omega = H \cdot \Delta J$, the scaling of $\Delta J$ by $5^{-1/3}$ results
Figure 3.10: Similar to Fig. 3.5, except the cluster mass 5 times less than those model.
in the same scaling for $\Delta \Omega$ and hence also for the extent of the tails ($\Delta \theta$), but does not affect the distributions of the directions of $\Delta \Omega$, which determines the shape of the tails.

### 3.4 Discussion and Summary

In this chapter, I have run simulations to show numerically that the disruption satellite in the thin-disc potential orbiting in low inclination forms a tidal ribbon. The analytical insight of the formation process of such a ribbon has been demonstrated for orbit with vertical extent in the range 0.4-4 kpc (Dehnen and Hasanuddin, 2018). This phenomenon is solely due to the properties of the Galactic gravitational field (and implied Hamiltonian), regardless of the properties of the satellite. For example, a less massive satellite in the simulation shows same spatial distribution except have less extent due to smaller drift rate $\propto M_{cl}^{1/3}$ (see Fig. 3.9).

These ribbons are hardly detectable using the stream finding methods, for example, Great Circle Cell Count (GC3) method (Johnston et al., 1996). The GC3 method assumes that the stars belonging to a tidal stream lie in a great circle band of the galactic sky-projection. This method is suitable for conventional streams which have a smaller region extent. On the contrary, the tidal ribbons which are two dimensional entities in six dimensional space spread in the larger regions. However, the ribbons in the wider solar vicinity should be detectable by searching the similarity of the actions in the distribution of the stream in the space of energy and two angular momentum which is called ‘integral of motion’ space and is used to find halo sub-structures (Helmi et al., 2017; Myeong et al., 2017).

Because the ribbons trace the vertical displacement and velocity of each $(R, \Phi)$ position in the disc, the detection of such a feature puts a strong constraint on the vertical structure of the Galactic mass distribution at each $(R, \Phi)$. Moreover, as shown in Fig. 3.11, the debris remarkably traces the surface of the constant vertical energy $E_z = \frac{1}{2} v_z^2 + \Phi(R, z)$. This allows one to estimate the $\Phi(R, z)$ from such a plot.
Figure 3.11: The distribution of tidal streams in the $Z - V_Z$ projection. The black curves denote the contours of the constant vertical energy.
Chapter 4

A Restricted $N$-body Model for Tidal Debris

4.1 Introduction

The observed tidal streams are an important tracer of the galactic potential since they are cold and hardly phase mixing. They also can be used to infer the shape of the Galactic halo (Koposov et al., 2010; Bowden et al., 2015; Bovy et al., 2016).

In order to interpret the observed stellar stream, some modelling are required. The most precise numerical models are $N$-body simulations of the satellite orbiting in a model for the Galactic potential (Dehnen et al., 2004; Fardal et al., 2013). Although this modelling is accurate, there is a high cost to pay off, especially when the investigation involves a large parameter space (of orbits and models for satellite and Galaxy). Therefore, alternative methods for modelling stellar streams are required.

Various authors have used methods based on action-angle formalism (Dehnen and Hasanuddin, 2018; Helmi and White, 1999; Eyre and Binney, 2011; Sanders and Binney, 2013a,b; Sanders, 2014; Bovy, 2014). This formalism has the advantage of providing a clear conceptual picture for the stream dynamics and thus enabling instructive insights. In the previous chapter, this formalism was used to give an analytical description of the formation of tidal ribbons in disc orbit. However, this approach has some shortcomings when it comes to practical modelling. First, it can be only applied to Galaxy potentials with regular orbits. If the system includes massive perturbers (e.g. the Magellanic clouds) and rotating components (spiral and bar), this modelling is unavailable. The second problem is the difficulty to find an appropriate distribution over actions and stripping times of the stream stars but see for some attempts (Eyre and Binney, 2011; Bovy, 2014).

To overcome the limitation of the angle-action approach to near-regular Galactic potentials, the orbit integration can be used instead. This approach is equivalent to the angle-
action formalism although it lacks a clear conceptual picture. However, this approach suffers from the second problem. In order to obtain the accurate distribution of debris stars (over action, initial phase, stripping time), we need to make a model of how the stars are stripped from their progenitor. The recent orbit integration method have been proposed by various authors with the nearly similar approach \cite{Lane2012b, Gibbons2014, Bonaca2014, Fardal2015, Amorisco2015}. The procedures are integrating the progenitor’s orbit and at same time releasing the star particles at regular intervals in its phase-space vicinity. The released star particles are then integrated further with the option of including the progenitor’s potential. Hence, this model governs a restricted $N$-body model of the debris\footnote{The term “restricted N-body model” is derived from “restricted 3-body model”, where a test particle (i.e. whose gravity on the other two is neglected) is added to two particles, orbiting on an elliptic orbit. Here, all the $N$ stars are such test particles, while the only gravitating ’particles’ are the cluster and galaxy, whose mutual orbit is integrated numerically.}. This approach is called the ’particle-sprinkling’ method. Crucially, in this technique the internal dynamics of the stream progenitor are not modelled, which make it much ($\sim 10^{3-4}$ times) faster than full $N$-body models.

Certainly, the self-gravity of the stellar stream is completely unimportant. The limitation of this method is the degree to which the particle sprinkling prescription reflects the actual stripping process. In practice, a leading and trailing stream are created by the release of particles at the two Lagrange points close to the satellite with velocity close to that of the satellite. This method is quite easy to implement, but requires several ad-hoc prescriptions for the details of the sprinkling process. In this prescription, one needs an exact way to choose the initial position and velocity of released particles. Moreover, one also needs to determine the rate of sprinkling which of course varies along the satellite orbit.

For many purposes, these details are unimportant and the particle sprinkling method is fully adequate. For example, when using tidal streams to constrain the overall shape and mass of the Milky Way’s dark-matter halo \cite{Gibbons2014, Bonaca2014}, only the large-scale properties of the streams are relevant but not their detailed properties. However, when comparing to the detailed structure of the observed streams, such as the distribution of density enhancements, particle-sprinkling models are inappropriate (although they have been used for this purpose), because such details are bound to be affected by the stripping process, which remains inadequately modelled.

By comparing to $N$-body simulations, one may calibrate the particle sprinkling to reflect the actual stripping process \cite{Lane2012b, Fardal2015}, but such calibration is only ever valid for the particular system modelled, since the stripping process can vary substantially between different satellite systems and orbits. For massive globular clusters, for example, the stripping is limited by the rate of evaporation of stars which are then
skimmed by the tidal field at moments of strong tides, i.e. during passages of the Galactic disc. For tidally shredded satellite galaxies, on the other hand, the stripping is much more dramatic in terms of the relative mass loss, occurs more evenly, though enhanced near peri-centre, and the decaying satellite mass implies that the initial action offset from satellite of the stream stars diminishes with time.

In this study, a novel method is presented which avoids the ad-hoc modelling of the stripping process inherent to particle sprinkling methods. Instead of sampling the stream stars outside the progenitor (hereafter simply ‘cluster’), i.e. after they have been stripped, we sample (potential) stream stars inside the cluster before they will be stripped. The stripping process is then modelled by the subsequent restricted $N$-body simulation, including the correct form of the time-variable tidal field. This method is conceptually very similar to the model of NCG 205 by (Howley et al., 2008). The main difference is that we only sample stars on weakly bound orbits from which they are liable to being stripped, which avoids the unnecessary (and costly) modelling of stars that are never stripped. Since the periods of such orbits within the progenitor are only slightly shorter than that of the satellite’s orbit within the host galaxy (by the condition for the balance of tidal and internal forces), the computational cost are only somewhat higher than for particle sprinkling and still considerably less than those for a full $N$-body simulation.

4.2 The Modelling Method

For modelling the stream that stripped from satellite, we need to model some properties and a mechanism of sampling. The ingredients of our model approach are:

1. a model $\Phi_{MW}$ for the gravitational potential of the Milky Way

2. a model $\Phi_{cl}$ for the gravitational potential of the satellite

3. a model $f_{cl}$ for the distribution function of stars within the satellite

4. a prescription for sampling potential tail stars

The first three are also required for the traditional $N$-body model, when $\Phi_{cl}$ and $f_{cl}$ are used to create the initial conditions. In our approach, $f_{cl}$ is used to for sampling potential tails stars, while $\Phi_{cl}$ is used in conjunction with $\Phi_{MW}$ to integrate the particles. In principle our model is not restricted to static model for Galaxy and satellite, but may include time dependency in both, for example due to tidal disruption of the satellite.
4.2.1 Model for Galaxy and Satellite Potential

From the potential theory, we can model the Galaxy as a smooth potential with different components (bulge, disc, and halo). I choose the Allen and Santillan model for this potential [Allen and Santillan (1991)]. In this model, the bulge and disc component are modelled by Plummer model and Miyamoto-Nagai model, respectively. The halo component is represented by spherical isothermal potential. In this potential, the circular speed at the sun galactic radius is 220 km/s.

I also include the satellite potential such that the debris can be recaptured and re-released again. For this potential, we use the ergodic plummer potential which has two parameters, total mass and scale radius.

4.2.2 Model for Distribution Function of Stars Within the Satellite

The globular clusters are generally in the spherical shape [Binney and Tremaine (2008)]. In this model, the stars initially sampled inside the cluster with isotropic Plummer distribution function [Dejonghe (1987)]

\[ f_{cl} = \frac{24 \sqrt{2}}{7\pi^3} \frac{b^2}{G^5 M^4} (-\varepsilon)^{7/2}, \]  

(4.1)

where \( G \) is the gravitational constant, \( M \) is the total cluster mass, \( b \) is the scale radius, and \( \varepsilon \) is the internal energy of stars \(^2\).

4.2.3 Sampling Methods

The sampled stars subsequently released inside the cluster with a constant rate on weakly bound energy \((\varepsilon > \varepsilon_{\text{cut}})\). \( \varepsilon_{\text{cut}} \) is a cluster internal energy threshold for sampling particle orbits. There are two opposing conditions for \( \varepsilon_{\text{cut}} \): (i) all orbits which may escape from the cluster should be sampled, but (ii) orbits which never escape (and do not contribute to the tails) should be avoided (to reduce unnecessary computations of their cluster-internal orbits).

The main procedure of sampling the stars inside the cluster are

1. draw \( \{x, v\} \) from \( f_{cl}(x, v) \) (see in appendix C);
2. if \( \varepsilon < \varepsilon_{\text{cut}} \) repeat step (1);
3. replace \( \{x, v\} \) with the peri-centre values of the same orbit;

\[^2\varepsilon \equiv \frac{1}{2} v^2 + \Phi_{cl}(x)\], the subscript cl refers to cluster.
Figure 4.1: The change of cluster-internal energy plotted against its initial value for all particles after a single disc crossing of a cluster of mass $M_{\text{cl}} = 10^6 M_\odot$ at $R = 5.9 \text{ kpc}$ (peri-centre of the orbit adopted for 47 Tuc) with $V_{Z,\text{cl}} = 145 \text{ kpc/Gyr}$. The blue line indicates $\Delta \varepsilon = -\varepsilon$, such that particles above this line become unbound, while the red circle marks $\varepsilon_{\text{esc}}$, the smallest value for $|\varepsilon|$ of any unbound particle.

(4) add the satellite’s current galacto-centric position and velocity.

A critical parameter in this procedure is $\varepsilon_{\text{cut}}$. I experimented with criteria involving angular momentum, but found a simple energy threshold to work best.

$$\varepsilon_{\text{cut}} = 2\varepsilon_{\text{esc}},$$

(4.2)

where $\varepsilon_{\text{esc}} < 0$ is the highest energy such that no star with $\varepsilon < \varepsilon_{\text{esc}}$ escapes the cluster over one orbital time scale. The justification of Equation (4.2) is postponed until section 4.2.4.

We can derive an estimation of $\varepsilon_{\text{esc}}$ that will be used in this model.

For disc-crossing orbits, the stripping is dominated by disc shocks since the duration of the crossing time is much less than the period of the stars within satellite. Due to the tidal force perturbed at this short crossing time, this phenomena also called tidal shock. The numeric value $\varepsilon_{\text{esc}}$ can be estimated using the impulse approximation. This approximation assumes that the internal motion of the stars in the satellite during disc crossing is
sufficiently small such that the vertical offset of stars from the cluster centre (z) is approximately constant (Ostriker et al., 1972). The vertical acceleration of the stars is

\[ \frac{dv_z}{dt} = g(Z) - g(Z_{cl}) \approx z \frac{dg}{dZ}(Z_{cl}), \]  

(4.3)

with \( v_z \) the vertical velocity with respect to cluster centre, \( g \) the vertical gravitational acceleration. This approximation also assumes that vertical velocity of cluster \( (V_{Z,cl}) \) is constant. Integrating equation (4.3) gives

\[ \Delta v_z = \frac{2zg_{\text{max}}}{V_{Z,cl}}, \]

(4.4)

with \( g_{\text{max}} \) the vertical acceleration at maximum Z. Hence (Ostriker et al., 1972),

\[ \Delta v_z \approx \frac{4\pi G\Sigma z}{V_{Z,cl}} \]

(4.5)

for the change in the vertical velocity of a cluster particle due to tides from a single disc crossing. Here, \( \Sigma \) is the surface mass density of the disc. The approximation (4.5) is independent of the vertical disc profile (i.e. the Galactic scale height) and valid as long as the time for crossing the disc is short compared to the cluster-internal orbital time, which is always the case near the outskirts of the cluster.

The change of internal energy is mainly because of the change in kinetic energy \( (\epsilon_{\text{kin}}) \) with respect to the cluster. The change of potential energy with respect to the cluster is negligibly small when using impulse approximation. The resulting change in the particle’s energy with respect to the cluster is then

\[ \Delta \epsilon \approx \Delta \epsilon_{\text{kin}} = v_z \Delta v_z + \frac{1}{2}(\Delta v_z)^2. \]  

(4.6)

The first term on the right hand side of this equation can have either sign, while the second term is always positive, i.e. unbinding. This is reflected in results from numerical integration, as shown in Fig. 4.1, at sufficiently large binding energies \(|\epsilon|\) the energy change is symmetric with respect to \( \Delta \epsilon = 0 \), while for weakly bound particles there is a tail to very large \( \Delta \epsilon > 0 \). For the open cluster, the tidal shock can increase the total internal energy and leads to expansion and disruption of the cluster (Spitzer, 1958). The critical energy for unbinding (the position of the red circle in Fig. 4.1) is in the regime where the contribution of the first term dominates that of the second on the right hand side of equation (4.6). Hence, we can obtain an estimate for \( \epsilon_{\text{esc}} \) from that term alone\(^3\). Assuming a point-mass potential for the outskirts of the cluster,

\[ |\Delta \epsilon| \approx \frac{4\pi G\Sigma zv_z}{V_{Z,cl}} \leq \frac{2^{3/2}\pi G^2\Sigma M_{cl}}{V_{Z,cl} \sqrt{|\epsilon|}}, \]  

(4.7)

\(^3\)We can also use both terms and approximate \( z^2 = (GM_{cl}/2\epsilon)^2 \), when the final estimate for \(|\epsilon_{\text{esc}}|\) is \(|(1 + \sqrt{2})/2|^{\frac{1}{3}} \approx 1.13 \) times larger than (4.8).
Figure 4.2: $|\varepsilon_{esc}|$, the cluster-internal binding energy of a star to the cluster above which no stars are unbound by a single disc crossing as obtained from numerical experiments, plotted versus $G^2 \Sigma M_{cl}/V_{Z,cl}$ for different values of $\Sigma$, $M_{cl}$, and $V_{z,cl}$. The line is not fit, but relation (4.8).
Figure 4.3: The numbers of stars sampled (red) and subsequently lost from the cluster (blue) as functions of their initial cluster-internal energy $\varepsilon$. The vertical lines are at $2\varepsilon_{\text{esc}}$ and $\varepsilon_{\text{esc}} = -31 \text{kpc}^2 \text{Gyr}^{-2}$ (for default simulation).

where we have used $z_{v_z} \lesssim \sqrt{GM_{\text{cl}}a}$ with $a = G M_{\text{cl}}/2|\varepsilon|$ and $M_{\text{cl}}$ denoting the cluster mass.

We can now obtain an estimate for $\varepsilon_{\text{esc}}$ by equating $|\varepsilon| = \max\{\Delta\varepsilon\}$, giving

$$|\varepsilon_{\text{esc}}| \approx \frac{2\pi^{2/3}}{3} \left(\frac{G^2 \Sigma M_{\text{cl}}/V_{Z,\text{cl}}}{\varepsilon_{\text{esc}}}\right)^{2/3}.$$ (4.8)

In Fig. 4.2 I plot the value obtained numerically (as indicated by the red point in Fig. 4.1) for $|\varepsilon_{\text{esc}}|$ against $G^2 \Sigma M_{\text{cl}}/V_{Z,\text{cl}}$ for various values of $\Sigma$, $M_{\text{cl}}$, and $V_{Z,\text{cl}}$. The line in that figure is the relation (4.8). Given the simplistic nature of that estimate, it is remarkably accurate at predicting the largest initial binding energy of any particle that is unbound by the tidal shock for a variety of parameter choices.

4.2.4 The Energy Threshold For Sampling

The crucial parameter $\varepsilon_{\text{cut}} < 0$ is the energy threshold above which the stars are sampled. The default $\varepsilon_{\text{cut}}$ in this model is written in equation (4.2).

To justify this, I run two simulations with shallower ($\varepsilon_{\text{cut}} = \varepsilon_{\text{esc}}$) and deeper ($\varepsilon_{\text{cut}} = 4\varepsilon_{\text{esc}}$) thresholds. In Fig. 4.3, the numbers of star sampled and the numbers of stars that escaped are plotted as a function of their initial energy ($\varepsilon_{\text{esc}}$). After first disc crossing, almost all stars with $\varepsilon > \varepsilon_{\text{esc}}$ escape from cluster. A fraction of stars with $\varepsilon$ deeper than $\varepsilon_{\text{esc}}$ are unable to escape but their internal energies change via random walk and probably will escape after several subsequent disc crossings due to the accumulation of energy changes. According to equation (4.7), the step size in this walk diminishes quickly for more bound stars and, consequently, so does the chance of escape, becoming negligible at $\varepsilon_{\text{cut}} \lesssim 4\varepsilon_{\text{esc}}$. 57
Equation (4.2) is chosen for $\varepsilon_{\text{cut}}$ corresponds to the energy $\varepsilon$ at which the probability of escape over Galactic times is $\sim 50\%$ (see to Fig. 4.3). This choice also as a compromise between the need to avoid the (time consuming) simulation of stars that never escape from the cluster and the requirement to adequately sample the distribution of escapees.

Simulation with $\varepsilon_{\text{cut}} = \varepsilon_{\text{esc}}$ overestimate the relative importance of escape due to just one disc crossing or tidal shock, as opposed to the accumulation of energy changes from many shocks. Such single-shock escapees (which were barely bound to the cluster) are more likely to have large offsets $|\Delta J|$ (see Fig. 4.4). Conversely, the distributions over $\Delta J$ are almost indistinguishable for simulations with $\varepsilon_{\text{cut}} = 2\varepsilon_{\text{esc}}$ or $4\varepsilon_{\text{esc}}$ (see Fig. 3.5 and 4.5).

Fig. 4.6 shows the spatial distribution of tail stars for simulations with $\varepsilon_{\text{cut}} = \varepsilon_{\text{esc}}$ and $4\varepsilon_{\text{esc}}$. Comparing these to the top two panels of Fig. 3.4, we can see that the spatial distribution is smoother along the azimuthal extent and more spread-out in radial and vertical direction for $\varepsilon_{\text{cut}} = \varepsilon_{\text{esc}}$. This can be understood to originate from the excess of stars with large $|\Delta J|$, which results in a wider distribution of directions $\Delta \Omega$ and hence a wider spread in space too.

The spatial distribution of the tail stars for the simulation with $\varepsilon_{\text{cut}} = 4\varepsilon_{\text{esc}}$ (bottom panel of Fig. 4.6) is very similar to that for default choice (Fig. 3.4), except that tails are more sparsely populated. This is simply because most simulated stars in this simulation never escape from the cluster (but require very short time steps to model their cluster-internal orbits).

### 4.3 The Sampling Distribution Function

All simulations presented so far have used the self-consistent Plummer distribution function for sampling the initial stellar positions and velocities with respect to the cluster. If two-body relaxation is the dominant process for re-filling the tidal loss region, a thermal distribution may be a more adequate model. To this end, the procedure of sampling is different. Firstly, since two-body relaxation predominantly occurs in the core (high density region), the initial position $\mathbf{x}$ is drawn from the core of the cluster with probability $\propto [\rho(|\mathbf{x}|) - \rho(s)]$, where $s$ denotes the scale radius and $\rho$ the Plummer density. Secondly, the velocity is drawn from a Maxwellian distribution with one-dimensional velocity dispersion $\sigma^2 = GM_{\text{cl}}/12s$, the velocity dispersion for an isotropic Plummer model at $r = s$.

The main difference from the default approach is the presence of a significant number of stars with $\varepsilon > 0$, i.e. which are already unbound initially and do not require help from the tides to escape the cluster.
Figure 4.4: The distribution of tails stars in the energy and angle offset for simulation with $\varepsilon_{\text{cut}} = \varepsilon_{\text{esc}}$. The actions are estimated via method described in Fig. 3.5.
Figure 4.5: The distribution of tails stars in the energy and angle offset for simulation with $\epsilon_{\text{cut}} = 4\epsilon_{\text{esc}}$. The actions are estimated via method described in Fig. 3.5.
Figure 4.6: As the top two panels of Fig.3.4 except for simulations with $\varepsilon_{\text{cut}} = \varepsilon_{\text{esc}}$ (top) and $\varepsilon_{\text{cut}} = 4\varepsilon_{\text{esc}}$ (bottom) instead of equation (4.2).
In Fig. 4.7, the simulated tidal tails looks similar to the previous approach, except the distribution of tidal tails is wider and smoother at the azimuthal extents. In the vertical projection, the debris spread and quickly fill the ribbons. This is because of the presence of already unbound stars in sampling, which result in a wide distribution in energy and actions offset (see Fig. 4.8).

Fig. 4.8 shows the distribution of tail stars over offsets from the cluster in orbital energy and the combined angular action for a Maxwellian velocity distribution function. The features of this distribution is similar to the distribution in Fig. 3.5 except the width of the orbital energy offset distribution in this simulation is 75% that of the previous model. Some unbound stars which is sampled at a constant rate in the simulation fill both edges of the orbital energy distribution (this can be seen clearly at the bottom panel of Fig. 4.8). Some extra-tidal stars with orbital energy very similar to that of cluster are also found in this simulation.

4.4 The Comparison to Streak-line and Sprinkling Method

For comparison, I also implement the streak-line and particle sprinkling methods. For this purpose, I closely follow Küpper et al. (2010a) and Lane et al. (2012b) and release
Figure 4.8: As Fig. 3.5 except for simulation with Maxwellian velocity distribution function.
particles with constant rate at positions \( \mathbf{X}_* = \mathbf{X}_{cl} \pm r_{tid} \hat{\mathbf{r}}_{cl} \), where the cluster’s instantaneous tidal radius is estimated as

\[
    r_{tid} = \left( \frac{G M_{cl}}{\Omega_{cl}^2 - \partial^2 \Phi_{MW}/\partial r^2} \right)^{1/3},
\]

with the cluster angular velocity \( \Omega_{cl} \equiv \mathbf{X}_{cl} \times \mathbf{V}_{cl}/X_{cl}^2 \) (King, 1962; Heggie and Hut, 2003; Küpper et al., 2010a) and \( \mathbf{X}_{cl} \) is the current galactic position of the cluster. Equation (4.9) is an attempt to generalize the concept of Lagrange points to non-circular orbits (for a recent compilation of similar efforts see van den Bosch et al. (2018)) but is poorly suited for non-spherical potentials and hardly describes appropriately the strong tides experienced during a crossing of the Galactic disc.

The particles’ initial velocity is set to the sum of the radial and angular velocities of the cluster and a random component

\[
    \mathbf{V}_* = \hat{\mathbf{X}}_{cl}(\hat{\mathbf{r}}_{cl} \cdot \mathbf{V}_{cl}) + \Omega_{cl} \times \mathbf{X}_* + \mathbf{V}_{\text{random}}(\sigma),
\]

where \( \mathbf{V}_{\text{random}} \) follows an isotropic normal distribution with one-dimensional dispersion \( \sigma \).

To generate a streakline, I set \( \sigma = 0 \) and increase the tidal radius by 1.2, which gives much better defined streaklines, since it largely avoids the capturing and later release of star particles which otherwise breaks the streak into disconnected pieces.

In this simulation, the cluster starts at the position \( \mathbf{X}_f = (6.75, -0.84, 0.35) \) kpc and velocity \( \mathbf{V}_f = (5.05, -188.96, 136.43) \) kpc/Gyr\(^{-1} \) from 5 Gyr in the past. The orbit is integrated to the current position \( \mathbf{X}_f = (-6.63, -2.31, -2.84) \) kpc and velocity \( \mathbf{V}_f = (-77.99, 164.62, 33.42) \) kpc/Gyr\(^{-1} \). This orbit represents the 47Tuc orbit which has eccentricity 0.12 with peri-center and apo-center are 5.9 kpc and 7.6 kpc, respectively. The inclination angle of this orbit is 23° with maximum vertical distance 3 kpc from the disc mid-plane. The radial period of the cluster orbit is 143 Myr.

In Fig. 4.9, the streak-line model is generated by releasing star particles at two sides with distance 1.2\( r_{tid} \) from cluster’s centre at constant rate of 260 particles per Myr. In the Galactic plane projection (\( R - \Phi \)), the tails form a continuous line which is not far from an

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\(^4\)Kupper, 2010 Küpper et al., 2010a and Lane et al., 2012 Lane et al., 2012b) actually implement a different formula in their computer code, which computes (via finite differences) \( \partial |\mathbf{V}_{\Phi_{MW}}|/\partial r \) instead of \( \partial^2 \Phi_{MW}/\partial r^2 \equiv -\partial(\hat{\mathbf{X}} \cdot \nabla \Phi_{MW})/\partial r \). This results in significant difference to the formula (4.9) only near the disc plane, where Galactic tides are strongest and the acceleration is dominated by its vertical component such that \( |\mathbf{V}_{\Phi_{MW}}| > |\hat{\mathbf{X}} \cdot \mathbf{V}_{\Phi_{MW}}| \). I tried this method (though avoiding finite differences) but found no big differences.

\(^5\)Lane et al., 2012b) state that they release each star with “exactly the angular velocity of the cluster” (plus a random component; as we do in equation (4.10), but in their code they actually implement \( \mathbf{V}_* = \mathbf{V}_{cl}(\mathbf{V}_d \pm \Omega_{cl} r_{tid}) \). The corresponding angular velocity deviates from that of the cluster by the factor \( (R_d \pm |\mathbf{X}_d|)/(R_d \pm r_{tid}) \), which is unity only if the cluster velocity has no radial component such that \( |\mathbf{X}_d| = 1 \), but in general is smaller for the trailing and larger for the leading tail.
Figure 4.9: Tidal tails formed by particle sprinkling model with release position at \( r_{\text{tide}} \) and velocity drawn from Gaussian distribution with \( \sigma = 0.43 \) kpc/Gyr. The red line shows the streak-line model with release position at \( 1.2r_{\text{tide}} \). This plot should be compared to Fig. 3.4.

orbit except the azimuthal extent of one radial period is slightly larger than those of orbit. In the vertical projection (\( Z - \Phi \)), the tails form a chaotic line where the average position of stars within the tails have \( \Omega_Z/\Omega_\Phi > (\Omega_Z/\Omega_\Phi)_{\text{cl}} \) since a small change in energy and action space is translated into a non-linear change in vertical frequency.

The tails formed in the particle sprinkling model are more diffuse than those the streak-line model. This is to be expected because of the added velocity dispersion. Comparing to streak-line model, the centroid of the tails is slightly miss-aligned. This is due to different sprinkling position of both models. Some clumping often happens along the azimuthal extent in this model (clearly seen in the bottom panel of Fig. 4.9) which is completely different from the tails in the restricted \( N \)-body model. The clumping is increasingly rare if the velocity dispersion is higher (see Fig. 4.10).

If we increase the position of sprinkling to \( 1.2r_{\text{tide}} \) in particle sprinkling model, the formed tails are not changed significantly, except the centroid of the star distribution is confined in those streak-line models. Also, similar effect of reduced clumps happens when the velocity dispersion is increased (see Fig. 4.12).
Figure 4.10: This plot is similar to Fig. 4.9 except the velocity is drawn from a Gaussian distribution with $\sigma = 2 \text{kpc/Gyr}$.

Figure 4.11: This plot similar to 4.9 except the particles are released at $1.2r_{\text{tide}}$.  

$r_{\text{sample}} = r_{\text{tide}}$ and $\sigma = 2 \text{kpc/Gyr}$

$r_{\text{sample}} = 1.2r_{\text{tide}}$ and $\sigma = 0.43 \text{kpc/Gyr}$
r_{sample} = 1.2r_{\text{tide}} \text{ and } \sigma = 2 \text{ kpc/Gyr}

Figure 4.12: This plot similar to 4.11 except the velocity drawn from Gaussian distribution with $\sigma = 2 \text{ kpc/Gyr}$.

4.5 Discussion

The restricted $N$–body model can be used to mimic the actual stripping process of tail stars without extravagant computations. The projection of tidal tails to observable sky of simulated tidal tails can be used for comparison to observation data.

A ribbon-like structure formed in the simulation can also seen on the area-preserving sky projection of tidal tails (see Fig. 4.13). In this projection, leading tail is misaligned with the future orbit of the satellite. On the other hand, the trailing tail is almost parallel to the cluster orbit. Some clumps or over-densities are clearly seen in the sky projection of the simulated tidal tails (Fig. 4.13). There are only one clump appears in the leading tails. The density of clumps is decreasing as their distance from center of the cluster increasing. In the leading tail, the streak-line model is not aligned with this model. Only in the trailing tail, the position of the clumps are close to the tails from streak-line model but with different angle. The restricted $N$-body model is incomplete inside the radius such that $\Phi_{\text{cl}}(r) = \varepsilon_{\text{cut}}$. Since the circle of this radius is smaller than the average circle of radius $1.2r_{\text{tide}}$, the restricted $N$–body model is more accurate than streakline model.

The detailed structure of the tidal tails in various parameters of the sprinkling model are different for the restricted $N$–body model. In the sky projection, the formed tails in the sprinkling model are thinner than that of the restricted $N$–body model (see Fig. 4.14 and 4.17).
Figure 4.13: Area-preserving sky projection of tidal tails formed in restricted N–body model for 47 Tuc (omitting 0.05 deg × 0.05 deg pixels with fewer than 5 particles). The blue curve shows the path of the cluster; the red curves are two streaklines with release positions at 1.2$r_{\text{tide}}$, the radius of the red circle. This model is incomplete inside the blue circle, whose radius satisfies $\Phi_{\text{cl}}(r) = \varepsilon_{\text{cut}}$. The green grid represents constant Galactic longitude $\ell$ or latitude $b$. 
Figure 4.14: Similar to Fig. 4.13 except the tidal tails formed by particle sprinkling methods with release position at $r_{\text{tide}}$ and $\sigma = 0.43$ kpc/Gyr.
In this simulation, the CPU time required for the simulations with the streak-line model is three times less than that of the restricted \( N \)–body model. Meanwhile, the CPU time required for the simulation with particle sprinkling model is half than that of the restricted \( N \)–body model. However, the factor of 2 for the increase in CPU time for the restricted \( N \)–body model compared to the particle sprinkling model is not very much since the restricted \( N \)-body model avoids various unjustified ad-hoc assumptions (as discussed in the introduction).

The simulation with a sprinkling model using release position at the \( r_{\text{tide}} \) and a velocity dispersion 0.43 kpc/Gyr show more clumps than the restricted \( N \)–body model (see Fig. 4.9). Moreover, the distribution is not smooth along the tails since the release distance from the cluster is too near resulting the high possibility of recapture of the stars. Increasing the velocity dispersion in the sprinkling method will make the tidal tails become more spread (see Fig. 4.14–4.17). This is due to increasing of the range of energy and actions offset. In the sprinkling model, increasing the distance of the releasing point from cluster centre makes the distribution of the tidal stars smoother. The stars released at \( r_{\text{tide}} \) have a chance to be recaptured again as a consequence of the presence of the cluster potential and make it sparsely distributed.

### 4.6 Summary

In this chapter, I have presented the details of a restricted model of tidal debris. The restricted \( N \)–body model is very useful to simulate the tidal tails of the Galactic satellite. The computational cost of this model are only two times higher than for particle sprinkling and still considerably less than those for a full \( N \)-body simulation. This integration based method complements the action-angle formalism and allows to understand the process of formation of the tidal debris. The striping process is modelled accurately to a certain degree depending on the computational cost adjustment in contrast to the particle-sprinkling model which has ad-hoc parameters. The detailed structure of tidal tails are recovered by using this method.

This method also has its own limitation. It will not work well, if the cluster potential itself is time-variable, for example if the cluster as a whole is strongly affected by tides and most stars in the cluster change their internal energy significantly, not just in the outskirts. In this case, only a complete \( N \)–body model is accurate.
Figure 4.15: Similar to Fig. 4.13 except the tidal tails formed by particle sprinkling methods with release position at $r_{\text{tidal}}$ and $\sigma = 2 \text{ kpc/Gyr.}$
Figure 4.16: Similar to Fig. 4.13 except the tidal tails formed by particle sprinkling methods with release position at $1.2r_{\text{tide}}$ and $\sigma = 0.43$ kpc/Gyr.
Figure 4.17: Similar to Fig. 4.13 except the tidal tails formed by particle sprinkling methods with release position at $1.2r_{\text{tide}}$ and $\sigma = 2 \text{kpc/Gyr}$. 
Chapter 5

Conclusions

5.1 Conclusion

In this thesis, I have studied several problems in astrophysics. Firstly, I studied the optimal time-step criteria for the systems where gauge invariant transformation of the physical quantity is needed. Secondly, I studied modelling the tidal streams from a galactic satellite to mimic the actual stripping of stars from their progenitor without suffering the expensive \( N \)-body model. Lastly, I studied the formation of the tidal ribbons from the simulation based on the restricted \( N \)-body model. The analysis uses instructive sight of action-angle (Hamiltonian) formalism.

In chapter 1, I rewound the beginning of the history of the astronomy from ancient to modern astronomy and the history of the development of law mechanics to explain the celestial problem and the structure of galactic systems. I also reviewed the discovery of the tidal streams and the previous modelling of tidal streams. I gave a short review of the history and development of the \( N \)-body simulations which is the tool for explaining many astrophysical systems.

In chapter 2, I reviewed the integrator, time-step criteria, and block-step scheme for \( N \)-body simulations. Then, I studied the new time-step criterion based on the tidal force. I showed the efficiency of the tidal time-step criterion compared to other time-step criteria. This test included the single orbit integration and \( N \)-body simulation and gave the result that tidal time-step criterion is more efficient than the others time-step criteria.

In chapter 3, I reviewed the action-angle variables concept and Hamiltonian formalism to analyze the structure of the tidal stream. Conventionally, the streams are thin and long structures in six-dimensional phase-space (Tremaine 1999). I presented the results from the simulation based on the restricted \( N \)-body model (given in the next chapter). If the satellite orbits are close to a Galactic disc, the structure of the tidal streams are no longer thin and long but spread in two dimensions forming a band-like structure called a tidalribbon.
ribbon. The formation of such streams due to the unharmonic motion of the stars in the approximately thin disc potential. This is can be analyzed by the action-angle variables and Hamiltonian formalism. The ribbon structure fades out when we increase the scale $b$ of the disc parameter (thicker disc and more spherical). I also showed the effect of the satellite potential (total mass of satellite) on the structure of the tidal tails. The ribbons structure is still forming when we varying the total mass of the progenitor except it has shorter azimuth extents. The observations of such tidal streams can be used to constrain the galactic potential and estimate the vertical profile of the galactic mass distribution.

In chapter 4 I gave the details of a restricted $N$–body model for tidal debris from a galactic satellite. This modelling consists of a model of galactic and satellite potential, a model for the distribution of stars within the satellite, and a prescription for sampling the stars. The crucial parameter in this model is cut-off energy which is the limit of deepest energy at which stars will escape from the progenitor. The main goals of this modelling are to avoid the ad-hoc parameters in the stripping process and to reduce the computational cost by sampling the less bound energy stars within the cluster but still maintaining the actual stripping process. I also compare the result with the streak-line and the particle-sprinkling model (using ad-hoc parameters phase-space point of release and sprinkling rate).

5.2 Future Works

In chapter 2 the tests of the tidal step criterion on the single orbit integration and the $N$–body simulation show the efficiency mainly due to tidal step function follows closely to the dynamical time of systems. The effect of the gauge invariant has not yet tested in these systems. The future works should apply the tidal time-step to systems which the acceleration of the particles in sub-system added by a constant acceleration. The challenge faced in this time-step criterion is the fluctuation of the time-step function due to the graininess of the norm of tidal force, especially in the high-density region. The future works should improve the time-stepping scheme by removing the fluctuation in the time-step function.

The simulation in chapter 3 uses the Allen and Santillan (1991) galactic potential with disc orbit (47Tuc orbit). The effect of various galactic potential (Dehnen and Binney, 1998; Wilkinson and Evans, 1999; Irrgang et al., 2013) should be explored to find the exciting results from the restricted $N$–body model. The various orbits, for example, halo orbit, pole orbit, etc., also should be experimented with to expand upon the results from these models.

This modelling is not only useful for tidal debris but also for the outskirts of the satellite since it properly models the region outside of a radius such that the potential of this radius
is equal to the cut-off energy which is not available in the streak-line and particle sprinkling method. The future works could be the study of this model in the distribution of stars in the outskirts of clusters.
Appendix A

Time-Step Function in the Spherical Potential

The potential in spherical symmetry depends on the radius, \( r \), only. Let define the variable of potential \( \psi(r) = -\Phi(r) \). The acceleration at the radius \( r \) is

\[
    a_i = \frac{\partial \psi}{\partial x_i} = \frac{x_i}{r} \psi' \tag{A.1}
\]

where \( \psi' = \frac{\partial \psi}{\partial r} \) is the norm of the acceleration. The tidal force or the gradient of the acceleration is

\[
    T_{ij} = \frac{\partial a_i}{\partial x_j} = \frac{\delta_{ij} \psi'}{r} - \frac{x_i x_j \psi'}{r^3} + \frac{x_i x_j \psi''}{r^2} \tag{A.2}
\]

where \( \psi'' = \frac{\partial^2 \psi}{\partial r^2} \).

The norm of the tidal force used here is the Frobenius norm or vector norm and defined as

\[
    \| T \| = \sqrt{\sum_{i,j} T_{ij}^2} \tag{A.3}
\]

The norm of tidal force in the spherical model is

\[
    \| T \| = \sqrt{\psi''^2 + 2 \frac{\psi'^2}{r^2}} \tag{A.4}
\]

The time-step function using the tidal force is

\[
    T_{\text{tide}} = \eta \frac{\| T \|}{\sqrt{\| T \|}} = \eta \left( \psi''^2 + 2 \frac{\psi'^2}{r^2} \right)^{1/4} \tag{A.5}
\]

In order to compare the time-step function to dynamical time, we need the analytical expression of mean density. The mean density of the spherical system is

\[
    \bar{\rho} = \frac{M(r)}{(4\pi/3)r^3} \tag{A.6}
\]
where $M(r)$ is the cumulative mass inside the radius $r$. The dynamical time is defined as

$$t_{\text{dyn}} \approx (G\rho)^{-1/2} \quad (A.7)$$

### A.1 Dehnen’s $\gamma$-Model

The observed luminosity of elliptical galaxy and bulge is well represented by the empirical formula of the de Vaucouleurs-1/4 profile. The density profile of this formula can be resembled by the spherical model of the Dehnen’s $\gamma$-model with parameter $\gamma = 3/2$ (Dehnen, 1993). The potential of this model is

$$\psi = \begin{cases} 
\frac{GM}{a} \frac{1}{2-\gamma} \left[ 1 - \left( \frac{r}{r+a} \right)^{2-\gamma} \right] & \text{for } \gamma \neq 2 \\
\frac{GM}{a} \ln \frac{r+a}{r} & \text{for } \gamma = 2 
\end{cases} \quad (A.8)$$

where $M$ is the total mass of model and $a$ is the scale radius. The range of parameter $\gamma$ is $[0, 3)$. The Jaffe model (Jaffe, 1983) and the Hernquist model (Hernquist, 1990) are the special case of parameter $\gamma = 2$ and $\gamma = 1$, respectively. The $\psi'$ and the $\psi''$ in this model are

$$\psi' = GM \frac{r^{1-\gamma}}{(r+a)^{3-\gamma}} \quad (A.9)$$

and

$$\psi'' = GM \frac{(1-\gamma)a - 2r}{r^{\gamma}(r+a)^{4-\gamma}} \quad (A.10)$$

The time-step function for this model is

$$T_{\text{ade}} = \eta \sqrt{\frac{r^{\gamma/2}(r+a)^{2-\gamma/2}}{GM \left[ 6r^2 + 4\gamma r + (3 - 2\gamma + \gamma^2)a^2 \right]^{1/4}}} \quad (A.11)$$

The resulting dynamical time of this model is

$$t_{\text{dyn}} \approx \sqrt{\frac{4\pi}{3GM}} r^{\gamma/2}(r+a)^{(3-\gamma)/2} \quad (A.12)$$

### A.2 Power Law Potential

The observed circular speed of stars and gas in the galaxies are often described by simple power law model and can be written as follow

$$v_{\text{circ}} = v_0 \left( \frac{r}{r_0} \right)^{\beta} \quad (A.13)$$
Here $v_0$ and $r_0$ are the scale of speed and scale of radius. The power law index $\beta$ is restricted to $-1/2 \leq \beta \leq 1$ with $\beta = 1$ is the harmonic potential and $\beta = -1/2$ is the potential generated by point mass at the center.

The $\psi'$ is derived from the circular speed and written as

$$\psi' = \frac{v_0^2}{r_0} \left( \frac{r}{r_0} \right)^{2\beta - 1}$$  \hspace{1cm} (A.14)

The second partial derivative of potential ($\psi''$) is

$$\psi'' = (2\beta - 1) \frac{v_0^2}{r_0^2} \left( \frac{r}{r_0} \right)^{2\beta - 1}$$  \hspace{1cm} (A.15)

Thus, the expression for the time step function is

$$T_{\text{ude}} = \frac{\eta r_0}{v_0} \left( \frac{r}{r_0} \right)^{1-\beta} \left( 4\beta^2 - 4\beta + 3 \right)^{-1/4}$$  \hspace{1cm} (A.16)

We can compare directly to the dynamical time in this system which is expressed as

$$t_{\text{dyn}} \approx \frac{\sqrt{4\pi/3} r_0}{v_0} \left( \frac{r}{r_0} \right)^{1-\beta}$$  \hspace{1cm} (A.17)
Appendix B

The Principle Component Analysis

The dimensionality of the stream can be seen from how they distributed in the frequency space. The thin and long stream shows that the frequency stretch in the one dimension and we can find the axis of this stretching by using principal component analysis (PCA).

PCA is a statistical procedure that uses an linear orthogonal transformation to convert the data of possibly correlated variables into another coordinate system of linearly uncorrelated variables called principal components \( \text{[Pearson, 1901]} \). This transformation is defined in such a way that the first principal component has the largest possible variance and the next principle component axis has the second largest possible variance with maintaining the orthogonality from the preceding component \( \text{[Jolliffe, 2002]} \).

Let the data \( \mathbf{X} \) is a \( m \times n \) matrix. Consider a map of \( \mathbf{X} \) to \( \mathbf{T} \) via linear orthogonal transformation of matrix \( \mathbf{V} \) such that data \( \mathbf{T} \) is the data in a new coordinate system with axes are the principal components.

\[ \mathbf{T} = \mathbf{X} \mathbf{V} \]  \hspace{1cm} (B.1)

In order to find the matrix \( \mathbf{V} \), we can use a procedure called singular value decomposition (SVD). SVD is the factorization of the matrix that generalized the eigen-decomposition of a positive semi-definite normal matrix (e.g. symmetric matrix) to any \( m \times n \) rectangular matrix. The decomposition of SVD is stated as

\[ \mathbf{X} = \mathbf{U} \Sigma \mathbf{V}^T \]  \hspace{1cm} (B.2)

where

1. \( \mathbf{U} \) is a \( m \times m \) unitary matrix and intuitively descibed as the rotation transformation matrix.
2. $\Sigma$ is a $m \times n$ diagonal matrix with non-negative real numbers on the diagonal. The diagonal entries of this matrix, $\sigma_i$, are known as singular value of matrix $X$ and uniquely determined by $X$. These singular values are the major semi-axes of the ellipsoid of data. The matrix $\Sigma$ is described as scaling transformation matrix.

3. $V^T$ is a $n \times n$ unitary matrix, a conjugate transpose of $V$ and also described as the rotation transformation matrix.

Some results from the combination of the matrix $X$ are

$$X^T X = V \left( \Sigma^T \Sigma \right) V^T \quad \text{(B.3a)}$$

$$XX^T = U \left( \Sigma \Sigma^T \right) U^T \quad \text{(B.3b)}$$

From the equation [B.3], $V$ and $U$ form the set of the eigenvector of the matrix $X^T X$ and $XX^T$, with the eigenvalues are the diagonal entries of matrix $\Sigma^T \Sigma$ and $\Sigma \Sigma^T$, respectively. The matrix $V$ can be solved using the equation [B.3a] and it represents the principal component axes. The transformed data can be expressed by

$$T = U \Sigma \quad \text{(B.4)}$$
Appendix C

Sampling the Phase-Space from Distribution Function

In chapter 4 section 4.2.3, I gave a procedure for sampling a star within the satellite. For the sake of clarity, I will state the details of for sampling the phase-space \((x, v)\) of a particle from distribution function \(f_{cl}(\varepsilon)\). The procedure is

1. randomly draw a radius \(r\) inside mass \(m < M\) (total mass of system),
2. calculate the potential \(\Phi\),
3. calculate the escape velocity \(v_{esc} = \sqrt{-2\Phi}\),
4. calculate the distribution function \(f_0 = f_{cl}(\Phi)\),
5. randomly draw a velocity \((v < v_{esc})\),
6. calculate \(\varepsilon = \frac{1}{2}v^2 + \Phi\),
7. repeat steps 5 and 6 until \(\varepsilon < 0\) (this step is to draw a bound star),
8. calculate distribution function \(f = f_{cl}(\varepsilon)\),
9. repeat steps 5-8 until \(f > \alpha f_0\) with \(\alpha\) is a random number between 0 and 1.


